

4° Enc. 7 m (2.

<36603217510011

<36603217510011

Bayer. Staatsbibliothek

ENCYCLOPÆDIA METROPOLITANA;

OR,

UNIVERSAL DICTIONARY OF KNOWLEDGE,

On an Original Plan:

COMPRISING THE TWOFOLD ADVANTAGE OF
A PHILOSOPHICAL AND AN ALPHABETICAL ARRANGEMENT,
WITH APPROPRIATE ENGRAVINGS.

EDITED BY

THE REV. EDWARD SMEDLEY, M.A.,
LATE FELLOW OF SIDNEY COLLEGE, CAMBRIDGE;

THE REV. HUGH JAMES ROSE, B.D.,
PRINCIPAL OF KING'S COLLEGE, LONDON;

AND

THE REV. HENRY JOHN ROSE, B.D.,
LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

VOLUME II.

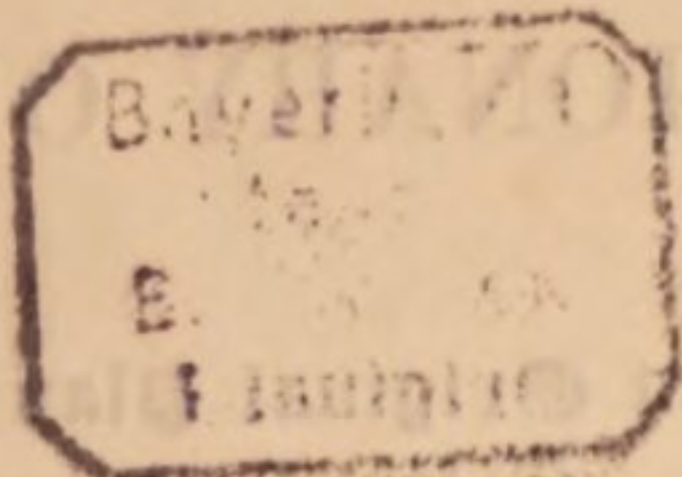
[PURE SCIENCES, VOL. 2.]

LONDON:

B. FELLOWES; F. AND J. RIVINGTON; DUNCAN AND MALCOLM; SUTTABY AND CO.; E. HODGSON; J. DOWDING;
G. LAWFORD; J. M. RICHARDSON; J. BOHN; T. ALLMAN; J. BAIN; S. HODGSON; F. C. WESTLEY; L. A. LEWIS;
T. HODGES; AND H. WASHBOURNE; ALSO J. H. PARKER, AND T. LAYCOCK, OXFORD;
AND J. AND J. J. DEIGHTON, CAMBRIDGE.

1845.

4th Enc. 7^m / 2



CONTAINING THE TWO-FOLD ADVANTAGE OF

A PHILOSOPHICAL AND AN ALPHABETICAL ARRANGEMENT

WITH APPROPRIATE ENGRAVINGS

EDITED BY

THE REV. EDWARD SNEYDER, M.A.

OF THE UNIVERSITY OF LEIDEN

THE REV. HUGH JAMES ROSE, B.D.

OF THE UNIVERSITY OF LEIDEN

THE REV. HENRY JOHN ROSE, B.D.

OF THE UNIVERSITY OF LEIDEN

VOLUME II.

[The Second Part, Vol. 2.]

NO. 2.

LONDON:—PRINTED BY WILLIAM CLOWES AND SONS, STAMFORD STREET.

58 gm



CONTENTS TO VOL. II.

	PAGE
INTEGRAL CALCULUS. By A. LEVY, Esq., M.A., F.G.S., Lecturer on Natural Philosophy and Mathematics in the University of Liege	1
CALCULUS OF VARIATIONS. By Rev. T. G. HALL, M.A., Professor of Mathematics in King's College, London	209
CALCULUS OF FINITE DIFFERENCES. By the same	227
CALCULUS OF FUNCTIONS. By A. DE MORGAN, Esq., Professor of Mathematics in University College, London, and Secretary of the Astronomical Society	305
THEORY OF PROBABILITIES. By the same	393
DEFINITE INTEGRALS. By Rev. H. MOSELEY, M.A., F.R.S., Professor of Natural Philosophy and Astronomy in King's College, London	491
MORAL AND METAPHYSICAL PHILOSOPHY. By Rev. FREDERICK DENISON MAURICE, Chaplain of Guy's Hospital, and Professor of English Literature and Modern History in King's College, London	545
LAW:—	
GENERAL PRINCIPLES OF LAW. By RICHARD JEBB, Esq., Lincoln's Inn	675
LAW OF NATIONS. By ARCHER POLSON, Esq., Lincoln's Inn	716
ROMAN AND CANON LAW. By J. T. GRAVES, M.A., F.R.S., Professor of Jurisprudence in University College, London	735
ENGLISH LAW. By ARCHER POLSON, Esq.	791
THEOLOGY:—	
CHAPTER I. By Rev. GEORGE ELWES CORRIE, B.D., Norrisian Professor of Divinity in the University of Cambridge	858
CHAPTER II. Rev. HENRY JOHN ROSE, B.D., late Fellow of St. John's College, Cambridge	864
CHAPTER III. Professor CORRIE	876
CHAPTER IV. Rev. HENRY JOHN ROSE	883

ENCYCLOPÆDIA METROPOLITANA;

OR,

UNIVERSAL DICTIONARY OF KNOWLEDGE.

First Division.

INTEGRAL CALCULUS.

Integral
Calculus.

Part III.

(110.) WE have supposed hitherto, in our consideration of the Differential and Integral Calculus, that the values of the differential coefficients of the sought functions were explicitly given in terms of the variables. We shall now proceed with the Integration of Differential Equations, and the investigation of the case in which we are to determine the values of the required functions, or rather the relations between the variables and the functions, from given relations, expressed by means of equations, between the variables, the functions themselves, and their differential coefficients. We shall consider this problem as completely resolved when we have reduced them to the determination of the functions by means of the values of their differential coefficients explicitly expressed in terms of the variables. The means of arriving at these reductions have as yet been discovered for a very limited number of cases. Among these, we can only mention here the most general and remarkable.

(111.) The simplest case is that in which it is required to find the relation between x and y , from a given equation between x , y and the first differential coefficient $\frac{dy}{dx}$. Such an equation is called a differential equation of the first order.

The equation between x and y which expresses the required relation is called, as it has been already stated, the integral of the proposed differential equation, and to integrate a differential equation, or to perform its integration, consists in the operation by means of which the integral is obtained.

Let us first suppose that $\frac{dy}{dx}$ enters the given equation algebraically. Its most general form may then be represented by the following equation,

$$\frac{dy^n}{dx^n} + P \frac{dy^{n-1}}{dx^{n-1}} + Q \frac{dy^{n-2}}{dx^{n-2}} + \dots T \frac{dy}{dx} + V = 0 \dots (a)$$

in which $P, Q, \dots, T, V$ may be any functions of x and y .

If $p_1, p_2, p_3, \dots, p_n$ represent the n values of $\frac{dy}{dx}$, derived from this equation, it may be put under the

form
$$\left(\frac{dy}{dx} - p_1\right) \left(\frac{dy}{dx} - p_2\right) \left(\frac{dy}{dx} - p_3\right) \dots \left(\frac{dy}{dx} - p_n\right) = 0.$$

Hence, if $f(x, y) = c$ is the integration of the equation $\frac{dy}{dx} p_1 = 0$, or of any other, formed by making equal to nothing one of the other factors, it will also be the integral of equation (a), since by differentiating $f(x, y) = c$, and substituting the value of $\frac{dy}{dx}$ obtained by this differentiation into the left side of (a) it will

reduce it to nothing. Thus it appears that in the supposition that the decomposition of the left side of (a) into factors may always be effected, the difficulty of integrating (a) is reduced to the integration of an equation of

the form $\frac{dy}{dx} - p_1 = 0$, in which p_1 may be any function of x and y . We shall, therefore, first examine the cases in which this integration may be effected.

(112.) Let $p_1 = -\frac{M}{N}$, the equation $\frac{dy}{dx} - p_1 = 0$ will become $M dx + N dy = 0$; under this form we see that if M and N are such that $\frac{dM}{dy} = \frac{dN}{dx}$, the left side will be the immediate differential of a function of two variables, which function may be found by the rules previously given. Let $f(x, y)$ be that function, then it is obvious that $f(x, y) = c$, c being an arbitrary constant will be the integral of the proposed differential equation; since by differentiating $f(x, y) = c$, we shall obtain a value for $\frac{dy}{dx}$ which will satisfy the equation $M dx + N dy = 0$.

Example. Let $(\sin y + y \cos x) dx + (\sin x + x \cos y) dy = 0$ be the proposed equation. Here $M = \sin y + y \cos x$ and $N = \sin x + x \cos y$, hence

$$\frac{dM}{dy} = \cos y + \cos x = \frac{dN}{dx};$$

therefore the left side may be considered as the immediate differential of a function of two variables; and by applying the known rules, we shall find that function equal to $x \sin y + y \sin x$, therefore the integral of the proposed equation is

$$x \sin y + y \sin x = c.$$

(113.) If M is a function of x alone, and if N contains only y , then it is obvious that the above criterion of integrability is satisfied, since $\frac{dM}{dy}$ and $\frac{dN}{dx}$ are both equal to nothing, and in that case the integral is clearly

$$\int M dx + \int N dy = c.$$

(114.) A very simple transformation will enable us to reduce to the preceding case the equation $M dx + N dy = 0$, if M be a function of y alone, and N a function of x ; or if, more generally, $M = PQ$, and $N = P'Q'$, P and P' being two functions of x , and Q and Q' two functions of y . In the first case, we shall divide both sides of the equation by MN , and in the second case by $Q P'$, and we shall obtain

$$\frac{dx}{N} + \frac{dy}{M} = 0 \quad \text{and} \quad \frac{P dx}{P'} + \frac{Q' dy}{Q} = 0.$$

In both of which the coefficient of dx contains only x , and that of dy only y . When a differential equation is reduced to such a form, the variables are said to be separated, and the integration of it is reduced to the determination of the integrals of the two terms of which it is composed.

(115.) This separation of the variables may easily be effected in some cases.

When M and N are homogeneous functions of x and y , it will be sufficient to assume $y = ux$, or $x = uy$, as may appear most convenient from the form of the equation. By this substitution M and N will become divisible by a power of x or y equal to their common degree, and be reduced to functions of u alone. Let $\phi(u)$ and $\psi(u)$ be those two functions, corresponding to the substitution $y = ux$, then the equation $M dx + N dy = 0$ will be transformed into

$$\phi(u) dx + \psi(u) (u dx + x du) = 0,$$

or

$$\frac{dx}{x} + \frac{\psi(u) du}{\phi(u) + \psi(u) \cdot u} = 0,$$

in which the variables are separated.

Example 1. Let $x dy - y dx = dx \sqrt{x^2 + y^2}$ be the proposed equation. We shall assume $y = ux$, and we shall get

$$x(u dx + x du) - ux dx = x dx \sqrt{1 + u^2}, \quad \text{or} \quad \frac{dx}{x} = \frac{du}{\sqrt{1 + u^2}},$$

integrating we have

$$lx = l(u + \sqrt{1 + u^2}) + c,$$

and putting now $\frac{y}{x}$ for u , we obtain $lx = l\left(\frac{y + \sqrt{x^2 + y^2}}{x}\right) + c$, or putting for the constant le , and reducing,

$$x^e = e^c + 2cy.$$

Example 2. Let the differential equation be $x dy - y dx = y l\left(\frac{y}{x}\right) dx$. We shall make $y = ux$, and we shall find successively

$$x(u dx + x du) - ux dx = u x l u dx,$$

$$\frac{dx}{x} = \frac{du}{u l u}, \quad lx = ll u + e = l(c l u),$$

$$x = c l u, \quad e^x = u^c = \left(\frac{y}{x}\right)^c, \quad \text{or } y^c = x^c e^x.$$

Example 3. Let the proposed equation be

$$(a + b x + c y) dx = (a' + b' x + c' y) dy.$$

In this case the coefficients of dx and dy are not homogeneous functions, but if we assume $x = e + x'$, and $y = f + y'$, e and f being two undeterminate quantities, they may be determined by the condition that in the transformed equation the constant parts of the coefficients dx and dy will vanish, and the transformed equation be, therefore, an homogeneous equation. The values of e and f satisfying these conditions, will be given by the two equations

$$a + b e + c f = 0, \quad a' + b' e + c' f = 0.$$

If $b c' = b' c$, the expressions for e and f become infinite; but in that case $b' x + c' y = \frac{c'}{c} (b x + c y)$, and consequently the equation becomes

$$a dx - a' dy + (b x + c y) \left(dx - \frac{c'}{c} dy \right) = 0.$$

It is sufficient to make in this equation $b x + c y = t$ to separate the variables, for then $dy = \frac{dt - b dx}{c}$, and these values being substituted give

$$dx = \frac{(a' c + c' t) dt}{a c^2 + a' b c + (c^2 + c' b) t}.$$

The integral of the right side of this equation will in general contain logarithms, except when $c^2 + c' b = 0$, in which case we have clearly

$$x = \frac{2 a' c t + c' t^2}{2 (a c^2 + a' b c)} + C.$$

Example 4. In some cases a similar method may be used to reduce to homogeneity equations more complicated than the preceding. Let

$$(6 x^2 - 4 x y + 2 y^2 - 4 x + 4 y + 2) dy + (3 x^2 + 5 y^2 + 10 y + 5) dx = 0$$

be the proposed equation; assume

$$x = m t + n u + p, \quad \text{and } y = m' t + n' u + p',$$

m, n, p, m', n', p' being undeterminate quantities the values of which are to be found by the condition that the coefficients of dt and du in the transformed equation will be homogeneous functions; that is to say, will not contain the first powers of t and u , nor any term constant. There are, therefore, six conditions and six quantities; but the equations which will express these conditions will be of a degree higher than the first, and therefore may lead to imaginary values for m, n, p, m', n', p' . In this example they are real, and we have

$$m = 1, \quad n = 1, \quad p = 1, \quad m' = 1, \quad n' = -1, \quad p' = -1.$$

The corresponding values of x and y being substituted, reduce the proposed equation to

$$(3 t^2 + t u + 5 u^2) dt + (t^2 - 3 t u - u^2) du = 0,$$

which is homogeneous, and in which the variables may readily be separated, as in the preceding examples.

(116.) The differential equations included under the form $\frac{dy}{dx} + P y = Q$, in which P and Q are functions of x alone, and which contains only the first power of the function y , and its first differential coefficient $\frac{dy}{dx}$, is called a linear equation of the first order. It may be easily integrated, by first separating the variables in the following manner.

The method employed is founded upon the propriety of the function e^x having its differential coefficient equal to the function itself multiplied by dx . Hence, $d e^{f P dx} = e^{f P dx} \cdot P dx$; therefore, if in the proposed equation $dy + P y dx = Q dx$, we assume $y = z e^{-f P dx}$, we shall have $dy = -z e^{-f P dx} \cdot P dx + e^{-f P dx} \cdot dz$; and these values being substituted, the first term of the value of dy will destroy the term $P y dx$, and the equation will be reduced to

$$e^{-f P dx} \cdot dz = Q dx, \quad \text{or } dz = e^{f P dx} \cdot Q dx,$$

in which the variables are separated. Integrating, we get

$$z = \int e^{f P dx} \cdot Q dx + c;$$

and, consequently,

$$y = e^{-f P dx} \left(\int e^{f P dx} \cdot Q dx + c \right).$$

Example 1. Let $dy = (a + b x + c y) dx$ be the proposed equation.

In this case

$$P = -c \text{ and } Q = (a + b x),$$

therefore

$$y = e^{cx} \left(\int e^{-cx} (a + b x) dx + c \right),$$

or

$$y = -\frac{a}{c} - \frac{b x}{c} - \frac{b}{c^2} + C e^{cx}, \text{ or } C e^{cx} = b + c(a + b x + c y).$$

Example 2. Let the proposed equation be

$$dy + y \frac{x dx}{1 - x^2} = \frac{a dx}{1 - x^2},$$

here

$$P dx = \frac{x dx}{1 - x^2} \text{ and } \int P dx = -\frac{1}{2} \log(1 - x^2), \text{ or } e^{\int P dx} = \sqrt{1 - x^2},$$

consequently

$$\int e^{\int P dx} \cdot Q dx + = \frac{ax}{\sqrt{1 - x^2}},$$

and

$$y = ax + e \sqrt{1 - x^2}.$$

(117.) The equation $dy + P y dx = Q y^{n+1} dx$, where P and Q are functions of x only, may easily be transformed into a linear equation of the first order; for if we assume $y = z^{-\frac{1}{n}}$, we get $dy = -\frac{1}{n} z^{-\frac{n+1}{n}} dz$, and $y^{n+1} = z^{-\frac{n+1}{n}}$. These values being substituted, the equation becomes

$$-\frac{1}{n} z^{-\frac{n+1}{n}} dz + P z^{-\frac{1}{n}} dx = Q z^{-\frac{n+1}{n}} dz,$$

and by dividing both sides by $z^{-\frac{n+1}{n}}$, and multiplying by $-n$,

$$dz - n P z dx = -n Q dx.$$

(118.) The more general equation,

$$n x^k y^k dy + m x^g y^h dx = l x^e y^f dx,$$

may also be transformed into a linear equation, when $\frac{h-f}{k-f+1} = 1$; and when the same quantity is equal 2, the variables may be separated, if $\frac{e-g}{g-i+1}$ is of the form $\frac{-4i}{2i \pm 1}$. In order to prove this, we shall

assume

$$y = z^{\frac{1}{k-f+1}} \text{ and } x = u^{\frac{1}{g-i+1}}.$$

Substituting for x and y these values, and those of their differentials dx , dy , which are derived from them, we get

$$dz + \frac{(k-f+1)m}{(g-i+1)n} z^{\frac{h-f}{k-f+1}} du = \frac{(k-f+1)l}{(g-i+1)n} u^{\frac{e-g}{g-i+1}} du.$$

Hence, if $\frac{h-f}{k-f+1} = 1$, the equation is linear. Let us now examine the case when the same quantity is equal 2; and, to simplify, assume $\frac{(k-f+1)m}{(g-i+1)n} = b$, $\frac{(k-f+1)l}{(g-i+1)n} = a$, and $\frac{e-g}{g-i+1} = m$, the last equation becomes

$$dz + b z^2 du = a u^m du.$$

First, when $m = 0$, the variables are immediately separated, and we get

$$\frac{dz}{a - b z^2} = du,$$

and

$$u = \frac{1}{2\sqrt{ab}} \log \left(\frac{\sqrt{a} + z\sqrt{b}}{\sqrt{a} - z\sqrt{b}} \right) + c.$$

If m be not equal to nothing, we shall assume $z = A u^p + u^q$, and we shall get by substitution

$$\left. \begin{aligned} &u^q dz + (q u^{q-1} + 2b A u^{p+q} + b u^{2q} z) z du \\ &+ (p A u^{p-1} + b A^2 u^{2p}) du \end{aligned} \right\} = a u^m du.$$

A , p , and q being indeterminate quantities, we may find values for them which will simplify this equation; it will be reduced to three terms if we have

$$p - 1 = 2p, \quad p A + b A^2 = 0, \quad q - 1 = p + q, \quad \text{and } q + 2b A = 0.$$

We have four equations and only three quantities to dispose of, but the first and third do not really differ from each other; they both give $p = -1$, and this being substituted in the second and fourth, we find $A = \frac{1}{b}$, and

$q = -2$. Therefore the supposition $z = \frac{1}{b u} + \frac{y}{u^2}$ will transform the equation $dz + b z^2 du = a u^m du$

into

$$dz + \frac{b z^2}{u^2} du = a u^{m+2} du.$$

This equation is homogeneous when $m = -2$, and if $m + 2 = -2$ or $m = -4$, the variables are separated; in that case the equation becomes

$$dz + (bz^2 - a) \frac{du}{u^2} = 0, \quad \text{or} \quad \frac{dz}{bz^2 - a} + \frac{du}{u^2} = 0.$$

These are not the only values of m for which the equation $dz + \frac{bz^2 du}{u^2} = a u^{m+2} \cdot du$, may be integrated.

Let now $z = \frac{1}{y}$ and $u^{m+2} = x$; substitute for z , dz , u , and du their values, after reduction we shall find

$$dy + \frac{a}{m+3} y^2 dx = \frac{b}{m+3} x^{-\frac{m+4}{m+3}} dx.$$

This equation has the same form as the equation from which it is derived, therefore it will be integrated by the same means when the exponent of x , or $-\frac{m+4}{m+3}$, will be equal to -2 or -4 ; that is to say, when m

will be equal to -2 or $-\frac{8}{3}$. The first of these two values shows nothing new, but the second is a new value

of m , for which the equation may be integrated.

$$dz + bz^2 du = a u^m du$$

We have thus found four different values of m , for which the integration may be performed; they are

$$m = 0, \quad m = -2, \quad m = -4, \quad m = -\frac{8}{3};$$

there are, besides, an infinite number of others, which may be found by following the same process. If in the

equation
$$dy + \frac{a}{m+3} y^2 dx = \frac{b}{m+3} x^{-\frac{m+4}{m+3}} dx,$$

we suppose $\frac{a}{m+3} = b'$, $\frac{b'}{m+3} = a'$, $-\frac{m+4}{m+3} = m'$, and $y = \frac{1}{y'}$ and $x^{m'+3} = x'$,

we shall get clearly
$$dy' + \frac{a'}{m'+3} y'^2 dx' = \frac{b'}{m'+3} x'^{-\frac{m'+4}{m'+3}} dx',$$

and this equation will be readily integrated when $-\frac{m'+4}{m'+3}$ will be equal to -2 or -4 . It is not necessary to consider the first value, since it corresponds to $m' = -2$, and consequently to $m = -2$; the second gives $m' = -\frac{8}{3}$ and $m = -\frac{12}{5}$.

Generalizing what precedes, we easily see, that, provided one of the terms of the series

$$m - \frac{m+4}{m+3}, \quad -\frac{m'+4}{m'+3}, \quad -\frac{m''+4}{m''+3}, \quad \&c.$$

in which each accented letter is the value of the preceding term, shall be equal to -4 , the equation

$$dz + bz^2 du = a u^m du$$

can be integrated.

To find the general term of this series, let us substitute in the third term for m' its value, that is the second term, and the value thus obtained for m'' in the fourth term; we shall have then for the terms of the series

$$m, \quad -\frac{m+4}{m+3}, \quad -\frac{3m+8}{2m+5}, \quad -\frac{5m+12}{3m+7}, \quad \&c.$$

and observing that in these the coefficients of m , in the numerators, form the series of the odd numbers, beginning with 1, and in the denominator the series of the natural numbers beginning with 1; that the constant parts in the numerators form the series of the successive multiple of 4, and in the denominator the series of the odd numbers beginning with 3; we shall conclude by analogy that the term preceded by i terms will be represented by the following expression,

$$-\frac{(2i-1)m+4i}{im+(2i+1)}.$$

We may verify the correctness of this induction by calculating the value of the following term, according to the general law; and observing that it is equal to the preceding expression, when for i , $i+1$ is substituted. We have for the following term

$$-\frac{\frac{(2i-1)m+4i}{im+2i+1} + 4}{\frac{(2i-1)m+4i}{im+2i+1} + 3} = -\frac{(2i+1)m+4(i+1)}{(i+1)m+2i+3}.$$

Hence the general term is expressed by

$$-\frac{(2i-1)m+4i}{im+2i+1};$$

therefore, according to what has been stated above, the proposed differential equation can be integrated. When the quantity $-\frac{(2i-1)m+4i}{im+2i+1}$ is equal to 0, to -2 , or to -4 . The first supposition gives $m = -$

$\frac{4i}{2i-1}$, the second $m = -2$, and the third $m = -\frac{4(i+1)}{2(i+1)}$. The last value is included in the first,

since it only differs from it by the substitution of $(i+1)$ to i . It results, therefore, from this investigation, that the equation

$$dz + bz^2 du = au^m du$$

can be integrated when m is equal to the expression $-\frac{4i}{2i-1}$, in which i may be nothing, infinity, or any integer

whatever. Moreover, if we make $z = \frac{1}{z'}$,

$$u^{m+1} = u', \quad \frac{a}{m+1} = b', \quad \frac{b}{m+1} = a', \quad -\frac{m}{m+1} = m',$$

the equation will be transformed into

$$dz' + b'z'^2 du' = a'x^{m'} du'.$$

This equation being similar to that from which it is derived may be integrated in the same cases, that is, when

$$m' = -\frac{4i}{2i-1}; \quad \text{but } m' = -\frac{m}{m+1},$$

therefore the equation

$$dz + bz^2 du = au^m du$$

may also be integrated when $-\frac{m}{m+1} = -\frac{4i}{2i-1}$, or when $m = \frac{-4i}{2i+1}$.

This equation was first considered by Count James Ricatti, and is known under the name of Ricatti's equation.

(119.) In many cases, besides the few we have examined, the variables may be separated by substitutions for which no general rule can be given. Numerous examples of these transformations are found in the works of Euler; and many more may be easily formed by assuming a differential equation in which the terms are homogeneous, or the variables separated, and substituting in it for one, or both variables, functions of other variables which will lead to a new differential equation, in which the variables may be separated by inverting the formulæ which have served to obtain it.

Let us take, for instance, the differential equation

$$du \left(1 - \frac{b}{1+u^2} \right) - ay dy = 0,$$

when the variables are separated; assuming $u = xy$, we shall have by substituting for u and du their values, and reducing

$$\{ (x - ay) dy + y dx \} (1 + x^2 y^2) = b (y dx + x dy),$$

which equation is not homogeneous, and in which the variables are not separated. It does not at first sight appear what substitution should be made in this equation to separate the variables; but it is obvious, from the manner

in which it has been obtained, that it is sufficient for that purpose to assume $x = \frac{u}{y}$.

We shall take for another example the equation

$$x dx + dy = m y^{\frac{1}{2}} dx.$$

To make it homogeneous, it will be sufficient to make $y = z^2$; it then becomes

$$x dx + 2z dz = m z dx,$$

assuming $z = tx$ to separate the variables, we find

$$dx + 2t(t dx + x dt) = m t dx,$$

or

$$\frac{dx}{x} + \frac{2t dt}{1 - mt + 2t^2} = 0,$$

or

$$\frac{dx}{x} + \frac{\frac{1}{2}(32t - 8m) dt}{(4t - m)^2 + 8 - m^2} + \frac{4m dt}{(4t - m)^2 + 8 - m^2} = 0,$$

Integral
Calculus.

integrating

$$lx + \frac{1}{2} l \{ 4t - m \}^2 + 8 - m^2 \} + \frac{m}{\sqrt{8 - m^2}} \tan^{-1} \frac{4t - m}{\sqrt{8 - m^2}} = lc,$$

and putting back for t its value $\frac{z}{x} = \frac{y^{\frac{1}{2}}}{x}$, and including $-l\sqrt{8}$ in the constant, we get

$$l \frac{c}{\sqrt{(x^2 - mx\sqrt{y+2y})}} = \frac{m}{\sqrt{8 - m^2}} \tan^{-1} \frac{4\sqrt{y - mx}}{x\sqrt{8 - m^2}}.$$

(120.) It is very remarkable that some equations, in which the variables are separated, admit of algebraical integrals, although the integral of each part considered separately is transcendental. The equation $\frac{dx}{x} + \frac{dy}{y} = 0$, for

instance, has for immediate integral $lx + ly = lc$, from which we deduce from the known properties of logarithmic functions $xy = c$, which is an algebraical equation. In this and some similar cases the algebraical integral may be found without the passage through the transcendental functions. If we multiply both sides of the above differential equation by xy , it becomes $ydx + xdy = 0$, which gives immediately $xy = c$ for its integral. When the algebraical integral may thus be obtained, it is of great importance, especially if the properties of the transcendental function are not known, for then it may be instrumental to discover them.

Let us take for example the differential equation

$$\frac{dx}{\sqrt{(a - bx + cx^2)}} + \frac{dy}{\sqrt{(a + by + cy^2)}} = 0.$$

To obtain an algebraical integral, we shall proceed in the following manner. We shall first square both sides, and we shall get

$$(a + bx + cx^2) dy^2 = (a + by + cy^2) dx^2.$$

If we differentiate this equation twice, we find

$$2 \frac{d^2 y}{dx^2} (a + bx + cx^2) + dy dx (b + 2cx) = dx^2 (b + 2cy),$$

and

$$2 \frac{d^2 y}{dx^2} (a + bx + cx^2) + 3 \frac{d^2 y}{dx^2} dx (b + 2cx) = 0,$$

the last equation, may be put under the following form

$$\frac{2 \frac{d^2 y}{dx^2}}{\frac{d^2 y}{dx^2}} + \frac{3(b + 2cx) dx}{a + bx + cx^2} = 0,$$

which being integrated gives

$$2l \frac{d^2 y}{dx^2} + 3l(a + bx + cx^2) = 2la,$$

a being an arbitrary constant, or

$$\left(\frac{d^2 y}{dx^2} \right)^2 (a + bx + cx^2)^3 = a^2;$$

substituting this value of $\frac{d^2 y}{dx^2}$ in the differential equation of the second order, we shall have

$$\frac{dy}{dx} = \frac{(b + 2cy) \sqrt{(a + bx + cx^2)} - 2a}{(b + 2cx) \sqrt{(a + bx + cx^2)}};$$

putting this value of $\frac{dy}{dx}$ in the proposed differential equation, we get for the integral under an algebraical form

$$(b + 2cy) \sqrt{(a + bx + cx^2)} + (b + 2cx) \sqrt{(a + by + cy^2)} = 2a \dots (1).$$

If we designate now by $f(x)$ the integral of $\frac{dx}{\sqrt{(a + bx + cx^2)}}$, we may represent the same integral by

$$f(x) + f(y) = \beta.$$

Hence we infer this property of the transcendant designated by f , that $f(x) + f(y)$ is equal to a constant when ever x and y are connected together by an equation such as (1).

To arrive at a result easy to verify, let us suppose $a = 1$, $b = 0$, $c = -1$, then the differential equation is

$$\frac{dx}{\sqrt{(1 - x^2)}} + \frac{dy}{\sqrt{(1 - y^2)}} = 0.$$

The integral we have obtained is reduced to

$$y \sqrt{(1 - x^2)} + x \sqrt{(1 - y^2)} = a,$$

and if we designate by $f(x)$ and $f(y)$ the integrals of $\frac{dx}{\sqrt{(1 - x^2)}}$ and $\frac{dy}{\sqrt{(1 - y^2)}}$, we shall have between these

transcendants the relation $f(x) + f(y) = \beta$. The constants a and β are not independent of each other, since one of the equations must be a transformation of the other. To find the relation which exists between them, we shall observe that, according to the first equation, when $y = 0$, $x = a$, substituting these values in the second,

Integral
Calculus.

we get

$$f(a) + f(o) = \beta.$$

But if we develop $\frac{dx}{\sqrt{1-x^2}}$ according to the powers of x , and integrate each term of the developement, it

will become evident that $f(o)$ is equal to nothing, therefore $\beta = f(a)$, and consequently, the two equations are

$$y\sqrt{1-x^2} + x\sqrt{1-y^2} = a, \quad f(x) + f(y) = f(a).$$

Let $f(x) = x'$, $f(y) = y'$, then $x = f^{-1}(x')$, $y = f^{-1}(y')$, $x' + y' = f(a)$, and $a = f^{-1}(x' + y')$. These values being substituted in the first equation, we get

$$f^{-1}(y')\sqrt{1-(f^{-1}(x'))^2} + f^{-1}(x')\sqrt{1-(f^{-1}(y'))^2} = f^{-1}(x' + y').$$

From this equation we could deduce all the properties of the transcendents $f^{-1}(x)$ and $f(x)$. Here these properties are already known, since $f(x) = \int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x$, and consequently, $f^{-1}(x) = \sin(x)$; but they are also

derivable from the above equation, which corresponds to the formula

$$\sin y' \cos x' + \sin x' \cos y' = \sin(x' + y').$$

(121.) We shall take for another example the equation

$$\frac{dx}{\sqrt{(a+bx+cx^2+ex^3+fx^4)}} + \frac{dy}{\sqrt{(a+by+cy^2+ey^3+fy^4)}} = 0.$$

To obtain an algebraical integral we shall use a method which may be applied in several analogous cases, and for which we are indebted to Lagrange and Euler.

The process consists in finding another differential equation of the first order, combined with the proposed steps to the required integral.

Let us suppose that x and y are functions of another variable t , and let

$$\frac{dx}{dt} = \sqrt{(a+bx+cx^2+ex^3+fx^4)}, \quad \frac{dy}{dt} = -\sqrt{(a+by+cy^2+ey^3+fy^4)}.$$

If we square both sides of each of these equations, and then differentiate, we find

$$\frac{2d^2x}{dt^2} = b + 2cx + 3ex^2 + 4fx^3, \quad \frac{2d^2y}{dt^2} = b + 2cy + 3ey^2 + 4fy^3,$$

and, by addition, $\left(\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2}\right) = b + c(x+y) + \frac{3}{2}e(x^2+y^2) + 2f(x^3+y^3)$;

making $x+y=p$ and $x-y=q$, we shall get by substitution

$$\frac{d^2p}{dt^2} = b + cp + \frac{3}{4}e(p^2+q^2) + \frac{1}{2}f(p^3+3pq^2).$$

Now from the values of p and q , we get

$$x = \frac{1}{2}(p+q), \quad y = \frac{1}{2}(p-q), \quad \text{and} \quad \frac{dx^2}{dt^2} - \frac{dy^2}{dt^2} = \frac{dpdq}{dt^2};$$

substituting in the last of these equations for $\frac{dx^2}{dt^2}$ and $\frac{dy^2}{dt^2}$ their values, and then for x and y their values in terms of p and q , we shall find

$$\frac{dpdq}{dt^2} = bq + cpq + \frac{1}{4}e(3p^2q+q^3) + \frac{1}{2}f(p^3q+pq^3);$$

this being subtracted from the value of $\frac{d^2p}{dt^2}$ multiplied by q , gives

$$q \frac{d^2p}{dt^2} - \frac{dpdq}{dt^2} = \frac{1}{2}eq^3 + fpq^3,$$

an equation which becomes integrable, when both sides are multiplied by $\frac{2dp}{q^3 dt}$, for it becomes then

$$\frac{2}{q^3} \frac{d^2p}{dt^2} \cdot \frac{dp}{dt} - \frac{2}{q^3} \frac{dp^2}{dt^2} \cdot \frac{dq}{dt} = e \frac{dp}{dt} + 2fp \frac{dp}{dt},$$

and the left side is the differential coefficient of $\frac{dp^2}{q^3 dt^2}$; we shall have, therefore, in integrating

$$\frac{dp^2}{q^3 dt^2} = ep + fp^2 + a^2, \quad \text{or} \quad \frac{dp}{dt} = q\sqrt{(a^2 + ep + fp^2)}.$$

But

$$\frac{dp}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = \sqrt{(a+bx+cx^2+ex^3+fx^4)} - \sqrt{(a+by+cy^2+ey^3+fy^4)};$$

Integral
Calculus.

 substituting this value as well as those of p and q in the preceding equation, we find

$$\sqrt{a+bx+cx^2+ex^3+fx^4} - \sqrt{a+by+cy^2+ey^3+fy^4} = (x-y)\sqrt{a^2+e(x+y)+f(x+y)^2}. \quad (1).$$

If we invert both sides of this equation, and then multiply by

$$(a+bx+cx^2+ex^3+fx^4) - (a+by+cy^2+ey^3+fy^4),$$

we shall obtain

$$\begin{aligned} & \sqrt{a+bx+cx^2+ex^3+fx^4} + \sqrt{a+by+cy^2+ey^3+fy^4} \\ &= \frac{b+c(x+y)+e(x^2+xy+y^2)+f(x^3+x^2y+xy^2+y^3)}{\sqrt{a^2+e(x+y)+f(x+y)^2}}. \end{aligned}$$

This equation being added to the preceding, and then both sides of it raised to the square, we shall have the integral under a rational form,

$$\begin{aligned} & b^2 - 4Aa^2 + (2bc - 4ad - 2ba^2)(x+y) + (c^2 - 4ae - 2ca^2 + a^4)(x^2+y^2) + 2(c^2 - 4ae - bd - a^4)xy \\ & + 2(cd - 2be - da^2)(x+y)xy + (d^2 - 4ea^2)x^2y^2 \} = 0. \end{aligned}$$

This equation is at the same time the integral of

$$\frac{dx}{\sqrt{a+bx+cx^2+ex^3+fx^4}} - \frac{dy}{\sqrt{a+by+cy^2+ey^3+fy^4}} = 0,$$

for the integral of this would first present itself under the form

$$\sqrt{a+bx+cx^2+ex^3+fx^4} + \sqrt{a+by+cy^2+ey^3+fy^4} = (x-y)\sqrt{a^2+d(x+y)+e(x+y)^2};$$

and it is obvious that if we were to deduce from this, by means similar to those used above, the value of the difference of the two radicals, and then by addition the value of the first, we should get precisely the same value we have had before.

 The integral we have obtained becomes $b = a^2$, when the quantity $a+bx+cx^2+ex^3+fx^4$ is a perfect square, that is, when we can suppose

$$a = a'^2, \quad b = 2a'b', \quad c = 2a'c' + b'^2, \quad e = 2b'c', \quad \text{and} \quad f = c'^2.$$

To obtain the integral in that supposition, assume

$$\sqrt{a+bx+cx^2+ex^3+fx^4} = \sqrt{(a'+b'x+c'x^2)^2 + k},$$

and

$$\sqrt{a+by+cy^2+ey^3+fy^4} = \sqrt{(a'+b'y+c'y^2)^2 + k},$$

 and develop the right side of each of these equations, according to the powers of k by the binomial theorem, we shall have

$$\sqrt{(a'+b'x+c'x^2)^2 + k} = (a'+b'x+c'x^2) + \frac{k}{2(a'+b'x+c'x^2)} + \&c.,$$

$$\sqrt{(a'+b'y+c'y^2)^2 + k} = (a'+b'y+c'y^2) + \frac{k}{2(a'+b'y+c'y^2)} + \&c.;$$

 substitute now these values in the general integral, we shall get, after dividing by $(x-y)$,

$$b' + c'(x+y) - \frac{(b' + c'(x+y))k}{2(a'+b'x+c'x^2)(a'+b'y+c'y^2)} = \sqrt{a^2 + 2bc(x+y) + c^2(x+y)^2}.$$

 Raising now both sides to the square, neglecting the square of k , and making $b^2 - a^2 = ka'$, we shall have, after reduction,

$$a'(a'+b'x+c'x^2)(a'+b'y+c'y^2) = \{b' + c'(x+y)\}^2;$$

which, therefore, is the integral under an algebraical and rational form of

$$\frac{dx}{a'+b'x+c'x^2} + \frac{dy}{a'+b'y+c'y^2} = 0.$$

Having thus arrived at the algebraical integral of

$$\frac{dx}{\sqrt{a+bx+cx^2+ex^3+fx^4}} + \frac{dy}{\sqrt{a+by+cy^2+ey^3+fy^4}} = 0,$$

we might, by determining the relation which exists between the constant contained in this integral, and that of the equation

$$\int \frac{dx}{\sqrt{a+bx+cx^2+ex^3+fx^4}} + \int \frac{dy}{\sqrt{a+by+cy^2+ey^3+fy^4}} = \beta,$$

 discover a fundamental property of the transcendant $\int \frac{dx}{\sqrt{a+bx+cx^2+ex^3+fx^4}}$

 (122.) We have already stated that $\int \frac{P dx}{\sqrt{a+bx+cx^2+ex^3+fx^4}}$ may be made to depend upon others,

which by proper transformations may be proved to be all included in the general expression

$$\int \frac{a + b \sin^2 \phi}{c + d \sin^2 \phi} \cdot \frac{d\phi}{\sqrt{(1 - e^2 \sin^2 \phi)}}.$$

Legendre has given to the functions represented by the integrals of this form, the name of elliptical transcendents, because some of them may be expressed by means of arcs of ellipses; he has discovered several remarkable properties of these functions, and deduced from them the means of calculating their approximate values. For all the details of these important investigations, we must refer those who may wish to know them to the work entitled *Exercices de Calcul Intégral*. In the sequel we shall have occasion to return again to this subject; for the present, we shall only determine the integral of the equation

$$\frac{d\phi}{\sqrt{(1 - e^2 \sin^2 \phi)}} + \frac{d\theta}{\sqrt{(1 - e^2 \sin^2 \theta)}} = 0,$$

by the method we have used above, and deduce from it a property of the transcendent $\int \frac{d\phi}{\sqrt{(1 - e^2 \sin^2 \phi)}}$.

We shall assume as before

$$\frac{d\phi}{dt} = \sqrt{(1 - e^2 \sin^2 \phi)}, \quad \frac{d\theta}{dt} = -\sqrt{(1 - e^2 \sin^2 \theta)};$$

squaring and differentiating these equations, we shall find

$$\frac{d^2 \phi}{dt^2} = -e^2 \sin \phi \cos \phi, \quad \frac{d^2 \theta}{dt^2} = -e^2 \sin \theta \cos \theta,$$

and consequently

$$\begin{aligned} \frac{d^2 \phi}{dt^2} + \frac{d^2 \theta}{dt^2} &= -e^2 (\sin \phi \cos \phi + \sin \theta \cos \theta) = -\frac{1}{2} e^2 (\sin 2\phi + \sin 2\theta), \\ \frac{d^2 \phi}{dt^2} - \frac{d^2 \theta}{dt^2} &= -e^2 (\sin \phi \cos \phi - \sin \theta \cos \theta) = -\frac{1}{2} e^2 (\sin 2\phi - \sin 2\theta). \end{aligned}$$

Let $p = \phi + \theta$, and $q = \phi - \theta$, we shall have

$$\phi = \frac{1}{2}(p + q), \quad \theta = \frac{1}{2}(p - q), \quad d^2 \phi + d^2 \theta = d^2 p, \quad d^2 \phi - d^2 \theta = d^2 q;$$

and hence

$$\begin{aligned} \frac{d^2 p}{dt^2} &= -\frac{1}{2} e^2 \{ \sin(p + q) + \sin(p - q) \} = -e^2 \sin p \cos q, \\ \frac{d^2 q}{dt^2} &= -\frac{1}{2} e^2 \{ \sin(p + q) - \sin(p - q) \} = -e^2 \cos p \sin q. \end{aligned}$$

But we have also

$$\begin{aligned} \frac{dp}{dt} \cdot \frac{dq}{dt} &= \frac{d\phi^2}{dt^2} - \frac{d\theta^2}{dt^2} = -e^2 (\sin^2 \phi - \sin^2 \theta) = \frac{1}{2} e^2 (\cos 2\phi - \cos 2\theta) \\ &= \frac{1}{2} e^2 \{ \cos(p + q) - \cos(p - q) \} = -e^2 \sin p \sin q. \end{aligned}$$

If we now divide the values of $\frac{d^2 p}{dt^2}$, $\frac{d^2 q}{dt^2}$, by that of $\frac{dp}{dt} \cdot \frac{dq}{dt}$, we find

$$\frac{d^2 p}{dp dq} = \frac{\cos q}{\sin q}, \text{ or } \frac{d^2 p}{dp} = \frac{dq \cos q}{\sin q}; \text{ and } \frac{d^2 q}{dp dq} = \frac{\cos p}{\sin p}, \text{ or } \frac{d^2 q}{dq} = \frac{\cos p \cdot dp}{\sin p}.$$

These results can be integrated without difficulty, and give

$$l \frac{dp}{dt} = l \sin q + l a, \text{ and } l \frac{dq}{dt} = l \sin p + l a',$$

or, which is the same thing,

$$\frac{dp}{dt} = a \sin q, \text{ and } \frac{dq}{dt} = a' \sin p.$$

The two constants a and a' are not independent of each other, for if we multiply together these two equations we get

$$\frac{dp}{dt} \cdot \frac{dq}{dt} = a a' \sin p \sin q;$$

but we have found

$$\frac{dp}{dt} \cdot \frac{dq}{dt} = -e^2 \sin p \sin q,$$

therefore $a a' = -e^2$, and one of the two equations

$$\frac{dp}{dt} = a \sin q, \quad \frac{dq}{dt} = a' \sin p,$$

III. Integral Calculus.

is an immediate deduction from the other. Let us substitute, now, in these for p and q , $\frac{dp}{dt}$ and $\frac{dq}{dt}$ their values in terms of ϕ and θ , we shall find

$$\begin{aligned}\sqrt{1 - e^2 \sin^2 \phi} - \sqrt{1 - e^2 \sin^2 \theta} &= a \sin(\phi - \theta), \\ \sqrt{1 - e^2 \sin^2 \phi} + \sqrt{1 - e^2 \sin^2 \theta} &= a' \sin(\phi + \theta),\end{aligned}$$

which are the integrals of the proposed equation put under two different forms. We may give it still a simpler form.

For that purpose we shall eliminate dt between the values of $\frac{dp}{dt}$ and $\frac{dq}{dt}$; we find $a \sin q \, dq = a' \sin p \, dp$, and

in integrating $a \cos q = a' \cos p + a''$.

To reduce to one the three constants a, a', a'' , let μ be the value of ϕ corresponding to $\theta = 0$; the two forms of the integral we have first obtained, will give, in that supposition,

$$a = \frac{-1 + \sqrt{1 - e^2 \sin^2 \mu}}{\sin \mu}, \quad a' = \frac{1 + \sqrt{1 - e^2 \sin^2 \mu}}{\sin \mu};$$

these values being substituted in the equation $a \cos q = a' \cos p + a''$, which in the hypothesis of $\theta = 0$, and

$\phi = \mu$, becomes $(a - a') \cos \mu = a''$, gives $a'' = -\frac{2 \cos \mu}{\sin \mu}$. This value, those of a and a' , and of p and q in

terms of ϕ and θ , being substituted in the equation $a \cos q = a' \cos p + a''$, will lead to the integral

$$\cos(\phi - \theta) \{ -1 + \sqrt{1 - e^2 \sin^2 \mu} \} = \cos(\phi + \theta) \{ 1 + \sqrt{1 - e^2 \sin^2 \mu} \} - 2 \cos \mu,$$

which will be successively transformed into

$$\cos(\phi - \theta) + \cos(\phi + \theta) + \{ \cos(\phi + \theta) - \cos(\phi - \theta) \} \sqrt{1 - e^2 \sin^2 \mu} = 2 \cos \mu,$$

or

$$\cos \phi \cos \theta - \sin \phi \sin \theta \sqrt{1 - e^2 \sin^2 \mu} = \cos \mu.$$

Now, if we designate by $f(\phi)$ the transcendant $\int \frac{d\phi}{\sqrt{1 - e^2 \sin^2 \phi}}$, and by $f(\theta)$ the value of $\int \frac{d\theta}{\sqrt{1 - e^2 \sin^2 \theta}}$, the proposed differential equation will immediately give $f(\phi) + f(\theta) = \beta$, and since when $\theta = 0$, $\phi = \mu$, we have $\beta = f(\mu)$, and, therefore, when

$$\cos \phi \cos \theta - \sin \phi \sin \theta \sqrt{1 - e^2 \sin^2 \mu} = \cos \mu,$$

we shall have

$$f(\phi) + f(\theta) = f(\mu).$$

If the proposed differential equation had been

$$\frac{d\phi}{\sqrt{1 - e^2 \sin^2 \phi}} - \frac{d\theta}{\sqrt{1 - e^2 \sin^2 \theta}} = 0,$$

by a similar process, we would have found that when

$$\cos \phi \cos \theta + \sin \phi \sin \theta \sqrt{1 - e^2 \sin^2 \mu} = \cos \mu,$$

we shall have

$$f(\phi) - f(\theta) = f(\mu).$$

It is from the combination of the equations we have just found that Legendre has deduced the properties of the function designated here by $f(\phi)$, to which we have alluded before.

(123.) When the left side of a differential equation $M dx + N dy = 0$, does not satisfy the criterion of integrability, that is, when $\frac{dM}{dy}$ does not equal $\frac{dN}{dx}$, it is possible, in some cases, to discover a factor such, that the left side of the equation, after having been multiplied by it, becomes the immediate differential of a function of two variables.

It is easy to see, that if we admit that every differential equation has an integral, such a factor must always exist. For if $f(x, y) = c$ be the integral of $M dx + N dy = 0$, c being the arbitrary constant, and if $P dx + Q dy = 0$, is the immediate differential of $f(x, y) = c$, the value of $\frac{dy}{dx} = -\frac{P}{Q}$ deduced from it,

must satisfy the equation $M dx + N dy = 0$; therefore $\frac{P}{Q}$ must be equal to $\frac{M}{N}$, and there is, consequently, a

certain factor by which the two terms of the fraction $\frac{M}{N}$, being multiplied, will become equal to $\frac{P}{Q}$, or in other words, a quantity by which the differential equation $M dx + N dy = 0$, being multiplied, will be transformed into the immediate differential $P dx + Q dy = 0$.

Let z be the factor by which it is necessary to multiply $M dx + N dy = 0$, to render the left side of it the immediate differential of a function of two variables. Then $Mz dx + Nz dy$ will satisfy the condition

$$\frac{dMz}{dy} = \frac{dNz}{dx}, \text{ or in developing this equation,}$$

Integral
Calculus.

or

$$M \frac{dz}{dy} + z \frac{dM}{dy} = N \frac{dz}{dx} + z \frac{dN}{dx},$$

$$M \frac{dz}{dy} - N \frac{dz}{dx} + \left(\frac{dM}{dy} - \frac{dN}{dx} \right) z = 0.$$

If in all the cases the value of z could be found by means of this last equation, then every differential equation of the form $M dx + N dy = 0$, could be integrated; or, at least, the difficulty would be reduced to the integration of a function of one variable. But, in general, the difficulty of determining the value of z , is of a higher degree than the integration of the proposed equation, since the equation from which it is to be obtained contains the partial differential coefficients of z with respect to two variables x and y .

The cases in which general rules may be given to find the value of the factor z are precisely those in which the integral of $M dx + N dy = 0$ may be obtained by some of the preceding rules. We shall not, in consequence, enter here into any details of the investigation of this factor. We shall only observe, that when we know a value of the factor z , we may easily deduce from it an infinite number of other factors which will satisfy the same condition.

Let u be the function of x and y , of which $z M dx + z N dy$ is the differential, then

$$du = z M dx + z N dy;$$

and if we multiply both sides of this equation by any function of u , which we shall designate by $\phi(u)$, we shall have

$$\phi u du = z \phi(u) M dx + z \phi(u) N dy,$$

which is an identical equation, the left side of which is obviously an exact differential; and, consequently, we infer that z multiplied by any function of u will be a factor by which $M dx + N dy$ being multiplied will become an immediate differential.

The preceding remark may be used to find the value of the factor when the proposed differential equation may be decomposed into two parts, for each of which a factor can be found. Let

$$p dx + q dy + p' dx + q' dy = 0$$

be the proposed equation, and let us suppose that factors which will make the two parts $p dx + q dy$ and $p' dx + q' dy$, separately, immediate differentials are known; and let them be represented by z and z' , so that $z p dx + z q dy = du$, and $z' p' dx + z' q' dy = du'$; hence $z \phi(u)$ and $z' \phi'(u')$, $\phi(u)$ and $\phi'(u')$ designating any two functions of u and u' , will include all the factors relative to the two parts $p dx + q dy$, and $p' dx + q' dy$; if, therefore, it is possible to determine the forms of the functions ϕ and ϕ' , in such a manner that $z \phi(u) = z' \phi'(u')$, we shall have one of the factors which will render $p dx + q dy + p' dx + q' dy$ an immediate differential.

Let us take, for example, the equation

$$a y dx + \beta x dy = x^m y^n (\gamma y dx + \delta x dy).$$

It is easily seen that the left side becomes an immediate differential when it is multiplied by $\frac{1}{xy}$, and the right side when multiplied by $\frac{1}{x^{m+1} y^{n+1}}$. In this case, therefore, $z = \frac{1}{xy}$ and $z' = \frac{1}{x^{m+1} y^{n+1}}$; and since when the two sides of the equation are multiplied by these factors they become

$$\frac{a dx}{x} + \frac{\beta dy}{y}, \quad \frac{\gamma dx}{x} + \frac{\delta dy}{y},$$

which are respectively equal to

$$d \cdot (a l x + \beta l y) = d \cdot l(x^a y^\beta) \text{ and } d(\gamma l x + \delta l y) = d l(x^\gamma y^\delta),$$

we shall have

$$u = l(x^a y^\beta) \text{ and } u' = l(x^\gamma y^\delta);$$

therefore

$$\frac{1}{xy} \phi(x^a y^\beta) \text{ and } \frac{1}{x^{m+1} y^{n+1}} \phi'(x^\gamma y^\delta)$$

will be the general forms of the factors of the two sides of the equation. Here there is no difficulty in making those two expressions equal; for if we suppose

$$\phi(x^a y^\beta) = x^{ka} y^{k\beta}, \text{ and } \phi'(x^\gamma y^\delta) = x^{k'\gamma} y^{k'\delta},$$

they become respectively

$$x^{ka-1} y^{k\beta-1}, \text{ and } x^{k'\gamma-m-1} y^{k'\delta-n-1};$$

they will, therefore, be equal, if

$$ka - 1 = k'\gamma - m - 1, \text{ and } k\beta - 1 = k'\delta - n - 1,$$

from which equations we derive the values

$$k = \frac{\gamma n - \delta m}{a\delta - \beta\gamma}, \quad k' = \frac{a n - \beta m}{a\delta - \beta\gamma}.$$

The required factor is, consequently,

$$\frac{x^{\frac{(\gamma n - \delta m)a}{a\delta - \beta\gamma} - 1}}{y^{\frac{(\gamma n - \delta m)\beta}{a\delta - \beta\gamma} - 1}};$$

and the proposed equation, being multiplied by that factor and integrated, gives

Integral
Calculus.

$$\frac{a\delta - \beta\gamma}{\gamma n - \delta m} x^{\frac{(\gamma n - \delta m)\alpha}{\alpha\delta - \beta\gamma}} y^{\frac{(\gamma n - \delta m)\beta}{\alpha\delta - \beta\gamma}} = \frac{a\delta - \beta\gamma}{a n - \beta m} x^{\frac{(\alpha n - \beta m)\alpha}{\alpha\delta - \beta\gamma}} y^{\frac{(\alpha n - \beta m)\beta}{\alpha\delta - \beta\gamma}} + c.$$

If $\gamma n = \delta m$, the factor is reduced to $x^{-1} y^{-1} = \frac{1}{xy}$, and in observing that $\delta = \frac{\gamma n}{m}$, the equation becomes, after the multiplication,

$$\frac{a dx}{x} + \frac{\phi dy}{y} = \frac{\gamma}{m} x^{m-1} y^{n-1} (m dx + n dy),$$

the integral of which is

$$l(x^m y^n) = \frac{\gamma x^m y^n}{m} + c.$$

If $a\delta = \beta\gamma$ the values of k and k' become equal to infinity; but in that case the equation is reduced to

$$a y dx + \beta x dy = \frac{\gamma}{a} x^m y^n (a y dx + \beta x dy),$$

or

$$(a y dx + \beta x dy) \left(1 - \frac{\gamma}{a} x^m y^n\right) = 0,$$

which gives two solutions, $x^m y^n = c$ and $1 - \frac{\gamma}{a} x^m y^n = 0$. The first is the true integral, since it contains an arbitrary constant; the second is a particular case of it, when $\beta m = a n$. We shall see in the sequel what is the meaning of such integrals.

(124.) In order to be able to try the preceding method in several cases, where one or both the parts into which the proposed differential equation has been decomposed are homogeneous, we shall investigate the value of the factor z , which makes the differential expression $M dx + N dy$ an exact differential, when M and N are homogeneous functions. In that supposition if we assume $y = ux$, and substitute, M and N will become divisible by x^m , m being the common degree of these quantities, so that the values of M and N may be represented as we have done before by $x^m \phi(u)$ and $x^m \psi(u)$, and we shall have

$$M dx + N dy = (\phi(u) + u \psi(u)) x^m dx + \psi(u) \cdot x^{m+1} du.$$

The right side of this equation becomes clearly an exact differential, if we multiply it by

$$\frac{1}{\{\phi(u) + u \psi(u)\} x^{m+1}},$$

since it then becomes

$$\frac{dx}{x} + \frac{\psi(u) du}{\{\phi(u) + u \psi(u)\}},$$

therefore

$$\frac{M dx + N dy}{\{\phi(u) + u \psi(u)\} x^{m+1}} = \frac{dx}{x} + \frac{\psi(u) du}{\{\phi(u) + u \psi(u)\}} = d \cdot F(x, u),$$

or putting again for u its value $\frac{y}{x}$, and for $\phi(u)$ and $\psi(u)$ their values $\frac{M}{x^m}$, and $\frac{N}{x^m}$, we get

$$\frac{M dx + N dy}{M x + N y} = d F\left(x, \frac{y}{x}\right) = d \cdot f(x, y).$$

Hence it appears that the factor which will render an homogeneous differential expression $M dx + N dy$ an immediate differential is $\frac{1}{M x + N y}$. This factor becomes infinite when $M x + N y = 0$, but then $N = -\frac{M x}{y}$ and $M dx + N dy = \frac{M}{y} (y dx - x dy)$, which becomes the exact differential of $\frac{y}{x}$, when multiplied by $\frac{1}{M y}$.

Let, for example, the equation be

$$y dy - x dx + \frac{b(y dx - x dy)}{x^6} = 0.$$

The left side is separated into two parts, each of which is homogeneous. The factor which will render the first part $y dy - x dx$ an exact differential, is, according to what precedes, $\frac{1}{y^2 - x^2}$; that corresponding to the second part is $-x^4$. After the multiplications, the two parts become respectively the differentials of $\frac{1}{2} \sqrt{y^2 - x^2}$, and of $\frac{y}{x}$; hence the general forms of the factors will be $\phi(y^2 - x^2)$, and $-x^4 \psi\left(\frac{y}{x}\right)$, ϕ and ψ designating any two functions. It remains now to examine if we can find the forms of these functions which

Integral
Calculus.

will make

$$\phi(y^2 - x^2) = -x^4 \psi\left(\frac{y}{x}\right)$$

This will not be very difficult, if we consider that $\phi(y^2 - x^2) = \phi\left\{x^2\left(\frac{y^2}{x^2} - 1\right)\right\}$, which may be supposed equal to $x^4\left(\frac{y^2}{x^2} - 1\right)^2$, and consequently that by taking $\psi\left(\frac{y}{x}\right) = \left(\frac{y^2}{x^2} - 1\right)^2$, we shall have

$$\phi(y^2 - x^2) = x^4 \psi\left(\frac{y}{x}\right).$$

The factor, which will make the left side of the proposed differential equation an immediate differential, is therefore $x^4\left(\frac{y^2}{x^2} - 1\right)^2$, or $(y^2 - x^2)^2$. Multiplying by this quantity, and integrating, we get

$$\frac{(y^2 - x^2)^3}{6} - \frac{b y^5}{5 x^5} + \frac{2 b y^3}{3 x^3} - \frac{b y}{x} = c.$$

(125.) Instead of proposing to determine, *à priori*, the value of the factor for a given differential equation, the problem may be reversed; and, assuming a certain form of the differential equation, and of the factor by which it is to be multiplied, in one or both of which there are undeterminate quantities, it may be proposed to find the relations which ought to subsist between the undeterminate quantities, in order that the assumed factor should render the assumed equation an immediate differential.

Let it be required, for instance, to determine the relations between the functions p , q , m , and N of x , in order that the equation $p y dx + (y + q) dy = 0$, when multiplied by the factor $\frac{1}{y^3 + m y^2 + n y}$, may satisfy the criterion of integrability. The equation given by this condition is

$$\frac{d \frac{p y}{y^3 + m y^2 + n y}}{d y} = \frac{d \frac{y + q}{y^3 + m y^2 + n y}}{d x},$$

which, being developed, gives

$$-2 p y^3 - p m y^2 = (y^3 + m y^2 + n y) \frac{d q}{d x} - (y + q) \left(y^2 \frac{d m}{d x} + y \frac{d n}{d x} \right),$$

$$\text{or} \quad \left(2 p + \frac{d q}{d x} - \frac{d m}{d x} \right) y^3 + \left(p m + m \frac{d q}{d x} - q \frac{d m}{d x} - \frac{d n}{d x} \right) y^2 + \left(n \frac{d q}{d x} - q \frac{d n}{d x} \right) y = 0;$$

and since the equation must be identical, we must have

$$2 p + \frac{d q}{d x} - \frac{d m}{d x} = 0, \quad p m + m \frac{d q}{d x} - q \frac{d m}{d x} - \frac{d n}{d x} = 0, \quad n \frac{d q}{d x} - q \frac{d n}{d x} = 0,$$

three equations, by means of which we may determine the values of three of the four quantities p , q , m , and n .

The last equation may be put under the form $\frac{d n}{n} = \frac{d q}{q}$, and gives by integration $n = a q$, a being an arbitrary constant; substituting this value in the two other equations they become

$$2 p dx + dq - dm = 0, \quad p m dx + m dq - q dm - a dq = 0,$$

and in eliminating p

$$(2 q - m) dm + (2 a - m) dq = 0.$$

If we consider m and q as the two variables of this equation, we may easily find the factor which will make the left side an immediate differential. Let us assume that it contains only m , and let it be represented by $\phi(m)$, then we must have

$$d \frac{\phi(m) \cdot (2 q - m)}{d q} = d \frac{\phi(m) \cdot (2 a - m)}{d m}, \text{ or } 2 \phi(m) = \frac{d \phi(m)}{d m} (2 a - m) - \phi(m),$$

and successively

$$\frac{d \phi(m)}{3 \phi(m)} = \frac{d m}{2 a - m}, \quad \frac{1}{3} \int \phi(m) = \int \frac{1}{(2 a - m)}, \quad \text{and } \phi(m) = \frac{1}{(2 a - m)^3}.$$

Having discovered this factor, we may now proceed to the integration of

$$(2 q - m) q m + (2 a - m) dq = 0.$$

The multiplication will transform the left side into $\frac{2 q - m}{(2 a - m)^3} dm + \frac{dq}{(2 a - m)^2}$; integrating this by the rules relative to functions of two variables, we get after reduction

$$q = m - a + b (2 a - m)^2;$$

but since $2 p dx + dq - dr = 0$, the preceding value of q will give

Integral
Calculus.

$$p = b(2a - m) \frac{dm}{dx};$$

taking, therefore, for m any function of x , we shall get

$$p = b(2a - m) \frac{dm}{dx}, \quad q = m - a + b(2a - m)^2,$$

and

$$n = a(m - a) + ab(2a - m)^2;$$

and the proposed equation will be changed into

$$by(2a - m)dm + (m - a + b(2a - m)^2 + y)dy = 0,$$

and will become an immediate differential by means of the factor

$$\frac{1}{y^3 + my^2 + (m - a + b(2a - m)^2)ay}$$

m being any function of x .

These kind of investigations, which were first undertaken by Euler, however, frequently involve differential equations which cannot be integrated by any known method; and it cannot be said that the cases in which they are successful, are of very great importance or extent.

In order to ensure this method all the success of which it is capable, it would require a very complete classification of the forms of differential equations of the first order, as well as a knowledge of the forms of the multipliers which are suited to each class. The immense extent, however, of this inquiry, and the great difficulties which are met with, even in the simplest cases, seems to preclude all hopes of its proving of much service in the general integration of differential equations of the first order.

(126.) We have now examined all the general cases in which differential equations of the form $Mdx + Ndy = 0$ can be integrated; and we have pointed out the means of determining in particular cases the value of the factor, which will make $Mdx + Ndy$ an immediate differential, in trying various forms of this factor with undeterminate coefficients and functions.

In number (123.) it has already been stated that the integration of differential equations of the first order, in which the differential coefficient $\frac{dy}{dx}$ enters in a degree higher than the first can be, in general reduced to the integration of equations of the form $Mdx + Ndy = 0$. Let us take a simple example of this case, which will give rise to an important remark.

Let $dy^2 - a^2 dx^2 = 0$ be the proposed equation. Decomposed into factors according to (123.), it becomes

$$\left(\frac{dy}{dx} - a\right)\left(\frac{dy}{dx} + a\right) = 0;$$

and it will be satisfied by the integral of either of the equations formed by rendering these factors equal to nothing; that is, by

$$y - ax - c = 0, \text{ or } y + ax - c' = 0.$$

It will also be satisfied by the product of these two equations, or

$$(y - ax - c)(y + ax - c') = 0.$$

This last integral appears more general than the other two, because it contains two arbitrary constants instead of one; but, it may be observed, that when an equation between x and y can be put under the form

$$X_1 \cdot X_2 \cdot X_3 \dots = 0,$$

X_1, X_2 , &c. being functions of x and y , each of which contains a constant c_1, c_2, c_3 , &c. not found in the others, all these constants can always be eliminated between the proposed equation and its differential of the first order. Hence, in the integral of a differential equation of the form

$$\left(\frac{dy}{dx} - p_1\right)\left(\frac{dy}{dx} - p_2\right) \&c. = 0,$$

obtained by multiplying together the integrals X_1, X_2 , &c. of each factor, the number of arbitrary constants it contains cannot be said to indicate a greater degree of generality; and we may, in general, assume any relation we please between these constants; we may, for instance, assume them to be all equal to each other. In that last supposition, the integral in the example we have chosen becomes

$$(y - c)^2 - a^2 x^2 = 0.$$

(127.) The method we have used in the last example, were it not for the difficulty of decomposing the proposed differential equation into factors, would be sufficient to make the integration depend upon that of equations of the form $Mdx + Ndy = 0$. But frequently the decomposition cannot be effected, and it is always best to avoid it when there is any possibility.

This can be done in several cases:

1st. When in the proposed equation

$$\frac{dy^n}{dx^n} + P \frac{dy^{n-1}}{dx^{n-1}} + Q \frac{dy^{n-2}}{dx^{n-2}} + \&c. = 0. \dots (1),$$

Integral
Calculus.

all the coefficients $P, Q, \&c.$ are constant quantities; for if we assume $dy = m dx$, we shall have to determine the value of m the equation Part III.

$$m^n + P m^{n-1} + Q m^{n-2} + \&c. = 0;$$

consequently m is a constant quantity; and since $dy = m dx$, we get by integration $y = mx + c$; eliminating m between this equation and the preceding, we shall have for the integral of the proposed equation

$$\left(\frac{y-c}{x}\right)^n + P \left(\frac{y-c}{x}\right)^{n-1} + Q \left(\frac{y-c}{x}\right)^{n-2} + \&c. = 0.$$

2dly. The resolution of the equation (1) may also be avoided when the coefficients $P, Q, \&c.$ contain only x , and that the value of x may be derived from it in function of $\frac{dy}{dx}$.

In that case make $\frac{dy}{dx} = p$, then by the equation we get $x = \phi(p)$. But $dy = p dx$, and consequently

$$y = px - \int x dp = px - \int \phi(p) dp;$$

eliminating p between this equation and the proposed, we shall obtain the required integral.

Example. Let the proposed equation be $\frac{dy^2}{dx^2} + \frac{x-1}{x} = 0$; making $\frac{dy}{dx} = p$, we get $x = \frac{1}{1+p^2}$, and therefore $y = px - \int \frac{p dp}{1+p^2} = px - \tan^{-1} p + c$; eliminating p , we find for the required integral

$$y = \sqrt{x^2 - x} - \tan^{-1} \sqrt{\frac{x-1}{x}} + c.$$

When the coefficients $P, Q, \&c.$ instead of x contain y , we must assume $\frac{dx}{dy} = p$, from which $x = py - \int y dp$, and then eliminate p as before.

Sometimes, instead of finding the value of x in terms of p , it is simpler to express both quantities in terms of another variable, the final elimination of which leads to the required integral.

Example. Let the equation be $x^3 + \frac{dy^3}{dx^3} = ax \frac{dy}{dx}$, or $x^3 + p^3 = xpx$; the values of x in terms of p , or of p in terms of x , would be rather complicated. Assume $p = ux$, then $x + u^3 x = au$, and hence

$$x = \frac{au}{1+u^3}, \quad p = \frac{au^2}{1+u^3};$$

substituting for p and for dx their values in the expression $y = \int p dx$, we find

$$y = a^2 \int \frac{u^2 du (1-2u^3)}{(1+u^3)^3} = \frac{1}{6} a^2 \frac{2u^3-1}{(1+u^3)^2} + \frac{1}{3} a^2 \frac{1}{1+u^3} + c,$$

and by eliminating u between this value of y and the equation $x = \frac{au}{1+u^3}$, we shall obtain the integral.

3dly. If in equation (1.) $P, Q, \&c.$ be homogeneous functions, we shall suppose $y = ux$, then x will disappear and the equation will contain only u and $\frac{dy}{dx} = p$. But since

$$y = ux, \quad dy = u dx + x du, \quad \text{that is } p dx = u dx + x du, \quad \text{or } \frac{dx}{x} = \frac{du}{p-u},$$

which gives

$$lx = \int \frac{du}{p-u}, \quad \text{or } x = e^{\int \frac{du}{p-u}} \quad \text{and } y = u e^{\int \frac{du}{p-u}};$$

and substituting in each of these, the value of u in terms of p , or of p in terms of u , and eliminating, we shall get the integral.

Example. Let the equation be $y dx - x dy = x \sqrt{dx^2 + dy^2}$, or $y - px = x \sqrt{1+p^2}$, the supposition of $y = ux$ will give $u = p + \sqrt{1+p^2}$; but $\frac{dx}{x} = \frac{du}{p-u}$, or integrating

$$lx = \int \frac{du - dp}{p-u} + \int \frac{dp}{p-u} = -l(p-u) + \int \frac{dp}{p-u};$$

substituting for u its value in p , we find

$$lx = -l \sqrt{1+p^2} + \int \frac{-dp}{\sqrt{1+p^2}} = -l \sqrt{1+p^2} - \log \{p + \sqrt{1+p^2}\} + lc,$$

and consequently

$$x = \frac{e^c}{\sqrt{1+p^2} \{p + \sqrt{1+p^2}\}} = \frac{-e^c}{\sqrt{1+p^2}} \{p - \sqrt{1+p^2}\}, \quad \text{and } y = ux = \frac{e^c}{\sqrt{1+p^2}};$$

Integral
Calculus.

Part III.

 eliminating p between the values of x and y , we get

$$x^2 + y^2 - 2cx = 0$$

for the integral.

(128.) There is still another case, pointed out by Clairaut, which is not of unfrequent occurrence, and which may be integrated without resolving the equation with respect to the differential coefficient. It is when the

 proposed differential equation is of the form $y = F(x, p)$, where $p = \frac{dy}{dx}$. We shall have by differentiating,

$$dy = R dx + S dp, \text{ or since } p dx = dy, p dx = R dx + S dp, \text{ or } (R - p) dx = S dp.$$

 If we could integrate this last equation, by eliminating p between the integral and the proposed equation, we would obtain the required relation between x , y and an arbitrary constant.

 If $y = px + P$ be the proposed equation, P being any function of p alone, this method succeeds. For if we differentiate we get $dy = p dx + d p \left(x + \frac{dP}{dp} \right)$, or simply $d p \left(x + \frac{dP}{dp} \right) = 0$, since $dy = p dx$. This

 last equation may be decomposed into two others, $dp = 0$ and $x + \frac{dP}{dp} = 0$. Eliminating p between the last

 and the equation $y = px + P$, we shall have an equation between x and y which will satisfy the proposed differential equation, but which cannot be considered as the complete integral, since it will not contain an arbitrary constant; it will belong to the class of integral equations which are designated under the name of *particular integrals*; the consideration of these integrals is of great importance, and we shall in the proper place return to that subject. The other equation $dp = 0$ gives $p = c$, or $dy = c dx$, or $y = cx + c'$, with two constants c and c' ; but these constants are not independent of each other; for in making in the proposed equation $p = c$, we get $y = cx + C$, c being what P becomes when in it c is written instead of p ; hence $c' = C$. We shall, therefore, obtain the complete integral of $y = px + P$, in substituting for p an arbitrary constant c .

 Example. Let the proposed equation be $y dx - x dy = a \sqrt{(dx^2 + dy^2)}$. It may be written

$$y = x \frac{dy}{dx} + a \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)} = px + a \sqrt{1 + p^2},$$

and therefore the general or complete integral will be

$$y = cx + a \sqrt{1 + c^2}.$$

 The same method will still be successful when the proposed equation is of the form $y = Px + P'$, P and P' being functions of p alone. If we differentiate, and put for dy its value $p dx$, we get

$$(P - p) dx + \left(x \frac{dP}{dp} + \frac{dP'}{dp} \right) dp = 0.$$

 This is a linear equation of the first order, and may be integrated by the method used in (116.); it will be sufficient to eliminate p between the result and the proposed equation $y = Px + P'$ to get the required integral.

 Example. Let the equation be $y = nx \frac{dy}{dx} + a \sqrt{\left(1 + \frac{dy^2}{dx^2}\right)}$, or $y = np x + a \sqrt{1 + p^2}$.

 Differentiating and putting $p dx$ for dy , we have

$$(n - 1) p dx + \left(nx + \frac{dp}{\sqrt{1 + p^2}} \right) dp = 0, \text{ or } dx + \frac{n}{(n - 1)p} x dp = - \frac{a dp}{(n - 1) \sqrt{1 + p^2}},$$

integrating we get

$$p^{\frac{n}{n-1}} x = - \frac{a}{n-1} \int \frac{p^{\frac{n}{n-1}} dp}{\sqrt{1 + p^2}},$$

 and the result of the elimination of p between this and the proposed equation will be the integral.

(129.) Besides the different general cases we have examined, there are many in which the integration may be effected by means of some transformations, or substitutions, for which no general rule can be given. We shall take two examples of these cases which do not come under any of the rules given in the preceding paragraphs.

 Example 1. Let $\frac{x dy - y dx}{\sqrt{dx^2 + dy^2}} = \sqrt{(x^2 + y^2 - a^2)}$ be the proposed equation. Assume $x = uz$ and $y = u$
 $\sqrt{1 - z^2}$, then $dx = u dz + z du$, $dy = - \frac{uz dz}{\sqrt{1 - z^2}} + du \cdot \sqrt{1 - z^2}$, the equation will become

$$\frac{-u^2 dz}{\sqrt{\{u^2 dz^2 + (1 - z^2) du^2\}}} = \sqrt{(u^2 - a^2)};$$

or in raising both sides to the square, and reducing, we find

$$\frac{dz}{\sqrt{1 - z^2}} = \frac{du \sqrt{u^2 - a^2}}{du},$$

where the variables are separated, and which being integrated gives

$$\sin^{-1} z = -\sin^{-1} \frac{\sqrt{(u^2 - a^2)}}{u} + \frac{\sqrt{(u^2 - a^2)}}{a} + C;$$

and substituting for z and u their values in x and y , which are respectively

$$z = \frac{x}{\sqrt{(x^2 + y^2)}}, \quad u = \sqrt{(x^2 + y^2)},$$

we shall find

$$\sin^{-1} \frac{x}{\sqrt{(x^2 + y^2)}} + \sin^{-1} \frac{\sqrt{(x^2 + y^2 - a^2)}}{\sqrt{(x^2 + y^2)}} = \frac{\sqrt{(x^2 + y^2 - a^2)}}{a} +$$

Example 2. Let the equation be $\frac{dy^a}{dx^k} = Ax^a + By^b$. Assume $x^a u^b = y^b$, we shall get in substituting for y and dy their values

$$x^{\frac{a}{b}} du + \frac{a}{\beta} x^{\frac{a-\beta}{b}} u dx = x^{\frac{a}{k}} dx (A + B u^{\frac{1}{k}}),$$

and if k is supposed equal to $\frac{a\beta}{a-\beta}$ we shall have

$$\beta x^{\frac{a}{b} - \frac{a}{k}} du + a u dx = \beta dx (A + B u^{\frac{1}{k}}),$$

and consequently

$$\frac{dx}{x^{\frac{a}{b} - \frac{a}{k}}} = \frac{\beta du}{\beta (A + B u^{\frac{1}{k}}) - a u},$$

where the variables are separated.

(130.) The difficulty of integration is much greater for differential equations of the second, third, or higher orders; that is, of equations in which are involved the differential coefficients of the second or third or higher orders, than for those of the first order.

The general process consists in deducing from the proposed equation another differential equation of an order less by one; from this another of an order less by two, and so on in succession till an equation between x and y , and without differential coefficients, be obtained. The first of this series of equations is called a *first integral* of the proposed; the second, a *second integral* of the proposed, &c.

(131.) It is manifest, from what has already been stated with respect to the elimination of constants between a proposed equation and its successive differentials, that a differential equation of the n^{th} order may be obtained with n constants less than the primitive equation. Hence the integral of a differential equation of the n^{th} order, ought to be as general as possible, to contain n arbitrary constants. It also results from the manner in which differential equations may be supposed to arise from the elimination of constants between the primitive equation and its differentials, that if the primitive equation contains n constants, there may be n distinct differential equations of the order $n - 1$, each containing only one of the n constants; every one of these equations will be a first integral of the differential equation of the n^{th} order; and therefore to a differential equation of the n^{th} order correspond n first integrals, each with a constant. To obtain the general integral of the proposed equation with n arbitrary constants, when we shall be able to find these n first integrals, it will be sufficient to eliminate between them the $n - 1$ differential coefficients which they contain.

Let, for instance, $y^2 = m(a^2 - x^2)$ be a primitive equation, with two constants m and a . A first differentiation gives $y dy = -mx dx$, where the constant a has disappeared; if we differentiate again, and eliminate m , we find for the differential equation of the second order, without constants,

$$y \frac{dy}{dx} - x \frac{dy^2}{dx^2} - xy \frac{d^2 y}{dx^2} = 0.$$

We can arrive at the same equation in a different manner; for, eliminating m between the proposed equation and the differential of the first order, we shall have

$$xy dx + dy(a^2 - x^2) = 0,$$

and then it will be sufficient to differentiate again, and eliminate a .

In this case, the two equations

$$y dy = -mx dx, \quad xy dx + dy(a^2 - x^2) = 0,$$

are the first integrals of

$$y \frac{dy}{dx} - x \frac{dy^2}{dx^2} - xy \frac{d^2 y}{dx^2} = 0,$$

and by eliminating $\frac{dy}{dx}$ between them, we find the primitive equation

$$y^2 = m(a^2 - x^2).$$

(132.) We shall now examine successively the cases in which the integrations of differential equations of higher orders may be effected.

Integral
Calculus.

Part III.

We need not consider the case in which the equation is of the form $\frac{d^n y}{dx^n} = X$, X being a function of x ; it is

evident that by n successive integrations we shall find the value of y .

It will always be easy to reduce to this case the integration of an equation containing only two successive differential coefficients of y and constants. Let us suppose, for instance, that the proposed equation contains only $\frac{d^3 y}{dx^3}$ and $\frac{d^2 y}{dx^2}$, if we assume $\frac{d^2 y}{dx^2} = q$ we shall have $\frac{d^3 y}{dx^3} = \frac{dq}{dx}$, and the equation will give $\frac{dq}{dx} = Q$,

Q being a function of q ; hence $dx = \frac{dq}{Q}$ and $x = \int \frac{dq}{Q} + c$; but since $\frac{d^2 y}{dx^2} = q$, we have

$$\frac{dy}{dx} = \int q dx + c' = \int \frac{q dq}{Q} + c', \text{ and } y = \int dx \left\{ \int \frac{q dq}{Q} + c' \right\} + c''.$$

The required integral will therefore be the result of the elimination of q between the two equations

$$x = \int \frac{dq}{Q} + c, \text{ and } y = \int dx \left\{ \int \frac{q dq}{Q} + c' \right\} + c'';$$

it will be the general integral, since it will contain three arbitrary constants c, c', c'' .

Example 1. Let $d^2 y = dx \sqrt{(dy^2 + dx^2)}$ be the proposed equation, and let $\frac{dy}{dx} = p$, then $\frac{d^2 y}{dx^2} = \frac{dp}{dx}$, and the equation becomes $\frac{dp}{dx} = \sqrt{(1 + p^2)}$, or $dx = \frac{dp}{\sqrt{(1 + p^2)}}$; hence $dy = p dx = \frac{p dp}{\sqrt{(1 + p^2)}}$; integrating we find

$$x = l(p + \sqrt{(1 + p^2)}) + c, \text{ and } y = \sqrt{(1 + p^2)} + c'.$$

The elimination of p from these equations will give for the required integral

$$x = l[c \{ y - c' + \sqrt{(y - c')^2 - 1} \}].$$

Example 2. Let the proposed equation be

$$a d^2 y d^3 y = dx^2 \sqrt{(dx^4 + d^2 y^2)}.$$

Make $\frac{d^2 y}{dx^2} = q$, and $\frac{d^3 y}{dx^3} = \frac{dq}{dx} = r$, the equation will become $a q r = \sqrt{(1 + q^2)}$, or $a r = \frac{\sqrt{(1 + q^2)}}{q}$, or again $\frac{a dq}{dx} = \frac{\sqrt{(1 + q^2)}}{q}$ and $dx = \frac{a q dq}{\sqrt{(1 + q^2)}}$ hence $x = a \int \frac{q dq}{\sqrt{(1 + q^2)}} + c$. Now

$$p \frac{dy}{dx} = \int q dx = \int \frac{a q^2 dq}{\sqrt{(1 + q^2)}} = \frac{a}{2} \sqrt{(1 + q^2)} - \frac{a}{2} l \{ q + \sqrt{(1 + q^2)} \} + c';$$

$$\text{but } dy = p dx = \frac{a^2 q^2 dq}{2} - \frac{a^2 q dq}{2 \sqrt{(1 + q^2)}} l \{ q + \sqrt{(1 + q^2)} \} + \frac{a c' q dq}{\sqrt{(1 + q^2)}},$$

$$\text{and consequently, } y = \frac{a^2 q^3}{6} - \frac{a^2}{2} \sqrt{(1 + q^2)} l \{ q + \sqrt{(1 + q^2)} \} + \frac{a^2}{2} q + a c' \sqrt{(1 + q^2)} + c''.$$

The elimination of q between the values of x and y will give the required relation between these two variables.

(133.) Let us next consider the case where the differential equation contains only two differential coefficients, the order of one of which is greater by two than the order of the other.

Let us suppose, for instance, that the differential equation contains only $\frac{d^4 y}{dx^4}$ and $\frac{d^2 y}{dx^2}$, and that the value of the first can be expressed in terms of the second.

Let $\frac{d^2 y}{dx^2} = q$, then $\frac{d^3 y}{dx^3} = \frac{dq}{dx}$ and $\frac{d^4 y}{dx^4} = \frac{d^2 q}{dx^2}$. The proposed equation will therefore assume the form $\frac{d^2 q}{dx^2} = Q$, when Q designates a function of q . If we multiply both sides of this equation by dq , we get

$$\frac{dq}{dx} \cdot \frac{d^2 q}{dx^2} = Q dq, \text{ and integrating } \frac{1}{2} \frac{d^3 q}{dx^3} = \int Q dq + c, \text{ or } \frac{d^3 q}{dx^3} = \sqrt{(2 \int Q dq + c)};$$

$$\text{or again, } dx = \frac{dq}{\sqrt{(2 \int Q dq + c)}}, \text{ and by integration } x = \int \frac{dq}{\sqrt{(2 \int Q dq + c)}} + c'.$$

But since $\frac{d^2 y}{dx^2} = q$, we have $\frac{dy}{dx} = \int q dx$, or substituting for dx its value

$$\frac{dy}{dx} = \int \frac{q dq}{\sqrt{(2 \int Q dq + c)}} + c'', \text{ and } dy = dx \left(\int \frac{q dq}{\sqrt{(2 \int Q dq + c)}} + c'' \right).$$

Integral
Calculus.

or replacing again dx by its value and integrating $y = \int \frac{dq}{\sqrt{(2 \int Q dq + c)}} \left\{ \int \frac{q dq}{\sqrt{(2 \int Q dq + c)}} + c'' \right\} + c'''$. Part III.

Eliminating then q between this value of y and the value of x , the required integral is obtained with four arbitrary constants.

Example 1. Let $a^2 d^2 y + y dx^2 = 0$ be the proposed equation. We have first $\frac{d^2 y}{dx^2} = -\frac{y}{a^2}$; multiplying by dy and integrating we find

$$\frac{dy^2}{dx^2} = -\frac{y^2}{a^2} + c^2, \text{ or } dx = \frac{a dy}{\sqrt{(a^2 c^2 - y^2)}}, \text{ and } x = a \sin^{-1} \frac{y}{ac} + c'$$

a value from which it is easy to derive

$$y = c \sin \frac{x}{a} + c' \cos \frac{x}{a},$$

c and c' being two new arbitrary constants.

Example 2. Let $\frac{a^2 d^4 y}{dx^4} = \frac{d^2 y}{dx^2}$ be the proposed equation. Making $\frac{d^2 y}{dx^2} = q$, we shall have successively $\frac{a^2 d^2 q}{dx^2} = q$, $\frac{a^2 dq}{dx} \frac{dq}{dx} = q dq$, $\frac{dq^2}{dx^2} = \frac{q^2 + c}{a^2}$, $dx = \frac{a dq}{\sqrt{(q^2 + c)}}$, $x = a l \left\{ \frac{q + \sqrt{(q^2 + c)}}{c'} \right\}$. Now $\frac{d^2 y}{dx^2} = q$, hence $\frac{dy}{dx} = \int q dx = \int \frac{a q dq}{\sqrt{(q^2 + c)}} = a \sqrt{(q^2 + c)} + c''$, $dy = \{a \sqrt{(q^2 + c)} + c''\} dx = a^2 dq + \frac{a c'' dq}{\sqrt{(q^2 + c)}}$,

$$y = a^2 q + a c' l \{q + \sqrt{(q^2 + c)}\} + c'''. \text{ The value of } x \text{ gives } c' e^{\frac{x}{a}} = q + \sqrt{(q^2 + c)},$$

$$\text{and } \frac{1}{c' e^{\frac{x}{a}}} = \frac{e^{-\frac{x}{a}}}{c'} = \frac{1}{q + \sqrt{(q^2 + c)}} = \frac{q - \sqrt{(q^2 + c)}}{-c}, \text{ or } -\frac{c e^{-\frac{x}{a}}}{c'} = q - \sqrt{(q^2 + c)};$$

by addition

$$q = \frac{c'}{2} e^{\frac{x}{a}} - \frac{c}{2c'} e^{-\frac{x}{a}};$$

this being substituted in the value of y , as well as $l \{q + \sqrt{(q^2 + c)}\} = \frac{x}{a} + l c'$, we get

$$y = \frac{a^2 c'}{2} e^{\frac{x}{a}} - \frac{a^2 c}{2c'} e^{-\frac{x}{a}} + c'' + c' l c' + c''' = c e^{\frac{x}{a}} + c' e^{-\frac{x}{a}} + c' x + c''''$$

(134.) When a differential equation of the n^{th} order contains only one of the variables, x and y , it can easily be reduced to another of the order $n - 1$.

If the variable which enters the equation is x , it is sufficient to make $\frac{dy}{dx} = p$, for then $\frac{d^2 y}{dx^2} = \frac{d^{n-1} p}{dx^{n-1}}$, and the proposed equation is transformed into an equation of the order $n - 1$, between the variables x and p . If it can be integrated and resolved with respect to p , the equation $y = \int p dx$ will give the relation between x and y ; if it can be resolved with respect to x , then $y = \int p dx$ will give the value of y in terms of p , and by eliminating this last quantity between the values of x and y , the integral will be obtained.

If instead of x it was y that entered the equation, we should in general substitute for the values of $\frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, &c. their values in terms of $\frac{dx}{dy}$, $\frac{d^2 x}{dy^2}$, &c. which have been given in the Differential Calculus, and then proceed as in the preceding case. We could obtain the integral in a different manner, if the proposed equation was only of the second order. Making $p = \frac{dy}{dx}$, we would have $\frac{d^2 y}{dx^2} = \frac{dp}{dx} = \frac{p dp}{dy}$, and in substituting these values the transformed equation would only contain p , dp , y and dy , if it could be integrated, and if we could derive from the integral the value of y in terms of p , or the value of p in terms of y , the values of x would be given by the formulæ

$$x + \frac{y}{p} = \int \frac{y dp}{p^2}, \quad x = \int \frac{dy}{p}.$$

We shall now apply these general notions to a few examples.

Example 1. Let the proposed equation be $\frac{(dx^2 + dy^2)^{\frac{3}{2}}}{dx dx^2 y} = X$, X being a function of X alone. Putting p

Integral Calculus. instead of $\frac{dy}{dx}$, it becomes $\frac{(1+p^2)^{\frac{3}{2}}}{dp} = X$, or $\frac{dx}{X} = \frac{dp}{(1+p^2)^{\frac{3}{2}}}$, and, consequently, $\int \frac{dx}{X} + c = \frac{p}{\sqrt{1+p^2}}$.

Let $\int \frac{dx}{X} + c = V$, then $p = \frac{V}{\sqrt{1-V^2}}$, and $y = \int p dx = \int \frac{V dx}{\sqrt{1-V^2}} + c$.

Example 2. Let the equation be $dx(dx^2 + dy^2) + xdyd^2y = ad^2y\sqrt{dx^2 + dy^2}$,
or $dx(1+p^2) + xpd p = ad p \sqrt{1+p^2}$.

If we divide both sides by $\sqrt{1+p^2}$, we find $dx\sqrt{1+p^2} + \frac{xpd p}{\sqrt{1+p^2}} = ad p$,

the left side is clearly the differential of $x\sqrt{1+p^2}$, and therefore $x\sqrt{1+p^2} = ap + c$,

$$x = \frac{ap+c}{\sqrt{1+p^2}}, \quad dx = \frac{adp(1+p^2) - (ap+c)pdp}{(1+p^2)^{\frac{3}{2}}} = \frac{(a-cp)dp}{(1+p^2)^{\frac{3}{2}}};$$

$$\text{but } dy = \int p dx, \text{ therefore } y = \int \frac{(ap - cp^2)dp}{(1+p^2)^{\frac{3}{2}}} = \frac{cp-a}{\sqrt{1+p^2}} - cl \left\{ \frac{p + \sqrt{1+p^2}}{c'} \right\},$$

and substituting for p its value derived from x , we find

$$y = \sqrt{a^2 + c^2 - x^2} - cl \left\{ \frac{c + \sqrt{a^2 + c^2 - x^2}}{c'(x-a)} \right\}.$$

Example 3. Let the equation be $(a^2 dy^2 + x^2 dx^2) d^2y = nx dx^3 dy$, or $(a^2 p^2 + x^2) dp = np x dx$. This equation is homogeneous, and if we assume $x = pu$, it becomes $\frac{dp}{p} = \frac{n u du}{a^2 + (1-n)u^2}$, and by integration

$$lp = \frac{n}{2(1-n)} l(a^2 + (1-n)u^2) + lc, \text{ hence } p = c \{ a^2 + (1-n)u^2 \}^{\frac{n}{2(1-n)}},$$

$$\text{or } x = pu = cu \{ a^2 + (1-n)u^2 \}^{\frac{n}{2(1-n)}}, \text{ but } y = \int x dp = c^2 u \{ a^2 + (1-n)u^2 \}^{\frac{n}{2(1-n)}} - nc \int u^2 du \{ a^2 + (1-n)u^2 \}^{\frac{2n-1}{2(1-n)}} + c'.$$

$$\text{If } n = 1, \text{ we have } \frac{dp}{p} = \frac{u du}{a^2}, \quad lp = \frac{u^2}{2a^2} + lc, \quad u = a \left(2l \frac{p}{c} \right)^{\frac{1}{2}},$$

$$\text{and } y = ap \sqrt{2l \frac{p}{c}}, \quad y = ap^2 \sqrt{2l \frac{p}{c}} - a \int p dp \sqrt{2l \frac{p}{c}} + c'.$$

Example 4. Let y be now the variable contained in the proposed equation, and let that equation be only of the second order, $ab d^2y = dx \sqrt{y^2 dx^2 + a^2 dy^2}$. In this case we substitute p for $\frac{dy}{dx}$, and $p \frac{dp}{dy}$ for $\frac{d^2y}{dx^2}$, the equation then becomes $\frac{ab p dp}{dy} = \sqrt{y^2 + a^2 p^2}$, an equation which is homogeneous with respect to q and p , the variables will be separated by assuming $y = pu$; this substitution gives after reduction

$$\frac{dy}{y} = \frac{ab du}{abu - u^2 \sqrt{a^2 + u^2}}.$$

To integrate the right side, make $\sqrt{a^2 + u^2} = su$, we shall have

$$u^2 = \frac{a^2}{s^2 - 1}, \quad \frac{du}{u} = \frac{-s ds}{s^2 - 1}, \quad \frac{dy}{y} = \frac{-bs ds}{bs^2 - as - b} = \frac{-s ds}{s^2 - 2ns - 1},$$

if $\frac{a}{b} = 2n$ the right side of this last equation may easily be decomposed into two fractions in observing that

$$s^2 - 2ns - 1 = (s - n - \sqrt{n^2 + 1})(s - n + \sqrt{n^2 + 1}),$$

$$\text{we shall get } \frac{2dy \sqrt{n^2 + 1}}{y} = \frac{-ds(n + \sqrt{n^2 + 1})}{s - n - \sqrt{n^2 + 1}} + \frac{ds(n - \sqrt{n^2 + 1})}{s - n + \sqrt{n^2 + 1}},$$

$$\text{and by integrating } y^{2\sqrt{n^2+1}} = \frac{c(s - n + \sqrt{n^2 + 1})^{n - \sqrt{n^2+1}}}{(s - n - \sqrt{n^2 + 1})^{n + \sqrt{n^2+1}}}.$$

But we have supposed $y = pu$, hence $dx = \frac{u dy}{y}$, and therefore by substituting for u and $\frac{dy}{y}$ their values in s , we

Integral
Calculus.

shall find $dx = \frac{-asds}{(s^2 - 2ns - 1)\sqrt{(s^2 - 1)}}$; integrating this formula, and eliminating s , we shall obtain the

relation between s and y .

(135.) Among the differential equations which contain both variables x and y , those which may be considered as homogeneous not only with respect to x and y , but also with respect to dx , dy , d^2y , &c., can be easily transformed into others which contain only one of the primitive variables. In this supposition the dimension of $\frac{dy}{dx}$ is 0, that of $\frac{d^2y}{dx^2}$ is -1, &c. Two examples will be sufficient to show what are the substitutions and transformations to be used in this case.

Example 1. Let the equation be $x^2 d^2y = x dx dy + 3y dx^2$, make $y = ux$, $\frac{dy}{dx} = p$, and $\frac{d^2y}{dx^2} = q = \frac{v}{x}$.

All the terms of the transformed will be divisible by the same power of x , and we shall find $xv = px + 3ux$,

or $v = p + 3u$; but since $\frac{v}{x} = q$, $\frac{dp}{dx} = \frac{v}{x}$, or $\frac{dx}{x} = \frac{dp}{v} = \frac{dp}{p + 3u}$, we have also $p dx = d(ux) = u dx + x du$,

and, consequently, $\frac{dx}{x} = \frac{du}{p - u}$, therefore $\frac{dp}{p + 3u} = \frac{du}{p - u}$, or $(p + 3u) du = (p - u) dp$. This is a differential equation of the first order; it is homogeneous and can easily be integrated; the integral is immediately obtained, if we observe, that by transposing the terms the equation becomes $p du + u dp = p dp - 3u du$,

each side of which is an exact differential; integrating we get $pu = \frac{p^2}{2} - \frac{3u^2}{2} + c$, from which $p = u + \sqrt{(c + 4u^2)}$,

putting c for $-2c$. Substituting this value of p in the equation $\frac{dx}{x} = \frac{du}{p - u}$, we shall find

$$\int \frac{dx}{x} = \int \frac{du}{\sqrt{(c + 4u^2)}} = \frac{1}{2} \int \frac{2u + \sqrt{(c + 4u^2)}}{c'} \quad \text{or} \quad \int \frac{dx}{x} = \frac{1}{2} \int \frac{u + \sqrt{(c + u^2)}}{c'} \quad \text{or} \quad \int \frac{dx}{x} = \frac{1}{2} \int \frac{u + \sqrt{(c + u^2)}}{c'}$$

putting $4c$ for c . Hence $x^2 = \frac{u + \sqrt{(c + u^2)}}{c'} = \frac{y + \sqrt{(cx^2 + y^2)}}{c'x}$, or $cx^2y = x^4 + c'$, by changing the constants.

Example 2. Let the equation be

$$(dx^2 + dy^2)^{\frac{3}{2}} = n dx d^2y \sqrt{(x^2 + y^2)}$$

and let

$$y = ux, \quad \frac{dy}{dx} = p, \quad \frac{d^2y}{dx^2} = q = \frac{v}{x},$$

the equation becomes

$$(1 + p^2)^{\frac{3}{2}} = nv \sqrt{(1 + u^2)}, \quad \text{or} \quad v = \frac{(1 + p^2)^{\frac{3}{2}}}{n \sqrt{(1 + u^2)}};$$

but we have as before $\frac{dp}{v} = \frac{du}{p - u}$; substituting the value of v , we shall have a differential equation of the first order in p and u ,

$$n(p - u) dp \sqrt{(1 + u^2)} = (1 + p^2)^{\frac{3}{2}} du.$$

In order to integrate this equation we shall make $p = \tan \theta$, $u = \tan \phi$, then $dp = \frac{d\theta}{\cos^2 \theta}$, $du = \frac{d\phi}{\cos^2 \phi}$,

$\sqrt{(1 + u^2)} = \frac{1}{\cos \phi}$, $(1 + p^2)^{\frac{3}{2}} = \frac{1}{\cos^3 \theta}$, $p - u = \tan \theta - \tan \phi = \frac{\sin(\theta - \phi)}{\cos \theta \cos \phi}$; these values being substituted give $n \sin(\theta - \phi) d\theta = d\phi$. Let $\theta = \phi + \psi$, and eliminate ϕ , we shall get $d\theta = \frac{d\psi}{1 - n \sin \psi}$. At the

same time we have

$$\frac{dx}{x} = \frac{du}{p - u} = \frac{d\phi \cos \theta}{\cos \phi \sin \psi} = \frac{d\phi \cos \psi}{\sin \psi} - \frac{d\phi \sin \phi}{\cos \phi},$$

or by putting $d\phi$ its value

$$d\theta - d\psi = \frac{d\psi}{1 - n \sin \psi} - d\psi = \frac{n \sin \psi d\psi}{1 - n \sin \psi},$$

we get

$$\frac{dx}{x} = \frac{n \cos \psi d\psi}{1 - n \sin \psi} - \frac{d\phi \sin \phi}{\cos \phi},$$

and in integrating

$$x = \frac{c \cos \phi}{1 - n \sin \psi}, \quad \text{and} \quad y = ux = \frac{c \sin \phi}{1 - n \sin \psi}.$$

These values joined to

$$\phi = \int \frac{n \sin \psi d\psi}{1 - n \sin \psi}$$

will give by the elimination of ψ the integral of the proposed equation.

Integral
Calculus.

(136.) The same kind of transformation will succeed if the proposed equation should become homogeneous by considering $y \frac{dy}{dx}$, $\frac{d^2 y}{dx^2}$, &c. and its differentials as being respectively of n , $n-1$, $n-2$, &c. dimensions. Part III.

In that case, supposing the proposed equation of the second order, we shall make

$$y = x^n u, \quad \frac{dy}{dx} = p = t x^{n-1}, \quad \text{and} \quad \frac{d^2 y}{dx^2} = q = v x^{n-2}.$$

These values being substituted will give an equation in which x will disappear, and by means of which we shall be able to obtain the value of v in function of t and u . Then we have

$$dy = x^n du + n x^{n-1} u dx = x^{n-1} t dx, \\ d^2 y = (n-1) x^{n-2} t dx^2 + x^{n-1} dt dx = v x^{n-2} dx^2.$$

consequently

The first of these equations gives

$$\frac{dx}{x} = \frac{du}{t - nu},$$

and the second

$$\frac{dx}{x} = \frac{dt}{v - (n-1)t};$$

and, therefore,

$$du(v - (n-1)t) = dt(t - nu);$$

and by putting for v its value in terms of t and u , we get a differential equation of the first order, which being integrated will easily lead to the integral of the proposed.

Example. Let the proposed equation be

$$x^4 d^2 y = x^3 dx dy + 2xy dx dy - 4y^2 dx^2, \quad \text{or} \quad x^4 q = x^3 p + 2xy p - 4y^2.$$

We shall make

$$y = x^2 u, \quad p = tx, \quad \text{and} \quad q = v,$$

the equation becomes

$$v = t + 2ut - 4u^2;$$

then we have

$$dy = x^2 du + 2ux dx = p dx = tx dx,$$

hence

$$\frac{dx}{x} = \frac{du}{t - 2u}.$$

We have also

$$dp = t dx + x dt = q dx = v dx \quad \text{and} \quad \frac{dx}{x} = \frac{dt}{v - t} = \frac{dt}{2ut - 4u^2},$$

therefore

$$\frac{du}{t - 2u} = \frac{dt}{2ut - 4u^2},$$

or

$$2u du(t - 2u) = dt(t - 2u).$$

This equation is satisfied by making $2u du = dt$, which gives $t = u^2 + c$, or by $t = 2u$. The first value of t being substituted in the expression of $\frac{dx}{x}$, we get

$$\frac{dx}{x} = \frac{du}{u^2 - 2u + c}.$$

The integral of the right side of this equation will assume three different forms according as c is less, equal, or greater than one. Let, first, $c = 1$; then

$$\frac{dx}{x} = \frac{du}{(u-1)^2} \quad \text{and} \quad lx = \frac{1}{1-u} + c', \quad \text{or} \quad lx = \frac{x^2}{x^2 - y} + c.$$

Let, secondly, $c = 1 - e^2$; then $\frac{dx}{x} = \frac{du}{(u-1)^2 - e^2}$ and $lx = -\frac{1}{2e} l\left(\frac{e+u-1}{e-u+1}\right) + l c'$,

or

$$x = c' \left\{ \frac{(e+1)x^2 - y}{(e-1)x^2 + y} \right\}^{\frac{1}{2e}}.$$

Thirdly, let $c = 1 + e^2$, and we shall get

$$l \frac{x}{c'} = \frac{1}{e} \tan^{-1} \frac{y - x^2}{e x^2}.$$

The second value of t , $t = 2u$ being substituted in the expression of $\frac{dx}{x}$, gives

$$du = 0, \quad \text{or} \quad u = c, \quad \text{that is} \quad y = c x^2.$$

This value of y satisfies the proposed equation; but as it cannot be derived from the complete integral we have obtained above by assuming a particular value for one of the arbitrary constants, it ought to be considered as a particular integral.

(137.) The order of a differential equation, in which y and its differential coefficients amount to the same dimensions in every term, may be reduced by only a very simple substitution. It is sufficiently evident, that if in such an equation we assume $y = e^{\int u dx}$, every term will become divisible by a power of $e^{\int u dx}$ equal to the

Integral
Calculus.

dimension of each term, since the values of dy , d^2y , &c. will be all multiplied by $e^{f'x}$; and besides that the expression of dy derived from $y = e^{\int u dx}$, containing only u , that of d^2y , only the first differential of u , &c. the order of the transformed equation will be less by one than that of the proposed. Part III.

Example. Let $xy dy^2 = y dx dy + x dy^2 + \frac{bx dy^2}{\sqrt{(a^2 - x^2)}}$ be the proposed equation. Make $y = e^{\int u dx}$, and the equation becomes

$$du + u^2 dx = \frac{u dx}{x} + u^2 dx + \frac{bu^2 dx}{\sqrt{(a^2 - x^2)}}$$

or

$$\frac{x du - u dx}{u^2} = \frac{bx dx}{\sqrt{(a^2 - x^2)}}$$

each of which equations is an exact differential, and which being integrated gives

$$u = \frac{x}{c + b\sqrt{(a^2 - x^2)}},$$

$$\text{and } \log y = \int u dx = \int \frac{x dx}{c + b\sqrt{(a^2 - x^2)}} = -\frac{\sqrt{(a^2 - x^2)}}{b} + \frac{c}{b^2} l. \left(\frac{c + b\sqrt{(a^2 - x^2)}}{c} \right).$$

(138.) We have given the name of *linear equation* of the first order to equations of the form

$$dy + P y dx = Q,$$

when P and Q are functions of x alone. There are also linear equations of all orders. Their general form is

$$d^n y + P d^{n-1} y dx + Q d^{n-2} y dx^2 + \dots + U y dx^n = V dx^n,$$

where $P, Q, \dots, U, V$ are functions of x alone.

This equation being homogeneous with respect to y and its differential coefficients dy, d^2y , and when $V = 0$, we may, in that case, as in the preceding paragraph, reduce the integration to that of an equation of an order less by one, by assuming $y = e^{\int t dx}$.

The proposed equation, in the supposition of $V = 0$, is

$$d^n y + P d^{n-1} y dx + Q d^{n-2} y dx^2 + \dots + U y dx^n = 0.$$

Let $y_1, y_2, y_3, \dots, y_n$ be n particular values of y , which satisfy this equation, then it may easily be proved that the general integral will be

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n,$$

$c_1, c_2, \dots, c_n$ being n arbitrary constants.

Since $y = y_1$ satisfies the proposed equation, we have

$$d^n y_1 + P d^{n-1} y_1 dx + Q d^{n-2} y_1 dx^2 + \dots + U y_1 dx^n = 0,$$

and in the same manner for the other values y_2, y_3 , &c.

$$d^n y_2 + P d^{n-1} y_2 dx + Q d^{n-2} y_2 dx^2 + \dots + U y_2 dx^n = 0,$$

$$\dots \dots \dots$$

$$d^n y_n + P d^{n-1} y_n dx + Q d^{n-2} y_n dx^2 + \dots + U y_n dx^n = 0.$$

Therefore if we multiply these equations respectively by $c_1, c_2, \dots, c_n$, and then add them, we shall have identically

$$\left. \begin{aligned} &c_1 (d^n y_1 + P d^{n-1} y_1 dx + Q d^{n-2} y_1 dx^2 + \dots + U y_1 dx^n) \\ &+ c_2 (d^n y_2 + P d^{n-1} y_2 dx + Q d^{n-2} y_2 dx^2 + \dots + U y_2 dx^n) \\ &\dots \dots \dots \\ &+ c_n (d^n y_n + P d^{n-1} y_n dx + Q d^{n-2} y_n dx^2 + \dots + U y_n dx^n) \end{aligned} \right\} = 0.$$

But the left side of this last equation is precisely the result we would obtain, if we were to substitute in the proposed equation the value

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n;$$

and since the result of this substitution is identically equal to nothing, and that the value of y contains n arbitrary constants, it may be considered as the complete integral of the proposed equation. The difficulty of integrating is, therefore, reduced to the determination of n particular values.

When $P, Q, \dots, U$ are constant quantities, these n particular values can easily be discovered. Make in that supposition $y = e^{mx}$, we may take m constant for since

$$\frac{dy}{dx} = e^{mx} \cdot m, \quad \frac{d^2y}{dx^2} = e^{mx} \cdot m^2, \dots, \frac{d^n y}{dx^n} = e^{mx} \cdot m^n.$$

We shall have to determine m after substituting these values in the proposed, and dividing by e^{mx} , the equation

$$m^n + P m^{n-1} + Q m^{n-2} + \dots + U = 0,$$

Integral Calculus. where $P, Q, \dots, U$ are constant quantities; therefore the value of m derived from it will be constant, as we have supposed it. This equation will give n values of m ; let us suppose, first, that they are all real and unequal, and let them be represented by $m_1, m_2, m_3, \dots, m_n$, then the n values of $y, e^{m_1 x}, e^{m_2 x}, \dots, e^{m_n x}$, are n particular solutions of the proposed equation, therefore

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$$

is the complete integral, $c_1, c_2, \dots, c_n$ being n arbitrary constants.

This integral may assume different forms. We shall make the transformations in the supposition of $n = 3$, and it will be easy to extend the same method to every case. We have, then,

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}.$$

Putting $e^{m_3 x}$ as a common factor, this value becomes

$$y = e^{m_3 x} (c_3 + c_2 e^{(m_2 - m_3)x} + c_1 e^{(m_1 - m_3)x});$$

substituting for the constants c_2 and c_1 other constants

$$\frac{c_2}{m_2 - m_3} = \frac{c_1}{(m_1 - m_2)(m_1 - m_3)},$$

and observing that

$$e^{(m_2 - m_3)x} = m_2 - m_3 \int e^{(m_2 - m_3)x} dx,$$

we get

$$y = e^{m_3 x} \left\{ c_3 + c_2 \int e^{(m_2 - m_3)x} dx - \frac{c_1}{(m_1 - m_2)(m_1 - m_3)} e^{(m_1 - m_3)x} \right\}.$$

But therefore

$$e^{(m_1 - m_3)x} = e^{(m_2 - m_3)x} \cdot e^{(m_1 - m_2)x}, \text{ and } (m_2 - m_3) = (m_1 - m_2) - (m_1 - m_3),$$

$$\begin{aligned} \frac{c_1}{(m_1 - m_2)(m_1 - m_3)} e^{(m_1 - m_3)x} &= \frac{c_1 (m_2 - m_3) e^{(m_1 - m_3)x}}{(m_1 - m_2)(m_1 - m_3)(m_2 - m_3)} = \frac{c_1 (m_1 - m_2) e^{(m_2 - m_3)x} \cdot e^{(m_1 - m_2)x}}{(m_1 - m_2)(m_1 - m_3)(m_2 - m_3)} \\ &\quad - \frac{c_1 (m_1 - m_3) e^{(m_2 - m_3)x} \cdot e^{(m_1 - m_2)x}}{(m_1 - m_2)(m_1 - m_3)(m_2 - m_3)}, \\ &= -c_1 \int e^{(m_2 - m_3)x} \cdot dx \times \int e^{(m_1 - m_2)x} \cdot dx + \dots + c_1 \int \left(\frac{e^{(m_2 - m_3)x} \cdot e^{(m_1 - m_2)x}}{(m_2 - m_3)} \cdot dx \right) \\ &= -c_1 \int \{ e^{(m_2 - m_3)x} \cdot \int e^{(m_1 - m_2)x} \cdot dx \}, \end{aligned}$$

which value being substituted, gives, finally,

$$y = e^{m_3 x} \left\{ c_3 + \int e^{(m_2 - m_3)x} \{ c_2 dx + \int e^{(m_1 - m_2)x} \{ c_1 dx^2 \right\}.$$

And for any value of n we shall have

$$y = e^{m_n x} \left\{ c_n + \int e^{(m_{n-1} - m_n)x} \{ c_{n-1} dx + \int e^{(m_{n-2} - m_{n-1})x} \{ c_{n-2} dx^2 + \dots \{ c_2 dx + \int e^{(m_1 - m_2)x} \{ c_1 dx^n \right\}.$$

We could have put as a common factor instead of $e^{m_n x}$ any other particular value of

$$y, e^{m_{n-1} x}, \text{ or } e^{m_{n-2} x}, \&c.;$$

to each factor would correspond a value of y , different in appearance from all the others, but in reality differing only by the values of the constants, since they are all deduced from the value

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}.$$

The new form of the value of y we have obtained is more complicated than the other, but it has the advantage of applying to the cases in which the equation

$$m^n + P m^{n-1} + Q m^{n-2} + \dots + U = 0$$

has equal roots, whilst the other does not without transformation. When $n = 3$, if the two roots m_1 and m_2 are equal, the formula gives

$$y = e^{m_3 x} \{ c_3 + \int e^{(m_2 - m_3)x} \{ c_2 dx + \int c_1 dx^2,$$

or performing the integrations

$$y = c''' e^{m_3 x} + c'' e^{m_2 x} + c' x e^{m_2 x},$$

in making

$$c''' = c_3, \quad c'' = \frac{c_2}{m_2 - m_3} - \frac{c_3}{(m_2 - m_3)^2}, \quad c' = \frac{c_1}{m_2 - m_3}.$$

We would have obtained a result differing only by the constants from the preceding, if we had assumed $m_2 = m_3$, instead of $m_1 = m_2$.

When the three roots m_1, m_2, m_3 are equal to one another, the same formula gives

$$y = c''' e^{m_3 x} + c'' x e^{m_2 x} + c' x^2 e^{m_1 x},$$

substituting c''' , c'' , c' for the constant quantities c_3 , c_2 , and $\frac{c_1}{2}$.

If among the roots of the equation

$$m^n + P m^{n-1} + Q m^{n-2} + \dots + U = 0,$$

there are some which are imaginary, the expressions we have obtained for the general integral contain terms affected with imaginary quantities; but it is easy to transform them into others containing only real terms, by substituting for the exponential their values in function of sines and cosines.

If m_1 and m_2 , for instance, are imaginary values, they will be of the form $a + \phi \sqrt{-1}$, $a - \phi \sqrt{-1}$, and the two terms $c_1 e^{m_1 x}$, $c_2 e^{m_2 x}$ of the integral which correspond to these roots will become

$$c_1 e^{(a+\phi\sqrt{-1})x} + c_2 e^{(a-\phi\sqrt{-1})x},$$

and by successive transformations

$$c_1 e^{ax} \cdot e^{\phi\sqrt{-1}x} + c_2 e^{ax} \cdot e^{-\phi\sqrt{-1}x},$$

$$c_1 e^{ax} (\cos \phi x + \sqrt{-1} \sin \phi x) + c_2 e^{ax} (\cos \phi x - \sqrt{-1} \sin \phi x),$$

$$e^{ax} \{ (c_1 + c_2) \cos \phi x + (c_1 - c_2) \sqrt{-1} \sin \phi x \};$$

and making

$$c_1 + c_2 = c', \quad \text{and} \quad (c_1 - c_2) \sqrt{-1} = c''$$

the aggregate of the two terms assumes the simple form

$$e^{ax} (c' \cos \phi x + c'' \sin \phi x).$$

Similar transformations would obviously apply to the case in which there should be equal imaginary roots, but we should in that case make use of the second form of the general integral we have given.

(139.) When the coefficients P , Q , &c. are not constant, there is no general process by means of which n particular values of the integral can be found. But if they can be discovered by any means, they not only give immediately the value of the general integral of the differential equation

$$d^n y + P d y^{n-1} dx + Q d^2 y dx^2 + \&c. \dots + U y dx^n = 0 \dots (a),$$

but from them may also be deduced the general integral of

$$d^n y + P d^{n-1} y dx + Q d^{n-2} y dx^2 + \dots + U y dx^n = V dx^n \dots (b).$$

We have already proved that $y_1, y_2, y_3, \dots, y_n$, being n particular values which satisfy equation (a), the integral general of that equation is

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n,$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants. Now it is obvious that whatever may be the general integral of equation (b), it may be supposed to be still represented by

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

provided we suppose $c_1, c_2, \dots, c_n$ to be functions of x instead of being arbitrary constants; and the question is thus reduced to discover what functions of x should be substituted for c_1, c_2, c_3 , in order that the value

$$y = c_1 y_1 + c_2 y_2 + \dots + c_n y_n$$

should satisfy equation (b). We shall show how this can be done for an equation of the third order,

$$d^3 y + P d^2 y dx + Q d y dx^2 + R y dx^3 = V dx^3,$$

and the method we shall make use of may without difficulty be extended to any order.

Let y_1, y_2, y_3 be the three particular values which satisfy the equation

$$d^3 y + P d^2 y dx + Q d y dx^2 + R y dx^3 = 0,$$

then we are to determine

$$c_1, c_2, c_3, \text{ so that } y = c_1 y_1 + c_2 y_2 + c_3 y_3$$

shall satisfy the proposed equation. We have thus only one condition to determine three quantities, and therefore we may choose ourselves two other conditions. We shall find, by differentiating the value of y ,

$$d y = c_1 d y_1 + c_2 d y_2 + c_3 d y_3 + y_1 d c_1 + y_2 d c_2 + y_3 d c_3.$$

For the first of the two conditions we may dispose of, we shall make

$$y_1 d c_1 + y_2 d c_2 + y_3 d c_3 = 0,$$

and the value of $d y$ will become

$$d y = c_1 d y_1 + c_2 d y_2 + c_3 d y_3.$$

Differentiating again, we find

$$d^2 y = c_1 d^2 y_1 + c_2 d^2 y_2 + c_3 d^2 y_3 + d y_1 d c_1 + d y_2 d c_2 + d y_3 d c_3;$$

and if we take for the second condition

Integral
Calculus.

this expression will be reduced to

$$d y_1 d c_1 + d y_2 d c_2 + d y_3 d c_3 = 0,$$

$$d^2 y = c_1 d^2 y_1 + c_2 d^2 y_2 + c_3 d^2 y_3.$$

A new differentiation will give

$$d^3 y = c_1 d^3 y_1 + c_2 d^3 y_2 + c_3 d^3 y_3 + d c_1 d^2 y_1 + d c_2 d^2 y_2 + d c_3 d^2 y_3.$$

 The values of y , $d y$, $d^2 y$, $d^3 y$, being now substituted in the proposed equation, it will become

$$\left. \begin{aligned} & c_1 (d^3 y_1 + P d^2 y_1 d x + Q d y_1 d x^2 + R y_1 d x^3) \\ & + c_2 (d^3 y_2 + P d^2 y_2 d x + Q d y_2 d x^2 + R y_2 d x^3) \\ & + c_3 (d^3 y_3 + P d^2 y_3 d x + Q d y_3 d x^2 + R y_3 d x^3) \\ & + d^2 y_1 d c_1 + d^2 y_2 d c_2 + d^2 y_3 d c_3 \end{aligned} \right\} = V d x^3.$$

This equation becomes simply

$$d^2 y_1 d c_1 + d^2 y_2 d c_2 + d^2 y_3 d c_3 = V d x^3,$$

 since y_1 , y_2 , y_3 are particular values which satisfy the equation

$$d^3 y + P d^2 y d x + Q d y d x^2 + R y d x^3 = 0.$$

 We have thus between the differentials $d c_1$, $d c_2$, $d c_3$, the three equations

$$y_1 d c_1 + y_2 d c_2 + y_3 d c_3 = 0,$$

$$d y_1 d c_1 + d y_2 d c_2 + d y_3 d c_3 = 0,$$

$$d^2 y_1 d c_1 + d^2 y_2 d c_2 + d^2 y_3 d c_3 = 0,$$

 from which we shall derive the values of these differentials in terms of x , when those of y_1 , y_2 , y_3 are known; if $X_1 d x$, $X_2 d x$, $X_3 d x$ are the values obtained by elimination, we shall get by integrating

$$c_1 = \int X_1 d x + E_1, \quad c_2 = \int X_2 d x + E_2, \quad c_3 = \int X_3 d x + E_3,$$

and consequently

$$y = y_1 (\int X_1 d x + E_1) + y_2 (\int X_2 d x + E_2) + y_3 (\int X_3 d x + E_3)$$

will be the complete integral of

$$d^3 y + P d^2 y d x + Q d y d x^2 + R y d x^3 = V d x^3.$$

 If only two particular values y_1 , y_2 were known, then in representing the general value of y by $c_1 y_1 + c_2 y_2$, the determination of c_1 and c_2 would require the integration of a differential equation of the second order, which may be reduced to a linear equation of the first order. In that case we have

$$d y = c_1 d y_1 + c_2 d y_2 + y_1 d c_1 + y_2 d c_2,$$

and since there are only two quantities to be determined, we can assume only one relation between them; if we make

$$y_1 d c_1 + y_2 d c_2 = 0,$$

 the value of $d y$ will simply be

$$c_1 d y_1 + c_2 d y_2,$$

and two successive differentiations will give

$$d^2 y = c_1 d^2 y_1 + c_2 d^2 y_2 + d y_1 d c_1 + d y_2 d c_2,$$

$$d^3 y = c_1 d^3 y_1 + c_2 d^3 y_2 + 2 d^2 y_1 d c_1 + 2 d^2 y_2 d c_2 + d y_1 d^2 c_1 + d y_2 d^2 c_2;$$

these values being substituted in the proposed equation, and reduced as in the preceding case, give

$$\left. \begin{aligned} & d y_1 d^2 c_1 + d y_2 d^2 c_2 + 2 d^2 y_1 d c_1 + 2 d^2 y_2 d c_2 \\ & + P d y_1 d c_1 d x + P d y_2 d c_2 d x \end{aligned} \right\} = V d x.$$

 From this equation we may eliminate $d c_2$ and $d^2 c_2$ by means of the equation

$$y_1 d c_1 + y_2 d c_2 = 0,$$

and its differential, we shall get then an equation containing only

$$d c_1, \quad d^2 c_1, \quad \text{and } x,$$

and therefore reducible to the first order.

 Lastly. If we know only one particular value, y_1 , the same method might still be used, and the general integral represented by $c_1 y_1$; but in that case the determination of c_1 would require the integration of a differential equation of the third order, which might be reduced to the second, since it would contain only x and the differentials of c_1 .

 The result we have thus obtained for the third order may easily be extended to any order n .

 (140.) When the coefficients P , Q , R , &c. are constant, we know how to find n particular values of the equation

$$d^n y + P d^{n-1} y d x + Q d^{n-2} y d x^2 + \dots + U y d x^n = 0$$

consequently by the preceding method we shall be able to find, in that case, the general integral of

$$d^n y + P d^{n-1} y dx + Q d^{n-2} y dx^2 + \dots + U y dx^n = V dx^n.$$

Let us take, for instance, the equation of the second order

$$d^2 y + P dy dx + Q y dx^2 = V dx^2,$$

and let m_1 and m_2 be the two roots of the equation $m^2 + Pm + Q = 0$, we assume that the required integral is $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$, and we shall have to determine the values of c_1 and c_2 , the two equations

$$e^{m_1 x} d c_1 + e^{m_2 x} d c_2 = 0, \quad e^{m_1 x} m_1 d c_1 + e^{m_2 x} m_2 d c_2 = V dx,$$

which give

$$d c_1 = \frac{V e^{-m_1 x} dx}{m_1 - m_2}, \quad d c_2 = \frac{V e^{-m_2 x} dx}{m_2 - m_1},$$

and, therefore,

$$y = e^{m_1 x} \left\{ \frac{\int V e^{-m_1 x} dx}{m_1 - m_2} + E_1 \right\} + e^{m_2 x} \left\{ \frac{\int V e^{-m_2 x} dx}{m_2 - m_1} \right\}.$$

The values of $d c_1$ and $d c_2$ we have just found are of the form $A_1 e^{-m_1 x} V dx$, $A_2 e^{-m_2 x} V dx$, A_1 and A_2 being constant quantities. It may easily be proved that it is the same with all orders, for if we assume

$$d c_1 = A_1 e^{-m_1 x} V dx, \quad d c_2 = A_2 e^{-m_2 x} V dx, \dots, d c_n = A_n e^{-m_n x} V dx,$$

the equations from which the values of $d c_1$, $d c_2$, &c. are to be derived will become, in substituting at the same time for y_1 , y_2 , y_3 , &c., their values $e^{m_1 x}$, $e^{m_2 x}$, &c.

$$\left. \begin{aligned} A_1 + A_2 + A_3 + \dots + A_n &= 0 \\ m_1 A_1 + m_2 A_2 + m_3 A_3 + \dots + m_n A_n &= 0 \\ m_1^2 A_1 + m_2^2 A_2 + m_3^2 A_3 + \dots + m_n^2 A_n &= 0 \\ \dots &\dots \\ m_1^{n-1} A_1 + m_2^{n-1} A_2 + m_3^{n-1} A_3 + \dots + m_n^{n-1} A_n &= 0 \end{aligned} \right\} (A),$$

which equations will be satisfied by constant values of $m_1, m_2, \dots, m_n$. Let

$$\begin{aligned} m_2 + m_3 + \dots + m_n &= -P' \\ m_2 m_3 + m_2 m_4 + \dots &= +Q' \\ m_2 m_3 m_4 + m_2 m_3 m_5 + \dots &= -R' \\ \dots &\dots \\ m_2 m_3 m_4 \dots m_n &= \pm U' \end{aligned}$$

Then the quantities $m_2, m_3, \dots, m_n$, are the roots of the equation

$$m'^{n-1} + P' m'^{n-2} + Q' m'^{n-3} + \dots + U' = 0.$$

This understood, multiply the last but one of the equations (A) by P' , the last but two by Q' , &c., and then add them all together, we shall get

$$\left. \begin{aligned} A_1 (m_1^{n-1} + P' m_1^{n-2} + Q' m_1^{n-3} + \dots + U') \\ + A_2 (m_2^{n-1} + P' m_2^{n-2} + Q' m_2^{n-3} + \dots + U') \\ + \dots \\ + A_n (m_n^{n-1} + P' m_n^{n-2} + Q' m_n^{n-3} + \dots + U') \end{aligned} \right\} = 0.$$

It is obvious that all the lines which compose this last equation are equal to nothing except the first, and consequently

$$A_1 = \frac{1}{m_1^{n-1} + P' m_1^{n-2} + Q' m_1^{n-3} + \dots + U'}$$

or

$$A_1 = \frac{1}{(m_1 - m_2)(m_1 - m_3) \dots (m_1 - m_n)}$$

The values of $A_2, A_3, \dots, A_n$ will be obtained in the same manner.

Hence

$$d c_1 = A_1 V e^{-m_1 x}, \quad d c_2 = A_2 V e^{-m_2 x}, \text{ &c.,}$$

Integral
Calculus, and

$$y = \frac{e^{m_1 x} (E_1 + \int V e^{-m_1 x} dx)}{(m_1 - m_2)(m_1 - m_3) \dots (m_1 - m_n)} + \frac{e^{m_2 x} (E_2 + \int V e^{-m_2 x} dx)}{(m_2 - m_1)(m_2 - m_3) \dots (m_2 - m_n)} + \dots + \frac{e^{m_n x} (E_n + \int V e^{-m_n x} dx)}{(m_n - m_1)(m_n - m_2) \dots (m_n - m_{n-1})}.$$

This value may be put under another form which applies without further transformations to the case in which the equation

$$m^n + P m^{n-1} + Q m^{n-2} + \dots + U = 0$$

has equal roots. It will be sufficient to show how the transformation can be effected in the case of the second order. Then

$$y = \frac{e^{m_1 x} (E_1 + \int V e^{-m_1 x} dx)}{m_1 - m_2} + e^{m_2 x} \frac{(E_2 + \int V e^{-m_2 x} dx)}{m_2 - m_1},$$

which becomes successively

$$y = \frac{E_2 e^{m_2 x}}{m_2 - m_1} + \frac{E_1 e^{m_1 x}}{m_1 - m_2} + \frac{e^{m_2 x} \int V e^{-m_2 x} dx}{m_2 - m_1} + \frac{e^{m_1 x} \int V e^{-m_1 x} dx}{m_1 - m_2},$$

$$\text{or } y = \frac{E_2 e^{m_2 x}}{m_2 - m_1} + e^{m_2 x} \int e^{(m_1 - m_2)x} \cdot E_1 + \frac{e^{m_2 x} \int e^{(m_1 - m_2)x} V e^{-m_1 x} dx}{m_2 - m_1} - \frac{e^{m_2 x} \cdot e^{(m_1 - m_2)x} \int V e^{-m_1 x} dx}{m_2 - m_1},$$

$$\text{or } y = e^{m_2 x} \left\{ \frac{E_2}{m_2 - m_1} + \int e^{(m_1 - m_2)x} \{ E_1 dx + \int V e^{-m_1 x} dx^2 \} \right\},$$

or putting c'' and c' for $\frac{E_2}{m_2 - m_1}$ and E_1 , we find

$$y = e^{m_2 x} \{ c'' + \int e^{(m_1 - m_2)x} \{ c' dx + \int V e^{-m_1 x} dx^2 \} \}$$

and generally, by similar steps, we should find

$$y = e^{m_n x} \{ c_n + \int e^{(m_{n-1} - m_n)x} \{ c_{n-1} dx + \int e^{(m_{n-2} - m_{n-1})x} \{ c_{n-2} dx^2 + \dots + \int e^{-m_1 x} V dx^{n-1} \} \}$$

If some of the roots are impossible quantities, the imaginary, which they will introduce in the integral, may be made to disappear by means analogous to those we have already used.

If, for instance, the two roots m_1 and m_2 are impossible, and respectively equal to $a + \phi \sqrt{-1}$, $a - \phi \sqrt{-1}$, the two first terms of the general expression will become

$$\frac{e^{(a+\phi\sqrt{-1})x} \int e^{-x(a+\phi\sqrt{-1})} dx}{A + B\sqrt{-1}} + \frac{e^{(a-\phi\sqrt{-1})x} \int e^{-x(a-\phi\sqrt{-1})} dx}{A - B\sqrt{-1}}.$$

Including the arbitrary constants E_1 and E_2 under the sign of integration, reducing to the same denominator, and substituting for the imaginary exponentials their values in function of sines and cosines, the expression becomes

$$e^{ax} \frac{(2A \cos \phi x + 2B \sin \phi x) \int e^{-ax} V dx \cos \phi x}{A^2 + B^2} + e^{ax} \frac{(2A \sin \phi x - 2B \cos \phi x) \int e^{-ax} V dx \sin \phi x}{A^2 + B^2}.$$

(141.) Linear equations, in which the coefficient P , Q , &c. are functions of x , can be integrated in very few cases. We shall only mention the following equation, where P , Q , &c. are constant quantities, and where a simple transformation succeeds:

$$(a + bx)^n d^n y + P(a + bx)^{n-1} d^{n-1} y + Q(a + bx)^{n-2} d^{n-2} y + \dots + U y dx^n = V dx^n.$$

Making $a + bx = bt$, we shall have $dx = dt$, and if W represents the value assumed by V when for x , $t - \frac{a}{b}$ is substituted, and P' , Q' , &c. the values of $\frac{P}{b}$, $\frac{Q}{b^2}$, &c., the equation becomes

$$t^n d^n y + P' t^{n-1} d^{n-1} y dt + Q' t^{n-2} d^{n-2} y dt^2 + \dots + U' y dt^n = \frac{W dt^n}{b^n};$$

the integration of which depends on the determination of n particular values of

$$t^n d^n y + P' t^{n-1} d^{n-1} y dt + Q' t^{n-2} d^{n-2} y dt^2 + \dots + U' y dt^n = 0.$$

These are easily obtained, if we observe that by assuming $y = t^m$, we shall have to determine m in the equation

$$m(m-1) \dots (m-n+1) + P' m(m-1) \dots (m-n+2) \dots + U' = 0,$$

Integral
Calculus.

which will be of the n^{th} degree, giving n values for m , and consequently n particular solutions, with which we shall proceed, as before, to obtain the integral of the proposed equation. Part III.

The equation $t^n d^n y + P' t^{n-1} d^{n-1} y dt + \&c. \dots + U' y dt^n = 0$ may be transformed into a linear equation of the n^{th} order with constant coefficients. It is sufficient for that purpose to make

$$\frac{dt}{t} = du, \quad \text{then } \frac{dy}{dt} = \frac{dy}{t du},$$

and
$$\frac{d^2 y}{dt^2} = \frac{1}{t^2} \left(\frac{d^2 y}{du^2} - \frac{dy}{du} \right), \quad \frac{d^3 y}{dt^3} = \frac{1}{t^3} \left(\frac{d^3 y}{du^3} - \frac{3 d^2 y}{du^2} + \frac{2 dy}{du} \right), \&c.,$$

and the transformed equation will have, obviously, constant coefficients.

On Particular Solutions of Differential Equations.

(142.) From what has already been stated respecting the integration of differential equations, it appears that to every differential equation corresponds an infinite number of primitive equations or integrals, differing only from each other by the particular values assigned to the arbitrary constants which enter into the *general integral*. But besides these *particular integrals* there exist, in a great many cases, equations between x and y , which satisfy the proposed differential equation without being included in the complete integral, that is without being deducible from it by giving particular values to the constants. These equations not being obtained by the process of integration are called *particular solutions* of the differential equations.

The *general integral* of $y dx - x dy = n \sqrt{(dx^2 + dy^2)}$, for instance, is $y - cx = n \sqrt{(1 + c^2)}$, and it may be easily verified that the equation $x^2 + y^2 = n^2$, which cannot be deduced from it by assigning any particular value to the constant, it satisfies, however, the proposed differential equation, and is, therefore, according to the preceding definition a *particular solution* of it.

It is frequently of as much importance to determine the *particular solutions* as the *general integral*, for, in many cases the true solution of the proposed problem is to be found amongst them, and not in the general integral.

(143.) Let us first investigate the circumstances upon which depends the existence of particular solutions. Let $f\left(x, y, \frac{dy}{dx}\right) = 0$ be any differential equation of the first order, and let $u = F(x, y, a) = 0$ be the general integral, a being the arbitrary constant introduced by the integration. Then $f\left(x, y, \frac{dy}{dx}\right) = 0$ will result of the elimination of a between $u = F(x, y, a) = 0$, and its immediate differential $\frac{du}{dx} dx + \frac{du}{dy} dy = 0$.

Now, if the proposed differential equation has any *particular solution*, it is obvious that although it cannot be deduced from the general integral by substituting for the constant a any particular value, it may still be considered as included in the equation $F(x, y, a) = 0$, by supposing that a instead of representing a constant quantity, is a function of x , or of x and y . In that hypothesis, the proposed differential equation $f\left(x, y, \frac{dy}{dx}\right) = 0$, must still result from a combination of the equation $F(x, y, a) = 0$, and its immediate differential equation, which, then, is $\frac{du}{dx} dx + \frac{du}{dy} dy + \frac{du}{da} da = 0$, for otherwise $F(x, y, a) = 0$ could not be a solution of it. But we have already said that $f\left(x, y, \frac{dy}{dx}\right) = 0$ is the result of the elimination of a between $F(x, y, a) = 0$, and $\frac{du}{dx} dx + \frac{du}{dy} dy = 0$, therefore, in the case where a is supposed variable, in order that it might also result from a combination of the two equations

$$F(x, y, a) = 0, \quad \frac{du}{dx} dx + \frac{du}{dy} dy + \frac{du}{da} da = 0,$$

it is necessary that a should be such that $\frac{du}{da} da = 0$.

This equation gives $da = 0$, or $\frac{du}{da} = 0$. From the first we get $a = \text{constant}$, and therefore this value corresponds to the general integral $F(x, y, a) = 0$. But in the other equation $\frac{du}{da} = 0$, $\frac{du}{da}$ will be generally a function of x, y , and a , therefore the value of a may be derived from it in terms of x and y , and this value being substituted in the equation $F(x, y, a) = 0$, will lead to a new equation in terms of x , and y , without an arbitrary constant, generally not susceptible of being deduced from the general integral $F(x, y, a) = 0$ by substituting

Integral
Calculus.

Part III.

any particular constant value for a , and yet satisfying the differential equation $f\left(x, y, \frac{dy}{dx}\right) = 0$. Therefore this equation, or its factors, when all the terms are on one side, will be the particular solution or solutions of the proposed differential equation.

(144.) Hence when the general integral $u = F(x, y, a) = 0$ of a proposed differential equation is known, to discover the particular solutions, if there be any, it will be sufficient to eliminate a between the two equations

$$u = 0, \text{ and } \frac{du}{da} = 0.$$

(145.) The equation to which this elimination will lead, will not always, however, belong to a particular solution. To ascertain whether it does or not, it will be necessary first to eliminate one of the variables x or y ,

between the two equations $u = 0$, and $\frac{du}{da} = 0$, and to determine the value of a by means of the resulting

equation. If this value be a function of the other variable, then it is clear that by substituting this value instead of a in $u = 0$, we shall have a particular solution. But, if the value be constant, it is obvious, on the contrary, that the

substitution of that value for a will only give a particular integral. The value of $\frac{du}{da}$ may also be independent of

a . This will happen when a enters into the general integral in the first degree. In that case, the general integral must have the form $P + Qa = 0$, P and Q being functions of x, y , and not containing a . We derive

from it $\frac{du}{da} = Q$, and the elimination of a between $P + Qa = 0$, and $Q = 0$, will give either $P = 0$, or $Q = 0$.

Both these are clearly particular integrals; the first corresponding to $a = 0$, and the other to $a = \infty$. If Q be a factor of P , the result of the elimination is only $Q = 0$, and the value of a corresponding is $\frac{0}{0}$. In that case

$Q = 0$ may be considered as a particular solution, for it satisfies the differential equation, which in that case is $PdQ - QdP = 0$. But such solutions can scarcely be ranked amongst particular solutions, since by multiplying the general integral $P + Qa = 0$ by any factor R not containing a , $R = 0$ would also become a particular solution. The same observation applies to the common divisors of M and N , in a differential equation of the form $Mdx + Ndy = 0$. Each of these common divisors made equal to nothing satisfies the differential equation, and if it cannot be derived from the general integral by giving a particular value to the arbitrary constant, it may be considered as a particular solution. Such particular solutions may be found without difficulty by the determination of the common divisor between M and N , but their number may be increased without limits by introducing new common factors to M and N .

(146.) We shall apply the rule derived from what precedes to find the particular solutions, when the general integral is known, to an example. We have already seen that the general integral of

$$ydx - xdy = n\sqrt{(dx^2 + dy^2)} \text{ is } y - cx = n\sqrt{(1 + c^2)};$$

we shall have, consequently, $\frac{du}{dc} = -\left(x + \frac{nc}{\sqrt{(1 + c^2)}}\right)$; eliminating c between the general integral and

$x + \frac{nc}{\sqrt{(1 + c^2)}} = 0$, we shall find $x^2 + y^2 = n^2$, which is the particular solution. There is no occasion here to

eliminate one of the variables, to ascertain whether the value of c is constant, for the equation $x + \frac{nc}{\sqrt{(1 + c^2)}} = 0$

shows at once that that quantity is a function of x .

(147.) The preceding considerations may easily be extended to the first integrals of equations containing differential coefficients of an order higher than the first.

Let $u = F\left(x, y, \frac{dy}{dx}, \dots, \frac{d^{n-1}y}{dx^{n-1}}, a\right) = 0$, be the first integral of $f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0$, a being the arbitrary constant introduced by the integration. By using the same reasoning as before, it will be obvious that the elimination of a between $u = 0$ and $\frac{du}{da} = 0$, will lead to a first integral of $f\left(x, y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}\right) = 0$ without an arbitrary constant, and which will be a particular solution, if the equation resulting from the elimination of one of the variables between $u = 0$ and $\frac{du}{da} = 0$ give a variable value for a . In the elimination of a between $u = 0$ and $\frac{du}{da} = 0$, several or all the differential coefficients might disappear, and in that case, we should obtain a particular solution of the differential equation, deprived of arbitrary constants and differential coefficients.

(148.) It is known by the general theory of differential equations that a differential equation of the n^{th} order

Integral
Calculus.

$$y + \left(\frac{a}{2} - a^2\right)x^2 - (1 - 2a)x \frac{dy}{dx} - a^2 - \frac{dy^2}{dx^2} = 0,$$

and

$$y - \frac{\left(b + \frac{dy}{dx}\right)}{2}x - \frac{\left(b - \frac{dy}{dx}\right)^2}{x^2} - b^2 = 0.$$

 If we differentiate the first with respect to a , we find

$$\left(\frac{1}{2} - 2a\right)x^2 + 2x \frac{dy}{dx} - 2a = 0,$$

which gives $a = \frac{x^2 + 4x \frac{dy}{dx}}{4(x^2 + 1)}$. This value being substituted in the first equation will give, after reduction, for the particular solution of the differential equation of the second order

$$(1 + x^2)y - \left(x + \frac{x^3}{2}\right) \frac{dy}{dx} - \frac{dy^2}{dx^2} + \frac{x^4}{16} = 0.$$

 If we differentiate now the second equation with respect to b , we find

$$\frac{x}{2} + \frac{2\left(b - \frac{dy}{dx}\right)}{x^2} + 2b = 0,$$

which gives $b = \frac{\frac{4}{dx} \frac{dy}{dx}}{4(1 + x^2)}$, and the substitution of this value gives precisely the same result as the substitution of a in the first.

(149.) When a particular solution of the $(n - 1)^{th}$ order has been found, for a differential equation of the n^{th} order, if the first general integral of that particular solution may be determined, the same process may be applied to it to find a particular solution of the first particular solution; it will then be a differential equation of the $(n - 2)^{th}$ order without any constant, but it will not always satisfy the proposed differential equation of the n^{th} order, and therefore cannot be considered as a particular solution of it. (See *Leçons sur le Calcul des Fonctions*, *Leçon 14me.*)

(150.) The particular solutions may be derived from the general integral by another process.

Let $F(x, y, a) = u = 0$ still represent the general integral of a differential equation of the first order, $f\left(x, y, \frac{dy}{dx}\right) = 0$. If in the general integral we consider a as a function of x and y , we shall get, in differentiating u successively with respect to x and to y ,

$$\frac{du}{dx} + \frac{du}{da} \cdot \frac{da}{dx} = 0, \quad \frac{du}{dy} + \frac{du}{da} \cdot \frac{da}{dy} = 0,$$

and hence

$$\frac{da}{dx} = \frac{du}{dx} \div \frac{du}{da}, \quad \frac{da}{dy} = \frac{du}{dy} \div \frac{du}{da}.$$

But it has been proved that the values of a which correspond to particular solutions, make $\frac{du}{da} = 0$, therefore these same values of a must make $\frac{da}{dx} = \infty$ and $\frac{da}{dy} = \infty$. If, therefore, there exist a particular solution, we shall find it by eliminating the quantity a between the general integral, and either of the equations $\frac{da}{dy} = \infty$, $\frac{da}{dx} = \infty$, derived from the general integral.

Example. The general integral of $x dx + y dy = dy \sqrt{(x^2 + y^2 - a^2)}$ is $x^2 - 2ay - a^2 - b = 0$. Here we have $\frac{da}{dx} = \frac{x}{a - y}$, $\frac{da}{dy} = \frac{a}{a - y}$, and the condition $\frac{da}{dx}$ or $\frac{da}{dy} = \infty$ gives $a - y = 0$, or $a = y$, which value being substituted in the general integral, leads to the particular solution $x^2 + y^2 = b$.

In this example there can be no doubt that the equation $x^2 + y^2 = b$, resulting from the elimination of a , is a particular solution, because the value of a is variable; but generally it will be necessary to determine, as before, by the elimination of x or y between the two equations, if a is not a constant quantity, in which case the equation obtained would only be a particular integral.

(151.) The manner we have just explained to find a particular solution, shows that when the general integral can be resolved with respect to the arbitrary constant a , and that the value of a expressed in terms of x and y , is either a fractional power less than one, of a rational function of x and y , or composed of a rational quantity and such an irrational part, the particular solution may be immediately obtained, by making the quantity raised

to a fractional power equal to zero. This is clearly true because in finding the value of $\frac{da}{dx}$, or $\frac{da}{dy}$, the quantity raised to a fractional power becomes a denominator. In the above example for instance, the value of a , deduced from the general integral, is $-y \pm \sqrt{(y^2 + x^2 - b)}$, and the quantity under the radical sign being equal to zero gives the particular solution $x^2 + y^2 = b$. In the example given (146.) the value of the constant is

$$\frac{yx \pm \sqrt{(y^2 x^2 - (y^2 - n^2)(x^2 - n^2))}}{x^2 - n^2},$$

and gives immediately for the particular solution $y^2 x^2 - (y^2 - n^2)(x^2 - n^2) = 0$, or $x^2 + y^2 = n^2$, as we found before. In this case the value of the constant c has for denominator a function of x , which will remain in the value of $\frac{dc}{dx}$, and consequently by making this denominator equal zero, we may have also a particular

solution; we find thus $x^2 - n^2 = 0$, but it is easy to see that this is only a particular integral corresponding to the value $c = \infty$. In other cases, however, by making the denominator equal to nothing we might get a particular solution.

(152.) Particular solutions have the remarkable property of rendering infinite, the factors by which differential equations ought to be multiplied to become an exact differential.

Let $M dx + N dy = 0$ be the differential equation, and let the general integral be $u = c$, supposing it to be resolved with respect to the constant. If $w = 0$ be a particular solution, it will not satisfy the immediate

differential of $u = c$, in the supposition of c being a constant, that is $\frac{du}{dx} dx + \frac{du}{dy} dy = 0$, since this particular solution corresponds to a variable value of c . Now if z be the factor which renders the proposed differential

equation $M dx + N dy = 0$ an immediate differential, the equation $\frac{du}{dx} dx + \frac{du}{dy} dy = 0$ is identical with

$z(M dx + N dy) = 0$. Therefore the particular solution $w = 0$ does not satisfy this last equation, but it satisfies the proposed differential equation $M dx + N dy$, consequently $w = 0$ must render z infinite.

The differential equation $x dx + y dy = y \sqrt{(x^2 + y^2 - a^2)}$ becomes an immediate differential

$$\frac{x dx + y dy}{\sqrt{(x^2 + y^2 - a^2)}} - dy = 0,$$

when it is multiplied by the factor $\frac{1}{\sqrt{(x^2 + y^2 - a^2)}}$, and the particular solution which we have found to be $x^2 + y^2 - a^2 = 0$ renders this function infinite.

(153.) It may be proved more generally, that if we equal to nothing a function by which multiplying or dividing a differential equation we render it an immediate differential, we have an equation which satisfies the proposed differential equation. This may easily be shown by means of the equation furnished by the condition

of integrability; for if we represent successively by z and $\frac{1}{z}$ the factor corresponding to a differential equation $M dx + N dy = 0$, and if we develop the equations of condition

$$\frac{dMz}{dy} = \frac{dNz}{dx}, \quad \frac{dM \frac{1}{z}}{dy} = \frac{dN \frac{1}{z}}{dx},$$

we get the following equations

$$\begin{aligned} \frac{M dz}{dy} - \frac{N dz}{dx} + z \left(\frac{dM}{dy} - \frac{dN}{dx} \right) &= 0, \\ \frac{M dz}{dy} - \frac{N dz}{dx} + z \left(\frac{dM}{dy} - \frac{dN}{dx} \right) &= 0, \end{aligned}$$

which the supposition of $z = 0$ reduces to $\frac{M dz}{dy} - \frac{N dz}{dx} = 0$; but if we differentiate $z = 0$, we get

$$\frac{dz}{dy} = - \frac{dz}{dx} \cdot \frac{dx}{dy},$$

and this value being substituted in the above equation gives the proposed differential equation $M dx + N dy = 0$, which therefore is satisfied since the value of z satisfies the preceding equations.

It has been proved in the preceding paragraph that every particular solution makes the factor equal to infinity, hence in the first of the two last suppositions the equation $z = 0$, which satisfies the proposed differential equation, cannot be a particular solution, and must be therefore a particular integral. It may be easily verified in observing that if $U = 0$ designate the general integral of $M dx + N dy = 0$, we have

Integral
Calculus.

Part III.

$$Z = \frac{1}{M} \frac{dU}{dx} = \frac{1}{N} \frac{dU}{dy}$$

from which we infer, that if the supposition $z = 0$ does not render M and N infinite, it will give $du = 0$, and consequently u equal to a constant quantity.

In the second case we have $z = M \cdot \frac{1}{\frac{du}{dx}} = N \cdot \frac{1}{\frac{du}{dy}}$, and the supposition $z = 0$ renders the functions $\frac{du}{dx}$ and $\frac{du}{dy}$ infinite, if it do not make M and N equal to nothing; this solution will besides be a particular solution, or a particular integral, according as it will give for u a value constant or variable.

It is necessary, however, to observe, that the equation $\frac{M dz}{dy} - \frac{N dz}{dx} = 0$ is no longer a necessary consequence of the supposition $z = 0$, when that supposition makes the functions $\frac{dM}{dy}$ and $\frac{dN}{dx}$ equal to infinity; but in the case where only one of the two functions M and N is equal to nothing, M for instance, the equation of condition gives $\frac{dz}{dx} = 0$, consequently z contains only y and therefore from $z = 0$ we deduce $y = a$ constant, an equation which clearly satisfies $M dx + N dy = 0$, since already M is equal to nothing.

The relations which exist between particular integrals, particular solutions, and the factors by which differential equations must be multiplied to become immediate differentials, may be useful to discover these factors. We cannot, however, enter into these investigations, and must refer the reader to the Papers of Trembley, in the *Mémoires de l'Académie de Turin*, for the years 1790 and 1791.

(154.) We have supposed hitherto that the general integral of the proposed differential equation was known, but it may also be required to find the particular solution without the help of the general integral. Let us first examine, how we may ascertain if an equation without a constant, satisfying a differential equation, is a particular integral or a particular solution.

Let the value of y derived from the proposed solution be X , and let the value derived from the general integral be represented by V . This last quantity contains necessarily an arbitrary constant, which we shall call c . Now, it is obvious that if the proposed solution be a particular solution, no constant value of c can make V equal to X ; but if it be a particular integral, there is a certain value c' of c which will give $V = X$; hence $V - X$ must be divisible by $c - c'$, and consequently $V - X$ may be represented by $A(c - c')^m$, where A is a quantity which becomes neither infinite nor equal to zero, when $c - c' = 0$, and where $m > 0$. Let $(c - c')^m = h$, then we may assume

$$V - X = V' h + V'' h^2 + \dots \text{ or } V = X + V' h + V'' h^2 + \&c.$$

Let the value of $\frac{dy}{dx}$, derived from the proposed differential equation, be represented by p , the equation

$dy = p dx$ ought to be satisfied by $y = v$, or $y = X + V' h + V'' h^2 + \&c.$ independently of h ; and if we designate this value of y by $X + K$, putting this value for y into p , we may represent the result of the substitution by

$$P + P' K^m + P'' K^n + \&c.$$

where the exponents $m, n, \&c.$ are ascending. They are, moreover, all positive, for p is not infinite when $K = 0$, since the value $y = X$ satisfies the differential equation $dy = p dx$, and gives $dX = P dx$. We have, therefore,

$$d(X + k) = dX + dk = (P + P' k^m + P'' k^n + \&c.) dx,$$

or,

$$dk = (P' k^m + P'' k^n + \&c.) dx.$$

Substituting for k its value, $V' h + V'' h^2 + \&c.$, we obtain

$$h dV' + h^2 dV'' + \&c. = \begin{cases} P' h^m (V' + V'' h^{n-1} + \dots)^m \\ + P'' h^n (V' + V'' h^{n-1} + \dots) \\ + \dots \end{cases}$$

This equation must be satisfied independently of any particular value of x , when $y = X$ is a particular integral; and if this be not possible, we may infer that $y = X$ is a particular solution. Comparing the terms, in both sides of the equation, containing the lowest exponents of h , we find first the equation

$$dV' = P' h^{m-1} V'^m dx,$$

which cannot be satisfied, independently of h , unless $m - 1 = 0$, or $m = 1$. In that supposition it becomes $dV' = P' V' dx$, from which we get $V' = e^{\int P' dx}$.

If $m > 1$, the first term $P' V'^m h^m dx$ of the right side, cannot be compared to the first term $h dV'$ of the left side; but $h dV'$ may disappear by supposing $dV' = 0$, that is V' equal to a constant quantity, if then we suppose $\mu = m$, we find $dV'' = P' dx$, or $V'' = \int P' dx$, and the other terms may be found in a similar way.

When $m < 1$ the two series cannot be identified by any supposition, therefore if m be a proper fraction $y = X$ is a particular solution, and if m be equal to or greater than one, $y = x$ is a particular integral.

Integral
Calculus.

(155.) It is easy to deduce from this investigation the means of discovering the particular solutions of differential equations of the first order, without knowing the general integral. We have proved that if $X + k$ be Part III.

substituted for y in the value of $\frac{dy}{dx}$ derived from the differential equation, $y = X$ being a particular solution, the second term of the developement contains a power of k less than one, consequently the value of $\frac{dp}{dy}$ corresponding to $y = X$, must be infinite. Hence if $\frac{dp}{dy} = \frac{K}{L}$, every particular solution must render $L = 0$, and must therefore be a divisor of it; and reciprocally every divisor of L , which is not at the same time a divisor of K , and which being made equal to nothing will verify the proposed differential equation, will be a particular solution of it.

(156.) This supposes that the differential equation is resolved with respect to the differential coefficient $\frac{dy}{dx}$, but this resolution may be avoided in observing that if $z = 0$ be this differential equation, z is a function of x , y and p ($p = \frac{dy}{dx}$), and gives

$$\frac{dz}{dx} dx + \frac{dz}{dy} dy + \frac{dz}{dp} dp = 0,$$

from which we derive

$$\frac{dp}{dy} = - \frac{\frac{dz}{dy}}{\frac{dz}{dp}}.$$

Hence if the equation $z = 0$ has been prepared in such a manner as to contain neither radical quantities nor denominators, it will be sufficient to make equal to nothing one of the divisors of $\frac{dz}{dp}$ which does not divide $\frac{dz}{dy}$ in order to render $\frac{dp}{dy}$ equal to infinity. If the differential equation $z = 0$ had not been cleared of radicals, the condition may also be satisfied by making equal to nothing one of the factors of $\frac{1}{\frac{dz}{dy}}$. The elimination of p

between the proposed differential equation, and the new equation thus obtained will give the particular solution.

This process would only give the particular solutions in which y enters; those having the form $x = a$ constant, could be obtained by considering in the proposed differential equation x as a function of y .

To give an example of the two preceding methods of discovering the particular solutions. Let

$$y - px - a(1 + p^2) = 0$$

be the proposed differential equation, p being equal to $\frac{dy}{dx}$. We derive from it

$$p = \frac{-x \pm \sqrt{x^2 + 4ay - y^2}}{2a}, \text{ and } \frac{dp}{dy} = \frac{1}{\sqrt{x^2 + 4ay - y^2}}.$$

The denominator of this expression being made equal to nothing gives $x^2 + 4ay - y^2 = 0$, which equation satisfies the differential equation, and is therefore a particular solution of it.

By the second method we shall obtain the same particular solution, without resolving the differential equation with respect to p . Here $z = y - px - a(1 + p^2)$ and $\frac{dz}{dp} = -x - 2ap$. Eliminating p between the proposed differential equation and $x + 2ap = 0$, we find as before $4ay - 4a^2 + x^2 = 0$.

(157.) From what has been already observed on the method of deducing singular solutions from general ones, (151.) it is obvious that whenever the value of p derived from the differential equation is composed of a rational function of x and y and of a power less than one of another rational function of these variables, by making this last function equal to nothing, the conditions $\frac{dp}{dy} = \infty$ will be satisfied, and therefore if the equation thus formed should satisfy the proposed differential equation it will be a particular solution of it. Thus, in the last example, p being equal to $\frac{-x \pm \sqrt{x^2 + 4ay - y^2}}{2a}$, we might have immediately obtained the particular solution by making the quantity under the radical sign equal to nothing.

(158.) When the differential coefficient p enters the proposed differential equation only in the first power, the differential equation has necessarily the form $K + pL = 0$, (K and L being functions of x and y without radicals

Integral
Calculus.

Part III.

and without denominators,) and consequently $\frac{dz}{dp}$ is in that case $= L$. This being made equal to nothing gives

the equation $L = 0$, and the elimination of p between this equation, and the proposed $K + pL = 0$ is no longer necessary, since L does not contain p . But the equation $L = 0$ cannot give a particular solution unless L , or one of the divisors of L , should be a divisor of K . Then, as it has already been stated, that divisor being made equal to nothing is a particular solution, if it cannot be derived from the general integral by giving a particular value to the constant.

(159.) Another consequence of the property which particular solutions possess of making $\frac{dz}{dp} = 0$ is that they render $\frac{dp}{dx}$ or $\frac{d^2p}{dx^2} = 0$, we have

$$\frac{dz}{dp} dp + \frac{dz}{dx} dx + \frac{dz}{dy} dy = 0,$$

from which we get

$$dp = \frac{d^2y}{dx^2} = - \frac{\left(\frac{dz}{dx} dx + \frac{dz}{dy} dy \right)}{\frac{dz}{dp}},$$

a fraction the numerator and denominator of which are both equal to nothing, since for particular solutions we have at the same time

$$\frac{dz}{dp} dp + \frac{dz}{dx} dx + \frac{dz}{dy} dy = 0, \quad \text{and} \quad \frac{dz}{dp} = 0.$$

(160.) It has been already stated that the common divisors of M and N in the differential equation $M dx + N dy = 0$, may be particular solutions. It may be proved, also, that every differential equation may be prepared in such a manner as to make its particular solution a factor of it.

Let $y = X$ be a solution of the differential equation $dy = p dx$, then $dX = P dx$, P being what p becomes when y is changed into X . If we subtract the last equation from the preceding, we find

$$\frac{dy}{dx} - \frac{dX}{dx} = p - P,$$

an equation which becomes identical by the supposition of $y = X$; hence we may suppose $p - P = Q(y - X)^m$, Q being a function which does not become infinite by the supposition of $y = X$, and m being a positive quantity.

We have, thus, $p = P + Q(y - X)^m$; but since the supposition $y = X$ makes $\frac{dp}{dy} = \infty$, m must be less than

one. Let us suppose $y - X = z$, we shall have then $p = P + Qz^m$, or $dy = (P + Qz^m) dx$. We have also $dy = dz + dX$, and hence $dz = Qz^m dx$, a result which may be written in the following manner,

$$z^m (z^{-m} dz - Q dx) = 0. \quad \text{If then we suppose } z = u^{\frac{1}{1-m}}, \text{ we have } z^m = u^{\frac{m}{1-m}}, \quad z^{-m} = u^{\frac{-m}{1-m}}, \text{ and } z^{-m} dz = \frac{du}{1-m},$$

which values being substituted in the equation, give

$$u^{\frac{m}{1-m}} \left(\frac{du}{1-m} - Q dx \right) = 0.$$

This differential equation has for factor $u^{\frac{m}{1-m}} = z = y - X$, that is the particular solution, and when it is divided by $u^{\frac{m}{1-m}}$ it becomes $\frac{du}{1-m} - Q dx = 0$, which is no longer satisfied by the particular solution $y - X = z = 0$.

(161.) The considerations which have led us to the preceding rules to obtain the particular solution of a differential equation of the first order by means of that equation may be extended to any order.

Let $\frac{dy}{dx} = y_1$, $\frac{d^2y}{dx^2} = y_2$, ..., $\frac{d^ny}{dx^n} = y_n$, and let us suppose the proposed differential equation of the n^{th} order to be resolved with respect to the differential coefficient of the $(n-1)^{\text{th}}$ order, and represented by

$$y_{n-1} = f(x, y, y_1, \dots, y_{n-2}, y_n).$$

Let $y_{n-1} = X$ be a differential equation of the $(n-1)^{\text{th}}$ order, without constant, satisfying the preceding, and let it be required to examine whether it is a particular integral or a particular solution of it. Let the general integral be $y_{n-1} = V$. If X be a particular integral, we shall have in using the same reasoning, and the same notation as in (154.)

$$y_{n-1} = X + V' h + V'' h^2 + \&c.$$

Differentiating both sides, we find

$$y_n = \frac{dX}{dx} + \frac{dX}{dy} y_1 + \frac{dX}{dy_1} y_2 + \dots + \frac{dX}{dy_{n-2}} (X + V' h + V'' h^2 + \&c.) + \frac{1}{dx} dV' \cdot h + \frac{1}{dx} dV'' h^2 + \&c.$$

Let X' be equal to the terms which do not contain h , and K the sum of those which are multiplied by the

Integral
Calculus.

powers of that quantity, then $y_n = x + k$, and that value substituted in the proposed differential equation must satisfy it independently of h . The substitution gives Part III.

$$y_{n-1} = x + v' h + v'' h^2 + \&c. = f(x, y, y_1, \dots, y_{n-2}, X') + P' K^m + P'' K^n + \&c.$$

but $X = f(x, y, y_1, \dots, y_{n-1}, X')$ since $y = X$ satisfies the proposed equation, therefore

$$v' h + v'' h^2 + \&c. = P' h^m \left\{ \frac{V' dX}{d y_{n-2}} + \frac{1}{d x} d V' + \left(V'' \frac{dX}{d y_{n-2}} + \frac{1}{d x} d V'' h^{n-1} \right) + \&c. \right\}^m + \&c.$$

Here it is when $m > 1$ that $y = X$ is a particular solution, for in order to complete the particular solution $y_{n-1} = X$, it is necessary that the function V' should not be equal to nothing, and it is precisely what ought to take place if $m > 1$, since the term $V' h$ would be the only one of that form.

In every other case, the value of V' may be determined. If $m = 1$, we have the equation

$$V' = P' \left(V' \frac{dX}{d y_{n-2}} + \frac{1}{d x} d V' \right),$$

from which the quantities $y, y_1, \dots, y_{n-2}$ may be eliminated by means of the general integral of $y_{n-1} = X$, and its differential equations down to the $(n-2)^{th}$ order.

When $m < 1$ the value of V' is determined by means of the equation

$$V' \frac{dX}{d y_{n-2}} + \frac{1}{d x} d V' = 0.$$

Now if $m > 1$ the value of $\frac{d y_{n-1}}{d y_n}$ derived from the proposed equation, and corresponding to the value $y_{n-1} = X$, must be equal to nothing. Hence a criterion by means of which the particular solutions of the $(n-1)^{th}$ order of a differential equation of the n^{th} order may be discovered.

It may happen that the value $y_{n-1} = X$ should render the right side of the proposed equation $y_{n-1} = f(x, y, y_1, y_2, \dots, y_{n-2}, y_n)$ equal to $\frac{0}{0}$. In that case this last equation may be put under the form

$$y_{n-1} = \frac{M(y_n - X')^n}{N(y_n - X')^n}, \text{ and is satisfied in taking } y_n = X'. \text{ This value agrees with } y_{n-1} = X + c \text{ when } X' \text{ does}$$

not contain X , that is to say when X does not contain y_{n-2} . In the contrary case, we find as above

$$x + v' h + \&c. = k^{n-1} (P + P' k + \&c.), \quad P \text{ being } \frac{M}{N},$$

and it is obvious that this equation cannot be satisfied independently of h unless $\mu = \nu$, $P = X$, and unless P' be not equal to nothing, in order that the term $v' h$ should remain. In every other case the equation $y_{n-1} = X$ is a particular solution.

(162.) If the proposed equation is under the form $y_n = p$, p being a function of $y, y_1, \dots, y_{n-1}$. We have then

$$d y_n = \frac{d p}{d x} d x + \frac{d p}{d y} d y + \frac{d p}{d y_1} d y_1 + \dots + \frac{d p}{d y_{n-1}} d y_{n-1},$$

from which

$$\frac{d y_{n-1}}{d y_n} = \frac{1}{\frac{d p}{d y_{n-1}}};$$

and since particular solutions satisfy the equation $\frac{d y_{n-1}}{d y_n} = 0$, they must render the quantity $\frac{d p}{d y_{n-1}}$ infinite. This

in the case where $n = 1$ is precisely the criterion we have found for particular solutions of differential equations of the first order.

(163.) When the differential equation is not resolved, if we represent it by $z = 0$, and differentiate it we find

$$\frac{d z}{d x} d x + \frac{d z}{d y} d y + \dots + \frac{d z}{d y_{n-1}} d y_{n-1} + \frac{d z}{d y_n} d y_n = 0$$

from which

$$\frac{d y_{n-1}}{d y_n} = - \frac{\frac{d z}{d y_n}}{\frac{d z}{d y_{n-1}}};$$

hence it follows that if z be a rational and integral function of y and of its differential coefficients, the particular solution $y_{n-1} = X$ will render equal to nothing the function $\frac{d z}{d y_n}$. If the quantity $\frac{d z}{d y_{n-1}}$ should vanish at the same

time in which case we would have $\frac{d y_{n-1}}{d y_n} = \frac{0}{0}$, we should find the true value of that fraction; and the only

Integral
Calculus.

Part III.

particular solutions would be those factors, which satisfying the proposed differential equation would, at the same time, render the true value of $\frac{dy_{n-1}}{dy_n}$ equal to nothing.

(164.) Let us consider now the case in which the equation without constants which satisfies the proposed differential equation is of the $(n-2)^{th}$ order. Let that equation be $y_{n-2} = X$, and the proposed differential equation by

$$y_{n-2} = f(x, y, y_1, \dots, y_{n-3}, y_{n-1}, y_n).$$

If, then, we represent the general integral by

$$y_{n-2} = X + V'h + V''h^2 + \&c.,$$

differentiate twice, and each time substitute for y_{n-2} its value, we find

$$\left. \begin{aligned} y_{n-1} &= \frac{dX}{dx} + \frac{dX}{dy} y_1 \dots + \frac{dX}{dy_{n-3}} (X + V'h + \&c.) \\ &\quad + \frac{1}{dx} dV'h + \&c. \end{aligned} \right\} = X' + w'h,$$

$$\left. \begin{aligned} y_n &= \frac{dX'}{dx} + \frac{dX'}{dy} y_1 \dots + \frac{dX'}{dy_{n-3}} (X + V'h + \&c.) \\ &\quad + \frac{1}{dx} dw'h. \end{aligned} \right\} = X'' + w'h,$$

X' and X'' being functions independent of h . The substitution of these expressions, in the proposed equation, and its development according to the powers of h , will easily be obtained, when we shall have the development of what becomes $f(x, y, y_1, \dots, y_{n-3}, y_{n-1}, y_n)$, when $y_{n-1} + k$, and $y_n + l$ are substituted for y_{n-1} and y_n . Let us suppose that this development has the following form,

$$f(x, y, y_1, \dots, y_{n-3}, y_{n-1}, y_n) + P'K^m + Q'l^n + \&c.$$

making now $y_{n-1} = x'$, $y_n = X''$, $k = w'h$, $l = w'h$, substituting in the proposed equation and reducing, it will remain

$$v'h + \&c. = P'w'^m h^m + Q'w'^n h^n + \&c.$$

an equation which cannot be satisfied independently of h , when m and n are > 1 , without making $v' = 0$, which shows that in that case no arbitrary constant can be introduced in the equation $y_{n-2} = X$, and therefore in that case it is a particular solution. Hence the two equations,

$$\frac{dy_{n-2}}{dy_{n-1}} = 0, \quad \frac{dy_{n-2}}{dy_n} = 0,$$

must be satisfied as well as the proposed differential equation, if $y_{n-2} = X$ be a particular solution.

By a similar process we would obtain the conditions which must be satisfied in order that an equation of the $(n-3)^{th}$ order should be a particular solution, and generally in order that an equation of the $(n-m)^{th}$ order, without arbitrary constant, should be a particular solution. If the proposed differential equation may be put under the form

$$y_{n-m} = f(x, y, y_1, \dots, y_{n-m-1}, y_{n-m+1}, \dots, y_n)$$

the particular solutions will satisfy the equations

$$\frac{dy_{n-m}}{dy_{n-m+1}} = 0, \frac{dy_{n-m}}{dy_{n-m+2}} = 0 \dots \frac{dy_{n-m}}{dy_{n-1}} = 0, \frac{dy_{n-m}}{dy_n} = 0$$

When the differential equation is not resolved with respect to y_{n-m} , if it be cleared of radicals and denominators, and represented by $z = 0$, the particular solutions of the order $(n-m)^{th}$ must satisfy the following equations,

$$\frac{dz}{dy_n} = 0, \frac{dz}{dy_{n-1}} = 0, \frac{dz}{dy_{n-2}} = 0 \dots \frac{dz}{dy_{n-m+1}} = 0.$$

(165.) The equation $z = 0$ gives

$$\frac{dz}{dx} dx + \frac{dz}{dy} dy + \frac{dz}{dy_1} dy_1 + \dots + \frac{dz}{dy_n} dy_n = 0,$$

and hence

$$dy_n = - \frac{\frac{dz}{dx} dx + \frac{dz}{dy} dy + \dots + \frac{dz}{dy_{n-1}} dy_{n-1}}{\frac{dz}{dy_n} dy_n};$$

therefore every particular solution must give $dy_n = 0$, since the substitution of the values of $y_n, y_{n-1}, \dots, y$, expressed in x , verifying the equation $dz = 0$, independently of its last term $\frac{dz}{dy_n} dy_n$, which the particular solution

Integral Calculus. makes equal to nothing, it is necessary that the remainder of the equation $dz = 0$, that is the numerator of the expression of dy_n should vanish also; hence every particular solution of a differential equation of the n^{th} order must make $\frac{d^{n+1}y}{dx^{n+1}} = 0$. Part III.

We shall take for example of the preceding rules, the following equation of the second order,

$$z = y - xy_1 + \frac{x^2}{2}y_2 - y_1y_2 + xy_2^2 = 0,$$

where as before $y_1 = \frac{dy}{dx}$ and $y_2 = \frac{d^2y}{dx^2}$. We find $\frac{dz}{dy_2} = \frac{x^2}{2} - y_1 + 2xy_2 = 0$; eliminating y_2 between this equation and the proposed we find

$$4xy - 3y_1x^2 - y_2^2 - \frac{x^4}{4} = 0,$$

which satisfies the proposed equation, and is consequently a particular solution of the first order.

This equation admits of no particular solution, containing x and y only, for to obtain it, we should eliminate y_1 and y_2 between $z = 0$, $\frac{dz}{dy_1} = 0$, and $\frac{dz}{dy_2} = 0$, that is between the proposed equation and $\frac{x^2}{2} - y_1 + 2xy_2 = 0$, and $x + y_2 = 0$. The elimination gives $y + \frac{x^3}{2} = 0$, which does not satisfy the proposed equation.

For more details upon the subject of particular solutions we must refer the reader to Lagrange's *Leçons sur le Calcul des Fonctions*; *Journal de l'Ecole Polytechnique*, 12^{me} cahier; *Mémoires de l'Académie des Sciences de Turin*, 1790—1791.

On the Integration of Differential Equations by Series.

(166.) The number of differential equations which can be completely integrated is very small, and, moreover most of the integrals are complicated and transcendental, and generally can be of little use to find the value of one of the variables in terms of the other, even by approximation. It is therefore of great importance to be able to derive from the differential equation the value of one of the variables, in a series of terms functions of the other, and easy to calculate. These series, when they are converging for the given value of the variable contained in the series, will give approximatively the value of the other.

(167.) The method which naturally presents itself to obtain these series is that of indeterminate coefficients. An example will show the manner in which it is employed.

Let the proposed equation be $dy + ydx = gx^m dx$, and assume

$$y = Ax^a + Bx^{a+1} + Cx^{a+2} + \&c.$$

We are to determine a , A , B , C , &c., by the condition that this value being substituted in the differential equation will satisfy it. The substitution gives

$$\left. \begin{aligned} aAx^{a-1} + (a+1)Bx^a + (a+2)Cx^{a+1} + (a+3)Dx^{a+2} + \&c. \\ -gx^m + Ax^a + Bx^{a+1} + Cx^{a+2} + \&c. \end{aligned} \right\} = 0;$$

and hence we find

$$a = m + 1, \quad A = \frac{g}{a}, \quad B = \frac{-g}{a(a+1)}, \quad C = \frac{+g}{a(a+1)(a+2)}, \quad D = \frac{-g}{a(a+1)(a+2)(a+3)}, \quad \&c.;$$

and, consequently,

$$y = g \left\{ \frac{x^{m+1}}{m+1} - \frac{x^{m+2}}{(m+1)(m+2)} + \frac{x^{m+3}}{(m+1)(m+2)(m+3)} - \&c. \right\}.$$

This value is not the general integral since it does not contain an arbitrary constant. But by a little modification of the same method the general integral may be obtained. For we shall suppose that to $x = a$ corresponds the value $y = b$, and we shall substitute in the differential equation $a + t$ for x , and $b + u$ for y , u being supposed to become equal to nothing when $t = 0$. We may therefore assume

$$u = At^s + Bt^{s+1} + Ct^{s+2} + \&c.;$$

substituting this value in the differential equation which becomes

$$du + (b + u)dt = g(a + t)^m dt,$$

we shall find

$$a = 1, \quad A = ga^m - b, \quad B = gm a^{m-1} - ga^m + b, \quad C = gm(m-1)a^{m-2} - gm a^{m-1} + ga^m - b, \quad \&c.$$

The equation $dy + xdy - mydx = 0$, treated in the same manner, leads to the following value of y .

Integral Calculus. $y = b \left\{ 1 + \frac{m(x-a)}{a+1} + \frac{m(m-1)}{1 \cdot 2} \frac{(x-a)^2}{(a+1)^2} + \&c. \right\} = b \left\{ 1 + \frac{x-a}{a+1} \right\}^m = \frac{b}{(a+1)^m} (1+x)^m = A(1+x)^m$; Part III.

a series which therefore agrees with the integral obtained by the separation of the variables in the differential equation.

(168.) We shall take for another example the equation $d^2 y + a y x^{m-2} dx = 0$, and we shall assume

$$y = A x^a + B x^{a+\delta} + c x^{a+2\delta} + \&c.;$$

substituting the value of y and $d^2 y$ in the equation, we find, by a comparison of the terms of the result, that we must make $a = 0$ or $a = 1$, and also $\delta = m$: from these two hypotheses we deduce two series for y , each of which contains an arbitrary constant; for in either hypothesis the value of y remains indeterminate, and consequently, being added together, gives the general integral. We find thus

$$y = A \left\{ 1 - \frac{a x^m}{(m-1)m} + \frac{a^2 x^{2m}}{1 \cdot 2 (m-1)(2m-1)m^2} - \frac{a^3 x^{3m}}{1 \cdot 2 \cdot 3 (m-1)(2m-1)(3m-1)m^3} + \&c. \right\} \\ + A' \left\{ x - \frac{a x^{m+1}}{(m+1)m} + \frac{a^2 x^{2m+1}}{1 \cdot 2 (m+1)(2m+1)m^2} - \frac{a^3 x^{3m+1}}{1 \cdot 2 \cdot 3 (m+1)(2m+1)(3m+1)m^3} + \&c. \right\}$$

These series cannot be used for every value of m ; they both fail when $m = 0$; the first when $m = 1$, and the second when $m = -1$, or more generally, i being an integer, the first when $m = \frac{1}{i}$, and the second when $m = -\frac{1}{i}$. We cannot give here this interesting discussion, and must refer the reader to the VIIth Chapter of

the IIId volume of Euler's *Integral Calculus*, or, after him, to Lacroix's *Integral Calculus*, in 4to., or to the *Collection of Examples of the Differential and Integral Calculus* by G. Peacock.

(169.) Another series may be derived from Taylor's Theorem. If we suppose the differential equation resolved with respect to the differential coefficient of the highest order, if we have, for instance,

$$\frac{d^n y}{dx^n} = f\left(x, y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, \dots, \frac{d^{n-1} y}{dx^{n-1}}\right),$$

it is obvious we shall find

$$y = A + A_1 \frac{(x-a)}{1} + A_2 \frac{(x-a)^2}{1 \cdot 2} + \dots + f(a, A, A_1, A_2, \dots, A_{n-1}) \frac{(x-a)^n}{1 \cdot 2 \cdot 3 \dots n} + \&c.,$$

where $A, A_1, A_2, \dots, A_{n-1}$ represent the values of $y, \frac{dy}{dx}, \frac{d^2 y}{dx^2}, \&c.$ when $x = a$ and are considered as arbitrary constants.

Let us take for example the differential equation $\frac{d^2 y}{dx^2} + \frac{y}{ex} = 0$, and let b and c be the values of y and $\frac{dy}{dx}$ corresponding to $x = a$. We shall have for $x = a$

$$\frac{d^2 y}{dx^2} = -\frac{b}{ea}, \quad \frac{d^3 y}{dx^3} = \frac{b-ac}{ea^2}, \quad \&c.$$

consequently

$$y = b + c \frac{(x-a)}{1} - \frac{b}{ea} \frac{(x-a)^2}{1 \cdot 2} + \frac{b-ac}{ea^2} \frac{(x-a)^3}{1 \cdot 2 \cdot 3} + \&c.$$

If $(x-a)$ be a small quantity, and if none of the differential coefficients as far as the series is carried becomes infinite, this series will easily give the value of y . If the value of x , corresponding to which the value of y is required, were much greater than a , we might suppose $x-a$ equal to nh , so that by taking n a large number we may always suppose h to be a very small quantity; then we might first calculate the value of y corresponding to $a+h$, and then, by means of it, the value of y corresponding to $x = a+2h$, and so on, by successive substitutions, we might find the required value. It must be observed, however, that this method frequently fails in giving the approximation with sufficient accuracy, in consequence of the accumulation of errors introduced after each operation.

(170.) Mr. Kramp has proposed another method. He first observes that if the general integral were known and the value of y expressed in function of x , it would always be possible, whatever be the nature of this function, to represent its value corresponding to any particular value a of x , or to any value differing but little from a by an expression of the form $A + Bx + Cx^2 + \dots + Rx^m$, and indeed that the difference between the value of y derived from the general integral, and the value of y derived from the equation $y = A + Bx + Cx^2 + \dots + Rx^m$, might be decreased at will by taking m sufficiently great. Let us see, now, if the values of the coefficients $A, B, C, \dots, R$, cannot be determined without the general integral. They must first satisfy the conditions relative to the determination of constants, which, in supposing the differential equation of the n^{th} order, will give n equations between $A, B, C, \dots, R$. It will remain only to find $m-n+1$ others.

Integral
Calculus.From the assumed value of y , we derive

$$\frac{dy}{dx} = B + 2Cx + 3Dx^2 + \dots + mRx^{m-1}$$

$$\frac{d^2y}{dx^2} = 2C + 6Dx + 12Ex^2 + \dots + m(m-1)Rx^{m-2}$$

.....
.....

These values must satisfy the differential equation, which I represent by $\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$. The

result of their substitution will be a certain function of $x, A, B, C, \dots, R$, which would be identically, or very nearly, equal to nothing when a , or values little differing from a , are substituted for x . If, therefore, we express that it is equal to nothing for $m+n-1$ such values of x , we shall get between the coefficients $A, B, C, \dots, R$, the necessary number of equations to complete their determination.

As the number m is an arbitrary number, we may always take it so that $m+n-1$ shall be an odd number $2p+1$; then it will be sufficient to substitute successively for $x, a, a \pm z, a \pm 2z, \dots, a \pm pz$, z being a very small quantity; or we may first substitute, in the proposed equation, $a+zx$ for x , and then we shall have only to make, in the equation of condition, $x=0, x=\pm 1, x=\pm 2, \dots, x=\pm p$.

If the proposed equation $\phi\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$ contain the differential coefficients in a degree

higher than the first, the substitutions of the values of $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, &c., derived from the assumed value of y , would

be laborious, and moreover the coefficients A, B, C would enter the equation in a degree higher than the first, and the equations, by means of which their values are to be found, would be difficult to resolve. To avoid this, the best way would be to differentiate the proposed equation a sufficient number of times, and to combine the differential equations together so as to obtain the simplest possible differential equation. It would become then necessary to add to the conditions already established for the determination of the constants, a number of others precisely equal to the excess of the order of differentiation of the new equation over the order of the proposed equation.

The same process of differentiating the proposed equation may be used in case it should contain transcendental functions, in order to obtain an algebraical equation.

We shall apply this method to a very simple example, but which will be sufficient to show the particular advantages of this method.

Let the equation be $x \frac{dy}{dx} = 1$; it is required to find the value of y corresponding to a given value a of x , supposing besides that the arbitrary constant, which ought to enter into the general integral, is to be determined by the condition that $y=0$ when $x=1$.

If we change x into $a+zx$, the proposed equation becomes $(a+zx) \frac{dy}{dx} - z = 0$, and the constant must be such that when $a+zx=1$, or $x=\frac{1-a}{z}$, y shall be equal to nothing. Let us first assume $y=A+Bx$,

we shall have $\frac{dy}{dx}=B$, and these values being substituted give $(a+zx)B - z = 0$; making $x=0$ we have

$aB - z = 0$; the condition relative to the constant gives, besides, $0 = A + \frac{(1-a)}{z}B$; eliminating b between

these two equations, from which z disappears, we shall find $A = \frac{a-1}{a}$, and $B = \frac{z}{a}$, therefore $y = \frac{a-1}{a} + \frac{Bz}{a}$;

and if we suppose $z=0$, simply $y = \frac{a-1}{a}$, or generally the value of y corresponding to any value x is equal to $\frac{x-1}{x}$. In this example, we may easily compare the exact value with the approximate one, for the differential

equation obviously gives $y = lx$, therefore we have approximatively $lx = \frac{x-1}{x}$. This formula is exact for

$x=0$ and for $x=1$, and for numbers very nearly equal to one it is sufficiently accurate.

To have a greater approximation, we shall assume

$$y = A + Bx + Cx^2 + Dy^3, \text{ then } \frac{dy}{dx} = B + 2Cx + 3Dx^2,$$

and these values being substituted, give

$$(a+zx)(B+2Cx+3Dx^2) - z = 0, \text{ or } (aB-z) + (Bz+2aC)x + (2Cz+3aD)x^2 + 3Dzx^3 = 0,$$

Integral
Calculus.

 and substituting successively for x , in that equation, the values $-1, 0, +1$, we find

$$0 = (aB - z) - (Bz + 2aC) + (2Cz + 3aD) + 3Dz,$$

$$0 = aB - z, \quad 0 = (aB - z) + (Bz + 2aC) + (2Cz + 3aD) + 3Dz,$$

from which we derive

$$B = +\frac{z}{a}, \quad C = -\frac{z^2}{2(a^2 - z^2)}, \quad D = +\frac{z^3}{3a(a^2 - z^2)}.$$

But the condition relative to the constant gives

$$A = \frac{(a-1)}{z} B - \frac{(a-1)^2}{z^2} C + \frac{(a-1)^3}{z^3} D;$$

 substituting for B, C, D their values, we shall find

$$A = \frac{a-1}{a} + \frac{(a-1)^2}{2(a^2 - z^2)} + \frac{(a-1)^3}{3a(a^2 - z^2)};$$

 and finally, in supposing $z = 0$, and changing a into x ,

$$y = lx = \frac{x-1}{x} + \frac{(x-1)^2}{2x^2} + \frac{(x-1)^3}{3x^3};$$

 a formula more exact than the preceding, but only for values of x differing but little from one

 In the same manner, if we assume $y = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5$, we shall find

$$lx = \frac{x-1}{x} + \frac{(x-1)^2}{2x^2} + \frac{(x-1)^3}{3x^3} + \frac{(x-1)^4}{4x^4} + \frac{(x-1)^5}{5x^5} +$$

 It is not necessary to carry the calculation any further to be led to suppose that the exact value of y would be

$$y = lx = \frac{x-1}{x} + \frac{1}{2} \left(\frac{x-1}{x} \right)^2 + \frac{1}{3} \left(\frac{x-1}{x} \right)^3 + \frac{1}{4} \left(\frac{x-1}{x} \right)^4 + \frac{1}{5} \left(\frac{x-1}{x} \right)^5 + \frac{1}{6} \left(\frac{x-1}{x} \right)^6 + \&c.,$$

a formula which is the known developement of the logarithm.

 For more details on this method see *Annales de Mathématiques de Gergonne*, vol. x. and xi.

 (171.) The value of y may also be obtained under the form of a continued fraction. Let the differential

 equation be $P + Qy + Ry^2 + S \frac{dy}{dx} = 0$, P, Q, R, S being any function whatever of x , we shall assume

$$y = \frac{Ax^a}{1 + Bx^b} \cdot \frac{1 + Cx^c}{1 + Dx^d} \cdot \frac{1 + Ex^e}{1 + \dots}$$

in which expression both the exponents and coefficients are indeterminate. Let

$$y''' = \frac{Dx^d}{1 + \&c.}, \quad y'' = \frac{Cx^c}{1 + y'''}, \quad y' = \frac{Bx^b}{1 + y''}, \quad \text{and } y = \frac{Ax^a}{1 + y'}.$$

 Let us suppose $Ax^a = X$, $Bx^b = X'$, $Cx^c = X''$, we shall have

$$y = \frac{X}{1 + y'}, \quad y' = \frac{X'}{1 + y''}, \quad y'' = \frac{X''}{1 + y'''} \&c.$$

 If then we substitute $\frac{X}{1 + y'}$ instead of y in the proposed equation, we shall obtain a result of the following form,

$$P' + Q'y' + R'y'^2 + S' \frac{dy'}{dx} = 0,$$

 in which $P' = P + QX + RX^2 + S \frac{dX}{dx}$, $Q' = 2P + QX + S \frac{dX}{dx}$, $R' = P$, $S' = -SX$.

 If now we put for y' its value $\frac{X'}{1 + y''}$, we shall have a result which shall be deduced from the preceding in the same manner as it was itself derived from the proposed equation, and therefore if we represent it by

$$P'' + Q''y'' + R''y''^2 + S'' \frac{dy''}{dx} = 0,$$

we shall have

$$P'' = P' + Q'X' + R'X'^2 + S' \frac{dX'}{dx}, \quad Q'' = 2P' + Q'X' + S' \frac{dX'}{dx}, \quad R'' = P', \quad S'' = -S'X'$$

We could obtain in the same manner the result of the substitution of $\frac{C x^\gamma}{1 + y''}$ for y'' , and so on for all the others.

Now to determine the value of X or $A x^a$, we shall first neglect y' , and therefore we shall simply substitute $A x^a$ and $a A x^{a-1}$ for y and $d y$ in the proposed equation, and determine A and a by the condition that the result of that substitution is an identical equation. This value of X once determined, we shall know the coefficients of

the equation resulting from the substitution of $\frac{X}{1 + y}$ for y , in the proposed equation; and since the value of y' is

reduced to X' when y'' is neglected, we shall be able to determine, by means of this transformed equation, the value of X' , in the same manner as we have determined the value of X by means of the proposed equation; thus, we shall find successively the terms of the continued fraction. If one of the quantities $P', P'', P''', \&c.$ which form the first term of the transformed equations, should become nothing, the continued fraction would have a limited number of terms

If we suppose, for instance, that $P''' = 0$, then the third transformed will be $Q''' y''' + R''' y''^2 + S''' \frac{d y''}{d x} = 0$,

which will be satisfied by $y''' = 0$. It is true that this value of y''' is only a particular solution or a particular integral of the preceding equation; but in this case the general integral or complete value of y''' could be

obtained, for by making $y''' = \frac{1}{z}$ the equation is transformed into $Q''' z + R''' - S''' \frac{d z}{d x} = 0$, which can be integrated.

We shall apply this general method to the following example. Let the differential equation be

$$m y + (1 + x) \frac{d y}{d x} = 0.$$

Let us first substitute $A x^a$ for y , we shall find $(m A + a A) x^a + a A x^{a-1} = 0$, or $(m + a) x + a = 0$. Neglecting as a first approximation the term multiplied by x , we find $a = 0$, and consequently A remains indeterminate, and will be the arbitrary constant of the integral. We have, therefore, in this case $X = A$, $P = 0$, $Q = m$, $R = 0$, $S = 1 + x$, and the first transformed equation will be

$$m A + m A y' - (1 + x) A \frac{d y'}{d x} = 0, \text{ or } m + m y' - (1 + x) \frac{d y'}{d x} = 0.$$

If we change now y' into $B x^b$, and $d y'$ into $\beta B x^{b-1} d x$, we find

$$m + m B x^b - \beta B x^{b-1} - \beta B x^b = 0,$$

an equation which is satisfied by making $\beta = 1$ and $B = m$, therefore $X' = m x$; and hence for the second transformed equation

$$(m - 1) x + (1 + (m - 1) x) y'' + y'^2 + (1 + x) x \frac{d y''}{d x} = 0,$$

putting $C x^\gamma$ for y'' , and $\gamma C x^{\gamma-1} d x$, we shall find $\gamma = 1$ or $\gamma = 0$; but the first of these values must be taken, for the exponent of x in the numerator of each of the fractions which follows $\frac{A x^a}{1}$ must be greater than one, since

all these fractions must vanish at the same time as x . We find then $C = -\frac{m-1}{2}$, and $X'' = -\frac{(m-1)x}{1 \cdot 2}$

These operations being carried as far as the sixth transformation, we shall find

$$X''' = \frac{(m+1)x}{3 \cdot 2}, X^{iv} = -\frac{(m-2)x}{3 \cdot 2}, X^v = \frac{(m+2)x}{5 \cdot 2}, X^{vi} = -\frac{(m-3)x}{5 \cdot 2}, \text{ and therefore}$$

$$y = \frac{A}{1 + \frac{m x}{1 - \frac{(m-1)x}{1 - \frac{(m+1)x}{1 - \frac{(m-2)x}{1 - \frac{(m+2)x}{1 - \frac{(m-3)x}{5 \cdot 2}}}}}}}$$

The proposed differential equation is easily integrated, and gives $y = C (1 + x)^{-m}$. When $x = 0$ this value of

Integral
Calculus

y is reduced to $y = C$, and for the same hypothesis, the value of y expressed by a continued fraction is reduced to A , therefore $A = C$, and we shall have consequently $(1+x)^{-m}$ equal to the above continued fraction, after having substituted 1 for A , or again $(1+x)^m$ equal to the denominator of A , carried indefinitely, if m be a fraction, or down to the numerator which becomes equal to nothing, if m be an integer. It is easy to derive from this expression the values of $l(1+x)$ and e^x , under the form of continued fractions. They may also be

found by integrating in the same manner the equations $1 - (1+x) \frac{dy}{dx} = 0$, $y - \frac{dy}{dx} = 0$

They are

$$l(1+x) = \frac{x}{1 + \frac{1}{1} \cdot \frac{x}{2}} \\ \frac{1}{1 + \frac{1}{3} \cdot \frac{x}{2}} \\ \frac{2}{1 + \frac{2}{3} \cdot \frac{x}{2}} \\ \frac{2}{1 + \frac{2}{5} \cdot \frac{x}{2}} \\ \frac{3}{1 + \frac{3}{5} \cdot \frac{x}{2}} \\ 1 + \&c$$

$$e^x = 1 + \frac{x}{1 - \frac{1}{1} \cdot \frac{x}{2}} \\ \frac{1}{1 + \frac{1}{3} \cdot \frac{x}{2}} \\ \frac{1}{1 - \frac{1}{3} \cdot \frac{x}{2}} \\ \frac{1}{1 + \frac{1}{5} \cdot \frac{x}{2}} \\ \frac{1}{1 - \frac{1}{5} \cdot \frac{x}{2}} \\ 1 + \&c$$

(172.) Another process to integrate by series differential equations is founded upon Newton's method of successive substitutions combined with the integration of differential equations of the first degree. Let the proposed

equation be $\frac{d^2 y}{dx^2} + y + a y^2 = b$, in which a is supposed to be a very small quantity, and assume

$$y = Y + a Y' + a^2 Y'' + a^3 Y''' + \&c. \dots Y, Y', Y'', Y''', \&c.$$

being unknown functions of x . This value being substituted in the proposed differential equation gives

$$\frac{d^2 y}{dx^2} + Y - b + a \left(\frac{d^2 Y'}{dx^2} + Y' + Y^2 \right) + a \left(\frac{d^2 Y''}{dx^2} + Y'' + 2 Y Y' \right) \\ + a^2 \left(\frac{d^2 Y'''}{dx^2} + Y''' + Y'^2 + 2 Y Y'' \right) + \&c. \dots = 0.$$

Making equal to nothing the coefficient of each power of a , we find as many differential equations of the second order and first degree as is necessary to find the values of all the quantities $Y, Y', Y'', \&c.$ These equations are

$$\frac{d^2 y}{dx^2} + Y = b, \quad \frac{d^2 Y'}{dx^2} + Y' = -Y^2, \quad \frac{d^2 Y''}{dx^2} + Y'' = -2 Y Y',$$

$$\frac{d^2 Y'''}{dx^2} + Y''' = -Y'^2 - 2 Y Y'', \&c.$$

The first will give the value of y , which will be substituted in the second in order to obtain that of y' , and so on. These equations may be integrated; and the values derived from the three first, are

$$Y = b + p \cos x + q \sin x$$

$$Y' = -\frac{1}{2}(2b^2 + p^2 + q^2) + b q x \cos x + \frac{1}{6}(p^2 - q^2) \cos 2x - b p x \sin x + \frac{1}{3} p q \sin 2x$$

$$Y'' = b(2b^2 + p^2 + q^2) - \left[\frac{1}{12} q (18b^2 + 5p^2 + 5q^2) x + \frac{1}{2} p b^2 x^2 \right] \cos x \\ + \left[\frac{1}{12} p (18b^2 + 5p^2 + 5q^2) x - \frac{1}{2} q b^2 x^2 \right] \sin x \\ - \left[\frac{1}{3} b (p^2 - q^2) - \frac{2}{3} b p q x \right] \cos 2x + \frac{1}{48} p (p^2 - 3q^2) \cos 3x \\ - \left[\frac{1}{3} b (p^2 - q^2) + \frac{2}{3} b p q x \right] \sin 2x + \frac{1}{48} q (3p^2 - q^2) \sin 3x$$

which being substituted in the assumed value of y , will give the approximate value of this quantity by neglecting only the powers of a higher than the second. The arbitrary constants introduced by the integration of the equations which give the values of Y' and Y'' are left out, because there are already two contained in the value of Y .

(173.) It is obvious that the same method may be applied to the equations of the first degree, having the form

Integral
Calculus.

$$\frac{d^n y}{d x^n} + a \frac{d^{n-1} y}{d x^{n-1}} + b \frac{d^{n-2} y}{d x^{n-2}} + \dots + h y = R + a S,$$

$a, b, \dots, h$ being constant coefficients, R a function of x alone, S a function of x and y , and a a very small quantity. We would assume as before $y = Y + a Y' + a^2 Y'' + \&c.$ substituting this value, and making the quantities which multiply each power of y separately equal to nothing, we shall get the following equations,

$$\frac{d^n Y}{d x^n} + a \frac{d^{n-1} Y}{d x^{n-1}} + b \frac{d^{n-2} Y}{d x^{n-2}} + \dots + h Y = R,$$

$$\frac{d^n Y'}{d x^n} + a \frac{d^{n-1} Y'}{d x^{n-1}} + b \frac{d^{n-2} Y'}{d x^{n-2}} + \dots + h Y' = S',$$

$$\frac{d^n Y''}{d x^n} + a \frac{d^{n-1} Y''}{d x^{n-1}} + b \frac{d^{n-2} Y''}{d x^{n-2}} + \dots + h Y'' = S'',$$

&c.

(174.) If we had several simultaneous differential equations of the same form, the same method might be used. Let us suppose, for instance, we had the three differential equations

$$\frac{d^2 x}{d t^2} + a^2 x = R + a S \quad \frac{d^2 y}{d t^2} + a^2 y = R_1 + a S_1, \quad \frac{d^2 z}{d t^2} + a^2 z = R_2 + a S_2,$$

in which R, R_1, R_2 are functions of t alone, S, S_1, S_2 functions where the variables t, x, y , and z are mixed, and a a very small quantity. We shall assume

$$x = X + a X' + a^2 X'' + \&c., \quad y = Y + a Y' + a^2 Y'' + \&c., \quad z = Z + a Z' + a^2 Z'' + \&c.,$$

and if we designate by

$$a S' + a^2 S'' + \&c., \quad a S_1' + a^2 S_1'' + \&c., \quad a S_2' + a^2 S_2'' + \&c.$$

what the quantities $a S, a S_1, a S_2$, become by this supposition, we shall find, in operating as before, that the quantities $X, Y, Z, X', Y', Z', \&c.$ are given by the equations

$$\frac{d^2 X}{d t^2} + a^2 X = R, \quad \frac{d^2 Y}{d t^2} + a^2 Y = R_1, \quad \frac{d^2 Z}{d t^2} + a^2 Z = R_2,$$

$$\frac{d^2 X'}{d t^2} + a^2 X' = S', \quad \frac{d^2 Y'}{d t^2} + a^2 Y' = S_1', \quad \frac{d^2 Z'}{d t^2} + a^2 Z' = S_2',$$

&c.

The functions S', S_1', S_2' , depend only on the variable t and on the functions $X, Y, Z, \&c.$, and consequently the integration of the equations which form the first line will lead to the integration of all the others.

(175.) Another method of integration is founded on the variation of arbitrary constants. Let $y = \phi(x, c_1, c_2, \dots, c_n)$

be the general integral of $\frac{d^n y}{d x^n} + U = 0$. This integral may easily be extended to $\frac{d^n y}{d x^n} + U = a V$, in con-

sidering the constants $c_1, c_2, \dots, c_n$ as functions of the variables. These quantities being n in number they may be chosen so as to satisfy, besides the proposed equation, $n - 1$ other conditions, so, for instance, as to make the values of the $(n - 1)$ first differential coefficients of y derived from the general integral, the same as if $c_1, c_2, \dots, c_n$ were constants. These conditions will give the following equations,

$$\frac{d y}{d c_1} d c_1 + \frac{d y}{d c_2} d c_2 + \dots + \frac{d y}{d c_n} d c_n = 0$$

$$\frac{d^2 y}{d x d c_1} d c_1 + \frac{d^2 y}{d x d c_2} d c_2 + \dots + \frac{d^2 y}{d x d c_n} d c_n = 0$$

$$\dots \dots \dots$$

$$\frac{d^{n-1} y}{d x^{n-2} d c_1} d c_1 + \frac{d^{n-1} y}{d x^{n-2} d c_2} d c_2 + \dots + \frac{d^{n-1} y}{d x^{n-2} d c_n} d c_n = 0.$$

In this manner the functions U and V , which in the most general case can only contain $x, y, \frac{d y}{d x}, \dots, \frac{d^{n-1} y}{d x^{n-1}}$,

will remain the same, and since when the quantities $c_1, c_2, \dots, c_n$ are not supposed to be variable we have

$$\frac{d^n y}{d x^n} + U = 0, \text{ it will remain simply after the substitution in the equation } \frac{d^n y}{d x^n} + U = a U,$$

$$\frac{d^n y}{d x^{n-1} d c_1} d c_1 + \frac{d^n y}{d x^{n-1} d c_2} d c_2 + \dots + \frac{d^n y}{d x^{n-1} d c_n} d c_n = a U d x.$$

This equation with the $(n - 1)$ preceding will give the values of $d c_1, d c_2, \dots, d c_n$. They will be of the form
 $d c_1 = a A_1 V d x, \quad d c_2 = a A_2 V d x, \dots, d c_n = a A_n V d x,$

Integral Calculus. $A_1, A_2, \dots, A_n$ containing as well as V , besides the variable x , the quantities $c_1, c_2, \dots, c_n$; but since these last quantities are constant when $a = 0$, we may suppose that their actual values are developed according to the powers of a in the following form, Part III.

$$c_1 = C_1 + a z_1 + a^2 z_1' + \&c.$$

$$c_2 = C_2 + a z_2 + a^2 z_2' + \&c.$$

$$\dots\dots\dots$$

$$c_n = C_n + a z_n + a^2 z_n' + \&c.$$

$c_1, c_2, \dots, c_n$ being constant quantities. Substituting these values in the preceding expressions of $d c_1, d c_2, \dots, d c_n$, and developing the right sides according to the powers of a , we shall obtain results of the following form,

$$a d z_1 + a^2 d z_1' + \&c. = (a p_1 + a^2 p_1' + \&c.) d x,$$

$$a d z_2 + a^2 d z_2' + \&c. = (a p_2 + a^2 p_2' + \&c.) d x,$$

$$\dots\dots\dots$$

$$a d z_n + a^2 d z_n' + \&c. = (a p_n + a^2 p_n' + \&c.) d x.$$

It is obvious that $p_1, p_2, \&c. p_n$, will only contain the variable x , and the constants $c_1, c_2, \dots, c_n$, consequently that the values

$$d z_1 = p_1 d x, d z_2 = p_2 d x, \dots, d z_n = p_n d x,$$

which are the first approximations, are the same as those which would have been obtained in considering in the equations

$$d c_1 = a A_1 V d x, d c_2 = a A_2 V d x, \dots, d c_n = a A_n V d x,$$

the quantities $c_1, c_2, \dots, c_n$ as constant, in the right sides.

The second approximation will depend on the equations

$$d z_1' = p_1' d x, d z_2' = p_2' d x, \dots, d z_n' = p_n' d x,$$

in the right sides of which, the only quantities which will enter will be the $z_1, z_2, \&c.$ determined by the first approximation. (See *Mémoires de l'Académie de Berlin*, 1781; *Mémoires de l'Institut, pour les Années 1808 et 1809*; *Journal de l'Ecole Polytechnique*, 15me Cahier; *Mécanique Analytique de Lagrange*, tome premier, seconde partie, cinquième section.)

Integration of Differential Equations containing several variables.

(176.) The rules to find a function of several variables when its complete differential, or one of its partial differential coefficients, is explicitly given, have been stated before. We shall now consider the case in which it is only relations expressed by means of equations, between the variables, and all, or some of the partial differential coefficients that are given, and in which it is the relations between the independent variables, and the variables which are functions of them, that are to be determined.

(177.) The simplest case is when the given equation contains all the partial differential coefficients, and each of them in the first power. It is then called a *total differential equation of the first order*. If it contains three variables, it will be of the form

$$P d x + Q d y + R d z = 0,$$

and if the left side of this equation be the exact differential of a function of three variables, the integral will be immediately obtained by the rules previously given, and this integral being made equal to nothing will give the required relation between x, y, z .

(178.) When the equation $P d x + Q d y + R d z = 0$ is not the immediate differential of a function of three variables, that is when the quantity $P d x + Q d y + R d z$ does not satisfy the conditions of integrability, it is sometimes possible to render it the immediate differential of a function of three variables by multiplying it by a factor. Let μ be this factor, and let us suppose that $\mu P d x + \mu Q d y + \mu R d z$ is an exact differential, we shall have then

$$\frac{d \mu R}{d y} = \frac{d \mu Q}{d z}, \quad \frac{d \mu R}{d x} = \frac{d \mu P}{d z}, \quad \frac{d \mu Q}{d x} = \frac{d \mu P}{d y}.$$

These equations being developed, give

$$\mu \left(\frac{d R}{d y} - \frac{d Q}{d z} \right) + R \frac{d \mu}{d y} - Q \frac{d \mu}{d z} = 0.$$

$$\mu \left(\frac{d R}{d x} - \frac{d P}{d z} \right) + R \frac{d \mu}{d x} - P \frac{d \mu}{d z} = 0.$$

$$\mu \left(\frac{d Q}{d x} - \frac{d P}{d y} \right) + Q \frac{d \mu}{d x} - P \frac{d \mu}{d y} = 0.$$

If we multiply the first by P , the second by $-Q$, and the third by R , and if we add the products, the quantity μ will be eliminated, and we find the equation

$$R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) = 0.$$

If the coefficients P, Q, R of the proposed equation cannot satisfy it, then it is impossible to find a factor μ which will render it an exact differential.

(179.) When the above condition is fulfilled, the integration of the proposed differential equation depends upon the integration of differential equations of two variables.

To prove it, let us suppose in the proposed equation $P dx + Q dy + R dz = 0$, z constant, it is then reduced to $P dx + Q dy = 0$, and if, in the same hypothesis, μ be the factor by which $P dx + Q dy$ ought to be multiplied to become an exact differential, if $U = \int \mu (P dx + Q dy)$, the integral of the equation $P dx + Q dy = 0$ would be $U = C$, C being an arbitrary constant. Now let us see if we could not determine a function of z , such as to make $U = \phi(z)$ the integral of $P dx + Q dy + R dz = 0$. If we differentiate $u = \phi(z)$, in considering x, y and z as variable quantities, we find

$$\mu (P dx + Q dy) + \frac{dU}{dz} dz = \frac{d\phi(z)}{dz} dz.$$

$$\text{But } \mu (P dx + Q dy) = -\mu R dz, \text{ hence } -\mu R dz + \frac{dU}{dz} dz = \frac{d\phi(z)}{dz} dz, \text{ or } -\mu R + \frac{dU}{dz} = \frac{d\phi(z)}{dz}.$$

To be able to determine $\phi(z)$, the left side of the last equation must reduce itself to a function of z by combining it, if necessary, with the equation $U = \phi(z)$. Therefore, if we suppose that we have taken the value of y or x in

the equation $U = \phi(z)$, and substituted it in the quantity $-\mu R + \frac{dU}{dz}$, the differential coefficient of that quantity with respect to x in the first supposition, or with respect to y in the second, must be equal to nothing.

Let us develop, for instance, $\frac{d\left(-\mu R + \frac{dU}{dz}\right)}{dy}$, we shall find, in considering x as a function of y ,

$$-\mu \left(\frac{dR}{dy} + \frac{dR}{dx} \cdot \frac{dx}{dy} \right) - R \left(\frac{d\mu}{dy} + \frac{d\mu}{dx} \cdot \frac{dx}{dy} \right) + \frac{d^2 U}{dy dz} + \frac{d^2 U}{dx dz} \cdot \frac{dx}{dy} = 0,$$

$$\text{but } \frac{d^2 U}{dy dz} = \frac{d}{dz} \cdot \frac{dU}{dy} = \frac{d\mu Q}{dz}, \quad \frac{d^2 U}{dx dz} = \frac{d}{dz} \cdot \frac{dU}{dx} = \frac{d\mu P}{dz}, \quad \frac{dx}{dy} = -\frac{Q}{P}.$$

We have also, according to the condition by which μ was determined,

$$\frac{d\mu P}{dy} = \frac{d\mu Q}{dx}, \text{ or } \frac{\mu dP}{dy} - \mu \frac{dQ}{dx} = \frac{Q d\mu}{dx} - \frac{P d\mu}{dy} = -P \left(\frac{d\mu}{dy} + \frac{d\mu}{dx} \cdot \frac{dx}{dy} \right);$$

and these values being substituted will give

$$R\left(\frac{dP}{dy} - \frac{dQ}{dx}\right) + Q\left(\frac{dR}{dx} - \frac{dP}{dz}\right) + P\left(\frac{dQ}{dz} - \frac{dR}{dy}\right) = 0,$$

which is precisely the same equation we have found (178.)

(180.) Hence results the means to integrate a total differential equation of three variables, in which the above condition is satisfied.

Let, for instance, the proposed equation be $dx (ay - bz) + dy (cz - ax) + dz (bx - cy) = 0$, in which the coefficients of dx, dy, dz render identical the above equation of condition. If we suppose z to be constant, the term multiplied by dz vanishes, and we have simply $dx (ay - bz) + dy (cz - ax) = 0$, or

$$\frac{dx}{cz - ax} + \frac{dy}{ay - bz} = 0.$$

Integrating, we find,

$$\frac{1}{a} \log \left(\frac{ay - bz}{cz - ax} \right) = c.$$

Let us assume now $c = \phi(z)$, we shall have to determine $\phi(z)$ the following equation, in observing that here

$$\mu = \frac{1}{(ay - bz)(cz - ax)}, \text{ and } R = bx - cy,$$

$$\frac{-(bx - cy)}{(ay - bz)(cz - ax)} + \frac{1}{a} \frac{d}{dz} \log \left(\frac{ay - bz}{cz - ax} \right) = \frac{d\phi(z)}{dz}.$$

Generally, it would be necessary to substitute in this equation for x or y their values derived from the equation

$\frac{1}{a} \log \left(\frac{ay - bz}{cz - ax} \right) = \phi(z)$, but in this case after having performed the differentiation which is indicated, it will be

found, that the left side is reduced to nothing. Hence it will follow,

$$\frac{d\phi(z)}{dz} = 0,$$

or $\phi(z)$ a constant quantity, consequently the integral of the proposed equation will be

$$\frac{1}{a} \log \left(\frac{ay - bz}{cz - ax} \right) = C,$$

or, by changing the arbitrary constant,

$$\frac{ay - bz}{cz - ax} = C.$$

It is easy to verify that result; for by differentiating the equation, and considering x, y , and z as variables, we find

$$\frac{a dx (ay - bz) + a dy (cz - ax) + a dz (bx - cy)}{(cz - ax)^2} = 0,$$

an equation which differs only from the proposed by the factor $\frac{a}{(cz - ax)^2}$.

We shall apply this method of integration to another example. The proposed equation is

$$(y^2 + yz + z^2) dx + (x^2 + xz + z^2) dy + (x^2 + xy + y^2) dz = 0.$$

If we consider z as a constant quantity, the last term disappears, and by dividing by the product of the coefficients of dx and dy , the variables are separated, and we have

$$\frac{dx}{x^2 + xz + z^2} + \frac{dy}{y^2 + yz + z^2} = 0.$$

Integrating, we find

$$\frac{2}{z\sqrt{3}} \left\{ \tan^{-1} \frac{z+2x}{z\sqrt{3}} + \tan^{-1} \frac{z+2y}{z\sqrt{3}} \right\} = C.$$

Considering, now C or its product by $\frac{2}{z\sqrt{3}}$, as a function of z , we shall have the tangent of the left side of the equation equal to the tangent of the right side, which will still be a function of z ; we shall represent by Z . Thus we find in including the factor $\frac{1}{z\sqrt{3}}$ in the function, Z ,

$$\frac{x + y + z}{z^2 - zx - zy - 2xy} = Z$$

z is to be determined by the condition that the last equation should satisfy the proposed differential equation. This last equation being differentiated gives

$$2(y^2 + yz + z^2) dx + 2(x^2 + xz + z^2) dy + (x^2 + y^2 - z^2 - 2xy - 2zy) dz = (z^2 - zx - zy - 2xy)^2 dZ,$$

which, being subtracted from the double of the proposed equation, becomes

$$(x^2 + y^2 + z^2 + 2xy + 2xz + 2yz) dz = (z^2 - zx - zy - 2xy)^2 dZ, \text{ or}$$

$$(x + y + z)^2 dz = - (z^2 - zx - zy - 2xy)^2 dZ$$

But we have found

$$(x + y + z) = Z(z^2 - zx - zy - 2xy);$$

by combining these two equations we shall have, therefore,

$$dz = \frac{-dz}{Z^2},$$

which being integrated gives

$$z + c = \frac{1}{Z}, \text{ or } Z = \frac{1}{z + c}.$$

Substituting this value of Z , we find for the integral of the proposed equation

$$\frac{x + y + z}{z^2 - zx - zy - 2xy} = \frac{1}{z + c}, \text{ or } c(x + y + z) + 2(xz + yz + xy) = 0,$$

or again, by putting zc instead of c ,

$$c(x + y + z) + xz + yz + xy = 0.$$

(181.) If the proposed differential equation contain the quantities dx, dy, dz in a degree higher than the first, it can only be integrated by the preceding method, when it can be decomposed into rational factors of the form $Pdx + Qdy + Rdz$; for it is obvious that whatever the integral may be, it is always possible to derive from it by differentiation an expression of the form $dz = pdx + qdy$, in which p and q designate any function whatever of x, y , and z .

If, for instance, the proposed equation be

$$P dx^2 + Q dy^2 + R dz^2 + 2S dx dy + 2T dx dz + 2V dy dz = 0,$$

we shall derive from it a value of dz containing a radical quantity, and unless the quantity under the radical be a perfect square, the decomposition into factors cannot take place. It is, therefore, necessary that this quantity, which in the present case is

$$(T^2 - PR) dx^2 + 2(TV - RS) dx dy + (V^2 - QR) dy^2,$$

should be a perfect square, which can only take place under the condition

$$(TV - RS)^2 - (T^2 - PR)(V^2 - QR) = 0.$$

(182.) It is easy to generalize these principles, and to obtain conditions of integrability for total differential equations of four or more variables. The number of equations of condition is, however, greater. If the number of variables be m , the number of equations of condition will be $(m-1)(m-2)$.

(183.) The number of equations of condition of integrability becomes also greater, when the proposed differential equations are of an order higher than the first. Let $V = 0$ be the differential equation, and let V be a function of $x, y, z, x_1, y_1, z_1, x_2, y_2, z_2$, the six last quantities being as in number (109.) respectively equal to $dx, d^2x, dy, d^2y, dz, d^2z$, &c. If, then, we designate by μ a function of x, y, z, x_1, y_1, z_1 , which will render μV an exact differential, we shall have, according to (109.), the following equations,

$$\frac{d \cdot \mu V}{dx} - d \frac{d \cdot \mu V}{dx_1} + d^2 \frac{d \cdot \mu V}{dx_2} = 0$$

$$\frac{d \cdot \mu V}{dy} - d \frac{d \cdot \mu V}{dy_1} + d^2 \frac{d \cdot \mu V}{dy_2} = 0.$$

$$\frac{d \cdot \mu V}{dz} - d \frac{d \cdot \mu V}{dz_1} + d^2 \frac{d \cdot \mu V}{dz_2} = 0.$$

Which being developed become

$$\mu \left(\frac{dV}{dx} - d \frac{dV}{dx_1} + d^2 \frac{dV}{dx_2} \right) - d\mu \left(\frac{dV}{dx_1} - 2d \frac{dV}{dx_2} \right) + d^2 \mu \frac{dV}{dx_2} = 0.$$

$$\mu \left(\frac{dV}{dy} - d \frac{dV}{dy_1} + d^2 \frac{dV}{dy_2} \right) - d\mu \left(\frac{dV}{dy_1} - 2d \frac{dV}{dy_2} \right) + d^2 \mu \frac{dV}{dy_2} = 0.$$

$$\mu \left(\frac{dV}{dz} - d \frac{dV}{dz_1} + d^2 \frac{dV}{dz_2} \right) - d\mu \left(\frac{dV}{dz_1} - 2d \frac{dV}{dz_2} \right) + d^2 \mu \frac{dV}{dz_2} = 0$$

Since μ does not contain x_2, y_2, z_2 , and that the terms multiplied by V and dV are reduced to nothing since $V = 0$. If we now eliminate μ and its differentials, between these equations we shall obtain two equations of condition, in which z_2 and z_3 may be replaced by their values derived from $V = 0$ and $dV = 0$.

Let the equation be

$$(x dx + z dz) d^2y - z dy d^2z - dy (dx^2 + dy^2 + dz^2) = 0.$$

We have

$$V = (xx_1 + zz_1)y_2 - zz_2y_1 - y_1(x^2 + y^2 + z^2).$$

The first of the above equations becomes useless, because dx is considered as a constant quantity, and we have simply

$$\frac{dV}{dy} = 0, \quad \frac{dV}{dy_1} = -zz_2 - x_1^2 - y_1^2 - z_1^2, \quad \frac{dV}{dy_2} = xx_1 + zz_1, \quad \frac{dV}{dz} = z_1y_2 - z_2y_1,$$

$$\frac{dV}{dz_1} = zy_2 - 2y_1z_1, \quad \frac{dV}{dz_2} = -zy_1.$$

These values substituted in the second and third of the above equations, give

$$(6z_1z_2 + 6y_1y_2 + 2zz_2)\mu + (3x_1^2 + 3y_1^2 + 3z_1^2 + 3zz_2)d\mu + (xx_1 + zz_1)d^2\mu = 0.$$

$$2y_2\mu + 3y_2d\mu + y_1d^2\mu = 0.$$

If first we eliminate $d^2\mu$, we shall find

$$(6y_1z_1z_2 + 6y_1^2y_2 + 2zz_2y_1 - 2xx_1y_2 - 2zz_1y_2)\mu + (3x_1^2y_1 + 3y_1^2 + 3y_1z_1^2 + 3y_1zz_2 - 3xx_1y_2 - 3zz_1y_2)d\mu = 0.$$

We need not eliminate μ and $d\mu$, for if we substitute in this last equation for z_2 and z_3 their values derived from $V = 0$ and $dV = 0$, the left side is reduced to nothing.

(184.) Let us determine, now, the conditions which ought to be satisfied in order that a differential equation containing three or a greater number of variables, and of an order superior to the first, should have a second, or third integral.

Let $V_2 = 0$ be a differential equation, and μ_2 the factor which renders $\mu_2 V_2$ an exact differential. According to what precedes, the following equations, the number of which is the same as that of the variables, will be satisfied.

Integral
Calculus.

Part III.

$$\left(\frac{dV_2}{dx} - d\frac{dV_2}{dx_1} + d^2\frac{dV_2}{dx_2} - \&c.\right)\mu_2 - \left(\frac{dV_2}{dx_1} - 2d\frac{dV_2}{dx_2} + \&c.\right)d\mu_2 + \left(\frac{dV_2}{dx_2} - \&c.\right)d^2\mu_2 - \&c.\dots = 0.$$

Secondly, V_1 being the integral of $\mu_2 V_2$, it will be necessary that there exist a factor μ_1 which can render $\mu_1 V_1$ an exact differential, in order that the proposed differential equation should have a second integral, and this cannot take place unless the following equations are satisfied,

$$\left(\frac{dV_1}{dx} - d\frac{dV_1}{dx_1} + d^2\frac{dV_1}{dx_2} - \&c.\right)\mu_1 - \left(\frac{dV_1}{dx_1} - 2d\frac{dV_1}{dx_2} + \&c.\right)d\mu_1 + \left(\frac{dV_1}{dx_2} - \&c.\right)d^2\mu_1 - \&c.\dots = 0$$

But since V_1 is the integral of $\mu_2 V_2$, we have

$$\frac{dV_1}{dx} = \frac{d \cdot \mu_2 V_2}{dx_1} - d\frac{d \cdot \mu_2 V_2}{dx_2} + d^2\frac{d \cdot \mu_2 V_2}{dx_3} - \&c. \quad \frac{dV_1}{dx_1} = \frac{d \cdot \mu_2 V_2}{dx_2} - d\frac{d \cdot \mu_2 V_2}{dx_3} + \&c. \\ \frac{dV_1}{dx_2} = \frac{d \cdot \mu_2 V_2}{dx_3} - \&c., \&c.$$

These values being substituted in the above equations will give

$$\left(\frac{d \cdot \mu_2 V_2}{dx_1} - 2d\frac{d \cdot \mu_2 V_2}{dx_2} + 3d^2\frac{d \cdot \mu_2 V_2}{dx_3} - \&c.\right)\mu - \left(\frac{d \cdot \mu_2 V_2}{dx_2} - 3d\frac{d \cdot \mu_2 V_2}{dx_3} + \&c.\right) \\ d\mu + \left(\frac{d \cdot \mu_2 V_2}{dx_3} - \&c.\right)d^2\mu - \&c.\dots = 0.$$

Developing these last equations, and observing that the differentials of the same order as V_2 do not enter in V_1 , and that all the terms which will be multiplied by V_2 , dV_2 , d^2V_2 , after the differentiations, will disappear, since we have $V_2 = 0$, $dV_2 = 0$, $d^2V_2 = 0$, &c., we shall find

$$\left(\mu_2 \frac{dV_2}{dx_1} - 2d\mu_2 \frac{dV_2}{dx_2} + 3d^2\mu_2 \frac{dV_2}{dx_3} - \&c.\right)\mu_1 - \left(\mu_2 \frac{dV_2}{dx_2} - 3d\mu_2 \frac{dV_2}{dx_3} + \&c.\right) \\ d\mu_1 + \left(\mu_2 \frac{dV_2}{dx_3} - \&c.\right)d^2\mu - \&c.\dots = 0.$$

Each variable will give a similar equation, which, with those relative to the first integral, will be all the equations of condition. In an analogous manner, the conditions relative to the existence of a third integral might be found.

(185.) When the proposed total differential equation is not an exact differential, nor can become so by the multiplication of a factor properly chosen, no primitive equation between the variables can satisfy it. In that case, it is always possible, however, to find a system of equations between the variables which, conjointly, satisfy the proposed equation.

In the equation $dz - ay dx - b dy = 0$, for instance, the left side is not an exact differential, and the equation of condition found (178.) not being satisfied, it is impossible to discover a factor which would make it so; but if

we assume $y = \frac{dX}{dx} = X'$, where X is an arbitrary function of x , the equation will become $dz = aX'dx + b dX'$,

and consequently $z = aX + bX'$, including the constant in the arbitrary function. Hence the system of equations $y = X'$ and $z = aX + bX'$, constitute the integral of the proposed equation.

(186.) Generally, let $Pdx + Qdy + Rdz = 0$ be the proposed equation, and integrate it, in supposing first z to be constant, in which case the equation is reduced to $Pdx + Qdy = 0$. Let the factor by which the left side of this equation ought to be multiplied to become an immediate differential, be represented by μ , and the integral by $U = C$. If we differentiate this last equation with respect to x , y , z and C , and if we subtract from

it the proposed equation multiplied by μ , we shall find $\frac{dC}{dz} = \frac{dU}{dz} - \mu R$. The right side of this equation is no

longer a function of z only, since the equation of condition found (178.) is not satisfied; but it is obvious that by supposing $C = \phi z$, an arbitrary function of z , the proposed differential equation will be satisfied by

$U = \phi(z)$, provided we have at the same time $\frac{d\phi(z)}{dz} = \frac{dU}{dz} - \mu R$. In other words, the system of the two

equations

$$U = \phi(z), \quad \frac{dU}{dz} - \mu R = \frac{d\phi(z)}{dz},$$

will constitute the integral of $P dx + Q dy + R dz = 0$.

This process being applied to the equation

$$\frac{dz}{z-c} = \frac{z dx + y dy}{x(x-a) + y(y-b)},$$

will give for the integral the two equations

$$x^2 + y^2 = \phi(z), \quad 2 \frac{x(x-a) + y(y-b)}{z-c} = \frac{d\phi(z)}{dz}.$$

(187.) It is easy to extend these considerations to equations containing more than three variables when the differentials are not in a degree higher than the first. If, for instance, we had an equation between four variables,

$$N du + P dx + Q dy + R dz = 0,$$

satisfying none of the conditions of integrability, we would first integrate it in considering two of the variables, u and x , as constants. The equation, in that supposition, becomes $Q dy + R dz = 0$; let μ be the factor which renders it an immediate differential, and let $\int \mu (Q dy + R dz) = U$, then $U = c$ would be the integral of the equation. If we differentiate it, in considering u, x, y, z and C as variables, and subtract it from the proposed equation multiplied by μ , we shall get

$$\frac{dU}{du} du + \frac{dU}{dx} dx - dC = \mu N du + \mu P dx.$$

To satisfy this equation, it is necessary to assume $C = \phi(u, x)$, ϕ designating an arbitrary function; and if we suppose

$$d\phi(u, x) = \phi_1(u, x) du + \phi_2(u, x) dx,$$

we shall have, in equalizing the coefficients of du , in both sides of the equation, as well as those of dx ,

$$\frac{dU}{du} - \phi_1(u, x) = \mu N, \quad \frac{dU}{dx} - \phi_2(u, x) = \mu P;$$

and, thus, the proposed equation will be satisfied by the system of the three equations

$$U = \phi(u, x), \quad \frac{dU}{du} - \mu N = \phi_1(u, x), \quad \frac{dU}{dx} - \mu P = \phi_2(u, x),$$

which contain an arbitrary function of two quantities.

There is a simpler means of satisfying the proposed equation; for if we eliminate the variables z and y between the three equations above, the resulting equation will contain only $x, u, \phi(x, u), \phi_1(x, u), \phi_2(x, u)$, that is will be an equation between x, u , the function ϕ , and its partial differential coefficients ϕ_1 and ϕ_2 . If by means of this equation the function ϕ can be determined, its value will contain, as will soon be shown, an arbitrary function of u and x ; and this value being substituted in the three equations, one of them may be suppressed, and the two remaining will satisfy conjointly the proposed differential equation.

Let us take as an example the differential equation

$$x du + y dx + u dy + z dz = 0,$$

which satisfies none of the conditions of integrability. We shall have

$$\mu = 2U = \int 2(u dy + z dz) = 2uy + z^2,$$

and consequently the integral may be represented by the system of the three equations

$$2uy + z^2 = \phi(u, x), \quad 2y - 2x = \phi_1(u, x), \quad -2y = \phi_2(u, x).$$

If we eliminate y between the two last of these equations, we get

$$\phi_1(u, x) + \phi_2(u, x) = -2x,$$

a partial differential equation, the integral of which, according to the method hereafter to be explained, is

$$\phi(u, x) + x^2 = \psi(u, x),$$

ψ being an arbitrary function. The value of $\phi(u, x)$ being substituted in the two first equations will give

$$x^2 + z^2 + 2uy = \psi(u - x), \quad 2y - 2x = \psi'(u - x).$$

ψ' representing the differential coefficient of ψ , which conjointly will satisfy the proposed differential equation.

(188.) The differential equations in which the differentials are in a degree higher than the first may be integrated in a similar manner, when the conditions of integrability are not satisfied. Let the equation be $dz^2 = a^2(dx^2 + dy^2)$. If we suppose y constant, it becomes $dz = m dy$, and gives consequently $z = my + C$. Let $C = \phi(x)$ and $dC = \phi'(x) dx$, and differentiate the equation $z = my + C$, in considering x, y , and z as variables, we shall find $dz = m dy + \phi'(x) dx$; and since this value ought to satisfy the proposed equation, we shall have

$$2dy = \frac{m dx}{\phi'(x)} - \frac{\phi'(x) \cdot dx}{m},$$

Part III.
Integral
Calculus.

and hence, by integration,

$$2y = m \int \frac{dx}{\phi'(x)} - \frac{1}{m} \phi(x).$$

The proposed equation, therefore, is satisfied by the system of the two equations

$$z = my + \phi(x), \quad 2y = m \int \frac{dx}{\phi'(x)} - \frac{1}{m} \phi(x) + A,$$

 A, being an arbitrary constant, and ϕ an arbitrary function of x .

Another system of two equations may be found by the following process. Let $\frac{dx}{dz} = a$, and $\frac{dy}{dz} = a$, a and a being constants, the proposed equation will be satisfied provided the constants a and a satisfy the equation $1 = m^2(a^2 + a^2)$. This gives $a = \frac{\sqrt{1 - m^2 a^2}}{m}$; substituting this value and integrating the two equations

$$\frac{dx}{dz} = a, \text{ and } \frac{dy}{dz} = \frac{\sqrt{1 - m^2 a^2}}{m},$$

we shall have

$$x = az + b, \quad y = \frac{z \sqrt{1 - m^2 a^2}}{m} + c,$$

 which contain three arbitrary constants, a, b, c , and satisfy the proposed equation. This is easy to verify, for if we differentiate each of these equations, the two constants b and c will disappear, and we shall get

$$dx = a dz, \quad dy = \frac{dz \sqrt{1 - m^2 a^2}}{m};$$

 and eliminating a between these two we shall find the proposed equation.

 It is also obvious that if a, b, c , instead of being constant, are variable quantities, the two equations written above will still satisfy the proposed equation, provided that equation may be obtained by the elimination of a, b, c . But if we differentiate the two equations, in the supposition of a, b, c , variables, we find

$$dx = a dz + z da + db, \quad dy = \frac{dz \sqrt{1 - m^2 a^2}}{m} - \frac{m z a da}{\sqrt{1 - m^2 a^2}} + dc.$$

 These values of dx and dy will be reduced to those we have found in the supposition of a, b , and c , constant, if we suppose

$$z da + db = 0, \quad - \frac{m z a da}{\sqrt{1 - m^2 a^2}} + dc = 0;$$

 and if we can find the variable values of a, b, c , which satisfy these equations, we shall obtain a new system of two equations satisfying the proposed equation. Let $a = \phi(b)$, and $c = \psi(b)$, then the two last equations and the two equations

$$x = az + b, \quad y = \frac{z \sqrt{1 - m^2 a^2}}{m} + c,$$

become

$$x = z \phi(b) + b, \quad 0 = z \phi'(b) + 1, \quad y = \frac{z \sqrt{1 - m^2 \phi(b)^2}}{m} + \psi(b), \quad 0 = \frac{-m z \phi(b) \phi'(b)}{\sqrt{1 - m^2 \phi(b)^2}} + \psi'(b),$$

which may be put under the following form,

$$\begin{aligned} \frac{x-b}{z} - \phi(b) = 0, \quad -\frac{1}{z} - \phi'(b) = 0, \quad \frac{y - \psi(b)}{z} - \frac{\sqrt{1 - m^2 \left(\frac{x-b}{z}\right)^2}}{m} = 0, \\ -\psi'(b) - \frac{m \left(\frac{x-b}{z}\right)}{\sqrt{1 - m^2 \left(\frac{x-b}{z}\right)^2}} = 0. \end{aligned}$$

The last of these equations may be identified with the fourth by taking a proper value for the arbitrary function ϕ , since they both express a rotation between the quantities $\frac{x-b}{z}$ and b ; we may, therefore, instead of the second equation, which is the differential of the first in considering b only as variable, by the differential of the fourth in the same supposition, or which is still the same thing, by the second differential of the third.

 If, therefore, we represent this third equation by $U = 0$, we shall have, instead of the four equations above,

$$U = 0, \quad \frac{dU}{db} = 0, \quad \frac{d^2 U}{db^2} = 0,$$

Integral
Calculus.

which contain only one arbitrary function, and if a particular value be assigned to this function, and then the quantity b eliminated between the three equations, we shall obtain two primitive equations satisfying the proposed. Part III.

This may be generalized, and the same process applied to any equation

$$F\left(\frac{dx}{dz}, \frac{dy}{dz}\right) = 0.$$

From this equation we may derive the value of $\frac{dy}{dz}$. Let $\frac{dy}{dz} = f\left(\frac{dx}{dz}\right)$, and assume $\frac{dx}{dz} = a$, we shall have

$\frac{dy}{dz} = f(a)$, which being integrated give

$$x = az + b, \quad y = z f(a) + c, \quad \text{or} \quad \frac{x-b}{z} = a = \frac{y-c}{z} = f\left(\frac{x-b}{z}\right).$$

If we assume then $a = \phi(b)$, $c = \psi(b)$, and if we differentiate with respect to b , we shall have

$$\frac{x-b}{z} - \phi(b) = 0, \quad -\frac{1}{z} - \phi'(b) = 0, \quad \frac{y-\psi(b)}{z} - f\left(\frac{x-b}{z}\right) = 0, \quad -\psi'(b) + f\left(\frac{x-b}{z}\right) = 0,$$

the first and last of which may be identified, and consequently the second replaced by the differential of the last, considering b only as variable. Let

$$\frac{y-\psi(b)}{z} - f\left(\frac{x-b}{z}\right) = U.$$

It is obvious that the above four equations will be reduced to the three following,

$$U = 0, \quad \frac{dU}{db} = 0, \quad \frac{d^2U}{db^2} = 0,$$

and it is moreover obvious that instead of U , we may write

$$F\left(\frac{x-b}{z}, \frac{y-\psi(b)}{z}\right)$$

In the example treated above, this last proposition is easily verified, in making the radical quantities disappear in the equation corresponding to $U = 0$, for then it becomes

$$1 = m \left\{ \left(\frac{x-b}{z} \right)^2 + \left(\frac{y-\psi(b)}{z} \right)^2 \right\},$$

which is precisely the proposed equation in which $\frac{x-b}{z}$, and $\frac{y-\psi(b)}{z}$ have been substituted respectively for

$$\frac{dx}{dz}, \quad \frac{dy}{dz}$$

(199.) Total differential equations of an order higher than the first, which do not satisfy the conditions of integrability, are treated like those of the first order.

We shall only consider here the equation $M d^2x + N d^2y + P dx^2 = Q dz^2$, in which dz is supposed to be constant, and M, N, P, Q to represent functions of x, y, z, dx, dy, dz , such as when the left side of the equation is multiplied by μ , it becomes an exact differential with respect to the factors x and y . If U' represent the first integral of that differential, and if it be of the n^{th} degree with respect to dx, dy, dz , we shall assume it equal to a function of z , multiplied by dz^n , and we shall have the equation $U' = \phi'(z) dz^n$. The differential being then taken in considering x, y, dx, dy , and z as variables, and compared with the proposed equation, will give

$$\phi''(z) dz^n - \frac{dU'}{dz} = \mu Q dz,$$

since by the supposition we have made

$$\frac{dU'}{dz} dz^2 x + \frac{dU'}{dz^2 y} dz^2 y + \frac{dU'}{dz} dx + \frac{dU'}{dz} dy = \mu M d^2x + \mu N d^2y + \mu P dx^2.$$

The proposed equation will, therefore, be satisfied by the equations

$$U' = \phi'(z) dz^n, \quad \frac{dU'}{dz} = \phi''(z) dz^n - \mu Q dz,$$

taken conjointly. To obtain the second integral of the proposed equation, without assuming any particular value for $\phi(z)$, it will be necessary to derive from the two last equations two equations which can be integrated in considering x, y, z , and $\phi(z)$ as distinct variables. Let us take for an example the equation

$$z dz d^2x - 2 dy d^2y = z dz^2$$

Integral
Calculus.

By the above process we shall find that it is satisfied by the equations

$$z dz dx - dy^2 = \phi'(z) dz^2, \quad dx = \phi''(z) dz - z dz;$$

the last of these being integrated, gives $2x = 2\phi'(z) - z^2$, and substituting in the first the value of dx derived from this integral, we shall find

$$dy = dz \sqrt{z\phi''(z) - \phi'(z) - z^2},$$

from which we shall conclude that the second integral of the proposed equation is represented by the two following equations,

$$2x = 2\phi'(z) - z^2, \quad y = \int dz \sqrt{z\phi''(z) - \phi'(z) - z^2}.$$

Integration of Partial Differential Equations.

(190.) The determination of functions of several variables by means of given relations or equations between the functions, the variables, and some of, or all, the partial differential coefficients of the functions with respect to the variables, is one of the most important subjects of the Integral Calculus. Hitherto very few of the numerous cases which this determination presents have been successfully treated, and what has been done is the result of the labours of the most celebrated Mathematicians.

We cannot enter here into all the details of this extensive branch of analysis; and we shall only be able, not to exceed our limits, to mention some of the simplest, most remarkable, and useful cases. For more information on the subject, we must refer the reader to the II^d and III^d volumes of Lacroix's complete treatise on the Differential and Integral Calculus, to the transactions of learned societies, the *Journal de l'Ecole Polytechnique*, *Les Annales de Mathématiques de Gergonne*, *Le Journal de Crelles*, and *Les Exercices Mathématiques de Cauchy*.

(191.) Before entering into the subject of partial differential equations, we shall first consider the following question.

Let $x, x_1, x_2, x_3, \dots, x_n$ be $(n+1)$ variables, and let it be required to determine the values of n of these variables, of $x_1, x_2, \dots, x_n$, for instance, in function of x , by means of n given differential equations between these variables, containing only the first differential coefficients of $x_1, x_2, \&c.$ with respect to x . In this supposition, the n given equations may always be reduced to the following form,

$$\frac{dx_1}{dx} = \frac{X_1}{X}, \quad \frac{dx_2}{dx} = \frac{X_2}{X}, \quad \dots \quad \frac{dx_n}{dx} = \frac{X_n}{X},$$

or written in the form of propositions, $dx : dx_1 : dx_2 : \dots : dx_n :: X : X_1 : X_2 : \dots : X_n$, where any one of the variables may be considered as the independent variable, and where $X, X_1, X_2, \dots, X_n$ are functions of $x, x_1, x_2, \dots, x_n$.

It is easy to eliminate between these n equations $n-1$ variables, for instance $x_2, x_3, \dots, x_n$. It is sufficient for that, to differentiate $n-1$ consecutive times the first equation $\frac{dx_1}{dx} = \frac{X_1}{X}$, and to substitute each time for $\frac{dx_2}{dx}, \frac{dx_3}{dx}, \&c.$ their values derived from the $n-1$ last equations. We shall thus obtain the values of

$$\frac{dx_1}{dx}, \quad \frac{d^2 x_1}{dx^2}, \quad \frac{d^3 x_1}{dx^3}, \quad \dots \quad \frac{d^n x_1}{dx^n},$$

in functions of $x, x_1, x_2, \dots, x_n$. Eliminating then between these n equations the $n-1$ variables $x_2, x_3, \dots, x_n$, we shall find a final equation of the form

$$\phi\left(x, x_1, \frac{dx_1}{dx}, \frac{d^2 x_1}{dx^2}, \dots, \frac{d^n x_1}{dx^n}\right) = 0.$$

This equation being of the n^{th} order has n first integrals of the $(n-1)^{\text{th}}$ order, containing each an arbitrary constant. Let these equations be

$$F_1\left(x, x_1, \frac{dx_1}{dx}, \frac{d^2 x_1}{dx^2}, \dots, \frac{d^{n-1} x_1}{dx^{n-1}}\right) = c_1,$$

$$F_2\left(x, x_1, \frac{dx_1}{dx}, \frac{d^2 x_1}{dx^2}, \dots, \frac{d^{n-1} x_1}{dx^{n-1}}\right) = c_2,$$

$$\dots \dots \dots$$

$$F_n\left(x, x_1, \frac{dx_1}{dx}, \frac{d^2 x_1}{dx^2}, \dots, \frac{d^{n-1} x_1}{dx^{n-1}}\right) = c_n,$$

where $c_1, c_2, \dots, c_n$ designate the arbitrary constants. If we substitute now in these equations for $\frac{dx_1}{dx}$,

$\frac{d^2 x_1}{dx^2}, \dots, \frac{d^{n-1} x_1}{dx^{n-1}}$, their values derived from the first equation $\frac{dx_1}{dx} = \frac{X_1}{X}$, we shall have the n required equa-

Part III.

Integral
Calculus.

tions between the n variables $x, x_1, \dots, x_n$, they will contain n arbitrary constants. They are the integrals of the n proposed differential equations. Part III.

(192.) The n equations last mentioned containing n arbitrary constants, may always be supposed to have been resolved with respect to the constants, and put under the following form,

$$\phi_1 = c_1, \quad \phi_2 = c_2, \dots, \phi_n = c_n.$$

If these are differentiated, the arbitrary constants disappear, and we get

$$\frac{d\phi_1}{dx} \cdot dx + \frac{d\phi_1}{dx_1} dx_1 + \dots + \frac{d\phi_1}{dx_n} dx_n = 0.$$

$$\frac{d\phi_2}{dx} \cdot dx + \frac{d\phi_2}{dx_1} dx_1 + \dots + \frac{d\phi_2}{dx_n} dx_n = 0.$$

$$\dots \dots \dots$$

$$\frac{d\phi_n}{dx} \cdot dx + \frac{d\phi_n}{dx_1} dx_1 + \dots + \frac{d\phi_n}{dx_n} dx_n = 0$$

These differential equations are satisfied by the equations

$$\phi_1 = c_1, \quad \phi_2 = c_2, \dots, \phi_n = c_n,$$

therefore they must agree with the proposed differential equations

$$\frac{dx_1}{dx} = \frac{X_1}{X}, \quad \frac{dx_2}{dx} = \frac{X_2}{X}, \dots, \frac{dx_n}{dx} = \frac{X_n}{X}, \text{ or } dx : dx_1 : dx_2 : \dots : dx_n :: X : X_1 : X_2 : \dots : X_n.$$

Consequently, if the values of $dx_1, dx_2, \dots, dx_n$ are taken in these last equations, and substituted in those with which they agree, we shall obtain the n following identical equations :

$$\left. \begin{aligned} \frac{d\phi_1}{dx} X + \frac{d\phi_1}{dx_1} X_1 + \frac{d\phi_1}{dx_2} X_2 + \dots + \frac{d\phi_1}{dx_n} X_n &= 0. \\ \frac{d\phi_2}{dx} X + \frac{d\phi_2}{dx_1} X_1 + \frac{d\phi_2}{dx_2} X_2 + \dots + \frac{d\phi_2}{dx_n} X_n &= 0. \\ \dots \dots \dots \\ \frac{d\phi_n}{dx} X + \frac{d\phi_n}{dx_1} X_1 + \frac{d\phi_n}{dx_2} X_2 + \dots + \frac{d\phi_n}{dx_n} X_n &= 0. \end{aligned} \right\} \quad (A)$$

The solution of the proposed question (191.), that is the integration of the n proposed differential equations, may therefore be considered as satisfying this other question : to find n different functions, each of which being substituted for ϕ , shall make the equation identical,

$$\frac{d\phi}{dx} \cdot X + \frac{d\phi}{dx_1} X_1 + \frac{d\phi}{dx_2} X_2 + \dots + \frac{d\phi}{dx_n} X_n = 0.$$

It is moreover easy to see that if the functions $\phi_1, \phi_2, \dots, \phi_n$ satisfy this condition, it will still be the same with respect to any compound function of these. For let $\pi(\phi_1, \phi_2, \dots, \phi_n)$ be such a function, if we multiply the equations (A) respectively by $\frac{d\pi}{d\phi_1}, \frac{d\pi}{d\phi_2}, \frac{d\pi}{d\phi_3}, \dots$, &c., and if we add them, we shall find

$$\frac{d\pi}{dx} X + \frac{d\pi}{dx_1} X_1 + \frac{d\pi}{dx_2} X_2 + \dots + \frac{d\pi}{dx_n} X_n = 0,$$

so that π satisfies also the condition.

(193.) The integration of partial differential equations of the first order, and of the first degree, results immediately from what precedes.

Let the proposed equation be

$$X = X_1 \frac{dx}{dx_1} + X_2 \frac{dx}{dx_2} + \dots + X_n \frac{dx}{dx_n},$$

where $\frac{dx}{dx_1}, \frac{dx}{dx_2}, \dots, \frac{dx}{dx_n}$, are the partial differential coefficients of x with respect to $x_1, x_2, \dots, x_n$,

$X, X_1, X_2, \dots, X_n$ functions of these same variables.

Let $\pi = 0$ be the required relation between x and the variables $x_1, x_2, \dots, x_n$, or the integral of the proposed equation. We shall have, in differentiating successively with respect to $x_1, x_2, \dots, x_n$,

$$\frac{d\pi}{dx} \cdot \frac{dx}{dx} + \frac{d\pi}{dx_1} = 0.$$

$$\frac{d\pi}{dx} \cdot \frac{dx}{dx_1} + \frac{d\pi}{dx_1} = 0.$$

.....

$$\frac{d\pi}{dx} \cdot \frac{dx}{dx_n} + \frac{d\pi}{dx_n} = 0.$$

And if we substitute the values derived from these in the proposed equation, it becomes

$$\frac{d\pi}{dx} X + \frac{d\pi}{dx_1} X_1 + \frac{d\pi}{dx_2} X_2 + \dots + \frac{d\pi}{dx_n} X_n = 0.$$

We have just proved (192.) that such an equation is satisfied in taking for π any function whatever of $\phi_1, \phi_2, \phi_3, \dots, \phi_n$.

Hence to integrate a partial differential equation of the first order, and of the first degree, such as

$$X = \frac{dx}{dx_1} X_1 + \frac{dx}{dx_2} X_2 + \dots + \frac{dx}{dx_n} X_n,$$

it will be sufficient to integrate the system of the following equations,

$$\frac{dx_1}{dx} = \frac{X_1}{X}, \quad \frac{dx_2}{dx} = \frac{X_2}{X} \dots \dots \frac{dx_n}{dx} = \frac{X_n}{X},$$

or a system of n equations furnished by the proportions $dx : dx_1 : dx_2 : \dots : dx_n :: X : X_1 : X_2 : \dots : X_n$; and then, if $\phi_1 = c_1, \phi_2 = c_2, \dots, \phi_n = c_n$ represent their integrals, $c_1, c_2, \dots, c_n$, being the arbitrary constants, and if we designate by π any function whatever of $\phi_1, \phi_2, \dots, \phi_n$, $\pi = 0$ will be the required integral.

(194.) These considerations may still be generalized, and it may be demonstrated that if, as above,

$$\phi_1 = c_1, \phi_2 = c_2, \phi_3 = c_3, \dots, \phi_n = c_n$$

are the integrals of

$$\frac{dx_1}{dx} = \frac{X_1}{X}, \quad \frac{dx_2}{dx} = \frac{X_2}{X} \dots \dots \frac{dx_n}{dx} = \frac{X_n}{X},$$

or of a system of n equations furnished by the proportions $dx : dx_1 : dx_2 : \dots : dx_n :: X : X_1 : X_2 : \dots : X_n$, a system of m equations such as

$$\left. \begin{aligned} X &= X_m \frac{dx}{dx_m} + X_{m+1} \frac{dx}{dx_{m+1}} + \dots + X_n \frac{dx}{dx_n}, \\ X_1 &= X_m \frac{dx_1}{dx_m} + X_{m+1} \frac{dx_1}{dx_{m+1}} + \dots + X_n \frac{dx_1}{dx_n}, \\ &\dots \dots \dots \\ X_{m-1} &= X_m \frac{dx_{m-1}}{dx_m} + X_{m+1} \frac{dx_{m-1}}{dx_{m+1}} + \dots + X_n \frac{dx_{m-1}}{dx_n}, \end{aligned} \right\} \text{(B)}$$

in which $x, x_1, x_2, \dots, x_{m-1}$ are considered as functions of $x_m, x_{m+1}, \dots, x_n$, will be satisfied by the equations

$$\pi_1 = 0, \quad \pi_2 = 0 \dots \dots \pi_m = 0,$$

$\pi_1, \pi_2, \dots, \pi_m$ being arbitrary functions of $\phi_1, \phi_2, \dots, \phi_n$.

Before demonstrating this proposition, it must be remarked that the characteristic property of the partial differential equations (B) is, that each equation contains only the partial differential coefficients of the same variable, and that in all the equations the partial differential coefficients taken with respect to the same variable are identical.

Now let $A, B, C, \dots, M$, be m quantities so determined as to satisfy the $m - 1$ following equations,

$$\begin{aligned} A \frac{d\pi_1}{dx_1} + B \frac{d\pi_2}{dx_1} + C \frac{d\pi_3}{dx_1} + \dots + M \frac{d\pi_m}{dx_1} &= 0. \\ A \frac{d\pi_1}{dx_2} + B \frac{d\pi_2}{dx_2} + C \frac{d\pi_3}{dx_2} + \dots + M \frac{d\pi_m}{dx_2} &= 0. \\ &\dots \dots \dots \\ A \frac{d\pi_1}{dx_{m-1}} + B \frac{d\pi_2}{dx_{m-1}} + C \frac{d\pi_3}{dx_{m-1}} + \dots + M \frac{d\pi_m}{dx_{m-1}} &= 0 \end{aligned}$$

Let us also assume, for the sake of brevity,

$$A \frac{d\pi_1}{dx} + B \frac{d\pi_2}{dx} + C \frac{d\pi_3}{dx} + \dots + M \frac{d\pi_m}{dx} = N.$$

It is easy to derive from the preceding equations, the following:

$$\begin{aligned} N \frac{dx}{dx_m} &= - \left\{ A \frac{d\pi_1}{dx_m} + B \frac{d\pi_2}{dx_m} + \dots + M \frac{d\pi_m}{dx_m} \right\}. \\ N \frac{dx}{dx_{m+1}} &= - \left\{ A \frac{d\pi_1}{dx_{m+1}} + B \frac{d\pi_2}{dx_{m+1}} + \dots + M \frac{d\pi_m}{dx_{m+1}} \right\}. \\ &\dots\dots\dots \\ N \frac{dx}{dx_n} &= - \left\{ A \frac{d\pi_1}{dx_n} + B \frac{d\pi_2}{dx_n} + \dots + M \frac{d\pi_m}{dx_n} \right\}. \end{aligned}$$

And we shall be able to form in a similar manner, the values of the partial differential coefficients of $x_1, x_2, \dots, x_n$. But since $\pi_1, \pi_2, \dots, \pi_m$, are arbitrary functions of $\phi_1, \phi_2, \dots, \phi_n$, we have, according to (192.), the identical equations

$$\begin{aligned} \frac{d\pi_1}{dx} X + \frac{d\pi_1}{dx_1} X_1 + \frac{d\pi_1}{dx_2} X_2 + \dots + \frac{d\pi_1}{dx_n} X_n &= 0. \\ \frac{d\pi_2}{dx} X + \frac{d\pi_2}{dx_1} X_1 + \frac{d\pi_2}{dx_2} X_2 + \dots + \frac{d\pi_2}{dx_n} X_n &= 0. \\ &\dots\dots\dots \\ \frac{d\pi_m}{dx} X + \frac{d\pi_m}{dx_1} X_1 + \frac{d\pi_m}{dx_2} X_2 + \dots + \frac{d\pi_m}{dx_n} X_n &= 0. \end{aligned}$$

If we multiply now these equations respectively by $A, B, C, \dots, M$, and if we add them, we shall obtain, after the reductions which will take place in consequence of the preceding equations,

$$X = X_m \frac{dx}{dx_m} + X_{m+1} \frac{dx}{dx_{m+1}} + \dots + X_n \frac{dx}{dx_n},$$

which is the first of the equations A. By a similar process the others will be obtained.

(195.) The application of these general principles to particular cases presents no difficulty. We shall enumerate some of the most useful, and give a few examples in which the calculation can be completed.

When the proposed differential equation contains only one of the partial differential coefficients of $x, \frac{dx}{dx_1}$ for

instance, its most general form is $X = \frac{dx}{dx_1} X_1$, or $X dx_1 - X_1 dx = 0$; and since the quantities we have, in

the general case, represented by $X_2, X_3, \dots, X_n$, are all equal to nothing, the other differential equations, besides $X dx_1 - X_1 dx = 0$, are simply $dx_2 = 0, dx_3 = 0, \dots, dx_n = 0$. Consequently they may be integrated separately, and give $x_2 = c_2, x_3 = c_3, \dots, x_n = c_n$. Hence in the equation $X dx_1 - X_1 dx = 0$, all the variables but x and x_1 may be considered as constant quantities, and, therefore, it may be integrated as a differential equation between two variables. Let that integral be resolved with respect to the arbitrary constant, and be represented by $\phi_1 = c_1$, then, according to the general rule, (193.) the integral of the proposed differential equation will be $\pi(\phi_1, x_2, x_3, \dots, x_n) = 0$, or $\phi_1 = \psi(x_2, x_3, \dots, x_n)$, the right side of the equation being an arbitrary function of all the variables it contains.

Example 1. Let $\sqrt{y^2 - z^2} = z \frac{dz}{dx}$ be the proposed equation. Integrating, in considering y as a constant quantity, we find $x + \sqrt{y^2 - z^2} = c$, and, consequently, $x + \sqrt{y^2 - z^2} = \psi(y)$, where $\psi(y)$ is an arbitrary function of y , is the integral required.

Example 2. The equation $\frac{dz}{dx} = \frac{y^2 + z^2}{y^2 + x^2}$ will lead by a similar process to the integral

$$\tan^{-1} \frac{z}{y} - \tan^{-1} \frac{x}{y} = \psi(y).$$

(196.) Let us suppose next that the proposed differential equation contains two, three, or generally m , of the partial differential coefficients, for instance, $\frac{dx}{dx_1}, \frac{dx}{dx_2}, \dots, \frac{dx}{dx_m}$, its general form will be

$$X = \frac{dx}{dx_1} X_1 + \frac{dx}{dx_2} X_2 + \dots + \frac{dx}{dx_m} X_m.$$

If then $\phi_1 = c_1, \phi_2 = c_2, \dots, \phi_m = c_m$, represent a system of primitive equations satisfying the equations $X dx_1 - X_1 dx = 0, X dx_2 - X_2 dx = 0, \dots, X dx_m - X_m dx = 0$, or which is the same a system of m equations furnished by the proportions $dx : dx_1 : dx_2 : \dots : dx_m :: X : X_1 : X_2 : \dots : X_m$, the integral of the proposed equation will be $\pi(\phi_1, \phi_2, \phi_3, \dots, \phi_m, x_{m+1}, x_{m+2}, \dots, x_n) = 0$.

(197.) Let us now take some examples of the case in which the proposed equation contains all the partial differential coefficients, and first, when there are only three variables, one of which is considered as a function of

the two others. The form of the equation is then $X = \frac{dx}{dx_1} X_1 + \frac{dx}{dx_2} X_2$. The two equations to integrate are

those furnished by the proportions $dx : dx_1 : dx_2 :: X : X_1 : X_2$, and if we represent by $\phi_1 = c_1$, and $\phi_2 = c_2$, a system of two equations satisfying them, the required integral is $\pi(\phi_1, \phi_2) = 0$, or $\phi_1 = \psi(\phi_2)$.

The proportions $dx : dx_1 : dx_2 :: X : X_1 : X_2$ give the three equations

$$X dx_1 - X_1 dx = 0, \quad X dx_2 - X_2 dx = 0, \quad \text{and} \quad X_1 dx_2 - X_2 dx_1 = 0.$$

When it will happen that in two of these equations only the variables of which they contain the differentials will enter, for instance, in the first, the variables x and x_1 , and in the second the variables x and x_2 , it will not be necessary to descend to the second order to discover the values of ϕ_1 and ϕ_2 . It is obvious that these two equations may then be integrated separately by the rules previously given for the integration of differential equations of two variables of the first order.

If only one of the three equations were in the preceding case, if the last, for instance, contained only the variables x_2 and x_1 , whilst the three variables x, x_1, x_2 , entered each of the two others, the determination of ϕ_1 and ϕ_2 might still be effected by the integration of differential equations of two variables, and of the first order. For the integral of this last equation being found the value of x_2 might be determined in function of x_1 , and by substituting it in the first equation, it will reduce it to a differential equation of two variables which may be integrated separately.

When $X = 0$, this last method may always be used, whatever be even the number of variables, for then the two first equations are both reduced to $dx = 0$, and give $x = c_1$, in the last equation $X_1 dx_2 - X_2 dx_1 = 0$, therefore x may be considered as a constant quantity, and it may be integrated as a differential equation of two variables. Let the integral be represented by $\phi_2 = c_2$, and then the integral of the partial differential equation proposed will be $x = \psi(\phi_2)$.

Example 1. Let the proposed equation be $Xp + Yq = Z$, in which X designates a function of x only, Y a function of y , Z a function of z , and p and q the partial differential coefficients $\frac{dz}{dx}, \frac{dz}{dy}$. The two equations to integrate are $X dz - Z dx = 0, Y dz - Z dy = 0$. They may be integrated separately, and give

$$\int \frac{dz}{Z} - \int \frac{dx}{X} = c_1, \quad \int \frac{dz}{Z} - \int \frac{dy}{Y} = c_2,$$

hence the required integral is

$$\int \frac{dz}{Z} - \int \frac{dx}{X} = \psi \left(\int \frac{dz}{Z} - \int \frac{dy}{Y} \right).$$

Example 2. Let the equation be $px + qy = nz$, p, q designating the same quantities as in the preceding example. The two equations are $x dz - nz dx = 0, y dz - nz dy = 0$, which may be integrated separately.

The integrals are $\frac{z}{x^n} = c_1, \frac{z}{y^n} = c_2$, and therefore the required integral is $\frac{z}{x^n} = \psi \left(\frac{z}{y^n} \right)$, which may easily be

transformed into $z = x^n \psi \left(\frac{y}{x} \right)$, which shows that z is equal to any homogeneous function of x and y . This

result might have been foreseen easily, for the equation from which we have derived it, is the expression of a property of homogeneous functions which has been proved before.

Example 3. Let $px + qz + y = 0$ be the proposed equation. The equations to be integrated are two of the three, $x dz + y dx = 0, z dz + y dy = 0$, and $x dy - z dx = 0$. The second equation is the only one which can be integrated as a differential equation of two variables. It leads to $z^2 + y^2 = a^2$, a^2 being the arbitrary constant. Substituting then the value of y derived from this in the first equation $x dz + y dx = 0$, it becomes $x dz + dx \sqrt{a^2 - z^2} = 0$, which is a differential equation of two variables. Integrating, we find,

$\sin^{-1} \frac{z}{a} + lx = lc_2$, or $xe \sin^{-1} \frac{z}{a} = c_2$, and in substituting for the constant a its value $\sqrt{z^2 + y^2}$, we get

$xe \sin^{-1} \frac{z}{\sqrt{z^2 + y^2}} = c_2$. The integral of $px + qz + y = 0$ is therefore $xe \sin^{-1} \frac{z}{\sqrt{z^2 + y^2}} = \psi(y^2 + z^2)$.

Example 4. Let $(y - bz)p - (x - az)q = bx - ay$ be the proposed equation. We have the three equations

$$(y - bz) dz - (bx - ay) dx = 0, \quad (x - az) dz + (bx - ay) dy = 0, \quad (y - bz) dy + (x - az) dx = 0.$$

Each of these equations contains the three variables x, y, z , and therefore it seems, at first, that in order to integrate any two of them, it is necessary to descend to the second order. This, however, may be avoided, for by substituting the value of z taken in the last equation in either of the two first, we get

Integral
Calculus.

$$dz + a dx + b dy = 0,$$

and by substituting in the last for $a dx + b dy$ its value $-dz$, derived from this, we get

$$x dx + y dy + z dz = 0.$$

The two equations we have just obtained, may therefore be taken instead of any two of the equations above. They are obviously satisfied by the two primitive equations

$$z + ax + by = c_1, \quad x^2 + y^2 + z^2 = c_2.$$

Therefore, the integral of the proposed equation is

$$x^2 + y^2 + z^2 = \psi(z + ax + by).$$

(198.) When the proposed equation is homogeneous with respect to the variables, the integration may always be made to depend upon that of an equation of two variables. Let $X = \frac{dx}{dx_1} X_1 + \frac{dx}{dx_2} X_2$ be the equation, and

make $x_1 = tx$, and $x_2 = ux$, the quantities X, X_1, X_2 will evidently assume the forms Yx^n, Y_1x^n, Y_2x^n , n being the sum of the dimensions of the variables in each term of the proposed equation. Hence the two equations $X dx_1 - X_1 dx = 0, X dx_2 - X_2 dx = 0$, become $(Yt - y_1) dx + Yx dt = 0, (Yu - Y_2) dx + Yx du = 0$,

and eliminating between these $\frac{dx}{x}$ we find $(Yt - y_1) du - (Yu - y_2) dt = 0$, an equation which contains

only t and u . This being integrated will give the value of t in function of u , which being substituted in one of the two equations $(Yt - y_1) dx + Yx dt = 0, (Yu - Y_2) dx + Yx du = 0$, will enable us to find the value of x , or the required integral.

As an example of this method, let the given equation be $pxz + qyz = xy$, the two equations to be integrated are $xz dz - xy dx = 0, yz dz - xy dy = 0$. Assuming $x = tz$, and $y = uz$, they become $(1 - ut) dz - uz dt = 0$,

and $(1 - ut) dz - tz du = 0$, and in eliminating between them $\frac{dz}{z}$ we find $t du - u dt = 0$, and, therefore,

$u = c_1 t$. Substituting this value in either of the two equations, we get $(1 - c_1^2) dz - c_1 t z dt = 0$, the integral of which is $z \sqrt{1 - c_1^2} = c_2$. But $t = \frac{x}{z}, \frac{u}{t} = c_1 = \frac{y}{x}$, therefore $\frac{y}{x} = c_1$, and $\sqrt{z^2 - xy} = c_2$, hence

the required integral is $\sqrt{z^2 - xy} = \psi\left(\frac{y}{x}\right)$.

(199.) Partial differential equations containing more than three variables, present various cases analogous to the preceding. We shall only give two examples.

First, let the proposed equation be

$$(x_2 + x_1) \frac{dx}{dx_3} + (x_3 + x_1) \frac{dx}{dx_2} + (x_2 + x_3) \frac{dx}{dx_1} = 0.$$

The equations to integrate are

$$dx = 0, \quad (x_2 + x_1) dx_2 = (x_3 + x_1) dx_3, \quad (x_2 + x_1) dx_1 = (x_2 + x_3) dx_3, \quad (x_3 + x_1) dx_1 = (x_2 + x_3) dx_2.$$

The first gives $x = c_1$, the others cannot be integrated separately, since they each contain more than two variables, but they may be transformed into others which lead immediately to primitive equations which satisfy them.

If we subtract the second from the first, we find $(x_2 + x_1)(dx_2 - dx_1) = (x_1 - x_2) dx_3$, and by adding them, and $(x_2 + x_1) dx_3$ to each side of the equation, we get $(x_2 + x_1)(dx_1 + dx_2 + dx_3) = 2 dx_3 (x_1 + x_2 + x_3)$.

Equalling the values of $\frac{dx_3}{x_2 + x_1}$, taken in each of these two last equations, we find

$$\frac{dx_2 - dx_1}{x_1 - x_2} = \frac{dx_1 + dx_2 + dx_3}{2(x_1 + x_2 + x_3)},$$

which is obviously satisfied by the primitive equation $(x_1 - x_2)^2 (x_1 + x_2 + x_3) = c_2$

Operating in the same manner on the two last equations to be integrated, we shall find

$$(x_3 - x_2)^2 (x_1 + x_2 + x_3) = c_3,$$

consequently the integral of the proposed equation is

$$\pi \{ z, (x_3 - x_2)^2 (x_1 + x_2 + x_3), (x_1 - x_2)^2 (x_1 + x_2 + x_3) \} = 0, \text{ or } z = \psi \{ (x_3 - x_2)^2 (x_1 + x_2 + x_3), (x_1 - x_2)^2 (x_1 + x_2 + x_3) \}.$$

Let, for the second example, the equation be

$$x_1 \frac{dx}{dx_1} + x_2 \frac{dx}{dx_2} + x_3 \frac{dx}{dx_3} + x_4 \frac{dx}{dx_4} = ax,$$

which contains five variables. The equations to integrate are

Part III.
Integral
Calculus.

Part III.

$$x_1 dx - ax dx_1 = 0, \quad x_2 dx - ax dx_2 = 0, \quad x_3 dx - ax dx_3 = 0, \quad x_4 dx - ax dx_4 = 0.$$

They may be integrated separately, and give

$$\frac{x}{x_1^a} = c_1, \quad \frac{x}{x_2^a} = c_2, \quad \frac{x}{x_3^a} = c_3, \quad \frac{x}{x_4^a} = c_4.$$

Instead of the three last equations we may take $\frac{x_2}{x_1} = b_2, \quad \frac{x_3}{x_1} = b_3, \quad \frac{x_4}{x_1} = b_4$, b_2, b_3, b_4 being arbitrary constants. Therefore the integral required is

$$x = x_1^a \psi \left(\frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1} \right).$$

That is to say, that x is an homogeneous function of the degree a of the four variables x_1, x_2, x_3, x_4 .

(200.) Hitherto we have only considered equations in which the partial differential coefficients entered in the first degree. Let us consider now an equation of a degree higher than the first, but still of the first order. Lagrange has given a method to reduce the integration of such an equation, between three variables, to the integration of a partial differential equation of the first degree between four variables. The determination of the arbitrary function, which enters in the integral, is made to depend on the integration of a total differential equation of the first order, and of the first degree, between three variables, by a system of two equations. This integration is always possible, and requires, in considering one of the variables as a constant, which is allowed, only the integration of a differential equation of the first order, and of the first degree, between two variables.

This may be generalized, and it may be proved that it is always possible to derive from a partial differential equation of the first order between $n+1$ variables, a system of partial differential equations of the first order, and of the first degree, which may be integrated by the preceding method. The determination of the arbitrary functions requires besides the integration of a total differential equation of the first order, between $2n-1$ variables by a system of n equations, or between $2n-2$ variables by a system of $n-1$ equations, in considering one of the variables as a constant quantity, which may always be done.

(201.) Let x be a function of $x_1, x_2, \dots, x_n$, and make $\frac{dx}{dx_1} = p_1, \quad \frac{dx}{dx_2} = p_2, \dots, \frac{dx}{dx_n} = p_n$. We shall

have for any integer κ and λ $\frac{d p_\kappa}{d x_\lambda} = \frac{d p_\lambda}{d x_\kappa}$. Let the proposed equation be

$$0 = \phi(x, x_1, x_2, \dots, x_n, p_1, p_2, \dots, p_n).$$

If we differentiate it with respect to x_1 , and if we substitute the values

$$\frac{dx}{dx_1} = p_1, \quad \frac{dp_2}{dx_1} = \frac{dp_1}{dx_2}, \quad \frac{dp_3}{dx_1} = \frac{dp_1}{dx_3}, \dots, \frac{dp_n}{dx_1} = \frac{dp_1}{dx_n},$$

we shall have

$$-p_1 \cdot \frac{d\phi}{dx} - \frac{d\phi}{dx_1} = \frac{d\phi}{dp_1} \cdot \frac{dp_1}{dx} + \frac{d\phi}{dp_2} \cdot \frac{dp_1}{dx_2} + \dots + \frac{d\phi}{dp_n} \cdot \frac{dp_1}{dx_n}.$$

Similar equations will be obtained in differentiating with respect to $x_2, x_3, \dots, x_n$. We have, besides, identically,

$$p_1 \frac{d\phi}{dp_1} + p_2 \frac{d\phi}{dp_2} + \dots + p_n \frac{d\phi}{dp_n} = \frac{d\phi}{dp_1} \cdot \frac{dx}{dx_1} + \frac{d\phi}{dp_2} \cdot \frac{dx}{dx_2} + \dots + \frac{d\phi}{dp_n} \cdot \frac{dx}{dx_n}.$$

Consequently, if we make

$$p_1 \frac{d\phi}{dp_1} + p_2 \frac{d\phi}{dp_2} + \dots + p_n \frac{d\phi}{dp_n} = P,$$

we shall have the $n+1$ following equations :

$$\left. \begin{aligned} P &= \frac{d\phi}{dp_1} \cdot \frac{dx}{dx_1} + \frac{d\phi}{dp_2} \cdot \frac{dx}{dx_2} + \dots + \frac{d\phi}{dp_n} \cdot \frac{dx}{dx_n} \\ -p_1 \frac{d\phi}{dx} - \frac{d\phi}{dx_1} &= \frac{d\phi}{dp_1} \cdot \frac{dp_1}{dx} + \frac{d\phi}{dp_2} \cdot \frac{dp_1}{dx_2} + \dots + \frac{d\phi}{dp_n} \cdot \frac{dp_1}{dx_n} \\ -p_2 \frac{d\phi}{dx} - \frac{d\phi}{dx_2} &= \frac{d\phi}{dp_1} \cdot \frac{dp_2}{dx_1} + \frac{d\phi}{dp_2} \cdot \frac{dp_2}{dx_2} + \dots + \frac{d\phi}{dp_n} \cdot \frac{dp_2}{dx_n} \\ &\dots \dots \dots \\ -p_n \frac{d\phi}{dx} - \frac{d\phi}{dx_n} &= \frac{d\phi}{dp_1} \cdot \frac{dp_n}{dx_1} + \frac{d\phi}{dp_2} \cdot \frac{dp_n}{dx_2} + \dots + \frac{d\phi}{dp_n} \cdot \frac{dp_n}{dx_n} \end{aligned} \right\} \quad (C)$$

These $n+1$ partial differential equations have the property of each containing only the partial differential

coefficients of the same variable, whilst the partial differential coefficients, relative to the same variable, in all the equations, have the same coefficients. These equations may, therefore, be integrated by means of the theorem demonstrated. (194.) To that effect we shall form the following $2n$ total differential equations: Part III.

$$\begin{aligned} P \frac{d x_1}{d x} &= \frac{d \phi}{d p_1}, & P \frac{d x_2}{d x} &= \frac{d \phi}{d p_2}, & \dots & P \frac{d x_n}{d x} &= \frac{d \phi}{d p_n}, \\ P \frac{d p_1}{d x} &= -p_1 \frac{d \phi}{d x} - \frac{d \phi}{d x_1}, & P \frac{d p_2}{d x} &= -p_2 \frac{d \phi}{d x} - \frac{d \phi}{d x_2}, & \dots & P \frac{d p_n}{d x} &= -p_n \frac{d \phi}{d x} - \frac{d \phi}{d x_n}. \end{aligned}$$

From these equations we derive, in multiplying the equations of the second line respectively by $\frac{d \phi}{d p_1}, \frac{d \phi}{d p_2},$

adding them, and substituting in the right side of the equation, for $p_1 \frac{d \phi}{d p_1} + p_2 \frac{d \phi}{d p_2} + \dots + p_n \frac{d \phi}{d p_n}$ its value P ,

$$\frac{d \phi}{d x} d x + \frac{d \phi}{d x_1} d x_1 + \frac{d \phi}{d x_2} d x_2 + \dots + \frac{d \phi}{d x_n} d x_n + \frac{d \phi}{d p_1} d p_1 + \frac{d \phi}{d p_2} d p_2 + \dots + \frac{d \phi}{d p_n} d p_n = 0,$$

which shows that the given equation $\phi = 0$ is one of the $2n$ integrals of these differential equations.

Let the $2n - 1$ others be

$$\phi_1 = c_1, \quad \phi_2 = c_2, \quad \dots \quad \phi_{2n-1} = c_{2n-1},$$

where $c_1, c_2, \dots, c_n$ are arbitrary constants, and $\phi_1, \phi_2, \dots$ functions which do not contain these constants. If then we designate by $\pi_1, \pi_2, \pi_3, \dots, \pi_n$, arbitrary functions of $\phi_1, \phi_2, \dots, \phi_n$, the integrals of the $n + 1$ equations will be

$$\phi = 0, \quad \pi_1 = 0, \quad \pi_2 = 0, \quad \pi_3 = 0, \quad \dots \quad \pi_n = 0.$$

(202.) The $n + 1$ equations (c) have been deduced from the given equation $\phi = 0$, in taking into consideration that the quantities $p_1, p_2, \dots, p_n$ are the partial differential coefficients of x . But we cannot conclude

reciprocally from this system of equations, that $p_1 = \frac{d x}{d x_1}, p_2 = \frac{d x}{d x_2}, \dots, p_n = \frac{d x}{d x_n}$. It is, therefore, neces-

sary to determine the n equations $\pi_1 = 0, \pi_2 = 0, \dots, \pi_n = 0$, so that they may lead to the values $p_1 = \frac{d x}{d x_1}, p_2 = \frac{d x}{d x_2}, \dots, p_n = \frac{d x}{d x_n}$, which may be included in the single equation

$$d x = p_1 d x_1 + p_2 d x_2 + \dots + p_n d x_n.$$

(203.) Since the problem is resolved by n equations, between the quantities $\phi_1, \phi_2, \dots, \phi_{2n-1}$, we may conclude that the equation

$$d x = p_1 d x_1 + p_2 d x_2 + \dots + p_n d x_n$$

may be transformed into another containing only the quantities $\phi_1, \phi_2, \dots, \phi_{2n-1}$. This proposition may also be demonstrated in a different manner.

Let us consider for that purpose $2n$ out of the $2n + 1$ quantities $x, x_1, x_2, \dots, x_n, p_1, p_2, \dots, p_n$, for instance all but x , as functions of that variable, and of the quantities $\phi_1, \phi_2, \dots, \phi_{2n-1}$. In this supposition the equation

$$d x = p_1 d x_1 + p_2 d x_2 + \dots + p_n d x_n$$

becomes

$$\begin{aligned} d x &= d x \left\{ p_1 \frac{d x_1}{d x} + p_2 \frac{d x_2}{d x} + \dots + p_n \frac{d x_n}{d x} \right\} \\ &+ d \phi_1 \left\{ p_1 \frac{d x_1}{d \phi_1} + p_2 \frac{d x_2}{d \phi_1} + \dots + p_n \frac{d x_n}{d \phi_1} \right\} \\ &+ d \phi_2 \left\{ p_1 \frac{d x_1}{d \phi_2} + p_2 \frac{d x_2}{d \phi_2} + \dots + p_n \frac{d x_n}{d \phi_2} \right\} \\ &\dots \dots \dots \\ &+ d \phi_{2n-1} \left\{ p_1 \frac{d x_1}{d \phi_{2n-1}} + p_2 \frac{d x_2}{d \phi_{2n-1}} + \dots + p_n \frac{d x_n}{d \phi_{2n-1}} \right\}. \end{aligned}$$

For the sake of brevity, we may write this value of $d x$ under the following form,

$$d x = K d x + P_1 d \phi_1 + P_2 d \phi_2 + \dots + P_{2n-1} d \phi_{2n-1}.$$

The partial differentials with respect to x may easily be found in considering that, to obtain them, we must suppose the quantities $\phi_1, \phi_2, \dots, \phi_{2n-1}$ to be constant. We shall have, therefore, for that case, the differential equations of (200.)

$$P d x_1 = \frac{d \phi}{d p_1} d x, \quad P d x_2 = \frac{d \phi}{d p_2} d x, \quad \dots \quad P d x_n = \frac{d \phi}{d p_n} d x,$$

Integral
Calculus.

$$P dp_1 = - \left\{ p_1 \frac{d\phi}{dx} + \frac{d\phi}{dx_1} \right\} dx \dots \dots P dp_n = - \left\{ p_n \frac{d\phi}{dx} + \frac{d\phi}{dx_n} \right\} dx$$

or

$$P = p_1 \frac{d\phi}{dp_1} + p_2 \frac{d\phi}{dp_2} + \dots \dots + p_n \frac{d\phi}{dp_n}.$$

We have, consequently,

$$\frac{P dx_1}{dx} = \frac{d\phi}{dp_1}, \quad \frac{P dx_2}{dx} = \frac{d\phi}{dp_2} \dots \dots \frac{P dx_n}{dx} = \frac{d\phi}{dp_n}.$$

$$\frac{P dx_1}{dx} = - \left\{ p_1 \frac{d\phi}{dx} + \frac{d\phi}{dx_1} \right\} \dots \dots \frac{P dx_n}{dx} = - \left\{ p_n \frac{d\phi}{dx} + \frac{d\phi}{dx_n} \right\}.$$

Therefore

$$K = p_1 \frac{dx_1}{dx} + p_2 \frac{dx_2}{dx} + \dots \dots + p_n \frac{dx_n}{dx} = 1.$$

 The equation $dx = K dx + P_1 d\phi_1 + P_2 d\phi_2 + \dots \dots + P_{2n-1} d\phi_{2n-1}$ is then reduced to

$$0 = P_1 d\phi_1 + P_2 d\phi_2 + \dots \dots + P_{2n-1} d\phi_{2n-1}.$$

 Now it may be proved that in $P_1, P_2, \dots \dots, P_{2n-1}$ the variable x can only enter as a common factor to all these quantities. For if we differentiate the equation

$$P_1 = p_1 \frac{dx_1}{d\phi_1} + p_2 \frac{dx_2}{d\phi_1} + \dots \dots + p_n \frac{dx_n}{d\phi_1}$$

 with respect to x , we find

$$\frac{dP_1}{dx} = d \cdot \frac{\left\{ p_1 \frac{dx_1}{dx} + p_2 \frac{dx_2}{dx} + \dots \dots + p_n \frac{dx_n}{dx} \right\}}{d\phi_1}$$

$$+ \frac{dp_1}{dx} \cdot \frac{dx_1}{d\phi_1} + \frac{dp_2}{dx} \frac{dx_2}{d\phi_1} + \dots \dots + \frac{dp_n}{dx} \cdot \frac{dx_n}{d\phi_1},$$

$$- \left\{ \frac{dp_1}{d\phi_1} \frac{dx_1}{dx} + \frac{dp_2}{d\phi_1} \frac{dx_2}{dx} + \dots \dots + \frac{dp_n}{d\phi_1} \cdot \frac{dx_n}{dx} \right\},$$

 the first line is equal to $\frac{dk}{d\phi_1}$, and is, consequently, equal to nothing, since $k = 1$; if in the two others we substitute the value of the partial differential with respect to x found (192.), we shall have

$$\frac{dP_1}{dx} = - \frac{d\phi}{dx} \frac{P_1}{P} - \frac{1}{P} \left\{ \frac{d\phi}{dx_1} \frac{dx_1}{d\phi_1} + \dots \dots + \frac{d\phi}{dx_n} \frac{dx_n}{d\phi_1} \right\} = - \frac{d\phi}{dx} \frac{P_1}{P},$$

$$+ \frac{d\phi}{dp_1} \frac{dp_1}{d\phi_1} + \dots \dots + \frac{d\phi}{dp_n} \frac{dp_n}{d\phi_1} \Bigg\} = - \frac{d\phi}{dx} \frac{P_1}{P},$$

 since the expression included between the brackets, and which is the complete differential of ϕ with respect to ϕ_1 , vanishes, because $\phi = 0$. We shall find in the same manner

$$\frac{dP_2}{dx} = - \frac{d\phi}{dx} \frac{P_2}{P} \dots \dots \frac{dP_{2n-1}}{dx} = - \frac{d\phi}{dx} \frac{P_{2n-1}}{P},$$

therefore

$$\frac{dP_1}{P_1 dx} = \frac{dP_2}{P_2 dx} \dots \dots = \frac{dP_{2n-1}}{P_{2n-1} dx} = - \frac{1}{P} \frac{d\phi}{dx}.$$

Integrating, we find

$$P_1 = F_1 e^{-\int \frac{1}{P} \frac{d\phi}{dx} dx}, \quad P_2 = F_2 e^{-\int \frac{1}{P} \frac{d\phi}{dx} dx} \dots \dots P_{2n-1} = F_{2n-1} e^{-\int \frac{1}{P} \frac{d\phi}{dx} dx},$$

 when $F_1, F_2, \dots \dots, F_{2n-1}$ do not contain x . Therefore, if we divide by $e^{-\int \frac{1}{P} \frac{d\phi}{dx} dx}$, the equation $dx = p_1 dx_1 + p_2 dx_2 + \dots \dots + p_n dx_n$ will be reduced to

$$0 = F_1 d\phi_1 + F_2 d\phi_2 + \dots \dots + F_{2n-1} d\phi_{2n-1},$$

 in which equation $F_1, F_2, \dots \dots, F_{2n-1}$ are functions of $\phi_1, \phi_2, \dots \dots, \phi_{2n-1}$ only. This last equation is now to be integrated by a system of n equations, and this can always be done.

(204.) The method of integrating an equation such as

$$0 = F_1 d\phi_1 + F_2 d\phi_2 + \dots \dots + F_{2n-1} d\phi_{2n-1},$$

 by a system of n equations, ought to find place here. But not to exceed our limits, we shall only consider the case of three variables.

 Let $F_1 d\phi_1 + F_2 d\phi_2 + F_3 d\phi_3 = 0$ be the proposed equation, in which it is not necessary that the conditions of integrability should be satisfied. If we consider ϕ_3 as a constant quantity $d\phi_3 = 0$, and the equation being reduced to $F_1 d\phi_1 + F_2 d\phi_2 = 0$ may be integrated as a function of two variables. Let $U = c$ represent that integral, in which, instead of the arbitrary constant c , we may put an arbitrary function of ϕ_3 , and then the integral

Integral Calculus. takes the form $U = \psi(\phi_3)$. If we differentiate it with respect to the three variables ϕ_1, ϕ_2, ϕ_3 , we find Part III.
 $\frac{dU}{d\phi_1} d\phi_1 + \frac{dU}{d\phi_2} d\phi_2 + \frac{dU}{d\phi_3} d\phi_3 = \psi'(\phi_3) d\phi_3$, which equation ought to agree with the proposed differential equation $F_1 d\phi_1 + F_2 d\phi_2 + F_3 d\phi_3 = 0$. Let μ be the factor by which the equation $F_1 d\phi_1 + F_2 d\phi_2 = 0$ ought to be multiplied to become an exact differential, then $\frac{dU}{d\phi_1} d\phi_1 + \frac{dU}{d\phi_2} d\phi_2 = \mu F_1 d\phi_1 + \mu F_2 d\phi_2$, or equal to $-\mu F_3 d\phi_3$, since by multiplying the proposed equation by μ , we get $\mu F_1 d\phi_1 + \mu F_2 d\phi_2 + \mu F_3 d\phi_3 = 0$. Substituting in the differential of $U = \psi(\phi_3)$ the value which has just been found for its two first terms, it becomes, after having been divided by $d\phi_3$, $\frac{dU}{d\phi_3} - \mu F_3 = \psi'(\phi_3)$. Consequently the system of the two equations

$$U = \psi(\phi_3), \quad \frac{dU}{d\phi_3} - \mu F_3 = \psi'(\phi_3),$$

will satisfy the proposed equation whatever be the form of the function ψ .

(205.) We may now apply the general method given (201.) (202.) (203.) to the integration of partial differential equations of the first order, and of a degree higher than the first. We shall take for first example the

equation $p_1 p_2 = x$, in which $p_1 = \frac{dx}{dx_1}$, $p_2 = \frac{dx}{dx_2}$. In this case, the quantity represented by P (201.) is equal to $2 p_1 p_2$, and the total differential equations of the same number are,

$$2 p_1 \frac{dx_1}{dx} = 1, \quad 2 p_2 \frac{dx_2}{dx} = 1, \quad 2 p_2 \frac{dp_1}{dx} = 1, \quad 2 p_1 \frac{dp_2}{dx} = 1.$$

These equations are easily integrated; the equations which satisfy them are clearly

$$p_2 - x_1 = c_1, \quad x_2 - p_1 = c_2, \quad \frac{p_2}{p_1} = c_3, \quad \text{and } p_1 p_2 = x.$$

From these we derive, in putting ϕ_1, ϕ_2, ϕ_3 , instead of c_1, c_2, c_3 ,

$$x_1 = -\phi_1 + \sqrt{\phi_3} x, \quad x_2 = \phi_2 + \frac{\sqrt{x}}{\phi_3}, \quad p_1 = \sqrt{\frac{x}{\phi_3}}, \quad p_2 = \sqrt{\phi_3} x.$$

Hence, since

$$\frac{dx_1}{d\phi_1} = -1, \quad \frac{dx_1}{d\phi_2} = 0, \quad \frac{dx_1}{d\phi_3} = \frac{\sqrt{x}}{2\sqrt{\phi_3}}, \quad \frac{dx_2}{d\phi_1} = 0, \quad \frac{dx_2}{d\phi_2} = 1, \quad \frac{dx_2}{d\phi_3} = -\frac{\sqrt{x}}{2\sqrt{\phi_3}},$$

the equation $P_1 d\phi_1 + P_2 d\phi_2 + P_3 d\phi_3 = 0$, or $F_1 d\phi_1 + F_2 d\phi_2 + F_3 d\phi_3 = 0$, will be simply

$$d\phi_1 - \phi_3 d\phi_2 = 0.$$

This integrated by the method given (203.) in considering first ϕ_1 as a constant quantity, will give the two equations

$$\phi_2 = \psi(\phi_1), \quad \frac{1}{\phi_3} = \psi'(\phi_1),$$

or, in substituting for ϕ_1, ϕ_2 , and ϕ_3 , their values,

$$x_2 - \frac{x}{p_2} = \psi(p_2 - x_1), \quad \frac{x}{p_2^2} = \psi'\left(\frac{p}{2} - x\right).$$

These equations conjointly satisfy the proposed equation $x = p_1 p_2$, and by assuming any particular value to ψ , and then eliminating p_2 , will give an equation between x, x_1 , and x_2 .

We shall take for the second and last example, the equation $p_1^2 + p_2^2 = 1$. Here we have $P = 2 p_1^2 + 2 p_2^2 = 2$, and the total differential equations to be integrated are

$$\frac{dx_1}{dx} = p_1, \quad \frac{dx_2}{dx} = p_2, \quad \frac{dp_1}{dx} = 0, \quad \frac{dp_2}{dx} = 0.$$

They are satisfied by the equations

$$x_1 = c_3 x + c_1, \quad x_2 = x \sqrt{1 - c_3^2} + c_2, \quad p_1 = c_3, \quad p_2 = \sqrt{1 - c_3^2}.$$

Hence

$$\frac{dx_1}{d\phi_1} = 1, \quad \frac{dx_1}{d\phi_2} = 0, \quad \frac{dx_1}{d\phi_3} = x, \quad \frac{dx_2}{d\phi_1} = 0, \quad \frac{dx_2}{d\phi_2} = 1, \quad \frac{dx_2}{d\phi_3} = -\frac{x\phi}{\sqrt{1 - \phi_3^2}},$$

and, therefore, the equation to be integrated is

$$\phi_3 d\phi_1 + \sqrt{1 - \phi_3^2} d\phi_2 = 0.$$

If we consider ϕ_3 as a constant quantity, the integral is represented by the two equations

$$\phi_3 \phi_1 + \sqrt{1 - \phi_3^2} \phi_2 = \psi(\phi_3), \quad \phi_1 - \frac{\phi_3 \phi_2}{\sqrt{1 - \phi_3^2}} = \psi'(\phi_3),$$

or in substituting for ϕ_1, ϕ_2, ϕ_3 , their values, we find for the required equations

$$p_1 x_1 - x + x \sqrt{1 - p_1^2} = \psi(p_1), \quad x_1 \sqrt{1 - p_1^2} - p_1 x_2 = \psi'(p_1).$$

(206.) We shall next consider partial differential equations of the second and higher orders.

A partial differential equation of the n^{th} order in its most general form ought to include the independent variables, the required function, and all its differential coefficients, from the first to the n^{th} order. No theorem relative to partial differential equations under this general form has hitherto been found. In some particular cases the difficulty may be reduced to the integration of differential equations of lower orders.

Equations between three variables of the form

$$F \left\{ x, y, \frac{d^n z}{d y^n}, \frac{d^{n+1} z}{d x d y^n}, \dots, \frac{d^{n+m} z}{d x^m d y^n} \right\} = 0,$$

may be reduced to the m^{th} order by assuming $\nu = \frac{d^n z}{d y^n}$, for then they become

$$F \left\{ x, y, \nu, \frac{d \nu}{d x}, \frac{d^2 \nu}{d x^2}, \dots, \frac{d^m \nu}{d x^m} \right\} = 0.$$

This equation may be considered as a differential equation between two variables ν and x , since all the differential coefficients relate to the variable x , and it may therefore be integrated in the supposition of y being a constant quantity, and consequently the m arbitrary constants contained in the integral replaced by m arbitrary functions of y . Having found by this process the value of ν , that of z will be obtained by means of the equation

$\frac{d^n z}{d y^n} = \nu$, in which x may be supposed to be a constant quantity, and in the integral of which to the n arbitrary

constants ought to be substituted n arbitrary functions of x .

(207.) Equations of the n^{th} order which include partial differential coefficients with respect to one variable only, may clearly be treated as differential equations of two variables. And the integral being obtained in that supposition, the arbitrary constants should be replaced by arbitrary functions of the variables, with respect to which no partial differential coefficient was to be found in the proposed differential equation.

The first method may be applied to the equation $\frac{d^2 z}{d x d y} + P \frac{d z}{d x} = Q$, where P and Q contain only the variables x and y .

If we make $\frac{d z}{d x} = \nu$, it becomes $\frac{d \nu}{d y} + P \nu = Q$, the integral of which is

$$\nu = e^{-\int P dy} \left(\int e^{\int P dy} \cdot Q dy + c \right).$$

Therefore $\frac{d z}{d x} = e^{-\int P dy} \left(\int e^{\int P dy} \cdot Q dy + \phi(x) \right)$; integrating this with respect to z and x alone, we find

$$z = \int dx e^{-\int P dy} \left\{ \int e^{\int P dy} Q dy + \phi(x) \right\} + \psi(y).$$

If $P = 0$, this value is reduced to

$$z = \int dx \int Q dy + \int dx \phi(x) + \psi(y),$$

or $z = \int dx \int Q dy +$ an arbitrary function of $x +$ an arbitrary function of y , a result the truth of which is obvious.

The second method may be applied to the equation $\frac{d^2 z}{d x^2} + P \frac{d z}{d x} = Q$, in which P and Q include x, y , and z .

The equation ought to be integrated as a differential equation of the second order, between the variables x and y , and instead of the two arbitrary constants, two arbitrary functions of y introduced. If $P = 0$, and Q contain only x and y , the integral is

$$z = \int dx \int Q dx + x \phi(y) + \psi(y).$$

(208.) We shall now explain by what process may be integrated a partial differential equation of the second order, of the form

$$\frac{d^2 z}{d x^2} + P \frac{d^2 z}{d x d y} + Q \frac{d^2 z}{d y^2} = R,$$

where P, Q , and R are supposed to be any function of $x, y, z, \frac{d z}{d x}$, and $\frac{d z}{d y}$.

Let $\frac{d z}{d x} = p, \frac{d z}{d y} = q, \frac{d^2 z}{d x^2} = r, \frac{d^2 z}{d x d y} = s$, and $\frac{d^2 z}{d y^2} = t$, the proposed equation will become

$$r + P s + Q t = R,$$

Integral
Calculus.

and we shall have also

$$dz = p dx + q dy, \quad dp = r dx + s dy, \quad dq = s dx + t dy.$$

Now, if we form the two systems of equations,

$$dy - k dx = 0, \quad k dp + Q dq - R k dx = 0, \quad \text{and} \quad dk - k' dx = 0, \quad k' dp + Q dq - R k' dx = 0,$$

where k and k' are the roots of the equation $k^2 - Pk + Q = 0$; if from either of the two above systems of equations we can deduce either separately, or by adding any multiple of one to any multiple of the other, two primitive equations $M = a$, $N = b$, then $N = \phi(M)$ will be a first integral of the proposed equation; the integral of this equation of the first order, determined by the ordinary process, will be the second or complete integral required.

(209.) To demonstrate the theorem just stated, it is necessary to prove that the proposed equation $r + Ps + Qt = R$, may be derived from the equation $N = \phi(M)$.

We have first, $dN = \phi'(M) dM$, and

$$dN = \frac{dN}{dx} dx + \frac{dN}{dy} dy + \frac{dN}{dp} dp + \frac{dN}{dq} dq + \frac{dN}{dz} dz,$$

$$dM = \frac{dM}{dx} dx + \frac{dM}{dy} dy + \frac{dM}{dp} dp + \frac{dM}{dq} dq + \frac{dM}{dz} dz.$$

Substituting for dy its value $k dx$, and for dz its value $p dx + q dy = p dx + q k dx$, we find

$$dN = \left\{ \frac{dN}{dx} + k \frac{dN}{dy} + p \frac{dN}{dz} + q k \frac{dN}{dz} \right\} dx + \frac{dN}{dp} dp + \frac{dN}{dq} dq,$$

$$dM = \left\{ \frac{dM}{dx} + k \frac{dM}{dy} + p \frac{dM}{dz} + q k \frac{dM}{dz} \right\} dx + \frac{dM}{dp} dp + \frac{dM}{dq} dq.$$

Taking the value of dq in the equation $k dp + Q dq - R k dx = 0$, and substituting it in the two preceding equations, we shall get

$$dN = \left\{ \frac{dN}{dx} + k \frac{dN}{dy} + p \frac{dN}{dz} + q k \frac{dN}{dz} + \frac{R k}{Q} \frac{dN}{dq} \right\} dx + \frac{\left(Q \frac{dN}{dp} - k \frac{dN}{dq} \right)}{Q} dp,$$

$$dM = \left\{ \frac{dM}{dx} + k \frac{dM}{dy} + p \frac{dM}{dz} + q k \frac{dM}{dz} + \frac{R k}{Q} \frac{dM}{dq} \right\} dx + \frac{\left(Q \frac{dM}{dp} - k \frac{dM}{dq} \right)}{Q} dp.$$

But since, by hypothesis, M and N are equal to constant quantities, their differentials are equal to nothing, and since, moreover, x and p are here considered as independent variables, the coefficients of dx and dp in the values of dN and dM are separately equal to nothing. Hence we have the four following equations:

$$Q \left(\frac{dN}{dx} + k \frac{dN}{dy} + p \frac{dN}{dz} + q k \frac{dN}{dz} \right) + R k \frac{dN}{dq} = 0, \quad Q \frac{dN}{dp} - k \frac{dN}{dq} = 0,$$

$$Q \left(\frac{dM}{dx} + k \frac{dM}{dy} + p \frac{dM}{dz} + q k \frac{dM}{dz} \right) + R k \frac{dM}{dq} = 0, \quad Q \frac{dM}{dp} - k \frac{dM}{dq} = 0.$$

If we take the values of $\frac{dN}{dx}$, $\frac{dM}{dx}$, $\frac{dN}{dp}$, and $\frac{dM}{dp}$, in these four equations, and substitute them in the

equation $dN = \phi'(M) dM$, or

$$\left(\frac{dN}{dx} dx + \frac{dN}{dy} dy + \frac{dN}{dp} dp + \frac{dN}{dq} dq + \frac{dN}{dz} dz \right) = \phi'(M) \left\{ \frac{dM}{dx} dx + \frac{dM}{dy} dy + \frac{dM}{dp} dp + \frac{dM}{dq} dq + \frac{dM}{dz} dz \right\},$$

we shall find for result, after substituting $p dx + q dy$ for dz ,

$$\left(\frac{dN}{dy} + q \frac{dN}{dz} \right) (dy - k dx) + \frac{1}{Q} \frac{dN}{dq} (k dp + Q dq - R k dx) = \phi'(M) \left\{ \left(\frac{dM}{dy} + q \frac{dM}{dz} \right) (dy - k dx) + \frac{1}{Q} \frac{dM}{dq} (k dp + Q dq - R k dx) \right\},$$

therefore

$$k dp + Q dq - R k dx = w (dy - k dx),$$

where

$$w = - \frac{\left(\frac{dN}{dy} + q \frac{dN}{dz} \right) - \phi'(M) \left(\frac{dM}{dy} + q \frac{dM}{dz} \right)}{\frac{1}{Q} \left(\frac{dN}{dq} - \phi'(M) \frac{dM}{dq} \right)}.$$

Part III.
Integral
Calculus.

Part III.

Substituting $r dx + s dy$ for dp , and $s dx + t dy$ for dq , we shall find an equation between the independent variables x and y , which will give the two equations

$$kr + Qs - Rk = -wk, \quad ks + Qt = w.$$

Eliminating w we obtain

$$kr + k^2s + Qs + Qkt - Rk = 0.$$

But we have $k^2 = Pk - Q$, substituting this value, we find

$$k(r + Ps + Qt - R) = 0, \quad \text{or } r + Ps + Qt - R = 0,$$

which is the proposed equation.

(210.) Let us take for example of this method the equation

$$\frac{d^2z}{dx^2} + \frac{2y}{x} \frac{dz}{dx} + \frac{y^2}{x^2} \frac{dz}{dy} = 0, \quad \text{or } r + \frac{2y}{x}s + \frac{y^2}{x^2}t = 0.$$

Here the two values of k are equal to one another, and we have only one system of two equations. They are

$$x dy - y dx = 0, \quad x dp + y dq = 0.$$

Integrating we find $\frac{y}{x} = c$, $p + \frac{y}{x}q = c'$, therefore the first integral of the proposed equation is $p + \frac{y}{x}q = \phi\left(\frac{y}{x}\right)$.

This partial differential equation of the first order may be integrated according to the process given, (194.) We shall have then to integrate the two equations

$$x dy - y dx = 0, \quad dz - dx \phi\left(\frac{y}{x}\right) = 0.$$

The first gives $\frac{y}{x} = c$, the second becomes $z - x \phi\left(\frac{y}{x}\right) = c'$, and the required integral is, therefore,

$$z = x \phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right).$$

(211.) The equation $A r + B s + C t = V$, in which A, B, C are constant quantities, and V a function of x and y only, is easily integrated by the same process. The two values of k are then given by the equation $A k^2 - B k + C = 0$. The two values of k are therefore constant, and if we represent them by k' and k'' , we shall have to integrate either of the two following systems of equations:

$$dy - k' dx = 0, \quad A k' dp + C dq - V k' dx = 0, \quad \text{and } dy - k'' dx = 0, \quad A k'' dp + C dq - V k'' dx = 0$$

They give

$$y - k' x = a, \quad A k' p + C q - k' \int V dx = b, \quad \text{and } y - k'' x = a', \quad A k'' p + C q - k'' \int V dx = b'$$

The integral $\int V dx$ depends only on one variable, since y may be eliminated by means of the relations $y - k' x = a$, or $y - k'' x = a'$. We thus obtain the first two integrals of the proposed equation,

$$A k' p + C q - k' \int V dx = \phi(y - k' x), \quad \text{and } A k'' p + C q - k'' \int V dx = \psi(y - k'' x).$$

Either of these equations being integrated will give the next integral.

To obtain the integral of the first, we must integrate the two equations

$$A k' dy - C dx = 0, \quad A k' dz - k' dx \int V dx - dx \phi(y - k' x) = 0.$$

The first gives $A k' y - C x = a$, and by substituting the value of y derived from this integral into the second

$$\text{equation, it becomes } A k' dz - k' dx \int V dx - dx \phi\left(\frac{C x + a}{A k'} - k' x\right) = 0, \quad \text{or } A k' dz - k' dx \int V dx -$$

$$\left(\frac{C}{A k'} - k'\right) dx \phi\left(\left(\frac{C}{A k'} - k'\right)x + \frac{a}{A k'}\right) = 0, \quad \text{for since } \phi \text{ is an arbitrary function, we may prefix to it the}$$

constant factor $\left(\frac{C}{A k'} - k'\right)$. Under this form the equation may be integrated, and it is obvious that the

integral of the last term will be an arbitrary function of $\left(\frac{C}{A k'} - k'\right)x + \frac{a}{A k'}$. We have thus $A k' z - k' \int dx \int V dx - \phi_1$

$$\int V dx - \phi_1\left(\left(\frac{C}{A k'} - k'\right)x + \frac{a}{A k'}\right) = b, \quad \text{or putting back for } a \text{ its value, } A k' z - k' \int dx \int V dx - \phi_1$$

$(y - k' x) = b$. The required integral is therefore

$$A k' z - k' \int dx \int V dx - \phi_1(y - k' x) = \psi_1(A k' y - C x).$$

But $C = A k' k''$, substituting this value, dividing both sides by $A k'$, and suppressing the constant factors of the arbitrary functions, we find

$$z = \frac{1}{A} \int dx \int V dx + \phi_1(y - k'x) + \psi_1(y - k''x).$$

In effecting the integrations indicated in this equation it is necessary to attend to the following remarks.

1. In determining the value of $\int V dx$, y should be replaced in V by $k'x + a$; and after the integration of $V dx$ has been effected, a should be replaced by $y - k'x$. Then, before effecting the integration of $dx \int V dx$, y should be replaced in $\int V dx$ by $\frac{cx + a}{A k'}$, or by $k''x + a'$, and after the integration has been effected, $y - k''x$ should be substituted for a' .

This method applied to the equation $\frac{d^2 z}{dx^2} = C^2 \frac{dz}{dy^2}$, or $r = C^2 t$, gives for the integral

$$z = \phi(y - Cx) + \psi(y + cx).$$

The same process applied to the equation $r + Bs + Ct = xy$, gives

$$z = \frac{1}{b} x^3 y + \frac{1}{4-b} (m' + m'') x^4 = \phi(y - m'x) + \psi(y - m''x),$$

where m' and m'' are the roots of the equation $m^2 - Bm + C = 0$.

(212.) The equations

$$s + Pp + Qq + \left\{ P Q + \frac{dP}{dx} \right\} z = V, \quad s + Pp + Qq + \left\{ P Q + \frac{dQ}{dy} \right\} z = V,$$

where P , Q , and V are functions of x and y only, may be integrated in making $\frac{dz}{dy} + Pz = v$, which gives

$\frac{dv}{dx} = \frac{d^2 z}{dx dy} + \frac{P dz}{dx} + \frac{dP}{dx} z = s + Pp + \frac{dP}{dx} z$, and, consequently, $\frac{dv}{dx} + Qq + PQz = V$, or $\frac{dv}{dx} + Qv = V$. This is a linear equation of the first order, and being integrated gives

$$v = e^{-\int Q dy} \left\{ \int e^{\int Q dy} V dx + c \right\} = \frac{dz}{dy} + Pz.$$

The same process being repeated on this last equation, we shall obtain the value of z with two arbitrary constants, one of which may be replaced by an arbitrary function of x , and the other by an arbitrary function of y .

(213.) Partial differential equations of the third and higher orders, in which the partial differential coefficients of the highest order enter only in the first degree, may be treated by a method analogous to that which has been used with those of the second order.

If, for instance, $A \frac{d^3 z}{dx^3} + B \frac{d^3 z}{dx^2 dy} + C \frac{d^3 z}{dx dy^2} + D \frac{d^3 z}{dy^3} = V$ is a partial differential equation of the third order, in which A , B , C , D , V are any functions of x , y , z , and of the partial differential coefficients of the second and first order, the following theorem may be proved.

If m be one of the roots of the equation $A m^3 - B m^2 + C m - D = 0$, and if $M = a$, $N = b$, be two primitive equations derived from the two equations

$$-m dx = 0, \quad A dr + (B - Am) ds + (C - Bm + Am^2) dt - V dx = 0 \dots \dots (1),$$

then $N = \phi(M)$ will be one of the first integrals of the proposed equation.

The same theorem may easily be extended to partial differential equations of the form

$$A \frac{d^n z}{dx^n} + B \frac{d^n z}{dx^{n-1} dy} + C \frac{d^n z}{dx^{n-2} dy^2} + \dots \dots + \frac{T d^n z}{dx dy^{n-1}} = V.$$

The integration of the equation will thus be reduced to that of two equations of a form analogous to equations (1.), and these can only be integrated simultaneously under certain conditions.

In the third order, for instance, the equations (1.) contain generally eight variables, namely, x , y , z , p , q , r , s , t , between which we have, besides, the three equations

$$dz = p dx + q dy, \quad dp = r dx + s dy, \quad dq = s dx + t dy.$$

If, therefore, we eliminate between these five equations as many variables as possible, the resulting equation will contain still four variables, and will not therefore be integrable in every case.

(214.) The preceding method may be applied with success in the case in which the coefficients A , B , C , &c. are constant; it leads generally to n first integrals, each containing an arbitrary function, which are themselves partial differential equations of the $(n-1)^{th}$ with constant coefficients, and accordingly may be treated in the same manner as the proposed equation, and by the repetition of the same process the general integral may be obtained.

When $V = 0$, the calculation is still simpler; but in that particular case it is easy to see that if $m_1, m_2, m_3, \dots, m_n$ are the n roots of the equation

$$A m^n - B m^{n-1} + C m^{n-2}, \text{ \&c. } = 0,$$

Integral Calculus. the general value of z will be

$$z = \phi(y - m_1 x) + \psi(y - m_2 x) + \dots \chi(y - m_n x).$$

For each of the values,

$$z = \phi(y - m_1 x), \quad z = \psi(y - m_2 x) \dots z = \chi(y - m_n x),$$

satisfies the proposed differential equation. This may easily be verified, since we have

$$\frac{d^n z}{dx^n} = \mp m_1^n \phi_n(y - m_1 x), \quad \frac{d^n z}{dx^{n-1} dy} = \pm m_1^{n-1} \phi_n(y - m_1 x), \quad \frac{d^n z}{dx^{n-2} dy^2} = \mp m_1^{n-2} \phi_n(y - m_1 x), \&c.,$$

where $\phi_n(y - m_1 x)$ designates the n^{th} differential coefficient of $\phi(y - m_1 x)$, and where the superior signs must be taken when n is an odd number, and the inferior signs when n is even. These values being substituted in the proposed equation, give

$$A m_1^n - B m_1^{n-1} + C m_1^{n-2} \&c. = 0,$$

which equation is satisfied, since m_1 is one of the roots of the equation

$$A m^n - B m^{n-1} + C m^{n-2} \&c. = 0.$$

When some of the roots of this last equation are equal to one another, the value of z obtained by this method does not contain n arbitrary functions, and seems to be less general than in the case in which all the roots are different.

Let $m_1 = m_2$, for instance, the two first terms of the value of z $\phi(y - m_1 x)$, and $\psi(y - m_2 x)$ become arbitrary functions of the same quantity. To avoid this difficulty, let us assume generally $m_2 = m_1 + h$, then

$$\psi(y - m_2 x) = \psi(y - m_1 x - hx) = \psi(y - m_1 x) - \frac{hx}{1} \psi'(y - m_1 x) + \frac{h^2 x^2}{1 \cdot 2} \psi''(y - m_1 x) - \&c. \psi' \psi'',$$

&c. being the first, second, and differential coefficients of $\psi(y - m_1 x)$. If we assume $-h \psi'(y - m_1 x) = \psi(y - m_1 x)$, we shall have $h^2 \psi''(y - m_1 x) = -h \psi_1'(y - m_1 x)$, consequently the two first terms of the value

$$\text{of } z \text{ will be } = \phi(y - m_1 x) + \psi(y - m_1 x) + x \psi_1(y - m_1 x) - \frac{h x^2}{1 \cdot 2} \psi_1'(y - m_1 x) + \&c. \text{ Assuming}$$

$\phi(y - m_1 x) + \psi(y - m_1 x) = \phi_1(y - m_1 x)$, and making $h = 0$, the two first terms of the value of z will become $\phi_1(y - m_1 x) + x \psi_1(y - m_1 x)$.

If three of the values of m were equal to each other, $m_1 = m_2 = m_3$, for instance, by a similar process it would be found that the three first terms of the value of z would be an arbitrary function of $y - m_1 x + x$, multiplied by another arbitrary function of the same quantity $+ x^2$ multiplied by another arbitrary function of the same quantity

(215.) Partial differential equations containing more than three variables, in which the differential coefficients of the highest order enter only in the first degree, may be treated in an analogous manner.

Let

$$A \frac{d^2 z}{du^2} + B \frac{d^2 z}{dx^2} + C \frac{d^2 z}{dy^2} + D \frac{d^2 z}{du dx} + E \frac{d^2 z}{du dy} + F \frac{d^2 z}{dx dy} = V$$

be a partial differential equation of the second order, containing four variables, A, B, C, D, E, F, V , being functions of the four variables u, x, y, z , and of the differential coefficients of the first order, we shall have

$$dz = \frac{dz}{du} du + \frac{dz}{dx} dx + \frac{dz}{dy} dy.$$

$$d \frac{dz}{du} = \frac{d^2 z}{du^2} du + \frac{d^2 z}{du dx} dx + \frac{d^2 z}{du dy} dy.$$

$$d \frac{dz}{dx} = \frac{d^2 z}{dx du} du + \frac{d^2 z}{dx^2} dx + \frac{d^2 z}{dx dy} dy.$$

$$d \frac{dz}{dy} = \frac{d^2 z}{dy du} du + \frac{d^2 z}{dy dx} dx + \frac{d^2 z}{dy^2} dy.$$

And if we make $\frac{dz}{du} = n$, $\frac{dz}{dx} = p$, $\frac{dz}{dy} = q$, we shall get

$$\frac{d^2 z}{du^2} = \frac{1}{du} \left(dn - \frac{d^2 z}{du dx} dx - \frac{d^2 z}{du dy} dy \right).$$

$$\frac{d^2 z}{dx^2} = \frac{1}{dx} \left(dp - \frac{d^2 z}{du dx} du - \frac{d^2 z}{dx dy} dy \right).$$

$$\frac{d^2 z}{dy^2} = \frac{1}{dy} \left(dq - \frac{d^2 z}{du dy} du - \frac{d^2 z}{dx dy} dx \right).$$

These values being substituted in the proposed equation, give

$$A d n d x d y + B d p d u d y + C d q d u d x - V d u d x d y = \begin{cases} \frac{d^2 z}{d u d x} (A d x^2 d y + B d u^2 d y - D d u d x d y); \\ + \frac{d^2 z}{d u d y} (A d x d y^2 + C d u^2 d x - E d u d x d y); \\ + \frac{d^2 z}{d x d y} (B d u d y^2 + C d u d x^2 - F d u d x d y); \end{cases}$$

assume, then,

$$\begin{aligned} A d x^2 d y + B d u^2 d y - D d u d x d y &= 0, & A d x d y^2 + C d u^2 d x - E d u d x d y &= 0, \\ B d u d y^2 + C d u d x^2 - F d u d x d y &= 0, & A d n d x d y + B d p d u d y + C d q d u d x - V d u d x d y &= 0. \end{aligned}$$

If we assume $d x = h d u$, $d y = k d u$, the three first of these equations will take the form

$$A h^2 - D h + B = 0, \quad A k^2 - E k + C = 0, \quad B k^2 - F h k + C h^2 = 0.$$

They only contain two unknown quantities, and therefore they will only agree when the equation resulting from the elimination of h and k between these three equations will be satisfied. This equation is

$$A F^2 + B E^2 + C D^2 = 4 A B C + D E F,$$

and when it will be satisfied, we shall have between the seven variables u, x, y, z, n, p, q , the following equations,

$$d x - h d u = 0, \quad d y - k d u = 0, \quad A h k d n + B k d p + C h d q - V h k d u = 0,$$

and it may be proved that if $L = a$, $M = b$, $N = c$, are three equations satisfying the preceding, the first integral of the proposed equation will be $N = \phi(L, M)$.

(216.) The integration of partial differential equations is sometimes affected by means of various transformations. The equation

$$R r + S s + T t + P p + Q q + N z = V,$$

where R, S, T, P, Q, N , and V , are functions of x and y , may always be reduced to the form

$$S_1 \frac{d^2 z}{d u d v} + P_1 \frac{d z}{d u} + Q_1 \frac{d z}{d v} + N_1 z = M_1.$$

For that purpose let us assume that x and y are functions of two new variables, u and v , then z will be a function of u and v , and we shall have

$$d u = \frac{d u}{d x} d x + \frac{d u}{d y} d y, \quad d v = \frac{d v}{d x} d x + \frac{d v}{d y} d y, \quad d z = \frac{d z}{d u} d u + \frac{d z}{d v} d v;$$

substituting in the last equation for $d u$ and $d v$, their values taken in the first, we find

$$d z = \left(\frac{d z}{d u} \frac{d u}{d x} + \frac{d z}{d v} \frac{d v}{d x} \right) d x + \left(\frac{d z}{d u} \frac{d u}{d y} + \frac{d z}{d v} \frac{d v}{d y} \right) d y,$$

and consequently,

$$p = \frac{d z}{d u} \frac{d u}{d x} + \frac{d z}{d v} \frac{d v}{d x}, \quad q = \frac{d z}{d u} \frac{d u}{d y} + \frac{d z}{d v} \frac{d v}{d y}.$$

Differentiating these values we get

$$\begin{aligned} r &= \frac{d^2 z}{d u^2} \cdot \frac{d u^2}{d x^2} + \frac{2 d^2 z}{d u d v} \frac{d u d v}{d x d x} + \frac{d^2 z}{d v^2} \frac{d v^2}{d x^2} + \frac{d z}{d u} \frac{d^2 u}{d x^2} + \frac{d z}{d v} \frac{d^2 v}{d x^2}, \\ s &= \frac{d^2 z}{d u^2} \frac{d u d u}{d x d y} + \frac{d^2 z}{d u d v} \frac{d u d v}{d x d y} + \frac{d^2 z}{d v^2} \frac{d v d v}{d x d y} + \frac{d z}{d u} \frac{d^2 u}{d x d y} + \frac{d z}{d v} \frac{d^2 v}{d x d y} + \frac{d^2 z}{d u d v} \frac{d u d v}{d y d x}, \\ t &= \frac{d^2 z}{d u^2} \frac{d u^2}{d y^2} + 2 \frac{d^2 z}{d u d v} \frac{d u d v}{d y d y} + \frac{d^2 z}{d v^2} \frac{d v^2}{d y^2} + \frac{d z}{d u} \frac{d^2 u}{d y^2} + \frac{d z}{d v} \frac{d^2 v}{d y^2}. \end{aligned}$$

These expressions being substituted in the equation

$$R r + S s + T t + P p + Q q + N z = M,$$

transform it into

$$\begin{aligned} &\left(R \frac{d u^2}{d x^2} + S \frac{d u d u}{d x d y} + T \frac{d u^2}{d y^2} \right) \frac{d^2 z}{d u^2} + \left(2 R \frac{d u d v}{d x d x} + S \left(\frac{d u d v}{d x d y} + \frac{d u d v}{d y d x} \right) + 2 T \frac{d u d v}{d y d y} \right) \frac{d^2 z}{d u d v} \\ &\quad + \left(R \frac{d v^2}{d x^2} + S \frac{d v d v}{d x d y} + T \frac{d v^2}{d y^2} \right) \frac{d^2 z}{d v^2} + N z \\ &+ \left(R \frac{d^2 u}{d x^2} + S \frac{d^2 u}{d x d y} + T \frac{d^2 u}{d y^2} + P \frac{d u}{d x} + Q \frac{d u}{d y} \right) \frac{d z}{d u} + \left(R \frac{d^2 v}{d x^2} + S \frac{d^2 v}{d x d y} + T \frac{d^2 v}{d y^2} + P \frac{d v}{d x} + Q \frac{d v}{d y} \right) \frac{d z}{d v} \end{aligned} = M$$

To simplify these equations we may suppose that u and v satisfy the two equations

$$\left. \begin{aligned} R \frac{d^2 u^2}{dx^2} + S \frac{du}{dx} \frac{du}{dy} + T \frac{du^2}{dy^2} &= 0 \\ R \frac{dv^2}{dx^2} + S \frac{dv}{dx} \frac{dv}{dy} + T \frac{dv^2}{dy^2} &= 0 \end{aligned} \right\} (a),$$

and then the equation will be reduced to the form

$$\frac{d^2 z}{du dv} + P_1 \frac{dz}{du} + Q_1 \frac{dz}{dv} + N_1 z = M_1.$$

The quantities S_1, P_1, Q_1, N_1 , and M_1 will be expressed in terms of u and v will have been determined by means of the equations (a). These equations become, by assuming $\frac{du}{dx} = m \frac{du}{dy}, \frac{dv}{dx} = n \frac{dv}{dy}$,

$$R m^2 + S m + T = 0, \quad R n^2 + S n + T = 0.$$

Hence m and n are the roots of one or the other of these two equations, and will only contain the variables x and y . Substituting then the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$ in the values of du and dv written above, we shall have

$$du = \frac{du}{dy} (dy + m dx), \quad dv = \frac{dv}{dy} (dy + n dx).$$

The values of $\frac{du}{dy}, \frac{dv}{dy}$ are still undeterminate, and may be considered as the factors by which the quantities $dy + m dx, dy + n dx$, ought to be multiplied to become differentials of functions of two variables. Let U and V represent the integrals of these differentials, then $u = U, v = V$, and by means of these equations the values of x and y may be determined in functions of u and v .

(217.) There are two cases in which the preceding transformation does not succeed. 1. When the roots of the equation $R m^2 + S m + T = 0$ are equal, that is when $4RT = S^2$. Then the values of u and v are, $u = \int \frac{du}{dy} (dy + m dx), v = \int \frac{dv}{dy} (dy + m dx)$, and consequently the variables u and v cannot be considered as independent variables. In that case it will be sufficient to assume x to be a function of u and y , that is to suppose $v = y$ in the preceding formulæ. We shall find first,

$$p = \frac{dz}{du} \frac{du}{dx}, \quad q = \frac{dz}{du} \frac{du}{dy} + \frac{dz}{dy};$$

$\frac{dz}{dy}$ in the last equation being not equal to q , but to the differential coefficient of z with respect to y after the transformation, we shall have also

$$\begin{aligned} r &= \frac{d^2 z}{du^2} \frac{du^2}{dx^2} + \frac{dz}{du} \frac{d^2 u}{dx^2}, & s &= \frac{d^2 z}{du^2} \frac{du}{dx} \frac{du}{dy} + \frac{d^2 z}{du dy} \frac{du}{dx} + \frac{dz}{du} \frac{du}{dx} \frac{du}{dy}, \\ t &= \frac{d^2 z}{du^2} \frac{du^2}{dy^2} + 2 \frac{d^2 z}{du dy} \frac{du}{dy} + \frac{d^2 z}{dy^2} + \frac{dz}{du} \frac{d^2 u}{dy^2}. \end{aligned}$$

By means of these values, and observing besides that $\frac{du}{dx} = m \frac{du}{dy}$, and $R m^2 + S m + T = 0$, or, in the present case, $S m + 2T = 0$, the proposed equation will take the form

$$S_1 \frac{d^2 z}{dy^2} + Q_1 \frac{dz}{dy} + P_1 \frac{dz}{du} + N_1 z = M_1.$$

2. The above method does not succeed when $S = 0$ and $T = 0$, but in that case the proposed equation becomes

$$R r + P p + Q q + N z = M,$$

that is to say, is precisely of that form to which it was proposed to reduce it.

(218.) In the general case the integration is therefore reduced to that of an equation of the form

$$\frac{d^2 z}{du dv} + P_1 \frac{dz}{du} + Q_1 \frac{dz}{dv} + N_1 z = M_1,$$

supposing that the whole equation has been divided by the coefficient of the first term. The two first terms of

this equation are equal to $\frac{d \left(\frac{dz}{dv} + P_1 z \right)}{du} - z \frac{d P_1}{du}$, and therefore it may be put under the form

$$\frac{d\left(\frac{dz}{dv} + P_1 z\right)}{du} + Q_1 \left\{ \frac{dz}{dv} + \frac{1}{Q_1} \left(N_1 - \frac{dP_1}{du} \right) z \right\} = M_1,$$

and it will be reduced to the first order, if

$$\frac{1}{Q_1} \left(N_1 - \frac{dP_1}{du} \right) = P_1, \text{ or } N_1 - P_1 Q_1 - \frac{dP_1}{du} = 0,$$

for then, if we make $\frac{dz}{dv} + P_1 z = z'$, it will be changed into

$$\frac{dz'}{du} + Q_1 z' = M_1$$

(219.) If the condition $N_1 - P_1 Q_1 - \frac{dP_1}{du} = 0$ is not satisfied, let $N_1 - P_1 Q_1 - \frac{dP_1}{du} = a$, and the value of N_1 derived from this equation be substituted in the equation

$$\frac{d\left(\frac{dz}{dv} + P_1 z\right)}{du} + Q_1 \left\{ \frac{dz}{dv} + \frac{1}{Q_1} \left(N_1 - \frac{dP_1}{du} \right) z \right\} = M_1,$$

we shall have

$$\frac{d\left(\frac{dz}{dv} + P_1 z\right)}{du} + Q_1 \left\{ \frac{dz}{dv} + P_1 z + \frac{a}{Q_1} z \right\} = M_1,$$

and if we assume $\frac{dz}{dv} + P_1 z = z'$, the preceding equation will be changed into

$$\frac{dz'}{du} + Q_1 z' + a z = M_1.$$

If we take now the value of z in this equation, and form the value of $\frac{dz}{dv}$, we shall have by substitution in the equation $\frac{dz}{dv} + P_1 z = z'$,

$$\frac{d^2 z'}{du dv} + P_1' \frac{dz'}{du} + Q_1 \frac{dz'}{dv} + N_1' z' = M_1' \dots\dots (b).$$

Making, for the sake of brevity,

$$P_1 - \frac{1}{a} \frac{da}{dv} = P_1', \quad a - \frac{Q_1}{a} \frac{da}{dv} + \frac{dQ_1}{dv} + P_1 Q_1 = N_1', \quad \frac{dM_1}{dv} - \frac{M_1}{a} \frac{da}{dv} + P_1 M_1 = M_1'.$$

The equation (b) may then be treated by the same process as the proposed equation, in assuming

$$\frac{dz'}{dv} + P_1' z' = z'', \quad N' - P_1' Q_1' - \frac{dP_1'}{du} = a',$$

and we shall derive from these suppositions

$$\frac{dz''}{du} + Q_1 z'' + a' z' = M_1';$$

and this equation can easily be integrated when $a' = 0$. If a' be not equal to nothing, then z' should be eliminated between this last equation and the equation (b). The result of this elimination would be an equation such as

$$\frac{d^2 z''}{du dv} + P_1'' \frac{dz''}{du} + Q_1 \frac{dz''}{dv} + N_1'' z'' = M_1''.$$

The same process ought to be applied to this equation, and being continued we shall form successively the values of $a, a', a'', \&c.$ if any one of them should be found equal to nothing. The integral of the corresponding equation in $z', z'', \&c.$ is immediately found, and successively the values of all the variables which have been introduced, and at last that of z .

(220.) Another transformation, which may sometimes be employed with advantage, is the following. Let

$$R \frac{d^2 z}{dx^2} + S \frac{d^2 z}{dx dy} + T \frac{d^2 z}{dy^2} = 0$$

be an equation in which the coefficients R, S, T are functions of p and q only. Then this equation may always be transformed into another of the form

$$R \frac{d^2 v}{dq^2} - S \frac{d^2 v}{dp dq} + T \frac{d^2 v}{dp^2} = 0,$$

where R, S, T are the same quantities as in the proposed equation.

Let $x dp + y dq = dv$; since $dz = p dx + q dy$, we have, in integrating by parts, $z = pq + qy - \int x dp + y dq$, therefore, $v = px + qy - z$, $x = \frac{dv}{dp}$, $y = \frac{dv}{dq}$. It is sufficient to know the value of v to be able to determine that of z .

To express now the value of $\frac{d^2 z}{dx^2}$ in functions of the new variables, it is necessary to observe that $\frac{d^2 z}{dx^2} = \frac{dp}{dx}$, y being considered as a constant quantity, in which hypothesis $d \frac{dv}{dq} = 0$, or $\frac{d^2 v}{dp dq} dp + \frac{d^2 v}{dq^2} dq = 0$. We have also $dx = \frac{d^2 v}{dp^2} dp + \frac{d^2 v}{dp dq} dq$; eliminating dq , we shall find

$$dx = \left(\frac{d^2 v}{dp^2} - \frac{\left(\frac{d^2 v}{dp dq} \right)^2}{\frac{d^2 v}{dq^2}} \right) dp.$$

Hence $\frac{dx}{dp} = \frac{\frac{d^2 v}{dp^2} \frac{d^2 v}{dq^2} - \left(\frac{d^2 v}{dp dq} \right)^2}{\frac{d^2 v}{dq^2}}$, and, consequently, $\frac{dp}{dx} = \frac{d^2 z}{dx^2} = \frac{\frac{d^2 v}{dq^2}}{\frac{d^2 v}{dp^2} \frac{d^2 v}{dq^2} - \left(\frac{d^2 v}{dp dq} \right)^2}$.

By similar operations we shall obtain

$$\frac{dz}{dx dy} = \frac{\frac{d^2 v}{dp^2} \frac{d^2 v}{dq^2} - \left(\frac{d^2 v}{dp dq} \right)^2}{\frac{d^2 v}{dp^2} \frac{d^2 v}{dq^2} - \left(\frac{d^2 v}{dp dq} \right)^2}, \quad \frac{dz}{dy^2} = \frac{\frac{d^2 v}{dp^2}}{\frac{d^2 v}{dp^2} \frac{d^2 v}{dq^2} - \left(\frac{d^2 v}{dp dq} \right)^2}.$$

These values being substituted in the proposed equation, will transform it into

$$R \frac{d^2 v}{dq^2} - S \frac{d^2 v}{dp dq} + T \frac{d^2 v}{dp^2} = 0.$$

(221.) Partial differential equations of the second or higher orders, which contain the differential coefficients in a degree higher than the first, present still greater difficulties to be integrated.

Any equations which contain only the differential coefficients of the second order, the general form of which is $r = f(s, t)$, may be transformed into another equation of the same order, containing the differential coefficients only in the first degree. To prove it, we have $dp = r dx + s dy$, $dq = s dx + t dy$, from which we derive $p = rx + sy - \int (x dr + y ds)$, $q = sx + ty - \int (x ds + y dt)$. If next we consider x and y as functions

of s and t , and if we assume $x ds + y dt = dv$, we find $x = \frac{dv}{ds}$, $y = \frac{dv}{dt}$. But the proposed equation being

differentiated, gives $dr = S ds + T dt$; by means of this value the function $x dr + y ds$ becomes $(Sx + y) ds + T x dt$, and, as it must be an exact differential, since $p = rx + sy - \int (x dr + y ds)$, we shall have the

equation $\frac{d \cdot (sx + y)}{dt} = \frac{d \cdot Tx}{ds}$, which being developed gives

$$S \frac{dx}{dt} + \frac{dy}{dt} = T \frac{dx}{ds}.$$

Substituting in this the values of $\frac{dx}{dt}$, $\frac{dx}{ds}$, $\frac{dy}{dt}$, we shall have the transformed equation

$$\frac{d^2 v}{dt^2} + S \frac{d^2 v}{ds dt} - T \frac{d^2 v}{ds^2} = 0,$$

which is only of the first degree with respect to the function v , since S and T do not contain the variables s and t . When v will become known, we shall derive from it, first the values of x and y in s and t , and then we shall find

$$q = sx + ty - v, \quad p = rx + sy - \int (x dr + y ds), \quad z = px + qy - \int (x dp + y dq);$$

and the integrations which are indicated will depend only on the integration of functions of one variable.

(222.) The first integral of the equation $rt - s^2 = 0$ may easily be found. It may be put under the form

$\frac{r}{s} = \frac{s}{t}$. Assume each of these ratios equal to w . Then $r = ws$, $s = wt$, and therefore, $r dx = ws dx$,

$s dy = wt dy$. Adding these two last equations, we get $r dx + s dy = w(s dx + t dy)$, or $dp = w dq$, or for the first integral of the proposed equation, $p = \phi(q)$.

Integral
Calculus.

(223.) Generally a particular integral may be obtained, the partial differential equations have the form $P = (rt - s^2)^n Q$, in which P designates a function of the differential coefficients p, q, r, s, t , and homogeneous with respect to the three last, n any positive exponent, and Q any function of the variables x, y, z , and of all the differential coefficients, such, however, that the supposition $rt - s^2 = 0$ does not make it infinite. Part III.

Making first $rt - s^2 = 0$, which gives $q = \phi(p)$, the above equation is reduced to $P = 0$; and from $q = \phi(p)$ we derive $s = r\phi'(p)$, $t = s\phi'(p) = r\phi'(p)^2$. Substituting these values in the equation $P = 0$, the differential coefficient r will become a common factor of the left side of the equation, which is homogeneous with respect to r, s, t .

There will remain, therefore, an equation between $p, \phi(p)$, and $\phi'(p)$, or between $p, q, \frac{dq}{dp}$, which may be used to

find the relation between the coefficients p and q , from which, by integration, we shall get an equation between x, y, z , containing an arbitrary function, and an arbitrary constant, and which ought to be considered as a particular integral of the proposed differential equation.

This method applied to the equation $t + 2ps + (p^2 - c^2)r = 0$, in which $Q = 0$, gives, by the substitutions indicated above,

$$\{\phi'(p)^2 + 2p\phi'(p) + p^2 - c^2\}r = 0, \quad \text{or } dq^2 + 2pdq + (p^2 - c^2)dp = 0.$$

Taking the value of dq in this last equation, we find

$$dq = -(p \mp c)dp, \quad q = -\frac{1}{2}p^2 \pm cp + a.$$

Substituting this value in the equation $dz = pdx + qdy$, we shall get the following system of equations,

$$z = px + (a \pm cp - \frac{1}{2}p^2)y + \phi(p), \quad 0 = x \pm cy - py + \phi'(p),$$

which will furnish two particular integrals of the proposed equation according to the sign of the constant c , and in taking different arbitrary functions.

(224.) The integral of partial differential equations may be represented by an infinite series, when none of the preceding methods can be used.

If the differential equation have the form

$$Rr + Ss + Tt + Pp + Qq + Nz = 0,$$

and if the quantities R, S, T, P, Q, N , are constant, it is obvious that we shall satisfy it by taking $z = A e^{ax+\phi y}$, provided a and ϕ satisfy the equation

$$Ra^2 + Sa\phi + T\phi^2 + Pa + Q\phi + N = 0.$$

Thus A remains undetermined as well as one of the exponents a, ϕ . Therefore if we take for a a series of arbitrary values, $a, a', a'', \&c.$ and if we designate by b and c, b' and c', b'' and $c'', \&c.$ the corresponding values of ϕ , the sum of the particular values of z will give

$$z = A e^{ax+by} + A' e^{a'x+b'y} + A'' e^{a''x+b''y} + \&c. \\ + B e^{ax+cy} + B' e^{a'x+c'y} + B'' e^{a''x+c''y} + \&c.$$

If we suppose $a = m + n\sqrt{-1}, a' = m - n\sqrt{-1}$, which give $b = p + q\sqrt{-1}, b' = p - q\sqrt{-1}$, the two first terms of the expression of z may be put under the form

$$e^{mx+py} \{A_1 \cos(nx + qy) + A'_1 \sin(nx + qy)\},$$

and an infinite series of such terms may be taken for the value of z .

(225.) The Theorem of Taylor may also be used to obtain in a series, arranged according to the powers of one of the independent variables, functions which are given by partial differential equations.

If z represent the unknown function, x one of the independent variables, and $Z, \frac{dZ}{dx}, \frac{d^2Z}{dx^2}, \&c.$ what $z, \frac{dz}{dx}, \frac{d^2z}{dx^2}, \&c.$ become when x is made equal to nothing, we shall have

$$z = Z + \frac{dZ}{dx}x + \frac{d^2Z}{dx^2} \cdot \frac{x^2}{1 \cdot 2} + \frac{d^3Z}{dx^3} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + \&c.$$

If the proposed equation be of the m^{th} order, the value of $\frac{d^m z}{dx^m}$, will be given by the equation, and all the

differential coefficients of higher orders will be found by the repeated differentiation of $\frac{d^m z}{dx^m}$ with respect to x only.

In the series the coefficients of all the powers of x less than m will remain undetermined, and arbitrary functions of all the variables but x may be taken to represent them.

Let us apply this method to the equation $\frac{d^2 z}{dx^2} = c^2 \frac{dz}{dy^2}$, which we have already integrated. (211.) In this case

Integral
Calculus.

Part III.

the first two terms of the series z and $\frac{dz}{dx}$ remain undetermined, and for the others we have by successive differentiations of the value of $\frac{d^2 z}{dx^2}$,

$$\begin{aligned}\frac{d^3 z}{dx^3} &= c^2 \frac{d^3 z}{dx dy^2} = c^2 \frac{d^2 \frac{dz}{dx}}{dy^2}, & \frac{d^4 z}{dx^4} &= c^2 \frac{d^4 z}{dx^2 dy^2} = c^2 \frac{d^3 \frac{d^2 z}{dx^2}}{dy^2} = c^4 \frac{d^4 z}{dy^4}, \\ \frac{d^5 z}{dx^5} &= c^2 \frac{d^5 z}{dx^3 dy^2} = c^2 \frac{d^4 \frac{dz}{dx}}{dy^2} = c^4 \frac{d^4 \frac{dz}{dx}}{dy^4}, \text{ \&c.}\end{aligned}$$

Designating by $\phi(y)$ the value of z when $x = 0$, $\phi'(y)$, or the differential coefficient of that function with respect to y , will be the corresponding value of $\frac{dz}{dy}$; let in the same supposition $\psi(y)$ represent the value of $\frac{dz}{dx}$, and we shall have

$$\begin{aligned}z &= \phi(y) + \psi(y) \frac{x}{1} + c^2 \phi''(y) \frac{x^2}{1 \cdot 2} + c^2 \psi''(y) \frac{x^3}{1 \cdot 2 \cdot 3} + c^4 \phi^{iv}(y) \\ &\quad \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + c^4 \psi^{iv}(y) \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \text{\&c.}\end{aligned}$$

The same series may be derived from the complete integral of the proposed differential equation which has been given. (211.) We have found

$$z = \phi(y + cx) + \psi(y - cx).$$

Developing each of these functions by Taylor's Theorem, we find

$$\phi(y + cx) = \phi(y) + \frac{cx}{1} \phi'(y) + \frac{c^2 x^2}{1 \cdot 2} \phi''(y) + \frac{c^3 x^3}{1 \cdot 2 \cdot 3} \phi'''(y) + \text{\&c.}$$

$$\psi(y - cx) = \psi(y) - \frac{cx}{1} \psi'(y) + \frac{c^2 x^2}{1 \cdot 2} \psi''(y) - \frac{c^3 x^3}{1 \cdot 2 \cdot 3} \psi'''(y) + \text{\&c.}$$

Adding these values, and substituting $\phi(y)$ to $\phi(y) + \psi(y)$, and $\psi(y)$ to $c\{\phi'(y) - \psi'(y)\}$, we shall obtain precisely the same series as we have found above.

(226.) The method of indeterminate coefficients may also be used to find the developement of the required function according to the powers of the independent variables. One or two examples will be sufficient to illustrate this process.

Let $\frac{d^2 u}{dx^2} + \frac{d^2 u}{dy^2} + \frac{d^2 u}{dz^2} = 0$ be the proposed equation, and assume

$$u = u_0 + u_1 z + u_2 z^2 + u_3 z^3 + \text{\&c.},$$

where $u_0, u_1, u_2, \text{\&c.}$ are functions of x and y . This value being substituted in the proposed equation, gives

$$\frac{d^2 u_0}{dx^2} + \frac{d^2 u_0}{dy^2} + 2u_2 + \left(\frac{d^2 u_0}{dx^2} + \frac{d^2 u_0}{dy^2} + 2 \cdot 3 \cdot u_3\right)z + \left(\frac{d^2 u_2}{dx^2} + \frac{d^2 u_2}{dy^2} + 3 \cdot 4 \cdot u_4\right)z^2 + \text{\&c.} \dots = 0,$$

and consequently

$$u_2 = -\frac{1}{2} \left(\frac{d^2 u_0}{dx^2} + \frac{d^2 u_0}{dy^2} \right), \quad u_3 = -\frac{1}{2 \cdot 3} \left(\frac{d^2 u_1}{dx^2} + \frac{d^2 u_1}{dy^2} \right),$$

$$u_4 = -\frac{1}{3 \cdot 4} \left(\frac{d^2 u_2}{dx^2} + \frac{d^2 u_2}{dy^2} \right) = -\frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \left(\frac{d^4 u_0}{dx^4} + 2 \frac{d^4 u_0}{dx^2 dy^2} + \frac{d^4 u_0}{dy^4} \right),$$

and finally

$$u = u_0 + \frac{u_1}{1} \cdot z - \left(\frac{d^2 u_0}{dx^2} + \frac{d^2 u_0}{dy^2} \right) \frac{z^2}{1 \cdot 2} + \left(\frac{d^4 u_0}{dx^4} + 2 \frac{d^4 u_0}{dx^2 dy^2} + \frac{d^4 u_0}{dy^4} \right) \frac{z^3}{1 \cdot 2 \cdot 3 \cdot 4} + \text{\&c.}$$

a series which contains two arbitrary functions u_0 and u_1 .

The equation $\frac{d^2 z}{dx^2} = \frac{dz}{dy}$, treated by the same process, in assuming

$$z = y_0 + y_1 x + y_2 x^2 + y_3 x^3 + \text{\&c.}$$

gives $y_2 = \frac{1}{1 \cdot 2} \frac{dy_0}{dy}, \quad y_3 = \frac{1}{2 \cdot 3} \frac{dy_1}{dy}, \quad y_4 = \frac{1}{3 \cdot 4} \frac{dy_2}{dy}, \quad y_5 = \frac{1}{4 \cdot 5} \frac{dy_3}{dy}, \text{\&c.}$

and consequently in making $y_0 = \phi(y)$ and $y_1 = \psi(y)$, we get

Integral
Calculus.

$$z = \phi(y) + \psi(y) \cdot x + \phi'(y) \cdot \frac{x^2}{1 \cdot 2} + \psi'(y) \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + \phi''(y) \cdot \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \psi''(y) \cdot \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c. \quad \text{Part III.}$$

This value of z contains two arbitrary constants, and therefore may be considered as the general integral. We may obtain by the same means the value of z in a series arranged according to the powers of y , and we shall find

$$z = \phi(x) + \phi'(x) \frac{y}{2} + \phi''(x) \cdot \frac{y^2}{1 \cdot 2} + \phi'''(x) \cdot \frac{y^3}{1 \cdot 2 \cdot 3} + \&c.$$

This integral is less general than the preceding for it contains only one arbitrary function. A similar reduction, in the number of arbitrary functions, must take place whenever the proposed equation does not contain at the same time the differential coefficients of the highest order with respect to each variable, since there remain fewer indeterminate terms in the series, arranged according to the powers of the variable with respect to which there has been a less number of differentiations. It may, moreover, easily be shown, in the last example, that the first value, obtained for z , may be transformed into another, identical with the second. For that purpose, let

$$\phi(y) = a + \beta \frac{y}{1} + \gamma \frac{y^2}{1 \cdot 2} + \delta \frac{y^3}{1 \cdot 2 \cdot 3} + \&c.$$

$$\psi(y) = a' + \beta' \frac{y}{1} + \gamma' \frac{y^2}{1 \cdot 2} + \delta' \frac{y^3}{1 \cdot 2 \cdot 3} + \&c.$$

Substituting these values of $\phi(y)$ and $\psi(y)$ in the value, and arranging the terms according to the powers of y , we shall find

$$\begin{aligned} z = & a + a'x + \beta \frac{x^2}{1 \cdot 2} + \beta' \frac{x^3}{1 \cdot 2 \cdot 3} + \gamma \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \gamma' \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c. \\ & + \left\{ \beta + \beta' \frac{x}{1} + \gamma \frac{x^2}{1 \cdot 2} + \gamma' \frac{x^3}{1 \cdot 2 \cdot 3} + \delta \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \delta' \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c. \right\} \frac{y}{1}, \\ & + \left\{ \gamma + \gamma' \frac{x}{1} + \delta \frac{x^2}{1 \cdot 2} + \delta' \frac{x^3}{1 \cdot 2 \cdot 3} + \&c. \right\} \frac{y^2}{1 \cdot 2}, \\ & + \&c. \end{aligned}$$

All the coefficients of x being undetermined, we may make

$$a + a' \frac{x}{1} + \beta \frac{x^2}{1 \cdot 2} + \beta' \frac{x^3}{1 \cdot 2 \cdot 3} + \gamma \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \gamma' \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \&c. = \phi(x),$$

ϕ designating a new arbitrary function; we find then, that the coefficient of $\frac{y}{1}$ is equal to $\frac{d^2 \phi(x)}{dx^2}$, or $\phi''(x)$, that of $\frac{y^2}{1 \cdot 2}$ equal to $\frac{d^4 \phi(x)}{dx^4}$, or $\phi^{(4)}(x)$, and so on with the others, and therefore the value of z becomes precisely the same as we have found before.

(227.) When the differential coefficient of the highest order that enters the equation has not been taken with respect to one variable only, the developement of the required function contains, not only arbitrary functions, but also arbitrary constants introduced by the integration of a series of differential equations of two variables. Let, for instance, the proposed equation be

$$\frac{d^2 z}{dx dy} + a \frac{dz}{dx} + b \frac{dz}{dy} + cz = 0,$$

and let

$$z = y_0 + y_1 x + y_2 x^2 + y_3 x^3 + \&c.$$

Substituting this value in the proposed equation, and making separately equal to nothing, the coefficients of the different powers of y , we shall have, to determine the values of $y_1, y_2, \&c.$, the following equations:

$$\frac{dy_1}{dy} + a y_1 + \frac{1}{1} \left(b \frac{dy_0}{dy} + c y_0 \right) = 0, \quad \frac{dy_2}{dy} + a y_2 + \frac{1}{2} \left(b \frac{dy_1}{dy} + c y_1 \right) = 0,$$

$$\frac{dy_3}{dy} + a y_3 + \frac{1}{3} \left(b \frac{dy_2}{dy} + c y_2 \right) = 0, \&c.$$

The function y_0 remains alone arbitrary, but the integration of the above differential equations will introduce an infinite number of constants.

If we suppose $a = 0, b = 0, c = -1$, the proposed equation will become $\frac{d^2 z}{dx dy} = z$, and we shall find

$$y_1 = \int y_0 dy, \quad y_2 = \frac{1}{2} \int y_1 dy, \quad y_3 = \frac{1}{3} \int y_2 dy + \&c.,$$

so that if we write $\phi(y)$ instead of y_0 , we shall have

Part III.
Integral
Calculus.

Part III.

$$z = \phi(y) + \frac{x}{1} \int \phi(y) dy + \frac{x^2}{1 \cdot 2} \int^2 \phi(y) dy^2 + \frac{x^3}{1 \cdot 2 \cdot 3} \int^3 \phi(y) dy^3 + \&c.$$

It may be proved that, in this case, all the arbitrary constants to be added to the integrals which are indicated, may be replaced by an arbitrary function. For that purpose, let us change $\phi(y)$ into $\phi(y) + A$, $\int \phi(y) dy$ into $\int \phi(y) dy + Ay + B$, $\int^2 \phi(y) dy^2$ into $\int^2 \phi(y) dy^2 + \frac{1}{2} Ay^2 + By + C$, &c. The terms which are not under the sign $\int$ being collected, will form the following expression,

$$A + Bx + \frac{1}{2} Cx^2 + \&c. + (Ax + \frac{1}{2} Bx^2 + \&c.)y + \left(\frac{1}{2} Ax^2 + \&c.\right)y^2 + \&c.,$$

which is the same thing as

$$\psi(x) + \frac{y}{1} \int \psi(x) dx + \frac{y^2}{1 \cdot 2} \int^2 \psi(x) dx^2 + \&c.,$$

for we may assume $A + Bx + Cx^2 + \&c. = \psi(x)$.

The general integral of the proposed equation will be then

$$z = \phi(y) + \frac{x}{1} \int \phi(y) dy + \frac{x^2}{1 \cdot 2} \int^2 \phi(y) dy^2 + \&c. \\ + \psi(x) + \frac{y}{1} \int \psi(x) dx + \frac{y^2}{1 \cdot 2} \int^2 \psi(x) dx^2 + \&c.$$

(228.) An important remark is that there are partial differential equations, of an order higher than the first, which have no first integral. This arises from the form of the primitive equation, which is such that neither of the arbitrary functions can be eliminated without proceeding to differential equations involving differential coefficients of the same order as the equation to be integrated. An example of this may be seen in the equation $rx - tx - 2p = 0$, the integral of which is $z = \phi(y+x) + \psi(y-x) - x\{\phi'(y+x) - \psi'(y-x)\}$, and to eliminate the arbitrary functions, it is necessary to proceed to the second order.

There are also equations the integrals of which do not admit a number of arbitrary functions equal to the exponent of their order, or for which the number of arbitrary functions, when equal to the exponent of the order, may be reduced. Sometimes the number of arbitrary functions is different according to the finite or infinite form given to the integral. It appears, therefore, that there is not a complete analogy between the arbitrary functions which enter in the integrals of partial differential coefficients, and the constants of the integrals of differential equations of two variables, since the number of the first is not always equal to the exponent of the order of the proposed partial differential equation.

If $V = 0$ represent a differential equation between the three variables x, y, z , we have the six equations

$$V = 0, \quad \frac{dV}{dx} = 0, \quad \frac{dV}{dy} = 0, \quad \frac{d^2V}{dx^2} = 0, \quad \frac{d^3V}{dx dx dy} = 0, \quad \frac{d^2V}{dy^2} = 0;$$

and between these we can eliminate five quantities, so that a partial differential equation of the second order may result from the elimination of five quantities. Generally, it is easy to see that a partial differential equation of the

order n may result from the elimination of $\frac{(n+1)(n+2)}{2} - 1$ quantities in a primitive equation.

If we suppose, first, that the quantities eliminated are constant, and if we call *complete integral* the primitive equation which contains them, to distinguish it from the general integral including arbitrary constants, it follows clearly from what precedes, that the first integral ought to contain only two arbitrary constants, the second five, the third nine, &c. We may also infer from the preceding considerations that there exist partial differential equations of the second order, which cannot have complete first integrals; for an integral of that kind can only contain in general two constants more than the partial differential equation from which it is derived, therefore it would be necessary that the five constants which are to be eliminated should enter V in such a manner as to disappear in two groups, the one composed of three constants, and the other of two. We shall illustrate these considerations by an example.

Let $z = f(a, b, a', b', c') + ax + by + a'x^2 + b'xy + c'y^2$

f designating a given function. We shall get in differentiating

$$p = a + 2a'x + b'y, \quad q = b + b'x + 2c'y, \quad r = 2a', \quad s = b', \quad t = 2c';$$

taking in these equations the values of the five constants a, b, a', b', c' , to substitute them in the first, we shall have

$$z = px + qy - \frac{1}{2}(rx^2 + 2sxy + ty^2) + f(p - rx - 5y, q - ty - 5x, \frac{1}{2}r, 5, \frac{1}{2}t);$$

and in order that this equation might be derived from an equation of the first order containing only two arbitrary

Integral
Calculus.

constants, three must disappear at the same time by the first differentiation of the proposed equation. Now if we subtract from this equation the values of p and q , multiplied respectively by x and y , we find Part III.

$$z - px - qy = f(a, b, a', b', c') - (a'x^2 + b'xy + c'y^2);$$

a result which will satisfy the required condition, only when the function f will not contain a', b', c' . In that supposition the quantity $a'x^2 + b'xy + c'y^2$ may be eliminated, and the resulting equation

$$2z - px - qy = 2f(a, b) + \alpha x + \beta y,$$

containing two arbitrary constants only, would be the required first integral of

$$z = px + qy - \frac{1}{2}(rx^2 + 2sxy + ty^2) + f(p - rx - 5y, q - ty - 5x).$$

(229.) Hitherto there are no means to determine, in general, the *complete integrals* of partial differential equations. The connection between the *complete integral* and the *general integral* is very simple, however, in the case of partial differential equations of the first order, and the second may be always deduced from the first, by making one of the constants an arbitrary function of the other; thus

$$z = ax + by + a^2 + b^2$$

is the complete solution of $z = px + qy + p^2 + q^2$. If we make $b = \phi(a)$, we get $z = ax + y\phi(a) + a^2 + \phi(a)^2$ and $x + y\phi'(a) + 2a + 2\phi(a)\phi'(a) = 0$; a system of equations from which the *general integral* is determined.

In the second and higher orders the passage from the *complete integral* to the *general integral* presents greater difficulties, and therefore very little advantage can be derived for the determination of the general integral, from the knowledge of the complete integral, which, in some cases, can be determined very readily.

(230.) Partial differential equations admit also of *particular solutions*. When the complete integral is known, the particular solutions may easily be found by the general method which has been explained. Let $V = 0$ be the complete integral of a partial differential equation of the first order, and let a and b be the arbitrary constants, then the system

$$V = 0, \quad \frac{dV}{da} da + \frac{dV}{db} db = 0,$$

will be the possible methods of satisfying the proposed differential equation; if we determine the values of a and b

by the two equations $\frac{dV}{da} = 0, \quad \frac{dV}{db} = 0$. These values substituted for a and b in the equation $V = 0$ will

give an equation without arbitrary constants, which will be a particular solution. The same reasoning, and the same process, will apply to equations of higher orders, when the complete integral is known.

(231.) Let us now investigate the characters by means of which it may be ascertained whether a given solution of a partial differential equation is included in the general solution or not. Let $Z = 0$ be the proposed equation, $z = V$ the given solution, and let $z = F(x, y, \phi(P), \psi(Q))$ represent the general integral, ϕ and ψ designating two arbitrary functions. If $z = V$ be a solution included in the general integral, then $\phi(P)$ and $\psi(Q)$ can take particular forms $f(P), f_1(Q)$, which will transform the general expression into V . This expression may be written in the following manner,

$$z = F\{x, y, f(P) + \phi(P) - f(P), \quad f_1(Q) + \psi(Q) - f_1(Q)\}.$$

The functions $\phi(P) - f(P), \psi(Q) - f_1(Q)$ being arbitrary may be represented by $a\phi(P)$ and $a\psi(Q)$. If we develop afterwards according to the powers of a , the result will take the form

$$z = V + a^m V \phi(P)^m + a^n W \psi(Q)^n + \&c.$$

or, more simply, $z = V + au$, u containing as many arbitrary functions as the general integral, and becoming neither infinite, nor equal to nothing when $a = 0$. We have by differentiation

$$p = \frac{dV}{dx} + a \frac{du}{dx}, \quad q = \frac{dV}{dy} + a \frac{du}{dy}, \quad r = \frac{d^2V}{dx^2} + a \frac{d^2u}{dx^2}, \quad s = \frac{d^2V}{dx dy} + a \frac{d^2u}{dx dy}, \quad t = \frac{d^2V}{dy^2} + a \frac{d^2u}{dy^2}.$$

These values, and the value of $z = V + au$, being substituted in the proposed equation $Z = 0$, we shall obtain, in observing that by supposition $z = V$ satisfies that equation,

$$\left. \begin{aligned} \frac{dz}{dz} u + \frac{dz}{dp} \frac{du}{dx} + \frac{dz}{dq} \frac{du}{dy} \\ + \frac{dz}{dr} \frac{d^2u}{dx^2} + \frac{dz}{ds} \frac{d^2u}{dx dy} + \frac{dz}{dt} \frac{d^2u}{dy^2} + \&c. \end{aligned} \right\} = 0.$$

If this substitution had made

$$\frac{dz}{dp} = 0, \quad \frac{dz}{dq} = 0, \quad \frac{dz}{dr} = 0, \quad \frac{dz}{ds} = 0, \quad \frac{dz}{dt} = 0, \quad \&c.$$

Integral
Calculus.

Part III.

the above equation would be reduced to $\frac{dz}{dz}u = 0$, and could only be verified in making $u = 0$, and, therefore,

there would be no means to introduce arbitrary functions in the expression $z = V$, and, consequently, $z = V$ ought to be considered as a particular solution.

If the differential coefficients $\frac{dz}{dp}$ and $\frac{dz}{dq}$ did not become equal to nothing, then the value of u would be

determined by the partial differential equation

$$\frac{dz}{dz}u + \frac{dz}{dp} \cdot \frac{du}{dx} + \frac{dz}{dq} \frac{du}{dy} = 0.$$

This equation being of the first order, the value of u would contain only one arbitrary function.

The mode of extending this rule to partial differential equations of all orders is obvious. Hence in the case of equations of the first order, for instance, if $V = 0$ be the result of the elimination of p and q from the equations

$V = 0$, $\frac{dV}{dp} = 0$, $\frac{dV}{dq} = 0$, the factors of $V = 0$ which satisfy these equations will be particular solutions; and

a similar method is applicable to equations of higher orders.

Let us take as an example, the equation

$$z = px + qy + a\sqrt{1+p^2+q^2}.$$

Here $Z = z - px - qy - a\sqrt{1+p^2+q^2}$, $\frac{dz}{dz} = 1$, $\frac{dz}{dp} = x - \frac{ap}{\sqrt{1+p^2+q^2}}$, $\frac{dz}{dq} = y - \frac{aq}{\sqrt{1+p^2+q^2}}$

Eliminating p and q between these two last equations and the proposed, we find

$$x^2 + y^2 + z^2 = a^2,$$

which satisfies them, and is consequently a particular solution.

Let us take for a second example, the equation

$$r^2 - 2qr\left(p - \frac{z}{1+x}\right) + \left(p - \frac{z}{1+x}\right)y = 0,$$

it gives $\frac{dz}{dr} = r - q\left(p - \frac{z}{1+x}\right)$, and since it contains neither s nor t , it is sufficient to combine the equation

$r - q\left(p - \frac{z}{1+x}\right) = 0$ with the proposed to determine the particular solution. By these means we find

$p - \frac{z}{1+x} = 0$, from which we derive $r = \frac{p}{1+x} - \frac{z}{(1+x)^2} = \frac{1}{1+x}\left(p - \frac{z}{1+x}\right)$, and which, consequently,

satisfies $Z = 0$, and $\frac{dZ}{dr} = 0$. It is therefore the particular solution required. Integrating it, we find

$z = (1+x)\phi(y)$.

(232.) The arbitrary functions which enter into the integral of partial differential equations are to be determined in each case by the conditions of the questions which led to the differential equations, in a manner analogous to that used for the arbitrary constants. This determination belongs, therefore, rather to the applications of the Differential and Integral Calculus, to Geometry, and Mechanics, than to the exposition of the rules of that calculus. In many cases besides, the conditions to be satisfied lead to equations expressing relations between unknown functions, the solution of which belongs to another calculus. We shall, therefore, briefly examine here, in what manner the arbitrary functions may be determined in some simple cases.

First let the integral of a partial differential equation be $M\phi(V) = 1$, where M and V are known functions of x , y , and z , and $\phi(V)$ an arbitrary function of V . Suppose that it is known, from the conditions of the question, that, besides the relation just found, there exist between the three variables x , y , z , two other relations expressed by the equations $F(x, y, z) = 0$, and $F_1(x, y, z) = 0$. Assume $V = t$, and by this, and the two preceding equations, let x , y , z be determined as functions of t . This being done, substitute those values of x , y , z , in

M , and let the result which will be a function of t be T , then we shall have $T \cdot \phi(t) = 1$, or $\phi(t) = \frac{1}{T}$, and the form of ϕ will thus be determined.

Let us next consider a case in which there are two arbitrary functions to be determined. Suppose that the integral is

$$M\phi(V) + N\psi(V) = 1,$$

and that the variables x , y , z must satisfy besides the conditions expressed by the two systems of equations,

$$F(x, y, z) = 0, \quad F_1(x, y, z) = 0, \quad \text{and} \quad F_2(x, y, z) = 0, \quad F_3(x, y, z) = 0.$$

Integral
Calculus.

As before, let $V = t$, and by this and the first pair of equations, let the values of x, y, z be determined as functions of t , and their values substituted in M and N . Then if T and T_1 represent the values of M and N after that substitution Part III.

$$T\phi(t) + T_1\psi(t) = 1.$$

If we determine in the same manner the values of x, y, z , as functions of t , by means of the equation $V = t$ and $F_2(x, y, z) = 0$, $F_3(x, y, z) = 0$, we shall have, in substituting these new values in M and N , and calling T_2 and T_3 the results of the substitutions,

$$T_2\phi(t) + T_3\psi(t) = 1.$$

From this equation and the other $T\phi(t) + T_1\psi(t) = 1$, the values of $\phi(t)$ and $\psi(t)$ may be determined as explicit functions of t , and therefore their forms will be known.

This process may easily be generalized, and applied to integrals containing a greater number of arbitrary functions, provided their form be similar to those we have considered.

Very frequently the arbitrary functions entering the integrals of partial differential equations remain entirely undetermined. The integrals, in this case, point out general properties belonging to the whole class of functions of x and y , represented by z .

APPLICATIONS OF THE DIFFERENTIAL AND INTEGRAL CALCULUS.

(233.) The Differential and Integral Calculus may be applied with great advantage to numerous and important questions. For many problems indeed it would be extremely difficult to obtain a solution without making use of that calculus. The development of functions of one or more variables in series of terms, proceeding according to certain laws, forms one of its most useful applications. Part III

(234.) Let y be a function of x which does not become infinite for $x = 0$, and let it be proposed to develop it according to the integral powers of x . Let us assume

$$y = A + Bx + Cx^2 + Dx^3 + \&c.$$

The problem is to determine the values of the coefficients $A, B, C, D, \&c.$ For that purpose let us differentiate several times this equation, we shall find successively,

$$\begin{aligned} \frac{dy}{dx} &= B + 2Cx + 3Dx^2 + 4Ex^3 + \&c. & \frac{d^2y}{dx^2} &= 2C + 3 \cdot 2Dx + 4 \cdot 3Ex^2 + \&c. \\ \frac{d^3y}{dx^3} &= 3 \cdot 2 \cdot D + 4 \cdot 3 \cdot 2 \cdot Ex + \&c. & \frac{d^4y}{dx^4} &= 4 \cdot 3 \cdot 2 \cdot E + \&c. \\ &\&c. & &\&c. \end{aligned}$$

Hence if we suppose in each of these equations $x = 0$, we shall easily find the values of $A, B, C, D, \&c.$ Let $Y, Y', Y'', \&c.$ be respectively the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \&c.$ when $x = 0$, we shall have

$$A = Y, \quad B = \frac{Y'}{1}, \quad C = \frac{Y''}{1 \cdot 2}, \quad D = \frac{Y'''}{1 \cdot 2 \cdot 3}, \quad E = \frac{Y''''}{1 \cdot 2 \cdot 3 \cdot 4}, \quad \&c.$$

and, consequently,

$$y = Y + Y' \cdot \frac{x}{1} + Y'' \cdot \frac{x^2}{1 \cdot 2} + Y''' \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + Y'''' \cdot \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.$$

This elegant expression of the development of a function of one variable in a series of terms arranged according to the powers of that variable, is known under the name of the *Theorem of Maclaurin*: it is of considerable use in the development of functions, though it does not always enable us to discover the general term of the resulting series.

(235.) More generally, it may be proved, in the same manner, that if $y = f(a + x)$ does not become infinite for $x = 0$, the development of y according to the integral powers of x will be

$$y = Y + Y' \frac{x}{1} + Y'' \frac{x^2}{1 \cdot 2} + Y''' \frac{x^3}{1 \cdot 2 \cdot 3} + Y'''' \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.$$

$Y, Y', Y'', Y''', \&c.$ being the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \&c.$ corresponding to $x = 0$, or which is exactly the same

thing, the values of $f(x), \frac{df(x)}{dx}, \frac{d^2f(x)}{dx^2}, \frac{d^3f(x)}{dx^3}, \&c.$ corresponding to $x = a$, so that the development of y may be written in the following manner,

$$y = f(a + x) = f(a) + \frac{df(a)}{da} \cdot x + \frac{d^2f(a)}{da^2} \cdot \frac{x^2}{1 \cdot 2} + \frac{d^3f(a)}{da^3} \cdot \frac{x^3}{1 \cdot 2 \cdot 3} + \&c.$$

This expression may also be deduced immediately from Taylor's Theorem.

The application of this method to particular cases does not present the least difficulty. We might easily make use of it to determine the developments of $a^x, \log(a + x), \sin x, \cos x, \&c.$ which have already been found by various means, (see (26.), (27.), (28.))

(236.) We shall take as examples the functions $\sin^{-1} x, \cos^{-1} x$, and $\tan^{-1} x$.

1. For the first we have

$$\begin{aligned} y &= \sin^{-1} x, & \frac{dy}{dx} &= (1 - x^2)^{-\frac{1}{2}}, & \frac{d^2y}{dx^2} &= x(1 - x^2)^{-\frac{3}{2}}, & \frac{d^3y}{dx^3} &= (1 - x^2)^{-\frac{3}{2}} + 3x^2(1 - x^2)^{-\frac{5}{2}}, \\ \frac{d^4y}{dx^4} &= 3 \cdot 3x(1 - x^2)^{-\frac{5}{2}} + 3 \cdot 5x^3(1 - x^2)^{-\frac{7}{2}}, & \frac{d^5y}{dx^5} &= 3 \cdot 3(1 - x^2)^{-\frac{5}{2}} + 3 \cdot 5 \cdot 6x^2(1 - x^2)^{-\frac{7}{2}} + 3 \cdot 5 \cdot 7x^4(1 - x^2)^{-\frac{9}{2}}, & \&c. \end{aligned}$$

Making $x = 0$, we shall find

$$Y = 0, \quad Y' = 1, \quad Y'' = 0, \quad Y''' = 1, \quad Y'''' = 0, \quad Y'''' = 3 \cdot 3, \quad \&c.$$

Integral
Calculus.

and, consequently,

$$y = \sin^{-1} x = x + \frac{x^3}{1.2.3} + \frac{3.3 x^5}{1.2.3.4.5} + \&c.$$

The general term of the series may be obtained by observing that $\frac{d^{n+1}y}{dx^{n+1}} = \frac{1}{dx^n} d^n (1-x^2)^{-\frac{1}{2}}$, and making use of the general formula given in *Example 6. (25.)* We shall find that the value of $d^n (1-x^2)^{-\frac{1}{2}}$ becomes equal to nothing by making $x=0$, whenever n is an odd number, and that when n is an even number it is reduced to

$$\frac{1.3.5.7 \dots (n-1)}{2.4.6.8 \dots n} n(n-1) \dots 1.$$

The general term $\frac{d^{n+1}y}{dx^{n+1}} \cdot \frac{x^{n+1}}{1.2 \dots n+1}$, will become then, putting for $\frac{d^{n+1}y}{dx^{n+1}}$ the value of $\frac{1}{dx^n} d^n (1-x^2)^{-\frac{1}{2}}$, which has just been found,

$$\frac{1.3.5.7 \dots (n-1) x^{n+1}}{2.4.6.8 \dots n(n+1)}.$$

It is obvious besides that $n+1$ will be an odd number. If then we write in the series $\sin y$ instead of x , we shall have

$$y = \sin y + \frac{(\sin y)^3}{2.3} + \frac{3.(\sin y)^5}{2.4.5} + \frac{3.5(\sin y)^7}{2.4.6.7} + \frac{3.5.7(\sin y)^9}{2.4.6.8.9} + \&c.$$

This series has been used by Newton to calculate the length of the circumference, the radius being 1. If we suppose y equal to a quadrant, we shall have $\sin y = 1$, and if we suppose y equal to an arc of 30° , we shall have $\sin y = \frac{1}{2}$. Substituting these numbers, we obtain the two series

$$\text{arc of } 90^\circ = 1 + \frac{1}{2.3} + \frac{3}{2.4.5} + \frac{3.5}{2.4.6.7} + \frac{3.5.7}{2.4.6.8.9} + \&c.$$

$$\text{arc of } 30^\circ = \frac{1}{2} + \frac{1}{2^4.3} + \frac{3}{2^6.4.5} + \frac{3.5}{2^8.4.6.7} + \frac{3.5.7}{2^{10}.4.6.8.9} + \&c.$$

The first being multiplied by 4, and the second by 12 will give the length of the circumference.

The developement of the arc in a series of terms arranged according to the powers of the cosine might be obtained by a similar process; but it is immediately found by substituting in the preceding series $\frac{\pi}{2} - y$ for y , and

observing that $\sin\left(\frac{\pi}{2} - y\right) = \cos y$. We get thus

$$y = \frac{\pi}{2} - \cos y - \frac{(\cos y)^3}{2.3} - \frac{3.(\cos y)^5}{2.4.5} - \frac{3.5(\cos y)^7}{2.4.6.7} - \frac{3.5.7(\cos y)^9}{2.4.6.8.9} - \&c.$$

Let $y = \tan^{-1} x$, we shall have by successive differentiations

$$\frac{dy}{dx} = (1+x^2)^{-1}, \quad \frac{d^2y}{dx^2} = -2x(1+x^2)^{-2}, \quad \frac{d^3y}{dx^3} = -2(1+x^2)^{-2} + 8x^2(1+x^2)^{-3},$$

$$\frac{d^4y}{dx^4} = 24x(1+x^2)^{-3} - 48x^3(1+x^2)^{-4}, \quad \frac{d^5y}{dx^5} = 24(1+x^2)^{-3} - 288x^2(1+x^2)^{-4} + 384x^4(1+x^2)^{-5}, \&c.$$

Making $x=0$, we find

$$Y=0, \quad Y=1, \quad Y''=0, \quad Y'''=-2, \quad Y^{IV}=0, \quad Y^V=2.3.4, \&c.$$

And we get therefore

$$y = \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \&c.$$

The law of the terms is very obvious, and if it were necessary, the general term might be obtained by referring to the general formula of *Example 6. (25.)* Substituting $\tan y$ for x , the series becomes

$$y = \tan y - \frac{(\tan y)^3}{3} + \frac{(\tan y)^5}{5} - \frac{(\tan y)^7}{7} + \&c.$$

(237.) The developement of $\tan x$ might be obtained by the same method, but it will be simpler to use another somewhat different, which will have besides the advantage to show how to proceed when the function to be developed is given implicitly by a differential equation.

Let $y = \tan x$, then $dx = \frac{dy}{1+y^2}$, and, consequently, $\frac{dy}{dx} = 1+y^2$; the value of $\frac{dy}{dx}$ depends then, in this case, upon y , and the differential coefficients, instead of being given explicitly in function of the independent variable, will be determined by means of each other. We shall have

$$\frac{dy}{dx} = 1+y^2, \quad \frac{d^2y}{dx^2} = 2y \frac{dy}{dx}, \quad \frac{d^3y}{dx^3} = 2 \frac{d^2y}{dx^2} + 2y \frac{d^2y}{dx^2}, \&c.$$

Integral
Calculus

To deduce from these equations the values of $Y, Y', Y'',$ &c. it is necessary to make $x = 0$ in each of them. Part III.
If we had not the equation between x and $y, y = \tan x$, the value of Y would remain indeterminate. Here we have $Y = 0$, and we shall find by means of this value

$$Y = 0, Y' = 1, Y'' = 0, Y''' = 2, Y^{iv} = 0, Y^v = 16, Y^vi = 0, Y^{vii} = 272, \&c.$$

and therefore

$$y = \tan x = \frac{x}{1} + \frac{2x^3}{1 \cdot 2 \cdot 3} + \frac{16x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \frac{272x^7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} + \&c.$$

In the same manner may be found the developement of $\sec x$. Let $y = \sec x$, we shall have $dx = \frac{dy}{y\sqrt{y^2-1}}$,

and $\frac{dy}{dx} = y\sqrt{y^2-1}$, or, raising both sides to the square, $\frac{dy^2}{dx^2} = y^4 - y^2$. Then, by successive differentiations,

we shall get

$$\begin{aligned} 2 \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} &= 4y^3 \frac{dy}{dx} - 2y \frac{dy}{dx}, \text{ or } \frac{d^2y}{dx^2} = 2y^3 - y, \quad \frac{d^3y}{dx^3} = (6y^2 - 1) \frac{dy}{dx}, \quad \frac{d^4y}{dx^4} = (6y^2 - 1) \frac{d^2y}{dx^2} \\ &+ 12y \frac{dy}{dx} \cdot \frac{d^3y}{dx^3} = (6y^2 - 1) \frac{d^3y}{dx^3} + 36y \frac{d^2y}{dx^2} \cdot \frac{dy}{dx} + 12 \frac{dy}{dx} \frac{d^3y}{dx^3}, \quad \frac{d^5y}{dx^5} = (6y^2 - 1) \frac{d^4y}{dx^4} + 48y \frac{d^3y}{dx^3} \cdot \frac{dy}{dx} \\ &+ 72 \frac{d^2y}{dx^2} \cdot \frac{d^3y}{dx^3} + 36y \left(\frac{d^2y}{dx^2} \right)^2 \&c. \end{aligned}$$

Hence, making $x = 0$, and observing that in that supposition $y = 1$, and $\frac{dy}{dx} = 0$ or $Y' = 0$, we have

$$Y'' = 1, \quad Y''' = 0, \quad Y^{iv} = 5, \quad Y^v = 0, \quad Y^{vi} = 61, \&c.$$

and

$$y = \sec x = 1 + \frac{x^2}{1 \cdot 2} + \frac{5x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{61x^6}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} + \frac{1385x^8}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8} + \frac{5021x^{10}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10} + \frac{2702765x^{12}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12} + \&c.$$

(238.) The same principles may be used with success to develop functions of several variables in series arranged according to the integral powers of these variables. Let, for instance, $u = f(x, y)$ represent a function of two variables x and y , and assume

$$\begin{aligned} u &= A + Bx + Dx^2 + Gx^3 + \&c. \\ &+ Cy + Exy + Hx^2y \\ &+ Fy^2 + Ixy^2 \\ &+ Ky^3. \end{aligned}$$

We shall have by successive differentiations, first with respect to x alone,

$$\begin{aligned} \frac{du}{dx} &= B + 2Dx + 3Gx^2, & \frac{d^2u}{dx^2} &= 2D + 6Gx, & \frac{d^3u}{dx^3} &= 6G, \&c. \\ &+ Ey + 2Hxy & & & & \\ &+ Ky & & & & \end{aligned}$$

which will give the values of A, B, D, G by making $x = 0$, and $y = 0$ in $u, \frac{du}{dx}, \frac{d^2u}{dx^2}, \frac{d^3u}{dx^3},$ &c. The same

values being substituted for x and y in the expressions of $\frac{du}{dy}, \frac{d^2u}{dy^2}, \frac{d^3u}{dy^3},$ &c. will give the values of $C, F, K,$ &c.

and the same substitutions being made in $\frac{d^2u}{dx dy}, \frac{d^3u}{dx^2 dy}, \frac{d^3u}{dx dy^2},$ &c. will determine the remaining coefficients $E, H, I,$ &c. Thus we shall find

$$\begin{aligned} f(x, y) &= u + \frac{1}{1} \left\{ \frac{du}{dx} x + \frac{du}{dy} y \right\} + \frac{1}{1 \cdot 2} \left\{ \frac{d^2u}{dx^2} x^2 + 2 \frac{d^2u}{dx dy} xy + \frac{d^2u}{dy^2} y^2 \right\} + \frac{1}{1 \cdot 2 \cdot 3} \left\{ \frac{d^3u}{dx^3} x^3 + 3 \frac{d^3u}{dx^2 dy} x^2 y \right. \\ &\quad \left. + 3 \frac{d^3u}{dx dy^2} xy^2 + \frac{d^3u}{dy^3} y^3 \right\} + \&c. \end{aligned}$$

Recollecting that in u , and in every differential coefficient, x and y are supposed to have been made equal to nothing.

It is easy to see that the same series will also represent the developement of $f(a+x, b+y)$, supposing that in u , and in every differential coefficient, a and b have been substituted for x and y . These results are also immediate consequences of Taylor's Theorem, extended to functions of several variables.

(239.) The developements known under the names of Lagrange's and Laplace's Theorems, are two of the most important applications of the Differential Calculus to the transformation of functions into series. We shall begin with the demonstration of the last of these two theorems, because the first may easily be deduced from it by a particular supposition. Part III.

Let $u = \psi(y)$ represent any function of y , let y be a function of the independent variable x , determined by the equation $y = F(a + x\phi(y))$, F and ϕ designating any function whatever, and let it be proposed to develop u in a series of terms containing the integral powers of x . For that purpose, according to Maclaurin's Theorem,

it is sufficient to find the values of u , $\frac{du}{dx}$, $\frac{d^2u}{dx^2}$, &c. for $x = 0$. To obtain them, let us differentiate the equation $y = F(a + x\phi(y))$ first with respect to x , and then with respect to a , we shall get

$$\frac{dy}{dx} = F'(a + x\phi(y)) \cdot \left\{ \phi(y) + x\phi'(y) \frac{dy}{dx} \right\}, \quad \frac{dy}{da} = F'(a + x\phi(y)) \cdot \left\{ 1 + x\phi'(y) \frac{dy}{da} \right\}$$

If we eliminate $F'(a + x\phi(y))$ between these two equations, we shall find after reductions

$$\frac{dy}{dx} - \phi(y) \frac{dy}{da} = 0;$$

but since u , or $\psi(y)$, contains only y , we shall have simply

$$\frac{du}{dx} = \psi'(y) \frac{dy}{dx}, \quad \frac{du}{da} = \psi'(y) \frac{dy}{da};$$

and hence, by the elimination of $\psi'(y)$,

$$\frac{du}{dx} \cdot \frac{dy}{da} - \frac{du}{da} \cdot \frac{dy}{dx} = 0;$$

substituting then for $\frac{dy}{dx}$ its value $\phi(y) \frac{dy}{da}$, and making, for the sake of simplicity, $\phi(y) = z$, we shall find

$$\frac{du}{dx} - z \frac{du}{da} = 0, \quad \text{or} \quad \frac{du}{dx} = z \frac{du}{da}.$$

Hence the value of $\frac{du}{dx}$ corresponding to $x = 0$ is the same as the value of $z \frac{du}{da}$ corresponding to the same value of x . If we differentiate now the preceding equation with respect to x , we shall have

$$\frac{d^2u}{dx^2} = \frac{d \cdot z \frac{du}{da}}{dx};$$

but the quantity $z \frac{du}{da} = \phi(y) \psi'(y) \frac{dy}{da}$, that is to say is equal to a certain function of y multiplied by $\frac{dy}{da}$, it may therefore be considered as the differential coefficient of a certain function of y , which we shall represent by u' , with respect to a , and we shall have

$$z \frac{du}{da} = \frac{du'}{da}, \quad \text{and consequently} \quad \frac{d^2u}{dx^2} = \frac{d \cdot \frac{du'}{da}}{dx} = \frac{d^2u'}{dx da} = \frac{d^2u'}{da dx} = \frac{d \cdot \frac{du'}{dx}}{da}.$$

Now since, u representing any function of y , we have found $\frac{du}{dx} = z \frac{du}{da}$, u' being also a function of y , we shall have likewise $\frac{du'}{dx} = z \frac{du'}{da}$. Substituting this value in the last expression we have found for $\frac{d^2u}{dx^2}$, we get

$$\frac{d^2u}{dx^2} = \frac{d \cdot z \frac{du'}{da}}{da} = \frac{d \cdot z^2 \frac{du}{da}}{da},$$

putting for $\frac{du}{da}$ its value $z \frac{du}{dx}$.

This last equation being differentiated with respect to x will give $\frac{d^3u}{dx^3} = \frac{d^2 z^2 \frac{du}{da}}{dx da}$; or making $z^2 \frac{du}{da} = \frac{du''}{da}$,

and inverting the order of differentiations $\frac{d^3u}{dx^3} = \frac{d^3u''}{da^2 dx} = \frac{d^2 \frac{du''}{dx}}{da^2}$; but u'' being a function of y , we have also

$$\frac{du''}{dx} = z \frac{du''}{da}, \quad \text{and consequently} \quad \frac{d^3u}{dx^3} = z^3 \frac{du''}{da^3}; \quad \text{therefore} \quad \frac{d^3u}{dx^3} = \frac{d^2 z^3 \frac{du}{da}}{da^3}.$$

And generally, if $\frac{d^{n-1}u}{dx^{n-1}} = \frac{d^{n-2} \cdot z^{n-1} \frac{du}{da}}{da^{n-2}}$, making $z^{n-1} \frac{du}{da} = \frac{du'' \dots (n-1)}{da}$, (where $(n-1)$ designates the number of accents and not an exponent), we shall find

$$\frac{d^n u}{dx^n} = \frac{d^n u'' \dots (n-1)}{dx da^{n-1}} = \frac{d^n u'' \dots (n-1)}{da^{n-1} dx}, \text{ and because}$$

$$\frac{d u'' \dots (n-1)}{dx} = z \frac{d u'' \dots (n-1)}{da} = z^n \frac{du}{da}, \text{ we have}$$

$$\frac{d^n u}{dx^n} = \frac{d^{n-1} z^n \frac{du}{da}}{da^{n-1}}.$$

This understood, we shall have, according to Maclaurin's Theorem,

$$u = U + z \frac{du}{da} \frac{x}{1} + \frac{d \cdot z^2 \frac{du}{da}}{da} \frac{x^2}{1 \cdot 2} + \frac{d^2 \cdot z^3 \frac{du}{da}}{da^2} \frac{x^3}{1 \cdot 2 \cdot 3} + \dots + \frac{d^{n-1} z^n \frac{du}{da}}{da^{n-1}} \frac{x^n}{1 \cdot 2 \cdot 3 \dots n},$$

x being supposed to be made equal to nothing in every coefficient, and U designating the value of u in the same supposition. When $x = 0$, the equation $y = F(a + x\phi(y))$ gives $y = F(a)$, and $\frac{dy}{da} = F'(a)$, values which ought to be substituted in z or $\phi(y)$, and in $\frac{du}{da}$, or $\psi'(y) \frac{dy}{da}$. In the particular case represented by the equation $y = a + x\phi(y)$, we have simply, when $x = 0$, $y = a$, $\frac{dy}{da} = 1$, and therefore $U, z, \frac{du}{da}$, will become respectively $\psi(a), \phi(a)$, and $\psi'(a)$, and, consequently, the developement of u arranged according to the powers of x will be simply

$$u = \psi(a) + \psi'(a) \phi(a) \frac{x}{1} + \frac{d \cdot \psi'(a) \phi(a)^2}{da} \frac{x^2}{1 \cdot 2} + \frac{d^2 \cdot \psi'(a) \phi(a)^3}{da^2} \frac{x^3}{1 \cdot 2 \cdot 3} + \dots + \frac{d^{n-1} \cdot \psi'(a) \phi(a)^n}{da^{n-1}} \frac{x^n}{1 \cdot 2 \dots n}.$$

(240.) Laplace has extended these researches to functions of any number of variables, analogous to that we have just considered. We shall only examine the case of two variables. Let u be any function of two quantities y and z determined by the two equations

$$y = F[a + t\phi(y, z)], \quad z = F_1[b + x\phi_1(y, z)],$$

F, F_1, ϕ and ϕ_1 designating any function whatever. It is obvious that y and z may be considered as functions of the four quantities a, t, b , and x . Let the first of the two equations above be differentiated with respect to each of these quantities, we shall find

$$\frac{dy}{da} = F'(a + t\phi) \left(1 + t \frac{d\phi}{da}\right), \quad \frac{dy}{dt} = F'(a + t\phi) \left(\phi + t \frac{d\phi}{dt}\right),$$

$$\frac{dy}{db} = F'(a + t\phi) \cdot t \frac{d\phi}{db}, \quad \frac{dy}{dx} = F'(a + t\phi) t \frac{d\phi}{dx},$$

where ϕ is written instead of $\phi(y, z)$; and, in the same manner, we shall write simply ϕ_1 instead of $\phi_1(y, z)$. It is also necessary to remember that the functions ϕ and ϕ_1 are not supposed to contain explicitly the quantities t and x . If in the preceding values we suppose $t = 0$ and $x = 0$, we find

$$\frac{dy}{da} = F'(a), \quad \frac{dy}{dt} = F'(a) \cdot \phi, \quad \frac{dy}{db} = 0, \quad \frac{dy}{dx} = 0.$$

In the same manner we shall find

$$\frac{dz}{da} = 0, \quad \frac{dz}{dt} = 0, \quad \frac{dz}{db} = F_1'(b), \quad \frac{dz}{dx} = F_1'(b) \phi_1,$$

and in the same supposition u will become a function of a and b alone, since then $y = F(a)$ and $z = F_1(b)$. Now, if we observe that u is an explicit function of y and z , and if we substitute for $\frac{dy}{dt}, \frac{dz}{dt}, \frac{dy}{dx}, \frac{dz}{dx}$, their values, we shall have

$$\frac{du}{dt} = \frac{du}{dy} \cdot \frac{dy}{dt} + \frac{du}{dz} \frac{dz}{dt} = \frac{du}{dy} F'(a) \phi, \quad \frac{du}{dx} = \frac{du}{dy} \frac{dy}{dx} + \frac{du}{dz} \frac{dz}{dx} = \frac{du}{dz} F_1'(b) \phi_1,$$

Integral
Calculus.

and because $F'(a) = \frac{dy}{da}$, and $F_1'(b) = \frac{dz}{db}$, we shall have, when $t = 0$ and $x = 0$,

$$\frac{du}{dt} = \frac{du}{dy} \frac{dy}{da} \cdot \phi = \frac{du}{da} \cdot \phi, \quad \frac{du}{dx} = \frac{du}{dz} \frac{dz}{db} \cdot \phi_1 = \frac{du}{db} \phi_1.$$

Let us suppose, for a moment, that the equation $z = F_1(b + x\phi_1)$ has been resolved with respect to z , and that this value of z has been substituted in the function u ; then this function will contain only y and x , and in the differentiations relative to t , it will be sufficient to take into consideration the first quantity y , since the second x is one of the independent variables, but if we substitute at the same time the value of z in the equation $y = F(a + t\phi)$, then the function ϕ , when we shall differentiate with respect to t , may be considered as containing only y ; therefore we shall have as in (239.)

$$\frac{d^m u}{dt^m} = \frac{d^{m-1} \cdot \phi^m \frac{du}{da}}{da^{m-1}},$$

and, consequently,

$$\frac{d^{m+n} u}{dt^m dx^n} = \frac{d^n \left(\frac{d^{m-1} \cdot \phi^m \frac{du}{da}}{da^{m-1}} \right)}{dx^n} = \frac{d^{m-1} \left(\frac{d^{n-1} \cdot \phi^m \frac{du}{da}}{dx^n} \right)}{da^{m-1}}.$$

It is clear that the developement of $d^n \cdot \phi^m \frac{du}{da}$, may be deduced from that of $d^n \phi \frac{du}{da}$, by substituting in it ϕ^m

to ϕ , $d \cdot \phi^m$ to $d\phi$, $d^2 \cdot \phi^m$ to $d^2\phi$, and so on; we may, therefore, write simply $\phi \frac{du}{da}$, instead of $\phi^m \frac{du}{da}$; and since this last quantity is equal to $\frac{du}{dt}$, when $t = 0$, we shall have

$$\frac{d^{m+n} u}{dt^m dx^n} = \frac{d^{m-1} \left(\frac{d^{n+1} u}{dx^n dt} \right)}{da^{m-1}}.$$

In a similar manner we shall have

$$\frac{d^n u}{dx^n} = \frac{d^{n-1} \cdot \phi_1^n \frac{du}{db}}{db^{n-1}}, \text{ which equation may be changed into } \frac{d^n u}{dx^n} = \frac{d^{n-1} \frac{du}{dx}}{db^{n-1}},$$

by writing ϕ_1 instead of ϕ_1^n , and substituting instead of $\phi_1 \frac{du}{db}$, its value $\frac{du}{dx}$, in the case where $x = 0$; but it will be necessary to replace ϕ_1 by ϕ_1^n in the result.

Let us substitute now this expression of $\frac{d^n u}{dx^n}$ in that of $\frac{d^{m+n} u}{dt^m dx^n}$, found above, and we shall find

$$\frac{d^{m+n} u}{dt^m dx^n} = \frac{d^{m+n-2} \left(\frac{d^2 u}{dt dx} \right)}{da^{m-1} db^{n-1}},$$

in which, it must not be forgotten, it is necessary to substitute ϕ^n for ϕ , ϕ_1^n for ϕ_1 , and after the differentiations to make $t = 0$ and $x = 0$.

The preceding transformations have enabled us to make the determination of the coefficient $\frac{d^{m+n} u}{dt^m dx^n}$ depend on that of the second order $\frac{d^2 u}{dt dx}$, and on differentiations relative to a and b , which present no difficulties when $t = 0$ and $x = 0$. It remains therefore only to determine the value of $\frac{d^2 u}{dt dx}$.

If we differentiate with respect to x , the equation $\frac{du}{dt} = \frac{du}{da} \phi$, we shall find

$$\frac{d^2 u}{dx dt} = \frac{d^2 u}{dx da} \phi + \frac{du}{da} \frac{d\phi}{dx};$$

and since we have $\frac{du}{dx} = \frac{du}{db} \phi_1$,

$$\frac{d^2 u}{dx dt} = \phi \frac{d \cdot \phi_1 \frac{du}{db}}{da} + \frac{du}{da} \frac{d\phi}{dx}.$$

Integral
Calculus.

 But ϕ representing, as well as u , any function whatever of y and z , there exists between $\frac{d\phi}{dx}$ and $\frac{d\phi}{db}$ the same Part III.

 relation as between $\frac{du}{dx}$ and $\frac{du}{db}$, therefore we shall have also $\frac{d\phi}{dx} = \phi_1 \frac{d\phi}{db}$, hence

$$\frac{d^2 u}{dx dt} = \phi \phi_1 \frac{d^2 u}{da db} + \phi \frac{d\phi_1}{da} \frac{du}{db} + \phi_1 \frac{d\phi}{db} \frac{du}{da},$$

 and since we must change ϕ into ϕ^m , and ϕ_1 into ϕ_1^n , we shall find

$$\frac{d^2 u}{dx dt} = \phi^m \phi_1^n \frac{d^2 u}{da db} + n \phi^m \phi_1^{n-1} \frac{d\phi_1}{da} \frac{du}{db} + m \phi^{m-1} \phi_1^n \frac{d\phi}{db} \frac{du}{da},$$

 when, it is well understood, ϕ and ϕ_1 as well as u will be reduced to functions of a and b , by putting instead of y and z their values, when $t = 0$, and $x = 0$. Having thus formed $\frac{d^2 u}{dx dt}$, which we shall represent by $U^{(m+n)}$,

we shall have for the general term of the developement of the proposed function

$$\frac{a^{m+n-2} U^{(m+n)}}{d a^{m-1} d b^{n-1}} \cdot \frac{t^m x^n}{1.2 \dots m.1.2 \dots n}.$$

 Each term will be obtained individually by taking for m and n all the integers 0, 1, 2, 3, &c.

 (241.) If in the last formulæ of (239.) we make $x = 1$, we shall have

$$y = a + \phi(y) \text{ and the series } \psi(y) = \psi(a) + \psi'(a) \phi(a) + \frac{1}{1.2} \frac{d\psi(a)}{da} \phi(a)^2 + \frac{1}{1.2.3} \frac{d^2 \psi(a)}{da^2} \phi(a)^3 + \&c.$$

which Lagrange obtained by way of induction, in developing the roots of literal algebraical equations.

 We shall propose for first example to find the expression of the n^{th} power of y , given by the equation $a - \beta y + \gamma y^m = 0$.

 This equation may be put under the form $y = \frac{a}{\beta} + \frac{\gamma}{\beta} y^m$, and if we compare it with $y = a + \phi(y)$, we have clearly

$$a = \frac{a}{\beta}, \quad \phi(y) = \frac{\gamma}{\beta} y^m, \quad \psi(y) = y^n.$$

 If we write, for the sake of brevity, a instead of $\frac{a}{\beta}$ we find

$$\phi(a) = a^m \frac{\gamma}{\beta}, \quad \psi(a) = a^n, \quad \psi'(a) \phi(a) = n a^{n+m-1} \frac{\gamma}{\beta}, \quad \psi'(a) \phi(a)^2 = n a^{n+2m-1} \frac{\gamma^2}{\beta^2}, \quad \psi'(a) \phi(a)^3 = n a^{n+3m-1} \frac{\gamma^3}{\beta^3}, \&c.$$

and, consequently

$$y^n = a^n + \frac{n}{1} a^{n+m-1} \frac{\gamma}{\beta} + \frac{n(n+2m-1)}{1.2} a^{n+2m-2} \frac{\gamma^2}{\beta^2} + \frac{n(n+3m-1)(n+3m-2)}{1.2.3} a^{n+3m-3} \frac{\gamma^3}{\beta^3} + \&c.$$

 or, substituting $\frac{a}{\beta}$ for a

$$y^n = \frac{a^n}{\beta^n} \left\{ 1 + \frac{n}{1} \frac{a^{m-1} \gamma}{\beta^m} + \frac{n(n+2m-1)}{1.2} \frac{a^{2m-2} \gamma^2}{\beta^{2m}} + \frac{n(n+3m-1)(n+3m-2)}{1.2.3} \frac{a^{3m-3} \gamma^3}{\beta^{3m}} + \&c. \right\}.$$

 The proposed equation ought to give, in general, as many values for y^n as it has roots; but the above developement expressing only one of these values it may be asked to which it corresponds. Lagrange has proved, generally, that a property of this series, when employed to express the root of an equation, is to be always equal to the least root. In the present case it is easy to verify the truth of this proposition, when $m = 2$. The equation is then $a - \beta y + \gamma y^2 = 0$, and the roots are

$$y = \frac{\beta \pm \sqrt{\beta^2 - 4a\gamma}}{2\gamma} = \frac{\beta}{2\gamma} \left\{ 1 \pm \sqrt{1 - \frac{4a\gamma}{\beta^2}} \right\}.$$

In the first place, we have

$$\left(1 - \frac{4a\gamma}{\beta^2} \right)^{\frac{1}{2}} = 1 - \frac{1}{2} \frac{4a\gamma}{\beta^2} - \frac{1}{2} \cdot \frac{1}{4} \frac{(4a\gamma)^2}{\beta^4} - \frac{1.1.3}{2.4.6} \frac{(4a\gamma)^3}{\beta^6} + \&c.$$

Hence

$$y = \frac{\beta}{2\gamma} \pm \frac{\beta}{2\gamma} \left\{ 1 - \frac{1}{2} \frac{4a\gamma}{\beta^2} - \frac{1}{2} \cdot \frac{1}{4} \frac{(4a\gamma)^2}{\beta^4} - \frac{1.1.3}{2.4.6} \frac{(4a\gamma)^3}{\beta^6} + \&c. \right\}$$

and if we take only the inferior sign, and make some numerical reductions, we find for the least root

$$y = \frac{a}{\beta} \left\{ 1 + \frac{a\gamma}{\beta^2} + \frac{2a^2\gamma^2}{\beta^4} + \&c. \right\}$$

 a series which is precisely identical with the general series found above for y^n , when we make in it $n = 1$ and $m = 2$.

Lagrange has shown, moreover, that by proper transformations of a proposed equation, when it is algebraical,

Integral
Calculus.

it is possible to derive from the above theorem the developement of each root. (See *Mémoires de l'Académie de Berlin*, 1768, et le *Traité de la Résolution des Equations Numériques*, note XI.) Part III.

(242.) From the theorems which have been demonstrated, we may easily deduce a general formula for the inversion of series. Let $z = a y + \beta y^2 + \gamma y^3 + \&c.$ be given, and let it be proposed to find the value of y in a series of terms arranged according to the powers of z . We shall first put the given equation under the form

$$\frac{z}{a} - y - \frac{\beta}{a} y^2 - \frac{\gamma}{a} y^3 + \&c. = 0,$$

and if then we compare it with the equation $a - y + \phi(y) = 0$, we have $a = \frac{z}{a}$, and $\phi(y) = -\frac{y^2}{a} (\beta + \gamma y + \delta y^2 + \&c.)$. Hence, if we put before the sign of differentiation the constant factor $-\frac{1}{a}$, which multiplies $y^2 (\beta + \gamma y + \delta y^2 + \&c.)$ we shall find

$$y^n = a^n - \frac{n}{1 \cdot a} a^{n+1} (\beta + \gamma a + \delta a^2 + \&c.) + \frac{n}{1 \cdot 2 \cdot a^2} d \frac{a^{n+2} (\beta + \gamma a + \delta a^2 + \&c.)}{d a} \\ + \frac{n}{1 \cdot 2 \cdot 3 \cdot a^3} d^2 \frac{a^{n+3} (\beta + \gamma a + \delta a^2 + \&c.)^2}{d a^2} + \&c.$$

And if we make $n = 1$, and perform the operations indicated, we shall find after reduction

$$y = \frac{1}{a} z - \frac{\beta}{a^2} z^2 + \left(\frac{2\beta^2 - a\gamma}{a^5} \right) z^3 - \left(\frac{5\beta^3 - 5a\beta\gamma + a^2\delta}{a^7} \right) z^4 \\ + \left(\frac{14\beta^4 - 21a\beta^2\gamma + 6a^2\beta\delta + 3a^2\gamma^2 - a^3\epsilon}{a^9} \right) z^5 + \&c.$$

(243.) Let us take for second example the equation

$$u + \frac{d u}{d x} h + \frac{d^2 u}{d x^2} \frac{h^2}{1 \cdot 2} + \frac{d^3 u}{d x^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \&c. \dots = 0,$$

and let it be proposed to find an expression for h in terms of u , and its differential coefficients.

If we put $p, q, r, s, \&c.$ for the differential coefficients of u , and compare the equation with $a - y + \phi(y) = 0$, we find $a = -\frac{u}{p}$, and $\phi(y) = -\frac{1}{p} \left\{ q \frac{h^2}{1 \cdot 2} + r \frac{h^3}{1 \cdot 2 \cdot 3 \cdot 4} + \&c. \right\}$ and therefore we shall get

$$h = -\left\{ \frac{u}{p} + \frac{q}{p^2} \cdot \frac{u^2}{1 \cdot 2} + \left(\frac{3q^2 - pr}{p^5} \right) \frac{u^3}{1 \cdot 2 \cdot 3} + \left(\frac{3 \cdot 5 q^3 - 2 \cdot 5 p q r + p^2 s}{p^7} \right) \frac{u^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c. \right\}.$$

This reverted series, which is used in the resolution of numerical equations, may be put under a very elegant and remarkable form. If we make $A = -\frac{d x}{d u} = -\frac{1}{p}$, then

$$h = A u + \frac{A d A}{d x} \cdot \frac{u^2}{1 \cdot 2} + \frac{A d (A d A)}{d x^2} \frac{u^3}{1 \cdot 2 \cdot 3} + \frac{A d \{ A d (A d A) \}}{d x^3} \cdot \frac{u^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.$$

To show the application of this to the resolution of numerical equations, let us suppose $u = x^4 - 2x^2 + 4x - 8$; we shall find by means of this series

$$h = -\left\{ \frac{1}{2^3 (x^3 - x + 1)} \cdot \frac{u}{1} + \frac{3x^2 - 1}{2^4 (x^3 - x + 1)^2} \frac{u^2}{1 \cdot 2} + \frac{(21x^4 - 12x^2 - 6x + 3)}{2^6 (x^3 - x + 1)^3} \frac{u^3}{1 \cdot 2 \cdot 3} + \&c. \right\};$$

if a be a root of the equation $u = 0$, and x be an approximate value of a , we shall find

$$a = x + h = x - \frac{1}{2^3 (x^3 - x + 1)} \cdot \frac{u}{1} - \frac{(3x^2 - 1)}{2^4 (x^3 - x + 1)^2} \frac{u^2}{1 \cdot 2} - \&c.$$

Thus if $x = \frac{3}{2}$, and therefore $u = -\frac{23}{16}$, we shall get $a = \frac{3}{2} + \frac{1}{8} - \frac{1}{64} + \frac{391}{135424} - \&c. = 1.61$ nearly.

If we substitute this new value of x , and confine ourselves to four terms of the series, as before, we shall find $a = 1.6117662$: and by a continuation of this process, the approximation may be carried to any required degree of accuracy.

(244.) These theorems possess especially great advantages with respect to equations involving transcendental functions. Let us take for example the equation $y = a + a \sin y$, which expresses a general relation of the first degree between an arc and its sine, and let it be proposed to find the developement of ly . The comparison between this equation and $a - y + \phi(y) = 0$ gives $\phi(y) = a \sin y$, and we have besides in this case $\psi(y) = ly$; therefore we shall find

$$ly = la + \frac{a}{1} \frac{\sin a}{a} + \frac{a^2}{1 \cdot 2} \frac{d \frac{(\sin a)^2}{a}}{d a} + \frac{a^3}{1 \cdot 2 \cdot 3} \frac{d^2 \frac{(\sin a)^3}{a}}{d a^2} + \&c.$$

Integral
Calculus.

In the same manner the equation $u = nt + e \sin u$, of so much use in Astronomy, gives, after substituting for the powers of the sines their values in function of the sines of the multiples of the arc, Part III.

$$u = nt + e \sin nt + \frac{e^2}{1.2.2} 2 \sin 2nt + \frac{e^3}{1.2.3.2^2} (3^2 \sin 3nt - 3 \sin nt) + \frac{e^4}{1.2.3.4.2^3} (4^3 \sin 4nt - 4 \cdot 2^3 \sin 2nt) + \&c.$$

(245.) Burmann has given on the developement of functions in series a very general theorem, from which that of Lagrange and a great many others may be deduced.

Let X designate any function of x , and let this last quantity be a function of u ; the problem to be resolved is to develop X according to the powers of u . If x were eliminated from X , then this quantity would be an explicit function of u , and Maclaurin's Theorem would give immediately

$$X = T + T' \frac{u}{1} + T'' \frac{u^2}{1.2} + T''' \frac{u^3}{1.2.3} + \&c.$$

a formula in which $T^{(n)} = \frac{d^n X}{d u^n}$ when u has been made equal to nothing after the differentiations.

The theorem of Burmann has for its object to transform the above formula into

$$\frac{d^n X}{d u^n} = \frac{d^{n-1} \left[\frac{d X}{d z} \left(\frac{z}{u} \right)^n \right]}{d z^{n-1}} \dots \dots (1),$$

u and z being two functions of x which vanish at the same time, but the ratio of which remains a finite quantity. The demonstration depends on the two following propositions.

1. If, after the differentiations have been performed, considering u as an implicit function of z , we make $z = 0$, we have

$$\frac{d^n u^r \left(\frac{z}{u} \right)^n}{d z^n} = n(n-1)(n-2) \dots (n-r+1) \frac{d^{n-1} \left(\frac{z}{u} \right)^{n-1}}{d z^{n-1}} \dots \dots (2),$$

for supposing $\frac{z}{u} = p$, if we replace u by its value in z , and if we develop p^{n-r} in a series of terms arranged according to the ascending powers of z , we shall find

$$u^r \left(\frac{z}{u} \right)^n = z^r \left(\frac{z}{u} \right)^{n-r} = z^r (A + A' z + A'' z^2 + \dots + A^{(m)} z^m + \&c.);$$

and taking the n^{th} differential coefficient of the general term, we shall obtain

$$(r+m)(r+m-1) \dots (r+m-n+1) A^{(m)} z^{r+m-n},$$

an expression which always vanishes when $z = 0$, except when $r+m-n = 0$, in which case we have

$$\frac{d^n u^r \left(\frac{z}{u} \right)^n}{d z^n} = n(n-1)(n-2) \dots 1 A^{(n-r)};$$

but $A^{(n-r)} = \frac{d^{n-r} p^{n-r}}{1.2.3 \dots n-r}$; therefore $\frac{d^n u^r \left(\frac{z}{u} \right)^n}{d z^n} = n(n-1)(n-2) \dots (n-r+1) \frac{d^{n-r} \left(\frac{z}{u} \right)^{n-r}}{d z^{n-r}}.$

Under the same condition

$$\frac{d^{n-1} u^r d p^n}{d z^n} = \frac{d^n u^r p^n}{d z^n} \dots \dots (3),$$

for

$$\frac{u^r d p^n}{d z} = n u^r p^{n-1} \frac{d p}{d z} = n u^r \left(\frac{z}{u} \right)^{n-1} \frac{d p}{d z} = n z^r \left(\frac{z}{u} \right)^{n-r-1} \frac{d p}{d z} = n z^r p^{n-r-1} \frac{d p}{d z} = \frac{n}{n-r} z^r \frac{d p^{n-r}}{d z},$$

putting for p^{n-r} its development, and performing the differentiation which is indicated, we find

$$\frac{u^r d p^n}{d z} = \frac{n}{n-r} z^r (A' + 2 A'' z + 3 A''' z^2 + \dots + m A^{(m)} z^{m-1} + \&c.);$$

differentiating $n-1$ times this equation, we shall have, according to the value of the general term $d^{n-1} \frac{m n}{n-r}$

$A^{(n-r-1)} z^{r+m-n-1}$, a result which will be equal to nothing as long as we have $r+m-1 > n-1$, and also when we have $r+m-1 < n-1$; thus we have simply

$$\frac{d^{n-1} u^r d p^n}{d z^n} = n(n-1)(n-2) \dots 1 A^{(n-r)} = \frac{d^n u^r p^n}{d z^n}.$$

This supposes that we have $r < n$; for when $r = n$, we have

$$\frac{u^r d \cdot p^n}{dz} = n u^n p^{n-1} \frac{dp}{dz} = n u z^{n-1} \frac{dp}{dz};$$

and when we shall have differentiated $n-1$ times this product, there will remain still in every term at least one of the factors z or u : therefore it will vanish when $z=0$. The same thing will happen if we have $r < n$, on account of the factor u , which will remain in every term.

2. This understood, let us consider the function $X p^n$, let us put in it the developement of X , indicated above, and let us take the differential coefficient of the n^{th} order with respect to z ; we shall obtain

$$\frac{d^n X p^n}{dz^n} = T \frac{d^n \cdot p^n}{dz^n} + \frac{T'}{1} \frac{d^n \cdot u p^n}{dz^n} + \frac{T''}{1 \cdot 2} \frac{d^n \cdot u^2 p^n}{dz^n} + \dots + \frac{T^{(r)}}{1 \cdot 2 \cdot 3 \dots r} \frac{d^n \cdot u^r p^n}{dz^n} + \&c.$$

and substituting for

$$\frac{d^n \cdot u p^n}{dz^n}, \quad \frac{d^n \cdot u^2 p^n}{dz^n}, \dots, \frac{d^n \cdot u^r p^n}{dz^n},$$

their values derived from equation (2), we shall have

$$\frac{d^n \cdot X p^n}{dz^n} = T \frac{d^n \cdot p^n}{dz^n} + \frac{n}{1} T' \frac{d^{n-1} \cdot p^{n-1}}{dz^{n-1}} + \frac{n(n-1)}{1 \cdot 2} \frac{d^{n-2} \cdot p^{n-2}}{dz^{n-2}} + \dots + n T^{(n-1)} \frac{dp}{dz} + T^{(n)},$$

an expression of which the numerical coefficients are those of the binomial formula, and which terminates in the same way because the exponent n is an integer.

If we operate in the same manner on the equation

$$\frac{d^{n-1} \cdot X d \cdot p^n}{dz^{n-1}} = T \frac{d^n \cdot p^n}{dz^n} + \frac{T'}{1} \frac{d^{n-1} \cdot u d \cdot p^n}{dz^n} + \frac{T''}{1 \cdot 2} \frac{d^{n-1} \cdot u^2 d \cdot p^n}{dz^n} + \&c$$

observing that by equations

$$\frac{d^{n-1} \cdot u^r d \cdot p^n}{dz^{n-1}} = \frac{d^n \cdot u^r p^n}{dz^n} = n(n-1) \dots (n-r+1) \frac{d^{n-r} \cdot p^{n-r}}{dz^{n-r}}$$

We shall find

$$\frac{d^{n-1} \cdot X d \cdot p^n}{dz^{n-1}} = T \frac{d^n \cdot p^n}{dz^n} + \frac{n}{1} T' \frac{d^{n-1} \cdot p^{n-1}}{dz^{n-1}} + \frac{n(n-1)}{1 \cdot 2} T'' \frac{d^{n-2} \cdot p^{n-2}}{dz^{n-2}} + \dots + \frac{n}{1} T^{(n-1)} \frac{dp}{dz},$$

an expression which is terminated one term before the preceding, because

$$\frac{d^{n-1} \cdot u^n d \cdot p^n}{dz^{n-1}} = 0.$$

Now, if we take the difference of these two expressions, we shall have

$$\frac{d^n \cdot X p^n}{dz^n} - \frac{d^{n-1} X d p^n}{dz^{n-1}} = T^n = \frac{d^n X}{d u^n};$$

and if we reduce to one the two terms of the left side of the equation, we shall find

$$\frac{d^{n-1} \left(\frac{d \cdot X p^n}{dz} - X \frac{d \cdot p^n}{dz} \right)}{dz^{n-1}} = \frac{d^{n-1} \left(\frac{d X}{dz} \cdot p^n \right)}{dz^{n-1}} = \frac{d^n X}{d u^n}$$

which is the theorem of Burmann.

(246.) By means of this theorem we first obtain that of Lagrange, by making

$$z = x - a, \text{ and } u = \frac{x-a}{\phi(x)}, \text{ hence } p = \phi(x), dz = dx, \text{ and } \frac{d^n X}{d u^n} = \frac{d^{n-1} \left(\phi^n \frac{d X}{dx} \right)}{d x^{n-1}} = \frac{d^{n-1} (\phi^n \psi')}{d x^{n-1}},$$

changing X into $\psi(x)$, and then making $z=0$ and $x=a$ after the differentiations; we shall observe afterwards that the equation $u = \frac{x-a}{\phi(x)}$ gives $x = a + u \phi(x)$; and if we make $u=1$, write y instead of x , we shall find the formula of Lagrange.

Let us assume $X = \psi(x)$, $u = \phi(x) - \phi(a)$, $z = x - a$, then we shall obtain

$$\psi(x) = \psi(a) + \frac{T'}{1} [\phi(x) - \phi(a)] + \frac{T''}{1 \cdot 2} [\phi(x) - \phi(a)]^2 + \&c. \dots +$$

an expression in which

$$T^{(n)} = \frac{d^{n-1}}{d x^{n-1}} \left\{ \left(\frac{\phi(x) - \phi(a)}{x-a} \right)^n \psi'(x) \right\}.$$

(247.) If we suppose $\phi(a) = 0$, the preceding formulæ are simplified, and become

$$\psi(x) = \psi(a) + \frac{T'}{1} \phi(x) + \frac{T''}{1 \cdot 2} \phi(x)^2 + \frac{T'''}{1 \cdot 2 \cdot 3} \phi(x)^3 + \&c. \text{ where } T^{(n)} = \frac{1}{d x^{n-1}} d^{n-1} \left[\left(\frac{\phi(x)}{x-a} \right)^n \psi'(x) \right].$$

Which remarkable formula is the developement of $\psi(x)$ in a series of terms arranged according to the powers

Part III.
Integral
Calculus.

 of $\phi(x)$; and since a designates a value of x which satisfies the equation $\phi(x) = 0$, there will be as many of these developements as this equation has roots. Let us take for an example

Part III.

$$\psi(x) = b^x, \quad \phi(x) = x c^x.$$

We have then

$$a = 0, \quad \psi'(x) = b^x l b, \quad T^{(n)} = \frac{1}{d x^{n-1}} d^{n-1} [c^{-nx} b^x l b] = l b (l b - n l c)^{n-1} c^{-n} b^x;$$

 and if we make $x = 0$, after the differentiations, we find

$$b^x = 1 + l b \frac{x c^x}{1} + l b (l b - 2 l c) \frac{x^2 c^{2x}}{1.2} + l b (l b - 3 l c)^2 \frac{x^3 c^{3x}}{1.2.3} + \&c.$$

 (See for more details on the developements of functions in series *Mémoire de M. Servois*, in the Vth Volume of the *Annales de Mathématiques*, and the last Chapter of the *Disquisitiones Analyticae* of Pfaff.)

 (248.) The first problem which Arbogast proposes to resolve in his *Traité des Dérivations*, is to develop according to the powers of x , an expression such as

$$\phi(a + \beta x + \gamma x^2 + \delta x^3 + \&c \dots),$$

 ϕ designating any function whatever. It is obvious that this developement is, with respect to Taylor's Theorem, exactly the same thing as the developement of the polynom

$$(a + \beta x + \gamma x^2 + \delta x^3 + \&c.)^m$$

with respect to the binomial theorem.

To apply Taylor's Theorem to the above proposed function, it is first necessary to put it under the form of a binomial quantity. Let us assume

$$a + \beta x + \gamma x^2 + \delta x^3 + \&c. = a + t x,$$

 so that $t = \beta + \gamma x + \delta x^2 + \&c. \dots$; considering then a as the variable, and $t x$ as its increment, we shall have by Taylor's Theorem, representing for the sake of brevity $\phi(a)$ by ϕ ,

$$\phi(a + t x) = \phi + \frac{d \phi}{d a} t x + \frac{d^2 \phi}{d a^2} \frac{t^2 x^2}{1.2} + \frac{d^3 \phi}{d a^3} \frac{t^3 x^3}{1.2.3} + \&c.$$

 and thus the developement depends on those of the powers of the algebraical function t .

 If it were required to find the coefficient of x^n , it would be necessary and sufficient to take the first term of the n^{th} power of t , the second of the $(n-1)^{\text{th}}$, and so on. Let us represent the developement of t^r by

$$T_0^r + T_1^r x + T_2^r x^2 + \dots + T_r^r x^r,$$

 so that the superior index of the letter T shows the degree of the power to which is raised the polynom t , and the inferior that of the letter x , according to this notation, the general term of the developement of $\phi(a + t x)$ will be

$$\left\{ \frac{d^n \phi}{1.2 \dots n d a^n} T_0^n + \frac{d^{n-1} \phi}{1.2 \dots (n-1) d a^{n-1}} T_1^{n-1} + \frac{d^{n-2} \phi}{1.2 \dots (n-2) d a^{n-2}} T_2^{n-2} + \dots + \frac{d \phi}{d a} T_{n-1}^1 \right\} x^n.$$

 Therefore if we had the general expression of the coefficient T_r , we should know the general term of the developement of $\phi(a + \beta x + \gamma x^2 + \dots)$; but this research cannot find place here, on account of its length, and, besides, it belongs rather to another branch of analysis, called *Combinatorial analysis*.

On the Values assumed, in certain Cases, by Differential Coefficients, and on the Value of Functions, which under certain Circumstances become $\frac{0}{0}$.

(249.) The formulæ of developement we have obtained in the preceding pages, depend all on the particular values which the differential coefficients of the function to be developed take, when nothing or certain constant quantities are substituted for the variables. They can, therefore, only be applied in those cases where the values of the differential coefficients are finite, a circumstance which does not always take place. The function

 $y = b + (x - a)^{\frac{1}{n}}$, n being supposed to be greater than one, offers a very simple example of it; for if we differentiate it, we find

$$\frac{d y}{d x} = \frac{1}{n} (x - a)^{\frac{1}{n} - 1} = \frac{1}{n (x - a)^{\frac{n-1}{n}}};$$

 and if we make $x = a$, a supposition which gives $y = b$, the value of $\frac{d y}{d x}$ becomes infinite. It will be the same

 thing with respect to the differential coefficients of superior order, for each differentiation increases the exponent of $(x - a)$ in the denominator. In the numbers (234.) and (235.) we have assumed that the functions which we have considered could be developed according to the integral powers of the variables; but this supposition may be wrong, and we shall know that it is so, when in calculating the coefficients of the developement, by

means of the values of differential coefficients of the function, corresponding to a particular value of x , we shall find that any one of them becomes infinite.

To make this clearer, let $y = f(x)$, and let us suppose that the developement of $f(a+x)$ according to the powers of x contains fractional exponents of x . Let

$$f(x+a) = f(a) + Px + Qx^2 + \dots + Tx^n + Vx^{\frac{m+1}{m}} + \&c.$$

If in the series no fractional exponent occurs before the term $Vx^{\frac{m+1}{m}}$, all the coefficients $P, Q, \dots, T$, may be calculated by means of the method explained in the numbers (234.) and (235.), and thus we shall have

$$P = \frac{dy}{dx}, \quad Q = \frac{d^2y}{dx^2}, \quad \dots \quad T = \frac{d^ny}{dx^n};$$

but if we try to employ the same method to find the value of the coefficient V , by taking the differential coefficient of the $(n+1)^{th}$ order, we obtain

$$\frac{d^{n+1}f(x+a)}{dx^{n+1}} = \left(\frac{mn+1}{m}\right) \left(\frac{mn+1}{m} - 1\right) \left(\frac{mn+1}{m} - 2\right) \dots 1 V x^{\frac{1-m}{m}} + \&c.$$

Now since the exponent $\frac{1-m}{m}$ of x is negative, the supposition $x=0$ will make the right side of the equation

infinite; therefore, this same supposition must also render equal to infinity the differential coefficient $\frac{d^{n+1}f(x+a)}{dx^{n+1}}$ and the value of V cannot be obtained by means of it. It will be the same with respect to the differential coefficients of superior order.

(250.) There is here, however, an apparent difficulty. We have observed before (231.) that the developement of $f(a+x)$ could be derived from the developement of $f(x+h)$, given by Taylor's Theorem, by substituting in that developement a for x , and x for h . It has been proved, besides, that the series of Taylor, as long as both x and h remain variable, can only contain integral powers of h . It might then be imagined that the developement of $f(a+x)$ derived from it, by giving a particular value to one of the variables, ought also to contain only integral powers of x , and be found in all cases as in number (235.)

It must be observed that the developement of $f(x+h)$ ought to have neither more nor less values than the function itself; and it is for that reason, as it has been explained before, that as long as x and h remain variables, the developement of $f(x+h)$ cannot contain a term with a fractional exponent of h . But the reasoning which has been used can no longer be applied, when one of the variables receives a particular value, because the number of values of the coefficients of the series may become less than the number of values of the function, and this deficiency must be made up by the terms containing fractional exponents.

Let us take, as an example, the function $y = b \pm \sqrt{x-a}$; by taking alternately the radical with the sign $+$ and with the sign $-$, we shall have

$$\frac{dy}{dx} = \pm \frac{1}{2} (x-a)^{-\frac{1}{2}}, \quad \frac{d^2y}{dx^2} = \mp (x-a)^{-\frac{3}{2}}, \quad \&c.$$

and if $y' = b \pm \sqrt{x+h-a}$, we shall find

$$y' = b + \sqrt{x-a} + \frac{1}{2} (x-a)^{-\frac{1}{2}} h - (x-a)^{-\frac{3}{2}} \cdot \frac{h^2}{1 \cdot 2} + \&c. \quad \text{or}$$

$$y' = b - \sqrt{x-a} - \frac{1}{2} (x-a)^{-\frac{1}{2}} h + (x-a)^{-\frac{3}{2}} \frac{h^2}{1 \cdot 2} - \&c.$$

These developements contain only integral powers of h , and we see a reason for it in this circumstance that no particular value being substituted for x , the coefficients of the series, which are the differential coefficients of the function, have as many values as y . But if we suppose $x=a$, the radical quantity $\sqrt{x-a}$ in the function y disappears, and the two values of y become equal to b . Therefore if the developement of y' were still of the form $y' + Ph + Qh^2 + Rh^3 + \&c.$ it would have only one value, whilst y' would have two, since it becomes then equal to $b \pm \sqrt{h}$. Hence the form of the developement must change, and the series must contain fractional powers of h . Analysis shows this change of form in giving infinite values for the differential coefficient $\pm \frac{1}{2} (x-a)^{-\frac{1}{2}}$ when $x=a$; and the truth of this reasoning is verified since, in that case, the developements of

the two values of y' are simply $b + h^{\frac{1}{2}}$ and $b - h^{\frac{1}{2}}$.

(251.) The methods explained (234.) and (235.) cannot therefore be used to develop $f(x)$ or $f(a+x)$ according to the powers of x , when these functions are such that the series which represent them ought to contain fractional powers of x . Neither can they be used if the series ought to contain negative powers of x , since we have assumed that the functions $f(x)$ and $f(a+x)$ did not become infinite for $x=0$. To develop the functions in these cases some other process must be used. The coefficients of the developement may be determined, for instance, in the following manner.

Let us assume generally, $f(a+x) = Ax^{-a} + Bx^{\beta} + Cx^{\gamma} + \&c.$ the exponents $a, \beta, \gamma, \&c.$ being supposed to be arranged in ascending order.

To find the value of a , let such a power of x be found as being multiplied by the given function $f(a+x)$ will give a product which becomes neither $=0$, nor infinite when $x=0$, the exponent of this power will be a .

Integral
Calculus.

For if not, let it be κ , then $f(a+x)x^\kappa = Ax^{\kappa-a} + Bx^{\kappa+\beta} + Cx^{\kappa+\gamma} + \&c.$ The exponents are arranged in an ascending order, therefore they are all positive if $\kappa - a$ is positive, and all the terms vanish when $h = 0$, which is contrary to hypothesis. If κ were $< a$, then the first exponent at least is negative, and the series would become infinite for $x = 0$, which is also contrary to hypothesis. Therefore $\kappa = a$. The exponent a being thus found, the coefficient A is what $f(a+x)x^a$ becomes when $x = 0$.

Having thus determined A and a , the first term of the developement becomes known, and being brought in the left side of the equation gives $f(a+x) - Ax^{-a} = Bh^\beta + Ch^\gamma + Dh^\delta + \&c.$ The left side of this equation being known, if b be negative, it may be found by the same process as that used to determine a . It may be known whether it be negative by determining if $x = 0$ render $f(a+x) - Ax^{-a}$ infinite. If $h = 0$ render this $= 0$, then β is > 0 , and we shall presently explain the method of determining it. If $h = 0$ do not render $f(a+x) - Ax^{-a}$ either $= 0$ or infinite, then $b = 0$, and the value of $f(a+x) - Ax^{-a}$ when $x = 0$ is the value of B . The other exponents, when negative, and coefficients may be determined in a similar way.

There is another method of finding the negative exponents when they are integers founded on the application of the Integral Calculus. Let $f(a+x) = Ax^{-a} + Bx^\beta + Cx^\gamma + \&c.$ Multiply both sides by dx , and integrate,

we find $\int f(a+x) dx = \frac{1}{1-a} Ax^{1-a} + \frac{1}{1+\beta} Bx^{1+\beta} + \frac{1}{1+\gamma} Cx^{1+\gamma} + \&c.$ Let $\int f(a+x) dx = f_1(a+x)$.

Multiplying again by dx , and integrating, we find $\int f_1(a+x) dx = \frac{1}{1-a} \cdot \frac{1}{2-a} Ax^{2-a} + \frac{1}{1-\beta} \cdot \frac{1}{2-\beta}$

$Bx^{2+\beta} + \&c.$ Let this process be continued until an integral be found, which will neither vanish nor be infinite when $x = 0$. If one be found which vanishes when $x = 0$, immediately next to another which becomes infinite for that supposition, then a is a fraction whose value is between the number of integrations and the integer next below it. If the value of the integral be finite, then a is equal to the number of integrations.

If $x = 0$ does not render infinite $f(a+x)$, the developement cannot contain negative powers of x , and the series will be $f(a+x) = f(a) + Bx^\beta + Cx^\gamma + Dx^\delta + \&c.$ Differentiating we get $\frac{df(a+x)}{dx} = \beta Bx^{\beta-1} + \gamma$

$Cx^{\gamma-1} + \delta Dx^{\delta-1} + \&c.$ Now, if $x = 0$ renders $\frac{df(a+x)}{dx}$ infinite, $-\beta - 1$ is negative, and therefore β is a fraction. To determine its value, let $f(a)$ be brought in the left side of the equation, we shall have $f(a+x) - f(a) = Bx^\beta + Cx^\gamma + Dx^\delta + \&c.$ and then let that power of x be found, by which $f(a+x) - f(a)$ being divided, the quotient will neither vanish nor become infinite when $x = 0$. The exponent of that power will obviously be equal to β .

If $x = 0$ did not render infinite neither $f(a+x)$ nor $\frac{df(a+x)}{dx}$, but $\frac{d^2f(a+x)}{dx^2}$, then we would have $\beta = 1$,

B equal to the value of $\frac{d^2f(a+x)}{dx^2}$ for $x = 0$, and γ would be a fraction > 1 whose value might be determined in a similar manner.

It follows, therefore, in general, that if $x = 0$ renders the n^{th} differential coefficient of $f(a+x)$ infinite, but none of the preceding ones, the methods given (235.) will give the true developement as far as the n^{th} term inclusive; but the exponent of x in the $(n+1)^{\text{th}}$ term is a fraction whose value is between the integers n and $n+1$, and which may be determined by the method explained above.

(252.) When for a particular value given to a variable x , a function of that variable takes the form $\frac{0}{0}$, it has always a determinate value which is either a finite quantity, or equal to nothing or infinity. The Differential Calculus gives, in most cases, very simple means to find the true value of the function. Let $\frac{X}{X'}$ be a fraction the numerator and denominator of which vanish for $x = a$. If we substitute $a+h$ for x , the functions X and X' may always be supposed to be developed in series arranged according to the ascending powers of h . Let these series be represented by $Ah^a + Bh^\beta + \&c.$, and $A'h^{a'} + B'h^{\beta'} + \&c.$, since they must become equal to nothing when $h = 0$, which answers to $x = a$. We shall have therefore $\frac{Ah^a + Bh^\beta + \&c.}{A'h^{a'} + B'h^{\beta'} + \&c.}$

instead of the proposed function. If in this fraction we suppose $h = 0$, we must find the same result as if we made $x = a$ in $\frac{X}{X'}$. Three cases may occur: we may have $a > a'$, $a = a'$, $a < a'$; in the two first the preceding expression may be written in the following manner:

$$\frac{Ah^{a-a'} + Bh^{\beta-a'} + \&c.}{A' + B'h^{\beta'-a'} + \&c.}$$

Under this form it is easy to see that if we have $a > a'$, the supposition $h = 0$ will make the fraction equal to nothing, and that it will be reduced to $\frac{A}{A'}$, when we shall have $a = a'$. On the contrary if we have $a < a'$ the fraction may be written as follows, dividing both numerator and denominator by h^a ,

Integral
Calculus.

$$\frac{A + B h^{a-a} + \&c.}{A' h^{a'-a} + B' h^{a'-a} + \&c.},$$

and will become infinite by the supposition of $h = 0$.

In each of the three cases, the true value of the fraction depends only upon the first term of each series.

(253.) To render the case more general we have supposed that the functions obtained by substituting $a + h$ for x into X and X' , could not be developed according to the integral powers of h , but if this can be done, then the fraction becomes (235.)

$$\frac{X_a + \frac{dX}{da} h + \frac{d^2 X}{da^2} \frac{h^2}{1.2} + \&c.}{X'_a + \frac{dX'}{da} h + \frac{d^2 X'}{da^2} \frac{h^2}{1.2} + \&c.},$$

$X_a, X'_a, \frac{dX}{da}, \frac{dX'}{da}, \&c.$ designating what $X, X', \frac{dX}{dx}, \frac{dX'}{dx}$ become when a is substituted for x . By hypothesis

$X_a = 0$, and $X'_a = 0$. If, at the same time, all the differential coefficients in the numerator up to the m^{th} order, and all those of the denominator up to the n^{th} order, become equal to nothing, the fraction will be reduced to

$$\frac{\frac{d^m X}{dx^m} \cdot \frac{h^m}{1.2 \dots m} + \&c.}{\frac{d^n X'}{da^n} \cdot \frac{h^n}{1.2 \dots n} + \&c.},$$

a quantity which will be equal to nothing if we have $m > n$, infinite if we have $m < n$, and equal to

$$\frac{\frac{d^m X}{dx^m}}{\frac{d^n X'}{da^n}},$$

if $m = n$. Hence, to obtain the true value of a function which becomes $\frac{0}{0}$ when a particular value is substituted for x , it is generally sufficient to find the successive differential coefficients of its numerator and its denominator, till we find for the one, or for the other, a result which does not become equal to nothing for the particular value of x ; the proposed function will become infinite in the first case, equal to nothing in the second; and if it have a finite value, the two differential coefficients of the numerator and denominator which do not vanish will be of the same order, and their quotient will be the value required.

This rule supposes that none of the coefficients which are successively calculated becomes infinite for the particular value of x ; if that were the case, it would indicate that one or both the functions resulting from the substitution of $a + h$ for x in X and X' cannot be developed according to the integral powers of h , and they should consequently be treated as in number (252.).

A few examples will elucidate the preceding rule.

Example 1. Let the proposed function be $\frac{a^3 - x^3}{a^2 - x^2}$, the supposition $x = a$ reduces it to the form $\frac{0}{0}$, it is required to find its true value.

$\frac{d(a^3 - x^3)}{dx} = -3x^2$ and for $x = a$ becomes $-3a^2$; $\frac{d(a^2 - x^2)}{dx} = -2x$, and for $x = a$ becomes $-2a$; therefore the true value is $\frac{3a^2}{2a} = \frac{3a}{2}$.

In the same manner the true value of $\frac{x^n - 1}{x - 1}$ for $x = 1$ is found to be equal to n .

Example 2.

$$\frac{ax^3 - 2acx + ac^3}{bx^3 - 2bcx + bc^3}, \text{ for } x = c.$$

$\frac{d(ax^3 - 2acx + ac^3)}{dx} = 2ax - 2ac$, which quantity becomes nothing for $x = c$; $\frac{d(bx^3 - 2bcx + bc^3)}{dx} = 2bx - 2bc$, and this quantity is also reduced to nothing when $x = c$. It is therefore necessary to differentiate again; we find then $\frac{d(2ax - 2ac)}{dx} = 2a$, and $\frac{d(2bx - 2bc)}{dx} = 2b$, therefore the true value of the fraction is $\frac{a}{b}$.

Example 3.

$$\frac{a^x - b^x}{x} \text{ for } x = 0.$$

$\frac{d(a^x - b^x)}{dx} = a^x \log a - b^x \log b$, $\frac{dx}{dx} = 1$. Therefore the true value is $\log a - \log b = \log\left(\frac{a}{b}\right)$.

Part III.
Integral
Calculus.

Part III.

In the same manner it will be found that the true value of $\frac{1 - \sin x + \cos x}{\sin x + \cos x - 1}$ for $x = \frac{\pi}{2}$ is equal to -1 .

Example 4. Sometimes it will be found that it is simpler, instead of calculating the differential coefficients, to develop the numerator and denominator after having substituted $a + h$ for x . The fraction

$$\frac{x^3 - 4ax^2 + 7a^2x - 2a^3 - 2a^2\sqrt{2ax - a^2}}{x^2 - 2ax - a^2 + 2a\sqrt{2ax - a^2}}$$

is an example of this case. It is only after differentiating four times the numerator and denominator that the true value, corresponding to $x = a$, is found. If instead of that process we substitute $a + h$ for x and reduce, we find first

$$\frac{2a^3 + 2a^2h - ah^2 + h^3 - 2a^2\sqrt{a^2 + 2ah}}{-2a^2 + h^2 + 2a\sqrt{a^2 - h^2}}.$$

Reducing in series the two radical quantities we shall find

$$\sqrt{a^2 + 2ah} = a + h - \frac{h^2}{2a} + \frac{h^3}{2a^2} - \frac{5h^4}{8a^3} + \&c. \quad \sqrt{a^2 - h^2} = a - \frac{h^2}{2a} - \frac{h^4}{8a^3} + \&c.$$

The substitution of these two series in the preceding fraction, and the supposition of $h = 0$, will give $-5a$ for the true value required.

Example 5. Let the proposed function be $\frac{(x^2 - a^2)^{\frac{3}{2}}}{(x - a)^{\frac{3}{2}}}$, it is required to find the true value for $x = a$. Here the

differential coefficients become infinite, and therefore it is necessary to use the general method. By substituting $a + h$ for x and developing, we find thus that the function becomes

$$\frac{(2ah + h^2)^{\frac{3}{2}}}{h^{\frac{3}{2}}} = (2a + h)^{\frac{3}{2}}.$$

Making then $h = 0$, we obtain for the true value $(2a)^{\frac{3}{2}}$.

(254.) A function may, for a particular value of x , take various indeterminate forms, differing in appearance from $\frac{0}{0}$, but which in reality come to the same thing.

1. The numerator and denominator of the fraction $\frac{X}{X'}$ may become both infinite at the same time. This case

may immediately be reduced to the one we have already examined by putting the fraction under the form $\frac{\left(\frac{1}{X'}\right)}{\left(\frac{1}{X}\right)}$.

2. In the product PQ of two functions of x , one of the factors may become equal to nothing for a particular value of x , and the other infinite; such is the case, for instance, for the expression $(1 - x) \tan \frac{\pi x}{2}$ when $x = 1$.

To reduce this case to the form $\frac{X}{X'}$, it is sufficient to make $Q = \frac{1}{R}$, then $PQ = \frac{P}{R}$, and the rules given may be applied.

3. It may be required to find the value of the difference $P - Q$ of two functions, when they both become infinite for a particular value of the variable. If the two functions are both rational, algebraical, and integral, the only value of x for which they can become infinite is x equal to infinity, and consequently the only case in which the expression $P - Q$ may have a finite value is when $P = Q + b$, b being a constant quantity. When P and Q are fractions the denominators of which vanish, it is easy to change the proposed function into another

which becomes $\frac{0}{0}$; it is sufficient for that to bring P and Q to the same denominator. In many cases the true

value of the function may also be obtained by substituting for x the particular value increased by h , and developing according to the powers of h . Let us take as an example the function $\frac{x}{x-1} - \frac{1}{l x}$, which when $x = 1$

becomes the difference of two infinite quantities. If we substitute $1 + h$ for x , it becomes $\frac{1+h}{h} - \frac{1}{l(1+h)}$.

But $l(1+h) = h - \frac{h^2}{2} + \frac{h^3}{3} - \&c$. Therefore

$$\frac{1+h}{h} - \frac{1}{l(1+h)} = \frac{(1+h)l(1+h) - h}{hl(1+h)} = \frac{\frac{h^2}{2} - \frac{h^3}{2 \cdot 3} + \&c.}{h^2 - \frac{h^3}{1 \cdot 2} + \&c.}$$

Integral Calculus Dividing both numerator and denominator by h^2 , and making afterwards $h = 0$, we find $\frac{1}{2}$ for the true value of Part III.
 $\frac{x}{x-1} - \frac{1}{lx}$.

In an analogous manner we should find for the true value of $\frac{1}{2x^2} - \frac{\pi}{2x \tan \pi x}$, corresponding to $x = 0, \frac{\pi}{6}$.

(255) We shall take for the last example the function $\frac{lx}{x}$, the numerator and denominator of which become infinite when x is supposed to be infinite. The rule given in the preceding number does not apply, for by preparing the function in the manner indicated, and then differentiating both the numerator and denominator, we find

$$\frac{\frac{1}{x}}{\frac{1}{lx}} \text{ and } \frac{d \cdot \frac{1}{x}}{d \cdot \frac{1}{lx}} = \frac{\frac{1}{x^2}}{\frac{1}{x(lx)^2}} = \frac{(lx)^2}{x},$$

a result from which no conclusion can be drawn. The simplest means to find the true value of the function or more generally of $\frac{lx}{x^n}$, is to develop x^n , considering it as an exponential, according to the powers of n ; we have then

$$\frac{lx}{x^n} = \frac{lx}{1 + lx \cdot \frac{n}{1} + (lx)^2 \cdot \frac{n^2}{1 \cdot 2} + (lx)^3 \cdot \frac{n^3}{1 \cdot 2 \cdot 3} + \&c.} = \frac{1}{\frac{1}{lx} + \frac{n}{1} + (lx)^2 \cdot \frac{n^2}{1 \cdot 2} + (lx)^3 \cdot \frac{n^3}{1 \cdot 2 \cdot 3} + \&c.}$$

The last expression having a constant quantity for its numerator, whilst its denominator increases with the value of x , and becomes infinite at the same time as this quantity, it follows that the value of $\frac{lx}{x^n}$ has for limit nothing.

Knowing the value of $\frac{lx}{x^n}$, when x is infinite, it is easy to find the values of other analogous functions.

1. The function $\frac{x^n}{lx}$ being equal to $\left(\frac{lx}{x^n}\right)^{-1}$ becomes clearly infinite when the last becomes nothing.
2. The function $x^n lx$, which becomes nothing multiplied by an infinite quantity when $x = 0$, has for true value nothing, for if we suppose $x = \frac{1}{y}$ it is transformed into $\frac{-ly}{y^n}$. The function $x^n (lx)^r$ has also the same value for $x = 0$, for by extracting the root of the degree r , it becomes $x^{\frac{n}{r}} lx$, and the value of the root for $x = 0$ must be the same as that of the function.

On the Maxima and Minima of Functions of One Variable.

(256.) Let u be a function of the variable x , and u', u'', u''' represent the three values of u corresponding respectively with $x = a - h$, $x = a$, $x = a + h$; then if u be a continuous function for values differing but little from a (see number (71.)), and if a be such a quantity that for any very small value of h , u' and u''' be both less than u'' , then this last quantity is said to be a *maximum* of the function u . It is said to be a *minimum* when under the same circumstances u' and u''' are both greater than u'' . In the same case, a is said to be a value of the variable corresponding to a maximum or to a minimum of the function. Hence it is clear, for instance, that the value $x = a$ corresponds to a maximum or a minimum of the function $b \mp (x - a)^2$, according as the second term $(x - a)^2$ is taken with the sign $-$ or $+$. It is also obvious, that the maximum or minimum of a function, from the definition given above, is not necessarily the same thing as the greatest or least value of the function. Thus, for instance, the value b^2 of the function $b^2(x - 1) + (x - 2)^2(x - 1)^2$, corresponding to $x = 2$, is a minimum, and nevertheless the value of the same function corresponding to $x = 1$ is equal to nothing. Another consequence of the definition which has been given of the maximum or minimum of a function, is that the same function may have several maxima and minima; such is the case for the function $a + b(x - 1)^2 + c(x - 2)^2(x - 1)^4$, the values of which corresponding to $x = 1$ and $x = 2$ are both minima.

(257.) Two questions present themselves relative to the maxima and minima of functions of one variable; they are not, however, essentially distinct, and the solution given for the first shows immediately how to resolve the second. The first is to determine if a given value of the variable does or does not correspond to a maximum or a minimum of the function, and the second to find all the values of the variable answering to maxima and minima.

(258.) Let $u = f(x)$ be the proposed function, and a the given value of x . It is required to determine whether it corresponds to a maximum or a minimum. Let $a + h$ be substituted for x , in the function, and let the result be developed according to the powers of h . The development may contain fractional powers of h ; let it be represented generally by

$$f(a + h) = f(a) + P h^\alpha + Q h^\beta + R h^\gamma + \&c.$$

where α, β, γ may be integral or fractional numbers, but where they are supposed arranged in an ascending order.

Integral
Calculus.

This understood, the developement of $f(a-h)$ may be deduced from that of $f(a+h)$ by writing $-h$ instead of h , and we shall have accordingly

$$f(a-h) = f(a) + P(-h)^a + Q(-h)^b + R(-h)^c + \&c.$$

Hence the differences, $f(a) - f(a+h)$, and $f(a) - f(a-h)$, will be respectively

$$-Ph^a - Qh^b - Rh^c - \&c., \quad -P(-h)^a - Q(-h)^b - R(-h)^c - \&c.$$

Now, if a correspond to a maximum, that is if $f(a)$ be a maximum of $f(x)$, both $f(a+h)$ and $f(a-h)$ must be less than $f(a)$ for any value of h sufficiently small, and consequently the two above differences should be positive. It is obvious, in a similar manner, that they should be both negative if a correspond to a minimum. This must be true however small be the value taken for h . By taking h sufficiently small, the sign of each series will be the same as that of the first term; hence a will correspond to a maximum if both Ph^a and $P(-h)^a$ are negative, and to a minimum if they are positive.

But the two quantities Ph^a and $P(-h)^a$ cannot be of the same sign unless a should be equal to an even integral number, or to a fraction which, reduced to its lowest terms, should have an even number for numerator. In both cases we shall have

$$f(a) - f(a+h) = -Ph^a - \&c., \quad f(a) - f(a-h) = -Ph^a - \&c.$$

and $f(a)$ will be a maximum if P be negative, and a minimum if it be positive.

To make use of this observation, we shall distinguish two cases, namely, when a is an integer, and when it is a fraction.

In the first case, we have (235.) $P = \frac{1}{1 \cdot 2 \dots a} \frac{d^a u}{d a^a}$, and since the exponent a must be an even number when $f(a)$ is a maximum or a minimum, it is first necessary that the value $x = a$ should reduce to nothing the first differential coefficient $\frac{d u}{d x}$, for otherwise the developement of $f(a+h)$ would begin by the term $\frac{d u}{d a} h$, that is to say by a term containing an odd power of h . It is necessary besides that this value $x = a$ should not reduce to nothing the differential coefficient $\frac{d^2 u}{d x^2}$, or if that happen, then it should, at the same time, reduce to nothing, the third differential coefficient $\frac{d^3 u}{d x^3}$, but not the fourth $\frac{d^4 u}{d x^4}$, and, generally, it is necessary that the first differential coefficient which the supposition $x = a$ does not make equal to nothing, should be of an even order. Then if this coefficient becomes a positive quantity when x is made equal to a , this value corresponds to a minimum, and if negative to a maximum of the function.

Let us examine now the second case, that is when a is a fractional number. This quantity may be either greater or less than one. If greater, it is necessary, as before, that the value of $\frac{d u}{d x}$ corresponding to $x = a$ should be equal to nothing, since otherwise the developement of $f(a+h)$ would begin by the term $\frac{d u}{d a} h$, and more generally it is necessary that the value $x = a$ should reduce to nothing all the differential coefficients, the orders of which are less than a .

And, finally, if we have $a < 1$, then $\frac{d u}{d x}$ must become infinite for the value $x = a$.

From what precedes it is easy to infer, that the values of x , corresponding to maxima and minima, can only be found among the roots of the equations $\frac{d u}{d x} = 0$, or $\frac{1}{\frac{d u}{d x}} = 0$. To find all the maxima and minima of the function each of these roots must be examined, to determine which of them, if any, satisfy the conditions stated above.

(259.) We shall now, to elucidate these rules, apply them to some examples.

Example 1. Let the proposed function $u = b + c(x-a)^n$, n being an integer. By making the first differential coefficient equal to nothing, we have the equation $\frac{d u}{d x} = nc(x-a)^{n-1} = 0$, which is only satisfied by the value $x = a$. This value reduces to nothing all the differential coefficients of u up to that of the n^{th} order exclusively. Hence it follows that if n be an odd number, the proposed function will have neither maximum nor minimum, but if n be an even number, the value $x = a$ will make u a maximum when c is negative, and a minimum when it is positive.

Example 2. Let $u = b + c(x-a)^{\frac{2}{3}}$, then $\frac{d u}{d x} = \frac{2}{3}c(x-a)^{-\frac{1}{3}}$. This quantity becoming infinite for $x = a$, there may be a maximum or a minimum of the function corresponding to $x = a$. We must, in this case, develope according to the powers of h the result of the substitution of $a+h$ for x in the function. By this substitution,



Integral
Calculus.

the function becomes $b + ch^{\frac{2}{3}}$, and, therefore, requires no further development. Moreover, the exponent of h being a fraction whose numerator is an even number, we immediately see that the value $x = a$ will correspond to a maximum if c be negative, and to a minimum if c be positive.

Example 3. To divide a number a into two parts such, that the product of the m^{th} power of the first, and the n^{th} power of the second, shall be a maximum or a minimum.

If x be one of the parts, the other is $a - x$, and, consequently, the function $u = x^m (a - x)^n$. We have

$$\frac{du}{dx} = (a - x)^{n-1} \cdot x^{m-1} \cdot \{ma - (m+n)x\} \text{ and } \frac{d^2u}{dx^2} = (a-x)^{n-2} \cdot x^{m-2} \{(ma - (m+n)x)^2 - m(a-x)^2 - nx^2\}.$$

The values of x which render $\frac{du}{dx} = 0$, are determined by the equations

$$a - x = 0, \quad x = 0, \quad ma - (m+n)x = 0,$$

which give $x = a$, $x = 0$, $x = \frac{ma}{m+n}$. The value $x = a$ renders the differential coefficient of the second order

equal nothing; and it is evident that since every differential coefficient of an order inferior to the n^{th} will have $(x - a)$ or some power of it as a factor, the same value of x will render all these $= 0$. The n^{th} differential coefficient will not have the factor $x - a$, and, therefore, by changing in it, x into a , the result will not be found $= 0$. If n be odd therefore, this value of x does not correspond to either a maximum or minimum, and if n be even, it will be found that the value $x = a$ renders the differential coefficient positive, and that, therefore, the function is a minimum.

Similar observations apply to the value $x = 0$. The value $x = \frac{ma}{m+n}$ being substituted for x in $\frac{d^2u}{dx^2}$, renders it negative, and therefore renders the function a maximum.

Example 4. Let $u = \frac{lx}{x^n}$. We get $\frac{du}{dx} = \frac{x^{n-1} - nx^{n-1}lx}{x^{2n}} = \frac{1 - nlx}{x^n + 1}$, and the value of x which makes $\frac{du}{dx} = 0$ is given by the equation $1 - nlx = 0$, or $lx = \frac{1}{n}$, or $x = \frac{1}{e^n}$. To find the value of $\frac{d^2u}{dx^2}$ corresponding

to that value, we may observe, that it is not necessary to differentiate the denominator of the value of $\frac{du}{dx}$, for it would be multiplied by the numerator which the value $x = \frac{1}{e^n}$ annihilates. Thus we shall have simply $\frac{-n}{x^{n+2}}$, a negative quantity, and which will remain so by the substitution of the value of x ; therefore this value corresponds to a maximum of the function.

Example 5. Let $u = \sqrt{x}$, or $u = x^{\frac{1}{2}}$. By taking the logarithms of both sides, we first find $lu = \frac{1}{x} lx$, and by differentiation $\frac{du}{u} = \frac{dx - dx lx}{x^2}$; hence $\frac{du}{dx} = \frac{u}{x^2} (1 - lx)$. This value of $\frac{du}{dx}$ may become equal to nothing in two different ways, either by making $1 - lx = 0$, that is $x = e$, or by making $\frac{u}{x^2} = 0$. In the first case the value of $\frac{d^2u}{dx^2}$ is reduced to $\frac{-u}{x^3}$, and indicates a maximum. In the second case, we may observe that $\frac{u}{x^2} = \frac{1}{x^{\frac{3}{2}}}$, and, under this form, it is obvious that the value of $\frac{u}{x^2}$ diminishes as x increases, and becomes equal to nothing, only when x is infinite.

Example 6. For the last example we shall propose to find in the line joining the centres of two spheres, the point from which the greatest portion of spherical surface is visible.

Let r and r' be the radii of the spheres A and A' , d the distance of their centres, and x the distance of the required point from the centre A ; then it is easy to find that the portion of spherical surface visible on the first sphere is equal to $\frac{2\pi r^2(x-r)}{x}$, and that visible on the second sphere to $\frac{2\pi r'^2(d-x-r')}{d-x}$; hence the function

of which the maximum is to be found is

$$\frac{r^2(x-r)}{x} + \frac{r'^2(d-x-r')}{d-x} = \frac{-(r^2+r'^2)x^2 + (d(r^2+r'^2) + r^3-r'^3)x - r^3d}{dx - x^2};$$

making the differential coefficient equal to nothing, we have the equation $(r^3 - r'^3)x^2 - 2r^3dx + r^3d^2 = 0$,

which gives $x = \frac{d(r^3 \pm \sqrt{r^3 r'^3})}{r^3 - r'^3}$, or $x = \frac{dr^{\frac{3}{2}}}{r^{\frac{3}{2}} + r'^{\frac{3}{2}}}$, and $x = \frac{dr^{\frac{3}{2}}}{r^{\frac{3}{2}} - r'^{\frac{3}{2}}}$. The first value corresponds to a

Integral
Calculus.

maximum of the function, and is the answer to the question, the second corresponds to a minimum, and must be considered as an analytical solution, since it places the required point not between the two centres, but on the line which joins them produced. The maximum of the function corresponding to the first value is

Part III.

$$2\pi \left\{ r^2 + r'^2 - \left(\frac{r^{\frac{3}{2}} + r'^{\frac{3}{2}}}{d} \right)^2 \right\}.$$

(260.) We shall next consider implicit functions of one variable. Let y represent a function of x given by the equation $y^2 - 2mxy + x^2 - a^2 = 0$. By differentiating we get $(y - mx) dy - (my - x) dx = 0$, and we find $\frac{dy}{dx} = \frac{my - x}{y - mx}$. This being made equal to nothing, gives $my - x = 0$, and to obtain the values of x corresponding to maxima and minima of the function y , this equation must be combined with the proposed equation. We shall have thus

$$y = \frac{x}{m}, \text{ and, by substitution, } \frac{x^2}{m^2} - x^2 - a^2 = 0,$$

from which we get $x = \frac{ma}{\sqrt{1-m^2}}$, $y = \frac{a}{\sqrt{1-m^2}}$. It remains now to examine if the value found for x correspond to a maximum or to a minimum; for that purpose we must find the value of the differential coefficient of the second order $\frac{d^2y}{dx^2}$. We get by differentiating the first differential equation,

$$(y - mx) \frac{d^2y}{dx^2} + \frac{dy^2}{dx^2} - 2m \frac{dy}{dx} + 1 = 0,$$

and this is reduced to $(y - mx) \frac{d^2y}{dx^2} + 1 = 0$, since we have supposed $\frac{dy}{dx} = 0$. Substituting then for y and x , the values $\frac{a}{\sqrt{1-m^2}}$, and $\frac{ma}{\sqrt{1-m^2}}$ we have obtained above, we find $\frac{d^2y}{dx^2} = \frac{-1}{a\sqrt{1-m^2}}$, and this result being negative, shows that the value of y is a maximum.

If we make equal to nothing the denominator of the value $\frac{dy}{dx}$ instead of the numerator, we find $y = mx$, a relation which renders also infinite $\frac{d^2y}{dx^2}$, and all the differential coefficients. It leads to $x = \frac{a}{\sqrt{1-m^2}}$ and $y = \frac{ma}{\sqrt{1-m^2}}$. These values do not differ from the preceding if we change x into y , and y into x , they indicate a maximum of x , which is found under this form, because the proposed equation is symmetrical with respect to its two variables.

(261.) We shall propose, for a second example, to find the maxima and minima of the function y given by the equation $x^3 - 3axy + y^3 = 0$. In this case, y having three values in x , ought to be considered as representing three distinct functions, which, however, are connected together by the character of roots of equations of the third degree; each of these functions may have maxima and minima, as we shall see by the following investigation.

We derive from the given equation $\frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$; making $\frac{dy}{dx} = 0$, we shall obtain a new equation $ay - x^2 = 0$, which ought to be combined with the proposed equation to determine the values of x and y . We shall find thus $x^3 - 2a^2x = 0$, from which we get $x^2 = 0$, and $x^3 - 2a^2x = 0$. When we make $x = 0$ we find $y = 0$, and $\frac{dy}{dx} = 0$; to know the signification of the roots of the equation $x^2 = 0$, we must examine what becomes $\frac{d^2y}{dx^2}$ for $x = 0$ and $y = 0$. By differentiating the first differential equation, we find

$$(y^2 - ax) \frac{d^2y}{dx^2} + 2y \frac{dy^2}{dx^2} - 2a \frac{dy}{dx} + 2x = 0.$$

The coefficient of $\frac{d^2y}{dx^2}$ vanishes, since we suppose $x = 0$ and $y = 0$, and the remaining equation of the second degree with respect to $\frac{dy}{dx}$, gives, in the same hypothesis, $\frac{dy}{dx} = 0$, and $\frac{dy}{dx}$ equal to infinity. If we differentiate again, in order to obtain the value of $\frac{d^2y}{dx^2}$, we find

$$(y^2 - ax) \frac{d^3y}{dx^3} + (6y \frac{dy}{dx} - 3a) \frac{d^2y}{dx^2} + 2 \frac{dy^3}{dx^3} + 2 = 0,$$

Integral
Calculus.

which equation, by making at the same time $x = 0$, $y = 0$, and $\frac{dy}{dx} = 0$, gives $\frac{d^2y}{dx^2} = \frac{2}{3a}$, which is a positive quantity, and shows, consequently, that the value $y = 0$ is a minimum value of the function. But if we suppress only, in the equation, the term multiplied by $\frac{d^2y}{dx^2}$, and if we make $\frac{dy}{dx} = p$, we find

$$\frac{d^2y}{dx^2} = -\frac{2 + 2p^3}{6yp - 3a} = -\frac{\frac{2}{p^3} + 2}{\frac{6y}{p^3} - \frac{3a}{p^3}},$$

which becomes infinite when we suppose $y = 0$, and p infinite: as these last values render the differential coefficients infinite, they ought, therefore, to be examined, by supposing x equal to a very small quantity, and determining the corresponding value of y , which we shall represent by k . We have then between h and k the

relation $h^3 - 3ahk + k^3 = 0$. From which we get for the three values of k $\frac{h^2}{3a}$, $+\sqrt{3ah}$ and $-\sqrt{3ah}$.

We see then, that the first of these remains a positive quantity whether $x = +h$ or $x = -h$, and, consequently, it has all the characters of a minimum when $h = 0$; it is not the same thing with respect to the other two, $+\sqrt{3ah}$, $-\sqrt{3ah}$, which become imaginary when a negative value is taken for h .

We have besides the equation $x^3 - 2a^3 = 0$, which gives $x = a\sqrt[3]{2}$ and $y = a\sqrt[3]{4}$ in consequence of the equation $ay - x^2 = 0$. If we substitute these values in the general expression of $\frac{d^2y}{dx^2}$, we shall find $\frac{d^2y}{dx^2} = -\frac{2}{a}$, a result showing that the function y has a maximum corresponding to $x = a\sqrt[3]{2}$.

On Maxima and Minima of Functions of several Variables.

(262.) Let us now consider a function of several variables, and let us suppose first, that it contains only two variables x and y . The definitions given (254.) may easily be extended to functions of more than one variable. With respect to functions of two variables, it is evident that we may consider one of them as a constant, and give to the other an infinite number of values, to each of which will correspond one or several values of the proposed function; among these some may be maxima and minima, and nothing is easier than to find them. Since we consider only one of the variables, it will be sufficient to equal to nothing the first differential coefficient with respect to that variable; thus, u being a function of x and y , if we suppose y constant, and if we make $\frac{du}{dx} = 0$, we shall obtain the values of x , which give the greatest and least values of u , among all those which answer to the same value of y . By eliminating x from the function u , by means of the equation $\frac{du}{dx} = 0$, we shall obtain a result which I shall represent by v , and which, containing still the variable y , may have maxima and minima which will be determined by means of the equation $\frac{dv}{dy} = 0$. We may obtain an equation answering the same purpose as $\frac{dv}{dy} = 0$, without eliminating x ; for that, it is necessary to observe that the equation $\frac{du}{dx} = 0$, furnished by the condition of maximum or minimum relative to x , establishes a relation between the variables x and y , so that the first may be considered as a function of the second. Differentiating u , in this hypothesis, we shall have $\frac{1}{dy} du = \frac{du}{dx} \cdot \frac{dx}{dy} + \frac{du}{dy}$, a result which is reduced to $\frac{du}{dy} = 0$, since we have already $\frac{du}{dx} = 0$; the greatest or least values of all the values of u , will answer therefore to some of the values of x and y given by the equations $\frac{du}{dx} = 0$, $\frac{du}{dy} = 0$. These considerations may easily be extended to any number of variables, and it will be proved, in a similar manner, that to obtain the absolute maxima and minima of these functions, it is necessary to equal separately to nothing the differential coefficients of the first order, taken with respect to each of the variables upon which they depend.

(263.) The distinction between maxima and minima presents more difficulties with respect to functions of several variables, than with those of one variable, although the principles upon which this distinction is founded are the same in both cases.

Let u be a function containing any number of variables x, y, z , &c.; if we call a, b, c , the values which answer to one of its maxima or minima, and if we represent by A, B, C, D , &c., F, G, H, I, K, L , &c., what the quantities $u, \frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, &c., $\frac{d^2u}{dx^2}, \frac{d^2u}{dxdy}, \frac{d^2u}{dy^2}, \frac{d^2u}{dxdz}, \frac{d^2u}{dydz}, \frac{d^2u}{dz^2}$, become, when we substitute in it a for x , b for y , c for z , &c., the result of the substitution of $a + h, b + k, c + l$, &c. for x, y, z , &c. in u , will be

$$u' = A + Bh + Ck + Dl + \&c.$$

$$+ \frac{1}{1.2} (Fh^2 + 2Ghk + Hk^2 + 2Ihl + 2Kkl + Ll^2 + \&c.)$$

$$+ \&c.$$

An expression of which the value ought to be less than A , when that quantity is a maximum of u , and less when it is a minimum, whatever be the signs of $h, k, l, \&c.$, provided however they are taken sufficiently small.

This understood, if we take $k = ah, l = \beta h, \&c.$ we shall find

$$u' - u = h(B + Ca + D\beta + \&c.) + \frac{h^2}{1.2} (F + 2Ga + Ha^2 + 2I\beta + 2Ka\beta + L\beta^2 + \&c.) + \&c.$$

We see that by taking h , and consequently kl , sufficiently small, it is possible to make $h(B + Ca + D\beta + \&c.)$ greater than all the rest of the series; but this quantity changing its sign at the same time as h , a reasoning similar to that used before, will show that $B + Ca + D\beta + \&c.$ ought to be equal to nothing in the case of a maximum or a minimum. Hence the values of $x, y, z, \&c.$ corresponding to a maximum or a minimum ought to

satisfy the equations $\frac{du}{dx} = 0, \frac{du}{dy} = 0, \frac{du}{dz} = 0, \&c.$, a condition which has already been established in the preceding number. It is necessary, moreover, that these same values should not render nothing, at the same time, all the terms of the second order, that is to say, that we should not have at the same time $F = 0, G = 0, \&c. \dots L = 0$. But this is not sufficient, the sign of this quantity of the second order should also be independent of the values and relations which may be established between the quantities $h, k, l, \&c.$, and of the signs they may have.

Now it is known by the general theory of equations, that when all the terms of an equation are on one side, the results of the substitutions of a series of particular values in that side cannot be of different signs, without there existing intermediate particular values, which render that side equal to nothing; and, consequently, if the equation have only imaginary or equal roots, whatever be the values substituted, the result of the substitution will always have the same sign. Let us first find the conditions which ought to be satisfied in order that the quantity

$$Fh^2 + 2Ghk + Hk^2 + 2Ihl + 2Kkl + Ll^2 + \&c.$$

being made equal to nothing and resolved as an equation with respect to one of the quantities $h, k, l, \&c.$ should have only imaginary roots. Let us take, for instance, the value of h , we shall have

$$h = \frac{-(Gk + Il + \&c.) \pm \sqrt{-(Hk^2 + 2Hkl + Ll^2 + \&c.)F + (Gk + Il + \&c.)^2}}{F},$$

this result will be imaginary, if the quantity under the radical be negative; but we may give to this quantity the form $-(Pk^2 + 2Qkl + Rl^2 + \&c.)$, by making $FH - G^2 = P, KF - GI = Q, IF - I^2 = R, \&c.$, and in order that it should not change sign, it will be necessary that being equalled to nothing, and resolved with respect to one of the letters $k, l, \&c.$, it should have only imaginary roots. By these operations, we shall get

$$K = \frac{-(Ql + \&c.) \pm \sqrt{(Rl^2 + \&c.)P + (Ql + \&c.)^2}}{P};$$

and again k can only be imaginary when the quantity under the radical is always negative; it may be put under the form $-(Tl^2 + \&c.)$ by making $PR - Q^2 = T, \&c.$, and then treated in the same manner as the two preceding. The same process may be continued till we come to the last but one of the quantities h, k, l . Let us suppose that there are only three variables in the function u , the operation will be terminated at the value of k ; and since T must be positive, it will be necessary that we should have $PR - Q^2$, or $T > 0$. This condition being fulfilled, the quantity $PK^2 + 2Qkl + Rl^2$ cannot change sign, and as it is reduced to PR^2 when $l = 0$, it is necessary in order that it should remain positive, as it is required by the nature of the question, that we should have $P > 0$, or $FH - G^2 > 0$. Hence we see that the coefficients F and H should be positive at the same time, or negative at the same time: in the first case, the quantity $Fh^2 + \&c.$ preserving always the positive sign it has when k and l are equal to nothing, indicates a minimum; it would be a maximum if that quantity were negative.

It may be easily verified that when the conditions we have just found are fulfilled, the terms of the second order of the series $A + Bh + Ck + \&c.$ will have always the same sign, whatever be the values of $h, k, l, \&c.$ It is sufficient to remark, for that purpose, that the equation of the second degree of which the roots are $t = -a \pm \sqrt{-\beta^2}$ is $(t + a)^2 + \beta^2 = 0$, and if we represent by $-Y^2, -Z^2$, the quantities which are under the radicals of the values of $h, k, \&c.$, we shall transform the expressions

$$\left. \begin{array}{l} Fh^2 + 2Ghk + \dots + Ll^2 + \&c. \\ Pk^2 + 2Qkl + Rl^2 + \&c. \dots \text{or } Y^2 \\ \&c. \end{array} \right\} \text{ into } \left\{ \begin{array}{l} F \left\{ \left(h + \frac{Gk + Il + \&c.}{F} \right)^2 + \frac{Y^2}{F^2} \right\} \\ P \left\{ \left(k + \frac{Ql + \&c.}{P} \right)^2 + \frac{Z^2}{P^2} \right\} \\ \&c. \end{array} \right.$$

Putting in Y^2 , for Z^2 its value $Tl^2 + \&c.$, and substituting the result in $F \left\{ \left(h + \frac{Gk + Il + \&c.}{F} \right)^2 + \frac{Y^2}{F^2} \right\}$, we

$$F\left(h + \frac{Gk + Il + \&c.}{F}\right)^2 + \frac{FP}{F^2}\left(k + \frac{Ql + \&c.}{P}\right)^2 + \frac{FPT}{F^2P^2}\left(l + \frac{\&c.}{T}\right)^2 + \&c.,$$

an expression which, it is obvious, will preserve the same sign whatever be the values of $h, k, l, \&c.$, and the sign of which will only depend on that of F , since P and T are positive.

If u contained only two variables, the coefficients $D, \dots, I, K, L, \&c.$ would be equal to nothing, and the conditions of maximum and minimum would be reduced to $FH - G^2 > 0$, the maximum taking place when F is negative, and the minimum in the contrary case. If the coefficients of the second order were equal to nothing at the same time as those of the first, there would be neither maximum nor minimum, unless the coefficients of the third order disappeared at the same time, and unless the terms of the fourth order formed a quantity whose sign would not depend in any way upon $h, k, l, \&c.$

(264.) We have hitherto omitted the case in which the quantity of the second order does not change sign, not because it has only imaginary roots, but because it has only equal roots. To simplify, we shall suppose that the function u contains only two variables, x and y . In that case, the function which ought not to change sign, whatever be the values of h and k , is $Fh^2 + 2Ghk + Hk^2$. It may be put under the following form,

$$h^2\left\{F + 2G\frac{k}{h} + H\frac{k^2}{h^2}\right\} = h^2\{F + 2Ga + Ha^2\} = Hh^2\left\{\left(\frac{G}{H} + a\right)^2 + \frac{FH - G^2}{H^2}\right\},$$

by making $\frac{k}{h} = a$. It will remain of the same sign as H , not only when we have $FH > G^2$, but also when

$FH = G^2$, because it becomes then $Hh^2\left(\frac{G}{H} + a\right)^2$. The condition $FH = G^2$ takes place when the equations

$\frac{du}{dx} = 0, \frac{du}{dy} = 0$, obtain in consequence of a common factor between the two quantities $\frac{du}{dx}$ and $\frac{du}{dy}$, in which

case the determination of the maxima and minima is an indeterminate problem, since the two equations $\frac{du}{dx} = 0$,

and $\frac{du}{dy} = 0$, by means of which the values of x and y are to be determined, are reduced to one. If we suppose

for instance $\frac{du}{dx} = PV, \frac{du}{dy} = QV$, it will be sufficient to make $V = 0$ to satisfy the two conditions of maximum or minimum. We shall have

$$\frac{d^2u}{dx^2} = V\frac{dP}{dx} + P\frac{dV}{dx}, \quad \frac{d^2u}{dy^2} = V\frac{dQ}{dy} + Q\frac{dV}{dy}, \quad \frac{d^2u}{dxdy} = V\frac{dP}{dy} + P\frac{dV}{dy}, \quad \frac{d^2u}{dxdy} = V\frac{dQ}{dx} + Q\frac{dV}{dx},$$

and by making $V = 0$, they are reduced to

$$F = P\frac{dV}{dx}, \quad H = Q\frac{dV}{dy}, \quad G = P\frac{dV}{dy}, \quad G = Q\frac{dV}{dx},$$

from which results the relation $FH = G^2$.

It is necessary to remark that when the equation $FH = G^2$ subsists at the same time as $V = 0$, the total differential $d^2u, d^3u, \&c.$ *ad infinitum*, vanish as soon as we introduce into it the relation given by the equation $dV = 0$. This is easily proved, for if we consider y as a function of x , we have

$$d^2u = \frac{d^2u}{dx^2}dx^2 + 2\frac{d^2u}{dxdy}dxdy + \frac{d^2u}{dy^2}dy^2 + \frac{du}{dy}d^2y,$$

and since in the present case $\frac{du}{dy} = 0$, this expression of d^2u will be reduced, in making use of the same letters as above, to

$$d^2u = Fdx^2 + 2Gdxdy + Hdy^2 = Hdx^2\left\{\left(\frac{G}{H} + \frac{dy}{dx}\right)^2 + \frac{FH - G^2}{H^2}\right\};$$

but $FH = G^2$, therefore $d^2u = Hdx^2\left(\frac{G}{H} + \frac{dy}{dx}\right)^2 = \frac{1}{H}(Gdx + Hdy)^2$. Substituting between the brackets the values of G and H in the supposition of $V = 0$, we find

$$d^2u = \frac{Q^2}{H}\left(\frac{dV}{dx}dx + \frac{dV}{dy}dy\right)^2 = \frac{Q^2}{H}dV^2,$$

an expression which becomes nothing when $dV = 0$.

It follows from what precedes, that the equation $V = 0$ renders the values of the function u constant; but without this condition d^2u is not equal to nothing, and its sign depends only upon that of H : thus for every value of the variables x and y , which differs from those which satisfy the equation $V = 0$, there will be a decrease of the value of u if H be negative, and increase if it be positive, therefore u will be a maximum in the first case, and a minimum in the second. Consequently there will exist an infinite number of systems of values of x and y which will render at the same time the function u either a maximum or a minimum.

(265.) There exist also with respect to functions of several variables, maxima and minima corresponding to

Integral Calculus. Part III. values of the variables which render the differential coefficients infinite. This is the case for the function

$u = b - (x^2 + y^2)^{\frac{1}{3}}$. It is obvious that whatever be the values substituted for the variables x and y , the result of the substitution will be less than that obtained by supposing $x = 0$ and $y = 0$; this last value $u = b$ is therefore a maximum, and the differential coefficients

$$\frac{du}{dx} = \frac{-2x}{3(x^2 + y^2)^{\frac{2}{3}}}, \quad \frac{du}{dy} = \frac{-2y}{3(x^2 + y^2)^{\frac{2}{3}}},$$

become infinite for $x = 0$ and $y = 0$.

Hence to complete the rules given for the research of maxima and minima with respect to functions of several variables, it would be necessary to add remarks analogous to those which have been made relatively to the determination of the maxima and minima of functions of one variable. But we shall simply observe that, in this case, the maxima and minima are known, because the decrease or increase of the proposed function produced by the substitution of $a + h$, and $b + k$ for x and y may be expressed in such a manner that the whole of it, or at least its most considerable part, does not change sign at the same time as the quantities h and k . In the example

given above, we have by that substitution $u' - u = -(h^2 + k^2)^{\frac{1}{3}}$, a quantity which does not alter its sign. We shall now give a few examples of the maxima and minima of functions of several variables.

Example 1. Let it be proposed to divide a quantity a into three parts x , y and $a - x - y$ such that the product $u = x^m y^n (a - x - y)^p$ shall be a maximum or a minimum.

The differential coefficients of the first order being made equal to nothing, give

$$\frac{du}{dx} = x^{m-1} y^n (a - x - y)^{p-1} \{ma - mx - my - px\} = 0, \quad \frac{du}{dy} = x^m y^{n-1} (a - x - y)^{p-1} \{na - nx - ny - py\} = 0;$$

the factors $ma - mx - my - px$, and $na - nx - ny - py$, being put equal to nothing, give two equations from which the values of x and y may be derived; and thus we find

$$x = \frac{ma}{m+n+p}, \quad y = \frac{na}{m+n+p}, \quad a - x - y = \frac{pa}{m+n+p}.$$

In order to discover whether these correspond to a maximum or a minimum, we must substitute them in the values of $\frac{d^2u}{dx^2}$, $\frac{d^2u}{dy^2}$, $\frac{d^2u}{dxdy}$, and if $m+n+p$ be represented by q , we shall get

$$F = -(m+p) \left(\frac{ma}{q}\right)^{m-1} \left(\frac{na}{q}\right)^n \left(\frac{pa}{q}\right)^{p-1}, \quad G = -\frac{mna}{q} \left(\frac{ma}{q}\right)^{m-1} \left(\frac{na}{q}\right)^{n-1} \left(\frac{pa}{q}\right)^{p-1}, \\ H = -(n+p) \left(\frac{ma}{q}\right)^m \left(\frac{na}{q}\right)^{n-1} \left(\frac{pa}{q}\right)^{p-1}.$$

The quantities F and H are both negative, and it will be easy to verify that they satisfy also the condition $FH - G^2 > 0$, and, therefore, the corresponding value of the function is a maximum.

Example 2. To find the greatest triangle which can be included within a given perimeter.

Let the perimeter be $2a$, the sides x , y and $2a - x - y$, we shall have in representing the area by u ,

$$u = \sqrt{a(a-x)(a-y)(x+y-a)},$$

and by taking the logarithms of both sides of this equation,

$$2 \log u = \log a + \log(a-x) + \log(a-y) + \log(x+y-a);$$

differentiating for x and y , we find

$$2 \frac{du}{u} = -\frac{dx}{a-x} + \frac{dx}{x+y-a}, \quad \text{from which we derive } \frac{du}{dx} = \frac{1}{2} u \frac{2a-2x-y}{(a-x)(x+y-a)}; \quad \text{in the same manner}$$

$$\text{we shall find } \frac{du}{dy} = \frac{1}{2} u \frac{2a-2y-x}{(a-y)(x+y-a)}. \quad \text{Putting equal to nothing these two partial differential coefficients,}$$

we get the two equations $2a - 2x - y = 0$, $2a - 2y - x = 0$, which being resolved, give $x = \frac{2}{3}a$,

$y = \frac{2}{3}a$, $a - x - y = \frac{2}{3}a$. These values being substituted in the differential coefficients of the second order,

will be found to satisfy the conditions of a maximum. Therefore they correspond to a maximum, and the required triangle is equilateral.

Example 3. Let the proposed function $u = x^4 + y^4 - ax^2y - axy^2 + c^2x^2 + c^2y^2$, then

$$\frac{du}{dx} = 4x^3 - 2axy - ay^2 + 2c^2x = 0, \quad \frac{du}{dy} = 4y^3 - 2axy - ax^2 + 2c^2y = 0.$$

If we subtract these equations from each other, we find $4(x^3 - y^3) + a(x^2 - y^2) + 2c^2(x - y) = 0$, of which equation $x - y$ is a factor: consequently, $y = x$, and $4x^3 - 3ax^2 + 2c^2x = 0$; which gives

Integral
Calculus.

$$x = 0, \text{ and } x = \frac{3a \pm \sqrt{(9a^2 - 32c^2)}}{8}.$$

If $x = 0$, and $y = 0$, we have $F = 2c^2$, $G = 0$, $H = 2c^2$, the condition $FH - G^2 > 0$ is satisfied, and F is a positive quantity, therefore the values $x = 0$ and $y = 0$ correspond to a minimum value of the function, which is equal to nothing.

$$\text{If } x = \frac{3a + \sqrt{(9a^2 - 32c^2)}}{8} = y, \text{ we find } F = \frac{21a^2 - 32c^2 + 7a\sqrt{(9a^2 - 32c^2)}}{8} = H, \text{ and}$$

$G = \frac{12a^2 + 4a\sqrt{(9a^2 - 32c^2)}}{8}$: these values give $FH - G^2 > 0$, since we suppose $9a^2 > 32c^2$. For the same reason F is positive, therefore the values of x and y correspond also to a minimum value of u , and we have

$$u = -\frac{27}{256}a^4 + \frac{9}{16}a^2c^2 - \frac{1}{2}c^4 - \frac{a}{256}(9a^2 - 32c^2)^{\frac{3}{2}}.$$

$$\text{If } x = \frac{3a - \sqrt{(9a^2 - 32c^2)}}{8} = y, \text{ we have } F = \frac{21a^2 - 32c^2 - 7a\sqrt{(9a^2 - 32c^2)}}{8} = H, \text{ and}$$

$$G = -\frac{12a^2 - 4a\sqrt{(9a^2 - 32c^2)}}{8}.$$

These values give $FH < G^2$; in this case, therefore, the value of u corresponding to the values of x and y is neither a maximum nor a minimum.

If we consider the other factor of $\frac{du}{dx} - \frac{du}{dy} = 0$, or $4x^3 + 4xy + 4y^3 - ax - ay + 2c^2 = 0$, and eliminate y from this equation, and the equation $\frac{du}{dx} = 0$, we shall get a final equation of six dimensions, the solution of which would be required in order to determine the other values of x and y .

Example 4. Let the proposed function be $u = \frac{2^4 acxyz}{(a+x)(x+y)(y+z)(z+c)}$. Considering $\log u = u'$, as the function, and making $\frac{du'}{dx} = 0$, $\frac{du'}{dy} = 0$, $\frac{du'}{dz} = 0$, we get $ay = x^2$, $xz = y^2$, $ey = z^2$, or $a : x :: x : y :: y : z :: z : e$ and $x = a \sqrt[4]{\frac{e}{a}} = am$, $y = am^2$, and $z = am^3$, if $m = \sqrt[4]{\frac{e}{a}}$. Again, if we put $a^2m(1+m^2) = A$, we find

$$F = \frac{d^2u'}{dx^2} = -\frac{2}{A}, \quad G = \frac{d^2u'}{dx dy} = \frac{1}{mA}, \quad H = \frac{d^2u'}{dy^2} = -\frac{2}{m^2A}, \quad K = \frac{d^2u'}{dy dz} = \frac{1}{m^2A},$$

$$L = \frac{d^2u'}{dz^2} = -\frac{2}{m^4A}, \quad I = \frac{d^2u'}{dx dz} = 0,$$

consequently $(FH - G^2)(LF - I^2) = \frac{12}{m^6A^4}$, and $KF - GI = -\frac{2a}{m^3A^3}$; therefore $(DF - E^2)(DH - K)$

$> (DG - EK)^2$, and the value of u' and therefore of u determined by this process is a maximum.

Example 5. Let the proposed function be $u = (x+1)(y+1)(z+1)$; the variables being connected by the equation $a^x b^y c^z = A$. Eliminating z we find $u = (x+1)(y+1) \left\{ 1 + \frac{\log A}{\log c} - \frac{y \log b}{\log c} - \frac{x \log a}{\log c} \right\}$; and

making $\frac{du}{dx} = 0$, and $\frac{du}{dy} = 0$, we get

$$\left\{ 1 + \frac{\log A}{\log c} - \frac{y \log b}{\log c} - \frac{x \log a}{\log c} \right\} - (x+1) \frac{\log a}{\log c} = 0, \text{ and } 1 + \frac{\log A}{\log c} - \frac{y \log b}{\log c} - \frac{x \log a}{\log c} - (y+1) \frac{\log b}{\log c} = 0.$$

These two equations give $a^{x+1} = b^{y+1}$, and in an analogous manner we should find $a^{x+1} = c^{z+1}$. These two equations combined with the equation $a^x b^y c^z = A$, after having multiplied both sides by abc , give

$$x = \frac{\log bcA - 2 \log a}{3 \log a}, \quad y = \frac{\log acA - 2 \log b}{3 \log b}, \quad z = \frac{\log abA - 2 \log c}{3 \log c}.$$

We have then

$$\frac{d^2u}{dx^2} = -2(y+1) \frac{\log a}{\log c}, \quad \frac{d^2u}{dx dy} = -(x+1) \frac{\log a}{\log c} + 1 + \frac{\log A}{\log c} - \frac{y \log b}{\log c} - \frac{x \log a}{\log c} - (y+1) \frac{\log b}{\log c},$$

$$\frac{d^2u}{dy^2} = -2(x+1) \frac{\log b}{\log c}.$$

Substituting in these the values of x and y , we shall find that the condition $FH - G^2 > 0$ is satisfied, and as F is negative, these values correspond to a maximum of $u = \frac{(\log \sqrt[3]{abcA})^3}{\log a \log b \log c}$.

(266.) The Differential and Integral Calculus furnishes methods to discover the properties of curve lines and curve surfaces, and to resolve geometrical problems, which are far more general and simple than those derived from geometrical considerations alone, or founded on common algebra. These methods, and the most interesting results to which they have led, will form the subject of the following pages.

We shall begin by the application of the Differential Calculus to the geometry of plane curves, and the first problem we shall propose to resolve will be to find the equation of a straight tangent, from a given point, to a curve given by its equation referred to rectangular coordinates.

By a tangent to a plane curve, we understand a straight line passing through one of the points of the curve which is called the point of contact, and such that no other straight line can be drawn through the same point, passing between the curve and the tangent, immediately after or immediately before the point of contact.

This understood, let AX, AY , (fig. 1,) be the axes of coordinates, MN the curve, M the point of contact, and MT the tangent. There are generally two different cases which may occur: the curve may turn its convexity towards the axis of x , as in fig. 1, and be, immediately about the point of contact, above its tangent with respect to that axis; or it may, on the contrary, turn its concavity towards the axis of x , and be under its tangent, immediately about its point of contact. Our reasonings will only apply to the first case, but it will be extremely simple to see how it ought to be modified to be applied to the second. The cases which are not included under these two will be considered afterwards separately. Let $y = f(x)$ be the equation of the curve, and $x' = AP$, and $y' = MP$, be the coordinates of the point of contact. Then $y' = f(x')$; and if we substitute $x' + h$ for x in the equation $y = f(x)$ and represent the corresponding value of y by $y' + k$, we shall have, in the supposition that $f(x' + h)$ may be developed according to the integral powers of h ,

$$y' + k = y' + \frac{dy'}{dx'} \cdot h + \frac{d^2y'}{dx'^2} \cdot \frac{h^2}{1.2} + \frac{d^3y'}{dx'^3} \cdot \frac{h^3}{1.2.3} + \&c.$$

$\frac{dy'}{dx'}$, $\frac{d^2y'}{dx'^2}$, $\frac{d^3y'}{dx'^3}$, designating the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$, &c. corresponding to $x = x'$. By taking h sufficiently small, $x' + h$ and $y' + k$ will represent the coordinates of a point of the curve taken as near the point of contact as we choose. Let N be that point, then $x' + h = AS$, $y' + k = NS$.

The equation of any straight line, such as $MN''n''$, drawn through the point M , will be of the form $y - y' = a(x - x')$, a being the tangent of the angle which it makes with the axis of the X . If in that equation we substitute $x' + h$ for x , the corresponding value of y will be the ordinate of the point where $MN''n''$ meets NS . Let N'' be that point, we shall have $N''S - y' = a(x' + h - x')$ or $N''S = y' + ah$. This value being compared with that of NS , gives

$$NS - N''S = \left(\frac{dy'}{dx'} - a \right) h + \frac{d^2y'}{dx'^2} \cdot \frac{h^2}{1.2} + \&c.$$

Now if MT be, as we have supposed, the tangent to the curve, the value of a which corresponds to it should be such as to render either the difference $NS - N''S = NN''$, which then becomes $NS - N'S = NN'$ less for any small value of h , positive or negative, than the value of this difference corresponding to any other value of a , or of a contrary sign to it. But it is obvious that the only way of satisfying this condition is to make the value

of a corresponding to $MT = \frac{dy'}{dx'}$, for then the difference $NS - N'S$ will be equal to $\frac{d^2y'}{dx'^2} \frac{h^2}{1.2} + \frac{d^3y'}{dx'^3} \frac{h^3}{1.2.3} + \&c.$

whilst for any other straight line, such as $MN''n''$, the difference will be $\left(\frac{dy'}{dx'} - a \right) h + \frac{d^2y'}{dx'^2} \cdot \frac{h^2}{1.2} + \frac{d^3y'}{dx'^3} \frac{h^3}{1.2.3} + \&c.$ This last series, by taking h sufficiently small, will always be greater than the first, or of a contrary sign to it. Hence the value of the tangent of the angle which MT makes with the axis of the X is equal to

$\frac{dy'}{dx'}$, and the equation of the tangent to the curve is $y - y' = \frac{dy'}{dx'} (x - x')$.

(267.) The consideration of infinitely small quantities, which in general considerably simplifies the applications of the Differential and Integral Calculus to Geometry, leads to the important result we have just obtained in a much easier manner. A curve, according to that doctrine, is considered as a polygon of an infinite number of infinitely small sides. Each of these sides is called an element of the curve, and one of these elements produced becomes a tangent to the curve. Hence if x', y' represent the coordinates of one of the extremities of an element $x' + dx', y' + dy'$ will be the coordinates of the other extremity, and the equation of the line passing

through these two points or the tangent, will therefore be $y - y' = \frac{dy'}{dx'} (x - x')$. If $\frac{dy'}{dx'} = 0$ the tangent is parallel

to the axis of x , if it be infinite, the tangent is perpendicular to the same axis.

(268.) We have found for the difference between the ordinate of a point of the curve very near the point of contact, and the corresponding ordinate of the tangent, the expression

$$\frac{d^2y'}{dx'^2} \frac{h^2}{1.2} + \frac{d^3y'}{dx'^3} \cdot \frac{h^3}{1.2.3} + \&c.$$

Integral
Calculus.

$x' + h$ being the abscissa of the point we consider. By taking h sufficiently small the sign of the whole series will only depend upon the sign of the first term, that is upon the sign of $\frac{d^2 y'}{d x'^2}$. Hence it is obvious that when this differential coefficient is positive, the curve is above its tangent with respect to the axis of x , and below when it is negative; it would be precisely the reverse if the ordinate of the point we consider were negative. Therefore by the sign of $\frac{d^2 y'}{d x'^2}$ we shall be able to determine whether, at the point whose coordinates are x' and y' , it is the concavity or convexity of the curve which is turned towards the axis of the x ; when y' and $\frac{d^2 y'}{d x'^2}$ have both

the same sign, it is the convexity, and when of contrary signs, it is the concavity which is turned towards that axis. (269.) The equation of the tangent being once determined, there is no difficulty in finding the equation of the normal MR , and the values of the subtangent PT , subnormal PR , as well as those of the lengths MT and MR of the tangent and normal comprised between the point of contact and the axis of x . We shall have

respectively, for the normal, $y - y' = -\frac{d x'}{d y'}(x - x')$; and $PT = y' \frac{d x'}{d y'}$, $PR = y' \frac{d y'}{d x'}$,

$$MT = y' \sqrt{1 + \frac{d x'^2}{d y'^2}}, \text{ and } MR = y' \sqrt{1 + \frac{d y'^2}{d x'^2}}.$$

(270.) The application of these general formulæ to particular examples does not present the least difficulty, for it will be sufficient, in each case, to substitute for $\frac{d y}{d x}$ and $\frac{d x}{d y}$, their values derived from the differential equation of the proposed equation. The general equation of the curves of the second degree is $y^2 = m x + n x^2$, the differential of this is $2 y d y = m d x + 2 n x d x$, and, therefore, $\frac{d y}{d x} = \frac{m + 2 n x}{2 y}$. Consequently the equations of the tangent and the normal in any point of which the coordinates are x' and y' , will be respectively

$$y - y' = \frac{m + 2 n x'}{2 y'}(x - x'), \quad y - y' = \frac{-2 y'}{m + 2 n x'}(x - x').$$

In the same manner we shall have for the values of the subtangent, subnormal, tangent, and normal, respectively, the following values

$$\frac{2(m x' + n x'^2)}{m + 2 n x'}, \quad \frac{m + 2 n x'}{2}, \quad \sqrt{\left\{m x' + n x'^2 + 4 \left(\frac{m x' + n x'^2}{m + 2 n x'}\right)^2\right\}}, \quad \sqrt{\left\{m x' + n x'^2 + \frac{1}{4}(m + 2 n x')^2\right\}}.$$

We have supposed in what precedes, that the axes of coordinates were at right angles, but it is easy to see that if they were not so, the equation of the tangent would still preserve the same form, and the only difference would be that $\frac{d y'}{d x'}$ instead of being equal to the trigonometrical tangent of the angle, which the tangent to the curve makes with the axis of the x , would be equal to the ratio of the sines of the angles which this same line makes with both the axis of x and the axis of y .

On Asymptote to plane Curve Lines.

(271.) When a straight line approaches indefinitely to a curve without ever meeting it, this straight line is called an *asymptote* to the curve. It is easy to find the asymptotes of a curve represented by an equation between two rectangular coordinates x and y . Let us first consider the asymptotes which are not parallel to the axis of the y , and let $y = k x + l$ be the equation of one of them. The ordinate corresponding to the abscissa x in the proposed curve, ought to reduce itself sensibly, for very large numerical values of x , to the ordinate of the asymptote, and to present itself under the form $y = k x + l \pm \epsilon$, $\pm \epsilon$ being a term which becomes nothing when

$\frac{1}{x} = 0$. This understood, since from the equation $y = k x + l \pm \epsilon$, we get $\frac{y}{x} = k + \frac{l \pm \epsilon}{x}$, and also $y - k x = l \pm \epsilon$,

it is obvious that for increasing values of x the limit of $\frac{y}{x}$ is equal to k , and the limit of $y - k x = l$. Hence, to deter-

mine the constant k , it will be sufficient to write $\frac{y}{x} = s$ or $y = s x$, in the equation of the curve, and to find the limit or limits of the variable s , for indefinitely increasing values of x . Then, the constant k having been thus determined, we shall obtain the value of l by making in the equation of the curve $y - k x = t$, or $y = k x + t$, and finding the limit of t for indefinitely increasing values of x . To each system of finite values of the quantities k and l will correspond an asymptote of the proposed curve.

By a similar reasoning, and changing x into y and y into x , we shall obviously find the asymptotes which are not parallel to the axis of x .

Let us take for first example the curve called *logarithmic*, the equation of which is $y = a^x$. We shall have

Part III.
Integral
Calculus.

Part III.

then $s = \frac{y}{x} = \frac{A^x}{x}$. For increasing positive values of x the expression $\frac{A^x}{x}$ has no limit, but on the contrary for increasing negative values of x the limit is clearly $= 0$. Therefore in this case $k = 0$; and we shall have simply $t = y = A^x$, the limit of which for increasing negative values of x is equal to nothing. Consequently the equation of the asymptote is $y = 0$, which equation belongs to the axis of the x , and it is even only the negative part of that axis which is asymptote to the curve. In the same manner it might be proved that the logarithmic $x = A^y$, has for asymptote the axis of the y .

Let us suppose now that the equation of the curve presents itself under the form $f(x, y) = c$, c being a constant quantity. The determination of the asymptotes will be very easy, if we can decompose $f(x, y)$ into several parts, each of which is an homogeneous function of x and y . Let m, n , &c. be the degrees of these homogeneous functions, and let

$$f(x, y) = x^m \phi\left(\frac{y}{x}\right) + x^n \psi\left(\frac{y}{x}\right) + \&c.$$

The numbers m, n , &c. being ranged in a decreasing order. If we make $\frac{y}{x} = s$, the value of s will be determined by the equation

$$x^m \phi(s) + x^n \psi(s) + \&c. = c, \quad \text{or } \phi(s) + \frac{1}{x^{m-n}} \psi(s) + \&c. = \frac{c}{x^m},$$

from which we shall get, by making $\frac{1}{x} = 0$, and $s = k$, $\phi(k) = 0$. Moreover, if we take for k one of the roots of the equation $\phi(k) = 0$, the corresponding value of $t = y - kx$ will be given by the formula

$$x^m \phi\left(k + \frac{t}{x}\right) + x^n \psi\left(k + \frac{t}{x}\right) + \&c. = c.$$

But since $\phi(k) = 0$, we may assume $\phi\left(k + \frac{t}{x}\right) = \frac{t}{x} \phi'(k + \theta \frac{t}{x})$, ϕ' being the characteristic of the function of k , which is the differential coefficient of $\phi(k)$, and θ being a number less than one. We shall have therefore

$$x^{m-1} t \phi'\left(k + \theta \frac{t}{x}\right) + x^n \psi\left(k + \frac{t}{x}\right) + \&c. = c, \quad \text{or } t \phi'\left(k + \theta \frac{t}{x}\right) + \frac{1}{x^{m-n-1}} \psi\left(k + \frac{t}{x}\right) + \&c. = \frac{c}{x^{m-1}}.$$

If $\phi'(k)$ and $\psi(k)$ have finite values, not equal to nothing, then by making $\frac{1}{x} = 0$, and $t = l$, we shall find by means of the preceding equation

$$1. \quad \text{when } n < m - 1, l = 0; \text{ for } n = m - 1, l = -\frac{\psi(k)}{\phi'(k)}; \text{ and for } n > m - 1, l = \pm \infty.$$

Consequently the root of the equation $\phi(k) = 0$, which has been taken for k , will correspond, in the first case, to an asymptote passing through the origin of the coordinates, which will have for its equation $y = kx$; in the second case to an asymptote having for its equation $y = kx - \frac{\psi(k)}{\phi'(k)}$; and in the third the asymptote being removed to an infinite distance will disappear entirely.

The elimination of the constant k between the equations $\phi(k) = 0$, and $y = kx$, gives $\phi\left(\frac{y}{x}\right) = 0$. This equation, in the first case, will give all the asymptotes not parallel to the axis of y ; instead of it we may use the equation $x^m \phi\left(\frac{y}{x}\right) = 0$; and this is precisely the one we should obtain, by trying to find, in the same case, the asymptotes which are not parallel to the axis of x . Therefore when the left side of the proposed equation of the curve, the right side being a constant quantity, may be decomposed into several homogeneous functions, and when the degree m of one of them is greater by more than one unity than the degrees of all the others, not only all the asymptotes pass through the origin, but they are all represented by the formula obtained in making equal to nothing the homogeneous function of the m^{th} degree.

Let us illustrate these rules by a few examples. The equation of an hyperbola referred to its axis is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$; it has, therefore, for asymptotes the two straight lines represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, or which is the same thing, by the two equations $\frac{x}{a} - \frac{y}{b} = 0$, $\frac{x}{a} + \frac{y}{b} = 0$.

In the same manner it is easy to see that, if we have $B^2 - Ac > 0$, the hyperbola having for its equation $Ax^2 + 2Bxy + Cy^2 = K$, will have for asymptotes the two straight lines represented by the equation $Ax^2 + 2Bxy + Cy^2 = 0$, that is to say, by the two equations $y = -\frac{Bx}{c} + \frac{x}{c} \sqrt{(B^2 - Ac)}$, $y = -\frac{Bx}{c} - \frac{x}{c} \sqrt{(B^2 - Ac)}$.

In the same manner, again, the curve represented by the equation $x^3 + y^3 + \sin \frac{y}{x} = 0$, will have for its asymptote the straight line having for equation $x + y = 0$, the ordinate of which is the only real value of y , which satisfies the equation $x^3 + y^3 = 0$.

In the case where $n = m - 1$, the equation $x^m \phi\left(\frac{y}{x}\right) = 0$ represents all the straight lines drawn through the origin, parallel to the asymptotes of the curve. Thus for instance, the curve called *folium of Descartes*, and represented by the equation $x^3 + y^3 = 3axy$, has an asymptote parallel to the straight line, of which the equation is $x + y = 0$, equation derived as above from $x^3 + y^3 = 0$; the equation of the asymptote is $y = -x + 1$.

When the quantities $\phi'(k)$ and $\psi(k)$ become infinite or equal to nothing, the quantity l may have values different from those given above; but to determine then the value of this quantity, it will always be sufficient to find the limit or the limits towards which the quantity t converges, whilst $\frac{1}{x}$ approaches to become equal to nothing for increasing values of x . Sometimes we may find for a single value of k several values of l . This will happen, for instance, if we consider the curve represented by the equation $y^2 = \cos\left(\frac{y}{x}\right)$. We have then $k = 0$; and, as the equation in t becomes $t^2 = \cos\left(\frac{t}{x}\right)$, we shall have by making $\frac{1}{x} = 0$, $t^2 = 1$, $t = \pm 1$. Consequently the proposed curve has two asymptotes parallel to the axis of the x , and represented by the equations $y = 1$ and $y = -1$.

It must, however, be observed, that it may happen that a value of k which verifies the equation $\phi'(k) = 0$ gives a single asymptote. Thus the curve represented by the equation $y^2(x^2 + y^2) = r^4$ has only one asymptote, which coincides with the axis of x .

(272.) The asymptotes of a plane curve may also be found, according to the remark of Ampère, by finding the positions which the tangent takes when the point of contact is situated at an infinite distance from the origin of the coordinates. Let us suppose, for instance, that the curve is an hyperbola represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The equation of the tangent to this hyperbola will be $\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1$, or if we substitute for $\frac{y'}{b}$ its value $\pm \frac{x'}{a} \sqrt{1 - \frac{a^2}{x'^2}}$, and multiply afterwards by $\frac{a}{x'}$, $\frac{x}{a} \pm \left(1 - \frac{a^2}{x'^2}\right) \frac{y}{b} = \frac{a}{x'}$. To express that the point of contact is situated at an infinite distance from the origin, it will be sufficient to make x equal to plus or minus infinity. Then the preceding formula will become $\frac{x}{a} \pm \frac{y}{b} = 0$, and will be the equations of the two asymptotes of the proposed hyperbola.

(273.) There is another method of determining whether a curve admits a rectilinear asymptote, still founded upon the principle that the tangent becomes an asymptote when the point of contact is infinitely removed. The equation of the tangent through the point whose coordinates are x' and y' , is $y - y' = \frac{dy'}{dx'}(x - x')$.

Let X be the part of the axis of x , intercepted between the origin and the point where the tangent meets the axis of x ; and let Y be the corresponding quantity with respect to the axis of y . It is evident that these quantities will be obtained by making y and x successively $= 0$ in the equation of the tangent. Hence we find $X = \frac{x'dy' - y'dx'}{dy'}$, $Y = \frac{y'dx' - x'dy'}{dx'}$. In these quantities let x' increase indefinitely. If the limits of X and Y with respect to the increase of x be finite, they will determine a rectilinear asymptote. If X have a limit, but Y none, then the asymptote is parallel to the axis of y at the distance X . If Y have a limit, but X none, then the asymptote is parallel to the axis of x at the distance Y . If neither have a limit, or if their values are rendered impossible by increasing x , then the curve has no rectilinear asymptote. If in the limit $X = 0$, $Y = 0$, then there may be an asymptote passing through the origin, and its direction may be found by determining the value of $\frac{dy}{dx}$ when x is infinite.

Let us take as an example of this method the equation $y^2 = mx + nx^2$. We shall easily find

$$X = \frac{xdy - ydx}{dy} = x - \frac{2y^2}{m + 2nx} = \frac{-mx}{m + 2nx}, Y = \frac{ydx - xdy}{dx} = y - \frac{mx + 2nx^2}{2y} = \frac{mx}{2\sqrt{mx + nx^2}}$$

These values may be put under the forms $\frac{-m}{\frac{m}{x} + 2n}$ and $\frac{m}{2\sqrt{\frac{m}{x} + n}}$: consequently their limits for increasing

values of x , or their values when x is infinite, are $-\frac{m}{2n}$, and $\frac{m}{2\sqrt{n}}$. If we had $n = 0$ these quantities, that is both the limits of X and Y would be infinite, and, consequently, in that case, the curve has no rectilinear asymptote, which might be foreseen, since then the curve is a parabola. If n be a negative quantity, one of the limits becomes imaginary, and therefore the curve has no rectilinear asymptote, and in fact in that case the curve is an ellipse.

Part III.
Integral
Calculus.

Part III.

(274.) The consideration of the various developements which may be derived from the equation of a curve, representing the value of the ordinate in a series of terms arranged according to decreasing powers of the abscissa, offers perhaps the most general and satisfactory means to determine the asymptotes of a curve. Let $F(x, y) = 0$, and $f(x, y) = 0$, be the equations of two plane curves having infinite branches, and let y and y_1 be the values of the ordinates in the two curves corresponding to the same abscissa. The distance between the curves measured in a direction parallel to the axis of y is $y - y_1$. If as x increases without limit, either positively or negatively, the distance $y - y_1$ diminishes constantly, but vanishes only when x becomes infinite, then the infinite branch of one of the curves is said to be an asymptote to the other.

In order that this should occur, it is clearly necessary that the quantity $y - y_1$, developed according to the powers of x , should contain only negative powers of x . For if it contained a positive power, $y - y_1$ would be rendered infinite by x becoming infinite, and if it contained a term independent of x it would be finite when x is

infinite. Hence the developement of $y - y_1$ must have the form $Ax^{-\alpha} + Bx^{-\beta} + \&c.$

It follows, therefore, that if the developement of y , according to the descending powers of x , contain any positive powers, or a term independent of x , all these must also occur in the developement of y_1 , in order that they may disappear by subtraction. Hence if the developement of y be $y = A'x^{\alpha'} + B'x^{\beta'} + \dots + M + Ax^{-\alpha} + Bx^{-\beta} + \&c.$

the developement of y_1 must be $y_1 = A'x^{\alpha'} + B'x^{\beta'} + \dots + M + A_1x^{-\alpha_1} + B_1x^{-\beta_1} + \&c.$

Since nothing determines the terms which follow M in the second series, it follows that there may be an infinite number of asymptotes to the same curve, and that each of these will be asymptotes to each other. The most simple asymptote which the curve admits, at least that the developement of which is the simplest, is the curve

represented by the equation $y_1 = A'x^{\alpha'} + B'x^{\beta'} + \dots + M$. The curve represented by $y_2 = A'x^{\alpha'} + B'x^{\beta'} + \dots + M + Ax^{-\alpha}$ is also an asymptote, and approaches closer to the curve than the former, since, by increasing x , it is manifest that y_2 approaches nearer to equality with y than y_1 does. In like manner, the curve represented by $y_3 = A'x^{\alpha'} + B'x^{\beta'} + \dots + M + Ax^{-\alpha} + Bx^{-\beta}$ has asymptotism of a still higher order with the curve.

(275.) Curves which admit asymptotes are sometimes divided into *hyperbolic* and *parabolic*. Hyperbolic are those which admit of rectilinear asymptotes; parabolic, those which do not.

All hyperbolic curves must, therefore, be involved in the class represented by the equation $y = A'x + B' + Ax^{-\alpha} + Bx^{-\beta} + \dots$. The equation of the rectilinear asymptote being $y = A'x + B'$.

To show the application of this method, let the proposed equation be $ax^4 - by^4 + c^2xy = 0$. If we investigate series for y in terms of x , by means of Lagrange's Theorem, we shall find one of them involving descending powers of x , which is

$$y = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}x + \frac{c^2}{4a^{\frac{1}{2}}b^{\frac{1}{2}}x} - \frac{c^2}{4^2 \cdot 1 \cdot 2b^{\frac{3}{4}} \cdot a^{\frac{5}{4}} \cdot x^3} + \&c.$$

If we confine ourselves to the first term, we shall have a linear equation $y = \frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}x$, which belong to the recti-

near asymptote of the curve. If we take two terms of the series, we shall have the equation of a curve, which in this case is an hyperbola, and is a curve asymptote to the proposed curves.

Let us take for a second example the equation $xy^2 - ey = ax^3 + bx^2 + cx + d$. The series which may be obtained for y are

$$y = x\sqrt{a} + \frac{b}{2\sqrt{a}} + \frac{4ac - b^2 + 4ac\sqrt{a}}{8ax\sqrt{a}} + \&c. \quad y = -x\sqrt{a} - \frac{b}{2\sqrt{a}} + \frac{4ac - b^2 - 4ac\sqrt{a}}{8ax\sqrt{a}} - \&c.$$

$$y = \frac{e}{x} + \frac{d}{e} + \frac{cx}{e} + \frac{bx^2}{e} + \&c. \quad \text{and} \quad y = -\frac{d}{e} - \frac{cx}{e} - \frac{bx^2}{e} - \&c.$$

the equations $y = x\sqrt{a} + \frac{b}{2\sqrt{a}}$, and $y = -x\sqrt{a} - \frac{b}{2\sqrt{a}}$ represent two asymptotes which cut the axis of x

at a distance from the origin of the coordinates equal to $\frac{b}{2a}$, and make angles with the same axis whose trigono-

metrical tangents are $\sqrt{a}$ and $-\sqrt{a}$. From the third series, it appears that y is infinite, when $x = 0$, and it is easy to see that the axis of y is likewise an asymptote to the curve. If $b = 0$, these three asymptotes pass through the origin of coordinates. If a be negative, the curve has only one asymptote, the equations for the two first involving imaginary quantities. If $a = 0$, $b = 0$, the two first series for y become

$$y = \sqrt{c} + \frac{e\sqrt{c+d}}{2x\sqrt{e+c}} + \&c. \quad y = -\sqrt{c} + \frac{e\sqrt{c-d}}{2x\sqrt{e+c}} - \&c.$$

and the equations for the two asymptotes are $y = \sqrt{c}$ and $y = -\sqrt{c}$, which correspond to two lines drawn parallel to the axis, at a distance $= \sqrt{c}$ above and below it. If $c = e$, these two asymptotes coincide with the axis, which is therefore an asymptote to the curve.

(276.) If the rules of the Differential Calculus are of little use to find the asymptotes of curves, they are on the contrary employed with some advantage to discover those peculiarities in a curve which are designated under the name of singular and remarkable points.

The points to be considered are: First. Those at which the tangent is parallel to the axis of x or to the axis of y . Secondly. The points at which the curve passes from one side of its tangent to the other, as in fig. 2; such a point is called a *point of inflexion*, and sometimes a *point of contrary flexure*. Thirdly. The points where two branches of a curve meet, being both tangent to the tangent passing through that point, and extending only on one side of the point of contact; such points are called *cusps*: they are *cusps of the first kind*, when the two branches lie at opposite sides of the common tangent, (fig. 3,) and they are *cusps of the second kind*, when the two branches lie at the same side of the tangent. (Fig. 4.) Fourthly. The points through which two or more branches of a curve pass, having all a common tangent at that point, but extending on each side of the point of contact, they may then have their convexities turned towards the same side, or contrary ways; (fig. 5;) such points are called *points of osculation*. Fifthly. The points through which pass two or more branches of a curve, which have not the same tangent at that point; (fig. 6;) such points are called *multiple points*. Sixthly. The points whose coordinates satisfy the equation of the curve, but which, however, are isolated, so that no point of the curve is found immediately near them, on either side; such points are called *conjugate points*. Seventhly. The points where a branch of a curve suddenly stops, extending only on one side of it; (fig. 7;) such points may be called *stop points*. And finally, the points where two or more branches of a curve meet and stop, extending only on one side of that point, and having each a particular tangent; (fig. 8;) such a point may be called a *booting point*.

(277.) The general method to discover whether a point whose coordinates are x' and y' is one of the singular points just mentioned, is to examine attentively the curve immediately before and immediately after the point. This may be done generally by substituting $x' + h$ for x in the equation of the curve, and developing the corresponding value of the ordinate in a series of terms arranged according to the powers of h . Two cases may occur; the developement may contain only integral powers of h , or it may contain fractional powers. In both cases, if we remember what has been said with respect to the developement of functions, it may be foreseen that the consideration of the values of the differential coefficients of the ordinate, corresponding to the coordinates x' and y' of the point under consideration, may be of great use.

(278.) When the value of the ordinate corresponding to $x' + h$, which we shall designate by y_1 , may be developed according to the integral powers of h , the developement is

$$y_1 = y' + \frac{dy'}{dx'} \cdot h + \frac{d^2y'}{dx'^2} \frac{h^2}{1 \cdot 2} + \dots \dots$$

We have proved that $\frac{dy'}{dx'}$ is the trigonometrical tangent of the angle which the tangent to the curve, at the point whose coordinates are x' and y' , makes with the axis of x ; and that, consequently, the value of the ordinate of the tangent corresponding to the abscissa $x' + h$ is $y' + \frac{dy'}{dx'} \cdot h$. Hence, for the points at which the tangent is parallel to the axis of the curve, we must have $\frac{dy'}{dx'} = 0$.

It does not follow, however, that when this condition is fulfilled the ordinate of the point of contact is always either a maximum or a minimum. If we had, at the same time, $\frac{d^2y'}{dx'^2} = 0$, and not $\frac{d^3y'}{dx'^3} = 0$, or, more generally, if several differential coefficients did vanish, and if the first that did not vanish were of an odd order, it is manifest that there would be neither maximum nor minimum. With respect to the maxima and minima of ordinates of curves, we have to make precisely the same observations as those which have already been made concerning maxima and minima of functions of one variable.

(279.) If we have $\frac{dy'}{dx'} = 0$, and $\frac{d^2y'}{dx'^2} = 0$, then the tangent is still parallel to the axis of x ; but the difference between the ordinate of the curve, and that of the tangent corresponding to the abscissa $x' + h$, being then $\frac{d^3y'}{dx'^3} \frac{h^3}{1 \cdot 2 \cdot 3} + \frac{d^4y'}{dx'^4} \frac{h^4}{1 \cdot 2 \cdot 3 \cdot 4} + \&c.$ it follows that the curve at the point of contact passes from one side of the tangent to the other, since the difference changes sign with h . In that case, therefore, the point $x' y'$ will be a point of *inflexion* or a point of *contrary flexure*. The same reasoning would still apply if we had not $\frac{dy'}{dx'} = 0$,

but simply $\frac{d^2y'}{dx'^2} = 0$, for the difference between the two ordinates of the tangent and curve would still be the same; only the tangent, in that case, would not be parallel to the axis of x . All that has been said would still subsist if the first differential coefficient which does not vanish, instead of being of the third order, was of any odd order.

(280.) Let $F(x, y) = 0$ be the equation of the curve, and let the first differential equation be represented by $N \frac{dy}{dx} + M = 0$. To obtain the value of the first differential coefficient $\frac{dy}{dx}$ corresponding to the point whose

Integral Calculus. coordinates are x', y' , we must substitute in the preceding equation x' for x and y' for y . Let the results of these substitutions in M and N be designated by M' and N' . If $M' = 0$, and N' be equal to a finite quantity, then

$\frac{dy'}{dx'} = 0$, and the tangent to the curve is parallel to the axis of x . Therefore to find points of the curve where this circumstance takes place, it is sufficient to find the systems of values of x and y which satisfy the two equations $F(x, y) = 0$ and $M = 0$, and to choose among them those which do not satisfy the equation $N = 0$: each of these systems will be the coordinates of one of the points required.

Let us next suppose $M' = 0$ and $N' = 0$, then the value of $\frac{dy'}{dx'}$ assumes the form $\frac{0}{0}$. To obtain the true value of this quantity, it will therefore be necessary to proceed to the differential equation of the second order.

Let $N \frac{d^2y}{dx^2} + R \frac{dy}{dx} + Q \frac{dy}{dx} + P = 0$ be this equation, by substituting x' for x and y' for y , and designating by R', Q', P' , the values assumed by R, Q, P , it will become $R' \frac{d^2y'}{dx'^2} + Q' \frac{dy'}{dx'} + P' = 0$, since $N' = 0$ by hypo-

thesis. If the coefficients of this equation be not equal to nothing, it will give generally two values for $\frac{dy'}{dx'}$. These values may be either real and unequal, real and equal, imaginary or infinite. If they be real and unequal, there being two unequal values of the first differential coefficient, corresponding to the same value of x and y , there may be two tangents to the curve at the corresponding point, and therefore two branches may intersect at that point, which, in that case, is a multiple point, and may be called a double point, since there are only two branches intersecting. If the roots be real and equal, there is but one value of the differential coefficient, and, consequently only one tangent. If the roots be imaginary, the developement of y_1 is imaginary, whether h be taken positively or negatively, and therefore the point whose coordinates produce this effect will generally stand alone, insulated, and be, according to the definition, a conjugate point. We shall examine afterwards the case in which the value of the differential coefficient is infinite.

If the values of $R', Q',$ and P' were all equal to nothing, it would be necessary to proceed one step further in the differentiation of the proposed equation, and we should obtain, to find the value of $\frac{dy'}{dx'}$, an equation of the form.

$W' \frac{d^3y'}{dx'^3} + V' \frac{d^2y'}{dx'^2} + T' \frac{dy'}{dx'} + S' = 0$. Then, if the roots of this equation be real and unequal, there may be three tangents at the corresponding point, and three branches of the curve may intersect at it. Therefore the point may be a multiple point or a triple point. If two of the roots be equal, there will be only two distinct values of $\frac{dy'}{dx'}$, and there may be at the point two distinct tangents. If two be imaginary, or if the three roots be equal, there will be only one real value of $\frac{dy'}{dx'}$, and, consequently, one tangent.

If we had also $W' = 0, V' = 0, T' = 0, S' = 0$, then to find the value of $\frac{dy'}{dx'}$ we should use the differential equation of the fourth order, which would give an equation of the fourth degree with respect to $\frac{dy'}{dx'}$. The discussion of the different cases which the roots of this equation might present, would lead to conclusions analogous to the preceding. Generally, it will be necessary to continue the differentiation until some equation is found of which the differential coefficients are not all equal to nothing. Then according to the number and nature of the values of $\frac{dy'}{dx'}$, inferences may be drawn with respect to the existence of a multiple or conjugate point of the curve corresponding to the coordinates x' and y' .

(281.) Let us suppose now that the coordinates x' and y' substituted for x and y in the general value of $\frac{d^2y}{dx^2}$ makes it assume the form $\frac{0}{0}$. Then the value or values, or $\frac{d^2y'}{dx'^2}$ will be obtained like those of $\frac{dy'}{dx'}$ by successive differentiations. If it have several real unequal values, there may be as many for the third term of the developement of y_1 , and therefore as many different branches of the curve may pass through the corresponding point. Since, however, the several values of y_1 would, in that case, agree as to the second term of the developements, they would have all a common tangent. Such a point, therefore, would be a point of osculation. If all the values of $\frac{d^2y'}{dx'^2}$ were found to be imaginary, then the corresponding point would be a conjugate point. Similar conclusions may be applied to the succeeding differential coefficients.

(282.) It is of great importance to add to the conclusions which may be drawn from what precedes that, in some cases, the equation of a curve may be put under such a form that the value of $\frac{dy}{dx}$ derived from it will not assume the form $\frac{0}{0}$ for the values of x and y corresponding to a multiple point, whilst in other circumstances,

Integral
Calculus.

Part III.

although it assumes that form, and the above method gives several values for the differential coefficient, yet the corresponding point is not a multiple point.

To give examples of these cases, let us first consider the curve corresponding to the equation $y = b + (x - a)\sqrt{x - c}$. It is obvious that the point corresponding to the abscissa $x = a$, is a multiple point, if by taking values for $x < \text{or} > a$, the quantity under the radical remains positive, for then there will be on both sides of the point where $x = a$, two distinct branches of the curve passing through that point. However, the value of $\frac{dy}{dx}$ derived from the above equation is $\sqrt{x - c} + \frac{x - a}{2\sqrt{x - c}}$, and does not assume the form $\frac{0}{0}$, but becomes equal to $\sqrt{a - c}$ when x is made equal to a .

On the contrary if we take the equation $ay + bx = xy$, which is that of an hyperbola passing through the origin of the coordinates, and put it under the form $\frac{a}{x} + \frac{b}{y} = 1$, we shall have by differentiation $\frac{dy}{dx} = -\frac{ay^2}{bx^2}$.

This value takes the form $\frac{0}{0}$ for the origin of the coordinates where $x = 0$ and $y = 0$. By differentiating again we find $2bx\frac{dy}{dx} + b x^2 \frac{d^2y}{dx^2} = -2ay\frac{dy}{dx}$. This equation cannot yet give the value of $\frac{dy}{dx}$ corresponding to $x = 0$ and $y = 0$, for by substituting these values for x and y every term of the equation vanishes. A new differentiation leads to the equation $2b\frac{dy}{dx} + 2bx\frac{d^2y}{dx^2} + 2bx\frac{d^2y}{dx^2} + b x^2 \frac{d^3y}{dx^3} = -2ay\frac{d^2y}{dx^2} - 2a\frac{dy^2}{dx^2}$,

which the supposition $x = 0$ and $y = 0$ reduces to $2b\frac{dy}{dx} = -2a\frac{dy^2}{dx^2}$. Hence we might infer that, at the origin, $\frac{dy}{dx}$ has two distinct values, $\frac{dy}{dx} = 0$ and $\frac{dy}{dx} = -\frac{b}{a}$, and, consequently, that the origin is a double point of the curve. However, this conclusion is obviously erroneous, for it is well known that an hyperbola has no multiple points. The right value of $\frac{dy}{dx}$, in this case, is $-\frac{b}{a}$, and is obtained immediately by differentiating the equation of the curve under its first form $ax + by = xy$.

It is also easy to see that the curve represented by the equation $y = b + (x - a)^m \sqrt{x - c}$ has a double point corresponding to $x = a$, when m is an even number, but that that point is simple when m is odd. In neither of these cases the value of $\frac{dy}{dx}$ derived from the above equation becomes $\frac{0}{0}$ for $x = a$. On the contrary, if the equation is first put under the form $(y - b)^m = (x - a)^m (x - c)$, it leads to a value of $\frac{dy}{dx}$ which assumes the form $\frac{0}{0}$ for $x = a$, when m is an even and when it is an odd number. It should be remarked, however, that if by

successive differentiations we try to determine the true value of $\frac{dy}{dx}$, we shall find that the differential equation which gives it has only two real values when m is an even number, and only one if it be odd.

From the preceding observations and examples, we may conclude that to establish with certainty the existence of a singular point indicated by the values assumed at that point, by the differential coefficients of the ordinate, it will always be necessary to examine with attention the values of the ordinates for a portion of the curve very near that point.

(283.) We shall now investigate the figure of a curve at a point whose coordinates render the first or any subsequent differential coefficient infinite.

In that case, x' and y' being still the coordinates of the point, and y_1 the value of the ordinate corresponding to $x' + h$, the developement of y_1 according to the powers of h will contain fractional exponents. Let

$$y_1 = y' + Ah^a + Bh^\beta + Ch^\gamma + \dots \&c.$$

If none of the exponents a, β, γ , &c. be a fraction with an even denominator, the value of y_1 is real whether h be taken positively or negatively. Hence the curve extends on both sides of the ordinate y' . The exponents a, β, γ , being supposed to be arranged in an ascending order, the value of $y_1 - y'$ will depend upon the first term only, for sufficiently small values of h . There are then two cases to examine, first, when the numerator of a is an odd number, and, secondly, when it is even.

If the numerator a be odd, the sign of Ah^a will change with that of h , and consequently at different sides of the point whose coordinates are x' and y' , the curve lies at different sides of a parallel to the axis of x passing through that point. If, in this case, we have $a > 1$, then $\frac{dy'}{dx'} = 0$, and the tangent at that point is parallel to the axis of x . Hence there is an inflexion, (such as in fig. 9,) the first when A is positive, and the second when it is negative. If a be < 1 , then $\frac{dy}{dx}$ is infinite, and the tangent is parallel to the axis of y . Since

the curve extends on both sides of y' , rising above the point of contact on one side, and descending below it on the other, there will be an inflexion such as in fig. 10. the first when A is positive, and the second when it is negative.

Let us next examine the case when the numerator of a is an even number. Then the sign of $A h^a$ does not change with that of h , and $A h^a$ has only one real value, since the denominator of a is necessarily odd. Therefore if a parallel to the axis of x be drawn through the point whose coordinates are x' and y' , the curve lies either above or below this parallel at both sides of the point, according as A is positive or negative. In this case, if a

be > 1 , we have $\frac{d y'}{d x'} = 0$, and, therefore, the parallel to the axis of x is a tangent, and the curve has one or the other of the two positions represented (fig. 11.) If a be < 1 , then $\frac{d y'}{d x'}$ is infinite, and the tangent is parallel to the axis of y . Hence the figure of the curve at the point in question, is one or the other of the two represented (fig. 12.)

If $a = 1$, and if β be a fraction having odd numbers for its numerator and denominator, then $A = \frac{d y'}{d x'}$, and the position of the tangent at the point whose coordinates are x' and y' is determined by the value of A . Moreover, since the sign of the second term $B h^\beta$ changes with the sign of h , and is equal to the first term of the difference between the ordinate of the curve and the ordinate of the tangent corresponding to the abscissa $x + h$, it follows that at different sides of the point of contact, the curve lies at different sides of the tangent, as represented in fig. 2. Hence, in this case, the point is a point of inflexion. In the same case the second differential coefficient $\frac{d^2 y'}{d x'^2} = 0$, if the exponent β be > 2 , and is infinite if β be < 1 . Thus, at a point of inflexion, the

second differential coefficient may be either nothing or infinite. If $a = 1$, and the numerator of β be even, the succeeding exponents not having any even denominator, the point is marked by no peculiarity.

(284.) If amongst the exponents a, β, γ , &c. is found a fraction with an even denominator, then a change in the sign of h changes y_1 from real to imaginary, or *vice versa*. If $+h$ render all the terms of the developement which are affected by such exponents real, and $-h$ imaginary, the curve extends only on the positive side of the ordinate y' , and is excluded from the negative side; and if $-h$ render them real, and $+h$ imaginary, it is just the reverse.

If some of the coefficients A, B, C , &c. are imaginary, and if some of the terms of the series remain imaginary whether h is taken positively or negatively, then the curve is excluded from both sides of the ordinate y' , and the point is a conjugate point.

If $+h$ or $-h$ render all the terms whose exponents have even denominators real, each of such terms will have two real values for every value of h , and, therefore, the number of branches of the curve emerging from the point in question may be double the number of combinations of such powers. The tangent to these branches will be determined by the value of the lowest exponent of h . If it be > 1 , the tangent is parallel to the axis of x , if < 1 , it is parallel to the axis of y , and if $= 1$, its position is determined by the value of the coefficient.

(285.) To illustrate these general principles. Let first $y_1 - y' = A h^{\frac{m}{2n}} + B h^\beta + C h^\gamma$, the first term of the series being the only one in which the exponent is supposed to have an even denominator. In this case $h^{\frac{m}{2n}}$ will be real if h be positive, and imaginary if it be negative. Hence if A be real, for every positive value of h , there are two real values of y_1 , and for any negative value of h , y is imaginary. Then, if $\frac{m}{2n} > 1$, $\frac{d y'}{d x'} = 0$, the tangent is parallel to the axis of x , and there is a cusp of the first kind at the point of contact. If $\frac{m}{2n} < 1$, then $\frac{d y'}{d x'}$ is infinite, the tangent is perpendicular to the axis of x , and is a limit of the curve. These two cases are represented by (fig. 13.) Next, let us suppose A to be an imaginary quantity, such that when $h^{\frac{m}{2n}}$ becomes imaginary by taking h negative, the product $A h^{\frac{m}{2n}}$ remains real; then, on the contrary, the first term of the series will be imaginary for a positive value of h . Consequently the curve will present the same figure as in the preceding case, according as $\frac{m}{2n}$ is greater or less than one, but it will be on the left instead of the right side of the ordinate. If A be an imaginary quantity such that $A h^{\frac{m}{2n}}$ remains imaginary for both $+h$ and $-h$, the point is conjugate.

(286.) Let us suppose, secondly, $y_1 - y' = A h^a + B h^{\frac{m}{2n}} + C h + \&c.$ the second term of the series being the only one in which the denominator is even. The direction of the tangent will be determined as before by the first term $A h^a$. If B is supposed to be a real quantity, as well as all the other coefficients, y_1 will have two real values if h be positive, and will be imaginary if h be negative. Hence there will be two branches of the curve passing through the point whose coordinates are x' and y' and extending only on the positive side of the ordinate

y' . If, on the contrary, B should be an imaginary quantity, such as to make $B h^{\frac{m}{2n}}$ real when h is taken negatively,

and if besides no other coefficient be imaginary, the curve will still have two branches, but both situated on the negative side of the ordinate y' . And, finally, the point will be conjugate, if B being imaginary, $B h^{\frac{m}{2n}}$ remains imaginary for both $+h$ and $-h$.

The two branches of the curve which pass through the point whose coordinates are x' and y' may form, in the case we have just examined, a cusp of the first or of the second kind, according to the value of a . First, if $a = 1$ the tangent common to both branches has an inclined position with respect to both axes, and the difference between the ordinates of the curve and tangent corresponding to the abscissa $x' + h$, beginning with the term $B h^{\frac{m}{2n}}$, shows that one of the branches is above the tangent and the other below, consequently there is a cusp of the first kind, on the right or on the left of y' , according as B is real or imaginary. (Fig. 3.) If a be > 1 , then $\frac{dy'}{dx} = 0$ the tangent is parallel to the axis of x , and the difference between the ordinates of the curve and tangent, corresponding to the abscissa $x' + h$, begins with the term $A h^a$. Therefore both branches will be above or below the tangent according as a is positive or negative, and there is a cusp of the second kind. (Fig. 14.) If B , in the same supposition, were imaginary, the two branches would be on the left of y' , both above the tangent if A were negative, and both below if A were positive, and if a were a fraction with an odd number for numerator; it would be the reverse if the numerator were even. Finally, if a be < 1 , then $\frac{dy'}{dx}$ is infinite, the tangent is parallel to the axis of y , and the two ordinates of the curve corresponding to $x' + h$ are both greater than y' if A is positive, and both less if A be negative, therefore there will be in both cases a cusp of the second kind. (Fig. 15.) If, in the same hypothesis, b be an imaginary quantity, both branches will be situated on the left side of y' , above the point or below the point according as A is positive or negative, and a a fraction whose numerator is an even or an odd number.

(287.) Let us for a last supposition, assume $y_1 - y' = A h^a + B h^b + C h^{\frac{m}{2n}}$, the third term of the series being the only one in which the denominator is even. The position of the tangent will depend on the values of a and A , and from the preceding investigation it obviously results, that the two branches of the curve will be on both sides of the curve represented by the equation $y_1 - y' = A h^a + B h^b$, and of the same side of the tangent, therefore there will be a cusp of the second kind at the point whose coordinates are x' and y' , on the right of the ordinate y' , if C be a real quantity, and on the left if C be an imaginary quantity, which renders $C h^{\frac{m}{2n}}$ real for $-h$, (fig. 16.) If C be an imaginary quantity not satisfying this condition, the point is clearly conjugate.

Generally, if the first and only term of the series, expressing the value of $y_1 - y'$, which has an even number for denominator, be the r^{th} , it is clear that if the coefficient R of that term be not imaginary, the curve will have a cusp of the second kind at the point whose coordinates are x' and y' , the two branches of which will be on the right side of the ordinate y' . There will still be a cusp, but on the other side of y' , if R being imaginary the product $R (-h)^{\frac{m}{2n}}$ be real; and the point will be conjugate if R being imaginary renders $R (\pm h)^{\frac{m}{2n}}$ imaginary.

(288.) The curves represented by the equations $y = x l x$, and $y = \frac{1}{l x}$ have each only one branch passing through the origin, but these branches stop suddenly at that point, where $x = 0$ and $y = 0$, since for a negative value of x , y becomes imaginary. The origin is, therefore, in each of these curves a *stop point*.

It may also be easily seen that the origin is a *shooting point*, in the curves represented by the equations $y = x \tan^{-1} x$, $y = \frac{x}{1 + e^x}$.

(289.) From the preceding investigations we may conclude, that we shall obtain generally an indication of the abscissa, corresponding to a single point of a curve, in determining in what cases the differential coefficients of the ordinate, considered as a function of the abscissa, become infinite, equal to nothing, or assume the form $\frac{0}{0}$. We shall find the species of the point, 1. by examining how many branches of the curve pass through that point, and whether they extend or not on both sides of the point; 2. by determining the position of their tangents; 3. by finding on what side they turn their concavity or convexity.

(290.) We shall now apply these general considerations to a few examples.

Example 1. Let $a y^2 - x^3 - b x^2 = 0$ be the equation of the curve. We easily find $\frac{dy}{dx} = \frac{3x^2 + 2bx}{2ay}$. This value assumes the form $\frac{0}{0}$ at the origin of coordinates where $x = 0$ and $y = 0$. Therefore the origin may be a multiple point. By using the differential equation of the second order, we find $\frac{d^2y}{dx^2} = \pm \sqrt{\frac{b}{a}}$, which indicates a double point at the origin; and by taking the value of y in the equation, we have $y = \pm \sqrt{\frac{x^3 + bx^2}{a}}$, which shows that the curve has in fact two branches passing through that point. If $b = 0$ the two values of $\frac{dy}{dx}$

become equal to each other and equal to nothing. Hence, in that case, the two tangents coincide with each other, and with the axis of x . If then we substitute for x , $0 + h$, or h , we find $y_1 = 0 + h^{\frac{3}{2}}$, from which we conclude that the origin is a cusp of the first kind. If b be negative, the values of $\frac{dy}{dx}$ become imaginary, and the origin is a conjugate point.

Example 2. Let $x^3 - axy - b^2y = 0$, be the equation of the curve. The differential equation is $3x^2dx - aydx - axdy - b^2dy = 0$, and gives $\frac{dy}{dx} = 0$ when $x = 0$ and $y = 0$, therefore the tangent at the origin is parallel to the axis of x . By a new differentiation we get $bxdx^2 - 2adydx - (ax + b^2)d^2y = 0$, and, consequently, at the origin we have also $\frac{d^2y}{dx^2} = 0$. Hence that point may be a point of inflexion or contrary flexure. To ascertain if it be so, we must examine if the value of $\frac{d^3y}{dx^3}$ is also equal to nothing, a new differentiation

gives $bdx^3 - 3ad^2ydx - (ax + b^2)d^3y = 0$, and for $x = 0$ and $y = 0$, $\frac{d^3y}{dx^3} = \frac{b}{b^2}$. Therefore the origin is a point of inflexion.

Example 3. Let $ax^3 + by^3 + c^4 = 0$. We have first $ax^2dx + by^2dy = 0$; and for $x = 0$, and the corresponding ordinate of the curve, $\frac{dy}{dx} = 0$. Hence the tangent at the point whose coordinates are $x = 0$ and $y = -c\sqrt[3]{\frac{c}{b}}$ is parallel to the axis of x . We see, in the same manner, that $\frac{dy}{dx}$ becoming infinite for $y = 0$, and the corresponding abscissa of the curve, the tangent is perpendicular at the axis of x , at the point whose coordinates are $y = 0$ and $x = -c\sqrt[3]{\frac{c}{a}}$. If we determine the values of $\frac{d^2y}{dx^2}$ and $\frac{d^3y}{dx^3}$ corresponding to the first point, we find $\frac{d^2y}{dx^2} = 0$, and $\frac{d^3y}{dx^3} = -\frac{2a}{b}$, therefore it is a point of inflexion. In the same manner, by examining the values of $\frac{d^2x}{dy^2}$ and $\frac{d^3x}{dy^3}$ corresponding to the second point, we shall verify that it is also a point of inflexion.

Example 4. Let $a^3y = x^4$ be the equation of the curve. By differentiation we find $a^3dy = 4x^3dx$, $a^3d^2y = 12x^2dx^2$, $a^3d^3y = 24xdx^3$, $a^3d^4y = 24dx^4$. Hence the values of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\frac{d^3y}{dx^3}$ are equal to nothing at the origin, and $\frac{d^4y}{dx^4} = 24$. Hence, the tangent at that point is parallel to the axis of x , but there is no point showing a geometrical inflexion, since the first differential coefficient, which does not become equal to nothing, is of an even order. Such points are sometimes designated under the name of points of double inflexion. It is likewise termed a point of *serpement* or of *undulation*; such points as these are points of inflexion, in an analytical rather than a geometrical sense, since the curve itself presents no visible character of a change in the nature of its curvature.

Example 5. Let there be a series of curves whose equations are

$$y = b + (x - a)^{\frac{2}{3}}, \quad y = b + (x - a)^{\frac{5}{3}}, \quad y = x + (x - a)^{\frac{4}{3}}, \quad y = x + (x - a)^{\frac{7}{3}},$$

and let it be required to find the species of the point corresponding in each to the abscissa $x = a$. In these cases some of the differential coefficients of y become infinite for $x = a$. Therefore the values of the ordinate in each curve, corresponding to $x = a + h$, developed in a series arranged according to the powers of h , will contain fractional powers. If we designate these ordinates by y_1 , we shall have, respectively,

$$y_1 = b + h^{\frac{2}{3}}, \quad y_1 = b + h^{\frac{5}{3}}, \quad y_1 = a + h + h^{\frac{4}{3}}, \quad y_1 = a + h + h^{\frac{7}{3}},$$

and, therefore, according to (284.), we may conclude that there is a point of inflexion in all these curves, corresponding to $x = a$; and also that, in the first, the tangent at the point of inflexion is perpendicular to the axis of x ; in the second, the tangent at the point of inflexion is parallel to the same axis; and in the third and fourth, that the tangent at the point of inflexion is inclined at an angle of 45° to the axis of x .

Example 6. Let $a^3y^2 = x^5$, the differential coefficient $\frac{dy}{dx}$ is equal to nothing at the origin, and, therefore, the tangent at that point is parallel to the axis of x . The third differential coefficient becomes infinite for the coordinates of the same point. The value of y corresponding to $0 + h$ is simply $\frac{h^{\frac{5}{2}}}{a^{\frac{3}{2}}}$, and, consequently, accord-

ing to (284.) the origin presents a cusp of the first kind. In a similar manner it is easy to see that for the point whose coordinates are $x = a$, and $y = b$, there is a cusp of the first kind.

Example 7. Let the equation of the curve be $y - a = (x - b)^{\frac{1}{3}} + (x - b)^{\frac{2}{3}}$. The value $x = b$ renders all the differential coefficients infinite. Hence the tangent at the point corresponding at that abscissa is perpendicular to the axis of x . Let $b + h$ be substituted for x , and we shall have for the corresponding value of y , $y_1 = a + h^{\frac{1}{3}} + h^{\frac{2}{3}}$. Hence (285.) there is a cusp of the second kind, at the point whose coordinates are a and b .

Example 8. Let the equation be $y = a + x + bx^2 + cx^{\frac{5}{2}}$. At the point whose coordinates are $x = 0$ and $y = a$ the value of the first differential coefficient is equal to 1. Therefore the tangent at that point is inclined at an angle of 45° to the axis. For the same values, the third differential coefficient becomes infinite. Substituting then $0 + h$ for x , we find $y_1 = a + h + bh^2 + ch^{\frac{5}{2}}$, and, consequently, (285.) there is at the point under consideration a cusp of the second kind.

Of the Contact of Curves and of Osculating Curves.

(291.) Let $y = f(x)$ and $y = f_1(x)$ be the equations of two curves passing through the same point. Let x' and y' be the coordinates of that point, and y_1, y_2 , the ordinates of the two curves corresponding to the same abscissa $x' + h$. Let us suppose also that none of the differential coefficients of the functions $f(x), f_1(x)$, up to the n^{th} order, become infinite for $x = x'$; and let us represent the values they assume by that substitution, for the first function by A_1, A_2, A_3 , &c. and for the second by B_1, B_2, B_3 , &c. Then we have

$$y_1 = y' + A_1 h + A_2 \frac{h^2}{1.2} + A_3 \frac{h^3}{1.2.3} + \&c., \quad y_2 = y' + B_1 h + B_2 \frac{h^2}{1.2} + B_3 \frac{h^3}{1.2.3} + \&c.$$

These series, expressing the values of y_1 and y_2 , have the same first term, which is the ordinate of the point common to the two curves; but generally the other terms differ from each other. Therefore by subtracting the second equation from the first, we have

$$y_1 - y_2 = (A_1 - B_1) h + (A_2 - B_2) \frac{h^2}{1.2} + (A_3 - B_3) \frac{h^3}{1.2.3} + \&c.$$

If $A_1 = B_1$, the first term of this difference vanishes; if, at the same time, $A_2 = B_2$, the two first terms vanish; if besides $A_3 = B_3$ the three first, and so on. In the first case, that is when the values of the differential coefficients of the ordinates of the two curves, corresponding to the abscissa of the point common to both, are equal to each other, the two curves are said to have a *contact of the first degree, or of the first order*; in the second case, when $A_1 = B_1$, and $A_2 = B_2$, they are said to have a *contact of the second degree, or of the second order*; in the third case, when $A_1 = B_1, A_2 = B_2$, and $A_3 = B_3$, they are said to have a *contact of the third degree or order*; and generally if the values of the coefficients of the n first powers of h in the developement of y_1 , are equal those of the coefficients of the n first powers of h in the developement of y_2 , the two curves are said to have a contact of the n^{th} order.

(292.) An immediate consequence of this definition is, that there is an infinite number of curves, which have, in a given point, and with a given curve, a contact of the n^{th} order. This becomes very obvious if we imagine the origin of coordinates placed at the given point; for, then, h represents the abscissa with respect to the new coordinates, $y_1 - y'$, which we shall call k , the ordinate. Consequently, the equation of the given curve referred to this new line is

$$k_1 = A_1 h + A_2 \frac{h^2}{1.2} + A_3 \frac{h^3}{1.2.3} + \&c. \dots \dots A_n \frac{h^n}{1.2.3 \dots n} + A_{n+1} \frac{h^{n+1}}{1.2.3 \dots n+1} + \&c.$$

and the equation of a curve having with it a contact of the n^{th} order with it, will be, if we designate by k_2 the ordinate

$$k_2 = A_1 h + A_2 \frac{h^2}{1.2} + A_3 \frac{h^3}{1.2.3} + \dots \dots A_n \frac{h^n}{1.2.3 \dots n} + B_{n+1} \frac{h^{n+1}}{1.2 \dots n+1} + \&c.,$$

where all the terms which follow $\frac{A_n h^n}{1.2.3 \dots n}$ remain entirely indeterminate.

(293.) When two curves have a contact of the n^{th} order, no curve, having with them a contact of a less degree, can be drawn through the point of contact, so as to pass between them immediately before or immediately after the point of contact. This theorem is demonstrated in a manner similar to that we have already used for the analogous proposition relative to the tangent.

Let the same notations be employed for the two curves having a contact of the n^{th} order as in number (290.) We have then for the ordinates of these two curves corresponding to the abscissa $x' + h$

$$y_1 = y' + A_1 \frac{h}{1} + A_2 \frac{h^2}{1.2} + A_3 \frac{h^3}{1.2.3} + \dots \dots + \frac{A_n h^n}{1.2.3 \dots n} + \frac{A_{n+1} h^{n+1}}{1.2.3 \dots n+1} + \&c.$$

$$y_2 = y' + A_1 \frac{h}{1} + A_2 \frac{h^2}{1.2} + A_3 \frac{h^3}{1.2.3} + \dots \dots + \frac{A_n h^n}{1.2.3 \dots n} + \frac{B_{n+1} h^{n+1}}{1.2.3 \dots n+1} + \&c.$$

Integral
Calculus.

Part III.

Hence $y_1 - y_2 = (A_{n+1} - B_{n+1}) \frac{h^{n+1}}{1.2.3 \dots n+1} + \&c.$ Let y_3 represent the ordinate of the third curve corresponding to the same abscissa $x' + h$. We shall have

$$y_3 = y' + A_1 \frac{h}{1} + A_2 \frac{h^2}{1.2} + \dots + \frac{A_m h^m}{1.2 \dots m} + \frac{C_{m+1} h^{m+1}}{1.2 \dots m+1} + \&c.$$

m being $< n$, since this third curve is supposed to have with the two others a contact of an order less than n .

We have therefore $y_1 - y_3 = (A_{m+1} - C_{m+1}) \frac{h^{m+1}}{1.2 \dots m+1} + \&c.$ Now, by taking h sufficiently small, we may compare, instead of the differences $y_1 - y_2$ and $y_1 - y_3$, the first terms of the series which express their values; and it is manifest that since m is $< n$, h may always be taken such as to make

$$(A_{n+1} - B_{n+1}) \frac{h^{n+1}}{1.2 \dots n+1} < (A_{m+1} - C_{m+1}) \frac{h^{m+1}}{m+1},$$

if the coefficients $A_{n+1} - B_{n+1}$, and $A_{m+1} - C_{m+1}$ are of the same sign. Hence it follows, that $y_1 - y_2$ is either less than, or of a contrary sign to $y_1 - y_3$, which was to be proved.

(294.) It may also be easily shown, that when two curves have a contact of an even degree, they intersect each other, one of the curves being, on one side of the point of contact above, and on the other side below the other curve. The contrary takes place when the contact is of an odd degree. In the first case, the difference of the series which represent the values of the ordinates of the curves corresponding to an abscissa $x + h$, x being the abscissa of the point of contact, begins with a term containing an odd power of h , consequently it changes sign at the same time as h , and therefore for sufficiently small values of h , if this difference be positive on one side of the point of contact, it will be negative on the other; in the second case, the first term of the difference contains an even power of h , consequently it does not change sign with h , and therefore the difference is positive or negative on both sides of the point of contact.

(295.) We have remarked (292.) that there is an infinite number of curves, which have, in a given point, and with a given curve a contact of the n^{th} order. Hence if the form and position of one of the curves be given, and if the form and position of the second curve may vary with the values of several constants, a, b, c , &c., contained in its equation, it may be proposed to determine these constants by the condition that the second curve shall have with the first a contact of a certain order.

Let $f(x, y) = 0$ be the equation of the first curve, and $\phi(x, y, a, b, c, \&c.) = 0$ that of the second curve. Let x be chosen for the independent variable, and let x, y represent the coordinates of a point taken on the first curve. We shall, generally, be able to determine the values of the constants a, b, c , &c. or of some of them, so that the values of the terms of the series $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \&c.$, corresponding to the abscissa of the point chosen on the first curve, will remain the same in passing from one curve to the other. Then the second curve will pass through the point (x, y) of the first, and will have with it at that point, a contact the order of which will be less by one than the number of the quantities $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \&c.$, which will not have changed their values. Let n be the number of the constants a, b, c , it is clear that they will all be determined if the n first quantities $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \&c.$ are made to have the same values in both curves, and that some of them will remain undeter-

minate if fewer than n of the quantities $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \&c.$ are made to have the same values in both curves. We

may therefore conclude, that among the systems of values which may be assigned to the constants a, b, c , &c. contained in the equation $\phi(x, y, a, b, c, \&c.) = 0$, there exists one generally for which the curve represented by this equation has with a given curve, at the point the abscissa of which is x , a contact of an order at least equal to the number $n - 1$, and an infinite number of systems of values of these constants, for which the contact between the two curves is of an order inferior to the same number.

To find, among the systems of values of a, b, c , &c., that which determines a contact of an order at least equal to $n - 1$, it is sufficient to combine between them, the formulæ which are obtained by substituting the values of

$$y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^{n-1}y}{dx^{n-1}}$$

derived from the equation of the given curve in the equation $\phi(x, y, a, b, c, \&c.) = 0$, and its differential equations of an order less than n .

If instead of the abscissa x , any function whatever of x and y were taken for the independent variable, it would be necessary to substitute the values of the quantities $x, y, dx, dy, d^2x, d^2y, \dots, d^{n-1}x, d^{n-1}y$ relative to the given curve, in the equation $\phi(x, y, a, b, c, \&c.) = 0$, and its differential equations of an order inferior to n , considering x and y as independent variables.

(296.) When there are only three constants, a, b, c , in the equation of a curve, it follows clearly from what precedes that an infinite number of systems of values may be assigned for the constants a, b, c , in order that that curve should be tangent to another given curve at a given point. To obtain these systems, it is sufficient to combine the two equations

$$\phi(x, y, a, b, c) = 0, \quad \frac{d\phi(x, y, a, b, c)}{dx} dx + \frac{d\phi(x, y, a, b, c)}{dy} dy = 0,$$

after having substituted the values of x, y, dx, dy , relative to the other given curve. By means of these equations two constants will be found in function of the third, which will remain an arbitrary quantity. The case will be different if it be required that the curve represented by the equation $\phi(x, y, a, b, c) = 0$ should have with the other curve, at the point (x, y) , a contact of an order at least equal to two, or in other words, if it be required that the point (x, y) should become for both curves a point of osculation. Then the three constants a, b, c will be determined by the system of the three equations

$$\begin{aligned} \phi(x, y, a, b, c) = 0, \quad \frac{d\phi(x, y, a, b, c)}{dx} dx + \frac{d\phi(x, y, a, b, c)}{dy} dy = 0, \\ \frac{d^2\phi(x, y, a, b, c)}{dx^2} dx^2 + \frac{2d^2\phi(x, y, a, b, c)}{dx dy} dx dy + \frac{d^2\phi(x, y, a, b, c)}{dy^2} dy^2 \\ + \frac{d\phi(x, y, a, b, c)}{dx} d^2x + \frac{d\phi(x, y, a, b, c)}{dy} d^2y = 0. \end{aligned}$$

If the equation $\phi(x, y, a, b, c)$ belongs to a circle, the three constants a, b, c are the coordinates of the centre and the radius; if we represent them respectively by $x', y',$ and ρ , the equation of the circle is $(x - x')^2 + (y - y')^2 = \rho^2$, and we have to determine the three quantities x', y' and ρ , the three equations

$$(x - x')^2 + (y - y')^2 = \rho^2, \quad (x - x') dx + (y - y') dy = 0, \quad (x - x') d^2x + (y - y') d^2y + dx^2 + dy^2 = 0.$$

The two first equations would determine the relation which exists between the abscissæ and ordinates of the centres of all the circles which touch the given curve at a given point. It is obvious, from the second equation, that all those centres are placed on the normal to the given curve, at the given point. We shall soon resume this question, or the determination of the coordinates of the centre and radius of the circle which has with a given curve a contact of the second order at a given point. But before leaving these general notions, we shall propose as an example to determine the n constants in the equation $y = a + bx + cx^2 + \dots + px^{n-2} + qx^{n-1}$, by the condition that the parabolic curve which it represents shall have with a given curve, at a given point, a contact of an order equal or superior to $n - 1$.

If we take x for the independent variable, we shall have by differentiating $n - 1$ times,

$$\begin{aligned} y &= a + bx + cx^2 + \dots + px^{n-2} + qx^{n-1}, \\ \frac{dy}{dx} &= b + 2cx + \dots + (n-2)px^{n-3} + (n-1)qx^{n-2}, \\ &\text{&c.} \qquad \qquad \qquad \text{&c.} \\ \frac{d^{n-2}y}{dx^{n-2}} &= 1.2.3 \dots (n-2)p + 2.3.4 \dots (n-1)qx, \\ \frac{d^{n-1}y}{dx^{n-1}} &= 1.2.3 \dots (n-1)q. \end{aligned}$$

These n equations, after having substituted for $x, y, \frac{dy}{dx}, \dots, \frac{d^{n-2}y}{dx^{n-2}}, \frac{d^{n-1}y}{dx^{n-1}}$, their values relative to the given curve and to the given point, will be sufficient to determine the values of the n constants $a, b, c, \dots, p, q$.

Let x_1, y_1 , be the coordinates of a point of the parabolic curve, which does not coincide with the given point (x, y) , this curve will then be represented by the equation resulting from the elimination of the n constants between the n above equations, and the equation

$$y_1 = a + bx_1 + cx_1^2 + \dots + px_1^{n-2} + qx_1^{n-1}.$$

But the right side of this equation may be developed according to the powers of $x_1 - x$, in observing that for any value whatever of the integer m we have

$$x_1^m = \{x + (x_1 - x)\}^m = x^m + \frac{m}{1} x^{m-1} (x_1 - x) + \frac{m(m-1)}{1.2} x^{m-2} (x_1 - x)^2 + \dots + (x_1 - x)^m.$$

Hence we shall find

$$\begin{aligned} y_1 &= a + bx + cx^2 + \dots + px^{n-2} + qx^{n-1}, \\ &+ \frac{b + 2cx + \dots + (n-2)px^{n-3} + (n-1)qx^{n-2}}{1} (x_1 - x), \\ &+ \text{&c.} \\ &+ \frac{1.2.3 \dots (n-2)p + 2.3.4 \dots (n-1)qx}{1.2.3 \dots (n-2)} (x_1 - x)^2, \\ &+ \frac{1.2.3 \dots (n-1)q}{1.2.3 \dots (n-1)} (x_1 - x)^{n-1}. \end{aligned}$$

In this development we see that the coefficients of the successive powers of $(x_1 - x)$ are precisely the values

Part III.
Integral
Calculus.

Part III.

we have found above for $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^{n-2}y}{dx^{n-2}}, \frac{d^{n-1}y}{dx^{n-1}}$, therefore by substituting, we shall find for the required equation of the parabolic curve

$$y_1 = y + \frac{dy}{dx} \frac{(x_1 - x)}{1} + \frac{d^2y}{dx^2} \frac{(x_1 - x)^2}{1.2} + \dots + \frac{d^{n-2}y}{dx^{n-2}} \frac{(x_1 - x)^{n-2}}{1.2.3 \dots (n-2)} + \frac{d^{n-1}y}{dx^{n-1}} \frac{(x_1 - x)^{n-1}}{1.2.3 \dots (n-1)},$$

when $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^{n-2}y}{dx^{n-2}}, \frac{d^{n-1}y}{dx^{n-1}}$ are the abscissa and ordinate of the given point on the given curve, and the differential coefficients of the ordinate corresponding to the given point derived from the equation of the given curve.

This simple and elegant formula might also be obtained very easily by first putting the equation of the parabolic curve under the form

$$y_1 - y = B(x_1 - x) + C(x_1 - x)^2 + \dots + P(x_1 - x)^{n-2} + Q(x_1 - x)^{n-1},$$

in which it is expressed that the curve passes through the point (x, y) , and then determining the constants $B, C, \dots, P, Q$ by the condition that the $(n-1)$ first differential coefficients of the ordinate derived from this

equation, and corresponding to $x_1 = x$ are equal to $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^{n-2}y}{dx^{n-2}}, \frac{d^{n-1}y}{dx^{n-1}}$. It is in this manner that the

equation of the tangent, or the parabolic line of the first degree having with a given curve in a given point a contact of the first order, has been found.

(297.) It will be necessary before we proceed to another important application of the theory of contact of curves, to determine the differential of an arc of a curve, the arc being considered as a function of the abscissa.

We have already had occasion to remark, that there are functions of one or more variables, which are such that although we cannot express the relation which exists between the function and the variables, either explicitly, or by means of equations, yet we can find the differential of the function. The arc considered as a function of the abscissa is an example of the case just mentioned.

Let us consider a plain curve represented by an equation between two rectangular coordinates x, y , and let s be the arc of this curve contained between a fixed point and the point (x, y) . If through this last point we draw a

tangent to the curve, the tangent of the acute angle which this tangent makes with the axis of x , is equal to $\frac{dy}{dx}$,

and therefore the secant of the same angle is equal to $\sqrt{1 + \frac{dy^2}{dx^2}}$. If, moreover, we draw an ordinate corresponding to the abscissa $x + h$, the portions of the curve and tangent, contained between the point (x, y) and

this ordinate may be represented the first by l , and the other is clearly equal to $h \sqrt{1 + \frac{dy^2}{dx^2}}$.

Let $y = f(x)$ be the equation of the curve under consideration, then we shall have $y + \kappa = f(x) + (f'(x + \theta h) - f'(x))h$, θ being a quantity less than one, and $f'(x)$ being the differential coefficient of $f(x)$. Hence we have $\kappa = \{f'(x + \theta h) - f'(x)\}h$.

Let us suppose now that the point whose coordinates are $x + h$ and $y + \kappa$, and the point (x, y) are so near each other that the two functions y and $\frac{dy}{dx}$, or $f(x)$ and $f'(x)$ are continuous functions in passing from one point

to the other, and that the second $\frac{dy}{dx} = f'(x)$ is constantly increasing or decreasing in all the values between x

and $x + h$. Then in the curvilinear triangle formed by the curve, the tangent, and the ordinate corresponding to the abscissa $x + h$, the second and third sides will be portions of straight lines, and the first a convex line. Therefore each side of this triangle will be inferior to the sum of the other two, and the difference between the two first sides will have a numerical value inferior to the third. Therefore the numerical values of the quantities

l and $h \sqrt{1 + \frac{dy^2}{dx^2}}$ will have a numerical value inferior to $(f'(x + \theta h) - f'(x))h$, and, in dividing these

three by h , we must conclude that the difference $\pm \frac{l}{h} - \sqrt{1 + \frac{dy^2}{dx^2}}$ will have a numerical value less than

$f'(x + \theta h) - f'(x)$. If we suppose that h converges towards zero, the expression $f'(x + \theta h) - f'(x)$ will

converge towards the same limit; but then $\frac{l}{h} = \frac{ds}{dx}$, therefore $\pm \frac{ds}{dx} - \sqrt{1 + \frac{dy^2}{dx^2}} = 0$, or

$ds = \pm \sqrt{1 + \frac{dy^2}{dx^2}} \cdot dx = \pm \sqrt{dx^2 + dy^2}$. The sign $+$ in this formula must be taken when the arc

increases at the same time as the abscissa, and the sign $-$ in the contrary case.

Hence if we designate by a the acute angle which the tangent to the curve at the point (x, y) makes with the axis of x , we shall have

$$\cos a = \pm \frac{dx}{ds}, \sin a = \pm \frac{dy}{ds}, \sec a = \pm \frac{ds}{dx}, \operatorname{cosec} a = \pm \frac{ds}{dy}.$$

Integral
Calculus.and the lengths of the normal and tangent between the point of contact and the axis of x will be respectively Part III.

$$\pm y \frac{ds}{dx}, \pm y \frac{ds}{dy}.$$

The chord of the arc we have above designated by l is the distance of the points (x, y) and $(x + h, y + \kappa)$, hence it is equal to $\sqrt{(h^2 + \kappa^2)}$. Consequently the ratio of the arc to its chord is $\frac{\pm l}{\sqrt{(l^2 + h^2)}}$, but, at the limit,

$$l = ds, \text{ and } \sqrt{(l^2 + h^2)} = \sqrt{(dx^2 + dy^2)}, \text{ therefore at the limit } \frac{\pm l}{\sqrt{(l^2 + h^2)}} = \frac{\pm ds}{\sqrt{(dx^2 + dy^2)}} = 1. \text{ Therefore}$$

when the arc of a plane curve becomes infinitely small the ratio of the arc to the curve becomes equal to one, or in other words, the limit of the ratio between the arc and the chord of a plane curve is equal to one.

Let a and b represent the angles which the chord joining the points (x, y) , $(x + h, y + \kappa)$ makes with the axis of x and y , each of these angles being comprised between 0° and 180° . Let also α and β be the limits towards which the angles a and b converged when the point $(x + h, y + \kappa)$ approaches indefinitely to the point (x, y) , that is to say, in other words, the angles formed by the tangent produced in the same direction as the chord with the

semi-axis of positive coordinates. We have $\cos a = \frac{h}{\sqrt{(h^2 + \kappa^2)}}$, $\cos b = \frac{\kappa}{\sqrt{(h^2 + \kappa^2)}}$, and at the limits we shall find

$$\cos \alpha = \frac{dx}{\sqrt{(dx^2 + dy^2)}}, \cos \beta = \frac{dy}{\sqrt{(dx^2 + dy^2)}}. \text{ But } \sqrt{(dx^2 + dy^2)} = \pm ds, \text{ therefore we shall have either}$$

$$\cos \alpha = \frac{dx}{ds}, \text{ and } \cos \beta = \frac{dy}{ds}, \text{ or } \cos \alpha = -\frac{dx}{ds}, \cos \beta = -\frac{dy}{ds}. \text{ The two first values are to be used when the}$$

tangent has been produced in the same direction as the arc s , and the two last when it has been produced in a contrary direction. This is obvious if we observe that the ratio $\frac{dx}{ds}$ being the limit of $\frac{h}{\kappa}$ will be positive when the

arc s increases at the same time as x , or, in other words, when the tangent, produced in the same direction as the arc, forms with the semi-axis of the positive abscissæ an acute angle, and consequently an angle the cosine of which is positive. If this angle become obtuse, its cosine and the ratio $\frac{dx}{ds}$ will become at the same time nega-

tive. Therefore the formulæ $\cos \alpha = \frac{dx}{ds}$, $\cos \beta = \frac{dy}{ds}$ must be preferred to the formulæ $\cos \alpha = -\frac{dx}{ds}$,

$\cos \beta = -\frac{dy}{ds}$, when the tangent is produced in the same direction as the arc.

(298.) Let us consider now two plane curves. Let x, y be the coordinates of the first, and s the arc of this curve comprised between a fixed point and the point (x, y) . Let x_1, y_1 and s_1 be the analogous quantities for the second curve. We shall have $ds^2 = dx^2 + dy^2$, $ds_1^2 = dx_1^2 + dy_1^2$. Let the tangents drawn to the first curve through the point (x, y) , and to the second curve through the point (x_1, y_1) be produced in the same direction as the arcs s , and s_1 , then they will form angles with the semi-axis of positive coordinates the cosines of which will be for the first tangent equal to $\frac{dx}{ds}$, $\frac{dy}{ds}$, and, for the second tangent, $\frac{dx_1}{ds_1}$, $\frac{dy_1}{ds_1}$. Hence

$$\text{if we call } \delta \text{ the angle of the two tangents, we shall have } \cos \delta = \frac{dx}{ds} \cdot \frac{dx_1}{ds_1} + \frac{dy}{ds} \cdot \frac{dy_1}{ds_1} = \frac{dx dx_1 + dy dy_1}{ds ds_1}.$$

$$\text{The two tangents will be parallel, when we shall have } \frac{dx_1}{ds_1} = \frac{dx}{ds}, \frac{dy_1}{ds_1} = \frac{dy}{ds}, \text{ or } \frac{dx_1}{ds_1} = -\frac{dx}{ds}, \frac{dy_1}{ds_1} = -\frac{dy}{ds}.$$

It is obvious that these two cases are included in the formula $\frac{dx_1}{dx} = \frac{dy_1}{dy}$, from which we derive

$$\frac{dx_1}{dx} = \frac{dy_1}{dy} = \pm \frac{\sqrt{(dx_1^2 + dy_1^2)}}{\sqrt{(dx^2 + dy^2)}} = \pm \frac{ds_1}{ds}.$$

If in the value we have just found for $\cos \delta$ we substitute for ds and ds_1 their values, we shall find

$$\cos \delta = \pm \frac{dx dx_1 + dy dy_1}{\sqrt{(dx^2 + dy^2)} \sqrt{(dx_1^2 + dy_1^2)}}. \text{ When in this last equation the sign of the right side is not deter-}$$

mined, it gives two values of δ which represent the acute and the obtuse angles formed by the two tangents produced on both sides of the points (x, y) , (x_1, y_1) .

When two curves meet, they are supposed to form the same angle as the tangents drawn through their common point. Then $x_1 = x$ and $y_1 = y$, and the angles formed by the two curves have for measures the values of δ we have found above.

Two curves are said to be normal to each other when they meet at right angles, that is to say, when their tangents drawn through their common point are perpendicular to each other, that is when the equation

$$1 + \frac{dy_1}{dx_1} \frac{dy}{dx} = 0 \text{ is verified for their point of intersection.}$$

Integral
Calculus.

Part III.

(299.) The following theorem may now be easily demonstrated. Two plane curves having a common point, and a common tangent at that point, if, on these curves, produced in the same direction, equal and very small lengths be taken from the point of contact, the straight line joining the extremities of these lengths will be very nearly parallel to the normal common to both curves.

Let us suppose that the equal lengths, which are taken on the first and second curve from the point of contact, have respectively for their extremities the points (x, y) and (x_1, y_1) . Let, moreover, s and s_1 be the lengths of the arcs of these two curves contained between two fixed points arbitrarily chosen on them, and the points (x, y) , (x_1, y_1) when the coordinates x, y, x_1, y_1 , will vary simultaneously, the difference $s_1 - s$ will remain constant, and, consequently, we shall have $ds_1 = ds$. Let α, β be the angles which the tangent common to both curves, and produced in the same direction as the arcs s and s_1 , forms with the semiaxis of the positive coordinates; let l be the distance of the two points (x, y) , (x_1, y_1) , and λ and μ the angles which the line joining these two points forms with the semiaxis of positive coordinates. We shall have

$$\cos \alpha = \frac{dx}{ds} = \frac{dx_1}{ds_1}, \quad \cos \beta = \frac{dy}{ds} = \frac{dy_1}{ds_1}, \quad l = \sqrt{(x - x_1)^2 + (y - y_1)^2}, \quad \cos \lambda = \frac{x - x_1}{l},$$

$$\cos \mu = \frac{y - y_1}{l}, \text{ and since } ds = ds_1, \quad dx^2 + dy^2 = dx_1^2 + dy_1^2, \text{ or } (dx + dx_1)(dx - dx_1)$$

$$+ (dy + dy_1)(dy - dy_1) = 0.$$

But the values of $\cos \alpha$ and $\cos \beta$ give $\frac{dx + dx_1}{\cos \alpha} = \frac{dy + dy_1}{\cos \beta} = 2 ds$; moreover, since for a value of x ,

differing but very little from a particular value of that variable which renders two given functions of x equal to nothing, the ratio between these functions differs very little from the ratios between their differential coefficients, and, consequently, from the ratio between their differentials, even though these differentials would become nothing for the same particular value of x , we see that the quantities $\cos \lambda, \cos \mu$, may be determined approximatively by the

following formulæ, $\cos \lambda = \frac{dx - dx_1}{dl}, \quad \cos \mu = \frac{dy - dy_1}{dl}$. Hence we shall have very nearly $\frac{dx - dx_1}{\cos \lambda} =$

$\frac{dy - dy_1}{\cos \mu} = dl$. The nearer the points (x, y) , (x_1, y_1) , are to the point of contact of the two curves, the more

exact is the last formula. If now we substitute in the formula $(dx + dx_1)(dx - dx_1) + (dy + dy_1)(dy - dy_1) = 0$, for the quantities $dx + dx_1$, and $dy + dy_1$, the quantities $\cos \alpha, \cos \beta$, which are proportional to them, and for the quantities $dx - dx_1, dy - dy_1$, the quantities $\cos \lambda$ and $\cos \mu$ which are proportional to them, we shall find $\cos \alpha \cos \lambda + \cos \beta \cos \mu = 0$. Therefore the straight line drawn through the points (x, y) , (x_1, y_1) will be very nearly perpendicular to the tangent, that is to say parallel to the normal.

(300.) Let r be the radius of a circle which touches a straight line in a given point. If we increase the length of r , the portion of the circumference about the point of contact, will differ less and less from a straight line, and will sensibly, within a small space about the point of contact, coincide with the straight tangent, when $\frac{1}{r}$ will nearly be equal to nothing. On the contrary the curvature of the circumference will increase more and more, when the radius decreases, or when the quantity $\frac{1}{r}$ increases. Hence it is natural to take the quantity $\frac{1}{r}$ for the

measure of what may be called the curvature of the circle. Let x, y , represent the coordinates of any point of the circumference, θ the angle which the tangent at that point makes with the axis of x , s the length of the arc between that point and a fixed point on the circumference, and let h, κ, t, l , be the respective increments of these variables when we pass from the point (x, y) to a point sufficiently near it that we may suppose the angle θ to have constantly increased or decreased in the interval. The length of the arc between the two last points is precisely $\pm l$, and the angle of the two radii, which are drawn to the extremities of the arc, is equal to the angle $\pm t$ formed by the two tangents. This understood, we shall have obviously $\pm l = \pm r t$, and, consequently, we shall have

for the curvature of the circle $\frac{1}{r} = \pm \frac{t}{l}$.

Let us conceive now that $x, y, x + h, y + \kappa$, are the two coordinates of two points not situated on the circumference of a circle, but on any curve whatever, and moreover, let them be near enough to each other that the angle θ may still be supposed to increase or decrease in a continuous manner in going from one point to the other. The ratio $\pm \frac{t}{l}$ will vary in general with the arc $\pm l$, and will be what may be called the *mean curvature* of that arc. If the

quantities h and κ are supposed to decrease indefinitely, or the point $(x + h, y + \kappa)$ to approach indefinitely to the point (x, y) , the quantities $\pm l, \pm t$, will converge towards the limit zero. But the limit towards which will,

at the same time, converge their ratio, is $\frac{d\theta}{ds}$, and in general will be a finite quantity, which we shall call the

curvature of the curve at the point (x, y) .

The angle $\pm t$ formed by the two tangents drawn at the extremities of the arc $\pm l$ is called generally the angle of contingence.

When the arc $\pm l$ is very small, the chord joining its extremities is very nearly perpendicular to the two normals drawn to the curve through the points (x, y) , $(x + h, y + \kappa)$; and the distance of the point (x, y) to the point

Integral
Calculus.

where the two normals meet is nearly equal to the radius of a circle, which would have the same curvature as the curve. For, let r be this distance, and s the limit towards which it converges. In the triangle formed by the two normals and the chord of the arc $\pm l$, the angle opposed to this chord will obviously be equal to the angle of contingence $\pm t$, whilst the angle opposed to the side r will differ very little from a right angle. Hence, if ϵ

Part III.

represent an infinitely small number, we shall have $\frac{\sin\left(\frac{\pi}{2} \pm \epsilon\right)}{r} = \frac{\sin(\pm t)}{\sqrt{(h^2 + k^2)}}$. And, consequently, at the limit

$\frac{1}{\rho} = \pm \frac{d\theta}{\sqrt{(dx^2 + dy^2)}}$, or $\frac{1}{\rho} = \pm \frac{d\theta}{ds}$. From this last formula it clearly results that ρ expresses the radius of a

circle the curvature of which is equal to $\pm \frac{d\theta}{ds}$. This radius is called the *radius of curvature* of the curve at the

point (x, y) , and if its length be taken from the point (x, y) , on the normal at that point, it determines on that normal the point which is called the *centre of curvature*, which may also be considered as the point where the two normals, drawn at the extremities of the infinitely small arc ds , meet. The circumference which has for its centre the centre of curvature, and for radius the radius of curvature, is called the *circle of curvature*, or the *osculating circle* at the point (x, y) .

(301.) The curvature, and the radius of curvature of a curve, may be expressed under various forms which are important to know, and which we shall now proceed to explain.

If we take x for the independent variable we shall have

$$\tan \theta = \pm \frac{dy}{dx}, \quad \theta = \pm \tan^{-1} \frac{dy}{dx}, \quad d\theta = \pm \frac{\frac{d^2y}{dx^2}}{1 + \frac{dy^2}{dx^2}} \cdot dx, \quad ds = \pm \sqrt{1 + \frac{dy^2}{dx^2}} \cdot dx,$$

and, consequently,

$$\frac{1}{\rho} = \pm \frac{\frac{d^2y}{dx^2}}{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}} = \pm \frac{d^2y}{dx^2} \cdot \frac{1}{\sec^3 \theta}.$$

The last formula shows immediately that the curvature becomes nothing, and the radius of curvature infinite, whenever $\frac{d^2y}{dx^2} = 0$. Then the osculating circle is a straight line and coincides with the tangent. This takes place, for instance, at the point of contrary flexure of the curve represented by the equation $y = x^3$, and generally for all the points of contrary flexure in the immediate vicinity of which the functions $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ remain continuous. If for a certain point the value $\frac{d^2y}{dx^2}$ be infinite, without the tangent being perpendicular to the axis of x , the curvature would be infinite, and the radius of curvature would vanish. Lastly, if the quantities $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ were both to be-

come infinite, the fraction $\pm \frac{\frac{d^2y}{dx^2}}{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}$ would assume an undeterminate form, but its true value might be

found by means of the rules previously given. Sometimes, whilst the abscissa x varies in a continuous manner, the value of the radius of curvature changes suddenly. This circumstance may occur not only for the points of curves which have been called *shooting or salient points*, when the function $\frac{dy}{dx}$ becomes discontinuous by changing suddenly of value, but also when this function remaining continuous, the value of $\frac{d^2y}{dx^2}$ is discontinuous. This takes place, for instance, in each of the curves represented by the equations $y = x\left(1 + \tan^{-1} \frac{1}{x}\right)$, $y = x^2\left(1 + \tan^{-1} \frac{1}{x}\right)$, at the point which coincides with the origin of the coordinates, which point is at the same time a shooting or salient point of the first curve, and a point of contact of the second curve with the axis of x . The radius of

curvature of the first curve is equal to $\pm \frac{\left\{\left(1 + \tan^{-1} \frac{1}{x} - \frac{x}{1+x^2}\right)^2 + 1\right\}^{\frac{3}{2}}}{2} (1+x^2)$; when the sign of x is

changed it becomes $\pm \frac{\left\{ \left(1 - \tan^{-1} \frac{1}{x} + \frac{x}{1+x^2} \right)^2 + 1 \right\}^{\frac{3}{2}} (1+x^2)}{2}$, and, consequently, when x vanishes previous

to changing its sign, the radius of curvature passes from the value $\frac{1}{2} \left\{ \left(\frac{\pi}{2} - 1 \right)^2 + 1 \right\}^{\frac{3}{2}}$ to the value $\frac{1}{2} \left\{ \left(\frac{\pi}{2} + 1 \right)^2 + 1 \right\}^{\frac{3}{2}}$. In the same circumstances the radius of curvature of the second curve passes from the value $\frac{1}{\pi - 2}$ to $\frac{1}{\pi + 2}$.

If in the formula $\frac{1}{\rho} = \pm \frac{d^2 y}{d x^2} \cdot \frac{1}{\sec^3 \theta}$, we substitute for $\sec \theta$ its value $\frac{y}{N}$, N being the length of the normal, we find $\rho = \pm \frac{N^3}{y^3} + \frac{1}{\frac{d^2 y}{d x^2}}$. By means of this expression the radii of curvature of several curves are very easily found.

We shall take for first example the curve represented by the equation $y^2 = 2 p x + q x^2$, where p is a positive quantity. This curve is an ellipse, a parabola, or an hyperbola, according as the quantity q is negative, nothing, or positive. Moreover, it is easy to see, 1. that the axis of x coincides with one of the axes of the curve, and the origin with one extremity of this axis; 2. that in the case where the constant q is greater than the quantity -1 , p represents the parameter, that is to say the ordinate drawn through one of the foci of the curve. Now if we differentiate the equation of the curve, we find $y \frac{d y}{d x} = p + q x$, and, consequently, $y^2 \frac{d^2 y}{d x^2} = p^2 + 2 p q x + q^2 x^2 = p^2 + q y^2$, $\frac{d^2 y}{d x^2} = \frac{p^2}{y^3} + q$, and then if we differentiate again, $\frac{d^3 y}{d x^3} = -\frac{p^2}{y^4}$. Thus we shall have $\rho = \frac{N^3}{p^2}$, and, consequently, in the curve represented by the equation $y^2 = 2 p x + q x^2$, the radius of curvature is equal to the cube of the normal, divided by the square of the length p . If we substitute for N its value $N = \sqrt{y^2 + y \frac{d y}{d x}} = \{ p^2 + (1 + q) y^2 \}^{\frac{1}{2}}$, we shall find $\rho = \frac{\{ p^2 + (1 + q) y^2 \}^{\frac{3}{2}}}{p^2} = \frac{\{ p^2 + (1 + q) (2 p x + q x^2) \}^{\frac{3}{2}}}{p^2}$. And lastly, when $q = 0$, or when the curve is a parabola, represented by the equation $y^2 = 2 p x$, we shall have simply $\rho = \frac{(p^2 + y^2)^{\frac{3}{2}}}{p^2} = p \left(1 + \frac{2 x}{p} \right)^{\frac{3}{2}}$. Both this last value, and the general value, become equal to p when $x = 0$,

in which case the value of y , or the ordinate of the curve, is also equal to nothing. Hence we may conclude that in the parabola, the ellipse, and the hyperbola, the radius of curvature corresponding to the summit, at one of the extremities of the greater axis, or of the real axis, is equal to the parameter.

Let us consider for a second example the ellipse or hyperbola, the axis of which is a and b , and the equation of which is $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \pm 1$. By two successive differentiations we shall find $\frac{x}{a^2} \pm \frac{y}{b^2} \frac{d y}{d x} = 0$, $\frac{d y}{d x} = \pm \frac{b^2}{a^2} \left(\frac{b^2}{y^2} \mp 1 \right)$, $\frac{d^2 y}{d x^2} = \mp \frac{b^4}{a^2 y^3}$, and $y^2 \cdot \frac{d^2 y}{d x^2} = \mp \frac{b^4}{a^2}$. Hence $\rho = \frac{N^3}{\left(\frac{b^2}{a^2} \right)^2}$, or by substituting for N its value

$$\rho = \frac{\left(\frac{b^2}{a^2} x^2 \pm a^2 \mp x^2 \right)^{\frac{3}{2}}}{a b} = \frac{\left(\frac{b^2}{a^2} x^2 + \frac{a^2}{b^2} y^2 \right)^{\frac{3}{2}}}{a b}.$$

If we consider in particular the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we shall find for the radius of curvature at the extremity of the axis $2 a$ $\rho = \frac{b^2}{a}$, and for the radius of curvature at the extremity of the axis $2 b$, $\rho = \frac{a^2}{b}$. Thus if $2 a$ represent the greater axis, $\frac{b^2}{a}$ will be the parameter designated in the preceding example by the constant p .

(302.) We shall take for the third example the cycloid. Let us suppose that from the origin of the coordinates we take on the axis of y , and in the direction of the positive ordinates, a length equal to r . Let us describe from the extremity of this length as a centre with a radius equal to r a circumference of a circle, and let this circumference be supposed to roll along the axis of x without ceasing to be tangent to it. The point of the circumference which at the first instant coincided with the origin of coordinates will describe the curve which is called *cycloid*. The curves described by a point within or without the generating circle, and moving with it, are called *trochoidal curves*.

Integral
Calculus.

Part III.

The equation of the cycloid may easily be found by the following considerations. Whilst the circle rolls on the axis of x , the centre moves parallel to the same axis, and the radius which in the first instant passed through the origin of coordinates, will turn round the centre in describing an angle which will continually increase. Let us designate by ω this angle; by x_1, y_1 , the coordinates of the centre; and by x, y the coordinates of the extremity of the moving radius, which will also be the coordinates of a point of the cycloid. The arc of circle between the extremity just mentioned, and the point where the circle touches the axis of x , will have obviously for measure the product $r\omega$; and since all the parts of this arc will have been applied successively one after the other, on parts of the axis of x , contained between the origin and the point of contact, equal to them in length, at any instant the distance of the point of contact to the origin of coordinates, or, which is the same thing, the abscissa of the centre of the circle, will have a length equal to $r\omega$. Hence, if we suppose that the circle moves in the direction of the positive x , we shall have $x_1 = r\omega, y_1 = r$. Moreover, it is obvious that the projections of the moving radius on the axis of x and y may be represented either by $x - x_1, y - y_1$, or by $-r \sin \omega, -r \cos \omega$. Therefore we shall have $x - x_1 = -r \sin \omega, y - y_1 = -r \cos \omega$, or $x = x_1 - r \sin \omega, y = y_1 - r \cos \omega$, and by substituting for x_1 and y_1 their values, we shall find $x = r(\omega - \sin \omega), y = r(1 - \cos \omega)$. It will be sufficient to extend these two last formulæ to the case where ω becomes negative, to obtain the coordinates x, y of the points of the cycloid situated on the side of the negative x , that is to say, the coordinates of the points with which coincides successively the extremity of the moving radius when the circle moves on that side. If now we take the value of ω in the equation $y = r(1 - \cos \omega)$ to substitute it in the value of x , we shall find, in supposing first the angle ω between

the limits 0 and π , $x = r \cos^{-1} \left(\frac{r-y}{r} \right) - \sqrt{(2ry - y^2)}$, and secondly, supposing the angle ω contained between the limits $\pm n\pi, \pm n\pi + \pi$, n being an integer, $x = r \cos^{-1} \left(\frac{r-y}{r} \right) \pm n\pi - \cos n\pi \sqrt{(2ry - y^2)}$,

which two formulæ may be replaced by $x = r \cos^{-1} \left(\frac{r-y}{r} \right) \pm \sqrt{(2ry - y^2)}$, where the radical must be taken with the sign $+$ or the sign $-$ according as the difference between the value taken for $\cos^{-1} \left(\frac{r-y}{r} \right)$, and the smallest value of that quantity, divided by π , gives an even or an odd number for quotient.

The basis of the cycloid is the straight line on which the generating circle is made to roll, and which has been taken for the axis of x .

It is easy to see by means of these equations, that the cycloid is composed of an infinite number of branches all equal to each other, and such that the extreme points situated on the axis of x answer to the abscissæ $x=0, x=\pm 2\pi r, x=\pm 4\pi r, x=\pm 6\pi r$ &c., and each of which is divided into two symmetrical parts by the ordinates corresponding to one of the abscissæ $x=\pm \pi r, x=\pm 3\pi r, x=\pm 5\pi r$, &c. If now we differentiate the values

of x and y found above, we get $dx = r(1 - \cos \omega) d\omega = y d\omega, dy = r \sin \omega d\omega$; and hence $\frac{dy}{dx} = \frac{r \sin \omega}{y} =$

$\pm \frac{\sqrt{(2ry - y^2)}}{y} = \pm \sqrt{\left(\frac{2r}{y} - 1 \right)}$. Therefore we shall have respectively for the lengths of the sub-normal, sub-tangent, normal and tangent, the following values:

$$\pm r \sin \omega = \sqrt{(2ry - y^2)}, \quad y \sqrt{\frac{y}{2r - y}}, \quad \sqrt{(2ry)}, \quad y \sqrt{\left(\frac{2r}{2r - y} \right)}$$

By comparing the value of $x_1 - x = r \sin \omega$ with the value of $\frac{dy}{dx} = \frac{r \sin \omega}{y}$, we find $x_1 - x = y \frac{dy}{dx}$, when x_1 is

the abscissa of the centre of the generating circle, or which is the same thing, the abscissa of the point of contact of this circle with the axis of x . Now, the value we have just found for this abscissa is precisely the same as that of the abscissa of the point where the tangent meets the axis of x . Hence it is easy to conclude that if we draw a diameter parallel to the axis of y in the generating circle of the cycloid, the directions of the normal and tangent to this curve will be those of the straight lines drawn through the point (x, y) , taken on the curve, to the extremities of the diameter of the generating circle in the corresponding position. The length of the normal will be the length of the chord corresponding in the circle to the angle ω , and to obtain the sub-normal of the cycloid, it will be sufficient to project the radius drawn from the point (x_1, y_1) to the point (x, y) on the axis of x .

Let us find now the radius of curvature. Since $\frac{dy^2}{dx^2} = \frac{2r}{y} - 1$, we shall have $\frac{d^2y}{dx^2} = -\frac{r}{y^2}$, and

$y^3 \frac{d^2y}{dx^2} = -ry = -\frac{1}{2} N^2$, N being the length of the normal, and, consequently, $\rho = \frac{N^3}{\frac{1}{2} N^2} = 2N$. Therefore, in the

cycloid, the radius of curvature is double the normal, when the basis of the curve is taken for the axis of x . Hence, the radius of curvature becomes nothing at the same time as the normal, and the curvature is infinite in all the points where the cycloid meets the basis, that is to say, at all the points where the curve offers a cusp of the first kind, whilst, in all the points furthest from the basis, the radius of curvature becomes equal to double the diameter of the generating circle.

(303.) Let us suppose now that the abscissa is not taken for the independent variable. We shall have

$$\tan \theta = \pm \frac{dy}{dx}, \quad \theta = \tan^{-1} \frac{dy}{dx}, \quad d\theta = \frac{dx \frac{d^2y}{dx^2} - dy \frac{d^2x}{dx^2}}{dx^2 + dy^2}, \quad ds = \pm \sqrt{dx^2 + dy^2},$$

and $\frac{\theta}{\rho} = \pm \frac{dx \frac{d^2y}{dx^2} - dy \frac{d^2x}{dx^2}}{(dx^2 + dy^2)^{\frac{3}{2}}} = \pm \frac{dx \frac{d^2y}{dx^2} - dy \frac{d^2x}{dx^2}}{ds^3}$. In the particular case where the arc s is considered as

the independent variable, the last formula may be transformed in such a manner as to contain only the differential coefficients of the second order of the variables x and y with respect to s . For that purpose let us differentiate the equation $dx^2 + dy^2 = ds^2$ in considering ds as a constant quantity, we shall have $dx \frac{d^2x}{ds^2} + dy \frac{d^2y}{ds^2} = 0$, and consequently

$$\frac{d^2y}{ds^2} = -\frac{d^2x}{ds^2} = \frac{dx \frac{d^2y}{dx^2} - dy \frac{d^2x}{dx^2}}{dx^2 + dy^2} = \pm \frac{\{(d^2x)^2 + (d^2y)^2\}^{\frac{1}{2}}}{(dx^2 + dy^2)^{\frac{3}{2}}}, \quad \text{and} \quad \frac{dx \frac{d^2y}{dx^2} - dy \frac{d^2x}{dx^2}}{ds^2} = \pm \frac{\{(d^2x)^2 + (d^2y)^2\}^{\frac{1}{2}}}{ds^3}.$$

Hence we shall find for the length of the radius of curvature

$$\frac{1}{\rho} = \frac{\{(d^2x)^2 + (d^2y)^2\}^{\frac{1}{2}}}{ds^3} = \left\{ \left(\frac{d^2x}{ds^2} \right)^2 + \left(\frac{d^2y}{ds^2} \right)^2 \right\}^{\frac{1}{2}}$$

If we take from the point (x, y) on the curve and on its tangent, produced in the same direction as the arc s , equal and infinitely small lengths represented by i , we shall find, for the coordinates of the extremity of one of

these lengths, $x + i \frac{dx}{ds}$, $y + i \frac{dy}{ds}$, and for the coordinates of the other

$$x + i \frac{dx}{ds} + \frac{i^2}{2} \left(\frac{d^2x}{ds^2} + I \right), \quad y + i \frac{dy}{ds} + \frac{i^2}{2} \left(\frac{d^2y}{ds^2} + J \right).$$

I and J designating infinitely small quantities. Therefore, if we call l the distance between the extremities of these two lengths, we shall have

$$l = \frac{i^2}{2} \left\{ \left(\frac{d^2x}{ds^2} + I \right)^2 + \left(\frac{d^2y}{ds^2} + J \right)^2 \right\}^{\frac{1}{2}}, \quad \text{and consequently} \quad \left\{ \left(\frac{d^2x}{ds^2} \right)^2 + \left(\frac{d^2y}{ds^2} \right)^2 \right\}^{\frac{1}{2}} = \lim \frac{2l}{i^2}.$$

This formula, compared to the value of $\frac{1}{\rho}$ found above, gives $\rho = \lim \frac{i^2}{2l}$. Therefore to obtain the radius of cur-

vature of a curve in a given point, it is sufficient to take on this curve and on its tangent, produced in the same direction, equal and infinitely small lengths, and to divide the square of one of them by the double of the distance between their extremities. The limit of the quotient is the exact value of the radius of curvature.

(304.) To apply the formulæ we have given to the determination of the curvature of a curve, it is necessary to begin to express the differential coefficients of the first and second order of the coordinates x, y , in function of these coordinates and the differential of the independent variable. This calculation may be dispensed with by means of the formulæ we are going to develop.

Let $u = 0$ be the equation of the given curve, u designating a function of the rectangular coordinates x, y . By differentiating this equation twice, we shall find

$$\frac{du}{dx} dx + \frac{du}{dy} dy = 0, \quad \frac{d^2u}{dx^2} dx^2 + \frac{d^2u}{dy^2} dy^2 = - \left(\frac{d^2u}{dx^2} dx^2 + 2 \frac{d^2u}{dx dy} dx dy + \frac{d^2u}{dy^2} dy^2 \right),$$

and, consequently,

$$\left(\frac{du}{dx} \right) = - \left(\frac{du}{dy} \right) = \frac{dy \frac{d^2x}{dx^2} - dx \frac{d^2y}{dx^2}}{\frac{d^2u}{dx^2} dx^2 + \frac{d^2u}{dy^2} dy^2} = \pm \frac{(dx^2 + dy^2)^{\frac{1}{2}}}{\left\{ \left(\frac{du}{dx} \right)^2 + \left(\frac{du}{dy} \right)^2 \right\}^{\frac{1}{2}}}.$$

Therefore we shall have

$$\begin{aligned} dx \frac{d^2y}{dx^2} - dy \frac{d^2x}{dx^2} &= \mp \frac{(dx^2 + dy^2)^{\frac{1}{2}}}{\left\{ \left(\frac{du}{dx} \right)^2 + \left(\frac{du}{dy} \right)^2 \right\}^{\frac{1}{2}}} \left\{ \frac{d^2u}{dx^2} dx^2 + \frac{d^2u}{dy^2} dy^2 \right\} \\ &= \pm \frac{(dx^2 + dy^2)^{\frac{1}{2}}}{\left\{ \left(\frac{du}{dx} \right)^2 + \left(\frac{du}{dy} \right)^2 \right\}^{\frac{1}{2}}} \left\{ \frac{d^2u}{dx^2} dx^2 + 2 \frac{d^2u}{dx dy} dx dy + \frac{d^2u}{dy^2} dy^2 \right\} \end{aligned}$$

This last formula, combined with the value $\frac{1}{\rho} = \frac{dx \frac{d^2y}{dx^2} - dy \frac{d^2x}{dx^2}}{ds^3}$, leads to

$$\frac{1}{\rho} = \pm \frac{\frac{d^2 u}{dx^2} dx^2 + 2 \frac{d^2 u}{dx dy} dx dy + \frac{d^2 u}{dy^2} dy^2}{(dx^2 + dy^2) \left\{ \left(\frac{du}{dx} \right)^2 + \left(\frac{du}{dy} \right)^2 \right\}^{\frac{1}{2}}},$$

and then replacing the differentials dx, dy by the quantities $\frac{du}{dy}, \frac{du}{dx}$, which are proportional to them, we shall have finally

$$\frac{1}{\rho} = \pm \frac{\left(\frac{du}{dy} \right)^2 \frac{d^2 u}{dx^2} - 2 \frac{du}{dx} \frac{du}{dy} \frac{d^2 u}{dx dy} + \left(\frac{du}{dx} \right)^2 \frac{d^2 u}{dy^2}}{\left\{ \left(\frac{du}{dx} \right)^2 + \left(\frac{du}{dy} \right)^2 \right\}^{\frac{3}{2}}}.$$

If the variables x, y were separated, that is to say, if the function u were composed of two parts, one containing the variable x alone, and the other the variable y , $\frac{d^2 u}{dx dy}$ would be equal to nothing, and the preceding value would be reduced to

$$\frac{1}{\rho} = \pm \frac{\left(\frac{du}{dx} \right)^2 \frac{d^2 u}{dy^2} + \left(\frac{du}{dy} \right)^2 \frac{d^2 u}{dx^2}}{\left\{ \left(\frac{du}{dx} \right)^2 + \left(\frac{du}{dy} \right)^2 \right\}^{\frac{3}{2}}}.$$

These two formulæ applied to the examples which have been chosen will give the same values as we have already found for the radii of curvature.

(305.) Before leaving this subject it is necessary that in the formula $\frac{1}{\rho} = \pm \frac{d\theta}{ds}$, we may substitute to θ any one of the values which verify the equation $\tan \theta = \frac{dy}{dx}$, that is to say, a value of the form $\pm n\pi + \tan^{-1} \frac{dy}{dx} = \pm n\pi \pm \theta$, n being an integer, and if we designate this general value by ψ , we shall have $\frac{1}{\rho} = \pm \frac{d\psi}{ds}$.

The centre of curvature corresponding to the point (x, y) will be placed with respect to this last point on the side of the positive y , or negative y , according as the value of the ratio $\frac{d\psi}{dx}$ will be positive or negative. For the centre of curvature being the point of intersection of two normals infinitely near each other, it is clear that the curve will always turn its concavity towards this same centre. Hence, if x be taken for the independent variable, it is obvious that the centre of curvature will be situated with respect to the point (x, y) on the side of the positive y if the value of $\frac{d^2 y}{dx^2}$ be positive, and on the side of the negative y in the contrary case. But we

have, x being still the independent variable, $\frac{d\psi}{dx} = \frac{\frac{d^2 y}{dx^2}}{1 + \frac{dy^2}{dx^2}}$, therefore the two quantities $\frac{d\psi}{dx}$ and $\frac{d^2 y}{dx^2}$ will be, at

the same time, positive or negative, and the sign of the first may be consulted instead of the sign of the other. If, moreover, we observe that the value and the sign of $\frac{d\psi}{dx}$ will remain the same whatever be the independent variable, we shall have demonstrated the theorem stated above.

(306.) Let ρ be as above the radius of curvature of a plane curve corresponding to the point (x, y) , and x_1, y_1 the coordinates of the centre of curvature. This centre being the extremity of the radius ρ taken on the normal from the point (x, y) , on the side towards which the curve is concave, the coordinates x_1, y_1 will obviously verify the two equations $(x_1 - x)^2 + (y_1 - y)^2 = \rho^2$, $(x_1 - x) dx + (y_1 - y) dy = 0$, which give

$$\frac{y_1 - y}{dx} = \frac{x_1 - x}{-dy} = \pm \frac{\left\{ (y_1 - y)^2 + (x_1 - x)^2 \right\}^{\frac{1}{2}}}{(dx^2 + dy^2)^{\frac{1}{2}}} = \pm \frac{\rho}{ds}.$$

From the remark made in the last number, it results, that if ψ be one of the angles determined by the equation $\tan \psi = \frac{dy}{dx}$, $y_1 - y$ and the ratio $\frac{d\psi}{dx}$ will be quantities of the same sign. Hence we shall have

$\frac{y_1 - y}{dx} = \frac{x_1 - x}{-dy} = \frac{1}{d\psi}$, and consequently $y_1 - y = \frac{dx}{d\psi}$, $x_1 - x = -\frac{dy}{d\psi}$. These equations, which are both

Integral Calculus. included in the preceding, are sufficient to determine the coordinates of the centre of curvature of any plane curve. Part III

If we take x for the independent variable, we shall find in multiplying by dx , and substituting for $\frac{d\psi}{dx}$ its value

$$\frac{\left(\frac{d^2 y}{dx^2}\right)}{1 + \left(\frac{dy}{dx}\right)^2}, \text{ we find } y_1 - y = \frac{x_1 - x}{-\left(\frac{dy}{dx}\right)} = \frac{1 + \left(\frac{d^2 y}{dx^2}\right)}{\left(\frac{d^2 y}{dx^2}\right)}, \text{ or which is the same thing, } y_1 - y = \frac{1 + \left(\frac{d^2 y}{dx^2}\right)}{\left(\frac{d^2 y}{dx^2}\right)};$$

$$x_1 - x = -\frac{dy}{dx} \frac{1 + \left(\frac{d^2 y}{dx^2}\right)}{\left(\frac{d^2 y}{dx^2}\right)}.$$

If x be not taken for the independent variable, the equation $\tan \psi = \frac{dy}{dx}$ will give $\frac{d\psi}{\cos^2 \psi} = d \frac{dy}{dx}$
 $= \frac{dx d^2 y - dy d^2 x}{dx^2}$, and $d\psi = \frac{1}{1 + \tan^2 \psi} \frac{dx d^2 y - dy d^2 x}{dx^2} = \frac{dx d^2 y - dy d^2 x}{dx^2 + dy^2}$, and therefore the formula
 $\frac{y_1 - y}{dx} = \frac{x_1 - x}{-dy} = \frac{1}{d\psi}$ will become $\frac{y_1 - y}{dx} = \frac{x_1 - x}{-dy} = \frac{dx^2 + dy^2}{dx d^2 y - dy d^2 x}$, or $y_1 - y = dx \frac{dx^2 + dy^2}{dx d^2 y - dy d^2 x}$,
 $x_1 - x = -dy \frac{dx^2 + dy^2}{dx d^2 y - dy d^2 x}$, from which it is easy to deduce $(x_1 - x) d^2 x + (y_1 - y) d^2 y = dx^2 + dy^2 = ds^2$,
 or which is the same thing, $(x_1 - x) \frac{d^2 x}{ds^2} + (y_1 - y) \frac{d^2 y}{ds^2} = 1$.

It is easy to find this last formula in a more direct manner. Let λ and μ be the angles which the normal, produced on the side towards which the curve turns its concavity, makes with the positive axis of x and y . We shall have clearly $\frac{x_1 - x}{\rho} = \cos \lambda$, $\frac{y_1 - y}{\rho} = \cos \mu$. Let us conceive also, that s is taken for the independent variable, and that from the point (x, y) we take on the normal and on the curve, produced in the same direction, equal and infinitely small lengths represented by i . Then if we call l the distance between the extremities of these two lengths, the distance l , reckoned from the tangent, will have for its algebraical projections on the axis quantities of the form $\frac{i^2}{2} \left(\frac{d^2 x}{ds^2} + I \right)$, $\frac{i^2}{2} \left(\frac{d^2 y}{ds^2} + J \right)$, I and J representing infinitely small quantities, and will form consequently with the same axis produced in the direction of the positive coordinates angles which will have for cosines $\frac{i^2}{2l} \left(\frac{d^2 x}{ds^2} + I \right)$, $\frac{i^2}{2l} \left(\frac{d^2 y}{ds^2} + J \right)$. But according to number (299.) the line l will be very nearly parallel to the normal; and as this length ought to be taken from the tangent, on the side where is the centre of curvature, we are allowed to conclude that the limits of the last expressions will be equivalent to the cosines of the angles λ and μ . This understood, if we reduce I and J to nothing, and substitute for $\frac{i^2}{2l}$ its limit ρ , we shall have $\rho \frac{d^2 x}{ds^2} = \cos \lambda$, $\rho \frac{d^2 y}{ds^2} = \cos \mu$. If the arc s were not taken for the independent variable, we ought to replace $\frac{d^2 x}{ds^2}$, $\frac{d^2 y}{ds^2}$ by $\frac{d \left(\frac{dx}{ds} \right)}{ds}$, $\frac{d \left(\frac{dy}{ds} \right)}{ds}$, and then the values of $\cos \lambda$ and $\cos \mu$ would be $\cos \lambda = \rho \left(\frac{d^2 x}{ds^2} - dx \frac{d^2 s}{ds^3} \right)$, $\cos \mu = \rho \left(\frac{d^2 y}{ds^2} - dy \frac{d^2 s}{ds^3} \right)$. If now we multiply the last equations by the two, $\cos \lambda = \frac{x_1 - x}{\rho}$, $\cos \mu = \frac{y_1 - y}{\rho}$, and if we add them, we shall find the equation which we proposed to demonstrate

$$(x_1 - x) \frac{d^2 x}{ds^2} + (y_1 - y) \frac{d^2 y}{ds^2} = 1.$$

In the particular case where s is taken for the independent variable, we have

$$x_1 - x = \rho^2 \frac{d^2 x}{ds^2} = \frac{dx^2 + dy^2}{(d^2 x)^2 + (d^2 y)^2} d^2 x, \quad y_1 - y = \rho^2 \frac{d^2 y}{ds^2} = \frac{dx^2 + dy^2}{(d^2 x)^2 + (d^2 y)^2} d^2 y.$$

The same formulæ are obtained by substituting in the formulæ

$$y_1 - y = dx \frac{dx^2 + dy^2}{dx d^2 y - dy d^2 x}, \quad x_1 - x = dy \frac{dx^2 + dy^2}{dx d^2 y - dy d^2 x},$$

Integral Calculus. the quantities $d^2 y$, $d^2 x$ to the quantities dx and $-dy$ to which they are respectively proportional, since $d^2 s$ or $d^2 x + d^2 y = 0$. Part III.

We have thus found between the quantities x, y, x_1, y_1 and ρ the three following equations:

$$(x_1 - x)^2 + (y_1 - y)^2 = \rho^2, (x_1 - x) dx + (y_1 - y) dy = 0, (x_1 - x) d^2 x + (y_1 - y) d^2 y - dx^2 - dy^2 = 0,$$

which are sufficient to determine in function of x the three unknown x_1, y_1 , and ρ . It is essential to remark that the second and third equations may be found by differentiating the first two consecutive times, and in considering x_1, y_1 , and ρ , as constant quantities.

(307.) When the point (x, y) changes its place on the curve, the centre of curvature changes at the same time. If the first point be supposed to move in a continuous manner, the second will describe a new curve. To obtain the equation of this last curve, it will obviously be sufficient to express in function of one variable only, x, y or s the values of x_1 and y_1 , and to eliminate this variable between the two values. The equation resulting from the elimination will contain only the two variables x_1 and y_1 , and will represent the line which is the locus of all the centres of curvature of the given curve. To discover the principal properties of this curve, we shall differentiate the two first of the three above equations. We shall have $(x - x_1) dx_1 + (y - y_1) dy_1 = \rho d\rho$, and $dx dx_1 + dy dy_1 = 0$. From the last of these two equations it results that the tangents drawn to the given curve through the point (x, y) and to the new curve through the point (x_1, y_1) are perpendicular to each other. Therefore the radius of curvature, which is reckoned on the normal drawn to the first curve through the point (x, y) , and which ends at the point (x_1, y_1) , will be, in this last point, tangent to the second curve. Moreover, if we call s_1 the arc of the new curve, contained between a fixed point and the point (x_1, y_1) , we shall have $dx_1^2 + dy_1^2 = ds_1^2$, and this combined with the above equations will give

$$\frac{dx_1}{x_1 - x} = \frac{dy_1}{y_1 - y} = \frac{(x_1 - x) dx_1 + (y_1 - y) dy_1}{\rho^2} = \pm \frac{\sqrt{dx_1^2 + dy_1^2}}{\rho} = \frac{d\rho}{\rho} = \pm \frac{ds_1}{\rho}.$$

Hence $d\rho = \pm ds_1$, or $d(\rho \pm s_1) = 0$. If, for more facility, we make $\rho \mp s_1 = \phi(x)$, we shall have $\frac{d\phi(x)}{dx} = 0$;

then if we designate by Δx a finite increment of x , and by $\Delta \phi(x)$ the corresponding increment of $\rho \mp s_1$, and by θ a number inferior to one, we shall have $\Delta \phi(x) = \phi(x + \Delta x) - \phi(x) = \frac{d \cdot (x + \theta \Delta x)}{dx} \cdot \Delta x = 0$. Hence

we shall have $\Delta(\rho \mp s_1) = 0$, or $\Delta\rho = \pm \Delta s_1$. This last equation shows that the arc $\pm \Delta s_1$ between two points of the new curve is equal to the difference of the radii of curvature which end at these points. The sign placed before the difference Δs_1 in the equation $\Delta\rho = \pm \Delta s_1$ will always be the same as that which will precede the

ratio $\frac{ds_1}{\rho}$ in the formula $\frac{dx_1}{x_1 - x} = \frac{dy_1}{y_1 - y} = \pm \frac{ds_1}{\rho}$, or the ratios $\frac{x_1 - x}{\rho}$, $\frac{y_1 - y}{\rho}$ in the two equations $\frac{dx_1}{ds_1} = \pm \frac{x_1 - x}{\rho}$, $\frac{dy_1}{ds_1} = \pm \frac{y_1 - y}{\rho}$. It results from this, that the arc s_1 and the radius of curvature ρ will increase

simultaneously if the tangent to the new curve, being produced in the same direction as the arc s_1 , coincide, not with the radius of curvature ρ , but with the producement of this radius beyond the point (x_1, y_1) , whilst in the contrary hypothesis, the radius ρ increasing, the arc s_1 will decrease. This understood, let us conceive that a thread is fixed by one of its extremities to the point (x_1, y_1) , let its length be equal to ρ , and let us first suppose that it is applied on the radius ρ . If then this thread, remaining constantly stretched, moves in such a manner that one part is wrapt up, the arc Δs_1 , contained between the point (x_1, y_1) and the point $(x_1 + \Delta x_1, y_1 + \Delta y_1)$, the other part which will remain straight, and will touch the new curve at the point $(x_1 + \Delta x_1, y_1 + \Delta y_1)$, will obviously have a length equal to the radius of curvature which ends at that point. Consequently, that of the extremities of the thread which first coincided with the point (x, y) will coincide now with the point $(x + \Delta x, y + \Delta y)$. As this last point is situated on the proposed curve, and that our reasoning holds whatever be the length of the arc $\pm \Delta s_1$, we may conclude that the extremity of the thread which is not fixed, will describe the proposed curve. The same curve will also be described, but in a contrary direction, if after having been applied on the arc $\pm \Delta s_1$, the thread moves so as to come back to its primitive position; and, in this retrograde motion, the portion of the thread unwrapped will be stretched in a straight line. From this remark are derived a few expressions, generally used, and which it is necessary to know. The new curve is called the *evolute* of the proposed curve, and the proposed curve, described by the free extremity of the thread, is called the *involute* of the new curve.

(308.) To show an application of the above formulæ, let us propose to find the locus of the centres of curvature of the cycloid, or, in other words, the evolute of that curve. The cycloid is represented by the two equations

$$x = r(\omega - \sin \omega), y = r(1 - \cos \omega) \quad (302.) \text{ from which we get } \tan \psi = \frac{dy}{dx} = \frac{r \sin \omega}{y} = \frac{\sin \omega}{1 - \cos \omega} = \cot \frac{\omega}{2},$$

and $\psi = \pm n\pi + \frac{\pi - \omega}{2}$, n designating any integer whatever. Hence we have $d\psi = -\frac{1}{2} d\omega$, and the formulæ

$$y_1 - y = \frac{dx}{d\psi}, x_1 - x = -\frac{dy}{d\psi} \text{ will give } x_1 = x + 2\frac{dy}{d\omega}, y_1 = y - 2\frac{dx}{d\omega}. \text{ If now we substitute for } x, y, \text{ and}$$

dx, dy their values, we shall find $x_1 = x + 2r \sin \omega = r(\omega + \sin \omega)$, $y_1 = -y = -r(1 - \cos \omega)$, and if we assume $\omega = \Omega + \pi$ we shall obtain $x_1 - \pi r = r(\Omega - \sin \Omega)$, $y_1 + 2r = r(1 - \cos \Omega)$, which represent the

Integral
Calculus.

Part III.

evolute of the cycloid. This curve will obviously pass through the point which has for its coordinates $x = \pi r$, $y = -2r$, and is the centre of curvature corresponding to the ordinate that is a maximum in the first branch. In order to place the origin of coordinates at this centre, it is sufficient to substitute $x_1 + \pi r$ for x , and $y_1 - 2r$ for y , so that the two equations become $x_1 = r(\Omega - \sin \Omega)$, $y_1 = r(1 - \cos \Omega)$. These two equations do not differ from the two equations which represent the given cycloid, therefore we may conclude that the evolute of a cycloid is another cycloid of the same form and of the same dimensions, the cusps of which coincide with the centres of curvature of the middle points of the several branches of the first. It is also obvious, that the cusps of the first are, at the same time, the middle points of the several branches of the second cycloid, and that the second cycloid being represented by the two equations $x_1 - \pi r = r(\Omega - \sin \Omega)$, $y_1 + 2r = r(1 - \cos \Omega)$, its basis coincides with the straight line which is represented by the equation $y_1 = -2r$.

The two equations $x_1 = x + 2r \sin \omega = r(\omega + \sin \omega)$, $y_1 = -y = -r(1 - \cos \omega)$ may be easily derived from the property we have demonstrated, namely, that in the cycloid, the radius of curvature is double the normal when the basis is taken for axis of x . In consequence of this proposition, the middle point of the radius of curvature,

that is to say, the point the coordinates of which are $\frac{x + x_1}{2}$, $\frac{y + y_1}{2}$, coincides with the point in which the basis is

touched by the generating circle, the abscissa of which is $r\omega = x + r \sin \omega$. We have, therefore, the equations

$$\frac{x + x_1}{2} = x + r \sin \omega, \quad \frac{y + y_1}{2} = 0, \text{ from which we may immediately derive the values given above for } x_1 \text{ and } y_1.$$

It is also possible, without using these formulæ, to determine the nature of the evolute of the cycloid. For that purpose, let us describe with the radius r two equal circles, having their centres placed upon the axis of y , the one above, the other below the axis of x , both touching this last axis, at the origin of coordinates. Let us conceive, moreover, that both these circles are made to roll, the first on the axis of x , and the second on the straight line $y = -2r$, parallel to it, so that the axis of x remains tangent to both. The radii which, in the first instant, passed through the origin of coordinates will turn round the centres, and describe, in the same time, equal angles; hence it results, 1. That these radii will be parallel in every position of the circles; 2. That the straight line joining their extremities will pass through the point of contact, and will be bisected at that point. But the part line inscribed, in the first circle, is equal to the normal of the proposed cycloid described by the extremity of the first radius, therefore the radius of curvature of this curve, which is equal to double the normal, will necessarily coincide with the whole line, and the centre of curvature with the extremity of the second radius. Hence, the locus of the centres of curvature, or the evolute of the first cycloid, will be precisely the second cycloid described by the extremity of the second radius.

(309.) Let again $u = 0$ be the equation of a plane curve, u being a function of the two coordinates x and y . By combining the following equations, which have been obtained,

$$(x_1 - x) dx + (y_1 - y) dy = 0, \quad (x_1 - x) d^2 x + (y_1 - y) d^2 y = ds^2, \quad \frac{du}{dx} dx + \frac{du}{dy} dy = 0,$$

$$\frac{du}{dx} d^2 x + \frac{du}{dy} d^2 y = - \left(\frac{d^2 u}{dx^2} dx^2 + 2 \frac{d^2 u}{dx dy} dx dy + \frac{d^2 u}{dy^2} dy^2 \right) = 0,$$

we shall find

$$\frac{x_1 - x}{\left(\frac{du}{dx}\right)} = \frac{y_1 - y}{\left(\frac{du}{dy}\right)} = \frac{(x_1 - x) d^2 x + (y_1 - y) d^2 y}{\frac{du}{dx} d^2 x + \frac{du}{dy} d^2 y} = - \frac{dx^2 + dy^2}{\frac{d^2 u}{dy^2} dy^2 + 2 \frac{d^2 u}{dx dy} dx dy + \frac{d^2 u}{dx^2} dx^2},$$

and then substituting for the differentials dx , dy , the quantities $\frac{du}{dy}$, $-\frac{du}{dx}$, which are respectively proportional to them, we shall find

$$\frac{x_1 - x}{\left(\frac{du}{dx}\right)} = \frac{y_1 - y}{\left(\frac{du}{dy}\right)} = \frac{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2}{\left(\frac{du}{dx}\right)^2 \frac{d^2 u}{dy^2} - 2 \frac{du}{dx} \cdot \frac{du}{dy} \cdot \frac{d^2 u}{dx dy} + \left(\frac{du}{dy}\right)^2 \frac{d^2 u}{dx^2}}.$$

If in the equation $u = 0$, the variables x and y are separated, we have $\frac{d^2 u}{dx dy} = 0$, and the preceding formula becomes

$$\frac{x_1 - x}{\left(\frac{du}{dx}\right)} = \frac{y_1 - y}{\left(\frac{du}{dy}\right)} = - \frac{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2}{\left(\frac{du}{dx}\right)^2 \frac{d^2 u}{dy^2} + \left(\frac{du}{dy}\right)^2 \frac{d^2 u}{dx^2}}.$$

To obtain the equation of the evolute of the curve represented by the equation $u = 0$, it will be sufficient to eliminate x and y between this equation and the two furnished by the preceding formula, or the one before, according as the variables x and y will or will not be separated in u . In several cases it is easy to resolve the two last equations, just mentioned, with respect to the variables x and y , and to deduce from them the values of x and y in function of the coordinates x_1 and y_1 . Then in substituting these values in the equation $u = 0$, the equation of the evolute is immediately obtained.

Integral
Calculus.(310.) Let us take for first example the ellipse represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. In this particular Part III.

case the variables are separated, and we may take $u = \frac{1}{2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right)$. Hence we shall have

$$\frac{du}{dx} = \frac{x}{a^2}, \quad \frac{du}{dy} = \frac{y}{b^2}, \quad \frac{d^2u}{dx^2} = \frac{1}{a^2}, \quad \frac{d^2u}{dy^2} = \frac{1}{b^2},$$

consequently,

$$\left(\frac{du}{dx} \right)^2 \frac{d^2u}{dy^2} + \left(\frac{du}{dy} \right)^2 \frac{d^2u}{dx^2} = \frac{1}{a^2 b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) = \frac{1}{a^2 b^2}.$$

And therefore, by means of the above formula, for the case in which the variables are separated,

$$a^2 \left(\frac{x_1}{x} - 1 \right) = b^2 \left(\frac{y_1}{y} - 1 \right) = - \left(b^2 \frac{x^2}{a^2} + a^2 \frac{y^2}{b^2} \right) = -a^2 + \frac{a^2 - b^2}{a^2} x^2 = -b^2 + \frac{b^2 - a^2}{b^2} y^2.$$

From this formula we derive $a^2 \frac{x_1}{x} = \frac{a^2 - b^2}{a^2} x^2$, $\frac{b^2 y_1}{y} = \frac{b^2 - a^2}{b^2} y^2$, and if we represent by A and B two positive quantities chosen in such a manner as to verify the formula $\pm (a^2 - b^2) = A a = B b$, we shall find $\frac{x_1}{A} = \pm \frac{x^2}{a^2}$,

$\frac{y_1}{B} = \mp \frac{y^2}{b^2}$, or, which is the same thing, $\frac{x}{a} = \pm \left(\frac{x_1}{A} \right)^{\frac{1}{2}}$, $\frac{y}{b} = \mp \left(\frac{y_1}{B} \right)^{\frac{1}{2}}$. If we substitute now these values in

the equation of the ellipse, we shall obtain for the equation of the evolute of that curve $\left(\frac{x_1}{A} \right)^{\frac{2}{3}} + \left(\frac{y_1}{B} \right)^{\frac{2}{3}} = 1$.

This equation may easily be transformed so as to contain only integral powers of the variables x and y . For that, it suffices to raise each side of the equation to the third power, we shall find

$$1 - \frac{x_1^2}{A^2} - \frac{y_1^2}{B^2} = 3 \left(\frac{x_1 y_1}{A B} \right)^{\frac{2}{3}} \left\{ \left(\frac{x_1}{A} \right)^{\frac{2}{3}} + \left(\frac{y_1}{B} \right)^{\frac{2}{3}} \right\} = 3 \left(\frac{x_1 y_1}{A B} \right)^{\frac{2}{3}}, \text{ and, finally, } \left(1 - \frac{x_1^2}{A^2} - \frac{y_1^2}{B^2} \right)^3 = 27 \frac{x_1^2}{A^2} \cdot \frac{y_1^2}{B^2}.$$

From either of these equations it is easy to conclude that the evolute of the ellipse is a curve completely shut, divided into four equal parts by two rectangular axes which coincide, as those of the ellipse, with the coordinate axis. Each part of the evolute is tangent to the axis where it meets them, and each of these points is, at the same time, a cusp of the first kind.

Let us next consider the hyperbola represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \pm 1$. By operating as in the preceding example, and designating by A and B two positive quantities, chosen in such a manner as to verify the

formula $a^2 \pm b^2 = A a = B b$, we shall find for the equation of the evolute of the hyperbola $\left(\frac{x_1}{A} \right)^{\frac{2}{3}} - \left(\frac{y_1}{B} \right)^{\frac{2}{3}} = \pm 1$,

or, which is the same thing, $\left\{ 1 \mp \left(\frac{x_1^2}{A^2} - \frac{y_1^2}{B^2} \right) \right\}^3 = -27 \frac{x_1^2}{A^2} \frac{y_1^2}{B^2}$. From either of these equations it is also easy to

conclude that the evolute of the hyperbola is a curve divided into four equal parts by the coordinate axis, and composed of two infinite branches separated from each other, each of which has a cusp situated on the production of the real axis $2a$, or $2b$, of the hyperbola, at the distance A or B from the origin.

Let us take for a last example the parabola represented by the equation $y^2 = 2px$, where p is supposed to be a positive quantity. In this case, we may take $u = \frac{1}{2} (2px - y^2)$, from which equation we derive $\frac{du}{dx} = p$,

$\frac{du}{dy} = -y$, $\frac{d^2u}{dx^2} = 0$, $\frac{d^2u}{dy^2} = -1$, and hence $\frac{x_1 - x}{p} = 1 - \frac{y_1}{y} = 1 + \frac{y^2}{p^2} = \frac{p + 2x}{p}$. We shall have, there-

fore, $x_1 - x = p + 2x$, $-\frac{y_1}{p} = \frac{y^2}{p^2}$, or, which is the same thing, $x_1 = p + 3x$, $y^2 = -p^2 \frac{y_1}{p}$, or $x = \frac{x_1 - p}{3}$,

$y = -p^{\frac{2}{3}} y_1^{\frac{1}{3}}$. Substituting these values in the equation of the parabola, and suppressing the factor p common

to both sides, we shall find $\frac{2}{3} (x_1 - p) = p^{\frac{1}{3}} y_1^{\frac{2}{3}}$. This equation, which may be put under the form $\frac{8}{27} (x_1 - p)^3$

$= p y_1^2$, represents the evolute of the parabola. It is easy to verify that this evolute extends infinitely on the side of the positive abscissæ, that it has for its axis the axis of the parabola, and that it meets this axis, in touching it, at the point the abscissa of which is p . This point, which is placed at the same distance from the focus as the summit of the parabola, is at the same time the summit of the evolute, and the only cusp it presents. To place the origin of coordinates at that point, it suffices to substitute $x_1 + p$ for x_1 in the preceding equation. And we shall have

$$x_1 = \frac{3}{2} p^{\frac{1}{3}} y_1^{\frac{2}{3}}, \quad y_1 = \left(\frac{2}{3}\right)^{\frac{3}{2}} p^{\frac{1}{2}} x_1^{\frac{3}{2}}.$$

This shows that any parabola of the second degree has for evolute a parabola of the degree $\frac{2}{3}$ or $\frac{3}{2}$.

(311.) When two curves having a common point, have, at the same time, not only the same tangent but also the same osculating circle, and, consequently, the same curvature, with their concavities turned towards the same side, the contact which exists between the two curves takes the name of *osculation*, and one of the curves is said to be an osculating curve of the other. This definition understood, the following proposition follows immediately.

Let us conceive that two plane curves are represented by two equations between the rectangular coordinates x, y , and that the abscissa x is taken for the independent variable. In order that one of the curves should be an osculating curve to the other in a common point, corresponding to the abscissa x , it will be necessary and sufficient that the ordinate y relative to that abscissa, and the first and second differential coefficients of y , that is to say, $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ should preserve, in passing from one curve to the other, the same numerical values and the same signs.

For, if these conditions are fulfilled, the two curves will obviously have a common point corresponding to the abscissa x . Moreover the two curves will have the same tangent, the same radius of curvature, and their convexities turned the same side. Reciprocally, if one of the curves is an osculating curve of the other, at the point the abscissa of which is x , not only the quantities $y, \frac{dy}{dx}$, and the sign of $\frac{d^2y}{dx^2}$ ought to remain the same in passing from one curve to the other, but it is clear that the same thing may be said with respect to the radius of curvature ρ , and the numerical value of $\frac{d^2y}{dx^2}$.

If x were not taken for the independent variable, the differential coefficient of the first order is still equal to $\frac{dy}{dx}$,

but the differential coefficient of the second order becomes equal $\frac{dx \frac{d^2y}{dx^2} - dy \frac{d^2x}{dx^2}}{dx^3} = \frac{d \cdot \left(\frac{dy}{dx}\right)}{dx}$. Let r be the independent variable, and let $r = f(x, y)$. By differentiating this equation two consecutive times, we find

$$1 = \frac{d \cdot f(x, y)}{dx} \cdot \frac{dx}{dr} + \frac{d \cdot f(x, y)}{dy} \cdot \frac{dy}{dr},$$

$$0 = \frac{d \cdot f(x, y)}{dx} \cdot \frac{d^2x}{dr^2} + \frac{d \cdot f(x, y)}{dy} \cdot \frac{d^2y}{dr^2} + \frac{d^2 f(x, y)}{dx^2} \frac{dx^2}{dr^2} + 2 \frac{d^2 f(x, y)}{dx dy} \frac{dx dy}{dr^2} + \frac{d^2 f(x, y)}{dy^2} \frac{dy^2}{dr^2}.$$

From these equations, it will be easy to derive the values of $\frac{dx}{dr}, \frac{dy}{dr}, \frac{d^2x}{dr^2}, \frac{d^2y}{dr^2}$, expressed in function of the quantities

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d \cdot f(x, y)}{dx}, \frac{d f(x, y)}{dy}, \frac{d^2 f(x, y)}{dx^2}, \frac{d^2 f(x, y)}{dx dy}, \frac{d^2 f(x, y)}{dy^2}.$$

And since these last quantities will preserve the same values, at the point of osculation of two plane curves in passing from one curve to another, it is clear that the same thing may be said of the quantities $\frac{dx}{dr}, \frac{dy}{dr},$

$$\frac{d^2x}{dr^2}, \frac{d^2y}{dr^2}.$$

If, for instance, we take for the independent variable the radius vector drawn through the origin to the point (x, y) we shall have

$$r = \sqrt{(x^2 + y^2)}, \quad 1 = \frac{x}{r} \frac{dx}{dr} + \frac{y}{r} \frac{dy}{dr}, \quad 0 = \frac{x}{r} \frac{d^2x}{dr^2} + \frac{y}{r} \frac{d^2y}{dr^2} + \frac{1}{r^3} \left(y \frac{dx}{dr} - x \frac{dy}{dr} \right)^2, \text{ and these combined with}$$

the formula $\frac{d^2y}{dx^2} = \frac{dx \frac{d^2y}{dx^2} - dy \frac{d^2x}{dx^2}}{dx^3}$, give

$$\frac{dx}{dr} = \frac{\sqrt{(x^2 + y^2)}}{x + y \left(\frac{dy}{dx}\right)}, \quad \frac{dy}{dr} = \frac{\frac{dy}{dx} \sqrt{(x^2 + y^2)}}{x + y \left(\frac{dy}{dx}\right)}, \quad \frac{d^2x}{dr^2} = - \frac{\left(y - x \frac{dy}{dx}\right)^2 + y \frac{d^2y}{dx^2} (x^2 + y^2)}{\left(x + y \frac{dy}{dx}\right)^3},$$

$$\frac{d^2y}{dr^2} = - \frac{\frac{dy}{dx} \left(y - x \frac{dy}{dx}\right)^2 - x \frac{d^2y}{dx^2} (x^2 + y^2)}{\left(x + y \frac{dy}{dx}\right)^3}.$$

Integral
Calculus.

If we take for the independent variable the arc s , between the point (x, y) and a fixed point, and if we suppose that this arc is produced in the same direction beyond the point of contact for both curves. We have then Part III.

$dx^2 + dy^2 = ds^2$, $dx d^2x + dy d^2y = 0$, which, combined with the value of $\frac{d^2y}{dx^2} = \frac{dx d^2y - dy d^2x}{dx^3}$, will give

$$\frac{dx}{ds} = \pm \frac{1}{\sqrt{1 + \frac{dy^2}{dx^2}}}, \quad \frac{dy}{ds} = \frac{\frac{dy}{dx}}{\sqrt{1 + \frac{dy^2}{dx^2}}}, \quad \frac{d^2x}{ds^2} = -\frac{\frac{dy}{dx} \cdot \frac{d^2y}{dx^2}}{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}, \quad \frac{d^2y}{ds^2} = \frac{\frac{d^2y}{dx^2}}{\left(1 + \frac{dy^2}{dx^2}\right)^{\frac{3}{2}}}.$$

The sign $+$ is to be taken when the arc s increases at the same time as the abscissa, and the sign $-$ in the contrary case. Hence we may conclude that for a point of osculation of two plane curves, the four quantities

$\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$ preserve the same numerical values and the same signs in passing from the first curve to

the second. When it is proposed to verify whether a point, common to two plane curves, is a point of osculation, we may substitute to the differential coefficients, the differentials themselves, and consequently, the following proposition is established.

Two plane curves being represented by two equations between the coordinates x, y , to ascertain whether one of the curves is an osculating curve of the other, in a given point, common to both, it will be sufficient to take for the independent variable, either a determinate function of the variables x, y , or the arc s , reckoned on each curve from a fixed point, and to examine if, for the given point, the same values of x, y, dx, dy, d^2x, d^2y , may be derived from the equations of the two curves.

If we suppose the second curve to be a circle represented by the equation $(x - x_1)^2 + (y - y_1)^2 = \rho^2$, the conditions which express that the point (x, y) is a point of osculation will suffice to determine the radius of the circle, and the coordinates of the centre, that is to say the three quantities x_1, y_1 and ρ . For if we take x for the

independent variable, the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$, derived from the equation of the first curve and its derived

equations, ought to satisfy to the equation of the circle, and to its differential equations of the first and second order,

that is to say, to the three formulæ $(x - x_1)^2 + (y - y_1)^2 = \rho^2$, $(x - x_1) + (y - y_1) \frac{dy}{dx} = 0$, $1 + \frac{dy^2}{dx^2} + (y - y_1) \frac{d^2y}{dx^2} = 0$.

When the values of the three unknown quantities x_1, y_1 and ρ , are determined by means of these

formulæ, the same results are found as we have obtained before.

If x were not taken for the independent variable, then the equation of the circle, and its two differential equations of the first and second order, might be presented under the form $(x - x_1)^2 + (y - y_1)^2 = \rho^2$, $(x - x_1) dx + (y - y_1) dy = 0$, $(x - x_1) d^2x + (y - y_1) d^2y + dx^2 + dy^2 = 0$, and ought to be verified by the values x, y, dx, dy, d^2x, d^2y , derived from the equation of the first curve. It is almost useless to remark that these last considerations respecting osculating curves, agree with the generalities, given before, respecting the contact of different orders which may exist between two curves.

(312.) In the preceding pages we have supposed, in general, that a plane curve was represented by an equation between two rectangular coordinates x, y . But it is often useful to substitute for these coordinates the polar coordinates r and p , r being the length of the radius vector drawn from the origin to the point, and p the angle which this line makes with the direction of the positive axis of x . The new coordinates are then connected with x and y by the two equations $x = r \cos p$, $y = r \sin p$. We shall now point out some of the results to which the consideration of polar coordinates may lead.

Let us first remark that the quantity r by which we designate the radius vector drawn from the origin or pole to any point, should always be considered as a positive quantity. As to the angle p formed by this radius vector and a given semi-axis, it may be positive or negative, and receive a numerical value inferior or superior to 2π . For more facility we shall call the semi-axis of the positive x from which the angle p is reckoned, semi-polar axis.

This understood, it is obvious, 1. That the equation of any circle having the origin for its centre will be $r = R$, R being a positive quantity equal to the radius of the circle; 2. That the equation of any semi-axis drawn from the origin will be $p = P$, P being a constant quantity positive or negative. It may be added that this equation $p = P$ will still represent the same semi-axis, if the quantity P is increased or diminished by any even multiple of π . Lastly, the semi-axis under consideration will be replaced by another in the same straight line, but in an inverse direction when the angle P is increased or diminished by an odd number of times π .

(313.) When a plane curve is represented by an equation between the rectangular coordinates x, y , we may easily, by means of the formulæ $x = r \cos p$, $y = r \sin p$, substitute for the variables x, y the polar coordinates, and obtain what is called the polar equation of the curve. Thus, for instance, if we call x_1, y_1 the rectangular coordinates of a fixed point, and R and P its polar coordinates, we have $x_1 = R \cos P$, $y_1 = R \sin P$. The straight line drawn through this point so as to form the angle ψ with the semi-axis of positive x , has for its equation, in rectangular

coordinates, $\frac{y - y_1}{x - x_1} = \tan \psi$, or $\frac{x - x_1}{\cos \psi} = \frac{y - y_1}{\sin \psi}$, and for the polar equation $\frac{r \cos p - R \cos P}{\cos \psi} = \frac{r \sin p - R \sin P}{\sin \psi}$,

or, which is the same thing, $r \sin(p - \psi) = R \sin(P - \psi)$. If the length R become equal to nothing, the

straight line will pass through the origin, and its polar equation will become $\sin(p - \psi) = 0$. This will be verified, in taking for n any integer whatever, by making $p = \psi \pm 2n\pi$, or $p = \psi \pm (2n + 1)\pi$. Each of these formulæ is similar to the equation $p = P$, and represents one of the semi-axes coinciding with the given straight line but in opposite directions, and both passing through the origin. We may easily, in the equation $r \sin(p - \psi) = R \sin(P - \psi)$, substitute the angle $p - P$ for the angle $p - \psi$. For, $p - \psi = (p - P) + (P - \psi)$, consequently $\sin(p - \psi) = \sin(p - P) \cos(P - \psi) + \sin(P - \psi) \cos(p - P)$, and this value being substituted, gives

$$\frac{r \cos(p - P) - R}{\cos(\psi - P)} = \frac{r \sin(p - P)}{\sin(\psi - P)}, \text{ or, which is the same thing, } \frac{r \cos(p - P) - R}{r \sin(p - P)} = \cot(\psi - P).$$

If, from the point (x_1, y_1) as a centre, with the radius ρ , we describe a circle, this circle, represented by the formula $(x - x_1)^2 + (y - y_1)^2 = \rho^2$, will have clearly for its polar equation $(r \cos p - R \cos P)^2 + (r \sin p - R \sin P)^2 = \rho^2$, or $r^2 - 2Rr \cos(p - P) + R^2 = \rho^2$. In the particular case where the centre coincides with the origin, R vanishes, and the equation is reduced to $r^2 = \rho^2$ or $r = \rho$. It is also very easy to verify that the two parabolas, and the ellipse or hyperbola, represented by the three equations $y^2 = 2ax$, $y^2 = -2ax$, $Ax^2 + 2Bxy + Cy^2 = \kappa$,

have respectively for their polar equations $r = \frac{2a \cos p}{\sin^2 p}$, $r = -\frac{2a \cos p}{\sin^2 p}$, $r^2 = \frac{\kappa}{A \cos^2 p + 2B \sin p \cos p + C \sin^2 p}$

If the equation $Ax^2 + 2Bxy + Cy^2 = \kappa$ takes the form $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

or $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the last polar equation becomes $r^2 = \frac{a^2 b^2}{a^2 \sin^2 p + b^2 \cos^2 p}$, or $r^2 = \frac{a^2 b^2}{b^2 \cos^2 p - a^2 \sin^2 p}$. Let us

suppose that a and b are positive, and that a is greater than b . The constant a will represent, in the parabolas, double the distance of the focus to the summit; in the ellipse half the greater or transverse axis; and in the hyperbola half the real or transverse axis. This understood, if we place the origin at the focus of the parabola, and if we

make $\frac{a}{2} = R$, the equation of the parabola in rectangular coordinates will take the form $y^2 = -4R(x - R)$, or

$x^2 + y^2 = (2R - x)^2$. And we shall conclude, if we observe that $R - x = \frac{y^2}{4R}$, that the quantity $R - x$, and

still more the quantity $2R - x$ must always remain positive. Hence $\sqrt{(x^2 + y^2)} = 2R - x$, or $x + \sqrt{(x^2 + y^2)} = 2R$. If now we substitute polar coordinates for the variables x and y we shall find for the polar equation of the parabola

$r + r \cos p = 2R$, or $r = \frac{2R}{1 + \cos p}$. When the same substitution is made in the equation $y^2 = -4R(x - R)$,

we find $r^2 = (2R - r \cos p)^2$, or $\{r(1 + \cos p) - 2R\} \{r(1 - \cos p) + 2R\} = 0$. Now, the quantities r and R being essentially positive, and the quantity $1 - \cos p$ being either positive or equal to nothing, it is clear that the factor $r(1 - \cos p) + 2R$ has always a finite value different from nothing. We may therefore suppress this factor, and by this means we find the same polar equation of the parabola as we had found before. In the ellipse, or the hyperbola, they give the name of *excentricity* to the ratio between the distance of a focus to the centre, and the half of the transverse axis or real axis. Let e be this ratio. The distance of the centre to one of the foci will be, for the ellipse, $ae = \sqrt{(a^2 - b^2)}$, and for the hyperbola $ae = \sqrt{(a^2 + b^2)}$. This understood, if we place the origin of coordinates, in the ellipse, at the focus situated on the side of the positive abscissæ, and in the hyperbola at the focus situated on the side of the negative abscissæ, the equations of these curves, in rectan-

gular coordinates, will present themselves under the form $\frac{(x + ae)^2}{a^2} + \frac{y^2}{b^2} = 1$, $\frac{(x - ae)^2}{a^2} - \frac{y^2}{b^2} = 1$, and if in

these equations we put for b its value derived from the equation $ae = \sqrt{(a^2 - b^2)}$, they will become respectively

$$y^2 = (1 - e^2) \{a^2 - (x + ae)^2\}, \quad y^2 = (e^2 - 1) \{(x - ae)^2 - a^2\}, \quad \text{or } x^2 + y^2 = \{a(1 - e^2) - ex\}^2, \\ \text{and } x^2 + y^2 = \{a(e^2 - 1) - ex\}^2.$$

Now, if in these last equations we substitute polar coordinates for rectangular coordinates, we shall find for the polar equation of the ellipse

$$\{r - a(1 - e^2) + er \cos p\} \{r + a(1 - e^2) - er \cos p\} = 0,$$

and for the polar equation of the hyperbola

$$\{r - a(e^2 - 1) + er \cos p\} \{r + a(e^2 - 1) - er \cos p\} = 0.$$

In the first equation $e = \sqrt{1 - \frac{b^2}{a^2}}$ being less than unity, the factor $r + a(1 - e^2) - er \cos p = r(1 - e \cos p) + a(1 - e^2)$ will have a positive value different from nothing. We may therefore suppress it, and reduce the

polar equation of the ellipse to the form $r(1 + e \cos p) - a(1 - e^2) = 0$, or $r = \frac{a(1 - e^2)}{1 + e \cos p}$. If in this last

equation we assume successively $p = 0$ and $p = \pi$, we shall have $r = a(1 - e)$ and $r = a(1 + e)$. These two values, the sum of which is $2a$, express the distances of the focus, taken for origin, to the two extremities of the transverse axis.

Integral
Calculus.

As to the equation of the hyperbola in which $\epsilon = \sqrt{1 + \frac{b^2}{a^2}}$ is evidently superior to one, it may be decomposed into two others, namely,

$$r(1 + \epsilon \cos p) - a(\epsilon^2 - 1) = 0, \text{ or } r = \frac{a(\epsilon^2 - 1)}{1 + \epsilon \cos p}, \text{ and } r(1 - \epsilon \cos p) + a(\epsilon^2 - 1) = 0, \text{ or } r = \frac{a(\epsilon^2 - 1)}{\epsilon \cos p - 1}.$$

Now, it is easy to verify that each of these equations represents only one of the two branches of the hyperbola; and that the equation from which we derive $r = a(\epsilon - 1)$ for $p = 0$, and $r = \infty$ for $p = \pm \cos^{-1} \frac{1}{\epsilon}$ belongs to the branch, the focus of which coincides with the origin, whilst the other equation from which we derive $r = a(\epsilon + 1)$ for $p = 0$, and $r = \infty$ for $p = \pm \cos^{-1} \frac{1}{\epsilon}$ belongs to the other branch. The two values of p , to

which correspond in each branch infinite values for r , indicate the directions of the two semi-axes which are the asymptots of that branch. As to the lengths $r = a(1 + \epsilon)$, $r = a(1 - \epsilon)$ the difference of which is equal to $2a$. They express the distances of the focus, taken for origin, to the two extremities of the real axis of the hyperbola.

(314.) Most of the preceding formulæ may be established in a direct manner, by geometrical considerations alone, in expressing by means of polar coordinates the known properties of the lines which these equations represent. Let us conceive, for instance, that P, R being the polar coordinates of a fixed point, and ψ any angle whatever positive or negative, it is required to find the polar equation of the straight line drawn through the point (P, R) parallelly to a semi-axis represented by the equation $p = \psi$. If, by the origin, we draw a radius vector r to any point whatever of the straight line, and if we call δ the acute or obtuse angle, which this radius vector, infinitely produced, makes with the semi-axis, the value of p corresponding to the radius vector under consideration will obviously be given by one of the four following formulæ, $p = \psi + \delta$, $p = \psi - \delta$, $p = \psi + \delta \pm 2n\pi$, $p = \psi - \delta \pm 2n\pi$, n being any integer whatever. It may be added that the first and third formulæ ought to be preferred, if a radius vector, turning round the axis, and setting off from the position in which it coincides with the semi-axis, is obliged to describe the angle δ to take a direct motion of rotation, whilst the second and fourth formulæ must be used in the contrary case. If now we project the radius vector r upon an axis perpendicular to the given straight line, we shall obtain for the projection the shortest distance from the origin to that straight line, and this shortest distance will be expressed by the product $r \sin \delta$, or $r \sin(p - \psi)$ or $r \sin(\psi - p)$. But the shortest distance just mentioned, being obviously a quantity independent of the variables r and p , we may conclude that the product $r \sin(p - \psi)$ or $r \sin(\psi - p)$ will not change when we replace in it p by P , and r by R . We shall have, therefore, $r \sin(p - \psi) = R \sin(P - \psi)$, or $r \sin(\psi - p) = R \sin(\psi - P)$. These formulæ are the same as those we have obtained before.

Let us try to find now the polar equation of the circle described from the point (P, R) as a centre with the radius ρ . If we call p, r the coordinates of any point of the circumference, and δ the angle contained between the radii r, R , we shall have $p = P \pm \delta$, or $p = P \pm \delta \pm 2n\pi$, and afterwards $\pm \delta = p - P \mp 2n\pi$, n being an integer, which may be supposed equal to nothing. Moreover, the angle δ being opposed to the radius ρ in the triangle, the sides of which are r, R and ρ , we shall have, by a well-known theorem of Trigonometry,

$$\cos \delta = \frac{R^2 + r^2 - \rho^2}{2Rr}.$$

If in this last formula we put for δ its values $p - P \mp 2n\pi$, we shall obtain the equation $\cos(p - P) = \frac{R + r^2 - \rho^2}{2Rr}$, which is the same as that obtained before.

Let us try now to find the polar equation of a parabola the summit of which coincides with the point (P, R) , and the focus with the origin. If we designate by p, r the coordinates of any point of the curve, and by δ the angle included between the radii r, R , we shall have still $\pm \delta = p - P \mp 2n\pi$. Moreover, the projection of the radius vector r on the axis of the parabola will have for its measure the numerical value of $r \cos \delta = r \cos(p - P)$, and as the distance of the focus to the directrix will be equal to $2R$, the distance of the point (p, r) to the directrix will obviously be equal to $2R - r \cos(p - P)$. But, according to a known property of the parabola, this distance ought also to be equal to the radius vector r . We shall have, therefore, $r = 2R - r \cos(p - P)$ or

$$r = \frac{2R}{1 + \cos(p - P)}.$$

To make this last equation coincide with the formula we have obtained before, it is sufficient to take for semi-polar axis that which, passing through the focus of the parabola, is directed towards the summit of that curve, and to make in consequence $P = 0$.

We might obtain with the same facility the equation of an ellipse, the focus of which would coincide with the origin, and one of the extremities of the transverse axis with the point (P, R) . For, let still p, r be the coordinates of any point of the curve, and δ the angle formed by the two radii R, r . Let a represent half the transverse axis, and ϵ the excentricity. In the triangle, the summits of which are the foci, and the point (p, r) of the curve, one of the sides, namely, the distance of the foci, will have for its measure the product of the transverse axis by the excentricity or the quantity $2a\epsilon$, and the two other sides, the sum of which, by a known property of the ellipse, is always equal to the transverse axis, will be represented by r and $2a - r$. In this triangle the angle opposed to the side $2a - r$ will be equal to δ , if the point (P, R) coincide with that extremity of the transverse axis which is most remote from the origin, and to the supplement of the angle δ , or to $\pi - \delta$ in the contrary case. If we adopt

Integral
Calculus.

Part III.

the second hypothesis, we shall find $\cos(\pi - \delta) = \frac{r^2 + (2a\epsilon)^2 - (2a - r)^2}{2(2a\epsilon)r} = \frac{1}{\epsilon} - \frac{a(1 - \epsilon^2)}{\epsilon r}$, or $r =$

$\frac{a(1 - \epsilon^2)}{1 + \epsilon \cos \delta}$; and then putting for δ its value derived from the equation $\pm \delta = p - P \mp 2n\pi$, we find

$r = \frac{a(1 - \epsilon^2)}{1 + \epsilon \cos(p - P)}$. This last equation contains with the angle P , two other constant quantities a, ϵ , one of

which might be replaced by the radius vector $R = a(1 - \epsilon)$. We may, moreover, remark that the last value we have obtained for r will become exactly the same as the value we have derived before from the equation of the ellipse in rectangular coordinates, if we take for semi-polar axis, that which passing through the origin of coordinates is directed towards the nearest extremity of the transverse axis, and if in consequence we make $P = 0$.

Let us consider lastly a branch of hyperbola, the focus of which is supposed to coincide with the origin, and the summit with the point (P, R) . Let p, r be the coordinates of any point of this branch, and δ the angle contained between the two radii r, R . Let $2a$ be the transverse or real axis of the hyperbola, and ϵ be the excentricity. In the triangle the summits of which are the foci, and the point (p, r) , one of the sides, namely, the distance of the foci, will be equal to the product $2a\epsilon$, and the two other sides, the difference of which is equal to $2a$, will be represented by r and $2a + r$. In this same triangle the angle opposed to the side $2a + r$ will be precisely the

angle δ . We shall have, therefore, $\cos \delta = \frac{r^2 + (2a\epsilon)^2 - (2a + r)^2}{2(2a\epsilon)r} = \frac{a(\epsilon^2 - 1)}{\epsilon r} - \frac{1}{\epsilon}$, or $r = \frac{a(\epsilon^2 - 1)}{1 + \epsilon \cos \delta}$,

and then putting for δ its value $\pm \delta = p - P \mp 2n\pi$, we shall have $r = \frac{a(\epsilon^2 - 1)}{1 + \epsilon \cos(p - P)}$. We might, in

this last equation, replace one of the constants a, ϵ by the radius $R = a(1 - \epsilon)$. We may moreover remark that if we take for semi-polar axis, that which, passing through the origin, is directed towards the summit of the branch we consider, and if accordingly we suppose $P = 0$, the last value we have obtained for r will become the same as that we have derived from the equation of the hyperbola referred to rectangular coordinates.

If the focus were placed, not at the focus of the branch of the hyperbola, the polar equation of which it is required to find, but at the focus of the other branch, it would obviously be necessary in the formula $r = \frac{a(\epsilon^2 - 1)}{1 + \epsilon \cos \delta}$ to

substitute for the length $2a + r$, the length $r - 2a$. We would have then $\cos \delta = \frac{r^2 + (2a\epsilon)^2 - (r - 2a)^2}{2(2a\epsilon)r}$

$= \frac{a(\epsilon^2 - 1)}{\epsilon r} + \frac{1}{\epsilon}$, or $r = \frac{a(\epsilon^2 - 1)}{\epsilon \cos \delta - 1}$, and, consequently, $r = \frac{a(\epsilon^2 - 1)}{\epsilon \cos(p - P) - 1}$. In supposing in this last

formula $P = 0$, we would find identically the result derived from the equation of the hyperbola in rectangular coordinates.

(315.) Polar coordinates may be used with advantage, not only to express, but also to discover the several properties of plane curves, and to determine their centres, their diameters, their singular points, &c. Let us suppose, for instance, that it is required to find the two axes, or the real axis of the ellipse or hyperbola represented

by the equation $r^2 = \frac{2\kappa}{A + C + 2B \sin 2p + (A - C) \cos 2p}$. For that purpose, it will obviously be sufficient

to double the maximum or minimum value of the radius vector r . It is clear, moreover, that this maximum or minimum value will correspond to a minimum or maximum of the expression $2B \sin 2p + (A - C) \cos 2p$, and

consequently, to a value of p determined by the formula $2B \cos 2p - (A - C) \sin 2p = 0$, or $\tan 2p = \frac{2B}{A - C}$,

which is obtained by making equal to nothing the differential coefficient of $2B \sin 2p + (A - C) \cos 2p$. Now the equation $\tan 2p = \frac{2B}{A - C}$ is satisfied by taking $p = \frac{1}{2} \tan^{-1} \frac{2B}{A - C} \pm n\pi$, or $p = \frac{1}{2} \tan^{-1} \frac{2B}{A - C} \pm$

$(n + \frac{1}{2})\pi$. These two last equations represent two straight lines at right angles to each other, which coincide with the axis of the curve under consideration.

From the formula $2B \cos 2p - (A - C) \sin 2p = 0$, we derive

$$\frac{\sin 2p}{2B} = \frac{\cos 2p}{A - C} = \pm \frac{1}{\sqrt{4B^2 + (A - C)^2}} = \frac{2B \sin 2p + (A - C) \cos 2p}{4B^2 + (A - C)^2},$$

and, consequently, we conclude that the maximum or minimum value of the expression $2B \sin 2p + (A - C) \cos 2p$ is $\pm \sqrt{4B^2 + (A - C)^2}$. We would arrive at the same conclusion in considering the equation

$$\{2B \sin 2p + (A - C) \cos 2p\}^2 + \{2B \cos 2p - (A - C) \sin 2p\}^2 = 4B^2 + (A - C)^2,$$

from which it results that the numerical value of the first quantity raised to the square, in the left side, is always inferior to the square root of the sum $4B^2 + (A - C)^2$, when the difference $2B \cos 2p - (A - C) \sin 2p$ has a value different from zero, and becomes equal to this square root, in the case where this same difference vanishes.

Integral
Calculus.

If now we substitute successively in the value of r the maximum and minimum values we have just found for a part of the denominator, we shall obtain the two equations Part III.

$$r^2 = \frac{\kappa}{A + C - \sqrt{\{4B^2 + (A - C)^2\}}}, \quad r^2 = \frac{\kappa}{A + C + \sqrt{\{4B^2 + (A - C)^2\}}}.$$

It is easy to see that these two values of r^2 are the two roots of the equation $\left(\frac{\kappa}{r^2} - A\right)\left(\frac{\kappa}{r^2} - C\right) = B^2$. Their

product, namely, $\frac{\kappa^2}{(AC - B^2)}$ has always the same sign as the difference $AC - B^2$, hence it is clear that they

will be both positive, if $AC - B^2$ being positive, A , C and K are quantities of the same sign. Then the maximum and minimum values of r^2 will give two real values maximum and minimum of r . On the contrary, one only of the values of r^2 will remain positive, and the corresponding value of r will alone be real if $AC - B^2$ become negative. We know, in fact, that the condition $AC - B^2 > 0$ is verified for the ellipse, and the condition $AC - B^2 < 0$ for the hyperbola, which has only one axis real.

(316.) Among the curves, the various properties of which may be more easily obtained by the use of polar coordinates than by the use of rectangular coordinates, we shall mention the curves named *spirals*, which generally form a great number, and even frequently an infinite number of revolutions round the centre, so as to approach more and more to the origin, or to become more and more distant from it. The spirals which have in particular been the object of the attention of geometers, are the *spiral of Archimedes*, the *hyperbolic spiral*, and the *logarithmic spiral*.

In the spiral of Archimedes, the radius vector r increases proportionally to the angle p . Therefore its polar equation is $r = ap$, a being a constant quantity. When this constant quantity is positive, only positive values must be taken for p , since r cannot be negative. In the same hypothesis, if we designate by R the value of r corresponding to $p = 2\pi$, we shall have $R = 2a\pi$, and if we make the angle p to increase from zero to infinity the radius vector r will vary between the same limits. Consequently the curve may be considered as described by a point which, setting off from the origin of coordinates, would turn an infinite number of times, with a direct motion, round that origin, and would get to an infinite distance from it. If in the equation $r = ap$, we

substitute for the constant a the radius vector R , this equation will become $r = \frac{Rp}{2\pi}$, and if we take R for the

unity of length we shall have simply $r = \frac{p}{2\pi}$.

(317.) The hyperbolic spiral is that curve the polar equation of which is obtained, in substituting the polar coordinates r and p in the place of the rectangular coordinates x and y in the equation $xy = a$, which represents an equilateral hyperbola referred to its asymptots. This spiral will, therefore, be represented by the formula

$rp = a$, or $r = \frac{a}{p}$. If we suppose that the constant a is positive, p must also be positive, and if we make p vary

between the limits zero and infinity, r will vary between the limits infinity and zero, so that we have $r = \frac{1}{0}$ for

$p = 0$, and $r = 0$ for $p = \frac{1}{0}$. Consequently the curve may be considered as described by a point which, setting

off from a position in which it would be placed at an infinite distance from the origin of coordinates, would make an infinite number of revolutions round the origin, with a direct motion, and would approach it indefinitely without ever reaching it, since r cannot be equal to nothing unless the number of revolutions is infinite. The hyperbolic spiral has for asymptot a straight line parallel to the axis of x , represented by the equation $y = a$. For, since

$y = r \sin p$, and $rp = a$, we have $y = \frac{a \sin p}{p}$, and, consequently, $y = a$ for $p = 0$, which value of p corresponds,

as it has already been remarked, with an infinite value for r .

(318.) The logarithmic spiral, sometimes called the *equiangular spiral*, is that in which the angle p is equal to the logarithm of the radius vector. If the basis of the logarithms is A , and if we have $Le = \frac{1}{lA} = a$, we shall

have for the polar equation of the logarithmic spiral $p = L(r) = alr$, or $r = e^{\frac{p}{a}}$. If $A > 1$, the constant a is positive. Then when p varies between the limits $-\frac{1}{0}$ and $+\frac{1}{0}$, r will remain positive, and will vary between the

limits $r = 0$ and $r = \infty$. It is also obvious that when $p = 0$, we have $r = 1$. This understood, we may clearly consider the logarithmic spiral as being described by a point, which being at first placed on the semi-polar axis, at a distance $= 1$ from the origin, would turn an infinite number of times round the origin, with a direct or retrograde motion, so as to become infinitely distant from the origin, in the first case, and to approach indefinitely the origin, in the second case, but without ever reaching it.

(319.) When a plane curve is represented by an equation in polar coordinates, to make it turn round the origin, in such a manner that each radius vector should describe an angle equal to P , it suffices to replace the coordinate p , by $p - P$, taking care to give to P a positive value when the motion of rotation is direct, and a negative value,

Integral
Calculus.

Part III.

in the contrary case. Operating in this manner on the equation $r = e^{\frac{p}{a}}$, we shall obtain the formula $r = e^{\frac{p-p}{a}}$, which will represent the logarithmic spiral in a new position. If then we call R the radius vector corresponding

to $p = 0$, we shall have $R = e^{\frac{-p}{a}}$, and the preceding equation becomes $r = R e^{\frac{p}{a}}$, or $p = a(lr - lR)$. If, in this last equation, we substitute for the coordinates r and p their values in x and y derived from the equations

$$x = r \cos p, \quad y = r \sin p, \quad \text{which are } p = \tan^{-1} \frac{y}{x}, \quad r = \sqrt{(x^2 + y^2)}, \quad \text{we shall find } \tan^{-1} \frac{y}{x} = a(l\sqrt{(x^2 + y^2)} - lR),$$

which is the equation of the logarithmic spiral referred to rectangular coordinates.

(320.) Polar coordinates may easily be introduced in the formulæ, which determine the direction of the tangent, the arc, or the radius of curvature of a plane curve.

We have found, ψ being the angle which the tangent at the point (x, y) makes with the semi-axis of positive abscissæ, $\tan \psi = \frac{dy}{dx}$. Now, since we have $x = r \cos p$, $y = r \sin p$, we shall find

$$\tan \psi = \frac{\sin p \, dr + r \cos p \, dp}{\cos p \, dr - r \sin p \, dp} = \frac{\tan p + \frac{r \, dp}{dr}}{1 - \tan p \frac{r \, dp}{dr}},$$

which formula gives

$$\frac{r \, dp}{dr} = \frac{\tan \psi - \tan p}{1 + \tan \psi \tan p} = \tan(\psi - p) \text{ or } \cot(\psi - p) = \frac{dr}{r \, dp}.$$

This last formula may easily be obtained in a direct manner. We have $\frac{\cos p}{x} = \frac{\sin p}{y}$, $\frac{\cos \psi}{dx} = \frac{\sin \psi}{dy}$, and, consequently,

$$\cot(\psi - p) = \frac{\cos(\psi - p)}{\sin(\psi - p)} = \frac{\cos p \cos \psi + \sin p \sin \psi}{\cos p \sin \psi - \sin p \cos \psi} = \frac{x \, dy + y \, dx}{x \, dy - y \, dx}.$$

But we have $r^2 = x^2 + y^2$, $p = \tan^{-1} \frac{y}{x}$, consequently $2r \, dr = 2x \, dx + 2y \, dy$, $dp = \frac{x \, dy - y \, dx}{x^2 + y^2} = \frac{x \, dy - y \, dx}{r^2}$,

or, which is the same thing, $x \, dx + y \, dy = r \, dr$, $r^2 \, dp = x \, dy - y \, dx$, and hence $\cot(\psi - p) = \frac{r \, dr}{r^2 \, dp} = \frac{dr}{r \, dp}$.

We might also obtain very easily this formula by the following considerations. We have found that $r \sin(p - \psi) = R \sin(P - \psi) = a$ constant quantity, was the polar equation of a straight line. In order that this line should be tangent to a plane curve, at a given point, it will be necessary that the values of the quantities

$p, r, \frac{dr}{dp}$ corresponding to that point should remain the same in passing from the straight line to the curve. But

by differentiating the equation of the straight line, we find $\sin(p - \psi) \, dr + \cos(p - \psi) \, r \, dp = 0$, which gives for $\cot(\psi - p)$ the same value we have found above.

(321.) By means of these formulæ, it is now easy to find the equations of the tangent and normal drawn to a given curve through a given point. Let us conceive, for instance, that p, r are the coordinates of the point of contact, and P, R the variable coordinates of the tangent. The polar equation of that line will be $R \sin$

$(P - \psi) = r \sin(p - \psi)$, the angle ψ being determined by the equation $\cot(\psi - p) = \frac{dr}{r \, dp}$. But the equation

$R \sin(P - \psi) = r \sin(p - \psi)$ may be written in the following manner; $\frac{R \cos(P - p) - r}{R \sin(P - p)} = \cot(\psi - p)$, and

therefore the polar equation of the tangent will be $\frac{R \cos(P - p) - r}{R \sin(P - p)} = \frac{dr}{r \, dp}$. To obtain the equation of the

normal, it will be sufficient to substitute $\psi \pm \frac{\pi}{2}$ to ψ . Hence we shall have, P and R being the variable coordi-

nates of the normal, $\frac{R \cos(P - p) - r}{R \sin(P - p)} = -\tan(\psi - p) = -\frac{r \, dp}{dr}$, for the equation of that line.

(322.) Let now v be the angle formed by the curve with the circle described from the origin as a centre with a radius r , that is to say the acute angle formed by the tangent with the perpendicular to this radius, $\frac{\pi}{2} - v$ will be the acute angle, which this same radius forms with the tangent, and consequently the least of the numerical values of $\psi - p$. Therefore, we shall have $\tan v = \cot\left(\frac{\pi}{2} - v\right) = \pm \cot(\psi - p)$, and afterwards $\tan v = \pm \frac{dr}{r \, dp}$

$= \pm \frac{dlr}{dp}$. Moreover, as the quantity $\tan v$ will be essentially positive, and as for very small numerical values of

Integral
Calculus.

Δp , the ratio $\frac{\Delta r}{\Delta p}$ will always have the same sign as its limit $\frac{dr}{dp}$, we shall take the sign + in the value of $\tan v$, if, setting off from the point (p, r) , the radius r increases with the angle p , and the sign - in the contrary case.

We shall for the sake of brevity suppose $\frac{dr}{dp} = r$, then the preceding value of $\tan v$ becomes $\pm \frac{r'}{r}$, and, consequently, we have

$$\cos v = \pm \frac{r}{\sqrt{(r^2 + r'^2)}}, \quad \sin v = \frac{r'}{\sqrt{(r^2 + r'^2)}}.$$

Let s represent the arc of a given curve reckoned from a fixed point on the curve, and let ρ be the radius of curvature. We have found $ds^2 = dx^2 + dy^2$, $\frac{1}{\rho} = \pm \frac{dx \, d^2y - dy \, d^2x}{(dx^2 + dy^2)^{\frac{3}{2}}}$. If we substitute in these formulæ the

values of x, y, dx, dy , &c. derived from the equations $x = r \cos p, y = r \sin p$, we shall have, after reduction, $ds^2 = dr^2 + r^2 dp^2$, $\frac{1}{\rho} = \pm \frac{r(dr \, d^2p - dp \, d^2r) + (2dr^2 + r^2 dp^2) dp}{(dr^2 + r^2 dp^2)^{\frac{3}{2}}}$, and, consequently, taking p for the

independent variable,

$$\frac{ds^2}{dp^2} = r^2 + r'^2, \quad \frac{1}{\rho} = \pm \frac{r^2 - r r'' + 2r'^2}{(r^2 + r'^2)^{\frac{3}{2}}},$$

where $r'' = \frac{d^2r}{dp^2}$. It is essential to remark that in using, instead of the equations $x = r \cos p, y = r \sin p$, the formulæ $x + y\sqrt{-1} = r e^{p\sqrt{-1}}, x - y\sqrt{-1} = r e^{-p\sqrt{-1}}$, which are equivalent to them, the calculation is much shortened, and there is no reduction to make. Thus, for instance, if we take x for the independent variable, we shall have

$$dx + dy\sqrt{-1} = (r' + r\sqrt{-1}) e^{p\sqrt{-1}} dp, \quad dx - dy\sqrt{-1} = (r' - r\sqrt{-1}) e^{-p\sqrt{-1}} dp,$$

$$d^2x + d^2y\sqrt{-1} = (r'' - r + 2r'\sqrt{-1}) e^{p\sqrt{-1}} dp^2, \quad d^2x - d^2y\sqrt{-1} = (r'' - r - 2r'\sqrt{-1}) e^{-p\sqrt{-1}} dp^2,$$

Now, if we multiply the two first equations, and then the second and third, we find immediately $dx^2 + dy^2 = (r^2 + r'^2) dp^2$, and we find also that $dx \, d^2y - dy \, d^2x$ is equal to the coefficient of $\sqrt{-1}$ in the product $(r' - r\sqrt{-1})(r'' - r + 2r'\sqrt{-1}) dp^3$, so that we have $dx \, d^2y - dy \, d^2x = (2r'^2 - r r'' + r^2) dp^3$. If we did not take p for the independent variable, we would have found

$$dx^2 + dy^2 = dr^2 + r^2 dp^2, \quad dx \, d^2y - dy \, d^2x = r(dr \, d^2p - dp \, d^2r) + (2dr^2 + r^2 dp^2) dp.$$

(323.) Let now x_1, y_1 be the rectangular coordinates of the centre of curvature corresponding to the point x, y , and P, R the polar coordinates of the same centre, so that $x_1 = R \cos P, y_1 = R \sin P$. We have found (306.)

$$\frac{y_1 - y}{dx} = -\frac{x_1 - x}{dy} = \frac{dx^2 + dy^2}{dx \, d^2y - dy \, d^2x},$$

from which we easily derive

$$\frac{y(y_1 - y) + x(x_1 - x)}{y \, dx - x \, dy} = \frac{x(y_1 - y) - y(x_1 - x)}{x \, dx + y \, dy} = \frac{dx^2 + dy^2}{dx \, d^2y - dy \, d^2x}$$

or, which is the same thing,

$$\frac{xx_1 + yy_1 - (x^2 + y^2)}{y \, dx - x \, dy} = \frac{xy_1 - yx_1}{x \, dx + y \, dy} = \frac{dx^2 + dy^2}{dx \, d^2y - dy \, d^2x}.$$

If then we substitute polar coordinates for rectangular coordinates by means of the formulæ we have already found, we shall have

$$\frac{R \cos(P - p) - r}{-r \, dp} = \frac{R \sin(P - p)}{dr} = \frac{dr^2 + r^2 dp^2}{r(dr \, d^2p - dp \, d^2r) + (2dr^2 + r^2 dp^2) dp}.$$

If we take p for the independent variable, we shall have simply,

$$1 - \frac{R}{r} \cos(P - p) = \frac{R}{r'} \sin(P - p) = \frac{r^2 + r'^2}{2r'^2 - r r'' + r^2}.$$

From this last formula, we derive

$$R \sin(P - p) = r' \frac{r^2 + r'^2}{2r'^2 - r r'' + r^2}, \quad R \cos(P - p) = r \left(1 - \frac{r^2 + r'^2}{2r'^2 - r r'' + r^2} \right) = r \frac{r'^2 - r r''}{2r'^2 - r r'' + r^2}$$

and, consequently,

$$\tan(P - p) = \frac{r'}{r} \frac{r^2 + r'^2}{r'^2 - r r''}, \quad R^2 = \frac{r'^2 (r^2 + r'^2)^2 + r^2 (r'^2 - r r'')^2}{(2r'^2 - r r'' + r^2)^2}.$$

It is useful to remark, that since $\frac{1}{\rho} = \pm \frac{r^2 - r r'' + 2r'^2}{(r^2 + r'^2)^{\frac{3}{2}}}$, we shall have

$$1 - \frac{R}{r} \cos (P - p) = \frac{R}{r} \sin (P - p) = \pm \frac{\rho}{\sqrt{(r^2 + r'^2)}},$$

and, consequently,

$$R \sin (P - p) = \pm \frac{r'}{\sqrt{(r^2 + r'^2)}} \rho = \pm \rho \sin v, \quad R \cos (P - p) - r = \mp \frac{r}{\sqrt{(r^2 + r'^2)}} \rho = \mp \rho \cos v.$$

(324.) These last formulæ may be obtained in a different manner. The circle described from the point (P, R) as a centre with a radius equal to ρ has for polar equation $r^2 - 2 R r \cos (P - p) + R^2 = \rho^2$. Now, let us suppose that this circle becomes the osculating circle of a given curve, in a certain point given on the curve. Not only the coordinates p and r but also the differentials dp, dr, d^2p, d^2r , corresponding to the coordinates of the point of contact ought to preserve the same values in passing from the circle to the curve. In other words, the value of p, dp, d^2p, r, dr, d^2r derived from the equations of the curve, must satisfy the equation of the circle, and its two differential equations of the first and second order. These three last equations may be written as follows:

$$\{R \sin (p - P)\}^2 + \{R \cos (p - P) - r\}^2 = \rho^2, \quad R r \sin (p - P) dp - \{R \cos (p - P) - r\} dr = 0,$$

$$R \sin (p - P) \{r d^2p + 2 dp dr\} - \{R \cos (p - P) - r\} (d^2r - r dp^2) = -(dr^2 + r^2 dp^2).$$

From which we immediately derive the formula

$$\begin{aligned} \frac{R \cos (p - P) - r}{r dp} &= \frac{R \sin (p - P)}{dr} = \pm \frac{\{[R \sin (p - P)]^2 + [R \cos (p - P) - r]^2\}^{\frac{1}{2}}}{\sqrt{(dr^2 + r^2 dp^2)}} = \pm \frac{\rho}{\sqrt{(dr^2 + r^2 dp^2)}} \\ &= \frac{R \sin (p - P) \cdot (r d^2p + 2 dp dr) - \{R \cos (p - P) - r\} (d^2r - r dp^2)}{r (dr d^2p - dp d^2r) + 2 (dr^2 + r^2 dp^2) dp} \\ &= \frac{dr^2 + r^2 dp^2}{r (dr d^2p - dp d^2r) + (2 dr^2 + r^2 dp^2) dp}. \end{aligned}$$

(325.) Most of the preceding formulæ may be established by Geometrical considerations alone.

Let us consider two points of a plane curve very near each other. Let p, r , and $p + \Delta p, r + \Delta r$ be the coordinates of these two points. Let s , as usual, be the length of the arc of the curve between the point (p, r) and a fixed point, and let v the acute angle formed by the curve, and the circle described from the origin as a centre, with a radius equal to r . The arcs of the curve and circle between the radii r and $r + \Delta r$, will be respectively Δs and $r \Delta p$. If we call ω the acute angle formed by the chords of these arcs, the limit of ω will be precisely v . The two chords and the length $\pm \Delta r$ will form a right angled triangle in which the side $\pm \Delta r$ will be opposed to the angle ω . In this same triangle, when the arc $\pm \Delta s$ will become infinitely small, the angle opposed to the chord of this arc will differ very little from a right angle, and may be represented consequently by $\frac{\pi}{2} \pm \epsilon$, $\pm \epsilon$ being a very small quantity. Hence, the third angle, opposed to the chord of the arc $\pm r \Delta p$, will be $\frac{\pi}{2} - \omega \mp \epsilon$, and if we compare two and two the sines of the angles to the sides opposed to them, we shall have

$$\pm \frac{\Delta r}{\Delta s} = \frac{\sin \omega}{\sin \left(\frac{\pi}{2} \pm \epsilon \right)}, \quad \pm \frac{r \Delta p}{\Delta s} = \frac{\sin \left(\frac{\pi}{2} - \omega \mp \epsilon \right)}{\sin \left(\frac{\pi}{2} \pm \epsilon \right)},$$

from which we shall derive, in substituting first $\cos (\omega \pm \epsilon)$ for $\sin \left(\frac{\pi}{2} - \omega \mp \epsilon \right)$, and then taking the limits,

$$\pm \frac{dr}{ds} = \sin v, \quad \pm \frac{r dp}{ds} = \cos v. \quad \text{If we divide the first of these equations by the second, we find as before,}$$

$$\tan v = \frac{dr}{r dp}; \quad \text{if we add them after having raised to the square both sides of each, we shall obtain the formula}$$

$$\frac{dr^2 + r^2 dp^2}{ds^2} = 1.$$

(326.) Let us consider now the triangle which has for its summits the origin of coordinates, the point (p, r) of the plane curve, and the centre of curvature corresponding to that point. If we call ρ the radius of curvature, and (P, R) the coordinates of the centre of curvature, the three sides of the triangle will be respectively r, R , and ρ . In the same triangle, the acute angle formed by r and ρ will clearly be equal to v , which is the angle between the tangent and the perpendicular to the radius vector, or to the supplement of that angle. If we call δ the angle formed by the radii r and R , we shall have clearly $\pm \delta = p - P \pm 2n\pi$, n being an integer, which may be equal to nothing. As to the third angle, its supplement will be the sum of the other two, that is $\delta + v$, or $\delta + \pi - v$.

This understood, if we compare the sines of the angles to the sides, we find $\frac{\sin v}{R} = \frac{\sin \delta}{\rho} = \frac{\sin v \pm \delta}{r}$. Hence,

$$R \sin \delta = \rho \sin v, \quad R \cos \delta - r = \mp \rho \cos v. \quad \text{If in these last formulæ we put for } \delta \text{ its value } p - P \pm 2n\pi, \text{ we shall find again the formulæ } R \sin (P - p) = \pm \rho \sin v, \quad R \cos (P - p) - r = \mp \rho \cos v.$$

(327.) We shall now show some applications of the general formulæ that have been demonstrated.

Integral
Calculus.

If, first, we consider the spiral of Archimedes represented by the equation $r = ap$, we shall have $r' = a$, $r'' = 0$, Part III.
and consequently, a being a positive quantity,

$$\tan v = \cot(\psi - p) = \frac{1}{p}, \quad \frac{1}{\rho} = \frac{2 + p^2}{(1 + p^2)^{\frac{3}{2}}} \cdot \frac{1}{a} = \left(1 + \frac{1}{1 + p^2}\right) \frac{1}{a \sqrt{1 + p^2}}.$$

At the same time the general values of $\tan(P - p)$ and R^2 given (321.) will become

$$\tan(P - p) = p + \frac{1}{p}, \quad R^2 = a^2 \frac{2p^2 + (1 + p)^2}{(2 + p^2)^2} = a^2 \left\{1 - \frac{1}{2 + p^2} - \frac{1}{(2 + p^2)^2}\right\}.$$

Hence for $p = 0$, we shall have $\tan v = \cot \psi = \tan P = \frac{1}{0}$, $v = \frac{\pi}{2}$, $\rho = R = \frac{a}{2}$; and for $p = \frac{1}{0}$, $\tan v = 0$,

$v = 0$, $\tan(P - p) = \frac{1}{0}$, $\frac{1}{\rho} = 0$, $R = a$. Consequently, 1. the spiral touches at the origin of coordinates the

polar axis; 2. for increasing values of p , the angle v , and the curvature $\frac{1}{\rho}$ decrease indefinitely in this same spiral,

whilst the value of R increases indefinitely, but in such manner as to be contained between the limits $\frac{a}{2}$ and a ;

3. for very large values of p , the radius R , drawn from the origin to the centre of curvature, becomes nearly equal to a , and very nearly also perpendicular to the radius r . If in the above value of $\tan(P - p)$, we substitute for p the real and positive value derived from the equation expressing the value of R , or which is the same thing, from

the equation $\frac{1}{2 + p^2} = \frac{1 \pm \sqrt{5 - \frac{4R^2}{a^2}}}{2}$, we shall find for the polar equation of the evolute of the spiral,

$$\tan \left\{ P - \frac{\left[\sqrt{\left(5 - \frac{4R^2}{a^2}\right)} - 3 + \frac{4R^2}{a^2} \right]^{\frac{1}{2}}}{2^{\frac{1}{2}} \left(1 - \frac{R^2}{a^2}\right)^{\frac{1}{2}}} \right\} = \frac{\frac{2R^2}{a^2} - 1 + \sqrt{\left(5 - \frac{4R^2}{a^2}\right)}}{2^{\frac{1}{2}} \left(1 - \frac{R^2}{a^2}\right)^{\frac{1}{2}} \left[\sqrt{\left(5 - \frac{4R^2}{a^2}\right)} - 3 + \frac{4R^2}{a^2} \right]^{\frac{1}{2}}}.$$

This evolute is clearly a new spiral which presents a *stop* point corresponding to the coordinates $R = \frac{a}{2}$, $P = \frac{\pi}{2}$;

it is perpendicular at that point to the circle described from the origin as a centre with the radius $\frac{a}{2}$, and which as it removes from this circle, makes an infinite number of revolutions round the origin, so as to approach more and more to the circumference of another circle, having the same centre as the first, and described with a radius twice as great. As this new spiral and this circumference will never meet, one of them may be said to be an asymptote of the other.

If instead of the spiral of Archimedes we consider the spiral represented by the equation $r = ap^n$, we find $r' = nap^{n-1} = \frac{nr}{p}$, $r'' = n(n-1)ap^{n-2} = \frac{n(n-1)r}{p^2}$, and, consequently, $\tan v = \cot(\psi - p) = \frac{n}{p}$,

$$\frac{1}{\rho} = \frac{n(n+1) + p^2}{(n^2 + p^2)^{\frac{3}{2}}} \cdot \frac{1}{ap^{n-1}}, \quad \tan(P - p) = p + \frac{n^2}{p},$$

$$R^2 = \frac{p^2 + (n^2 + p^2)^2 n^2 a^2 p^{2n-2}}{(n(n+1) + p^2)^2}.$$

If in the preceding formulæ, we suppose that the constant n is positive, we shall find again for $p = 0$, $\tan v = \cot \psi = \frac{1}{0}$, $v = \frac{\pi}{2}$, and for $p = \frac{1}{0}$, $\tan v = 0$, $v = 0$, $\tan(P - p) = \frac{1}{0}$. Hence we shall infer that the curve touches at the origin the polar axis, and becomes for large values of p very nearly perpendicular to the radius vector r . In the same hypothesis, the values of ρ and R corresponding to $p = 0$, or $p = \frac{1}{0}$, will be as this angle nothing or infinite unless we suppose $n = 1$, that is to say, unless the curve becomes the spiral of Archimedes.

Let us consider next the hyperbolic spiral represented by the equation $rp = a$. We shall find $r' = -\frac{a}{p^2}$, $r'' = \frac{2a}{p^3}$, and supposing that the constant a is a positive quantity, we shall have $\tan v = \cot(\psi - p) = \frac{1}{p}$,

$$\frac{1}{\rho} = -\frac{p^2}{(1 + p^2)^{\frac{3}{2}}} \cdot \frac{1}{a}, \quad \tan(P - p) = p + \frac{1}{p}, \quad R^2 = \left\{ \frac{1}{p^2} + \left(1 + \frac{1}{p^2}\right)^2 \right\} \frac{a^2}{p^4}.$$

Then, for $p = 0$, we have $\tan v = \cot \psi = \tan P = \frac{1}{0}$, $v = \frac{\pi}{2}$, $\rho = R = \frac{1}{0}$, and for $p = \frac{1}{0}$, $\tan v = 0$, $v = 0$, $\tan(P - p) = \frac{1}{0}$, $\rho = R = 0$.

Integral
Calculus.

Part III.

These results show, 1. that the angle v is nearly a right angle and the curvature very nearly nothing, for very small values of p , that is to say in that part of the hyperbolic spiral which is very remote from the origin, and which nearly coincides with the asymptote of that curve; 2. that for increasing values of the angle p , the angle

v decreases from $v = \frac{\pi}{2}$ to $v = 0$, and that the same thing may be said with respect to the radius of curvature ρ ,

and the radius vector R , the values of which, at first very large, at last become equal to nothing.

It is clear that by eliminating p between the value of $\tan(P-p)$ and that of R^2 we shall find the polar equation of the evolute of the hyperbolic spiral. It will show that this evolute is a spiral, which possesses also the remarkable property of approaching indefinitely to the centre without ever reaching it.

Lastly, let us consider the logarithmic spiral represented by the equation $r = e^{\frac{p}{a}}$. We shall find $r' = \frac{r}{a}$, $r'' = \frac{r}{a^2}$,

$\tan v = \cot(\psi - p) = \frac{1}{a} \rho = \left(1 + \frac{1}{a^2}\right)^{\frac{1}{2}} e^{\frac{p}{a}} = \left(1 + \frac{1}{a^2}\right)^{\frac{1}{2}} r = r \sec v$. The value of $\tan v$ shows that the

tangent and the normal form constantly the same angles with the radius vector r , which remark gives the means to draw the tangents and normals to the curve through a given point.

We have also in this case, $R \sin(P-p) = r' = \frac{r}{a}$, $R \cos(P-p) = 0$, and, consequently, $\cos(P-p) = 0$, $\sin(P-p) = 1$,

$R = \frac{r}{a}$. The smallest value of P which verifies these equations being $P = p + \frac{\pi}{2}$, it may be affirmed, that the

radii r , R of the curve and its evolute meet at right angles, or, in other words, that r and R are the two sides of a right angled triangle, the hypotenuse of which is the radius of curvature. If we eliminate p between the

equations $r = e^{\frac{p}{a}}$, $R = \frac{r}{a}$, and $P = p + \frac{\pi}{2}$, we shall find for the polar equation of the evolute, $R = \frac{1}{a} e^{\frac{1}{a}(P - \frac{\pi}{2})}$

$= e^{\frac{1}{a}(P - \frac{\pi}{2} - a \log a)}$. In substituting in this last formula r for R , and $p + \frac{\pi}{2} + a \log a$ for P , we would find for result the

equation $r = e^{\frac{p}{a}}$. Hence 1. the logarithmic spiral and its evolute are two curves having the same form and dimensions; 2. to obtain the evolute it is sufficient to make the involute turn round the origin, so as to make each

radius vector describe, by a direct motion of rotation, an angle equal to $\frac{\pi}{2} + a \log a$.

On Curves represented by two Equations between three Variables.

(328.) We shall now consider any curve whatever represented by two equations between three rectangular coordinates x, y, z . The results which we shall obtain will therefore be applicable to any curve, whether it be a plane curve, or a curve of double curvature, that is a curve which is not all in the same plane.

Let us represent the coordinates of any point of that curve by x, y, z , and by $x+h, y+\kappa, z+l$, the coordinates of another point belonging to the same curve. Then, if we join these two points, and if we designate by a, b, c , the angles which the line of junction makes respectively with the three axes, x, y, z , we shall have (See ANALYTICAL GEOMETRY)

$$\cos a = \frac{h}{\sqrt{h^2 + \kappa^2 + l^2}}, \quad \cos b = \frac{\kappa}{\sqrt{h^2 + \kappa^2 + l^2}}, \quad \cos c = \frac{l}{\sqrt{h^2 + \kappa^2 + l^2}},$$

and hence $\frac{\cos a}{h} = \frac{\cos b}{\kappa} = \frac{\cos c}{l}$. If, besides, we represent by x_1, y_1, z_1 , the coordinates of any point of the chord under consideration, it is easy to see that we shall have also

$$\frac{x_1 - x}{\cos a} = \frac{y_1 - y}{\cos b} = \frac{z_1 - z}{\cos c}, \quad \text{and, consequently,} \quad \frac{x_1 - x}{h} = \frac{y_1 - y}{\kappa} = \frac{z_1 - z}{l}.$$

Let us now imagine that the point the coordinates of which are $x+h, y+\kappa, z+l$, approaches nearer and nearer, and indefinitely to the point the coordinates of which are x, y, z . The secant which joins the two points will, at the same time, approach nearer and nearer to coincide with a certain line which is called a *tangent* to the given curve, and which touches it at the point (x, y, z) . To obtain the angles α, β, γ , which the producement of this tangent makes with the three axes x, y, z , it will be sufficient to find the limits towards which the angles a, b, c converge

whilst the quantities h, κ, l become less and less, and infinitely small quantities. But at the limit the ratios $\frac{\kappa}{h}, \frac{l}{h}$,

become equal to the quantities $\frac{dy}{dx}, \frac{dz}{dx}$, therefore

$$\cos \alpha = \frac{dx}{\sqrt{(dx)^2 + (dy)^2 + (dz)^2}}, \quad \cos \beta = \frac{dy}{\sqrt{(dx)^2 + (dy)^2 + (dz)^2}}, \quad \cos \gamma = \frac{dz}{\sqrt{(dx)^2 + (dy)^2 + (dz)^2}},$$

Integral
Calculus.

Part III.

and hence $\frac{\cos \alpha}{dx} = \frac{\cos \beta}{dy} = \frac{\cos \gamma}{dz}$. If, moreover, we designate by x_1, y_1, z_1 , the coordinates of any point of the tangent, we have

$$\frac{x_1 - x}{\cos \alpha} = \frac{y_1 - y}{\cos \beta} = \frac{z_1 - z}{\cos \gamma}, \text{ and therefore } \frac{x_1 - x}{dx} = \frac{y_1 - y}{dy} = \frac{z_1 - z}{dz}.$$

These equalities are verified for any point of the tangent; they are true even in case the axis of coordinates should be oblique, and when these axes are rectangular they are the equations of the projections of the tangent upon the three coordinate planes. They are also true whatever be the independent variable.

(329.) A plane is said to be tangent to a curve in a given point, when it passes through the tangent at that point. Hence it is obvious that through a given point, on a given curve, an infinite number of tangent planes may be drawn.

(330.) A straight line is said to be normal to a curve in a given point, when it is perpendicular to the tangent at that point. Hence it is obvious that through a given point it will be possible to draw an infinite number of normals to a curve. All these normals will be situated in a plane perpendicular to the tangent, and this plane is called a normal plane to the curve. If we represent still by α, β, γ , the angles which the tangent to the curve makes with the coordinate axes, and by x_1, y_1, z_1 , the coordinates of any point of the normal plane, it will clearly follow from what precedes that the equation of the normal plane will be

$$(x_1 - x) \cos \alpha + (y_1 - y) \cos \beta + (z_1 - z) \cos \gamma = 0,$$

or by using the relations we have found between $\cos \alpha, \cos \beta, \cos \gamma$, and dx, dy, dz ,

$$(x_1 - x) dx + (y_1 - y) dy + (z_1 - z) dz = 0,$$

a result which is true whatever be the independent variable.

(331.) Let now $u = 0, v = 0$ be the two equations of the given curve, that is the equations of two surfaces which contain it, so that u and v designate two functions of three variables. We shall have by differentiation

$$\frac{du}{dx} dx + \frac{du}{dy} dy + \frac{du}{dz} dz = 0, \quad \frac{dv}{dx} dx + \frac{dv}{dy} dy + \frac{dv}{dz} dz = 0,$$

and consequently since $\frac{\cos \alpha}{dx} = \frac{\cos \beta}{dy} = \frac{\cos \gamma}{dz}$, we find

$$\frac{du}{dz} \cos \alpha + \frac{du}{dy} \cos \beta + \frac{du}{dx} \cos \gamma = 0, \quad \frac{dv}{dx} \cos \alpha + \frac{dv}{dy} \cos \beta + \frac{dv}{dz} \cos \gamma = 0.$$

These two last equations, together with the equation $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$ will be sufficient to determine the values of $\cos \alpha, \cos \beta$, and $\cos \gamma$. We shall find

$$\begin{aligned} \frac{\cos \alpha}{\frac{du}{dy} \cdot \frac{dv}{dz} - \frac{du}{dz} \cdot \frac{dv}{dy}} &= \frac{\cos \beta}{\frac{du}{dz} \cdot \frac{dv}{dx} - \frac{du}{dx} \cdot \frac{dv}{dz}} = \frac{\cos \gamma}{\frac{du}{dx} \cdot \frac{dv}{dy} - \frac{du}{dy} \cdot \frac{dv}{dx}} = \\ &= \pm \frac{1}{\left[\left(\frac{du}{dy} \cdot \frac{dv}{dz} - \frac{du}{dz} \cdot \frac{dv}{dy} \right)^2 + \left(\frac{du}{dz} \cdot \frac{dv}{dx} - \frac{du}{dx} \cdot \frac{dv}{dz} \right)^2 + \left(\frac{du}{dx} \cdot \frac{dv}{dy} - \frac{du}{dy} \cdot \frac{dv}{dx} \right)^2 \right]}. \end{aligned}$$

These values being substituted, will give for the equations of the tangent, and of the normal plane,

$$\begin{aligned} \frac{x_1 - x}{\frac{du}{dy} \cdot \frac{dv}{dz} - \frac{du}{dz} \cdot \frac{dv}{dy}} &= \frac{y_1 - y}{\frac{du}{dz} \cdot \frac{dv}{dx} - \frac{du}{dx} \cdot \frac{dv}{dz}} = \frac{z_1 - z}{\frac{du}{dx} \cdot \frac{dv}{dy} - \frac{du}{dy} \cdot \frac{dv}{dx}}, \text{ and} \\ \left(\frac{du}{dy} \cdot \frac{dv}{dz} - \frac{du}{dz} \cdot \frac{dv}{dy} \right) (x_1 - x) &+ \left(\frac{du}{dz} \cdot \frac{dv}{dx} - \frac{du}{dx} \cdot \frac{dv}{dz} \right) (y_1 - y) + \left(\frac{du}{dx} \cdot \frac{dv}{dy} - \frac{du}{dy} \cdot \frac{dv}{dx} \right) (z_1 - z) = 0. \end{aligned}$$

Moreover it may be observed that instead of the equations of the projections of the tangent upon the coordinate planes, we may use to represent this line the two following equations,

$$(x_1 - x) \frac{du}{dx} + (y_1 - y) \frac{du}{dy} + (z_1 - z) \frac{du}{dz} = 0, \quad (x_1 - x) \frac{dv}{dx} + (y_1 - y) \frac{dv}{dy} + (z_1 - z) \frac{dv}{dz} = 0. \dots (a).$$

These are immediately obtained by combining the differential equations of the two surfaces with the relations

$$\frac{x_1 - x}{dx} = \frac{y_1 - y}{dy} = \frac{z_1 - z}{dz}, \text{ which have been obtained above, and they represent two planes the intersection of}$$

which is the tangent to the curve. By comparing them with the differential equations of the curve, we find that in order to obtain the equations of the tangent to any curve whatever, it is sufficient to substitute, in the differential equations of the curve, $x_1 - x, y_1 - y, z_1 - z$, for dx, dy , and dz .

Integral
Calculus.

 (332.) When u is a function of x and y only, the first of the two last equations is reduced to $(x_1 - x) \frac{du}{dx} +$ Part III.

$(y_1 - y) \frac{du}{dy} = 0$. In that case $u = 0$ is the equation of the projection of the curve on the plane of the x, y , and the equation $(x_1 - x) \frac{du}{dx} + (y_1 - y) \frac{du}{dy} = 0$ is, at the same time, the equation of the projection of the tangent to the curve in space on the same plane, and the equation of the tangent to the projection of the curve. Hence it appears that the projection of the tangent to any curve whatever, upon a given plane, coincides with the tangent to the projection of the curve on the same plane.

(333.) We shall now give a few applications of the preceding formulæ. Let us first consider the ellipse resulting from the intersection of a cylinder, having a circle for its base, with a plane. Let $x^2 + y^2 = r^2$ be the equation of the cylinder, and $ax + by + cz = d$ be the equation of the plane. The differential equations of this ellipse being $x dx + y dy = 0$, $adx + bdy + cdz = 0$, the angles α, β, γ , formed by the tangent with the axes x, y, z , will be determined by means of the equations

$$x \cos \alpha + y \cos \beta = 0, \quad A \cos \alpha + B \cos \beta + C \cos \gamma = 0, \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

Hence we shall find

$$\frac{\cos \alpha}{y} = \frac{\cos \beta}{-x} = \frac{\cos \gamma}{\frac{1}{c}(bx - ay)} = \pm \frac{1}{\left[r^2 + \left(\frac{bx - ay}{c}\right)^2\right]^{\frac{1}{2}}},$$

and we shall have for the equations of the tangent

$$x(x_1 - x) + y(y_1 - y) = 0, \quad a(x_1 - x) + b(y_1 - y) + c(z_1 - z) = 0, \quad \text{or } xx_1 + yy_1 = r^2, \\ ax_1 + by_1 + cz_1 = d.$$

In the same manner the equation of the normal plane will be found to be

$$c(x_1 y - y_1 x) + (bx - ay)(z_1 - z) = 0.$$

The equations of the tangent represent, as might have been expected, the first, the tangent to the circle which is the base of the cylinder, and the second, the plane of the curve itself.

We shall take for a second example the intersection of a conic surface, having a circle for its base, with a plane.

Let $x^2 + y^2 = a^2 z^2$ be the equation of the cone, and $ax + by + cz = d$ that of the plane. We shall find for the equations of the tangent

$$xx_1 + yy_1 = a^2 z z_1, \quad ax_1 + by_1 + cz_1 = d,$$

and for the equation of the normal plane

$$c(x_1 y - y_1 x) + (bx_1 - ay_1)a^2 z + (bx - ay)(z_1 - z - a^2 z) = 0.$$

The equations of the tangent represent, as might have been foreseen, the two planes, the first passing through the origin of the coordinates and one of the edges of the cone, and the other coinciding with the plane of the curve.

Let, for another example, $x^2 + y^2 + z^2 = r^2$, and $\left(x - \frac{r}{2}\right)^2 + z^2 = \frac{r^2}{4}$, be the equations of the curve. The first

belongs obviously to a sphere having its centre at the origin, and for radius r , and the second to a right cylinder having for its base a circle drawn in the plane of the x, z , passing through the origin of the coordinates, and the diameter of which is equal to the radius of the sphere. This curve will have for its projection on the plane of the x, y the parabola represented by the equation $y^2 = r(r - x)$, for its projection on the plane of the z, x , the circle which is the base of the cylinder, and for its projection on the plane of the z, y , the *lemniscata* represented by the equation $r^2 z^2 = y^2(r^2 - y^2)$. Therefore the tangent to the given curve will have for its projections on the coordinate planes the tangents to the parabola, the circle and the lemniscata, that is to say, the three lines represented by the equations

$$yy_1 + \frac{rx_1}{2} = r\left(r - \frac{x}{2}\right), \quad \left(x - \frac{r}{2}\right)x_1 + zz_1 = \frac{rx}{2}, \quad r^2 z z_1 - (r^2 - 2y^2)yy_1 = y^4.$$

As to the equation of the normal plane it may be put under the form

$$\frac{x_1}{x} - \frac{z_1}{z} = \frac{r}{2x} \left(\frac{y_1}{y} - \frac{z_1}{z}\right),$$

and then it is easy to verify that this plane contains the line drawn from the origin to the point of contact, since it passes through the origin of the coordinates.

We shall take for the last example the *helix*, or the curve of double curvature formed by a thread, wrapped round the surface of a cylinder, so as always to make the same angle with the axis. Let us suppose that the axis of the cylinder coincides with the axis of z , and that the basis of the cylinder is the circle represented by the equation $x^2 + y^2 = r^2$. If we consider this cylinder as being generated by the uniform revolution of a rectangular parallelogram round one of its sides, the *helix* may be conceived to be traced out by a point moving uniformly upon

Integral
Calculus.

Part III.

the other. Hence if we call p the angle which the radius, which we may imagine to describe the base of the cylinder by its rotation round the origin, makes in any position with the axis of x , and if we suppose that the point describing the helix is in the plane of the x, y , and on the axis of x when the radius begins its revolution, we shall have, in representing by x, y, z , the coordinates of the point of the helix corresponding to the angle p , $x = r \cos p$, $y = r \sin p$, and $z = arp$, a being a certain constant quantity depending on the ratio of the velocities of the radius and describing point. By eliminating p between these three equations we would find the two equations of the helix. But we may find the equations of the tangent and normal plane, without proceeding to that elimination, since the formulæ we have obtained are true whatever may be the independent variable. By differentiation, we find

$$dx = -r \sin p \, dp, \quad dy = r \cos p \, dp, \quad dz = ar \, dp.$$

Hence we shall get

$$\cos \alpha = \frac{\sin p}{\sqrt{1+a^2}}, \quad \cos \beta = \frac{\cos p}{\sqrt{1+a^2}}, \quad \cos \gamma = \frac{a}{\sqrt{1+a^2}}.$$

The last expression shows that the angle γ is a constant quantity, that is to say, that the angle which the tangent to the curve makes with the axis of z , or the axis of the cylinder, is constant. The equations of the tangent will be

$$\frac{x_1 - x}{-\sin p} = \frac{y_1 - y}{\cos p} = \frac{z_1 - z}{a},$$

and the equation of the normal plane $(y_1 - y) \cos p - (x_1 - x) \sin p + a(z_1 - z) = 0$.

(334.) The asymptotes of curves in space may be determined by means analogous to those which have been used to find the asymptotes of a plane curve. Let us first suppose that it is required to find the asymptotes not parallel to the plane of y, z , and let $y = ax + b$, $z = cx + d$ be the equations of one of them. Then if we derive from the equations of the curve the values of y and z in function of x , these values, for very great values of x , must become respectively equal to $ax + b$, and $cx + d$. Let $y = ax + b + l$ and $z = cx + d + m$ be the equations of the curve, l and m designating two functions of x which will vanish when x is infinite. Hence, for increasing values of x ,

$$\lim \frac{y}{x} = a, \quad \lim \frac{z}{x} = c, \quad \lim (y - ax) = b, \quad \lim (z - cx) = d.$$

Therefore to find the constant a and c it will suffice to make in the equations of the curve $y = px$, and $z = qx$, and then to find the limit, or limits, towards which converge p and q whilst the values of x increase indefinitely. These limits will be the values of a and c . After having found them, the values $ax + r$, $cx + s$ must be substituted for y and z in the equations of the curve and the limits, towards which converge the values of r and s , for increasing values of x be determined. These limits will be the values of b and d . To each system of finite values of the quantities a, b, c, d , will correspond an asymptote of the proposed curve. By a similar reasoning, but by exchanging between them the coordinates x, y, z , we might obtain the asymptotes which are not parallel to the plane of z, x , or to the plane of z, y .

The asymptotes of curves in space might also be determined by observing that their projections upon each of the three coordinate planes are, in general, the asymptotes of the projection of the curve. It must only be observed that the projection of one of the first asymptotes would be reduced to a point, if that asymptote were perpendicular to the plane. Then this point would be a *stop point* of the projection of the curve.

If we take for an example the hyperbola which is the intersection of a plane and of an hyperboloid represented by the two equations

$$fx + gy + hz = 0, \text{ and } Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = L,$$

we shall have by substituting $y = px$ and $z = qx$

$$(f + gp + hq)x = 0, \quad A + Bp^2 + Cq^2 + 2Dpq + 2Eq + 2Fp = \frac{L}{x^2}.$$

The limits towards which converge p and q whilst the values of x increase indefinitely, or the values of p and q corresponding to $x = \infty$ will be determined by the two equations

$$f + gp + hq = 0, \quad A + Bp^2 + Cq^2 + 2Dpq + 2Eq + 2Fp = 0.$$

The values of p and q being found, and designated by a and c , we shall substitute $ax + r$ and $cx + s$ for y and z in the two equations of the curve. Then we shall have to determine the values of r and s corresponding to $x = \infty$, recollecting that a and b are the values of p and q which satisfy the two equations above,

$$gr + hs = 0, \quad 2Bar + 2Ccs + 2D(ar + cs) + 2Es + 2Fr = 0.$$

These two equations give immediately $r = 0$ and $s = 0$. Hence, if the curve have asymptotes their equations will be simply of the form $y = px$ and $z = qx$, that is to say that they will pass through the origin of the coordinates. If, moreover, we consider the form of the equations by means of which the values of p and q are to be determined it will become evident that the curve has for its asymptote the straight lines represented by the equations

$$fx + gy + hz = 0, \quad Ax^2 + By^2 + Cz^2 + 2Dyz + 2Ezx + 2Fxy = 0.$$

(335.) We shall not enter here into any discussions about the singular or remarkable points of a curve in space, because, in general, these points have for their projections, on the three coordinate planes, points which present the same circumstances with respect to the projections of the curve.

(336.) Let $u = 0$ be the equation of a curve surface, u designating a function of three rectangular coordinates x, y, z . If through a given point on this surface, the coordinates of which are represented by x, y, z , we imagine another surface to pass, we shall have a certain curve resulting from the intersection of the two surfaces. The tangent drawn to that curve through the given point will be the line of intersection of two planes represented by the equations (a) given in number (331.) Now it is easy to see that one of these planes, namely, that of which the variable coordinates satisfy the equation

$$(x_1 - x) \frac{du}{dx} + (y_1 - y) \frac{du}{dy} + (z_1 - z) \frac{du}{dz} = 0,$$

will be independent of the nature of the second surface, and also consequently of the nature of the curve of intersection. Hence all the curves drawn on the first surface and passing through the given point (x, y, z) will have their tangents in that point contained in one plane. This plane, represented by the above equation, is the *tangent plane* to the given surface, at the given point.

The reasonings which have just been used would still subsist, and consequently the equation of the tangent plane would preserve the same form, if, in the function u , the variables x, y, z were referred to three oblique instead of three rectangular axes.

If the equation of the surface $u = 0$ be differentiated in considering u as a function of three independent variables, x, y, z , we shall find

$$\frac{du}{dx} dx + \frac{du}{dy} dy + \frac{du}{dz} dz = 0,$$

and if we compare this equation with the equation of the tangent plane,

$$(x_1 - x) \frac{du}{dx} + (y_1 - y) \frac{du}{dy} + (z_1 - z) \frac{du}{dz} = 0,$$

we see that to obtain this equation, it suffices to replace, in the differential equation of the surface, the differentials dx, dy, dz , by the differences $x_1 - x, y_1 - y, z_1 - z$.

(337.) If through the point (x, y, z) on the surface, we draw a straight line in a direction perpendicular to the tangent plane, this line will be the *normal* at the given point. Let λ, μ , and ν be the angles formed by this normal with the three axes. The tangent plane being perpendicular to the normal, its equation may be put under the form

$$(x_1 - x) \cos \lambda + (y_1 - y) \cos \mu + (z_1 - z) \cos \nu = 0.$$

But we have found for the equation of the same plane,

$$(x_1 - x) \frac{du}{dx} + (y_1 - y) \frac{du}{dy} + (z_1 - z) \frac{du}{dz} = 0,$$

therefore, in order that these two equations should agree, we must have

$$\frac{\cos \lambda}{\cos \nu} = \frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dz}\right)}, \quad \frac{\cos \mu}{\cos \nu} = \frac{\left(\frac{du}{dy}\right)}{\left(\frac{du}{dz}\right)}.$$

These equations, combined with the relation $\cos^2 \lambda + \cos^2 \mu + \cos^2 \nu = 1$, give

$$\frac{\cos \lambda}{\left(\frac{du}{dx}\right)} = \frac{\cos \mu}{\left(\frac{du}{dy}\right)} = \frac{\cos \nu}{\left(\frac{du}{dz}\right)} = \pm \frac{1}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}}.$$

If to shorten the equation we make the denominator of this last expression equal to r , we shall have

$$\cos \lambda = \frac{1}{r} \frac{du}{dx}, \quad \cos \mu = \frac{1}{r} \frac{du}{dy}, \quad \cos \nu = \frac{1}{r} \frac{du}{dz}.$$

This understood, it becomes very easy to find the two equations of the normal drawn through the point (x, y, z) . If we designate by x_1, y_1, z_1 , the variable coordinates of any point of this straight line, we shall have

$$\frac{x_1 - x}{\cos \lambda} = \frac{y_1 - y}{\cos \mu} = \frac{z_1 - z}{\cos \nu};$$

and then, if we replace the quantities $\cos \lambda, \cos \mu, \cos \nu$, by their values in function of the partial differential coefficients $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, we find

$$\frac{(x_1 - x)}{\left(\frac{du}{dx}\right)} = \frac{(y_1 - y)}{\left(\frac{du}{dy}\right)} = \frac{(z_1 - z)}{\left(\frac{du}{dz}\right)},$$

which are the two required equations of the normal.

Integral
Calculus.

(338.) The acute angle formed by the tangent plane at the point x, y, z , with the plane of x, y , is called the *inclination of the tangent plane*, or the inclination of the surface at that point. This same angle is obviously equal to the acute angle formed by the normal with the axis of z , therefore if we designate it by τ we shall have

$$\cos \tau = \pm \frac{1}{r} \frac{du}{dz}, \text{ or } \sec \tau = \pm \frac{r}{\left(\frac{du}{dz}\right)}.$$

(339.) If the given surface were represented by the equation $u = c$ instead of $u = 0$, c being a constant, the equation of the tangent plane and normal would still be

$$(x_1 - x) \frac{du}{dx} + (y_1 - y) \frac{du}{dy} + (z_1 - z) \frac{du}{dz} = 0 \quad \frac{x_1 - x}{\left(\frac{du}{dx}\right)} = \frac{y_1 - y}{\left(\frac{du}{dy}\right)} = \frac{z_1 - z}{\left(\frac{du}{dz}\right)}.$$

If, in the first of these equations, we consider the coordinates x_1, y_1, z_1 as constant quantities, and the coordinates x, y, z as variable, it will no longer represent the equation of the tangent plane to the surfaces $u = 0$, or $u = c$, but it will be the equation of another surface which is the locus of all the points where the surfaces derived from the equation $u = c$, by substituting successively different values for the constant c , are touched by the plane passing through the point (x_1, y_1, z_1) .

In the same manner, if in the other equations, those of the normal, the coordinates x, y, z are supposed to be variable, and x_1, y_1, z_1 to be constant, they will belong to a curve which will be the locus of all the points where the surfaces just mentioned are met by the normal drawn to them through the point x_1, y_1, z_1 .

(340.) When the equation of the surface is resolved with respect to one of the variables, that is to say, is of the form $z = f(x, y)$ or $f(x, y) - z = 0$, we have $u = f(x, y) - z$.

Let $\frac{df(x, y)}{dx} = p, \frac{df(x, y)}{dy} = q$, we shall have clearly $\frac{du}{dx} = p, \frac{du}{dy} = q, \frac{du}{dz} = -1$, and the differential equation of the surface will become $p dx + q dy - dz = 0$, or $dz = p dx + q dy$. The equations of the tangent plane and of the normal become also respectively

$$z_1 - z = p(x_1 - x) + q(y_1 - y) \text{ and } x_1 - x + p(z_1 - z) = 0, y_1 - y + q(z_1 - z) = 0.$$

In this case, we shall have, also, $\cos \tau = \frac{1}{\sqrt{p^2 + q^2 + 1}}$, or $\sec \tau = \sqrt{1 + p^2 + q^2}$.

(341.) Let us suppose now that the curve surface is represented by an equation of the form

$$u + v + w + \dots = c, \quad u, v, w, \dots$$

designating several functions of the variables x, y, z . The equation of the tangent plane may then be put under the following form

$$\left(\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \dots\right)x_1 + \left(\frac{du}{dy} + \frac{dv}{dy} + \frac{dw}{dy} + \dots\right)y_1 + \left(\frac{du}{dz} + \frac{dv}{dz} + \frac{dw}{dz} + \dots\right)z_1 =$$

$$x \frac{du}{dx} + y \frac{du}{dy} + z \frac{du}{dz} + x \frac{dv}{dx} + y \frac{dv}{dy} + z \frac{dv}{dz} + x \frac{dw}{dx} + y \frac{dw}{dy} + z \frac{dw}{dz} + \dots$$

Let us assume that u, v, w , &c. are integral and homogeneous functions, the first of the m^{th} degree, the second of the degree $m - 1$, the third of the degree $m - 2$, &c., the surface will then be a surface of the m^{th} degree. In this case we shall have by the theorem relative to homogeneous functions,

$$x \frac{du}{dx} + y \frac{du}{dy} + z \frac{du}{dz} + x \frac{dv}{dx} + y \frac{dv}{dy} + z \frac{dv}{dz} + \dots = mu + (m - 1)v + (m - 2)w + \dots,$$

and consequently the equation of the tangent plane becomes

$$\left(\frac{du}{dx} + \frac{dv}{dx} + \frac{dw}{dx} + \dots\right)x_1 + \left(\frac{du}{dy} + \frac{dv}{dy} + \frac{dw}{dy} + \dots\right)y_1 + \left(\frac{du}{dz} + \frac{dv}{dz} + \frac{dw}{dz} + \dots\right)z_1 =$$

$$z_1 = mu + (m - 1)v + (m - 2)w + \dots = mc - v - 2w - \&c.$$

If in this last formula we consider the coordinates x_1, y_1, z_1 as constant quantities, and the coordinates x, y, z as variables, it will represent the equation of a surface of the $(m - 1)$ degree, which will contain the points of contact of the first with the tangent planes drawn through the point x_1, y_1, z_1 . Hence if the first surface is of the second degree the second will be of the first. This result may be stated in the following manner.

If through a given point, tangent planes are drawn to a surface of the second degree, all the points of contact will be situated in one plane.

If we suppose $v = 0, w = 0$, that is, if we suppose the equation of the surface reduced to $u = c$, u designating an integral and homogeneous function of the degree m , the above equation of the tangent plane becomes simply

$$\frac{du}{dx}x' + \frac{du}{dy}y' + \frac{du}{dz}z' = mc.$$

(342.) The tangent plane drawn to a surface through a given point (x, y, z) may have only that point common with the surface, or it may touch it in all the points of one or several lines, or it may also intersect it. In these

Integral Calculus. two last cases, if we designate by $f(x, y, z) = 0$ the equation of the proposed surface, and by x_1, y_1, z_1 , the coordinates of any point of the line or lines common to the surface and to the tangent plane, these coordinates must clearly verify the two equations

Part III.

$$f(x_1, y_1, z_1) = 0, \quad \frac{d \cdot f(x, y, z)}{d x} (x_1 - x) + \frac{d \cdot f(x, y, z)}{d y} (y_1 - y) + \frac{d \cdot f(x, y, z)}{d z} (z_1 - z) = 0.$$

When it is possible to draw, on the surface, through a point (x, y, z) one or several straight lines, each of these lines coincides necessarily with its tangent, and, consequently, is contained in the tangent plane at that point. Then, the elimination of z_1 between the two above equations produces an equation between x_1 and y_1 , which is satisfied by taking for y_1 a linear function of x_1 . If this take place not only for a particular point of the surface under consideration, but for every point of it, this surface will be of the number of those which can be generated by a straight line, and which are denominated by the French analysts *surfaces réglées*. Among these must be noticed those which are touched by each tangent plane at every point of the generating line common to the plane and to the surface; they admit of developement upon a plane, and are called by the French *surfaces développables*; then those which do not admit of developement upon a plane, are called by the French *surfaces gauches*. Cylindrical and conical surfaces belong to the first division. A cylindrical surface, in general, may be considered as generated by a straight line which in every position is parallel to the same straight line. A conical surface, in general, may be considered as generated by a straight line which, in every position, passes through the same point.

We shall now give a few examples of the preceding formulæ.

1. Let the proposed surface be a cylindrical surface, the generating line of which is parallel to the axis of z . This surface will have for its basis a certain curve drawn in the plane of the x, y , and if $f(x, y) = 0$ be the equation of this curve, it will also be the equation of the cylindrical surface. Let us suppose, for brevity sake,

$$\frac{d \cdot f(x, y)}{d x} = \phi(x, y) \text{ and } \frac{d f(x, y)}{d y} = \psi(x, y), \text{ then the equation of the tangent plane will become } (x_1 - x) \phi$$

$(x, y) + (y_1 - y) \psi(x, y) = 0$, and those of the normal $(x_1 - x) \psi(x, y) - (y_1 - y) \phi(x, y) = 0$ and $z_1 = z$. These equations show that the tangent plane is parallel to the axis of z , and the normal perpendicular to the same axis, and moreover that the tangent plane will touch the surface all along the generating line passing through the point of contact.

2. Let us consider, in the next place, a conical surface, the summit or centre of which, that is the point where all the generating lines meet, coincides with the origin of coordinates. Let us conceive besides that the basis of this surface is a curve, situated in a plane parallel to the plane of x, y , which meets the axis of z at a distance of the origin $= 1$. Let x' and y' be the coordinates of any point of that curve, and $f(x', y') = 0$ its equation. The

generating line passing through the point (x', y') will be represented by the two equations $\frac{x}{z} = x', \frac{y}{z} = y'$. If

then we eliminate between these two equations, and the equation $f(x', y') = 0$, the two quantities x' and y' , it is

clear that the equation $f\left(\frac{x}{z}, \frac{y}{z}\right) = 0$, which we shall obtain, will be true for any point of any of the generating

lines, and therefore that it will be the equation of the conical surface. This understood, if we assume, as in the

preceding example, $\frac{d \cdot f(x, y)}{d x} = \phi(x, y)$, and $\frac{d f(x, y)}{d y} = \psi(x, y)$, we shall have for the differential equation of the conical surface

$$\phi\left(\frac{x}{z}, \frac{y}{z}\right) dx + \psi\left(\frac{x}{z}, \frac{y}{z}\right) dy - \frac{1}{z} \left\{ x \phi\left(\frac{x}{z}, \frac{y}{z}\right) + y \psi\left(\frac{x}{z}, \frac{y}{z}\right) \right\} dz = 0.$$

If now we designate by x_1, y_1, z_1 , the coordinates of any point of the tangent plane or of the normal at the point (x, y, z) , we shall find for the equation of the tangent plane

$$(x_1 - x) \phi\left(\frac{x}{z}, \frac{y}{z}\right) + (y_1 - y) \psi\left(\frac{x}{z}, \frac{y}{z}\right) - \frac{1}{z} \left\{ x \phi\left(\frac{x}{z}, \frac{y}{z}\right) + y \psi\left(\frac{x}{z}, \frac{y}{z}\right) \right\} (z_1 - z) = 0, \text{ or after reduction}$$

$$x_1 \phi\left(\frac{x}{z}, \frac{y}{z}\right) + y_1 \psi\left(\frac{x}{z}, \frac{y}{z}\right) = \left\{ \frac{x}{z} \phi\left(\frac{x}{z}, \frac{y}{z}\right) + \frac{y}{z} \psi\left(\frac{x}{z}, \frac{y}{z}\right) \right\} z_1 = 0.$$

The equation of the normal will be

$$\frac{x_1 - x}{\phi\left(\frac{x}{z}, \frac{y}{z}\right)} = \frac{y_1 - y}{\psi\left(\frac{x}{z}, \frac{y}{z}\right)} = \frac{z(z - z_1)}{x \phi\left(\frac{x}{z}, \frac{y}{z}\right) + y \psi\left(\frac{x}{z}, \frac{y}{z}\right)}.$$

It may be remarked that since the equation of the tangent plane remains the same when x, y , and z are made to

vary in such a manner as not to change the values of the ratios $\frac{x}{z}, \frac{y}{z}$, it follows that each tangent plane will touch

the surface at every point of the generating line passing through the given point of contact. Here the two equations which are to be verified by linear functions of z_1 , substituted for y_1 and z_1 , are

$$f\left(\frac{x_1}{z_1}, \frac{y_1}{z_1}\right) = 0 \left(\frac{x_1}{z_1} - \frac{x}{z}\right) \phi\left(\frac{x}{z}, \frac{y}{z}\right) + \left(\frac{y_1}{z_1} - \frac{y}{z}\right) \psi\left(\frac{x}{z}, \frac{y}{z}\right) = 0.$$

It is obvious that by taking $\frac{x_1}{z_1} = \frac{x}{z}$ and $\frac{y_1}{z_1} = \frac{y}{z}$, these equations are satisfied, since we must always have $f\left(\frac{x}{z}, \frac{y}{z}\right) = 0$, and that $\frac{x_1}{z_1} = \frac{x}{z}$, and $\frac{y_1}{z_1} = \frac{y}{z}$ represent a straight line passing through the origin or the centre of the conical surface. Hence this straight line is common to the tangent plane and to the surface.

3. Let the equation of the surface be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2\frac{z}{c}$, that is to say, let the surface be an *elliptical paraboloid*. We shall find for the equation of the tangent plane

$$\frac{x}{a^2}(x_1 - x) + \frac{y}{b^2}(y_1 - y) = \frac{1}{c}(z_1 - z), \text{ or } \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{z_1 + z}{c}.$$

The equations of the normal will be

$$\frac{a^2(x_1 - x)}{x} = \frac{b^2(y_1 - y)}{y} = -c(z_1 - z) = \frac{x(x_1 - x) + y(y_1 - y) + 2z(z_1 - z)}{0}, \text{ or}$$

$$\frac{a^2(x_1 - x)}{x} = \frac{b^2(y_1 - y)}{y}, \quad x(x_1 - x) + y(y_1 - y) + 2z(z_1 - z) = 0.$$

To ascertain whether the tangent plane has more than one point common with the surface, it suffices to examine if several systems of real values of x_1, y_1, z_1 may be found to verify the two equations

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{2z_1}{c}, \quad \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{z + z_1}{c}.$$

From these two equations, and from the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2z}{c}$ we easily derive $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xx_1}{a^2} -$

$\frac{2yy_1}{b^2} = 0$, or $\left(\frac{x_1 - x}{a}\right)^2 + \left(\frac{y_1 - y}{b}\right)^2 = 0$, an equation which can only be satisfied by making $x_1 = x$ and $y_1 = y$,

and these values give $z_1 = z$. Therefore the point (x, y, z) is the only point that is at the same time on the tangent plane and on the surface.

4. Let us next consider the *hyperbolic paraboloid* represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$. All sections of this surface parallel to the plane of x, y are hyperbolas, and all sections by planes perpendicular to the axis of x or to the axis of y are parabolas. The section by the plane of the x, y is represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, and consequently is composed of two straight lines passing through the origin of the coordinates, and represented by the two equations $\frac{x}{a} - \frac{y}{b} = 0$, $\frac{x}{a} + \frac{y}{b} = 0$.

The equation of the tangent plane is

$$\frac{x(x_1 - x)}{a^2} - \frac{y(y_1 - y)}{b^2} = \frac{z_1 - z}{c}, \text{ or } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{z_1 + z}{c},$$

and the equations of the normal are

$$\frac{a^2(x_1 - x)}{x} - \frac{b^2(y_1 - y)}{y} = 0, \quad x(x_1 - x) - y(y_1 - y) + 2z(z_1 - z) = 0.$$

To obtain the lines which, passing through the point (x, y, z) , are common to the surface and to the tangent plane, we must find the values of or relations between x_1, y_1 , and z_1 , which satisfy the two equations

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} = \frac{2z_1}{c}, \quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{z + z_1}{c}.$$

These being combined with $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$, give

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} + \frac{x^2}{a^2} - \frac{y^2}{b^2} - 2\left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2}\right) = 0, \text{ or } \left(\frac{x_1 - x}{a}\right)^2 - \left(\frac{y_1 - y}{b}\right)^2 = 0$$

and therefore

$$\frac{x_1 - x}{a} = \frac{y_1 - y}{b}, \text{ or } \frac{x_1 - x}{a} = -\frac{y_1 - y}{b}.$$

Integral
Calculus.

Part III.

Each of these two last equations, together with the equation $\frac{x x_1}{a^2} - \frac{y y_1}{b^2} = \frac{z + z_1}{c}$, represent a straight line. There-

fore there are two straight lines passing through the point x, y, z , which are at the same time on the hyperbolic paraboloid, and in the tangent plane. But, in this case, the tangent plane touches the surface only at the point (x, y, z) and not in every point where it meets the surface. Hence the hyperbolic paraboloid does not admit of developement upon a plane, and belongs to the class of *surfaces gauches*. We shall not demonstrate this proposition here, because in the sequel we shall give a general rule to ascertain whether or not a tangent plane touches a surface in every point of the line or lines where it meets it.

In the same manner we would find respectively for the equations of the tangent planes and normals.

1. To the ellipsoid represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$,

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} + \frac{z z_1}{c^2} = 1, \quad \frac{a^2 (x_1 - x)}{x} = \frac{b^2 (y_1 - y)}{y} = \frac{c^2 (z_1 - z)}{z}.$$

2. To the hyperboloid of two surfaces, or the hyperboloid of two summits (the *hyperboloïde à deux nappes* of the French) represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$,

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} - \frac{z z_1}{c^2} = -1, \quad \frac{a^2 (x_1 - x)}{x} - \frac{b^2 (y_1 - y)}{y} = \frac{c^2 (z_1 - z)}{-z}.$$

3. To the hyperboloid of one surface, or the hyperboloid of four summits (the *hyperboloïde à une nappe* of the French) represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$,

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} - \frac{z z_1}{c^2} = 1, \quad \frac{a^2 (x_1 - x)}{x} = \frac{b^2 (y_1 - y)}{y} = \frac{c^2 (z_1 - z)}{-z}.$$

We shall find that for this last hyperboloid that there are also two straight lines, passing through the point of contact, which are, at the same time, on the surface and on the tangent plane, but that this plane touches the surface only in one point, and consequently that the surface is a *surface gauche*.

For the last example we shall take the curve surface which is generated by a horizontal line passing constantly through the vertical axis of z , and whose extremity is always found in the helix of the last example, given (335.).

If a plane pass through the axis of z , it will meet the generating line in one and even in an infinite number of its positions; and if any point whatever be taken in this line, whose coordinates are z, x, y , then z will bear the same ratio to the angle described by the plane from its first position, as in the case of the helix, which guides the generating line, and we shall have for the equation of the surface,

$$z = a r \tan^{-1} \frac{y}{x}, \quad \text{or } y = x \tan \frac{z}{a r},$$

and for the differential equation

$$dz = a r \frac{x dy - y dx}{x^2 + y^2}, \quad \text{or } \frac{x^2 + y^2}{a r} dz = x dy - y dx.$$

Hence the equations of the tangent plane and normal will be

$$x y_1 - y x_1 = \frac{x^2 + y^2}{a r} (z_1 - z), \quad \frac{a r (z_1 - z)}{x^2 + y^2} = \frac{y_1 - y}{-x} = \frac{x_1 - x}{y}.$$

By means of these equations we may easily ascertain various properties of the surface. It may be proved, for instance, that there are an infinite number of straight lines common to the surface and to the tangent plane, one of which is the generating line passing through the point of contact, or the perpendicular drawn through the point of contact to the axis of z . It may also be remarked, that the axis of z is entirely contained in the surface under consideration, since the equation of the surface is satisfied for $x = 0$, and $y = 0$, independently of any particular value of z . To have the equation of the tangent plane drawn through one of the points of the axis of z , we must write, in the general equation, $x = 0$ and $y = 0$, or rather $x = 0$ in the

equation $y_1 - x_1 \tan \frac{z}{a r} = \frac{x}{a r} (z_1 - z) \sec^2 \frac{z}{a r}$, resulting from the elimination of y between the equation of the surface and the equation of the tangent plane. Thus we find for the equation of the tangent plane in one

point of the axis of z , $\frac{y_1}{x_1} = \tan \frac{z}{a r}$. Therefore this plane will contain the axis of z , and all the generating lines

which make the same angle with the axis of x . It may also be remarked, that if x or y be very large, the normal is perpendicular to the plane of x, y , and the tangent plane parallel to it.

This is the surface presented by the superior and inferior surface of a staircase, attached to a vertical column round which it winds, or to the concave wall of a circular tower, its thickness being supposed to be uniform, and no regard being paid to irregularities caused by the form of the steps.

(343.) There are sometimes points upon curve surfaces through which an infinite number of tangent planes.

Integral
Calculus.

Part III.

may be drawn. For each point of this kind the values of $\cos \lambda$, $\cos \mu$, $\cos \nu$, must become undeterminate, and this can only take place in two cases, 1. when one at least of the quantities $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$, becomes undeterminate ;

2. when these three quantities become at once equal to nothing or infinity. In both these cases the substitution of the values of x , y , z , transform the equation of the tangent plane to an identical equation, or at least to an equation which contains an arbitrary constant.

Let us consider, for instance, the centre of the conical surface represented by the equation $x^2 + y^2 = r^2 z^2$. Since this centre is placed at the origin of coordinates, the corresponding values of x , y , and z will be equal to nothing, and being substituted in the equation of the tangent plane $x_1 x + y_1 y = r^2 z_1 z$, will make both sides vanish. But if we introduce polar coordinates ρ and p instead of x and y , by making $x = \rho \cos p$, and $y = \rho \sin p$, the equation of the surface becomes $z = \pm \frac{\rho}{r}$, and the equation of the tangent plane $x_1 \cos p + y_1 \sin p = r z_1$. This last

equation contains only one of the polar coordinates, namely p , and this angle, which is determined for every point of the surface other than the centre, is undeterminate for the centre itself, and then changes itself into an arbitrary constant. To the various values which this arbitrary constant may receive correspond an infinite number of planes represented by the equation $x_1 \cos p + y_1 \sin p = r z_1$.

Centres and Diameters of Curve Surfaces, and Curves in Space, and Axes of Curve Surfaces.

(344.) Generally the centre of a curve, or a curve surface, is a point such that the radii drawn from it to the curve, or curve surface, are equal to two and two and in opposite direction. When a curve, or a curve surface, has a centre, and that this point has been taken for the origin of coordinates, no alteration is produced in the equations, or equation, of this curve, or curve surface by changing x into $-x$, y into $-y$, z into $-z$, &c. When the centre coincides with the point which has for its coordinates a , b , c , these equations are not altered neither by substituting $2a - x$ for x , $2b - y$ for y , $2c - z$ for z . These propositions are obvious consequences of the definition of a centre.

It is necessary to remark, that a curve, or a curve surface, may have an infinite number of centres. Thus, for instance, if we have a curve situated in a plane perpendicular to a straight line, the centre of which is situated on that straight line, if then we imagine a cylindrical surface having that curve for its base, and the generating line of which should be parallel to the straight line just mentioned, it is obvious that every point of that line will be a centre of the cylindrical surface.

Any straight line drawn through the centre of a curve, or a curve surface, is a diameter of this curve, or curve surface.

An *axis* of a curve surface is a straight line drawn in such a manner as to divide in two symmetrical parts each of the plane curves, which are obtained by cutting the surface by planes which contain this same straight line. Such an axis is necessarily a normal to the curve surface at the points where it meets it, unless the tangent plane drawn through that point of meeting should become undeterminate. Moreover, each point of the axis is obviously a centre of the curve obtained by cutting the surface by a plane drawn perpendicularly to the axis through that point. Hence if this axis coincide with the axis of x , and if the coordinate lines be rectangular, the equation of the surface will not be altered by substituting $-y$ for y , and $-z$ for z .

(345.) Let now $u = f(x, y, z) = 0$ be the equation of a surface limited on all sides, and let the point which we suppose to be the only centre of the surface be designated by 0. If this surface admit one or several axes the point 0 will be situated upon each of them, for the intersection of the surface with planes drawn through this point, in directions perpendiculars to the axis, will be curves having for only centre the point 0. If for greater simplicity we place the origin of coordinates at the point 0, a line drawn through the origin to the point of the surface, the coordinates of which are x , y , z , will make angles with the three coordinate axes the cosines of which are proportional to x , y , z ; and the normal drawn through the same point will make angles with the same lines the cosines of which are proportional to $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$. This understood, let us imagine that the line drawn from the point 0 to the

point x , y , z , coincides with one of the axes of the curve surface. In that case, the normal at the point (x, y, z) coincides with the axis of the curve surface, therefore the quantities $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$, are proportional to the quantities

x , y , z , and we shall have, consequently, $\frac{(\frac{du}{dx})}{x} = \frac{(\frac{du}{dy})}{y} = \frac{(\frac{du}{dz})}{z}$. This result would still be true if the proposed surface were represented by the equation $u = c$ instead of the equation $u = 0$, c designating a constant quantity.

The three equations $u = c$, $\frac{(\frac{du}{dx})}{x} = \frac{(\frac{du}{dy})}{y} = \frac{(\frac{du}{dz})}{z}$, will be sufficient to determine the coordinates x , y , z , of the points where the normals drawn through the origin to the surface $u = c$ meet this same surface. To each of these normals answer a particular system of values of the variables x , y , z , and these values being substituted in the formula

$$\frac{\cos \lambda}{x} = \frac{\cos \mu}{y} = \frac{\cos \nu}{z} = \pm \frac{1}{\sqrt{x^2 + y^2 + z^2}},$$

give the angles λ , μ , and ν , which the normal makes with the coordinate axis. If then it be required to determine whether the normal under consideration is an axis of the surface $u = c$, it will be sufficient to cut this surface by planes drawn in a direction perpendicular to the axis, and to examine whether each curve of intersection has a centre situated on the normal itself. But if we take on the normal a point situated at a distance κ from the origin, the coordinates of this point will be $\kappa \cos \lambda$, $\kappa \cos \mu$, $\kappa \cos \nu$, and the plane drawn perpendicularly to the normal through the same point will be represented by the equation $x \cos \lambda + y \cos \mu + z \cos \nu = \kappa$; therefore we shall have only to examine whether the system of the two equations $x \cos \lambda + y \cos \mu + z \cos \nu = \kappa$, and $f(x, y, z) = c$ is altered by the substitution of $2\kappa \cos \lambda - x$, $2\kappa \cos \mu - y$, $2\kappa \cos \nu - z$, for x, y, z ; and since the first of these two equations will not be changed by that substitution, it will be sufficient to verify whether the equation $f(2\kappa \cos \lambda - x, 2\kappa \cos \mu - y, 2\kappa \cos \nu - z) = c$ may result, whatever be the value of κ , of the combination of the two equations $x \cos \lambda + y \cos \mu + z \cos \nu = \kappa$, and $u = c$.

In the case where u is an homogeneous function of the m^{th} degree, we have

$$x \frac{du}{dx} + y \frac{du}{dy} + z \frac{du}{dz} = mu,$$

and since $\frac{(du)}{dx} = \frac{(du)}{dy} = \frac{(du)}{dz}$, each of these quantities will also be equal to $\frac{mu}{x^2 + y^2 + z^2}$. Consequently, if

we designate by r the radius vector drawn through the origin to the point (x, y, z) , that is, if we make

$x^2 + y^2 + z^2 = r^2$, we shall get $\frac{(du)}{dx} = \frac{(du)}{dy} = \frac{(du)}{dz} = \frac{mc}{r^2}$. It is essential to observe that the equations

$\frac{(du)}{dx} = \frac{(du)}{dy} = \frac{(du)}{dz}$ are those which determine for the surface $u = 0$, or $u = c$, the values maxima and

minima of the function $x^2 + y^2 + z^2$, that is, the values maxima and minima of the radius vector. Therefore when a curve surface has one or several axes, which pass through the origin of the coordinates, the radius vector must coincide with one of these axes to become a maximum or a minimum.

On the Differential of the Arc of a Curve in Space.

(346.) Let x, y, z be the coordinates of a point of a curve in space represented by two equations between these quantities, and let s designate the length of an arc of this curve reckoned from that point. Two of the coordinates x, y, z are functions of the third, and it is obvious that the arc s is also a function of the same independent variable. Let it be proposed now to find the differential of that unknown function. For that purpose let $x + h, y + \kappa, z + l$, be the coordinates of another point of the curve situated at a very small distance from the point (x, y, z) . The length of the chord joining these two points is clearly $\sqrt{h^2 + \kappa^2 + l^2}$. Let also m represent the length of

the corresponding arc. Then the ratio $\frac{m}{\sqrt{h^2 + \kappa^2 + l^2}}$, that is, the ratio of the arc to the chord, will be greater

than one, but the nearer the second point will be to the first, the nearer will the ratio be equal to one, or, in other words, one is the limit of that ratio. Moreover, at the limit, we have $m = ds$, $h = dx$, $\kappa = dy$, and $l = dz$, therefore

$$1 = \frac{ds}{\sqrt{dx^2 + dy^2 + dz^2}}, \quad \text{or } ds = \sqrt{dx^2 + dy^2 + dz^2}.$$

If x be taken for the independent variable, and if we make $\frac{dy}{dx} = p$, and $\frac{dz}{dx} = q$, we shall have $ds = dx \sqrt{1 + p^2 + q^2}$.

This value of $\sqrt{dx^2 + dy^2 + dz^2}$ being substituted in the expressions which have been found (330.) for the cosines of the angles which a tangent to a curve in space makes with the axes of the three coordinates, they will become

$$\cos \alpha = \frac{dx}{ds}, \quad \cos \beta = \frac{dy}{ds}, \quad \cos \gamma = \frac{dz}{ds}.$$

The acute angle formed by the tangent at the point (x, y, z) with the axis of x , is called the *inclination of the tangent*, or the *inclination of the curve* with respect to that axis, it is obviously equal to, or supplement of, the angle α .

(347.) Let us consider now any two curves. Let x, y, z be the coordinates of any point of the first, and s its distance to a certain point on the same curve. Let x_1, y_1, z_1 and s_1 represent the analogous quantities with respect to the second curve. We shall have

$$ds^2 = dx^2 + dy^2 + dz^2, \text{ and } ds_1^2 = dx_1^2 + dy_1^2 + dz_1^2.$$

The tangent to the first curve, at the point (x, y, z) , will make with the axis certain angles the cosines of which are $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, and the tangent to the second curve, at the point (x_1, y_1, z_1) , will make, with the same axis, angles the cosines of which are $\frac{dx_1}{ds_1}, \frac{dy_1}{ds_1}, \frac{dz_1}{ds_1}$. Therefore if we call δ the angle of the two tangents, we shall have

$$\cos \delta = \frac{dx}{ds} \cdot \frac{dx_1}{ds_1} + \frac{dy}{ds} \cdot \frac{dy_1}{ds_1} + \frac{dz}{ds} \cdot \frac{dz_1}{ds_1} = \frac{dx \cdot dx_1 + dy \cdot dy_1 + dz \cdot dz_1}{ds \cdot ds_1}$$

If the two tangents are parallel, we have $\frac{dx}{ds} = \frac{dx_1}{ds_1}, \frac{dy}{ds} = \frac{dy_1}{ds_1}, \frac{dz}{ds} = \frac{dz_1}{ds_1}$, or $\frac{dx}{dx_1} = \frac{dy}{dy_1} = \frac{dz}{dz_1} = \frac{ds}{ds_1}$;

and if they are perpendicular, $dx dx_1 + dy dy_1 + dz dz_1 = 0$. When two curves have a common point they are said to form the same angle as their tangents at that point. They are said to be normal to each other when the angle of their tangents is a right angle, and that they are tangent to each other when that angle is equal to nothing, that is when their tangents at that point coincide. In the first case the coordinates of the point common

to the curve must verify the equation $1 + \frac{dy}{dx} \cdot \frac{dy_1}{dx_1} + \frac{dz}{dx} \cdot \frac{dz_1}{dx_1} = 0$, and in the second case the two equations $\frac{dy_1}{dx_1} = \frac{dy}{dx}, \frac{dz_1}{dx_1} = \frac{dz}{dx}$. Hence if x be the abscissa of the point common to the curve, when they will be tangent the values of y and z , and $\frac{dy}{dx}, \frac{dz}{dx}$, corresponding to that abscissa, must be the same for both curves.

(348.) We shall now proceed to demonstrate a theorem of great use in the theory of the contact of curves. It may be stated in the following manner. Two curves which are tangent to each other being given, if from the point of contact we take upon each of these curves, produced in the same direction, two very small and equal lengths, the straight line joining the extremities of these lengths will sensibly be perpendicular to the tangent, common to the two curves at the point where they meet.

Let the coordinates of the extremities of these equal lengths be, for the first curve (x, y, z) , and for the second (x_1, y_1, z_1) ; let also s and s_1 designate respectively the length of an arc of the first curve contained between a point chosen on this curve and the point (x, y, z) , and the length of an arc of the second curve contained between a point of that curve and the point (x_1, y_1, z_1) . If the coordinates x, y, z, x_1, y_1, z_1 , are made to vary simultaneously, then the difference $s_1 - s$ will remain a constant quantity, and therefore we shall have $ds_1 = ds$.

Let α, β, γ , be the cosines of the angles which the tangent, common to both curves, makes with the three axes, let l be the length of the line joining the two points $(x, y, z), (x_1, y_1, z_1)$, and λ, μ, ν the cosines of the angles it makes with the axis. We shall have

$$\cos \alpha = \frac{dx}{ds} = \frac{dx_1}{ds_1}, \cos \beta = \frac{dy}{ds} = \frac{dy_1}{ds_1}, \cos \gamma = \frac{dz}{ds} = \frac{dz_1}{ds_1}, \text{ and } l = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2},$$

$$\cos \lambda = \frac{x - x_1}{l}, \cos \mu = \frac{y - y_1}{l}, \cos \nu = \frac{z - z_1}{l}. \text{ But since } ds = ds_1, \text{ we have also}$$

$$dx^2 + dy^2 + dz^2 = dx_1^2 + dy_1^2 + dz_1^2, \text{ or } (dx + dx_1)(dx - dx_1) + (dy + dy_1)(dy - dy_1) + (dz + dz_1)(dz - dz_1) = 0.$$

From the above values of $\cos \alpha, \cos \beta, \cos \gamma$, we derive also

$$\frac{dx + dx_1}{\cos \alpha} = \frac{dy + dy_1}{\cos \beta} = \frac{dz + dz_1}{\cos \gamma} = ds + ds_1 = 2ds.$$

If, moreover, we observe, that the quantities $x - x_1, y - y_1, z - z_1$, and l are equal to nothing at the point of contact, we shall conclude, that for values of the coordinates differing but very little from those of that point, the ratios of these functions are very nearly equal to the ratios of their differentials. Therefore the following equations will be very nearly exact.

$$\cos \lambda = \frac{x - x_1}{l} = \frac{dx - dx_1}{dl}, \cos \mu = \frac{y - y_1}{l} = \frac{dy - dy_1}{dl}, \cos \nu = \frac{z - z_1}{l} = \frac{dz - dz_1}{dl}.$$

Thus we have very nearly $dx - dx_1 = dl \cos \lambda, dy - dy_1 = dl \cos \mu, dz - dz_1 = dl \cos \nu$, but we have found $dx + dx_1 = 2ds \cos \alpha, dy + dy_1 = 2ds \cos \beta, dz + dz_1 = 2ds \cos \gamma$. Hence by multiplication and addition we shall get

$$(dx - dx_1)(dx + dx_1) + (dy - dy_1)(dy + dy_1) + (dz - dz_1)(dz + dz_1) = 2dl \cdot ds (\cos \alpha \cos \lambda + \cos \beta \cos \mu + \cos \gamma \cos \nu).$$

Integral Calculus. The quantity in the first side of this equation has been found equal to nothing, therefore we shall have very nearly Part III.
 $\cos \alpha \cos \lambda + \cos \beta \cos \mu + \cos \gamma \cos \nu = 0$, that is to say that the tangent, common to the two curves, is very nearly perpendicular to the line joining the two points (x, y, z) , (x_1, y_1, z_1) .

(349.) It is clear that in the preceding theorem, instead of one of the two curves we might take the tangent to the other, and consequently, if from the point of contact we take upon a curve and its tangent, produced in the same direction, two equal and very small lengths, the straight line joining the extremities of these lengths will very nearly be perpendicular to the tangent, that is to say to the normal plane. Let i represent the very small lengths taken upon the curve and tangent, the less i is, the nearer is the line joining the extremities of the tangent and curve to a parallel to a certain normal of the curve. This normal, the direction of which is the limit of the direction of the line of junction, has been designated by the name of the *principal normal*; its equations may be found in the following manner.

Let x, y, z be the coordinates of the point of the curve through which the tangent passes. Let s be the arc reckoned on the curve between the point (x, y, z) , and another point situated in such a manner that the length i should be the producement of the arc s . Let α, β, γ be the angles which the tangent, at the point (x, y, z) , and produced in the same direction as the arc s makes with the three coordinate axes. If we take the arc s for the independent variable, the extremity of the length i , on the curve, will obviously have for its coordinates the three expressions

$$x + i \frac{dx}{ds} + \frac{i^2}{2} \left(\frac{d^2x}{ds^2} + I \right), \quad y + i \frac{dy}{ds} + \frac{i^2}{2} \left(\frac{d^2y}{ds^2} + J \right), \quad z + i \frac{dz}{ds} + \frac{i^2}{2} \left(\frac{d^2z}{ds^2} + K \right),$$

where I, J, K are quantities which vanish when $i = 0$. In the same manner the coordinates of the extremity of the length i , on the tangent, will be

$$x + i \cos \alpha = x + i \frac{dx}{ds}, \quad y + i \cos \beta = y + i \frac{dy}{ds}, \quad z + i \cos \gamma = z + i \frac{dz}{ds}.$$

This understood, if we call k the distance of the two extremities, and λ, μ, ν the angles formed by the line joining them with the coordinate axis, we shall have clearly

$$k = \frac{i^2}{2} \sqrt{\left(\frac{d^2x}{ds^2} + I \right)^2 + \left(\frac{d^2y}{ds^2} + J \right)^2 + \left(\frac{d^2z}{ds^2} + K \right)^2},$$

$$\cos \lambda = \frac{\frac{i^2}{2} \left(\frac{d^2x}{ds^2} + I \right)}{k}, \quad \cos \mu = \frac{\frac{i^2}{2} \left(\frac{d^2y}{ds^2} + J \right)}{k}, \quad \cos \nu = \frac{\frac{i^2}{2} \left(\frac{d^2z}{ds^2} + K \right)}{k},$$

and, consequently,

$$\frac{\cos \lambda}{\frac{d^2x}{ds^2} + I} = \frac{\cos \mu}{\frac{d^2y}{ds^2} + J} = \frac{\cos \nu}{\frac{d^2z}{ds^2} + K} = \frac{1}{\sqrt{\left(\frac{d^2x}{ds^2} + I \right)^2 + \left(\frac{d^2y}{ds^2} + J \right)^2 + \left(\frac{d^2z}{ds^2} + K \right)^2}}.$$

If now we suppose i to converge towards the limit zero, the values of I, J, K will decrease indefinitely, and at the limit we shall have

$$\frac{\cos \lambda}{\frac{d^2x}{ds^2}} = \frac{\cos \mu}{\frac{d^2y}{ds^2}} = \frac{\cos \nu}{\frac{d^2z}{ds^2}} = \frac{1}{\sqrt{\left(\frac{d^2x}{ds^2} \right)^2 + \left(\frac{d^2y}{ds^2} \right)^2 + \left(\frac{d^2z}{ds^2} \right)^2}}.$$

or, what is the same thing,

$$\frac{\cos \lambda}{\frac{d^2x}{ds^2}} = \frac{\cos \mu}{\frac{d^2y}{ds^2}} = \frac{\cos \nu}{\frac{d^2z}{ds^2}} = \frac{\pm 1}{\sqrt{\left(\frac{d^2x}{ds^2} \right)^2 + \left(\frac{d^2y}{ds^2} \right)^2 + \left(\frac{d^2z}{ds^2} \right)^2}}.$$

It would be very easy to verify that the line passing through the point (x, y, z) and forming with the coordinate axis angles determined by the above formulæ, is one of the normals passing through that point. For if we differentiate the equation $ds^2 = dx^2 + dy^2 + dz^2$, in considering s as the independent variable, we shall find $dx \frac{d^2x}{ds^2} + dy \frac{d^2y}{ds^2} + dz \frac{d^2z}{ds^2} = 0$, which is the same thing as $\cos \alpha \cos \lambda + \cos \beta \cos \mu + \cos \gamma \cos \nu = 0$, α, β, γ being the angles with the tangent to the curve, at the point (x, y, z) makes with the axis.

The angles λ, μ, ν , being once determined by means of the above formulæ, it is clear that the equation of the principal normal will be, in representing by x_1, y_1, z_1 the coordinates of any point of that line,

$$\frac{x_1 - x}{\cos \lambda} = \frac{y_1 - y}{\cos \mu} = \frac{z_1 - z}{\cos \nu},$$

or in supposing that s is taken for the independent variable,

$$\frac{x_1 - x}{\frac{d^2x}{ds^2}} = \frac{y_1 - y}{\frac{d^2y}{ds^2}} = \frac{z_1 - z}{\frac{d^2z}{ds^2}}$$

and if s were not taken for the independent variable,

$$\frac{x_1 - x}{d \cdot \left(\frac{dx}{ds} \right)} = \frac{y_1 - y}{d \cdot \left(\frac{dy}{ds} \right)} = \frac{z_1 - z}{d \cdot \left(\frac{dz}{ds} \right)}.$$

(350.) When two curve surfaces have a common point, they are supposed to form, at that point, the same angle as their tangent planes. Two surfaces in particular are said to be normal to each other when their tangent planes are perpendicular to each other, and they are said to be *tangent* or to *touch* each other when these planes coincide. In the last case the normals coincide also. This understood, let $u = 0$, $v = 0$, be the equations of two surfaces referred to three rectangular axes, and touch each other at the point (x, y, z) . Then it is obvious from what precedes that we shall have

$$\frac{du}{dx} : \frac{dv}{dx} :: \frac{du}{dy} : \frac{dv}{dy} :: \frac{du}{dz} : \frac{dv}{dz}.$$

Reciprocally, if this condition be satisfied for the point (x, y, z) , they will have at that point a common normal and be tangent to each other.

If the equations of the two surfaces were resolved with respect to z , their differential equations would be of the form $dz = p dx + q dy$. Then the two surfaces would be tangent to each other, if in the passage of the first to the second, the two quantities p and q preserved the same values.

On the Osculating Plane of a Curve of Double Curvature.

(351.) Let us consider upon a given curve any point P , and let us conceive that a tangent to the curve has been drawn through that point. The plane passing through that tangent and the principal normal, deserves a particular notice, and is called the *osculating plane* of the curve. To obtain this plane, it is sufficient to draw a tangent plane to the curve through the point P , and containing, at the same time, a second point Q of the curve, and to find the position of this plane when the second point approaches indefinitely to the first. Let i be the length of the arc PQ comprised between the two points, and let us suppose that the tangent being produced on the same side as the arc PQ , the extremity of the length i upon the tangent is a point R . The straight line QR , contained in the plane passing through the tangent and the point Q , will be for very small values of i very nearly parallel to the principal normal, therefore this plane will approach, for decreasing values of i , nearer and nearer to coincide with the osculating plane.

Let, as before, α, β, γ represent the angles which the tangent to the curve makes with the axis, and λ, μ, ν those which the principal normal makes with the same lines. Let us suppose besides that the arc s is taken for the independent variable. If x, y, z are the coordinates of P , we shall have

$$\frac{\cos \alpha}{dx} = \frac{\cos \beta}{dy} = \frac{\cos \gamma}{dz}, \text{ and } \frac{\cos \lambda}{d^2 x} = \frac{\cos \mu}{d^2 y} = \frac{\cos \nu}{d^2 z}.$$

This understood, let us imagine that through the point (x, y, z) we draw a line perpendicular to the osculating plane. This line will be at the same time perpendicular to the tangent, and to the principal normal. Therefore, if we call l, m, n the angles it forms with the coordinate axis, we shall have the two equations

$$\cos \alpha \cos l + \cos \beta \cos m + \cos \gamma \cos n = 0, \quad \cos \lambda \cos l + \cos \mu \cos m + \cos \nu \cos n = 0,$$

which the above formulæ will reduce to

$$\cos l \cdot dx + \cos m \cdot dy + \cos n \cdot dz = 0, \quad \cos l \cdot d^2 x + \cos m \cdot d^2 y + \cos n \cdot d^2 z = 0.$$

These last equations may be replaced by the formula

$$\frac{\cos l}{dy d^2 z - dz d^2 y} = \frac{\cos m}{dz d^2 x - dx d^2 z} = \frac{\cos n}{dx d^2 y - dy d^2 x}.$$

And since $\cos^2 l + \cos^2 m + \cos^2 n = 1$, we have

$$\frac{\cos l}{dy d^2 z - dz d^2 y} = \frac{\cos m}{dz d^2 x - dx d^2 z} = \frac{\cos n}{dx d^2 y - dy d^2 x} = \pm 1$$

$$= \frac{\pm 1}{\sqrt{\{(dy d^2 z - dz d^2 y)^2 + (dz d^2 x - dx d^2 z)^2 + (dx d^2 y - dy d^2 x)^2\}}} \\ = \frac{\pm 1}{\sqrt{\{(dx^2 + dy^2 + dz^2)((d^2 x)^2 + (d^2 y)^2 + (d^2 z)^2) - (dx d^2 x + dy d^2 y + dz d^2 z)^2\}}}.$$

If, moreover, we recollect, that $dx^2 + dy^2 + dz^2 = ds^2$, and $dx d^2 x + dy d^2 y + dz d^2 z = 0$, we shall find

$$\frac{\cos l}{dy d^2 z - dz d^2 y} = \frac{\cos m}{dz d^2 x - dx d^2 z} = \frac{\cos n}{dx d^2 y - dy d^2 x} = \frac{\pm 1}{ds \sqrt{\{(d^2 x)^2 + (d^2 y)^2 + (d^2 z)^2\}}}.$$

It is not useless to remark, that for the relations we have found, may be substituted the following:

$$\frac{\cos l}{dy^2 \cdot d\left(\frac{dz}{dy}\right)} = \frac{\cos m}{dz^2 \cdot d\left(\frac{dx}{dz}\right)} = \frac{\cos n}{dx^2 \cdot d\left(\frac{dy}{dx}\right)}, \text{ or}$$

$$\frac{\cos l}{\cos \beta \cdot d \cos \gamma - \cos \gamma \cdot d \cos \beta} = \frac{\cos m}{\cos \gamma \cdot d \cos \alpha - \cos \alpha \cdot d \cos \gamma} = \frac{\cos n}{\cos \alpha \cdot d \cos \beta - \cos \beta \cdot d \cos \alpha}.$$

These last equations are obtained by remarking that since $\cos \alpha = \frac{dx}{ds}$, $\cos \beta = \frac{dy}{ds}$, $\cos \gamma = \frac{dz}{ds}$, we have

Integral Calculus. $\frac{d \cos \alpha}{ds} = \frac{d^2 x}{ds^2}, \frac{d \cos \beta}{ds} = \frac{d^2 y}{ds^2}, \frac{d \cos \gamma}{ds} = \frac{d^2 z}{ds^2}$. But we have also $\frac{\cos \lambda}{d^2 x} = \frac{\cos \mu}{d^2 y} = \frac{\cos \nu}{d^2 z}$, therefore $\frac{\cos \lambda}{d \cos \alpha} = \frac{\cos \mu}{d \cos \beta} = \frac{\cos \nu}{d \cos \gamma}$. These equations combined with the two $\cos \alpha \cos l + \cos \beta \cos m + \cos \gamma \cos n = 0$,

$\cos \lambda \cos l + \cos \mu \cos m + \cos \nu \cos n = 0$ lead obviously to those above. They may also be found in a different way. Let l, m, n designate the angles formed by the coordinate axis with a line perpendicular to a plane, which passing through the tangent at the point (x, y, z) is at the same time parallel to a tangent at a point placed at a very small distance from the point (x, y, z) . Let also $\Delta x, \Delta y, \Delta z, \Delta \cos \alpha, \Delta \cos \beta, \Delta \cos \gamma$, be the increments which the quantities $x, y, z, \cos \alpha, \cos \beta, \cos \gamma$, take in passing from one point to the other. In this case, the values of l, m, n will clearly be obtained by means of the following equations

$$\cos \alpha \cos l + \cos \beta \cos m + \cos \gamma \cos n = 0, (\cos \alpha + \Delta \cos \alpha) \cos l + (\cos \beta + \Delta \cos \beta) \cos m + (\cos \gamma + \Delta \cos \gamma) \cos n = 0.$$

The last, in consequence of the first, is reduced to $\Delta \cos \alpha \cos l + \Delta \cos \beta \cos m + \Delta \cos \gamma \cos n = 0$. But if the second point approaches indefinitely to the first, the differences $\Delta \cos \alpha, \Delta \cos \beta, \Delta \cos \gamma$, will sensibly be proportional to the differential $d \cos \alpha, d \cos \beta, d \cos \gamma$. Therefore at the limit we shall have $\cos l \cdot d \cos \alpha + \cos m \cdot d \cos \beta + \cos n \cdot d \cos \gamma = 0$, and this combined with $\cos \alpha \cos l + \cos \beta \cos m + \cos \gamma \cos n = 0$, will give as above

$$\frac{\cos l}{\cos \beta \cdot d \cos \alpha - \cos \alpha \cdot d \cos \beta} = \frac{\cos m}{\cos \gamma \cdot d \cos \alpha - \cos \alpha \cdot d \cos \gamma} = \frac{\cos n}{\cos \alpha \cdot d \cos \beta - \cos \beta \cdot d \cos \alpha}.$$

Hence the angles l, m, n determined in the last hypothesis, are equal to the angles l, m, n determined in the first. Therefore the plane which contains the tangent at the point (x, y, z) , and which is at the same time parallel to a tangent drawn through a point infinitely near the first, coincides with the osculating plane.

If the arc s were not taken for the independent variable, some of the formulæ found in this paragraph would be no longer exact. For instance, we have then $dx d^2 x + dy d^2 y + dz d^2 z = ds d^2 s$ and not equal to nothing. Consequently, in this case,

$$\frac{\cos l}{dy d^2 z - dz d^2 y} = \frac{\cos m}{dz d^2 x - dx d^2 z} = \frac{\cos n}{dx d^2 y - dy d^2 x} = \frac{\pm 1}{ds \sqrt{\{(d^2 x)^2 + (d^2 y)^2 + (d^2 z)^2 - (d^2 s)^2\}}}.$$

In the particular case where x is taken for the independent variable, if y', z', y'', z'' designate the differential coefficients of the first and second order of y and z , the preceding formula becomes

$$\frac{\cos l}{y' z'' - z' y''} = \frac{\cos m}{-z''} = \frac{\cos n}{y''} = \frac{\pm 1}{\sqrt{\{y''^2 + z''^2 + (y' z'' - z' y'')^2\}}}.$$

(352.) The angles l, m , and n being once determined it becomes easy to obtain the equation of the osculating plane. If we represent by x_1, y_1 , and z_1 , the coordinates of any point of that plane, we shall have for that equation

$$(x_1 - x) \cos l + (y_1 - y) \cos m + (z_1 - z) \cos n = 0,$$

or by substituting for $\cos l, \cos m, \cos n$, the values we have found,

$$(x_1 - x) (dy d^2 z - dz d^2 y) + (y_1 - y) (dz d^2 x - dx d^2 z) + (z_1 - z) (dx d^2 y - dy d^2 x) = 0.$$

If x were considered as the independent variable, the equation of the osculating plane would be

$$(x_1 - x) (y' z'' - z' y'') - z'' (y_1 - y) + y'' (x_1 - x) = 0.$$

(353.) Let us represent by Δs , the increment which the variable s takes when the coordinates x, y, z become $x + \Delta x, y + \Delta y, z + \Delta z$. The angle of the two tangents drawn through the extremities of the arc Δs is called the *angle of contingence*. Let us designate by ω that angle, and by Ω the infinitely small angle formed by the two osculating planes which correspond to the same extremities of the arc, or, what is the same thing, by the perpendiculars to these planes. The quantities ω and Ω will be equal to nothing, the first when the proposed curve is changed to a straight line, and the second when it is a plane curve. But, in general, ω and Ω will have

finite values, as well as the limits of the ratios $\frac{\omega}{\Delta s}, \frac{\Omega}{\Delta s}$, when Δs decreases indefinitely. These limits, which will

be equal if the curve be plane, the one to the curvature of the curve, and the other to nothing, will be used to measure in every case what may be called the *first* and *second* curvature of the proposed curve. It is in consequence of these two curvatures that a curve which is not entirely contained in one plane is called a curve of *double curvature*.

If we represent by $\frac{1}{\rho}$ and $\frac{1}{r}$ these two curvatures ρ and r will be the radii of two circles which will have the same curvatures, and we shall have according to what precedes

$$\frac{1}{\rho} = \lim \left(\frac{\omega}{\Delta s} \right), \quad \frac{1}{r} = \lim \left(\frac{\Omega}{\Delta s} \right).$$

When the arc Δs is very small, its chord $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$ is very nearly perpendicular to the two normal planes to the curve drawn through the points, the coordinates of which are (x, y, z) and $(x + \Delta x, y + \Delta y, z + \Delta z)$; and the shortest distance between the point (x, y, z) and the line of intersection of the two planes is very nearly equal to the radius ρ . Let ρ_1 be this shortest distance. If we conceive a plane passing through the length ρ_1 , and at the same time perpendicular to the two normal planes, the chord $\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$ will form a very small angle with that plane; that is to say, that it will form a very small angle with its projection on that plane. Hence this projection will be equal to the chord multiplied by a cosine very nearly

Integral
Calculus.

equal to one, and therefore may be represented by a product such as $(1 - i) \sqrt{(\Delta x^2 + \Delta y^2 + \Delta z^2)}$, i designating an infinitely small quantity. Moreover in the triangle formed by the length ρ_1 , and by the projection of the chord, the angle opposite to the side ρ_1 will be very nearly a right angle, and the angle opposite to the projection of the chord will be precisely the angle of the normal planes, that is the angle ω formed by the tangents drawn through the extremities of the arc Δs . We shall have, therefore,

$$\frac{\sin\left(\frac{\pi}{2} + \epsilon\right)}{\rho_1} = \frac{\sin \omega}{(1 - i) \sqrt{(\Delta x^2 + \Delta y^2 + \Delta z^2)}} = \frac{\sin \omega}{(1 - i) \omega} \cdot \frac{\Delta s}{\sqrt{(\Delta x^2 + \Delta y^2 + \Delta z^2)}} \cdot \frac{\omega}{\Delta s}$$

where ϵ is supposed to be an infinitely small number. Hence at the limit we shall find $\frac{1}{\lim \rho_1} = \lim \frac{\omega}{\Delta s} = \frac{1}{\rho}$, or $\lim \rho_1 = \rho$.

It is necessary to remark that the osculating plane passing through the point (x, y, z) being very nearly parallel to the tangent drawn through the point $x + \Delta x, y + \Delta y, z + \Delta z$, will also be very nearly perpendicular to the two normal planes drawn through the extremities of the arc Δs , and to their line of intersection. Therefore the normal which is perpendicular to this line of intersection, will very nearly coincide with the normal situated in the osculating plane, that is to say with the principal normal. From this remark, and from what has been said above, it follows obviously that, in order to obtain the radius ρ , it is sufficient to draw the principal normal corresponding to the point (x, y, z) , and to find the portion of the length of this line contained between the point (x, y, z) and a normal plane infinitely near to the line itself. The radius ρ , measured in this manner, on the principal normal, is what has been called the *radius of curvature* of the proposed curve, relative to the point (x, y, z) ; and the *centre of curvature* is that extremity of the radius of curvature which may be considered as the point where the principal normal, and a normal plane very near it meet. The circle which has for centre this last point, and for radius the radius of curvature, is called circle of curvature, or osculating circle. It touches the proposed curve, and has the same curvature. We may add that, if by the tangent at the point (x, y, z) and by the perpendicular to the osculating plane, we conceive a new plane to pass, the centre of curvature will obviously be situated, with respect to the new plane on the same side as the point $(x + \Delta x, y + \Delta y, z + \Delta z)$, and consequently this centre will always coincide with one of the points of the line the direction of which is determined by the angles λ, μ, ν which verify the formulæ of (349.)

(354.) If in the value of $\frac{1}{\rho}$, found above, we replace the infinitely small quantity ω by the angles α, β, γ , or rather by the differentials of their cosines, it will be sufficient to use the following formulæ:

$$\cos \omega = \cos \alpha (\cos \alpha + \Delta \cos \alpha) + \cos \beta (\cos \beta + \Delta \cos \beta) + \cos \gamma (\cos \gamma + \Delta \cos \gamma),$$

$$1 = \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma, \quad 1 = (\cos \alpha + \Delta \cos \alpha)^2 + (\cos \beta + \Delta \cos \beta)^2 + (\cos \gamma + \Delta \cos \gamma)^2;$$

from which we get

$$2(1 - \cos \omega) = \cos^2 \alpha - 2 \cos \alpha (\cos \alpha + \Delta \cos \alpha) + (\cos \alpha + \Delta \cos \alpha)^2 + \cos^2 \beta - 2 \cos \beta (\cos \beta + \Delta \cos \beta) + (\cos \beta + \Delta \cos \beta)^2 + \cos^2 \gamma - 2 \cos \gamma (\cos \gamma + \Delta \cos \gamma) + (\cos \gamma + \Delta \cos \gamma)^2,$$

or which is the same thing, $\left(2 \sin \frac{\omega}{2}\right)^2 = (\Delta \cos \alpha)^2 + (\Delta \cos \beta)^2 + (\Delta \cos \gamma)^2$. Dividing by Δs^2 both sides of this last equation, we shall get

$$\left(\frac{\sin \frac{1}{2} \omega}{\frac{1}{2} \omega}\right)^2 \left(\frac{\omega}{\Delta s}\right)^2 = \left(\frac{\Delta \cos \alpha}{\Delta s}\right)^2 + \left(\frac{\Delta \cos \beta}{\Delta s}\right)^2 + \left(\frac{\Delta \cos \gamma}{\Delta s}\right)^2.$$

Then if we suppose that Δs converges towards a limit equal to nothing, we shall find at the limit

$$\frac{1}{\rho} = \sqrt{\left\{\left(\frac{d \cos \alpha}{d s}\right)^2 + \left(\frac{d \cos \beta}{d s}\right)^2 + \left(\frac{d \cos \gamma}{d s}\right)^2\right\}}.$$

In quite a similar manner, it might be proved that the equation $\frac{1}{r} = \lim \left(\frac{\Omega}{\Delta s}\right)$ leads to

$$\frac{1}{r} = \sqrt{\left\{\left(\frac{d \cos l}{d s}\right)^2 + \left(\frac{d \cos m}{d s}\right)^2 + \left(\frac{d \cos n}{d s}\right)^2\right\}}.$$

From these formulæ it obviously results that the first curvature $\frac{1}{\rho}$ is generally equal to nothing when the angles

α, β, γ become constant, and the second curvature $\frac{1}{r}$, when the angles l, m, n are constant. If in the value of $\frac{1}{\rho}$ we substitute for the cosines of α, β, γ their values, we shall have

$$\frac{1}{\rho} = \sqrt{\left\{\left(\frac{d \cdot \left(\frac{d x}{d s}\right)}{d s}\right)^2 + \left(\frac{d \cdot \left(\frac{d y}{d s}\right)}{d s}\right)^2 + \left(\frac{d \cdot \left(\frac{d z}{d s}\right)}{d s}\right)^2\right\}} \quad \text{and developing, and observing that } dx^2 + dy^2 + dz^2 = ds^2$$

Integral Calculus. $= ds d^2s$, we find $\frac{1}{\rho} = \frac{\sqrt{((d^2x)^2 + (d^2y)^2 + (d^2z)^2 - (d^2s)^2)}}{ds^2}$, or again,

$$\frac{1}{\rho} = \frac{\left[\frac{(dx^2 + dy^2 + dz^2) \{ (d^2x)^2 + (d^2y)^2 + (d^2z)^2 \} - (dx d^2x + dy d^2y + dz d^2z)^2}{ds^3} \right]^{\frac{1}{2}}}{\sqrt{\{ (dy d^2z - dz d^2y)^2 + (dz d^2x - dx d^2z)^2 + (dx d^2y - dy d^2x)^2 \}}}{(dx^2 + dy^2 + dz^2)^{\frac{3}{2}}}.$$

If s is the independent variable, we have simply

$$\frac{1}{\rho} = \frac{\sqrt{\{ (d^2x)^2 + (d^2y)^2 + (d^2z)^2 \}}}{ds^2}.$$

But if we take x for the independent variable, we find

$$\frac{1}{\rho} = \frac{(y' z'' - y'' z')^2 + z'^2 + y'^2}{(1 + y'^2 + z'^2)^{\frac{3}{2}}}.$$

And, finally, if from the point (x, y, z) we take on the curve and on the tangent produced in the same direction, two equal and infinitely small lengths represented by i , and if we call κ the distance between the extremities of these two lengths, we shall have, according to what has been proved (349.),

$$\sqrt{\left\{ \left(\frac{d^2x}{ds^2} \right)^2 + \left(\frac{d^2y}{ds^2} \right)^2 + \left(\frac{d^2z}{ds^2} \right)^2 \right\}} = \lim \frac{2\kappa}{i^2},$$

s being the independent variable. Hence, since we have found $\frac{1}{\rho} = \frac{\sqrt{\{ (d^2x)^2 + (d^2y)^2 + (d^2z)^2 \}}}{ds^2}$, we shall have $\rho = \lim \frac{i^2}{2\kappa}$. Therefore to obtain the radius of curvature of a curve in a given point, it is sufficient to take

upon this curve and its tangent, produced in the same direction, two equal and infinitely small lengths, and to divide the square of one of them by the double of the distance between their extremities. The limit of the quotient is the exact value of the radius of curvature.

If we suppose that the plane of (x, y) becomes parallel to the osculating plane, the values of $\cos l$ and $\cos m$ will vanish while those of $\cos n$ will become equal to ± 1 . Then we shall have $dz = 0$, $d^2z = 0$, and

$$\frac{1}{\rho} = \frac{dx d^2y - dy d^2x}{(dx^2 + dy^2)^{\frac{3}{2}}}. \text{ This value is precisely the same as we have obtained before for the radius of curvature}$$

of a plane curve, and it proves, therefore, that the radius of curvature of a curve in space is the same, with respect to its magnitude and direction, as the radius of curvature of the projection of the curve on the osculating plane.

The remarks which have been made with respect to the radius of curvature of plane curves, may obviously be extended to any curve whatever, and it may happen that for certain points situated upon a curve of double curvature, the radius of curvature should become nothing or infinite, or change suddenly its value. The second case will occur whenever the projection of the curve on the osculating plane has a point of contrary flexure.

(355.) We shall now apply the preceding formulæ to a particular case, and we shall take for an example the helix represented by the equations

$$x = r \cos p, \quad y = r \sin p, \quad z = arp.$$

Let us take p for the independent variable, we shall find

$$dx = -r \sin p dp, \quad dy = r \cos p dp, \quad dz = ar dp, \quad d^2x = -r \cos p dp^2, \quad d^2y = -r \sin p dp^2, \quad d^2z = 0,$$

or, which is the same thing,

$$\frac{dx}{-\sin p} = \frac{dy}{\cos p} = \frac{dz}{a} = r dp, \quad \frac{d^2x}{\cos p} = \frac{d^2y}{\sin p} = \frac{d^2z}{0} = -r dp^2.$$

Hence

$$ds = \pm (1 + a^2)^{\frac{1}{2}} r dp, \quad d^2s = 0, \quad \text{and} \quad \frac{\cos \alpha}{-\sin p} = \frac{\cos \beta}{\cos p} = \frac{\cos \gamma}{a} = \pm \frac{1}{\sqrt{1 + a^2}}.$$

Since $d^2s = 0$, whatever be the value of p , it follows that we may employ the formulæ in which the arc s has been taken for the independent variable. Thus we shall have

$$\frac{\cos \lambda}{\cos p} = \frac{\cos \mu}{\sin p} = \frac{\cos \nu}{0} = \pm 1, \quad \text{and} \quad \frac{\cos l}{-a \sin p} = \frac{\cos m}{a \cos p} = \frac{\cos n}{-1} = \pm \frac{1}{\sqrt{1 + a^2}}.$$

We shall find also

$$\frac{1}{\rho} = \frac{1}{\sqrt{1 + a^2}} \left\{ \left(\frac{d \sin p}{ds} \right)^2 + \left(\frac{d \cos p}{ds} \right)^2 \right\}^{\frac{1}{2}} = \pm \frac{1}{\sqrt{1 + a^2}} \frac{dp}{ds} = \frac{1}{(1 + a^2)r}, \text{ and hence } \rho = (1 + a^2)r.$$

Integral
Calculus.

Moreover, the equation of the osculating plane will become

$$a \{ (x_1 - x) \sin p - (y_1 - y) \cos p \} + z_1 - z = 0,$$

which may be written in the following manner, $z_1 - z = a (y_1 \cos p - x_1 \sin p)$. Finally, we shall find, in representing by r_1 the radius of the second curvature which before we have represented by r

$$\frac{1}{r_1} = \frac{a}{\sqrt{(1+a^2)}} \left\{ \left(\frac{d \sin p}{ds} \right)^2 + \left(\frac{d \cos p}{ds} \right)^2 \right\}^{\frac{1}{2}} = \frac{a}{(1+a^2)r}, \text{ and hence } r_1 = \frac{1+a^2}{a} r.$$

If in these formulæ we substitute for the quantity a the angle γ determined by the equation $\cos \gamma = \frac{a}{\sqrt{1+a^2}}$, they become respectively,

$$\frac{\cos l}{\sin p} = \frac{\cos m}{-\cos p} = \frac{\cos n}{\tan \gamma} = \pm \cos \gamma, \quad \rho = \frac{r}{\sin^2 \gamma} = r \operatorname{cosec}^2 \gamma, \quad r_1 = \frac{r}{\sin \gamma \cos \gamma} = 2r \operatorname{cosec} 2\gamma.$$

From the values we have obtained for $\cos \lambda$, $\cos \mu$, $\cos \nu$, it obviously results that the principal normal of the helix, at the point (x, y, z) , coincides with the perpendicular drawn through this point on the axis of z , or which

is the same thing, with the generating line of the surface represented by the equation $z = ar \tan^{-1} \frac{y}{x}$. Hence the

plane which contains this line and the tangent to the helix, or, in other words, the plane which touches that surface at the point (x, y, z) , will coincide with the osculating plane of the helix. This might also be seen by substituting in the equation of the tangent plane to that surface which we have found (342.), $x = r \cos p$, $y = r \sin p$, for then it becomes identical with the equation of the osculating plane we have just found.

The value of the radius of curvature ρ , might also be found by means of the rule given (343.). For that, let us conceive that from the point (x, y, z) , we take on the curve and on its tangent produced in the same direction, two equal and infinitely small lengths designated by i . Let κ be the distance of the extremities of these two lengths, and let us suppose that the first is the point of the helix, the coordinates of which are $x + \Delta x$, $y + \Delta y$, $z + \Delta z$; the second extremity will be on the tangent in a point which will still have for one of its coordinates $z + \Delta z$. Hence the length taken upon the tangent will have for its projection upon the plane of (x, y) , $\tan \gamma \cdot \Delta z$, γ being the angle of that tangent with the axis of z . Let I designate this same projection, since we have, generally, $z = arp$,

$\tan \gamma = \frac{1}{a}$, we shall have also $I = r \Delta p$. Therefore the projection I will be equal to the projection of the arc

Δs , that is to say, to the projection of the length $i = \Delta s$, taken on the helix. The first projection will be reckoned on the tangent to the circle represented by the formula $x^2 + y^2 = r^2$. It is moreover obvious, that the distance κ between two points which have the same z will be parallel to the plane (x, y) , and will not differ from its projection on that plane. Therefore, if, in that same plane, we take from the point (x, y) , on the circle and on its tangent produced in the same direction, two infinitely small lengths equal to i , κ will be the distance of the two extremities of these lengths. This understood, since the radius of curvature of a circle is precisely the radius

of the circle, we shall have, according to a theorem previously demonstrated, $r = \lim \frac{I^2}{2\kappa}$, and $\rho = \lim \frac{i^2}{2\kappa}$,

therefore $\frac{\rho}{r} = \lim \frac{i^2}{I^2}$. But we have just found $i = \Delta s$, $I = r \Delta p$, therefore $\frac{\rho}{r} = \lim \left(\frac{\Delta s}{r \Delta p} \right)^2 = \frac{ds^2}{r^2 dp^2} = 1 + a^2 = \operatorname{cosec}^2 \gamma$ which agrees with the value found above.

Determination of the Centre of Curvature.

(356.) Let, as before, ρ represent the radius of curvature of any curve whatever corresponding to the point (x, y, z) , and let x_1, y_1, z_1 be the coordinates of the extremity of that radius which is called the centre of curvature,

and λ, μ, ν the angles formed by the radius with the three axes, we shall have $\frac{x_1 - x}{\rho} = \cos \lambda$, $\frac{y_1 - y}{\rho} = \cos \mu$,

$\frac{z_1 - z}{\rho} = \cos \nu$, the angles λ, μ , and ν being determined by the formula

$$\frac{\cos \lambda}{\frac{d^2 x}{ds^2}} = \frac{\cos \mu}{\frac{d^2 y}{ds^2}} = \frac{\cos \nu}{\frac{d^2 z}{ds^2}} = \frac{1}{\sqrt{\left\{ \left(\frac{d^2 x}{ds^2} \right)^2 + \left(\frac{d^2 y}{ds^2} \right)^2 + \left(\frac{d^2 z}{ds^2} \right)^2 \right\}}},$$

or, since we have found $\frac{1}{\rho} = \frac{\sqrt{\{ (d^2 x)^2 + (d^2 y)^2 + (d^2 z)^2 \}}}{ds^2}$, by the formula

$$\frac{\cos \lambda}{\frac{d^2 x}{ds^2}} = \frac{\cos \mu}{\frac{d^2 y}{ds^2}} = \frac{\cos \nu}{\frac{d^2 z}{ds^2}} = \frac{\rho}{ds^2}, \text{ from which we get } \cos \lambda = \rho \frac{d^2 x}{ds^2}, \cos \mu = \rho \frac{d^2 y}{ds^2}, \cos \nu = \rho \frac{d^2 z}{ds^2},$$

and, consequently,

$$x_1 - x = \rho^2 \frac{d^2 x}{ds^2}, \quad y_1 - y = \rho^2 \frac{d^2 y}{ds^2}, \quad z_1 - z = \rho^2 \frac{d^2 z}{ds^2}.$$

If s were not taken for the independent variable, it would be necessary to substitute for

$$\frac{d^2 x}{ds^2}, \frac{d^2 y}{ds^2}, \frac{d^2 z}{ds^2}, \text{ the quantities } \frac{d \cdot \left(\frac{dx}{ds} \right)}{ds}, \frac{d \cdot \left(\frac{dy}{ds} \right)}{ds}, \frac{d \cdot \left(\frac{dz}{ds} \right)}{ds},$$

and thus we would have

$$\begin{aligned} \cos \lambda &= \rho \frac{d \left(\frac{dx}{ds} \right)}{ds}, \quad \cos \mu = \rho \frac{d \left(\frac{dy}{ds} \right)}{ds}, \quad \cos \nu = \rho \frac{d \left(\frac{dz}{ds} \right)}{ds}, \\ x_1 - x &= \rho^2 \frac{d \left(\frac{dx}{ds} \right)}{ds}, \quad y_1 - y = \rho^2 \frac{d \left(\frac{dy}{ds} \right)}{ds}, \quad z_1 - z = \rho^2 \frac{d \left(\frac{dz}{ds} \right)}{ds}. \end{aligned}$$

But when s is not taken for the independent variable, we have found

$$\frac{1}{\rho} = \frac{\sqrt{\{ (dy d^2 z - dz d^2 y)^2 + (dz d^2 x - dx d^2 z)^2 + (dx d^2 y - dy d^2 x)^2 \}}}{ds^3},$$

and therefore we shall have

$$\begin{aligned} x_1 - x &= \frac{ds d^2 x - dx d^2 s}{(dy d^2 z - dz d^2 y)^2 + (dz d^2 x - dx d^2 z)^2 + (dx d^2 y - dy d^2 x)^2} ds^3 \\ &= \frac{dy (dy d^2 x - dx d^2 y) + dz (dz d^2 x - dx d^2 z)}{(dy d^2 z - dz d^2 y)^2 + (dz d^2 x - dx d^2 z)^2 + (dx d^2 y - dy d^2 x)^2} (dx^2 + dy^2 + dz^2) \\ y_1 - y &= \frac{ds d^2 y - dy d^2 s}{(dy d^2 z - dz d^2 y)^2 + (dz d^2 x - dx d^2 z)^2 + (dx d^2 y - dy d^2 x)^2} ds^3 \\ &= \frac{dz (dz d^2 y - dy d^2 z) + dx (dx d^2 y - dy d^2 x)}{(dy d^2 z - dz d^2 y)^2 + (dz d^2 x - dx d^2 z)^2 + (dx d^2 y - dy d^2 x)^2} (dx^2 + dy^2 + dz^2) \\ z_1 - z &= \frac{ds d^2 z - dz d^2 s}{(dy d^2 z - dz d^2 y)^2 + (dz d^2 x - dx d^2 z)^2 + (dx d^2 y - dy d^2 x)^2} ds^3 \\ &= \frac{dx (dx d^2 y - dy d^2 x) + dy (dy d^2 z - dz d^2 y)}{(dy d^2 z - dz d^2 y)^2 + (dz d^2 x - dx d^2 z)^2 + (dx d^2 y - dy d^2 x)^2} (dx^2 + dy^2 + dz^2). \end{aligned}$$

In the particular case where x is taken for the independent variable, these formulæ become

$$\begin{aligned} x_1 - x &= - \frac{y' y'' + z' z''}{(1 + y'^2 + z'^2)^{3/2}} \rho^2 = - \frac{y' y'' + z' z''}{(y' z'' - z' y'')^2 + z'^2 + y'^2} (1 + y'^2 + z'^2), \\ y_1 - y &= \frac{z' (z' y'' - y' z'') + y''}{1 + y'^2 + z'^2} \rho^2 = \frac{z' (z' y'' - y' z'') + y''}{(y' z'' - z' y'')^2 + z'^2 + y'^2} (1 + y'^2 + z'^2), \\ z_1 - z &= \frac{y' (y' z'' - y'' z') + z''}{1 + y'^2 + z'^2} \rho^2 = \frac{y' (y' z'' - y'' z') + z''}{(y' z'' - y'' z')^2 + z'^2 + y'^2} (1 + y'^2 + z'^2). \end{aligned}$$

These formulæ will enable us to determine the coordinates of the centre of curvature. These coordinates must also, in general, verify the equations

$$\begin{aligned} (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 &= \rho^2, \quad (x_1 - x) dx + (y_1 - y) dy + (z_1 - z) dz = 0, \\ (x_1 - x) d^2 x + (y_1 - y) d^2 y + (z_1 - z) d^2 z - dx^2 - dy^2 - dz^2 &= 0. \end{aligned}$$

The two first are obviously satisfied by the coordinates of all the points situated in the normal plane at a distance ρ from the point (x, y, z) . As to the last equation it may be obtained by multiplying the values of $x_1 - x$, $y_1 - y$, $z_1 - z$, by $d^2 x$, $d^2 y$, and $d^2 z$, and then adding them. It is, moreover, essential to observe, that the second and third equations may be found by differentiating the first and second, in the supposition that x_1 , y_1 , and z_1 , are constant quantities.

(357.) To every point of the curve in space corresponds a centre of curvature, and all these centres form a curve the equations of which will easily be obtained. It will be sufficient for that purpose to express the values of x_1 , y_1 , and z_1 in function of only one variable x , y or s , and then to eliminate that variable between the three values. The two equations resulting from that elimination will only contain the three variables x_1 , y_1 , and z_1 , and will represent the line formed by the centres of curvature of the given curve. To discover the principal properties of this curve, we shall differentiate the two first of the three preceding equations, in considering each of the quantities this curve, we shall find thus $(x_1 - x) dx_1 + (y_1 - y) dy_1 + (z_1 - z) dz_1 = \rho d\rho$, and they contain as a variable. We shall find thus $(x_1 - x) dx_1 + (y_1 - y) dy_1 + (z_1 - z) dz_1 = \rho d\rho$, and $dx dx_1 + dy dy_1 + dz dz_1 = 0$. From the last equation it follows that the tangent drawn to the new curve through the point (x_1, y_1, z_1) forms a right angle with the tangent to the given curve at the point (x, y, z) . Hence the tangent to the new curve is contained in the normal plane to the proposed curve. Moreover if we call

Integral Calculus. s_1 the arc of the new curve, contained between a fixed point and the point (x_1, y_1, z_1) , we shall have $ds_1^2 = dx_1^2 + dy_1^2 + dz_1^2$, and we shall derive from this value, and the first of the two preceding equations, the relation Part III.

$$\frac{d\rho}{ds_1} = \frac{x_1 - x}{\rho} \frac{dx_1}{ds_1} + \frac{y_1 - y}{\rho} \frac{dy_1}{ds_1} + \frac{z_1 - z}{\rho} \frac{dz_1}{ds_1}. \quad \text{This formula shows that the ratio } \frac{d\rho}{ds_1} \text{ is equal to the cosine}$$

of the obtuse or acute angle formed by the radius of curvature ρ with the tangent to the new curve. When the proposed curve is a plane curve, this tangent coincides with the radius of curvature, or with its producement,

and consequently the ratio $\frac{d\rho}{ds_1}$ is reduced to the cosine of an angle equal to nothing or to π , which is ± 1 .

Therefore we have in that case $d\rho = \pm ds$. From this result we may conclude, as we have done before, that in the case of a plane curve the arc Δs_1 is the difference of the radii of curvature corresponding to the two extremities.

But this is no longer true when the curve is not a plane curve, and in that case the ratio $\frac{d\rho}{ds_1}$ obtains a numerical value which generally differs from one.

(358.) Let us conceive now that an inextensible thread, of a known length, is fixed by one of its extremities in a certain point of the proposed curve, and that this thread, first applied on the tangent drawn to the curve through the point to which it is attached, moves in remaining always stretched, so that a part of it is bent on the arc of the curve contained between the fixed point, and the point (x, y, z) that is supposed to move. The extremity of the other part of the thread which will remain straight, and will touch the curve at the point (x, y, z) , will describe a new curve. The given curve is called the *evolute*, and the new curve the *involute*. Their relative properties may be easily established in the following manner.

Let x_1, y_1, z_1 be the coordinates of a point of the involute which corresponds to the point (x, y, z) of the evolute, and g the distance between these two points. We shall have

$$\frac{x_1 - x}{dx} = \frac{y_1 - y}{dy} = \frac{z_1 - z}{dz} = \frac{g}{ds}, \text{ or } \frac{x_1 - x}{dx} = \frac{y_1 - y}{dy} = \frac{z_1 - z}{dz} = -\frac{g}{ds}.$$

The first formula must be used when the length g is reckoned from the point (x, y, z) of the evolute, on the tangent produced in the same direction as the arc s , and the other formula in the contrary case. In the first hypothesis we have $g + s = c$ or $dg = -ds$; and in the second $g - s = c$ or $dg = ds$, c designating a

constant quantity. Hence we shall have in both cases $x_1 - x = -g \frac{dx}{dg}$, $y_1 - y = -g \frac{dy}{dg}$, $z_1 - z = -g \frac{dz}{dg}$.

If we differentiate these last equations, we shall find $dx_1 = -g d\left(\frac{dx}{dg}\right)$, $dy_1 = -g d\left(\frac{dy}{dg}\right)$, $dz_1 = -g d\left(\frac{dz}{dg}\right)$,

and as we have besides $dg^2 = ds^2 = dx^2 + dy^2 + dz^2$, or $\left(\frac{dx}{dg}\right)^2 + \left(\frac{dy}{dg}\right)^2 + \left(\frac{dz}{dg}\right)^2 = 1$, or again $\frac{dx}{dg} \cdot d\left(\frac{dx}{dg}\right) + \frac{dy}{dg} \cdot d\left(\frac{dy}{dg}\right) + \frac{dz}{dg} \cdot d\left(\frac{dz}{dg}\right) = 0$, we shall find by multiplying the values of dx_1, dy_1, dz_1 , respectively by dx, dy, dz , and adding them, $dx dx_1 + dy dy_1 + dz dz_1 = 0$. From this last formula it follows

that the tangents drawn through the points (x_1, y_1, z_1) and (x, y, z) to the involute and evolute are at right angles to each other. Therefore the tangent to the evolute is always normal to the involute. This proposition, which has already been proved for plane curves, is also true for curves of double curvature. When the evolute is known, it is sufficient to obtain the equations of the involute to substitute in the values we have found for $x_1 - x, y_1 - y, z_1 - z$, the values of x, y, z in function of s , to replace g by $c \mp s$, and to eliminate s between the three equations thus obtained.

(359.) Let us suppose now that it is proposed to find not the involute, but an evolute of the curve to which belong the coordinates x, y, z . If we call x_1, y_1, z_1 the coordinates of a point of this evolute, which corresponds to the point (x, y, z) of the given curve, and if we still designate by g the distance between these two points, we shall have by exchanging in the formulæ obtained in the preceding paragraph x, y, z into x_1, y_1 , and z_1 , and

reciprocally, $\frac{x - x_1}{dx_1} = \frac{y - y_1}{dy_1} = \frac{z - z_1}{dz_1} = \frac{-g}{dg}$, and consequently, $dx_1 = \frac{(x_1 - x) dg}{g}$, $dy_1 = \frac{(y_1 - y) dg}{g}$,

$dz_1 = \frac{(z_1 - z) dg}{g}$. Moreover we shall have $(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 = g^2$. If we differentiate

three times this last equation, in taking s for the independent variable, and if we make use of the values we have just obtained for dx_1, dy_1, dz_1 , we shall find $(x_1 - x) dx + (y_1 - y) dy + (z_1 - z) dz = 0$, $(x_1 - x) d^2x + (y_1 - y) d^2y + (z_1 - z) d^2z = ds^2$, $(x_1 - x) d(g d^2x) + (y_1 - y) d(g d^2y) + (z_1 - z) d(g d^2z) = 0$; from which we shall conclude

$$\frac{x_1 - x}{dy dg (g d^2z) - dz dg (g d^2y)} = \frac{y_1 - y}{dz dg (g d^2x) - dx dg (g d^2z)} = \frac{z_1 - z}{dx dg (g d^2y) - dy dg (g d^2x)} = \frac{ds^2}{d^3x \{ dy dg (g d^2z) - dz dg (g d^2y) \} + d^3y \{ dz dg (g d^2x) - dx dg (g d^2z) \} + d^3z \{ dx dg (g d^2y) - dy dg (g d^2x) \}}$$

$$\begin{aligned} \text{Integral Calculus.} &= \pm \frac{g}{\sqrt{\{d y d(g d^2 z) - d z d(g d^2 y)\}^2 + \{d z d(g d^2 x) - d x d(g d^2 z)\}^2 + \{d x d(g d^2 y) - d y d(g d^2 x)\}^2}} \\ &= \pm \frac{g}{\sqrt{[d s^2 \{[d(g d^2 x)]^2 + [d(g d^2 y)]^2 + [d(g d^2 z)]^2\} - \{d x d(g d^2 x) + d y d(g d^2 y) + d z d(g d^2 z)\}^2}}. \end{aligned}$$

From this last formula we shall derive by inverting two of the fractions, and raising each of them to the square,

$$\begin{aligned} \frac{d s^2 \{[d(g d^2 x)]^2 + [d(g d^2 y)]^2 + [d(g d^2 z)]^2\} - \{d x d(g d^2 x) + d y d(g d^2 y) + d z d(g d^2 z)\}^2}{g^2} = \\ \frac{g^2}{d s^4} \{d x d^2 y d^3 z - d x d^2 z d^3 y + d y d^2 z d^3 x - d y d^2 x d^3 z + d z d^2 x d^3 y - d z d^2 y d^3 x\}^2, \end{aligned}$$

or which is the same thing

$$\begin{aligned} &\left[\left(\frac{d^2 x}{d s^2}\right)^2 + \left(\frac{d^2 y}{d s^2}\right)^2 + \left(\frac{d^2 z}{d s^2}\right)^2\right] \frac{d g^2}{d s^2} + 2 \left[\frac{d^2 x}{d s^2} \cdot \frac{d^2 y}{d s^2} + \frac{d^2 y}{d s^2} \cdot \frac{d^2 z}{d s^2} + \frac{d^2 z}{d s^2} \cdot \frac{d^2 x}{d s^2}\right] g \frac{d g}{d s} \\ &+ \left[\left(\frac{d^3 x}{d s^3}\right)^2 + \left(\frac{d^3 y}{d s^3}\right)^2 + \left(\frac{d^3 z}{d s^3}\right)^2 - \left(\frac{d x}{d s} \frac{d^3 x}{d s^3} + \frac{d y}{d s} \frac{d^3 y}{d s^3} + \frac{d z}{d s} \frac{d^3 z}{d s^3}\right)^2\right] \\ &g^2 = \frac{(d x d^2 y d^3 z - d x d^2 z d^3 y + d y d^2 z d^3 x - d y d^2 x d^3 z + d z d^2 x d^3 y - d z d^2 y d^3 x) g^4}{d s^3}. \end{aligned}$$

(360.) If in this equation we substitute for x, y, z their values expressed in function of s , it will only contain the variable s and the unknown g , with the differential coefficient $\frac{d g}{d s}$. It will be a differential equation of the first order. The integral will contain an arbitrary constant, and there is therefore an infinite number of functions of s , corresponding to the various values of the arbitrary constant, which satisfy this equation. If, after having determined one of these functions, we substitute in the three equations

$$\begin{aligned} (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 &= g^2, \quad (x_1 - x) d x + (y_1 - y) d y + (z_1 - z) d z = 0, \\ (x_1 - x) d^2 x + (y_1 - y) d^2 y + (z_1 - z) d^2 z &= d s^2, \end{aligned}$$

for the quantities x, y, z, g , their values in function of s , we shall have only to eliminate s between these formulæ to obtain between the coordinates x_1, y_1 and z_1 two equations to represent an evolute of the proposed curve. This understood, it is clear that this curve will have an infinite number of evolutes which correspond to the various values of g , or which is the same thing, to the various values of the arbitrary constant. All these evolutes are situated on the surface represented by the equation resulting from the elimination of s between the equations $(x_1 - x) d x + (y_1 - y) d y + (z_1 - z) d z = 0$, $(x_1 - x) d^2 x + (y_1 - y) d^2 y + (z_1 - z) d^2 z = d s^2$. This surface, or the locus of all the evolutes of the proposed curve, possesses some remarkable properties which are easily deduced from the two preceding equations. These may be put under the form

$$(x_1 - x) \frac{d x}{d s} + (y_1 - y) \frac{d y}{d s} + (z_1 - z) \frac{d z}{d s} = 0, \quad (x_1 - x) \frac{d^2 x}{d s^2} + (y_1 - y) \frac{d^2 y}{d s^2} + (z_1 - z) \frac{d^2 z}{d s^2} = 1.$$

First, it is obvious that if we take for s and the quantities which depend upon it, that is to say, for $x, y, z, \frac{d x}{d s}, \frac{d y}{d s}, \frac{d z}{d s}, \frac{d^2 x}{d s^2}, \frac{d^2 y}{d s^2}, \frac{d^2 z}{d s^2}$, determinate values, the two equations above, in which x_1, y_1 , and z_1 will alone remain variables, will represent two planes the common section of which will be a straight line contained in the surface in question. Therefore this surface will contain an infinite number of straight lines corresponding to the various values of s , and will be consequently a *surface réglée*.

It may be observed that the first of the two last equations is precisely that of the normal plane drawn through the point (x, y, z) to the given curve. As to the second equation, it may be transformed by means of the equations

$$\cos \lambda = \rho \frac{d^2 x}{d s^2}, \quad \cos \mu = \rho \frac{d^2 y}{d s^2}, \quad \cos \nu = \rho \frac{d^2 z}{d s^2}, \quad \text{into } (x_1 - x) \cos \lambda + (y_1 - y) \cos \mu + (z_1 - z) \cos \nu = \rho;$$

and then we see immediately that it is satisfied by the values of x_1, y_1 , and z_1 , which belong to the centre of curvature, and that it represents a plane drawn through this same centre perpendicularly to the radius of curvature the direction of which forms with the axis the angles λ, μ , and ν . If we assume $x = \phi(s), y = \chi(s), z = \psi(s)$, the two equations above will become respectively

$$\begin{aligned} \{x_1 - \phi(s)\} \phi'(s) + \{y_1 - \chi(s)\} \chi'(s) + \{z_1 - \psi(s)\} \psi'(s) &= 0, \\ \{x_1 - \phi(s)\} \phi''(s) + \{y_1 - \chi(s)\} \chi''(s) + \{z_1 - \psi(s)\} \psi''(s) &= 1. \end{aligned}$$

The first equation will represent the surface above mentioned if we suppose s to be determined by the second. This understood, it is easy to find the differential equation of the surface. For that let us differentiate the first equation, in considering x, y, z , and s , as variables, then we shall have after reduction, by means of the second equation, and of the relation $(\phi'(s))^2 + (\chi'(s))^2 + (\psi'(s))^2 = 1$, $\phi'(s) d x_1 + \chi'(s) d y_1 + \psi'(s) d z_1 = 0$.

Integral
Calculus.

This equation shows that the normal drawn through the point (x_1, y_1, z_1) to the *surface réglée* makes angles with the axis the cosines of which are proportional to the derived functions $\phi'(s) = \frac{dx}{ds}$, $\chi'(s) = \frac{dy}{ds}$, $\psi'(s) = \frac{dz}{ds}$. Part III.

Therefore this normal and the tangent drawn through the point (x, y, z) to the given curve are two parallel lines. Hence, since the tangent plane drawn through the point (x_1, y_1, z_1) to the *surface réglée* must contain the generating line of this surface, and consequently the centre of curvature of the proposed curve, it will necessarily coincide with the plane drawn through this centre perpendicularly to the two straight lines, that is to say, with the normal plane to the curve. From these observations it follows, that each normal plane to the curve will touch the *surface réglée* at every point of the generating line through which it passes. Hence the surface in question, that is the locus of all the evolutes of the curve, will be a *surface développable*.

It is essential to remark, that the equation $\phi'(s) dx_1 + \chi'(s) dy_1 + \psi'(s) dz_1 = 0$ is the same as the equation $dx dx_1 + dy dy_1 + dz dz_1 = 0$, we have found before, in which for dx , dy , and dz , their values $\phi'(s) ds$, $\chi'(s) ds$, $\psi'(s) ds$, have been substituted.

It is also necessary to observe, that in the case where the variable s is no longer an independent variable, in order to obtain the equation $(x_1 - x) d^2 x + (y_1 - y) d^2 y + (z_1 - z) d^2 z = ds^2$, it is sufficient to differentiate the equation $(x_1 - x) dx + (y_1 - y) dy + (z_1 - z) dz = 0$, and to reduce it, in taking into consideration the equation itself, and the values which have been given for dx_1 , dy_1 , dz_1 . Hence we may infer, that the equation in x_1, y_1, z_1 , resulting from the elimination of s between the two equations just mentioned, will represent in every case the surface which is the locus of all the evolutes of the given curve. And, moreover, as these two equations are identically the same as the two last of the three which we have said are verified by the coordinates of any point of the curve which is the locus of the centres of curvature, it may be affirmed, that this last curve is situated upon the *surface réglée*.

It may be proved easily, and without the help of calculation, that a *surface développable* which is tangent to every normal plane to any given curve, is, at the same time, the locus of all the evolutes of that curve. This, in fact, may be demonstrated by means of the following considerations.

Let us imagine, that the plane tangent to a *surface développable* is made to roll upon this surface, with several straight lines drawn through a point taken at random in this plane, this point will describe a curve and each straight line will draw on the surface an evolute of this curve. Hence the *surface développable* will be the locus of all the evolutes; and the tangent plane which will contain the tangents to the evolutes, or, in other terms, several straight lines normal to the curve described by the point, will necessarily coincide with the normal plane to the curve. If now it is required that the curve described by the point should be a given curve, it will be sufficient to make the *surface développable* to coincide with that which touches every normal plane to the proposed curve, and to choose properly the point.

(361.) To show an application of the preceding formulæ, we shall consider again the helix represented by the equations $x = r \cos p$, $y = r \sin p$, $z = arp$. If we take for the independent variable the arc s , or which is the same thing, the angle p , we shall have

$$dx = -r \sin p dp, \quad dy = r \cos p dp, \quad dz = ar dp, \quad d^2 x = -r \cos p dp^2, \quad d^2 y = -r \sin p dp^2, \quad d^2 z = a dp^2, \\ d^3 x = r \sin p dp^3, \quad d^3 y = -r \cos p dp^3, \quad d^3 z = 0.$$

We have, besides, already found $ds = \pm (1 + a^2)^{\frac{1}{2}} r dp$, $\rho = (1 + a^2) r$. This understood, we shall have for the coordinates of the centre of curvature

$$x_1 = -a^2 r \cos p, \quad y_1 = -a^2 r \sin p, \quad z_1 = arp.$$

If we eliminate p between these last equations, we shall obtain between the coordinates x_1, y_1, z_1 , two equations which will represent the line of the centres of curvatures. We shall have thus for these equations $\frac{y_1}{x_1} = \tan \frac{z_1}{ar}$, $x_1^2 + y_1^2 = a^4 r^2$. Hence it is clear, that this line will be a second helix situated as well as the first on the surface

represented by the equation $\frac{y}{x} = \tan \frac{z}{ar}$, and on a right cylinder having for its base, in the plane x, y , a circle the radius of which is equal to $a^2 r$. This result might have been easily foreseen, since we know that to obtain the centre of curvature corresponding to a point (x, y, z) of the helix, it suffices to take on the generating line of the helicoidal surface, from the point (x, y, z) the length $\rho = (1 + a^2) r$, and consequently from the axis of the cylinder the length $\rho - r = a^2 r$.

The equations which determine the evolute,

$$(x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2 = g^2, \quad (x_1 - x) dx + (y_1 - y) dy + (z_1 - z) dz = 0, \\ (x_1 - x) d^2 x + (y_1 - y) d^2 y + (z_1 - z) d^2 z = ds^2, \quad (x_1 - x) d(g^2 x) + (y_1 - y) d(g^2 y) + (z_1 - z) d(g^2 z) = 0,$$

will become in this example

$$x_1^2 + y_1^2 - 2r(x_1 \cos p + y_1 \sin p) + r^2 + (z_1 - arp)^2 = g^2, \quad x_1 \sin p - y_1 \cos p = a(z_1 - arp), \\ x_1 \cos p + y_1 \sin p = -a^2 r(x_1 \cos p + y_1 \sin p - r) dg + (y_1 \cos p - x_1 \sin p) g dp = 0.$$

If after having raised to the square both sides of the second and third equation, we add them, we find

$$x_1^2 + y_1^2 = a^4 r^2 + a^2 (z_1 - arp)^2, \text{ and consequently, } p = \frac{z_1}{ar} \mp \left\{ \frac{x_1^2 + y_1^2}{a^4 r^2} - 1 \right\}^{\frac{1}{2}}. \text{ Substituting then this value}$$

Integral
Calculus.

Part III.

 of p in the equation $x_1 \cos p + y_1 \sin p = -a^2 r$, we find

$$\left(x_1 \cos \frac{z_1}{ar} + y_1 \sin \frac{z_1}{ar}\right) \cos \left(\frac{x_1^2 + y_1^2}{a^4 r^2} - 1\right)^{\frac{1}{2}} + a^2 r = \pm \left(y_1 \cos \frac{z_1}{ar} - x_1 \sin \frac{z_1}{ar}\right) \sin \left(\frac{x_1^2 + y_1^2}{a^4 r^2} - 1\right)^{\frac{1}{2}}.$$

This last equation, being that which results from the elimination of p , represents the *surface développable* which is tangent to every normal plane to the helix, and which is the locus of all the evolutes of that curve. To obtain the equations of one of these evolutes, it will be sufficient to join to the equation of the *surface développable* that which results from the elimination of p between the two equations $x_1^2 + y_1^2 - 2r(x_1 \cos p + y_1 \sin p) + r^2 + (z_1 - arp)^2 = g^2$,

and $p = \frac{z_1}{ar} \pm \left\{ \frac{x_1^2 + y_1^2}{a^4 r^2} - 1 \right\}^{\frac{1}{2}}$. The first of these two last equations may be replaced in the present case by $\left(\frac{x_1^2 + y_1^2}{a^2} + r^2\right)(1 + a^2) = g^2$, which is obtained by combining it with $x_1^2 + y_1^2 = a^4 r^2 + a^2(z_1 - arp)^2$.

Moreover the differential equation of the first order between s and g will become $\frac{dg^2}{dp^2} + \frac{a^2}{1+a^2} g^2 = \frac{a^2}{(1+a^2)^3} \frac{g^4}{r^2}$,

which, being integrated, gives $\frac{a}{\sqrt{1+a^2}} p \pm \cos^{-1} \frac{r(1+a^2)}{g} = c$, c being an arbitrary constant, and consequently $g = \frac{r(1+a^2)}{\cos\left(c - \frac{ap}{\sqrt{1+a^2}}\right)}$. This value being put in the equation $\left(\frac{x_1^2 + y_1^2}{a^2} + r^2\right)(1+a^2) = g^2$, gives

$$\frac{r(1+a^2)}{\cos\left(c - \frac{ap}{\sqrt{1+a^2}}\right)} = g.$$

after the substitution of the value of p found above

$$\frac{a^2(1+a^2)r^2}{x_1^2 + y_1^2 + a^2 r^2} = \left\{ \cos\left(c - \frac{z_1}{r\sqrt{1+a^2}} \pm \frac{\sqrt{x_1^2 + y_1^2 - a^4 r^2}}{ar\sqrt{1+a^2}}\right) \right\}^2.$$

This equation joined to the equation of the *surface développable* determines for each value of the constant c an evolute of the helix.

It is not useless to remark that the smallest value of g is $r(1+a^2)$, that is to say equal to the radius of curvature, and corresponds to an infinite number of values of the angle p , one of which is clearly $\frac{c\sqrt{1+a^2}}{a}$, for then

$$\cos\left(c - \frac{ap}{\sqrt{1+a^2}}\right) = 1. \text{ If we assume } \frac{c\sqrt{1+a^2}}{a} = p_1, \text{ then we shall have } g = \frac{r(1+a^2)}{\cos\left(\frac{a(p_1-p)}{\sqrt{1+a^2}}\right)}, \text{ and}$$

consequently $x_1^2 + y_1^2 = a^2 r^2 \left\{ \frac{1+a^2}{\cos\left(\frac{a(p_1-p)}{\sqrt{1+a^2}}\right)} - 1 \right\}$, and this last equation combined with $x_1^2 + y_1^2 = a^4 r^2 + a^2(z_1 - arp)^2$

$(z_1 - arp)^2$ will give $z_1 - arp = \pm (1+a^2)^{\frac{1}{2}} r \tan \frac{a(p_1-p)}{\sqrt{1+a^2}}$. From this last equation, and from the two

$x_1 \sin p + y_1 \cos p = a(z_1 - arp)$, $x_1 \cos p + y_1 \sin p = -a^2 r$ we may derive the values of x_1 , y_1 , and z_1 in function of p , and we shall obtain thus three relations, which are contained in the formulæ

$$\frac{x_1 + a^2 r \cos p}{a} = \frac{y_1 + a^2 r \sin p}{-a} = \frac{z_1 - arp}{1} = \pm (1+a^2)^{\frac{1}{2}} r \tan \frac{a(p_1-p)}{\sqrt{1+a^2}},$$

which alone may represent each of the evolutes of the helix. These formulæ show that the coordinates x_1 , y_1 , and z_1 of each evolute become infinite whenever the angle p obtains a value of the form $p = p_1 \pm (2n+1)\frac{\pi}{2}$

$\frac{\sqrt{1+a^2}}{a}$, n being any integer whatever. Moreover whilst the angle p converges towards one of these values, the

point (x_1, y_1, z_1) is always situated on the *surface développable* whose equation we have found, and approaches indefinitely that generating line of the same surface to which the equations $x_1 \sin p - y_1 \cos p = a(z_1 - arp)$, $x_1 \cos p + y_1 \sin p = -a^2 r$, when for p the above value has been substituted. Therefore each evolute will be composed of an infinite number of infinite branches, each of which will have for its asymptotes two of the generating lines of the surface.

If we call r_2 and p_2 the polar coordinates of the projection of one of these evolutes on the plane (x, y) , we shall have $x_1 = r_2 \cos p_2$, $y_1 = r_2 \sin p_2$, and consequently we shall have

$$r_2 \sin(p - p_2) = \pm a(1+a^2)^{\frac{1}{2}} r \tan \frac{a(p_1-p)}{\sqrt{1+a^2}}, \quad r_2 \cos(p - p_2) = -a^2 r.$$

These two equations combined with $z_1 - arp = \pm (1+a^2)^{\frac{1}{2}} r \tan \frac{a(p_1-p)}{\sqrt{1+a^2}}$ may be used with advantage to

Integral
Calculus.

discover the properties of the evolutes of the helix. If for more simplicity we assume $p_1 = 0$, these equations become Part III.

$$z_1 - arp = \mp (1 + a^2)^{\frac{1}{2}} r \tan \frac{ap}{\sqrt{1 + a^2}}, \quad r_2 \sin(p - p_2) = \mp a(1 + a^2)^{\frac{1}{2}} r \tan \frac{ap}{\sqrt{1 + a^2}},$$

$$r_2 \cos(p - p_2) = -a^2 r,$$

and will represent that evolute upon which is situated the centre of curvature corresponding to the point of the helix, which coincides with the origin of coordinates. To return from these last formulæ to those we mentioned just before, it is obviously sufficient to replace the quantities p , p_2 and z_1 by the differences $p - p_1$, $p_2 - p_1$, $z_1 - arp_1$. But the substitution of $p - p_1$ for p , and of the ordinate $z_1 - arp_1$ for z_1 , is precisely that which it is necessary to perform in order that the curve to which belong the three coordinates p_2 , r_2 , and z_1 should displace itself in space, in such a manner that each point should describe first, in turning round the axis of z , with a direct motion of rotation, if p_1 be positive, the angle $+p_1$, or with a retrograde motion of rotation, if p_1 be negative, the angle $-p_1$, and afterwards should move upon a line parallel to the axis of z of a quantity equal to arp , in the direction of the positive values of z , or of a quantity equal to $-arp$, in the direction of the negative values of z . Therefore this displacement is sufficient to transform the evolute we have last considered in any one of the other evolutes of the helix. Therefore these evolutes are curves which may be made to coincide with each other. If

we should take for p_1 , one of the values contained in the formula $p_1 = \pm (2n + 1) \frac{\pi}{2} \frac{\sqrt{1 + a^2}}{a}$, it will be the

various branches of the first evolute, which will be made to coincide by the displacement we have just indicated. Therefore all the branches of one of the evolutes are equal, and may be made to coincide with each other. These remarkable properties of the evolutes of the helix, depend upon the regular form of this curve, which may be made to coincide with itself by moving it, at the same time, round the axis of z and parallel to that axis.

(362.) When two curves of double curvature have a common point, and when they have, at this point, not only the same tangent, but also the same osculating circle, and consequently the same osculating plane, the same principal normal and the same curvature, then one of the curves is said to be an *osculating curve* of the other, or the contact which exists between the two curves is called an *osculation*. This understood, the following proposition results obviously from what precedes.

In order that a curve of double curvature should be an osculating curve of another curve of double curvature, at a point common to both corresponding to an abscissa equal x , it will be necessary and sufficient that the ordinates y and z relative to that abscissa, and their differential coefficients of the first and second order, that is to

say, the six quantities $y, y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}, z, z' = \frac{dz}{dx}, z'' = \frac{d^2z}{dx^2}$, should preserve the same values in passing from one curve to the other. And reciprocally when one of the curves is an osculating curve of the other, these conditions must be satisfied.

An immediate consequence of this proposition is clearly that when one of the curves is an osculating curve of the other, their projections upon the coordinate planes, and generally upon any plane whatever, have the same relation of contact as the curves themselves.

From this proposition it also follows that, two curves of double curvature being represented by two equations between the coordinates x, y, z to determine whether one of the curves is an osculating curve of the other, it will be sufficient to take for the independent variable, either a given function of the variables x, y, z , or the arcs s , reckoned upon each curve from a fixed point, and to examine whether for the given point the same values of $x, y, z, dx, dy, dz, d^2x, d^2y, d^2z$, may be derived from the equations of the two curves.

If one of the curves be the intersection of the sphere and plane represented by the equations

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = \rho^2, \quad (x - x_1) \cos l + (y - y_1) \cos m + (z - z_1) \cos n = 0,$$

the conditions to express that the point (x, y, z) is a point of osculation will suffice to determine the radius of the circle and the coordinates of the centre, that is to say, the four quantities x_1, y_1, z_1 , and ρ , with two of the three angles l, m, n , between which we have the relation $\cos^2 l + \cos^2 m + \cos^2 n = 1$. Let us suppose that x is taken for the independent variable, then the values of y, y', y'', z, z', z'' , derived from the equations of the other curve and its differential equations, must satisfy the equations of the circle and its differential equations, that is to say, the six following formulæ :

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = \rho^2, \quad x - x_1 + (y - y_1) y' + (z - z_1) z' = 0,$$

$$1 + y'^2 + z'^2 + (y - y_1) y'' + (z - z_1) z'' = 0, \quad (x - x_1) \cos l + (y - y_1) \cos m + (z - z_1) \cos n = 0,$$

$$\cos l + y' \cos m + z' \cos n = 0, \quad y'' \cos m + z'' \cos n = 0.$$

When by means of these formulæ joined to the equation $\cos^2 l + \cos^2 m + \cos^2 n = 1$, the values of l, m, n, x_1, y_1, z_1 , and ρ are determined, we find identically the same values we have obtained before.

If x were not taken for the independent variable, then the equations of the circle and its differential equations of the first and second order might be represented by

$$(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = \rho^2, \quad (x - x_1) dx + (y - y_1) dy + (z - z_1) dz = 0,$$

$$(x - x_1) d^2x + (y - y_1) d^2y + (z - z_1) d^2z = -ds^2, \quad (x - x_1) \cos l + (y - y_1) \cos m + (z - z_1) \cos n = 0,$$

$$\cos l dx + \cos m dy + \cos n dz = 0, \quad \cos l d^2x + \cos m d^2y + \cos n d^2z = 0,$$

Integral Calculus. and ought to be verified by the values of $x, y, z, dx, dy, dz, d^2x, d^2y, d^2z$, determined by means of the equations of the other curves. Part III.

(363.) Let us now consider a curve surface represented by the equation $u = 0$, in which u designates, as usual, a function of the three coordinates x, y, z . If through a given point (x, y, z) on this surface, we conceive a normal plane, the intersection of this plane with the surface will be a certain curve, which we shall call a *normal section*. Let ρ be the radius of curvature of this curve relative to the point (x, y, z) , and x_1, y_1, z_1 the coordinates of the corresponding centre of curvature. We shall have $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = \rho^2$, and since the centre

of curvature is situated on the normal, we shall have $\frac{x - x_1}{\left(\frac{du}{dx}\right)} = \frac{y - y_1}{\left(\frac{du}{dy}\right)} = \frac{z - z_1}{\left(\frac{du}{dz}\right)}$. Let us observe, now, that the

equation $(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 = \rho^2$, when x, y , and z alone are supposed to be variables, is one of the equations of the osculating circle of the normal section under consideration, and, consequently, that the differential equation of the second order of that same equation,

$$(x - x_1) d^2x + (y - y_1) d^2y + (z - z_1) d^2z + dx^2 + dy^2 + dz^2 = 0,$$

ought to be satisfied by the values of $x, y, z, dx, dy, dz, d^2x, d^2y, d^2z$, derived from the equations of the normal section.

If we designate by s the arc of this curve, we have $ds^2 = dx^2 + dy^2 + dz^2$, and the above equation is then reduced to $(x - x_1) d^2x + (y - y_1) d^2y + (z - z_1) d^2z = -ds^2$. But if we differentiate twice the equation of the surface, we have

$$\begin{aligned} \frac{du}{dx} d^2x + \frac{du}{dy} d^2y + \frac{du}{dz} d^2z + \frac{d^2u}{dx^2} dx^2 + \frac{d^2u}{dy^2} dy^2 + \frac{d^2u}{dz^2} dz^2 + \frac{2d^2u}{dydz} dydz \\ + \frac{2d^2u}{dzdx} dzdx + \frac{2d^2u}{dxdy} dxdy = 0. \end{aligned}$$

Let, for brevity sake,

$$q = \frac{d^2u}{dx^2} \frac{dx^2}{ds^2} + \frac{d^2u}{dy^2} \frac{dy^2}{ds^2} + \frac{d^2u}{dz^2} \frac{dz^2}{ds^2} + \frac{2d^2u}{dydz} \frac{dydz}{ds^2} + \frac{2d^2u}{dzdx} \frac{dzdx}{ds^2} + \frac{2d^2u}{dxdy} \frac{dxdy}{ds^2},$$

then the equation above will be reduced to

$$\frac{du}{dx} d^2x + \frac{du}{dy} d^2y + \frac{du}{dz} d^2z = -ds^2.$$

If we assume

$$r = \sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2},$$

we shall have

$$\begin{aligned} \frac{x - x_1}{\left(\frac{du}{dx}\right)} = \frac{y - y_1}{\left(\frac{du}{dy}\right)} = \frac{z - z_1}{\left(\frac{du}{dz}\right)} = \pm \frac{\left\{ (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \right\}^{\frac{1}{2}}}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}} = \pm \frac{\rho}{r} \\ = \frac{(x - x_1) d^2x + (y - y_1) d^2y + (z - z_1) d^2z}{\frac{du}{dx} d^2x + \frac{du}{dy} d^2y + \frac{du}{dz} d^2z} = \frac{1}{q}, \end{aligned}$$

and consequently, $\frac{1}{\rho} = \pm \frac{q}{r}$. It is essential to observe that the quantity r represents a known function of the

quantities x, y, z . As to the quantity q it may be expressed in function of the coordinates x, y, z , and the angles which the tangent to the normal section through the point (x, y, z) makes with the coordinate axis. Let α, β, γ be these angles, we shall have

$$\cos \alpha = \frac{dx}{ds}, \cos \beta = \frac{dy}{ds}, \cos \gamma = \frac{dz}{ds}, \text{ or else } \cos \alpha = -\frac{dx}{ds}, \cos \beta = -\frac{dy}{ds}, \cos \gamma = -\frac{dz}{ds},$$

and in both cases we shall have

$$q = \frac{d^2u}{dx^2} \cos^2 \alpha + \frac{d^2u}{dy^2} \cos^2 \beta + \frac{d^2u}{dz^2} \cos^2 \gamma + 2 \frac{d^2u}{dydz} \cos \beta \cos \gamma + 2 \frac{d^2u}{dzdx} \cos \gamma \cos \alpha + 2 \frac{d^2u}{dxdy} \cos \beta \cos \alpha.$$

Now, it is clear that the second member of this last equation, is a known function of the variables $x, y, z, \alpha, \beta, \gamma$; and, therefore, if with the point (x, y, z) the tangent drawn through this point to a normal section is also given, it will be easy by means of the preceding formulæ to find the radius and centre of curvature of that normal section.

When we pass from a normal section to another without displacing the point (x, y, z) the angles α, β, γ take other values, and the same thing takes place, at least generally speaking, with respect to the quantity q and the variables x, y, z , and ρ which depend upon it. If in this passage it should happen that the quantity ρ change

its sign, then the radius of curvature of one of the sections would be determined by the equation $\frac{1}{\rho} = -\frac{q}{r}$.

Integral
Calculus.

and that of the other section by the equation $\frac{1}{\rho} = \frac{-q}{r}$, since the quantities ρ and r are essentially positive. On the other side, the quantities $\frac{du}{dx}, \frac{du}{dy}, \frac{du}{dz}$, being independent of the angles α, β, γ , it may be affirmed that if in

the passage of one section to another, the quantity q changes its sign, the differences $x - x_1, y - y_1, z - z_1$, will also change theirs. Therefore the centres of curvature of the two sections will be situated with respect to the point (x, y, z) , on different sides, and both on the normal drawn through this point to the given surface; so that the radii of curvature will be in opposite directions.

The radii of curvature of the various normal sections which pass through the same point, present properties which deserve to be remarked. To discover these relations, it is first necessary to discover the law according to which the radius of curvature ρ varies with the angles α, β, γ . This law may be made very obvious by means of a geometrical construction, which consists in taking on the tangent to each normal section, on both sides of the point (x, y, z) two lengths equal to the radius of curvature corresponding, or to a positive power of this radius. The curve which will pass through the extremities of these lengths will clearly be a plane curve, contained in the tangent plane; and in the same manner as the radius vector drawn through the point (x, y, z) to this curve will vary, will the radius of curvature of the normal section tangent to that radius vector increase or decrease, so that it is clear that the nature of this curve will show the law according to which the radius of curvature varies. In the particular case where it is supposed that the length taken upon each tangent is equal to the square root of the radius of curvature corresponding, the curve we have just mentioned is reduced to a curve of the second degree.

For if we designate by x_1, y_1, z_1 , the coordinates of the extremity of a length equal to $\rho^{\frac{1}{2}}$ taken upon the tangent which forms with the axis the angles α, β, γ , we shall find $x_1 - x = \rho^{\frac{1}{2}} \cos \alpha, y_1 - y = \rho^{\frac{1}{2}} \cos \beta, z_1 - z = \rho^{\frac{1}{2}} \cos \gamma$; and if we place the origin of coordinates at the point (x, y, z) , we shall find $x_1 = \rho^{\frac{1}{2}} \cos \alpha, y_1 = \rho^{\frac{1}{2}} \cos \beta, z_1 = \rho^{\frac{1}{2}} \cos \gamma$. We have besides $q\rho = \pm r$, or substituting for q its value, we have

$$\frac{d^2 u}{dx^2} \cos^2 \alpha + \frac{d^2 u}{dy^2} \cos^2 \beta + \frac{d^2 u}{dz^2} \cos^2 \gamma + \frac{2 d^2 u}{dy dz} \cos \beta \cos \gamma + \frac{2 d^2 u}{dz dx} \cos \gamma \cos \alpha + \frac{2 d^2 u}{dy dx} \cos \beta \cos \alpha = \pm r.$$

Therefore if for $\cos \alpha, \cos \beta, \cos \gamma$, the values $\frac{x_1}{\rho^{\frac{1}{2}}}, \frac{y_1}{\rho^{\frac{1}{2}}}, \frac{z_1}{\rho^{\frac{1}{2}}}$, are substituted, we shall have

$$\frac{d^2 u}{dx^2} x_1^2 + \frac{d^2 u}{dy^2} y_1^2 + \frac{d^2 u}{dz^2} z_1^2 + \frac{2 d^2 u}{dy dz} y_1 z_1 + \frac{2 d^2 u}{dz dx} x_1 z_1 + \frac{2 d^2 u}{dy dx} y_1 x_1 = \pm r.$$

This last equation, when x_1, y_1 , and z_1 , are considered as the only variables, represents a surface of the second degree which contains the plane curve above mentioned. It may be remarked, that this surface has for its centre the new origin of coordinates, that is the point (x, y, z) ; and as the plane of the curve passes also through the same point, we may conclude that the curve in question is a line of the second degree which has for its centre the point (x, y, z) . To obtain the two equations of this curve, it is sufficient to join to the above equation of the surface of the second degree, the equation of the tangent plane, at the point (x, y, z) , when the origin is placed

at that point, that is to say, the equation $x_1 \frac{du}{dx} + y_1 \frac{du}{dy} + z_1 \frac{du}{dz} = 0$. But if λ, μ, ν represent the angles

which the normal makes with the axis, we have $\frac{\cos \lambda}{\left(\frac{du}{dx}\right)} = \frac{\cos \mu}{\left(\frac{du}{dy}\right)} = \frac{\cos \nu}{\left(\frac{du}{dz}\right)}$, and, consequently, the equation of

the plane containing the curve of the second degree takes the form $x_1 \cos \lambda + y_1 \cos \mu + z_1 \cos \nu = 0$.

From all that precedes it follows, that the curve drawn in the tangent plane according to the construction indicated, will be reduced, in general, to an ellipse, or to the system of two conjugated hyperbolas; but that in certain particular cases it may be transformed into a circle, or in a single point, or in a system of two parallel straight lines, equally distant from the point (x, y, z) , or in a system of two straight lines drawn through that point. The same conclusion results also from the form which takes the equation

$$\frac{d^2 u}{dx^2} x_1^2 + \frac{d^2 u}{dy^2} y_1^2 + \frac{d^2 u}{dz^2} z_1^2 + 2 \frac{d^2 u}{dy dz} y_1 z_1 + 2 \frac{d^2 u}{dz dx} x_1 z_1 + 2 \frac{d^2 u}{dy dx} y_1 x_1 = \pm r,$$

when the plane of (x, y) is parallel to the tangent plane drawn through the point (x, y, z) . Then the equation of the plane of the curve is reduced to $z_1 = 0$, and in combining this with the above equation, it reduces it to

$$\frac{d^2 u}{dx^2} x_1^2 + \frac{d^2 u}{dy^2} y_1^2 + 2 \frac{d^2 u}{dy dx} y_1 x_1 = \pm r. \text{ Now it is well known that this equation will represent an ellipse}$$

when the difference $\frac{d^2 u}{dx^2} \cdot \frac{d^2 u}{dy^2} - \left(\frac{d^2 u}{dy dx}\right)^2$ is positive, two conjugate hyperbolas when this difference becomes negative, and two parallel lines if this difference is equal to nothing. The ellipse becomes a circle when

Integral Calculus. $\frac{d^2 u}{d x^2} = \frac{d^2 u}{d y^2}$, and $\frac{d^2 u}{d x d y} = 0$, and a point when $r = 0$, in this last case the two conjugate hyperbolas are transformed into two straight lines. Since the plane of (x, y) may be chosen arbitrarily, the preceding reasoning

may obviously be applied to any point of the surface. In order that the equation $x_1 \frac{d u}{d x} + y_1 \frac{d u}{d y} + z_1 \frac{d u}{d z} = 0$

should become, as we have just supposed, $z_1 = 0$, it is necessary that of the three quantities $\frac{d u}{d x}, \frac{d u}{d y}, \frac{d u}{d z}$, the two

first should vanish, or the third become infinite. In the first case $r = \pm \frac{d u}{d z}$, and the equation of the curve

becomes $\frac{d^2 u}{d x^2} x_1^2 + 2 \frac{d^2 u}{d x d y} x_1 y_1 + \frac{d^2 u}{d y^2} y_1^2 = \pm \frac{d u}{d z}$. Since the radius of curvature ρ of a normal section is the

square of the radius vector $\sqrt{(x_1^2 + y_1^2)}$ drawn through the point (x, y, z) to a point of the line of the second degree drawn in the tangent plane, it is obvious that each of the normal planes which pass through the axis of this line produces a normal section, the radius of curvature of which is a *maximum* or a *minimum*. We shall call principal sections, and principal radii of curvature, the two normal sections in question, and their radii of curvature. This understood, it is clear that the planes of the principal sections are always at right angles to each other, and that the principal radii of curvature are in the same or opposite directions according to the conditions above stated; the first case will take place when the curve drawn in the tangent plane is an ellipse, the second when this curve is a system of two conjugate hyperbolas. In the first case these radii of curvature will represent the one a maximum, the other a minimum of the variable ρ , and in the second case two values minima of the same variable. In other words, if the curve drawn in the tangent plane be an ellipse, the two principal sections will be the two normal sections of greatest and least curvature; but if this curve be a system of two conjugate hyperbolas, both principal sections will be normal sections of greatest curvature, only their curvatures will be in opposite directions. In the same hypothesis the normal sections, the planes of which would contain the asymptotes common to both hyperbolas, would have curvatures equal to nothing corresponding to values of ρ equal to infinity. Hence the planes of the two normal sections, of which the curvatures are equal to nothing, will form equal angles with the planes of the principal sections.

When the ellipse becomes a circle, all the normal sections have equal curvatures, and then the name of principal sections may be given to any two normal sections the planes of which are at right angles to each other.

When the difference $\frac{d^2 u}{d x^2} \cdot \frac{d^2 u}{d y^2} - \left(\frac{d^2 u}{d x d y} \right)^2 = 0$, the line in the tangent plane is reduced to two parallel

straight lines, which may be considered as representing an ellipse the greater axis of which has become infinite. In this case the principal sections correspond to a minimum value and to an infinite value of ρ , so that one of the sections has a curvature equal to nothing.

If the quantity r should vanish, all the normal sections would have radii of curvature equal to nothing or infinity.

It may easily be proved that if by the centre of an ellipse or of two conjugate hyperbolas, two radii are drawn to the curve at right angles to each other, the lengths of which are represented by r_1 and r_2 , the quantity $\frac{1}{r_1^2} + \frac{1}{r_2^2}$ is a constant quantity equal to $\frac{1}{a^2} + \frac{1}{b^2}$, a and b being the two axes of the ellipse, or the real axis of the two

conjugate hyperbolas, provided, however, that, in the case of the conjugate hyperbolas, if we have agreed to take with the sign $+$ one of the quotients relative to one of the hyperbolas, the other, when it will refer to the second hyperbola, should be taken with the sign $-$.

From this proposition it clearly follows that if we draw through a point (x, y, z) of a surface two planes at right angles to each other, and normal to the surface, and if we divide successively unity by each of the radii of curvature of the two lines of intersection, the sum of the quotients will be a constant quantity, provided that in this sum we always take with the sign $+$ the radii drawn in a certain direction, and with the sign $-$ the radii in the opposite direction. Consequently this sum will be equal, without paying attention to the sine, to the sum or to the difference of the quotients relative to the principal radii of curvature.

(364.) We may also easily find the relation which exists between the radius of curvature of any normal section, and the angles formed by the plane of that section with the principal sections. To obtain it, let us first observe that, in the case where the tangent plane to the proposed surface is parallel to the plane of (x, y) , we have for

the tangent to each normal section $\cos \gamma = 0$, $\cos^2 \alpha + \cos^2 \beta = 1$. Hence we shall have in that case $q = \frac{d u}{d x^2}$

$\cos^2 \alpha + 2 \frac{d^2 u}{d x d y} \cos \alpha \cos \beta + \frac{d^2 u}{d y^2} \cos^2 \beta$. This value of q becomes still simpler, when we suppose the planes

of x, z , and y, z parallel to the planes of the principal sections. Then the equation of the curve drawn in the tangent plane, belonging to a curve of the second degree referred to its axis, cannot contain the product of the

Integral
Calculus.

Part III.

coordinates x_1 and y_1 , and must be reduced to $\frac{d^2 u}{d x^2} x_1^2 + \frac{d^2 u}{d y^2} y_1^2 = \pm r$. Therefore we have $\frac{d^2 u}{d y d x} = 0$, and,

consequently, the value of q becomes $q = \frac{d^2 u}{d x^2} \cos^2 \alpha + \frac{d^2 u}{d y^2} \cos^2 \beta$. But we have also $\cos^2 \beta = 1 - \cos^2 \alpha =$

$\sin^2 \alpha$, hence the value of q may be written $q = \frac{d^2 u}{d x^2} \cos^2 \alpha + \frac{d^2 u}{d y^2} \sin^2 \alpha$. This understood, the formula $\frac{1}{\rho} = \pm$

$\frac{q}{r}$ will give $\frac{1}{\rho} = \pm \frac{1}{r} \left\{ \frac{d^2 u}{d x^2} \cos^2 \alpha + \frac{d^2 u}{d y^2} \sin^2 \alpha \right\}$. Let now ρ_0 and ρ_1 be the principal radii of curvature. To

obtain their values it will be sufficient to assume successively in the preceding formula $\alpha = 0$ and $\alpha = \frac{\pi}{2}$. We

shall find thus $\frac{1}{\rho_0} = \pm \frac{1}{r} \frac{d^2 u}{d x^2}$, $\frac{1}{\rho_1} = \pm \frac{1}{r} \frac{d^2 u}{d y^2}$, and, consequently, $\frac{1}{\rho} = \pm \frac{1}{\rho_0} \cos^2 \alpha \pm \frac{1}{\rho_1} \sin^2 \alpha$. It is essential

to observe that $\frac{d^2 u}{d x^2}$, $\frac{d^2 u}{d y^2}$ are quantities of the same sign in the case where the curve drawn in the tangent plane is an ellipse, and of contrary sign when it is two conjugate hyperbolas. Hence in the first case, we shall have

$\frac{1}{\rho} = \frac{1}{\rho_0} \cos^2 \alpha + \frac{1}{\rho_1} \sin^2 \alpha$, and in the second case $\pm \frac{1}{\rho} = \frac{1}{\rho_0} \cos^2 \alpha - \frac{1}{\rho_1} \sin^2 \alpha$, the sign $+$ in the left side being

taken when the radius of curvature ρ is in the same direction as ρ_0 , and the sign $-$ when it is in the same direction

as the radius ρ_1 . If the differential coefficients $\frac{d^2 u}{d x^2}$, $\frac{d^2 u}{d y^2}$ are equal, the curve described in the tangent plane

becomes a circle. At the same time we shall have $\rho_1 = \rho_0$ and $\rho = \rho_0$.

If one of the two quantities $\frac{d^2 u}{d x^2}$, $\frac{d^2 u}{d y^2}$ is equal to nothing, the line drawn in the tangent plane becomes the

system of two parallel lines. One of the radii ρ_0 , ρ_1 then becomes infinite; let $\frac{d^2 u}{d y^2} = 0$, we shall have $\rho_1 = \infty$,

and the formula will give $\frac{1}{\rho} = \frac{1}{\rho_0} \cos^2 \alpha$. In this case ρ_0 is the radius of curvature of the normal section which has for its tangent a straight line perpendicular to the two parallel straight lines drawn through the point (x, y, z) .

If we add to the value of $\frac{1}{\rho}$, what becomes that value when for α , $\alpha + \frac{\pi}{2}$ is substituted, we find $\pm \frac{1}{\rho_0} \pm \frac{1}{\rho_1}$, a result agreeing with the theorem stated at the end of the preceding paragraph.

(365.) Let us suppose now, that it is required to find in space for any point (x, y, z) of the given surface the directions of the tangents to the principal sections, and the principal radii of curvature. It will evidently be sufficient to find the maximum and minimum, or the two minima of the radius of curvature $\rho = x_1^2 + y_1^2 + z_1^2$, the coordinates x_1 , y_1 , and z_1 being connected by the two equations

$$\frac{d^2 u}{d x^2} x_1^2 + \frac{d^2 u}{d y^2} y_1^2 + \frac{d^2 u}{d z^2} z_1^2 + \frac{2 d^2 u}{d y d z} y_1 z_1 + \frac{2 d^2 u}{d z d x} z_1 x_1 + \frac{2 d^2 u}{d x d y} x_1 y_1 = \pm r, \quad x_1 \frac{d u}{d x} + y_1 \frac{d u}{d y} + z_1 \frac{d u}{d z} = 0.$$

Hence the values of x_1 , y_1 , and z_1 , corresponding to the principal radii of curvature, must satisfy the equation $x_1 d x_1 + y_1 d y_1 + z_1 d z_1 = 0$, after having eliminated $d x_1$, $d y_1$, $d z_1$, by means of the two equations

$$\left(\frac{d^2 u}{d x^2} x_1 + \frac{d^2 u}{d x d y} y_1 + \frac{d^2 u}{d x d z} z_1 \right) d x_1 + \left(\frac{d^2 u}{d x d y} x_1 + \frac{d^2 u}{d y^2} y_1 + \frac{d^2 u}{d y d z} z_1 \right) d y_1 + \left(\frac{d^2 u}{d x d z} x_1 + \frac{d^2 u}{d y d z} y_1 + \frac{d^2 u}{d z^2} z_1 \right) d z_1 = 0.$$

$$\frac{d u}{d x} d x_1 + \frac{d u}{d y} d y_1 + \frac{d u}{d z} d z_1 = 0.$$

If then we add the three last equations after having multiplied the first and last by two indeterminate coefficients $-s$, $-t$, it is clear that we may determine the values of s and t so as to render equal to nothing, in the resulting equation, the coefficients of $d x_1$, $d y_1$, $d z_1$, since it is obvious that these three quantities may be eliminated from the three equations. Thus the values of s and t may be determined so as to satisfy the three equations,

$$\frac{d^2 u}{d x^2} x_1 + \frac{d^2 u}{d x d y} y_1 + \frac{d^2 u}{d x d z} z_1 = s x_1 + t \frac{d u}{d x}, \quad \frac{d^2 u}{d x d y} x_1 + \frac{d^2 u}{d y^2} y_1 + \frac{d^2 u}{d y d z} z_1 = s y_1 + t \frac{d u}{d y},$$

$$\frac{d^2 u}{d x d z} x_1 + \frac{d^2 u}{d y d z} y_1 + \frac{d^2 u}{d z^2} z_1 = s z_1 + t \frac{d u}{d z}.$$

If we add these equations after having multiplied them respectively by x_1 , y_1 , z_1 , we shall have, $\pm r = s \rho$, or

Integral Calculus. $s = \pm \frac{r}{\rho} = q$, since we have already between the various quantities which enter into these equations, the relations Part III.

$$\frac{d^2 u}{dx^2} x_1^2 + \frac{d^2 u}{dy^2} y_1^2 + \frac{d^2 u}{dz^2} z_1^2 + \frac{2 d^2 u}{dz dy} z_1 y_1 + \frac{2 d^2 u}{dz dx} z_1 x_1 + \frac{2 d^2 u}{dy dx} y_1 x_1 = \pm r,$$

$$x_1 \frac{du}{dx} + y_1 \frac{du}{dy} + z_1 \frac{du}{dz} = 0, \text{ and } x_1^2 + y_1^2 + z_1^2 = \rho.$$

Now, since the quantity s is the same as the quantity q , the three above equations may be put under the following form,

$$\left(\frac{d^2 u}{dx^2} - q \right) x_1 + \frac{d^2 u}{dx dy} y_1 + \frac{d^2 u}{dx dz} z_1 = t \frac{du}{dx}, \quad \frac{d^2 u}{dx dy} x_1 + \left(\frac{d^2 u}{dy^2} - q \right) y_1 + \frac{d^2 u}{dy dz} z_1 = t \frac{du}{dy},$$

$$\frac{d^2 u}{dx dz} x_1 + \frac{d^2 u}{dy dz} y_1 + \left(\frac{d^2 u}{dz^2} - q \right) z_1 = t \frac{du}{dz}.$$

From which we shall derive the values of x, y, z . We shall find

$$x_1 = \kappa \left\{ \frac{du}{dx} \left[\left(\frac{d^2 u}{dy^2} - q \right) \left(\frac{d^2 u}{dz^2} - q \right) - \left(\frac{d^2 u}{dy dz} \right)^2 \right] + \frac{du}{dy} \left[\frac{d^2 u}{dz dx} \cdot \frac{d^2 u}{dy dz} - \left(\frac{d^2 u}{dz^2} - q \right) \frac{d^2 u}{dx dy} \right] \right.$$

$$\left. + \frac{du}{dz} \left[\frac{d^2 u}{dy dz} \cdot \frac{d^2 u}{dx dy} - \left(\frac{d^2 u}{dy^2} - q \right) \frac{d^2 u}{dz dx} \right] \right\},$$

$$y_1 = \kappa \left\{ \frac{du}{dx} \left[\frac{d^2 u}{dz dx} \cdot \frac{d^2 u}{dy dz} - \left(\frac{d^2 u}{dz^2} - q \right) \frac{d^2 u}{dx dy} \right] + \frac{du}{dy} \left[\left(\frac{d^2 u}{dz^2} - q \right) \left(\frac{d^2 u}{dx^2} - q \right) - \left(\frac{d^2 u}{dz dx} \right)^2 \right] \right.$$

$$\left. + \frac{du}{dz} \left[\frac{d^2 u}{dx dy} \frac{d^2 u}{dz dx} - \left(\frac{d^2 u}{dx^2} - q \right) \frac{d^2 u}{dy dz} \right] \right\},$$

$$z_1 = \kappa \left\{ \frac{du}{dx} \left[\frac{d^2 u}{dy dz} \frac{d^2 u}{dx dy} - \left(\frac{d^2 u}{dy^2} - q \right) \frac{d^2 u}{dz dx} \right] + \frac{du}{dy} \left[\frac{d^2 u}{dx dy} \frac{d^2 u}{dz dx} - \left(\frac{d^2 u}{dx^2} - q \right) \frac{d^2 u}{dy dz} \right] \right.$$

$$\left. + \frac{du}{dz} \left[\left(\frac{d^2 u}{dx^2} - q \right) \left(\frac{d^2 u}{dy^2} - q \right) - \left(\frac{d^2 u}{dx dy} \right)^2 \right] \right\}.$$

In these formulæ κ designates a quantity the value of which may be derived from the relation $x_1^2 + y_1^2 + z_1^2 = \rho$.

We find thus $\kappa = \frac{\pm \rho^{\frac{1}{2}}}{u}$, u^2 being the sum of the squares of the coefficients of κ in the values of x_1, y_1 , and z_1 .

Moreover, if we substitute these same values in the formulæ $x_1 \frac{du}{dx} + y_1 \frac{du}{dy} + z_1 \frac{du}{dz} = 0$, we find

$$0 = \left(\frac{du}{dx} \right)^2 \left[\left(\frac{d^2 u}{dy^2} - q \right) \left(\frac{d^2 u}{dz^2} - q \right) - \left(\frac{d^2 u}{dy dz} \right)^2 \right] + \left(\frac{du}{dy} \right)^2 \left[\left(\frac{d^2 u}{dz^2} - q \right) \left(\frac{d^2 u}{dx^2} - q \right) - \left(\frac{d^2 u}{dz dx} \right)^2 \right]$$

$$+ \left(\frac{du}{dz} \right)^2 \left[\left(\frac{d^2 u}{dx^2} - q \right) \left(\frac{d^2 u}{dy^2} - q \right) - \left(\frac{d^2 u}{dx dy} \right)^2 \right] + 2 \frac{du}{dy} \frac{du}{dz} \left[\frac{d^2 u}{dx dy} \frac{d^2 u}{dz dx} - \left(\frac{d^2 u}{dx^2} - q \right) \frac{d^2 u}{dy dz} \right]$$

$$+ 2 \frac{du}{dz} \frac{du}{dx} \left[\frac{d^2 u}{dy dz} \frac{d^2 u}{dx dy} - \left(\frac{d^2 u}{dy^2} - q \right) \frac{d^2 u}{dz dx} \right] + 2 \frac{du}{dy} \frac{du}{dx} \left[\frac{d^2 u}{dz dx} \frac{d^2 u}{dy dz} - \left(\frac{d^2 u}{dz^2} - q \right) \frac{d^2 u}{dx dy} \right].$$

This equation is of the second degree with respect to q , and the two values it gives for that quantity are those corresponding to the two principal radii of curvature. Knowing these values of q , the equation $\rho = \pm \frac{r}{q}$ will give the values of the two principal radii of curvature, and the above values of x_1, y_1, z_1 , together with the equation $\kappa = \pm \frac{\rho^{\frac{1}{2}}}{u}$, will give the coordinates of four points situated on the tangents to the principal sections.

To determine the directions of these tangents, we have the equations

$$x_1 = \rho^{\frac{1}{2}} \cos \alpha, \quad y_1 = \rho^{\frac{1}{2}} \cos \beta, \quad z_1 = \rho^{\frac{1}{2}} \cos \gamma.$$

When the variables x, y, z are separated in the equation $u = 0$, that is to say, when the function u is divided into three parts, each of which contains only one of the variables x, y, z , we have for all the points of the given surface $\frac{d^2 u}{dy dz} = 0, \frac{d^2 u}{dz dx} = 0, \frac{d^2 u}{dx dy} = 0$. Then the equations which, in the general case, give the values of x_1, y_1 , and z_1 , become

$$\left(\frac{d^2 u}{dx^2} - q \right) x_1 = t \frac{du}{dx}, \quad \left(\frac{d^2 u}{dy^2} - q \right) y_1 = t \frac{du}{dy}, \quad \left(\frac{d^2 u}{dz^2} - q \right) z_1 = t \frac{du}{dz},$$

Integral
Calculus.and by substituting the values of x_1 , y_1 , and z_1 , derived from these formulæ, in the equation

$$x_1 \frac{du}{dx} + y_1 \frac{du}{dy} + z_1 \frac{du}{dz} = 0,$$

we find

$$\frac{\left(\frac{du}{dx}\right)^2}{\frac{d^2u}{dx^2} - q} + \frac{\left(\frac{du}{dy}\right)^2}{\frac{d^2u}{dy^2} - q} + \frac{\left(\frac{du}{dz}\right)^2}{\frac{d^2u}{dz^2} - q} = 0.$$

These same formulæ combined with the equation $\rho = x_1^2 + y_1^2 + z_1^2$, will give

$$t = \pm \rho^{\frac{1}{2}} \left\{ \left(\frac{\frac{du}{dx}}{\frac{d^2u}{dx^2} - q} \right)^2 + \left(\frac{\frac{du}{dy}}{\frac{d^2u}{dy^2} - q} \right)^2 + \left(\frac{\frac{du}{dz}}{\frac{d^2u}{dz^2} - q} \right)^2 \right\}^{-\frac{1}{2}},$$

and then, by means of the formulæ $x_1 = \rho^{\frac{1}{2}} \cos \alpha$, $y_1 = \rho^{\frac{1}{2}} \cos \beta$, $z_1 = \rho^{\frac{1}{2}} \cos \gamma$, we get

$$\frac{\left(\frac{d^2u}{dx^2} - q\right) \cos \alpha}{\frac{du}{dx}} = \frac{\left(\frac{d^2u}{dy^2} - q\right) \cos \beta}{\frac{du}{dy}} = \frac{\left(\frac{d^2u}{dz^2} - q\right) \cos \gamma}{\frac{du}{dz}} = t.$$

These formulæ joined to the relation $\frac{1}{\rho} = \pm \frac{q}{r}$ will be sufficient, in the present hypothesis, to determine the directions of the tangents to the principal sections, and the principal radii of curvature.

We shall apply these formulæ to a few examples.

1. Let the proposed surface be the ellipsoid represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. We may assume $u = \frac{1}{2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right)$, and then we shall find

$$\frac{du}{dx} = \frac{x}{a^2}, \quad \frac{du}{dy} = \frac{y}{b^2}, \quad \frac{du}{dz} = \frac{z}{c^2}, \quad \frac{d^2u}{dx^2} = \frac{1}{a^2}, \quad \frac{d^2u}{dy^2} = \frac{1}{b^2}, \quad \frac{d^2u}{dz^2} = \frac{1}{c^2}.$$

Then the equation which determines the value of q will become

$$\frac{x^2}{a^2(1 - qa^2)} + \frac{y^2}{b^2(1 - qb^2)} + \frac{z^2}{c^2(1 - qc^2)} = 0.$$

We shall have also

$$\frac{1}{\rho} = \pm q \left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} \right)^{-\frac{1}{2}},$$

and

$$\frac{(1 - qa^2) \cos \alpha}{x} = \frac{(1 - qb^2) \cos \beta}{y} = \frac{(1 - qc^2) \cos \gamma}{z} = \pm \left\{ \left(\frac{x}{1 - qa^2} \right)^2 + \left(\frac{y}{1 - qb^2} \right)^2 + \left(\frac{z}{1 - qc^2} \right)^2 \right\}^{-\frac{1}{2}}.$$

These formulæ will be sufficient to determine in each point of the ellipsoid the directions of the tangents to the principal sections, and the two principal radii of curvature. Moreover, if we subtract from the equation of the ellipsoid the equation which is used to determine the values of q , we get

$$\frac{x^2}{a^2 - \frac{1}{q}} + \frac{y^2}{b^2 - \frac{1}{q}} + \frac{z^2}{c^2 - \frac{1}{q}} = 1. \text{ Which equation}$$

shows, that if, after having calculated one of the maximum or minimum values of the variable q , a new ellipsoid is constructed, the semi-axis of which, designated by a_1 , b_1 , c_1 are determined by the equations

$$a_1^2 = a^2 - \frac{1}{q}, \quad b_1^2 = b^2 - \frac{1}{q}, \quad c_1^2 = c^2 - \frac{1}{q},$$

this new ellipsoid will also pass through the point (x, y, z) . It may be remarked, that the sections made by the coordinate planes in the proposed ellipsoid and in the new ellipsoid will have the same foci, since

$$a^2 - b^2 = a_1^2 - b_1^2, \quad a^2 - c^2 = a_1^2 - c_1^2, \quad b^2 - c^2 = b_1^2 - c_1^2.$$

2. Let us conceive that after having drawn, in the plane of x, y , a curve represented by the equation $y = f(x)$, we make this curve turn round the axis of x . It will generate a surface of revolution, in which the distance of the point (x, y, z) from the axis of x , which is $\sqrt{y^2 + z^2}$, will be equal, without taking into consideration the sign, to the ordinate of the generating curve. Therefore we shall have for every point of the surface

$$\sqrt{y^2 + z^2} = \pm f(x), \text{ or } y^2 + z^2 = (f(x))^2.$$

Integral Calculus.

We may assume $u = \frac{1}{2} \{ y^2 + z^2 - (f(x))^2 \}$, and we shall find then

$$\frac{du}{dz} = -f(x)f'(x), \frac{du}{dy} = y, \frac{du}{dz} = z, \frac{d^2u}{dx^2} = -\{ (f'(x))^2 + f(x)f''(x) \}, \frac{d^2u}{dy^2} = 1, \frac{d^2u}{dz^2} = 1.$$

Then the equation to determine the value of q , in the case where the variables x, y, z are separated, will become

$$\frac{\{f(x)f'(x)\}^2}{q + (f'(x))^2 + f(x)f''(x)} + \frac{y^2 + z^2}{q - 1} = 0. \text{ If then we substitute for } y^2 + z^2 \text{ its value } (f(x))^2, \text{ we shall}$$

$$\text{find } q = -\frac{f(x)f''(x)}{1 + (f'(x))^2}. \text{ We shall have besides, } r = \sqrt{\{y^2 + z^2 + (f(x)f'(x))^2\}} = \pm f(x) \sqrt{1 +}$$

$$(f'(x))^2\}. \text{ Consequently, } \rho = \pm \frac{\{1 + (f'(x))^2\}^{\frac{3}{2}}}{f''(x)}. \text{ This value of } \rho \text{ is the radius of curvature of the}$$

generating curve, which in fact coincides with one of the principal sections of the surface of revolution. As to the radius of curvature of the other principal section, it will be sufficient to determine it to take the three equations in q and t , which in the present case will be reduced to

$$\{q + (f'(x))^2 + f(x)f''(x)\}x_1 = t f(x)f'(x), (1 - q)y_1 = t y, (1 - q)z_1 = t z.$$

These three equations are verified by taking $q = 1, t = 0, x_1 = 0$, therefore the formula $\frac{1}{\rho} = \pm \frac{q}{r}$ will give $\rho =$

$\pm r = \pm f(x) \{1 + (f'(x))^2\}^{\frac{1}{2}}$, a value which is precisely that of the normal to the generating curve. Besides, since $x_1 = 0$, we shall have $\cos \alpha = 0$; and therefore it is obvious that the principal section corresponding to this radius of curvature will have for its tangent a straight line contained in a plane perpendicular to the axis of x .

(366.) The general formulæ we have obtained become simpler when the equation of the surface is resolved with respect to z , and reduced to the form $z = f(x, y)$.

Let us assume then

$$\frac{df(x, y)}{dx} = p, \frac{df(x, y)}{dy} = q, \frac{d^2f(x, y)}{dx^2} = r, \frac{d^2f(x, y)}{dx dy} = s, \frac{d^2f(x, y)}{dy^2} = t.$$

We shall have $u = f(x, y) - z$, and, consequently,

$$\frac{du}{dx} = p, \frac{du}{dy} = q, \frac{du}{dz} = -1, \frac{d^2u}{dx^2} = r, \frac{d^2u}{dy^2} = t, \frac{d^2u}{dz^2} = 0, \frac{d^2u}{dy dz} = 0, \frac{d^2u}{dz dx} = 0, \frac{d^2u}{dx dy} = s.$$

These values being substituted in the preceding formulæ will give $r_1 = \sqrt{1 + p^2 + q^2}$, $q = r \cos^2 \alpha + 2s \cos \alpha \cos \beta + t \cos^2 \beta$, $p \cos \alpha + q \cos \beta = \cos \gamma$, $r x_1^2 + 2s x_1 y_1 + t y_1^2 = \pm r$, $p x_1 + q y_1 = z_1$, r_1 and q_1 being the same quantities we have hitherto represented by r and q . The two last will be in the present hypothesis the two equations of the plane curve drawn in the tangent plane. Since the first of these two equations contains only the two coordinates x_1 and y_1 , it represents the projection of the curve on the plane of x, y , and as this equation belongs either to an ellipse, to two conjugate hyperbolas, to a system of two parallel lines, or to a system of two straight lines passing through the point (x, y, z) , it may be assumed that the curve in question will present the same circumstances.

As to the principal radii of curvature, they will be determined by means of the formula $x_1 dx_1 + y_1 dy_1 + z_1 dz_1 = 0$, from which the quantities dx_1, dy_1, dz_1 must be eliminated by means of the differential equations

$$(r x_1 + s y_1) dx_1 + (s x_1 + t y_1) dy_1 = 0, p dx_1 + q dy_1 = dz_1.$$

But if we first eliminate dz_1 between the first and the third of the three last equations, we find $(x_1 + p z_1) dx_1 + (y_1 + q z_1) dy_1 = 0$; and this being compared to the second equation gives

$$\frac{r x_1 + s y_1}{x_1 + p z_1} = \frac{s x_1 + t y_1}{y_1 + q z_1} = \frac{x_1(r x_1 + s y_1) + y_1(s x_1 + t y_1)}{x_1(x_1 + p z_1) + y_1(y_1 + q z_1)}$$

and then, since we have $r x_1^2 + 2s x_1 y_1 + t y_1^2 = \pm r_1$, $p x_1 + q y_1 = z_1$ and $\rho = x_1^2 + y_1^2 + z_1^2$, we shall get

$$\frac{r x_1 + s y_1}{(1 + p^2) x_1 + p q y_1} = \frac{s x_1 + t y_1}{(1 + q^2) y_1 + p q x_1} = \pm \frac{r_1}{\rho} = q_1;$$

or, which is the same thing,

$$[r - (1 + p^2) q_1] x_1 + (s - p q q_1) y_1 = 0, (s - p q q_1) x_1 + \{t - (1 + q^2) q_1\} y_1 = 0. \dots (a).$$

To obtain from these formulæ the value of q_1 , it is sufficient to eliminate x_1 and y_1 . We shall find thus,

$$(r - (1 + p^2) q_1) \{t - (1 + q^2) q_1\} - (s - p q q_1)^2 = 0, \text{ or } (1 + p^2 + q^2) q_1^2 - \{ (1 + p^2) t - 2 p q s + (1 + q^2) r \} q_1 + r t - s^2 = 0.$$

We have besides, $\frac{1}{\rho} = \pm \frac{q_1}{\sqrt{1 + p^2 + q^2}}$, and by combining the equation (a) with the equation $p x_1 + q y_1 = z_1$,

we find

$$\frac{x_1}{s - p q q_1} = \frac{y_1}{(1 + p^2) q_1 - r} = \frac{z_1}{p s - q r + q q_1} = \frac{\rho^{\frac{1}{2}} \sqrt{1 + p^2}}{\sqrt{\{ (1 + p^2) s - p q r \}^2 + (1 + p^2 + q^2) [(1 + p^2) q_1 - r]^2}};$$

Integral Calculus. and, consequently,

$$\frac{\cos \alpha}{s - p q q_1} = \frac{\cos \beta}{(1 + p^2) q_1 - r} = \frac{\cos \gamma}{p s - q r + q q_1} = \pm \frac{\sqrt{(1 + p^2)}}{\sqrt{\{(1 + p^2) s - p q r\}^2 + (1 + p^2 + q^2) [(1 + p^2 + q^2) q_1 - r]^2}}.$$

The two values of q_1 derived from the equation of the second degree, found above, are

$$q_1 = \frac{(1 + p^2) t - 2 p q s + (1 + q^2) r \pm \sqrt{\{(1 + p^2) t - 2 p q s + (1 + q^2) r\}^2 - 4 (1 + p^2 + q^2) (r t - s^2)}}{2 (1 + p^2 + q^2)}.$$

But the quantity under the radical is identically equal to

$$\frac{\{(1 + p^2) [(1 + p^2) t - (1 + q^2) r] + 2 p q [p q r - (1 + p^2) s]^2\} + 4 (1 + p^2 + q^2) [p q r - (1 + p^2) s]^2}{(1 + p^2)^2};$$

it follows that the two values of q_1 are essentially real. When $r t - s^2 > 0$, the equation from which those two values are derived shows that these values are of the same sign. Therefore in that case the two principal radii of curvature are in the same direction, and represent a maximum and a minimum of the variable ρ . If we had $r t - s^2 < 0$, the two values of q_1 would have contrary signs, and the principal radii of curvature would have opposite directions, and represent two minima of the variable ρ . Lastly, if we had $r t - s^2 = 0$ one of the values of $q_1 = 0$, and the maximum value of ρ becomes infinite; therefore the curvature of one of the principal sections would be equal to nothing, that is to say, that one of the principal sections would be a straight line. This takes place in each point of a *surface développable*; consequently, the values of r, s, t derived from the equation of such a surface will verify the formula $r t - s^2 = 0$, whatever be the values of x, y, z .

The two values of q_1 become equal in the case where the quantity under the radical becomes equal to nothing. This cannot take place unless we have, at the same time, $p q r - (1 + p^2) s = 0$, and $(1 + p^2) t - (1 + q^2) r = 0$,

or which is the same thing, $\frac{r}{1 + p^2} = \frac{s}{p q} = \frac{t}{1 + q^2}$. But in this case $q_1 = \frac{r}{1 + p^2} = \frac{s}{p q} = \frac{t}{1 + q^2}$. Therefore

all the values of the variables q_1 and ρ_1 become equal. In the same supposition the values of $\cos \alpha, \cos \beta, \cos \gamma$ become indeterminate.

In order that the principal radii of curvature should be equal and in opposite directions, it is necessary that the coefficient of the first power of q_1 should be equal to nothing, or $(1 + p^2) - 2 p q s + (1 + q^2) r = 0$.

It was Euler who first established the theory of the curvature of surfaces, and showed the relations which exist between the radii of curvature of the several sections made in a surface by normal planes. (See *Mémoires de l'Académie de Berlin*, 1760.)

(367.) We shall investigate now the values of the radii of curvature of the curves which may be drawn upon a given surface,

Let $u = 0$ be the equation of the surface, and let us suppose that on this surface a curve is drawn, passing through the point, the coordinates of which are represented by x, y, z . If s be the length of the arc of this curve, and ρ its radius of curvature corresponding to the point x, y, z , the cosines of the angles formed by this radius with

the axes of coordinates will be according to (316.) $\rho \frac{d^2 x}{d s^2}, \rho \frac{d^2 y}{d s^2}, \rho \frac{d^2 z}{d s^2}$. If through the same point (x, y, z) we

draw a normal to the surface, and if, for brevity sake, we suppose $r = \sqrt{\left\{\left(\frac{d u}{d x}\right)^2 + \left(\frac{d u}{d y}\right)^2 + \left(\frac{d u}{d z}\right)^2\right\}}$ the cosines of the angles formed by the normal, produced in a certain direction, and the axes will be respectively $\frac{1}{r} \frac{d u}{d x}, \frac{1}{r} \frac{d u}{d y}, \frac{1}{r} \frac{d u}{d z}$. Let now δ be the angle formed by the direction of the normal with the direction of the

radius of curvature, we shall have

$$\cos \delta = \frac{\rho}{r} \frac{\frac{d u}{d x} \frac{d^2 x}{d s^2} + \frac{d u}{d y} \frac{d^2 y}{d s^2} + \frac{d u}{d z} \frac{d^2 z}{d s^2}}{d s^2}.$$

Let, also, α, β, γ be the angles which the tangent to the curve forms with the three axes, we shall have, in making

$$q = \frac{d^2 u}{d x^2} \cos^2 \alpha + \frac{d^2 u}{d y^2} \cos^2 \beta + \frac{d^2 u}{d z^2} \cos^2 \gamma + 2 \frac{d^2 u}{d y d z} \cos \beta \cos \gamma + 2 \frac{d^2 u}{d z d x} \cos \gamma \cos \alpha + 2 \frac{d^2 u}{d x d y} \cos \beta \cos \alpha,$$

$$\frac{d u}{d x} \frac{d^2 x}{d s^2} + \frac{d u}{d y} \frac{d^2 y}{d s^2} + \frac{d u}{d z} \frac{d^2 z}{d s^2} = - q d s^2,$$

and, consequently,

$$\cos \delta = \frac{\rho}{r} q, \text{ or } \frac{\cos \delta}{\rho} = - \frac{q}{r}, \text{ or } \rho = - \frac{r}{q} \cos \delta.$$

The quantity r contained in the right side of this equation depends only on the position of the point (x, y, z) on the surface; the quantity q depends not only on this position but also on the direction of the tangent to the curve drawn upon the surface; and δ represents one of the two angles formed by the osculating plane of the curve, or,

Integral Calculus. which is the same thing, the angle formed by the principal normal to the curve with the normal to the surface. Part III. This understood, it is obvious that if together with the point, (x, y, z) the tangent to the curve and the osculating plane are given, the quantities q and r will be known, as well as the numerical value of $\cos \delta$, and the radius

of curvature will be determined by the above formula $\rho = -\frac{r}{q} \cos \delta$. Hence, since the quantities ρ and r are

quantities which are essentially positive, we may conclude that $\cos \delta$ and q will always be quantities of different signs. This remark will enable us to determine first the sign of $\cos \delta$, and consequently in what direction, on the principal normal, the radius of curvature should be taken.

From what precedes, results, first, that, if two curves drawn on the same surface through the same point, have at that point the same osculating plane, they will also have the same radius of curvature; secondly, that if these two curves have the same tangent without having the same osculating plane, their radii of curvature will be proportional to the cosines of the angles which their principal normals form with the normal to the surface. In the particular case where the osculating plane of a curve becomes a normal plane to the surface upon which the curve is drawn, we have $\cos \delta = \pm 1$, the sign being determined by the direction of the radius of curvature; and

we have therefore $\rho = \mp \frac{r}{q}$, which is the same formula we have already obtained in (363.) By comparing this value

of ρ with the general value $\rho' = -\frac{r}{q} \cos \delta$, we obtain a theorem which may be stated in the following manner.

Any curve whatever being drawn on a surface, if through the tangent to this curve, at a given point, (x, y, z) we conceive a normal plane to the surface, the radius of curvature of the curve will be equal to the product of the radius of curvature of the section corresponding to the normal plane, by the cosine of the angle formed by that plane and the osculating plane of the curve.

It is easy to verify the preceding theorem in several simple cases. Thus, for instance, if through a given point on a sphere, a great circle and a small circle are drawn, having, at that point, the same tangent, it is obvious that the radius of the small circle is equal to the radius of the sphere multiplied by the cosine of the acute angle formed by the two planes.

Let us consider for a second example the surface generated by the revolution, round the axis of x , of a curve drawn in the plane of x, y , and represented by the equation $y = f(x)$; and let it be required to find the radius of curvature ρ , corresponding to the point (x, y, z) in the curve of intersection of that surface, and of a normal plane to it, drawn through that point, perpendicularly to the generating curve passing through the same point. The acute angle which this normal plane makes with the plane passing through the point (x, y, z) perpendicularly to the axis of x , will clearly be equal to the angle τ of the generating curve, at the point (x, y, z) with the axis of x . Therefore, since the intersection of the second plane with the surface of revolution will be a circle, the radius of this circle will be equal to $\rho \cos \tau$. It is obvious, moreover, that this radius is equal in length to the ordinate $f(x)$ of the generating curve, and consequently, we shall have $\rho \cos \tau = \pm f(x)$; or, since

$$\cos \tau = \frac{1}{\sqrt{1 + (f'(x))^2}}, \quad \rho = \pm f(x) \sqrt{1 + (f'(x))^2},$$

which is a formula already obtained in Example 2. (365.)

Let us suppose now the equation of the surface resolved with respect to z , and reduced to the form $z = f(x, y)$, then if we assume $df(x, y) = p dx + q dy$, and $d^2f(x, y) = r dx^2 + 2s dx dy + t dy^2$, we shall have,

according to the preceding formulæ $\frac{\cos \delta}{\rho} = \frac{r \cos^2 \alpha + 2s \cos \alpha \cos \beta + t \cos^2 \beta}{\sqrt{1 + p^2 + q^2}}$, when δ will be the angle included

between the radius ρ and the normal, this normal being produced in such a manner as to form with the axis the angles λ, μ, ν , determined by the equations

$$\cos \lambda = \frac{p}{\sqrt{1 + p^2 + q^2}}, \quad \cos \mu = \frac{q}{\sqrt{1 + p^2 + q^2}}, \quad \cos \nu = \frac{r}{\sqrt{1 + p^2 + q^2}}.$$

(368.) Two surfaces are said to have a point of osculation, when, having a common point, they have at that point, not only the same tangent plane, but also their principal sections, for that point, contained in the same normal planes, and their principal radii of curvature equal, and in the same direction.

This understood, if we conceive any plane passing through the point of osculation of two surfaces, it is easy to conclude from what has been demonstrated in the preceding paragraph, that the two curves of intersection of that plane with the two surfaces, will have, at their common point, the same radii of curvature. Consequently the

formula $\frac{r \cos^2 \alpha + 2s \cos \alpha \cos \beta + t \cos^2 \beta}{\sqrt{1 + p^2 + q^2}}$ will have the same value for both surfaces, for any values of α and β

which satisfy the equation $\cos^2 \alpha + \cos^2 \beta + (p \cos \alpha + q \cos \beta)^2 = 1$. Moreover, the two surfaces having the same tangent plane at their common point, the quantities p and q , and consequently the denominator $\sqrt{1 + p^2 + q^2}$ will be the same for both surfaces; hence, the numerator $r \cos^2 \alpha + 2s \cos \alpha \cos \beta + t \cos^2 \beta$ must retain the same value for both surfaces; and as for the supposition $\cos \alpha = 0$, this numerator is reduced to $t \cos^2 \beta$, and for the supposition $\cos \beta = 0$, to $r \cos^2 \alpha$; it is easy to see that r, t , and s must have the same values for both surfaces. Therefore two surfaces, represented by two equations between rectangular coordinates

Integral
Calculus.

x, y, z have a point of osculation, the values $z, p = \frac{dz}{dx}, q = \frac{dz}{dy}, r = \frac{d^2z}{dx^2}, s = \frac{d^2z}{dx dy}, t = \frac{d^2z}{dy^2}$, have for the

coordinates of the common point, the same values in both surfaces. In general, the inverse of this proposition is obviously true; however, it might not be so, if the values of some of the quantities, p, q, r, s, t , were infinite.

(369.) From the preceding number, and from what has been demonstrated (363.), it clearly results, that in order that a point belonging to two surfaces should be a point of osculation, it is sufficient, 1. that the two surfaces should have, at that point, the same tangent plane; 2. that the conic section drawn on the tangent plane, any radius vector of which is equal to the radius of curvature of the normal section, made in the first surface, through the same radius vector of the conic section, should coincide with the analogous curve drawn on the same tangent plane, corresponding to the second surface. But a conic section is completely determined when the centre is known, as well as three lines drawn from the centre to three points of the curve. Moreover, when the radius of curvature of the curve of intersection of a surface by an oblique plane passing through one of its points is known, it is easy to find the radius of curvature of the normal section, which has the same tangent, at that point, as the preceding curve, and, consequently, the length of the radius vector for one of the points of the above-mentioned conic section. Hence, in order that the common point of two surfaces, which have at that point the same tangent plane, should be a point of osculation, it is sufficient that this point should be a point of osculation of the three pairs of curves of intersection of the surfaces by any three planes drawn through the same point. This condition will be satisfied, for instance, if the radii of curvature of the sections made in the first and second surfaces by three planes parallel to the coordinates planes are the same in length and direction.

(370.) Let $u = 0, v = 0$ be the equations of two curve surfaces, we shall have, by differentiating successively with respect to x and y , $\frac{du}{dx} + \frac{du}{dz} \frac{dz}{dx} = 0, \frac{du}{dy} + \frac{du}{dz} \frac{dz}{dy} = 0$; and by differentiating again each of these equations with respect to x and y ,

$$\begin{aligned} \frac{d^2u}{dx^2} + 2 \frac{d^2u}{dx dz} \frac{dz}{dx} + \frac{d^2u}{dz^2} \left(\frac{dz}{dx}\right)^2 + \frac{du}{dz} \frac{d^2z}{dx^2} &= 0, \\ \frac{d^2u}{dx dy} + \frac{d^2u}{dy dz} \frac{dz}{dx} + \frac{d^2u}{dx dz} \frac{dz}{dy} + \frac{d^2u}{dz^2} \frac{dz}{dx} \frac{dz}{dy} + \frac{du}{dz} \frac{d^2z}{dx dy} &= 0, \\ \frac{d^2u}{dy^2} + 2 \frac{d^2u}{dy dz} \frac{dz}{dy} + \frac{d^2u}{dz^2} \left(\frac{dz}{dy}\right)^2 + \frac{du}{dz} \frac{d^2z}{dy^2} &= 0; \end{aligned}$$

or, which is the same thing,

$$\begin{aligned} \frac{du}{dx} + p \frac{du}{dz} &= 0, & \frac{du}{dy} + q \frac{du}{dz} &= 0, & \frac{d^2u}{dx^2} + 2p \frac{d^2u}{dx dz} + p^2 \frac{d^2u}{dz^2} + q \frac{du}{dz} &= 0, \\ \frac{d^2u}{dy^2} + 2q \frac{d^2u}{dy dz} + q^2 \frac{d^2u}{dz^2} + t \frac{du}{dz} &= 0, & \frac{d^2u}{dx dy} + p \frac{d^2u}{dy dz} + q \frac{d^2u}{dx dz} + pq \frac{d^2u}{dz^2} + s \frac{du}{dz} &= 0 \end{aligned}$$

and, consequently,

$$\begin{aligned} p &= -\frac{\left(\frac{du}{dx}\right)}{\left(\frac{du}{dz}\right)}, & q &= -\frac{\left(\frac{du}{dy}\right)}{\left(\frac{du}{dz}\right)}, & r &= -\frac{\frac{d^2u}{dx^2} \left(\frac{du}{dz}\right)^2 - 2 \frac{d^2u}{dx dz} \frac{du}{dz} + \frac{d^2u}{dz^2} \left(\frac{du}{dx}\right)^2}{\left(\frac{du}{dz}\right)^3}, \\ s &= -\frac{\frac{d^2u}{dx dy} \left(\frac{du}{dz}\right)^2 - \frac{d^2u}{dy dz} \frac{du}{dz} \frac{du}{dx} - \frac{d^2u}{dx dz} \frac{du}{dy} \frac{du}{dz} + \frac{d^2u}{dz^2} \frac{du}{dx} \frac{du}{dy}}{\left(\frac{du}{dz}\right)^3}, \\ t &= -\frac{\frac{d^2u}{dy^2} \left(\frac{du}{dz}\right)^2 - 2 \frac{d^2u}{dy dz} \frac{du}{dz} \frac{du}{dy} + \frac{d^2u}{dz^2} \left(\frac{du}{dy}\right)^2}{\left(\frac{du}{dz}\right)^3}. \end{aligned}$$

By substituting v for u in these formulæ we shall have the values of p, q, r, s, t for the second surface, and by equalling those last values to the preceding five equations which will express the conditions which ought to be satisfied in order that the point (x, y, z) should be a point of osculation of the two surfaces.

(371.) When two curves in space have a common point, and have, at that point, a common tangent, they are said, as it has already been stated, to touch each other. Now the contact between two curves in space which touch each other, is said to be of various degrees or orders according to circumstances we are going to examine, and which are analogous to those which establish the different orders of contact between two plane curves, and which have been discussed (291.) and following.

If from the point of contact of two curves in space, touching each other, as a centre, with an infinitely small radius, the length of which shall be designated by i , we conceive a sphere to be described, the surface of this

Part III.
Integral
Calculus.

Part III.

sphere will meet the two curves in two points very near each other. The distance of these two points will be the distance of the two curves at a distance i of the point of contact. This distance is clearly measured by the chord of the arc of a great circle of the described sphere, contained between the two curves. The radii drawn from the centre of the sphere to the extremities of this small arc will include a small angle, which we shall call w , and each of these radii will form a very small angle with the common tangent to both curves. The arc of the great circle, above mentioned, will be equal to $i w$, and the chord to $2 i \sin \frac{w}{2}$.

It is obvious now that the less the length of the chord $2 i \sin \frac{w}{2}$ is, the nearer to each other will be the two curves at the distance i from the point of contact. But the degree of smallness of the chord $2 i \sin \frac{w}{2}$, for the same value of i , will depend on the degree of smallness of $\sin \frac{w}{2}$, or of w . Hence, it is natural to take the degree of smallness of w , considered as a function of i , to indicate the order of contact of the two curves.

Let us then suppose that w , considered as a function of i , is an infinitely small quantity of the order α . Since

the ratio $\frac{\sin \frac{1}{2} w}{\frac{1}{2} w}$ has unity for its limit, the product $w \cdot \frac{\sin \frac{w}{2}}{\frac{w}{2}} = 2 \sin \frac{w}{2}$, will also be an infinitely small quantity

of the order α , and consequently the expressions $i w$, and $2 i \sin \frac{w}{2}$, infinitely small quantities of the order $\alpha + 1$.

We shall, therefore, say, that the order of contact of two curves in space, which touch each other at a point P , is less by one than the order of the infinitely small quantity which represents the distance between two points Q and R , situated on the two curves, at equal distances from the point of contact, and the distance of which to that point is an infinitely small quantity of the first order.

It is necessary to remark, that the line $Q R$, being the base of an isosceles triangle, opposite to the very small angle w of that triangle, will nearly be perpendicular to the sides $P Q$, $P R$, and consequently to the common tangent to the two curves. It is also obvious that the surface of this small triangle is equal to $\frac{i^2}{2} \sin w$, that is to an infinitely small quantity of the order $\alpha + 2$.

Let us imagine now that the two curves and the triangle $P Q R$ are projected on a plane not very nearly perpendicular to the plane of the triangle $P Q R$. The projections of the two curves will have a common tangent, and therefore will touch each other. Let the three points which are the projections of P , Q , and R be denoted by $P' Q' R'$. Let δ be the angle of the two planes $P Q R$, $P' Q' R'$, and ϕ , χ , and ψ be the angles which the lines $P Q$, $P R$, $Q R$ make respectively with their projections $P' Q'$, $P' R'$, $Q' R'$. We shall have

$$P' Q' = P Q \cos \phi = i \cos \phi, \quad P' R' = P R \cos \chi = i \cos \chi, \quad Q' R' = Q R \cos \psi = 2 i \sin \frac{w}{2} \cos \psi,$$

and also by a well-known theorem, the surface of

$$P' Q' R' = P Q R \cos \delta = \frac{1}{2} i^2 \sin w \cos \delta = i^2 \sin \frac{1}{2} w \cos \frac{1}{2} w \cos \delta.$$

Moreover, the sine of the angle $P' Q' R'$ is equal to the double of the surface of the triangle $P' Q' R'$ divided by the rectangle $P' Q' \times Q' R'$ of the sides including the angle. Hence, $\sin P' Q' R' = \frac{\cos \frac{1}{2} w \cos \delta}{\cos \phi \cos \psi}$, and the product of this sine by the line $P' Q'$, or the length of the perpendicular drawn from the P' on the line $Q' R'$, will be represented by $\frac{i \cos \frac{1}{2} w \cos \delta}{\cos \psi}$. But the value of w being very small, and those of the angles ϕ , χ , ψ ,

being sensibly different from $\frac{\pi}{2}$, the quantities $\cos \frac{1}{2} w$, $\cos \delta$, $\cos \phi$, $\cos \chi$, $\cos \psi$, will have values which will not be infinitely small. Consequently, the line $Q' R'$ will be an infinitely small quantity of the order $\alpha + 1$, the angle $P' Q' R'$, which it forms with the line $P' Q'$, will be sensibly different from nothing, and the distance of the point P' to the line $Q' R'$ an infinitely small quantity of the first order. Besides, the common tangent to the projections of the two curves will nearly coincide with the line $P' Q'$, and will therefore form with the secant $Q' R'$ a finite angle. Consequently, the projections of the two curves will have, as well as the curves in space, a contact of the order α .

If the plane of the triangle $P' Q' R'$ were sensibly perpendicular to the plane of the triangle $P Q R$, and if, at the same time, the angles of the lines $P Q$, $P R$, and, consequently, the angle of the common tangent to the two curves with this same plane $P' Q' R'$, were sensibly different from nothing, the projections of the two curves on the plane $P' Q' R'$ would have a constant of an order equal or superior to α . For then, the distances $P' Q'$, $P' R'$, would still be infinitely small quantities of the first order, and the distance $Q' R'$ an infinitely small quantity of the order $\alpha + 1$, or of a superior order. If, therefore, in this hypothesis, we imagine that a sphere be described having P' for its centre, and $P' Q'$ for its radius, and that it cut the projection of the second curve in a point s' ,

Integral
Calculus.

the line $R'S'$ will very nearly be parallel to the common tangent to the projections of the two curves in space, since its two extremities will be situated on one of these curves at infinitely small distances from P' . On the contrary, the line $Q'S'$, basis of the small isosceles triangle $P'Q'S'$, will nearly be perpendicular to the same tangent. Therefore the triangle $Q'R'S'$ will nearly be right-angled at S' , and consequently the line $Q'S'$ nearly equal to $Q'R' \cos R'Q'S'$, that is to say, to an infinitely small quantity of the order $a+1$, or even of an order superior to that number; and, hence, the order of contact of the two projections will be equal or superior to the number a .

Let us suppose now that the two curves in space are projected on the planes of the x, y , and x, z , and, moreover, that the angle of the common tangent to the curves and the axis of x differs sensibly from a right angle. This tangent will then be neither nearly perpendicular to the plane of x, y , nor to the plane of x, z . These coordinate planes, besides, cannot be at the same time nearly perpendicular to the plane of the triangle PQR , for if it were so, their line of intersection, that is to say the axis of x , would necessarily form an angle nearly equal to 90° with the lines PQ, PR , and, consequently, with the common tangent to the two curves. This understood, it results from the principles above established, that the contact of the projections of the two curves on the plane of the x, y , and that of the projections of the two curves on the plane of the x, z , will always be of the order a , or of an order superior to the number a , and on one of the two coordinate planes, at least, of an order equal to a .

Hence, to obtain the order of contact of two curves which touch each other in a point where the common tangent to the two curves does not form a right angle with the axis of x , it is sufficient to find the numbers which indicate the orders of contact of the projections of the two curves on the planes of x, y , and on the plane of x, z . Each of these numbers, if they be equal, or the smallest of the two, if they be unequal, will indicate the order of contact of the two curves in space. This is equally true whether x, y, z represent rectangular or oblique coordinates.

(372.) Let us suppose that two curves, each represented by two equations between the three rectangular or oblique coordinates x, y, z , have a common point, corresponding to the abscissa x , and, at that point, a common tangent not perpendicular to the axis of x , with a contact of the order a . Let us, moreover, take x for the independent variable, and let $y', y'', y''', \&c.$, and $z', z'', z''', \&c.$, designate the successive derived functions of y and z , considered as functions of x . Then, according to the preceding principles, and to what has been before stated with respect to the contact of two plane curves, the quantities $y, y', y'', \dots y^{(n)}, z, z', z'', \dots z^{(n)}$, will assume the same values, for the value of x , corresponding to the point of contact in both curves, n being equal to a , or the integer immediately superior to a , whilst the values of $y^{(n+1)}$, and $z^{(n+1)}$, or at least the values of one of them, will be different.

Hence, when the common tangent to the two curves is not perpendicular to the axis of x , and that it is known that the order of contact is an integer, it is sufficient to determine this order to find the greatest value which can be taken for n , in choosing this number in such a manner that the values of the quantities $y, y', y'', \dots y^{(n)}, z, z', z'', \dots z^{(n)}$, corresponding to the abscissa of the point of contact, should be the same in passing from one curve to the other. This value of n precisely indicates the required order of contact. If the common tangent to the two curves formed a right angle with the axis of x , it could not be, at the same time, perpendicular to the plane of x, y , and to the plane of x, z . Consequently, it would form an angle differing from 90° with the axis of y , and with the axis of z , or, at least, with one of these two axes. Therefore to find, in this hypothesis, the order of contact of the two given curves, it would be sufficient to substitute the axis of y , or the axis of z for the axis of x , and the plane of y, z for the plane of x, z or x, y .

(373.) The order of contact of two curves may also be obtained by a different process, which has on the preceding the advantage of presenting no restriction.

Let P be the common point of the two curves which touch each other. Let also Q and R be two other points situated on the first and second curve, at equal distances from the point of contact, which distance is infinitely small and represented by i . Moreover, from the point P , on the second curve, let an arc PS be taken, having the same length and direction as the arc PQ , and terminating at a point S . The secant QS , as well as the secant QR , will nearly be perpendicular to the common tangent; and the chord RS nearly be parallel to the same tangent. Consequently, in the triangle QRS , the sides QR and QS will form angles with the third side RS which will very little differ from a right angle. Therefore the ratio between the two first sides, or, which is the same thing, the ratio between the sines of the opposite angles, will very little differ from one, and, designating by I an infinitely small quantity, we shall have $QS = QR(1 + I) = (1 + I) 2i \sin \frac{w}{2}$. But the limit of

the ratio between the arc PQ and the chord PQ is also equal to one, therefore if j designate another infinitely small quantity we shall have $\text{arc } PQ = (1 + j)i$. This understood, let us suppose that the two curves have a contact of the order a , and that i is an infinitely small quantity of the first order. It is clear that the arc PQ will also be an infinitely small quantity of the first order, whilst the distance QS will be an infinitely small quantity of the order $a+1$; and the order of this distance will not vary, if we suppose that the arc PQ , or a quantity such as the arc PQ , remains an infinitely small quantity of the first order.

Hence, to obtain the order of contact of two curves which touch each other. It suffices to find the number which represents the order of the infinitely small distance, between the extremities of two equal lengths taken on the two curves from the point of contact, in the case where these same lengths become infinitely small quantities of the first order. This number minus one indicates in every case the order of contact.

Let i be the infinitely small quantity which represents each of the two equal lengths above mentioned. Let x, y, z , and x_1, y_1, z_1 , be the coordinates of the extremities of those lengths, and l the length of the line which joins

Integral Calculus. them. Then $l = \sqrt{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}$. If i be an infinitely small quantity of the first order, and if α be the order of contact of the two curves, according to (371.) l will be an infinitely small quantity of the order $\alpha + 1$, and consequently, $l^2 = (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2$ an infinitely small quantity of the order $2\alpha + 2$. Therefore each of the differences $x - x_1, y - y_1, z - z_1$ must be infinitely small quantities of the order $\alpha + 1$, or, at least, one of them, the others being then infinitely small quantities of higher orders. The same conclusion results also from the remark that the quantities $x - x_1, y - y_1, z - z_1$, are the projections of l on the three axes, and that, generally, an infinitely small distance and its projection on any axis whatever are quantities of the same order, only the order of the projection may be higher than the order of the distance in the case where that distance is nearly perpendicular to the axis, and it is obvious that this circumstance cannot occur at the same time for the three axes of x, y, z .

Let n represent the integer equal or immediately superior to α . Then since i being considered as an infinitely small quantity of the first order, one of the three differences $x - x_1, y - y_1, z - z_1$, must be of the order $\alpha + 1$; the two others being of the same or of an higher order, it follows that if we take i for the independent variable.

$$x - x_1, \frac{d(x - x_1)}{di}, \frac{d^2(x - x_1)}{di^2}, \dots, \frac{d^n(x - x_1)}{di^n}, \quad y - y_1, \frac{d(y - y_1)}{di}, \frac{d^2(y - y_1)}{di^2}, \dots, \frac{d^n(y - y_1)}{di^n},$$

$$z - z_1, \frac{d(z - z_1)}{di}, \frac{d^2(z - z_1)}{di^2}, \dots, \frac{d^n(z - z_1)}{di^n},$$

will vanish with the quantity i , whilst, on the contrary, each of the quantities

$$\frac{d^{n+1}(x - x_1)}{di^{n+1}}, \quad \frac{d^{n+1}(y - y_1)}{di^{n+1}}, \quad \frac{d^{n+1}(z - z_1)}{di^{n+1}},$$

or at least one of them, will not vanish for $i = 0$. Let s and s_1 be the arcs of the first and second curve beginning at the point of contact, and terminating respectively at the points x, y, z , and x_1, y_1, z_1 . Let us also suppose that these arcs are in the same direction as the arc i . Since the three variables i, s , and s_1 , will differ from each other by constant quantities, we shall have $di = ds = ds_1$, and, with respect to the first curve, we may take for the independent variable s instead of i , and with respect to the second curve s_1 instead of i . This understood, the quantities written above, which were said to vanish for $i = 0$, will become

$$x - x_1, \frac{dx}{ds} - \frac{dx_1}{ds_1}, \frac{d^2x}{ds^2} - \frac{d^2x_1}{ds_1^2}, \dots, \frac{d^nx}{ds^n} - \frac{d^nx_1}{ds_1^n}, \quad y - y_1, \frac{dy}{ds} - \frac{dy_1}{ds_1}, \frac{d^2y}{ds^2} - \frac{d^2y_1}{ds_1^2}, \dots, \frac{d^ny}{ds^n} - \frac{d^ny_1}{ds_1^n},$$

$$z - z_1, \frac{dz}{ds} - \frac{dz_1}{ds_1}, \frac{d^2z}{ds^2} - \frac{d^2z_1}{ds_1^2}, \dots, \frac{d^nz}{ds^n} - \frac{d^nz_1}{ds_1^n},$$

and

$$\frac{d^{n+1}x}{ds^{n+1}} - \frac{d^{n+1}x_1}{ds_1^{n+1}}, \quad \frac{d^{n+1}y}{ds^{n+1}} - \frac{d^{n+1}y_1}{ds_1^{n+1}}, \quad \frac{d^{n+1}z}{ds^{n+1}} - \frac{d^{n+1}z_1}{ds_1^{n+1}}.$$

In making each of the differences in the first line equal to nothing, we shall form the equations

$$x = x_1, \quad \frac{dx}{ds} = \frac{dx_1}{ds_1}, \quad \frac{d^2x}{ds^2} = \frac{d^2x_1}{ds_1^2}, \dots, \frac{d^nx}{ds^n} = \frac{d^nx_1}{ds_1^n}, \quad y = y_1, \quad \frac{dy}{ds} = \frac{dy_1}{ds_1}, \quad \frac{d^2y}{ds^2} = \frac{d^2y_1}{ds_1^2}, \dots, \frac{d^ny}{ds^n} = \frac{d^ny_1}{ds_1^n},$$

$$z = z_1, \quad \frac{dz}{ds} = \frac{dz_1}{ds_1}, \quad \frac{d^2z}{ds^2} = \frac{d^2z_1}{ds_1^2}, \dots, \frac{d^nz}{ds^n} = \frac{d^nz_1}{ds_1^n},$$

which must be verified by the coordinates of the point of contact of the two given curves, whilst, for the same point, each of the three differences between the partial differential coefficients of the order $n + 1$, or, at least, one of them, must obtain a value different from zero.

Hence we may conclude that, when two curves have a common point, and touch each other at that point, if the coordinates x, y, z , of each of them be considered as functions of the arc s , taken for the independent variable, and if we suppose that this arc s is taken on each curve, in such a manner that its prolongation beyond the point of contact be in the same direction in both cases; not only for that point of contact, the variables x, y, z , and

their first differential coefficients $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$, will not change their values in passing from the first curve to the

second, but it will also be the same thing with respect to the differential coefficients $\frac{d^2x}{ds^2}, \frac{d^2y}{ds^2}, \frac{d^2z}{ds^2}, \dots, \frac{d^2y}{ds^2}, \frac{d^2z}{ds^2}, \dots$

$\frac{d^3x}{ds^3}, \frac{d^3y}{ds^3}, \frac{d^3z}{ds^3}, \dots$, up to that the order of which will be indicated by the integer equal or immediately superior to the order of contact. These differential coefficients will be the last which will verify the above-stated condition, so that the three following of the next order, or at least one of the three following, will change value in passing from one curve to another.

Hence, without repeating almost the whole of the preceding paragraph, it is easy to see what will happen when the two curves will have a contact of the n^{th} order, n designating an integer.

(374.) Let $r = f(x, y, z)$ be any function whatever of the three variables x, y, z . If we consider these variables as functions of s , r will also be a function of s , and we shall find

Integral
Calculus.

$$\begin{aligned}\frac{dr}{ds} &= \frac{d \cdot f(x, y, z)}{dx} \cdot \frac{dx}{ds} + \frac{d \cdot f(x, y, z)}{dy} \cdot \frac{dy}{ds} + \frac{d \cdot f(x, y, z)}{dz} \cdot \frac{dz}{ds}, \\ \frac{d^2 r}{ds^2} &= \frac{d \cdot f(x, y, z)}{dx} \frac{d^2 x}{ds^2} + \frac{d \cdot f(x, y, z)}{dy} \frac{d^2 y}{ds^2} + \frac{d \cdot f(x, y, z)}{dz} \frac{d^2 z}{ds^2} + \frac{d^2 f(x, y, z)}{dx^2} \cdot \frac{dx^2}{ds^2} + \frac{d^2 f(x, y, z)}{dy^2} \cdot \frac{dy^2}{ds^2} \\ &\quad + \frac{d^2 f(x, y, z)}{dz^2} \cdot \frac{dz^2}{ds^2} + \frac{2 d^2 f(x, y, z)}{dy dz} \cdot \frac{dy dz}{ds^2} + \frac{2 d^2 f(x, y, z)}{dx dz} \cdot \frac{dx dz}{ds^2} + \frac{2 d^2 f(x, y, z)}{dy dx} \cdot \frac{dy dx}{ds^2}, \\ &\dots\dots\dots \\ \frac{d^n r}{ds^n} &= \frac{d \cdot f(x, y, z)}{dx} \cdot \frac{d^n x}{ds^n} + \frac{d \cdot f(x, y, z)}{dy} \cdot \frac{d^n y}{ds^n} + \frac{d \cdot f(x, y, z)}{dz} \cdot \frac{d^n z}{ds^n} + \&c.\end{aligned}$$

From these equations, and what precedes, it easily results, that if the two curves have a contact of the n^{th} order, each of the quantities

$$r = f(x, y, z), \quad \frac{dr}{ds} = \frac{df(x, y, z)}{ds}, \quad \frac{d^2 r}{ds^2} = \frac{d^2 \cdot f(x, y, z)}{ds^2}, \dots, \quad \frac{d^n r}{ds^n} = \frac{d^n \cdot f(x, y, z)}{ds^n},$$

will have the same values, for the point of contact, in passing from one curve to the other. It is what will happen, for instance, if we take for r the radius drawn from the origin of coordinates to the point x, y, z ; in which case $r = \sqrt{(x^2 + y^2 + z^2)}$. At the same time the value of $\frac{d^{n+1} r}{ds^{n+1}} = \frac{d^{n+1} f(x, y, z)}{ds^{n+1}}$, determined by the equation

$$\frac{d^{n+1} f(x, y, z)}{ds^{n+1}} = \frac{d \cdot f(x, y, z)}{dx} \cdot \frac{d^{n+1} x}{ds^{n+1}} + \frac{d \cdot f(x, y, z)}{dy} \cdot \frac{d^{n+1} y}{ds^{n+1}} + \frac{d \cdot f(x, y, z)}{dz} \cdot \frac{d^{n+1} z}{ds^{n+1}} + \&c.$$

will generally change in passing from one curve to the other, because this will be the case for at least one of the quantities $\frac{d^{n+1} x}{ds^{n+1}}, \frac{d^{n+1} y}{ds^{n+1}}, \frac{d^{n+1} z}{ds^{n+1}}$. Nevertheless, the contrary might take place in some particular cases,

of, for instance, the values of x, y, z , relative to the point of contact, reduced to nothing, in the value of $\frac{d^{n+1} r}{ds^{n+1}}$, the coefficients of the three above coefficients of the $n+1^{\text{th}}$ order, or, at least, the coefficient of that, among the three, the value of which changes in passing from one curve to the other. The same remark applies to the differential

coefficients $\frac{d^{n+2} r}{ds^{n+2}}, \frac{d^{n+3} r}{ds^{n+3}}$.

(375.) Let us suppose now that it is required to take r instead of s for the independent variable. Then we may conceive that the coordinates x, y, z , being always functions of s , s becomes a function of r ; and we shall obtain by successive differentiations of the equation $r = f(x, y, z)$,

$$\begin{aligned}1 &= \frac{df(x, y, z)}{ds} \cdot \frac{ds}{dr}, \quad 0 = \frac{d \cdot f(x, y, z)}{ds} \cdot \frac{d^2 s}{dr^2} + \frac{d^2 f(x, y, z)}{ds^2} \cdot \left(\frac{ds}{dr}\right)^2, \\ &\&c. \\ 0 &= \frac{d \cdot f(x, y, z)}{ds} \cdot \frac{d^n s}{dr^n} + \&c.\end{aligned}$$

These formulæ, combined with the preceding rule to find the order of contact of two curves in space, show that if these curves have a contact of the n^{th} order, the quantities $\frac{ds}{dr}, \frac{d^2 s}{dr^2}, \&c. \dots, \frac{d^n s}{dr^n}$, will have the same values for the coordinates of the point of contact in passing from one curve to the other. If, therefore, we substitute the values of these quantities in the following formulæ,

$$\begin{aligned}\frac{dx}{dr} &= \frac{dx}{ds} \cdot \frac{ds}{dr}, \quad \frac{d^2 x}{dr^2} = \frac{dx}{ds} \cdot \frac{d^2 s}{dr^2} + \frac{d^2 x}{ds^2} \cdot \left(\frac{ds}{dr}\right)^2, \quad \&c. \dots, \quad \frac{d^n x}{dr^n} = \frac{dx}{ds} \cdot \frac{d^n s}{dr^n} + \&c., \\ \frac{dy}{dr} &= \frac{dy}{ds} \cdot \frac{ds}{dr}, \quad \frac{d^2 y}{dr^2} = \frac{dy}{ds} \cdot \frac{d^2 s}{dr^2} + \frac{d^2 y}{ds^2} \cdot \left(\frac{ds}{dr}\right)^2, \quad \&c. \dots, \quad \frac{d^n y}{dr^n} = \frac{dy}{ds} \cdot \frac{d^n s}{dr^n} + \&c., \\ \frac{dz}{dr} &= \frac{dz}{ds} \cdot \frac{ds}{dr}, \quad \frac{d^2 z}{dr^2} = \frac{dz}{ds} \cdot \frac{d^2 s}{dr^2} + \frac{d^2 z}{ds^2} \cdot \left(\frac{ds}{dr}\right)^2, \quad \&c. \dots, \quad \frac{d^n z}{dr^n} = \frac{dz}{ds} \cdot \frac{d^n s}{dr^n} + \&c.,\end{aligned}$$

it will become evident that, if the two curves have a contact of the order n , and if we take $r = f(x, y, z)$ for the independent variable, not only the coordinates x, y, z , but also their derived functions,

$$\frac{dx}{dr}, \frac{d^2 x}{dr^2}, \dots, \frac{d^n x}{dr^n}, \quad \frac{dy}{dr}, \frac{d^2 y}{dr^2}, \dots, \frac{d^n y}{dr^n}, \quad \frac{dz}{dr}, \frac{d^2 z}{dr^2}, \dots, \frac{d^n z}{dr^n},$$

will have the same values, corresponding to the point of contact, in passing from one curve to the other. It is, however, necessary to exclude from this result some particular cases, which might occur, and in which the values

Integral
Calculus.

Part III.

of the above derived functions would present themselves, for both curves, under the form $\frac{0}{0}$, or $\frac{\infty}{\infty}$, and would,

however, vary in passing from one curve to the other. In every other case the variable r may be, as it has just been demonstrated, substituted to the variable s . Only, it is necessary to remark, that it is not possible then to affirm

that, for the point of contact, one, at least, of the three differential coefficients, $\frac{d^{n+1}x}{dr^{n+1}}$, $\frac{d^{n+1}y}{dr^{n+1}}$, $\frac{d^{n+1}z}{dr^{n+1}}$, will change value in passing from one curve to the other.

(376.) Supposing again that the order of contact of the two curves is n , and that r is taken for the independent variable, let p, q , &c. represent new functions of the coordinates x, y, z . We shall have

$$\begin{aligned}\frac{dp}{dr} &= \frac{dp}{dx} \cdot \frac{dx}{dr} + \frac{dp}{dy} \cdot \frac{dy}{dr} + \frac{dp}{dz} \cdot \frac{dz}{dr}, \\ \frac{d^2p}{dr^2} &= \frac{dp}{dx} \cdot \frac{d^2x}{dr^2} + \frac{dp}{dy} \cdot \frac{d^2y}{dr^2} + \frac{dp}{dz} \cdot \frac{d^2z}{dr^2} + \frac{d^2p}{dx^2} \cdot \frac{dx^2}{dr^2} + \frac{d^2p}{dy^2} \cdot \frac{dy^2}{dr^2} + \frac{d^2p}{dz^2} \cdot \frac{dz^2}{dr^2} \\ &\quad + 2 \frac{d^2p}{dydz} \cdot \frac{dydz}{dr^2} + 2 \frac{d^2p}{dzdx} \cdot \frac{dzdx}{dr^2} + 2 \frac{d^2p}{dxdy} \cdot \frac{dxdy}{dr^2}, \\ &\text{&c.} \dots \dots \dots\end{aligned}$$

$$\frac{d^n p}{dr^n} = \frac{dp}{dx} \frac{d^n x}{dr^n} + \frac{dp}{dy} \frac{d^n y}{dr^n} + \frac{dp}{dz} \frac{d^n z}{dr^n} + \text{&c.}$$

Hence it is obvious that, in passing from one curve to the other, the differential coefficients $\frac{dp}{dr}, \frac{d^2p}{dr^2}, \dots, \frac{d^n p}{dr^n}$,

and even the differentials $dp, d^2p, \dots, d^n p$, will have the same values. Similar conclusions would be obtained by substituting the function q for the function p . We might also take p or q for the independent variable. Thus, for instance, we might prove, that the differentials $dq, d^2q, \dots, d^n q$, and $dr, d^2r, \dots, d^n r$, taken with respect to p considered as the independent variable, do not vary, for the coordinates of the point of contact, in passing from one curve to the other.

If we suppose $r = x, p = y, q = z$, these last conditions would become identical with those found above.

We may now remark, that it is always possible to choose the axis of x in such a manner as it should form with the common tangent to the two curves an angle differing from a right angle. This understood, from what has been demonstrated in the last paragraphs, and from what has been stated (311.), it readily follows that if a curve have a contact of the second order, or of a higher order, with another curve, each of them is an osculating curve of the other, and reciprocally that two curves which have an osculation have a contact, at least of the second order.

(377.) Let us consider now two curved surfaces which touch each other at a point P . If through this point we imagine a normal plane to the two surfaces, the two lines of intersection will be tangents to each other; and if we imagine this plane to turn round the normal, the two lines will generally change both their position and form. As to the number which will represent the order of contact of these two lines, it may remain constantly the same, or change value with the position of the normal plane. Now, this number, when it is constant, or its minimum value, in the contrary case, is used to measure what is called the order of contact of the two surfaces. Let a be this order; and let us suppose that the two surfaces being cut by any plane passing through the normal, Q and R represent the points where the curves of intersection are met by an arc of a circle having P for its centre, and an infinitely small quantity of the first order, designated by i , for its radius. The distance QR , which varies with the position of the normal plane, will be an infinitely small quantity, the order of which will be equal to a constant or variable number, the minimum of which will be $a + 1$.

Let us conceive now that through the point Q , situated on the first surface, a secant is drawn in a direction parallel to a line which forms with the plane, tangent at both surfaces, an angle δ having a value sensibly different from 0, but inferior, or at most equal to $\frac{\pi}{2}$. Let us, moreover, suppose, that this secant meets the other surface

in a point s . In the triangle QRS , the side RS will nearly be parallel to the tangent plane, since it joins two points of the second surface, situated very near the point of contact. Hence it will form with the sides QR, QS , finite angles, the first of which will very little differ from a right angle, whilst the second will be equal or superior to δ . Therefore, if we designate by I an infinitely small quantity, and by Δ an angle greater

than δ , and less than 90° , we shall have $QS = \frac{\sin(\frac{\pi}{2} + I)}{\sin \Delta} QR$. From this last formula it results that the

infinitely small distance QS will be, for all the positions of the normal plane, of the same order as the distance QR . Moreover, since the ratio between the perpendicular drawn from the point P on the line QS and the line $PQ = i$ will be the sine of the angle PQS , formed by the line QS with a line PQ sensibly parallel to the tangent plane, and consequently a finite quantity, this perpendicular will clearly be an infinitely small quantity of the first order. From these various remarks the following proposition is readily deduced.

The order of contact of two surfaces which touch each other at a point P , is inferior by one unity to the value or to the minimum value of the number which represents the order of the infinitely small distance of the points Q and S , where those surfaces are met by a secant, which forms with the tangent plane, common to the two

surfaces, a finite angle, the distance of the point of contact to that secant being considered as an infinitely small quantity of the first order.

(378.) Let us suppose now that a plane passing through the point P forms a finite angle with the tangent plane. The intersections of this plane with the two surfaces will be two new curves. We may imagine that the secant above mentioned, coincides with a line situated in this same plane, and then it will easily follow, that when two surfaces have, at a given point, a contact of the order a , the intersections with both surfaces of every normal or oblique plane which forms a finite angle with the tangent plane, common to both surfaces, are two curves which have a contact of the order a , or of a superior order.

It is important to remark here, that not only the two sections above mentioned may have a contact of an order much higher than the number a , but that even, in certain cases, they may coincide. Then the number which represents the order of contact of the two sections becomes infinite. These two sections, in some cases, are reduced to one straight line. We may take as an instance the straight line common to two conical or cylindrical surfaces which touch each other.

(379.) If the two surfaces are represented by two equations between the coordinates x, y, z , and if the tangent plane drawn through the point of contact of the two surfaces is not very nearly parallel to the axis of z , then supposing the secant QS parallel to this same axis, we shall immediately conclude the following proposition.

To obtain the order of contact of two surfaces which touch each other at a given point where the tangent plane to the two surfaces is not parallel to the axis of z , it is sufficient to draw an ordinate very near the point of contact, and to find the value, or the minimum value of the number constant, or variable, which represents the order of the infinitely small portion of ordinate contained between the two surfaces, supposing that the distance of the point of contact to that ordinate is considered as an infinitely small quantity of the first order. This value, or this minimum value decreased by one order indicates the order of contact.

Let $z = f(x, y)$ and $z = F(x, y)$ be the equations of the two curved surfaces. They will have a common point corresponding to a system of given values of the variables x, y , and, at that point, a tangent plane common to both, not parallel to the axis of z , if for the proposed values of x and y the equations $z = f(x, y)$ and $z = F(x, y)$ give equal and finite values, not only for the ordinate z , but also for its partial differential coefficients $p = \frac{dz}{dx}$, $q = \frac{dz}{dy}$, so that the equations

$$f(x, y) = F(x, y), \quad \frac{df(x, y)}{dx} = \frac{dF(x, y)}{dx}, \quad \frac{df(x, y)}{dy} = \frac{dF(x, y)}{dy},$$

should be verified, and that both sides of each should take finite values. In this hypothesis the difference $F(x, y) - f(x, y)$ will vanish for the values of x and y relative to the point of contact, and will become infinitely small when the variables x and y will receive infinitely small increments $\Delta x, \Delta y$; and if we consider the distance $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ as being an infinitely small quantity of the first order, the order of the infinitely small quantity which will represent the new value of $F(x, y) - f(x, y)$ will be higher by one unity than the order of contact of the two surfaces. It is necessary, moreover, to remark that the expression $\sqrt{(\Delta x)^2 + (\Delta y)^2}$ will be an infinitely small quantity of the first order, if each of the increments $\Delta x, \Delta y$, is an infinitely small quantity of that order, or if one of them be of that order, the other being nothing or of a higher order.

If the two surfaces touch each other in a point of the axis of z , but in such a manner that the axis of z should not be contained in the tangent plane drawn through the point of contact, it will be sufficient, according to what has just been stated, to determine the order of contact, to find the number which indicates the order of the difference $F(x, y) - f(x, y)$, in considering the two variables x and y as infinitely small quantities of the first order and to diminish this number by one. By operating according to this method, it will be easy to ascertain that the four surfaces represented by the four equations $z = x^2 + y^2$, $z = x^3 + y^3$, $z = x^2 + y^3$, $z = x^3 + y^2$, have at the origin of coordinates a contact of the first order, whilst, at the same point, the two surfaces $z = x^n + y^n$, $z = x^{n+1} + y^{n+1}$, have a contact of the order n , and the two surfaces $z = x^{\frac{5}{3}} + y^{\frac{5}{3}}$, $z = x^{\frac{5}{4}} + y^{\frac{5}{4}}$, a contact of the order $\frac{5}{4} - 1 = \frac{1}{4}$.

Let us suppose, again, that the two surfaces have, in a common point, the coordinates of which are x and y , the same tangent plane, not parallel to the axis of z , with a contact of the order a , and let, besides, n be the integer equal or immediately superior to a . The difference $F(x, y) - f(x, y)$ will vanish; and, if we designate by Δx and Δy infinitely small increments of the first order, given to the coordinates x and y , the expression $F(x + \Delta x, y + \Delta y) - f(x + \Delta x, y + \Delta y)$ will be an infinitely small quantity of the order $a + 1$. In order that the increments Δx and Δy should be infinitely small, it is sufficient to take $\Delta x = \alpha dx$, $\Delta y = \alpha dy$, α designating an infinitely small quantity of the first order, and to give to dx and dy finite values. Then we shall have $F(x + \alpha dx, y + \alpha dy) - f(x + \alpha dx, y + \alpha dy)$, an infinitely small quantity of the order $a + 1$, whatever be the values given to dx and dy . Let us suppose, for the sake of brevity,

$$F(x + \alpha dx, y + \alpha dy) - f(x + \alpha dx, y + \alpha dy) = \psi(\alpha).$$

Then $\psi^{n+1}(\alpha)$ will be the first of the derived functions $\psi(\alpha)$, $\psi'(\alpha)$, $\psi''(\alpha)$, &c. which will not vanish at the same time as α . We shall have, therefore, $\psi(0) = 0$, $\psi'(0) = 0$, $\psi''(0) = 0 \dots \psi^{(n)}(0) = 0$. But we have, for the coordinates of the point of contact,

$$\psi(0) = F(x, y) - f(x, y), \quad \psi'(0) = dF(x, y) - df(x, y), \quad \psi''(0) = d^2F(x, y) - d^2f(x, y), \quad \&c. \dots \psi^{(n)}(0) = d^n F(x, y) - d^n f(x, y).$$

Integral
Calculus.

Therefore, for the coordinates of the point of contact of the two surfaces, we shall have

Part III.

$$F(x, y) = f(x, y), \quad dF(x, y) = df(x, y), \quad d^2F(x, y) = d^2f(x, y), \quad \&c. \dots d^n F(x, y) = d^n f(x, y),$$

or which is the same thing,

$$F(x, y) = f(x, y), \quad \frac{dF(x, y)}{dx} dx + \frac{dF(x, y)}{dy} dy = \frac{df(x, y)}{dx} dx + \frac{df(x, y)}{dy} dy,$$

$$\frac{d^2F(x, y)}{dx^2} dx^2 + 2 \frac{d^2F(x, y)}{dx dy} dx dy + \frac{d^2F(x, y)}{dy^2} dy^2 = \frac{d^2f(x, y)}{dx^2} dx^2 + 2 \frac{d^2f(x, y)}{dx dy} dx dy + \frac{d^2f(x, y)}{dy^2} dy^2,$$

$$\&c. \dots$$

$$\frac{d^n F(x, y)}{dx^n} dx^n + \frac{n}{1} \frac{d^n F(x, y)}{dx^{n-1} dy} dx^{n-1} dy + \dots + \frac{d^n F(x, y)}{dy^n} dy^n = \frac{d^n f(x, y)}{dx^n} dx^n$$

$$+ \frac{n}{1} \frac{d^n f(x, y)}{dx^{n-1} dy} dx^{n-1} dy + \dots + \frac{d^n f(x, y)}{dy^n} dy^n.$$

 Since these last formulæ must be verified whatever be the finite values given to dx and dy , they will obviously lead to the following equations,

$$F(x, y) = f(x, y), \quad \frac{dF(x, y)}{dx} = \frac{df(x, y)}{dx}, \quad \frac{dF(x, y)}{dy} = \frac{df(x, y)}{dy}, \quad \frac{d^2F(x, y)}{dx^2} = \frac{d^2f(x, y)}{dx^2},$$

$$\frac{d^2F(x, y)}{dx dy} = \frac{d^2f(x, y)}{dx dy}, \quad \frac{d^2F(x, y)}{dy^2} = \frac{d^2f(x, y)}{dy^2}, \quad \&c. \dots \frac{d^n F(x, y)}{dx^n} = \frac{d^n f(x, y)}{dx^n},$$

$$\frac{d^n F(x, y)}{dx^{n-1} dy} = \frac{d^n f(x, y)}{dx^{n-1} dy}, \quad \dots \frac{d^n F(x, y)}{dx dy^{n-1}} = \frac{d^n f(x, y)}{dx dy^{n-1}}, \quad \frac{d^n F(x, y)}{dy^n} = \frac{d^n f(x, y)}{dy^n}.$$

Consequently when two surfaces touch each other at a point where the tangent plane to both surfaces is not parallel to the axis of z , not only, for that point, the ordinate z considered as a function of the two independent variables x and y , and its partial differential coefficients of the first order $\frac{dz}{dx}, \frac{dz}{dy}$, have the same values in passing from one curve to the other, but the same thing takes place with respect to the partial differential coefficients

$$\frac{d^2z}{dx^2}, \frac{d^2z}{dx dy}, \frac{d^2z}{dy^2}, \frac{d^3z}{dx^3}, \frac{d^3z}{dx^2 dy}, \frac{d^3z}{dx dy^2}, \frac{d^3z}{dy^3}, \dots$$

up to these the order of which is an integer equal or immediately superior to the order of contact. In other words, if we designate by n this integer, the ordinate z and its total differentials of all the orders up to the order n , that is to say, the quantities $dz, d^2z, d^3z, \&c. \dots d^nz$, will have the same values in passing from one surface to the other, whatever be the values given to the differentials dx, dy of the independent variables.

(380.) If, the two surfaces having a contact of the order a , the tangent plane were parallel to the axis of z , then by giving to the values of the coordinates x, y , relative to the point of contact, infinitely small increments of the first order, the difference $F(x, y) - f(x, y)$ would not generally be found an infinitely small quantity of the order $a + 1$. Nevertheless we might still determine the order of contact by the method we have made use of, in substituting one of the variables x or y for the variable z . Thus, for instance, to show that the two surfaces $z = x^{\frac{1}{3}}(1 - y^2)^{\frac{1}{3}}, z = x^{\frac{1}{4}}(1 - y^2)^{\frac{1}{4}}$, which touch the plane of y, z , at the origin of coordinates, have, at that point, a contact of the second order, it is sufficient to remark that the two equations resolved with respect to x assume the forms $x = \frac{z^3}{1 - y^2}, x = \frac{z^4}{1 - y^2}$, and that the difference $\frac{z^4}{1 - y^2} - \frac{z^3}{1 - y^2}$, is an infinitely small quantity of the third order, when y and z are considered as infinitely small quantities of the first order. As to the difference $F(x, y) - f(x, y)$, it is reduced in this example to $x^{\frac{1}{4}}(1 - y^2)^{\frac{1}{4}} - x^{\frac{1}{3}}(1 - y^2)^{\frac{1}{3}}$, and when we consider x and y as infinitely small quantities of the first order, it is an infinitely small quantity, not of the second order, but of the order $\frac{1}{4}$ only.

(381.) When the tangent plane to both surfaces is not parallel to the axis of z , and the order of contact is an integer, it suffices to determine this order to find which is the last of the equations

$$F(x, y) = f(x, y), \quad dF(x, y) = df(x, y), \quad d^2F(x, y) = d^2f(x, y), \quad \&c.$$

which is verified for the point of contact, independently of the values given to dx and dy . The order of the total differentials contained in this last equation, will be precisely the order required.

The axis of z may always be chosen so as not to be parallel to the tangent plane drawn through the point of contact of the two surfaces. This understood, from what precedes it results that, generally, when a surface has with another a contact of the second or a higher order, one of them is always an osculating surface of the other; and reciprocally when two surfaces have a point of osculation, they have at that point, at least a contact of the second order.

(382.) The principal object of the preceding pages has been to show the application of Differential Calculus to Geometry. But it is only, as it may have been observed, those geometrical questions which relate to the position of lines and surfaces with respect to each other, that Differential Calculus can be instrumental to resolve. To that other and most important part of Geometry, which has for its object to measure lines, surfaces, and bulks, Differential Calculus can be of little use, but to that part, on the contrary, Integral Calculus can be applied with the greatest advantage. These applications will form the subject of the following pages.

We shall first propose a curve being given by its equation, or equations, to find the length of any portion of it contained between two given points.

Let the curve be a plane curve, represented by an equation between the rectangular coordinates x, y , or a curve of double curvature, represented by two equations between the three rectangular coordinates x, y, z . Let s be the length of the arc contained between a point A on this curve, and another point of the curve, the abscissa of which is x ; and let it be agreed, moreover, that s will be taken with the sign $+$ or $-$ according to the side of A the length s is supposed to be taken on the curve. If τ represent the angle of the curve with the axis of x , at the extremity of the arc s , we shall have $ds = \pm \sec \tau dx$, the first sign being taken when the arc s increases at the same time as the abscissa x , and the second sign in the contrary case. This understood, let P and Q be two points of the curve, respectively determined, the first by the coordinates x_0, y_0 , and the second by the coordinates x_1, y_1 . Let us also suppose that the abscissa x increases or decreases constantly from the point P to the point Q. And, finally, let s_0 and s_1 be the two values of s corresponding to $x = x_0$ and $x = x_1$. Then since $ds = \pm \sec \tau dx$, we shall have by integrating both sides of this equation, and supposing the integrals to begin with $x = x_0$,

$s - s_0 = \pm \int_{x_0}^x \sec \tau dx$, and supposing, in this last expression $x = x_1$, we get $s_1 - s_0 = \pm \int_{x_0}^{x_1} \sec \tau dx$. This formula will give the value of the arc PQ, the sign $+$ being taken when the differences $s_1 - s_0, x_1 - x_0$, have the same sign, and the sign $-$ in the contrary case. If the point P be supposed to coincide with the point A, then if we suppose the quantities $s, s_1, x - x_0, x_1 - x_0$, to be positive, the above formulæ would give $s = \int_{x_0}^x \sec \tau dx$, $s_1 = \int_{x_0}^{x_1} \sec \tau dx$. When the abscissa x is considered as the independent variable, we have

$$\sec \tau = \sqrt{1 + \frac{dy^2}{dx^2}} = \pm \frac{n}{y},$$

n being the normal. Therefore $s = \int_{x_0}^x \sqrt{1 + \frac{dy^2}{dx^2}} dx$. In the same manner for a curve of double cur-

vature, $\sec \tau = \sqrt{1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}}$, and consequently $s = \int_{x_0}^x \sqrt{1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}} dx$.

If the angle τ were constant, which is the case when the proposed curve is a straight line, or an helix drawn upon a cylinder the axis of which is the axis of x , then we have $s = \sec \tau \int_{x_0}^x dx = (x - x_0) \sec \tau$. But $x - x_0$ is equal to the projection of the arc s on the axis of x ; hence it appears, that when a curve, at every point, is equally inclined to the axis of x , the length of a portion of this line is equal to the product of its projection on the axis of x , by the secant of the inclination.

Let us now apply the preceding formula to a few examples.

Example 1. Let the curve be the circumference represented by the equation $x^2 + y^2 = r^2$. We shall have $n = r$, and consequently $\sec \tau = \pm \frac{n}{y} = \frac{r}{\sqrt{r^2 - x^2}}$. Therefore $s = r \int_{x_0}^x \frac{dx}{\sqrt{r^2 - x^2}} = r \left(\sin^{-1} \frac{x}{r} - \sin^{-1} \frac{x_0}{r} \right)$, a result which might easily have been foreseen.

Example 2. Let the curve be the ellipse represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; we have then

$$\frac{dy^2}{dx^2} = \frac{b^4 x^2}{a^4 y^2} = \frac{b^2 x^2}{a^2 (a^2 - x^2)}, \text{ and } \sec \tau = \sqrt{\frac{a^4 - (a^2 - b^2)x^2}{a^2 (a^2 - x^2)}}.$$

Let e be the eccentricity of that ellipse, then $ae = \sqrt{a^2 - b^2}$, and $\sec \tau = \sqrt{\frac{a^2 - e^2 x^2}{a^2 - x^2}}$. Therefore we shall have $s = \int_{x_0}^x \sqrt{\frac{a^2 - e^2 x^2}{a^2 - x^2}} dx$. If, in this last formula, we make $x_0 = 0$ and $x = a$, we have for the fourth part of the perimeter of the ellipse $s = \int_0^a \sqrt{\frac{a^2 - e^2 x^2}{a^2 - x^2}} dx$, and consequently four times that quantity for the whole perimeter. If we suppose $x = at$, we have the whole perimeter of the ellipse

$$= 4a \int_0^1 \sqrt{\frac{1 - e^2 t^2}{1 - t^2}} dt.$$

This definite integral cannot be obtained in finite terms, but various methods give the means to calculate the values of it, with as great an approximation as may be desired.

Example 3. Let the curve be the hyperbola represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we shall have

Integral Calculus. $\sec \tau = \sqrt{\frac{(a^2 + b^2)x^2 - a^4}{a^2(x^2 - a^2)}}$. Let ϵ be the eccentricity of the curve, then $a\epsilon = \sqrt{a^2 + b^2}$, and Part III.

$\sec \tau = \sqrt{\frac{\epsilon^2 x^2 - a^2}{x^2 - a^2}}$, therefore $s = \int_{x_0}^x \sqrt{\frac{\epsilon^2 x^2 - a^2}{x^2 - a^2}} dx$. This formula is the same as that obtained for the length of an arc of ellipse, but, in the last case, the quantity ϵ is less than one, whilst in the case of the hyperbola it is greater.

Example 4. Let the curve be the parabola represented by the equation $y^2 = 2px$. We shall have $\sec \tau = \sqrt{1 + \frac{p}{2x}}$, and $s = \int_{x_0}^x \sqrt{1 + \frac{p}{2x}} dx$.

To find the value of this integral, let us make $\sqrt{1 + \frac{p}{2x}} = t$, or $x = \frac{p}{2(t^2 - 1)}$, and we shall find,

$$\begin{aligned} \int \sqrt{1 + \frac{p}{2x}} dx &= \int t dx = tx - \int x dt = tx - \frac{p}{2} \int \frac{dt}{t^2 - 1} = tx - \frac{p}{4} l \left(\frac{t-1}{t+1} \right) \\ &+ C = x \sqrt{1 + \frac{p}{2x}} - \frac{p}{4} l \left\{ \frac{\sqrt{1 + \frac{p}{2x}} - 1}{\sqrt{1 + \frac{p}{2x}} + 1} \right\} + C. \end{aligned}$$

If we suppose $x_0 = 0$, we shall have for the value of the arc of parabola contained between the summit and the point corresponding to the abscissa x ,

$$s = x \sqrt{1 + \frac{p}{2x}} - \frac{p}{4} l \left\{ \frac{\sqrt{1 + \frac{p}{2x}} - 1}{\sqrt{1 + \frac{p}{2x}} + 1} \right\}$$

Example 5. Let the curve be the logarithmic represented by the equation $y = alx$. We shall have $\tan \tau = \frac{a}{x}$, $x = a \tan \tau$, $dx = -\frac{a d\tau}{\sin^2 \tau}$. This understood, if we designate by τ_0 , the value of τ corresponding to $x = x_0$, we shall have $s = -a \int_{\tau_0}^{\tau} \frac{d\tau}{\cos \tau \sin^3 \tau}$. But we have $\frac{1}{\cos \tau \sin^3 \tau} = \frac{\cos^2 \tau + \sin^2 \tau}{\cos \tau \sin^3 \tau} = \frac{\cos \tau}{\sin^3 \tau} + \frac{1}{\cos \tau}$, and, moreover,

$$\int \frac{\cos \tau d\tau}{\sin^3 \tau} = -\frac{1}{\sin \tau} + C, \quad \int \frac{d\tau}{\cos \tau} = \int \frac{\frac{1}{2} d\tau}{\sin \left(\frac{\pi}{4} + \frac{\tau}{2} \right) \cos \left(\frac{\pi}{4} + \frac{\tau}{2} \right)} = l \tan \left(\frac{\pi}{4} + \frac{\tau}{2} \right) + C,$$

hence we shall have

$$s = a \left\{ \frac{1}{\sin \tau} - \frac{1}{\sin \tau_0} - l \tan \left(\frac{\pi}{4} + \frac{\tau}{2} \right) + l \tan \left(\frac{\pi}{4} + \frac{\tau_0}{2} \right) \right\}.$$

It would now be easy to introduce in this formula the abscissa x instead of the angle τ .

Example 6. Let us next consider the curve represented by the equation $y = a \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}$ which is the catenary, that is, the curve formed by a heavy and flexible chain attached by its two ends. We shall have for this curve $y' = \frac{e^{\frac{x}{a}} - e^{-\frac{x}{a}}}{2}$, consequently, $\sec \tau = \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}$, and hence

$$s = \int_0^x \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2} dx = a \frac{e^{\frac{x}{a}} - e^{-\frac{x}{a}}}{2} = ay'.$$

Moreover, it is easy to see that in this curve the point corresponding to the abscissa $x = 0$ is precisely the lowest point. Consequently, in the catenary, the length of an arc, beginning at the lowest point, is proportional to $y' = \tan \tau$, that is to say, to the trigonometrical tangent of the inclination corresponding to the extremity of the same arc.

(383.) If in the general formula $ds = \sec \tau dx$, we change x into y , we must, at the same time, change τ into $\frac{\pi}{2} - \tau$. We shall have, therefore, $ds = \pm \operatorname{cosec} \tau dy$. This understood, we shall have $s_1 - s_0 = \int_{y_0}^{y_1} \operatorname{cosec} \tau dy$ if the differences $s_1 - s_0$ and $y_1 - y_0$ are quantities of the same sign, and $s_1 - s_0 = -\int_{y_0}^{y_1} \operatorname{cosec} \tau dy$, if

Integral
Calculus.

Part III.

these same differences have contrary signs. When the curve is plane, $\operatorname{cosec} \tau = \sqrt{1 + \frac{1}{y^2}}$, and, consequently,

supposing $s_0 = 0$, we shall have $s = \int_y^y \sqrt{1 + \frac{1}{y^2}} dy$. To show an application of this last formula, let us

suppose that the arc s is the arc of a cycloid, represented by the equation $x = r \cos^{-1} \frac{r-y}{r} - \sqrt{(2ry - y^2)}$, the origin of the arc being supposed to be the origin of the coordinates. We shall have then

$$dx = \sqrt{\left(\frac{y}{2r-y}\right)} dy, \quad y' = \sqrt{\left(\frac{2r-y}{y}\right)}, \quad 1 + \frac{1}{y'^2} = \frac{2r}{2r-y},$$

$$s = \sqrt{2r} \int_0^y \frac{dy}{\sqrt{(2r-y)}} = 2\sqrt{2r} \{ \sqrt{2r} - \sqrt{2r-y} \},$$

and consequently $4r - s = 2\sqrt{2r(2r-y)}$. This last equation gives the arc s of a cycloid in function of the ordinate y , provided the extremity of the arc is situated between the origin of coordinates and the summit of the first branch of the curve. If in the value of s we suppose $y = 2r$, we shall have the value of the whole arc $= 4r$, and by doubling it, the whole length of any branch of a cycloid, between two consecutive cusps.

(384.) The formula $s_1 - s_0 = \pm \int_{s_0}^{s_1} \sec \tau dx$ is no longer exact, when it is not possible to go from one of the extremities P of the arc to the other extremity Q, without making the abscissa increase and decrease alternately. Let us suppose that $s_0, s', s'', \dots, s^{(m)}, s_1$ are values of the arc s forming a series of increasing or decreasing terms. Let us also suppose that the abscissa x increases or decreases constantly, from the extremity of s_0 to the extremity of the arc s' from this last extremity to the extremity of the arc s'' , and, finally, from the extremity of the arc $s^{(m)}$ to the extremity Q. Then the whole length of the arc P Q cannot be determined by means of the above formula, but this formula may clearly be used to determine the length of each of the differences $s' - s_0, s'' - s', \dots, s_1 - s^{(m)}$, and it will be sufficient to add them together to obtain the value of $s_1 - s_0$, that is the length of the arc P Q.

In the same manner, to apply the formula $s_1 - s_0 = \pm \int_{y_0}^{y_1} \operatorname{cosec} \tau dy$ to the determination of an arc P Q, it would be sufficient to choose the arcs $s', s'', \dots, s^{(m)}$, in such a manner that the ordinate y should continually increase or decrease from the extremity of one of these arcs to the extremity of the next.

To make an application of what precedes, let us propose to find the length of an arc s contained between the origin and any point whatever of the cycloid represented by the equation $x = \cos^{-1} \left(\frac{r-y}{r} \right) \pm \sqrt{(2ry - y^2)}$, in which the radical quantity ought to be taken with the sign $+$ or $-$ according as the angle ω , determined by the formula $\omega = \cos^{-1} \left(\frac{r-y}{r} \right)$, has for its sine a negative or positive quantity. Let us, moreover, suppose $s_0 = 0$, the arc s_1 to be a positive quantity, and the coordinates of the extremity of this arc to be designated by x_1 and y_1 . In this case, we shall take for the extremities of the arcs $s', s'', \dots, s^{(m)}$, the summits and cusps of the cycloid which are between the origin and the point (x_1, y_1) . Then, it is obvious that each of the arcs $s', s'' - s', s''' - s'', \dots, s^{(m)} - s^{(m-1)}$ will be contained between two points, one corresponding to the ordinate $y = 0$, and the other to the ordinate $y = 2r$, and the arc $s_1 - s^{(m)}$ between a point corresponding to one of these two ordinates, and the point, the coordinates of which are x_1 and y_1 . From the above equation of the curve we find

$$dx = \pm \frac{y}{\sqrt{(2ry - y^2)}} dy, \quad \frac{dy}{dx} = \pm \sqrt{\frac{2r-y}{y}}, \quad \operatorname{cosec} \tau = \sqrt{\frac{2r}{2r-y}}.$$

This understood, if we suppose that s is a positive quantity, we shall find, according to the formula

$$s_1 - s_0 = \int_{y_0}^{y_1} \operatorname{cosec} \tau dy,$$

$$s' = s''' - s'' = s^{(m)} - s^{(m-1)} = \&c. \dots = \int_0^{2r} \sqrt{\left(\frac{2r}{2r-y}\right)} dy = 4r;$$

and, according to the formula $s_1 - s_0 = \int_{y_0}^{y_1} \operatorname{cosec} \tau dy$,

$$s'' - s' = s^{(m)} - s^{(m-1)} = \&c. \dots = \int_{2r}^0 \sqrt{\left(\frac{2r}{2r-y}\right)} dy = 4r.$$

As to the value of $s_1 - s^{(m)}$ which must be determined by the first of the two last formulæ mentioned if m be an even number, and by the last if it be odd, it will be in the first case

$$s_1 - s^{(m)} = \int_0^{y_1} \sqrt{\left(\frac{2r}{2r-y}\right)} dy = 4r - 2\sqrt{[2r(2r-y)]};$$

and, in the second,

$$s_1 - s^{(m)} = - \int_{2r}^{y_1} \sqrt{\left(\frac{2r}{2r-y}\right)} dy = 2\sqrt{[2r(2r-y)]}.$$

Integral Calculus. If we add now the arcs $s', s'' - s', s''' - s'', \dots, s^{(m)} - s^{(m-1)}, s_1 - s^{(m)}$, we shall find for the even values of m , $s_1 = 4(m+1)r - 2\sqrt{2r(2r-y)}$, and for the odd values of m , $s' = 4mr + 2\sqrt{2r(2r-y)}$, results which were both easy to foresee. Part III.

(385.) When the integrals contained in the general formulæ cannot be obtained in finite terms, it is then necessary to make use of methods of approximation. One of the simplest consists in developing the values of

$\sec \tau = \sqrt{1 + \frac{dy^2}{dx^2}}$, or $\operatorname{cosec} \tau = \sqrt{1 + \frac{dx^2}{dy^2}}$ in a converging series, each term of which multiplied by dz or dy should be easily integrable. This is obtained in various cases in making use of one of the two following formulæ.

$$\sec \tau = \left(1 + \frac{dy^2}{dx^2}\right)^{\frac{1}{2}} = 1 + \frac{1}{2} \frac{dy^2}{dx^2} - \frac{1}{2} \cdot \frac{1}{4} \frac{dy^4}{dx^4} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \frac{dy^6}{dx^6} - \&c.$$

$$\operatorname{cosec} \tau = \left(1 + \frac{dx^2}{dy^2}\right)^{\frac{1}{2}} = 1 + \frac{1}{2} \frac{dx^2}{dy^2} - \frac{1}{2} \cdot \frac{1}{4} \frac{dx^4}{dy^4} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{6} \frac{dx^6}{dy^6} - \&c.$$

The first of which is converging when $\frac{dy}{dx}$ is less than one, and the other when it is superior to one. Thus, for instance, by means of these formulæ, we may develop in a converging series an arc of ellipse or hyperbola, provided no point of the curve corresponding to $\frac{dy^2}{dx^2} = 1$ should be between the extremities of this arc.

(386.) Another method by means of which an arc of ellipse or hyperbola may easily be obtained, consists in developing $\sqrt{\frac{a^2 - e^2 x^2}{a^2 - x^2}}$, or $\sqrt{\frac{e^2 x^2 - a^2}{x^2 - a^2}}$, according to the increasing or decreasing powers of the eccentricity. Let us suppose first, that it is required to find the length of an arc of ellipse. If to obtain it we make use of the formula $s = \int_{x_0}^x \sqrt{\frac{a^2 - e^2 x^2}{a^2 - x^2}} dx$, it is clear that the numerical value of x will always be less than a , we may therefore suppose $x = a \cos \phi$, and if ϕ_0 be the value of ϕ corresponding to $x = x_0$, the preceding formula will become $s = -a \int_{\phi_0}^{\phi} \sqrt{1 - e^2 \cos^2 \phi} d\phi$. If now we develop according to the increasing powers of e the radical, we shall find

$$\sqrt{1 - e^2 \cos^2 \phi} = 1 - \frac{e^2}{2} \cos^2 \phi - \frac{1}{2} \cdot \frac{e^4}{4} \cos^4 \phi - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{e^6}{6} \cos^6 \phi - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} e^8 \cos^8 \phi - \&c.,$$

a developement which is always converging since by hypothesis e is less than one. Hence we shall find

$$s = a(\phi_0 - \phi) + \frac{e^2}{2} \int_{\phi_0}^{\phi} \cos^2 \phi d\phi + \frac{1}{2} \cdot \frac{e^4}{4} \int_{\phi_0}^{\phi} \cos^4 \phi d\phi + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{e^6}{6} \int_{\phi_0}^{\phi} \cos^6 \phi d\phi + \&c.$$

But we have $\cos \phi = \frac{1}{2}(e^{\phi \sqrt{-1}} + e^{-\phi \sqrt{-1}})$, and also n being any integer whatever,

$$\cos 2n\phi = \frac{1}{2^{2n}}(e^{\phi \sqrt{-1}} + e^{-\phi \sqrt{-1}})^{2n},$$

or, which is the same thing,

$$\cos^{2n} \phi = \frac{1}{2^{2n-1}} \left\{ \cos 2n\phi + \frac{2n}{1} \cos (2n-2)\phi + \frac{2n(2n-1)}{1 \cdot 2} \cos (2n-4)\phi + \dots \right. \\ \left. + \frac{1}{2} \cdot \frac{1 \cdot 2 \cdot 3 \dots 2n}{1 \cdot 2 \dots n \cdot 1 \cdot 2 \dots n} \right\},$$

and in multiplying by $d\phi$, and integrating, the integral beginning with $\phi = \phi_0$, we find

$$\int_{\phi_0}^{\phi} \cos^{2n} \phi d\phi = \frac{1}{2^{2n-1}} \left\{ \frac{\sin 2n\phi - \sin 2n\phi_0}{2n} + \frac{2n \sin (2n-2)\phi - \sin (2n-2)\phi_0}{1} \right. \\ \left. + \dots + \frac{1}{2} \cdot \frac{1 \cdot 2 \cdot 3 \dots 2n}{1 \cdot 2 \dots n \cdot 1 \cdot 2 \dots n} (\phi - \phi_0) \right\}.$$

From this last expression it results that it is possible to express by means of a finite number of terms each term of the infinite series which represents the value of s and therefore to find s by means of a converging series each term of which may be calculated exactly.

It is not unnecessary to remark that in the preceding formula, the angle ϕ , determined by the equation $\cos \phi = \frac{x}{a}$, is precisely the angle included between the positive axis of x , and the radius drawn from the origin to

Integral Calculus. the point where the ordinate corresponding to the abscissa x meets the circumference of the circle which has for its diameter the transverse axis of the ellipse equal to $2a$. Part III.

If it were required to find the arc s equal to the fourth part of the whole perimeter of the ellipse, the limits of the integral relative to x would be 0 and a , and consequently those of the integral relative to ϕ , 0 and $\frac{\pi}{2}$, then the above formula gives

$$\int_0^{\frac{\pi}{2}} \cos^{2n} \phi d\phi = - \int_0^{\frac{\pi}{2}} \cos^{2n} \phi d\phi = - \frac{1}{2n} \cdot \frac{1.2.3 \dots 2n}{1.2.3 \dots n.1.2.3 \dots n} \frac{\pi}{2} = - \frac{1.2.3 \dots 2n}{(2.4 \dots 2n)(2.4 \dots 2n)} \frac{\pi}{2} \\ = - \frac{1.3.5 \dots 2n-1}{2.4.6 \dots 2n} \frac{\pi}{2},$$

and, consequently,

$$s = a \int_0^{\frac{\pi}{2}} \sqrt{1 - \epsilon^2 \cos^2 \phi} d\phi = \frac{a\pi}{2} \left\{ 1 - \left(\frac{1}{2}\epsilon\right)^2 - \frac{1}{3} \left(\frac{1}{2} \cdot \frac{3}{4}\epsilon^2\right)^2 - \frac{1}{5} \left(\frac{1.3.5}{2.4.6}\epsilon^3\right)^2 - \frac{1}{7} \left(\frac{1.3.5.7}{2.4.6.8}\epsilon^4\right)^2 - \&c. \right\}.$$

When the ellipse becomes a circle, then $\epsilon = 0$, and $s = \frac{a\pi}{2}$.

It may be remarked, that the different products included between parentheses, in the right side of this equation, are precisely the different terms which compose the developement of $(1 - \epsilon^2)^{\frac{1}{2}}$.

Let us suppose now that it is required to calculate an arc of hyperbola. Then x is always greater than a , and we shall assume $x = \frac{a}{\cos \phi}$. Designating, as before, by ϕ_0 , the value of ϕ corresponding to $x = x_0$, we shall

have $s = a \int_{\phi_0}^{\phi} \sqrt{\epsilon^2 - \cos^2 \phi} \frac{d\phi}{\cos^2 \phi}$. And by developing, according to the decreasing powers of ϵ we shall obtain the equation

$$\frac{\sqrt{\epsilon^2 - \cos^2 \phi}}{\cos^2 \phi} = \frac{\epsilon}{\cos^2 \phi} - \frac{1}{2\epsilon} - \frac{1}{2} \cdot \frac{1}{4\epsilon^3} \cos^2 \phi - \frac{1.3}{2.4} \frac{1}{6\epsilon^5} \cos^4 \phi - \frac{1.3.5}{2.4.6} \frac{1}{8\epsilon^7} \cos^6 \phi$$

the left side of which is a converging series, since, in the hyperbola, we have $\epsilon > 1$. Hence we shall find

$$s = a\epsilon [\tan \phi - \tan \phi_0] - \frac{a}{2\epsilon} (\phi - \phi_0) - \frac{1}{2} \frac{a}{4\epsilon^3} \int_{\phi_0}^{\phi} \cos^2 \phi d\phi - \frac{1.3}{2.4} \frac{a}{6\epsilon^5} \int_{\phi_0}^{\phi} \cos^4 \phi d\phi - \frac{1.3.5}{2.4.6} \frac{a}{8\epsilon^7} \int_{\phi_0}^{\phi} \cos^6 \phi d\phi - \&c.$$

We have seen above that each of the integrals remaining in this series may be expressed in finite terms, therefore we have the value of s by means of a converging series, each term of which may be easily calculated. If we suppose that the origin of the arc coincides with the summit of the hyperbola corresponding to the abscissa $x = a$, we must suppose $\phi_0 = 0$, and the formula will be simply reduced to the following:

$$s = a\epsilon \tan \phi - \frac{a}{2\epsilon^3} \phi - \frac{1}{2} \frac{a}{4\epsilon^3} \int_0^{\phi} \cos^2 \phi d\phi - \frac{1.3}{2.4} \frac{a}{6\epsilon^5} \int_0^{\phi} \cos^4 \phi d\phi - \frac{1.3.5}{2.4.6} \frac{a}{8\epsilon^7} \int_0^{\phi} \cos^6 \phi d\phi - \&c.$$

It may be remarked, that the angle ϕ is precisely the angle contained between the positive axis of x , and the radius drawn through the origin to the point where the circle, having its centre at the origin, and for radius a , is touched by a tangent which cuts the axis of x at the point, the abscissa of which is x .

Since the asymptotes of the hyperbola are represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$, it is clear that if we call r the distance between the origin and a point of one of the asymptotes corresponding to the abscissa x , we shall

have $r^2 = x^2 + \frac{b^2 x^2}{a^2} = \frac{a^2 + b^2}{a^2} x^2 = \epsilon^2 x^2$, and, consequently, if we suppose x to be a positive quantity,

$r = \epsilon x = \frac{a\epsilon}{\cos \phi}$. If from the preceding value of r we subtract the arc s , determined by the above formula, we shall find

$$r - s = a\epsilon \left(\frac{\cos \phi}{\sin \phi} + \frac{1}{2\epsilon} \phi + \frac{1}{2} \frac{a}{4\epsilon^3} \int_0^{\phi} \cos^2 \phi d\phi + \frac{1.3}{2.4} \frac{a}{6\epsilon^5} \int_0^{\phi} \cos^4 \phi d\phi + \frac{1.3.5}{2.4.6} \frac{a}{8\epsilon^7} \int_0^{\phi} \cos^6 \phi d\phi + \&c. \right)$$

Integral
Calculus.

If now we suppose $\phi = \frac{\pi}{2}$, the last equation will give the value of $r - s$ corresponding to $x = \frac{a}{\cos \frac{\pi}{2}} = \infty$, that Part III.

is very nearly the difference between two very great lengths, taken, the first from the origin on the asymptote, the second on the hyperbola from the summit, in such a manner that the extremities should correspond to the same abscissa. Therefore if we call D this difference, we shall have very nearly, when x will be very large,

$$D = \frac{a\pi}{4e} \left\{ 1 + \frac{1}{2} \left(\frac{1}{2} \cdot \frac{1}{e} \right)^2 + \frac{1}{3} \left(\frac{1.3}{2.4} \cdot \frac{1}{e^3} \right)^2 + \frac{1}{4} \left(\frac{1.3.5}{2.4.6} \frac{1}{e^5} \right)^2 + \dots \right\}.$$

(387.) The determination of the length of an arc of ellipse may also be obtained by other means. We have clearly

$$\sec^2 \tau = \frac{1}{\cos^2 \tau} = \frac{e^2 x^2 - a^2}{x^2 - a^2}, \text{ and consequently, } x^2 = \frac{a^2 (1 - \cos^2 \tau)}{1 - e^2 \cos^2 \tau}, \text{ or } x = \pm \frac{a \sin \tau}{\sqrt{1 - e^2 \cos^2 \tau}},$$

$$dx = \pm \frac{a (1 - e^2) \cos \tau d\tau}{(1 - e^2 \cos^2 \tau)^{\frac{3}{2}}}, \text{ and } ds = \pm \sec \tau dx = \pm \frac{a (1 - e^2) d\tau}{(1 - e^2 \cos^2 \tau)^{\frac{3}{2}}}.$$

This understood, if we admit that, for increasing values of τ , the arc s increases in the ellipse and decreases in the hyperbola, and if we designate by τ_0 the value of τ corresponding to the origin of the arc s , we shall find for

$$\text{the ellipse } s = a (1 - e^2) \int_{\tau_0}^{\tau} \frac{d\tau}{(1 - e^2 \cos^2 \tau)^{\frac{3}{2}}}, \text{ and for the hyperbola } s = -a (1 - e^2) \int_{\tau_0}^{\tau} \frac{d\tau}{(1 - e^2 \cos^2 \tau)^{\frac{3}{2}}}.$$

The quantities under the sign of integration may be developed in converging series according to the increasing or decreasing powers of the eccentricity. If we develop, for instance, the first formula, and if we suppose the

integrals taken between the limits $\tau = 0$ and $\tau = \frac{\pi}{2}$, we shall obtain a value of s equal to the fourth part of the perimeter of an ellipse, and hence we shall find the whole perimeter P ,

$$P = 2a\pi (1 - e^2) \left\{ 1 + 3 \left(\frac{1}{2} e^2 \right) + 5 \left(\frac{1.3}{2.4} e^4 \right) + 7 \left(\frac{1.3.5}{2.4.6} e^6 \right) + \&c. \dots \right\}$$

which value, it is easy to ascertain, agrees with the value already found above.

If, in the equation $x = \frac{a^2 (1 - \cos^2 \tau)}{1 - e^2 \cos^2 \tau}$ we substitute for x its value $a \cos \phi$, we find

$$1 \cos^2 \phi - \cos^2 \tau + e^2 \cos^2 \phi \cos^2 \tau = 0,$$

and consequently, when $\cos \phi$ is a positive quantity,

$$\cos \phi = \frac{\sin \tau}{\sqrt{1 - e^2 \cos^2 \tau}}.$$

This understood, we shall have

$$d(e^2 \cos \phi \cos \tau) = d \left\{ \frac{e^2 \cos \tau \sin \tau}{\sqrt{1 - e^2 \cos^2 \tau}} \right\} = -\sqrt{1 - e^2 \cos^2 \tau} d\tau + \frac{(1 - e^2) d\tau}{(1 - e^2 \cos^2 \tau)^{\frac{3}{2}}},$$

and hence

$$(1 - e^2) \int_{\tau_0}^{\tau} \frac{d\tau}{(1 - e^2 \cos^2 \tau)^{\frac{3}{2}}} = e^2 (\cos \phi \cos \tau - \cos \phi_0 \cos \tau_0) + \int_{\tau_0}^{\tau} \sqrt{1 - e^2 \cos^2 \tau} d\tau.$$

We shall therefore be able to put the value of s under the following form :

$$s = a e^2 (\cos \phi \cos \tau - \cos \phi_0 \cos \tau_0) + a \int_{\tau_0}^{\tau} \sqrt{1 - e^2 \cos^2 \tau} d\tau.$$

This formula supposes $\tau > \tau_0$. Then the product $a \int_{\tau_0}^{\tau} \sqrt{1 - e^2 \cos^2 \tau} d\tau$, is positive, and represents the arc of the ellipse, the extremities of which correspond to the abscissæ $a \cos \tau$ and $a \cos \tau_0$. If this arc be designated by s' we shall have $s = a e^2 (\cos \phi \cos \tau - \cos \phi_0 \cos \tau_0) + s'$, or $s - s' = a e^2 (\cos \phi \cos \tau - \cos \phi_0 \cos \tau_0)$, or again if x' and x'_0 be the abscissæ of the extremities of s' , $s - s' = e^2 \frac{x x' - x_0 x'_0}{a}$.

Hence the difference which exists between two arcs of ellipse so situated that the inclinations of the tangents drawn by the extremities of one of these arcs be respectively equal to the inclinations of the two radii drawn through the centre of the ellipse to the points where the circumference described on the transverse axis as a diameter, is met by the ordinates of the extremities of the second arc, may be expressed in finite terms.

When we suppose $\tau_0 = 0$, we have clearly $\phi_0 = \frac{\pi}{2}$ and $x_0 = 0$, and we find $s - s' = e^2 \frac{x x'}{a}$, which equation is

Integral
Calculus.

the expression of a theorem discovered by Count Fagnano, an Italian Geometer. Let us observe, moreover, Part III. that, in every case, the abscissæ x and x' relative to the extremities of the arcs s and s' should verify the equation $a^4 - a^2(x^2 + x'^2) + \epsilon^2 x^2 x'^2 = 0$, which is derived from the equation $1 - \cos^2 \phi - \cos^2 \tau + \epsilon^2 \cos^2 \phi \cos^2 \tau = 0$

by putting $\frac{x}{a}$ and $\frac{x'}{a}$ for $\cos \phi$ and $\cos \tau$.

(388.) We shall next consider the quadrature of plane surfaces, that is to say, that we shall propose to find the value of a plane surface limited by curves the equations of which are given.

Let $y = f(x)$ be the equation of a plane curve referred to rectangular coordinates. Let P and Q be two points on this curve corresponding, the first to the abscissa x_0 , and the second to the abscissa x_1 ; let p be any point of the curve corresponding to the abscissa x , and let it be required to find the area included between the curve the axis of x , and the two ordinates corresponding to the abscissæ x_0 and x . This area will be a function of x , which we shall designate by u , and which will vanish for $x = x_0$. Moreover, if we suppose $x > x_0$, and if we call Δx a positive and very small increment of x , the corresponding increment of u or Δu will represent the element of surface contained between the curve, the axis of x , and the two ordinates $f(x)$, $f(x + \Delta x)$. This understood, let q be the point of the curve corresponding to the abscissa $x + \Delta x$. The ordinate of any point of the arc $p q$ will necessarily be of the form $f(x + \theta \Delta x)$, θ being a number inferior to one. Hence the ordinates of two points situated on the same arc, the one at the least distance from the axis of x , the other at the greatest distance, will be of the form $f(x + \theta_1 \Delta x)$, $f(x + \theta_2 \Delta x)$, θ_1 and θ_2 being numbers less than one. The numerical values of these ordinates will clearly represent the heights of the inscribed and circumscribed rectangles to the element of surface Δu , whilst the area of these rectangles will be measured by the numerical values of the products $\Delta x \cdot f(x + \theta_1 \Delta x)$, $\Delta x \cdot f(x + \theta_2 \Delta x)$. Hence, if we use the notation $M(a, a', a'', \&c.)$ to represent a mean quantity between the quantities $a, a', a'', \&c.$, that is to say, a quantity the value of which is greater than the least, and less than the greatest of the quantities $a, a', a'', \&c.$ we shall have,

$$\frac{\Delta u}{\Delta x} = \pm M[f(x + \theta_1 \Delta x), f(x + \theta_2 \Delta x)],$$

and taking the limits $\frac{d u}{d x} = \pm f(x) = \pm y$, or $d u = \pm y d x$. If therefore we integrate, in the supposition

that the integral begins with $x = x_0$, we shall have $u = \pm \int_{x_0}^x y d x$, and if we call u_1 , the value of u corre-

sponding to $x = x_1$, $u_1 = \pm \int_{x_0}^{x_1} y d x$. According to the hypothesis the function u increasing at the same time

as the abscissa x , it will be necessary to take the right side of the preceding formula with the sign $+$ when the ordinate $y = f(x)$ will be positive, and with the sign $-$ in the contrary case. Let us now apply this formula to some examples.

Example 1. Let the curve be the circumference represented by the equation $x^2 + y^2 = r^2$, we shall have for the positive value of y , $y = \sqrt{r^2 - x^2}$, and hence $u = \int_{x_0}^x \sqrt{r^2 - x^2} d x$. But according to the rules of integration $\int \sqrt{r^2 - x^2} = \frac{1}{2} x \sqrt{r^2 - x^2} - \frac{1}{2} r^2 \tan^{-1} \frac{\sqrt{r^2 - x^2}}{x} + \text{const.}$ If we suppose $x_0 = 0$, we shall have, therefore, $u = \frac{1}{2} x \sqrt{r^2 - x^2} + \frac{1}{2} r^2 \left(\frac{\pi}{2} - \tan^{-1} \frac{\sqrt{r^2 - x^2}}{x} \right) = \frac{1}{2} x \sqrt{r^2 - x^2} + \frac{1}{2} r^2 \tan^{-1} \frac{x}{\sqrt{r^2 - x^2}}$, or, which is the same thing, $u = \frac{1}{2} x y + \frac{1}{2} r^2 \tan^{-1} \frac{x}{y}$, a result which it is easy to verify. If in this last formula we suppose $x = r$, we must suppose at the same time $y = 0$, and the value of u becomes $\frac{\pi}{4} r^2$, therefore the whole surface of the circle is equal to πr^2 .

Example 2. Let the curve be the ellipse represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. We get for the positive value of y , $y = \frac{b}{a} \sqrt{a^2 - x^2}$, and consequently $u = \frac{b}{a} \int_{x_0}^x \sqrt{a^2 - x^2} d x$; but

$$\int \sqrt{a^2 - x^2} d x = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}},$$

therefore we shall have

$$u = \frac{1}{2} \frac{b}{a} x \sqrt{a^2 - x^2} + \frac{1}{2} a b \tan^{-1} \frac{x}{\sqrt{a^2 - x^2}},$$

or, which is the same thing,

$$u = \frac{1}{2} x y + \frac{1}{2} a b \tan^{-1} \frac{b x}{a y}.$$

Part III.
Integral
Calculus.

If from the surface u we subtract the surface of the right-angled triangle, having for the two sides of the right angle x and y , that is to say, $\frac{1}{2}xy$, the remainder,

$$\frac{1}{2}ab \tan^{-1} \frac{bx}{ay} = \frac{1}{2}ab \tan^{-1} \left(\frac{\frac{x}{a}}{\frac{y}{b}} \right),$$

will represent the elliptical sector contained between the radius drawn in the direction of the positive y , and the radius drawn from the centre to the point the coordinates of which are x and y . The surface of the fourth part of the ellipse will be the value of the preceding expression of u , corresponding to $x = a$ and $y = 0$. Therefore this surface $= \frac{\pi}{4}ab$, and that of the whole ellipse will be πab .

Example 3. Let us consider the hyperbola represented by one of the equations $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, a and b being two positive quantities. We shall find for the positive value of y , $y = \frac{b}{a}\sqrt{x^2 \mp a^2}$, and, consequently $u = \frac{b}{a} \int_{x_0}^x \sqrt{x^2 \mp a^2} dx$. But

$$\int \sqrt{x^2 \mp a^2} dx = \frac{1}{2}x \sqrt{x^2 \mp a^2} \mp \frac{1}{8}a^2 l \left\{ \frac{x + \sqrt{x^2 \mp a^2}}{x - \sqrt{x^2 \mp a^2}} \right\} + c.$$

Therefore if we consider the hyperbola the transverse axis of which is $2a$, and if we take $x_0 = a$, and suppose x to be a positive quantity, we shall have

$$u = \frac{1}{2} \frac{b}{a} x \sqrt{x^2 - a^2} - \frac{1}{4} ab l \left\{ \frac{x + \sqrt{x^2 - a^2}}{x - \sqrt{x^2 - a^2}} \right\},$$

or, which is the same thing,

$$u = \frac{1}{2}xy - \frac{1}{4}ab l \left(\frac{\frac{x}{a} + \frac{y}{b}}{\frac{x}{a} - \frac{y}{b}} \right) = \frac{1}{2}xy - \frac{1}{2}ab l \left(\frac{x}{a} + \frac{y}{b} \right).$$

Such is the value of the area contained between the axis of x , the ordinate y , and the arc of hyperbola the two extremities of which coincide, the first with the summit of the curve corresponding to $x = a$, and the second with the point the coordinates of which are x and y . If we subtract this area from the surface of the right-angled triangle the sides of which are x and y , the remainder,

$$\frac{1}{2}ab l \left(\frac{x}{a} + \frac{y}{b} \right) = \frac{1}{2}ab l \left(\frac{1}{\frac{x}{a} - \frac{y}{b}} \right),$$

will represent the hyperbolic sector contained between the radius drawn in the direction of the positive x , and the radius drawn from the centre to the point the coordinates of which are x and y . In the case where this last point is situated at an infinite distance, that is to say, in the case where x and y are infinite, the expression of the surface of the sector becomes also infinite.

If we designate by x_1 and y_1 the coordinates of a point of the asymptote corresponding to the branch of curve we consider, we shall have $\frac{y_1}{b} = \frac{x_1}{a}$; therefore if we suppose $y_1 = y$, we shall have $\frac{y}{b} = \frac{x_1}{a}$, and the value of the

hyperbolic sector above mentioned will become $\frac{1}{2}ab l \left(\frac{a}{x - x_1} \right) = -\frac{1}{2}ab l \left(\frac{x - x_1}{a} \right)$. Therefore this surface

is equal, but of contrary sign to half the product of the semi-axis a and b by the hyperbolic logarithm of the ratio obtained in comparing with half the transverse axis the length $x - x_1$; reckoned upon a parallel to this axis, between the curve and its asymptote.

In the case of the equilateral hyperbola, represented by the equation $x^2 - y^2 = r^2$, the surface of the hyperbolic sector would be $\frac{1}{2}r^2 l \left(\frac{x+y}{r} \right)$. If the same hyperbola were referred to its asymptotes, its equation would be

$$xy = \frac{1}{2}r^2, \text{ or } y = \frac{1}{2} \frac{r^2}{x}, \text{ and we would have } u = \frac{1}{2}r^2 \int_{x_0}^x \frac{dx}{x} = \frac{1}{2}r^2 l \frac{x}{x_0}.$$

Example 4. Let the curve be the parabola represented by the equation $y^2 = 2px$, the positive value of y will be $y = 2p^{\frac{1}{2}}x^{\frac{1}{2}}$, and we shall have therefore $u = (2p)^{\frac{1}{2}} \int_{x_0}^x x^{\frac{1}{2}} dx = \frac{2}{3}(2p)^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}xy$. Hence the area con-

tained between the axis of the parabola, an arc of this curve which has the summit for origin, and an ordinate perpendicular to the axis, is equal to the $\frac{2}{3}$ of the circumscribed rectangle. From this we may immediately conclude, that the area contained between the arc of the parabola, the tangent drawn through the summit, and a straight line parallel to the axis, has for its measure the third of the surface of the circumscribed rectangle.

Example 5. Let the curve be the parabola of the degree a represented by the equation $y = Ax^a$, A and a being two real and constant quantities. We shall have $u = A \int_0^x x^a dx = \frac{A}{a+1} x^{a+1}$ or, which is the same thing, $u = \frac{xy}{a+1}$. Hence in a parabola of the degree a the surface contained between an arc beginning at the summit

or at the point of inflexion, the tangent drawn through that point, and a straight line perpendicular to this tangent, is to the surface of the circumscribed rectangle as one to $1+a$. If a were less than one, the ratio between these two surfaces would be that of the two numbers a and $a+1$. If we suppose $a=2$, or $a=1$, this general theorem gives results already known.

Example 6. Let the curve be the logarithmic represented by the equation $y = al(x)$ in which a designates a positive and constant quantity, we shall derive from the general formula, in supposing $x_0 = 1$ and $x > 1$,

$u = a \int_1^x l x dx = a(xlx - x + 1)$. To find the area included between the negative axis of y , and the part of the logarithmic which has this axis for asymptote, we should make use of the general formula $u = \pm \int_{x_0}^{x_1} y dx$. Then by taking the left side with the sign $-$, and supposing $x_0 = 0$, and $x_1 = 1$, we would

find for the required area $u = -a \int_0^1 l x dx = a$. This area, therefore, is not infinite as that contained between the hyperbola and its asymptote; but it is equal to the number a , that is to say, that this number represents the limit of the area contained between the axis of x the part of the curve produced on the side of the negative y , and an ordinate of the same curve, supposing this ordinate to approach nearer and nearer to the axis of y .

Example 7. Let the curve be the catenary represented by the equation $y = a \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}$. If we suppose $x_0 = 0$ we shall find $u = a^2 \frac{e^{\frac{x}{a}} - e^{-\frac{x}{a}}}{2}$. Such is the expression of the surface contained between the axis of x , the catenary, and the two ordinates a and y , corresponding to the two abscissæ 0 and x . If besides, we designate by s , the arc contained between these two ordinates, we have found before $s = a \frac{e^{\frac{x}{a}} - e^{-\frac{x}{a}}}{2}$. Consequently $u = as$. Hence

the area contained between the axis of abscissa, the arc s beginning at the lowest point, and the two ordinates of the extremities of this arc, is equal to the rectangular parallelogram, having for its base the arc s , and for its height the ordinate of the lowest point.

(389.) In several cases the valuation of surfaces may be more easily obtained by substituting for the variable x another variable. Let it be supposed that u represents the surface contained between the axis of x , and the cycloid represented by the equations $x = r(w - \sin w)$, $y = r(1 - \cos w)$. Then $dx = y dw$, and if we designate by

w_0 the value of w corresponding to $x = x_0$, we have $u = \int_{w_0}^w y^2 dw = r^2 \int_{w_0}^w (1 - \cos w)^2 dw$. Let us suppose that

the area begins at the origin of coordinates, then $w_0 = 0$, and we have

$$u = \int_0^w (1 - \cos w)^2 dw = r^2 \int_0^w (1 - 2 \cos w + \cos^2 w) dw = r^2 \int_0^w \left(\frac{3}{2} - 2 \cos w + \frac{1}{2} \cos 2w \right) dw \\ = r^2 \left\{ \frac{3}{2} w - 2 \sin w + \frac{1}{4} \sin 2w \right\}.$$

And, since w designates the angle described by the generating circle of the cycloid, it is obvious that by making $w = 2\pi$ in the last formula, we shall have the area contained between the axis of x , and the first branch of the cycloid. We find thus $u = 3\pi r^2$, therefore the area contained between the basis of the cycloid and one of the branches of the curve is equal to three times the area of the generating circle.

(390.) Let u now represent the area contained between two curves represented by the equations $y = f(x)$ and $y = F(x)$, and secondly, between the ordinates corresponding to the abscissæ x_0 and x . Then it is easy to prove that $du = \pm \{F(x) - f(x)\} dx$, the sign $+$ being taken when the difference $F(x) - f(x)$ is positive, and the sign $-$ when it is negative. If we suppose $\phi(x)$ equal to the numerical value of the difference $\pm \{F(x) - f(x)\}$,

Integral
Calculus.

we shall have $du = \phi(x) dx$, and $u = \int_{x_0}^x \phi(x) dx$. This formula is similar to that we have used before, only

$\phi(x)$ does not represent here the ordinate of a curve, but the length of the linear section made in the surface u by a plane perpendicular to the axis of x , and corresponding to the abscissa x . If the curves represented by the equations $y = f(x)$ and $y = F(x)$, intersected in two points corresponding to the abscissæ x_0 and x , then u would designate the area contained between the two curves. It would also be the area contained between the two curves, if they touch each other at the points corresponding to the abscissæ x_0 and x . Lastly, the values $y = f(x)$ and $y = F(x)$ may be two values derived from the same equation, $f(x, y) = 0$, representing a curve limited all

round. Then to derive from the formula $u = \int_{x_0}^x \phi(x) dx$ the value of the area inclosed by this curve, it will be

sufficient to substitute for x and x_0 the greatest and smallest abscissæ of the points of the curve, or, which is the same thing, the abscissæ of the two points of the axis of x which include between them the projection of the curve on this axis. In most cases, these abscissæ will be those of the points of the curve when the tangent becomes parallel to the axis of y .

Let us take for example the curve represented by the equation $x^{2m} + y^{2m} = 1$, m being any integer whatever. We shall have two values of the ordinate

$$y = -(1 - x^{2m})^{\frac{1}{2m}}, y = (1 - x^{2m})^{\frac{1}{2m}}, \text{ and consequently, } \phi(x) = 2(1 - x^{2m})^{\frac{1}{2m}}.$$

The equation of the curve gives besides $\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{2m-1}$, and it is obvious that the tangent will become parallel

to the axis of y , at the points where $y = 0$, and consequently $x = -1$, or $x = +1$. It is easy to verify that these two values of x are the maximum and minimum of all the values corresponding to the different points of the

curve. This understood, we shall have for the whole area of the curve $u = 2 \int_{-1}^1 (1 - x^{2m})^{\frac{1}{2m}} dx$, or which is the same thing, $u = 4 \int_0^1 (1 - x^{2m})^{\frac{1}{2m}} dx$. If $m = 1$ the curve becomes a circle, and we find $u = \pi$.

Let us next suppose that the curve is represented by the equation $x^{\frac{2m}{2n+1}} + y^{\frac{2m}{2n+1}} = 1$. We shall have for the area inclosed by the curve $u = 2 \int_{-1}^1 \left(1 - x^{\frac{2m}{2n+1}}\right)^{\frac{2n+1}{2m}} dx$. But if we suppose m to be inferior, or at most equal, to n , we shall find that the abscissæ -1 and $+1$ belong to points where the curve presents cusps, and where the tangent, instead of being parallel to the axis of y , is parallel to the axis of x .

If we suppose $m = 1$, we find $u = 2 \int_{-1}^1 \left(1 - x^{\frac{2}{2n+1}}\right)^{\frac{2n+1}{2}} dx = 4 \int_0^1 \left(1 - x^{\frac{2}{2n+1}}\right)^{\frac{2n+1}{2}} dx$. To find this value let us suppose $x = \sin^{\frac{2n+1}{2}} \phi$, we shall get

$$u = 4(2n+1) \int_0^{\frac{\pi}{2}} \cos^{2n+2} \phi \sin^{\frac{2}{2n+1}} \phi d\phi = (8n+4) \int_0^{\frac{\pi}{2}} \cos^{(2n+2)} \phi (1 - \cos^2 \phi)^{\frac{n}{2}} d\phi;$$

developing, and making use of the formula

$$\int_0^{\frac{\pi}{2}} \cos^{2n} \phi d\phi = -\frac{1.3.5 \dots 2n-1}{2.4.6 \dots 2n} \frac{\pi}{2},$$

we find

$$u = (4n+2) \pi \left\{ \frac{1.3.5 \dots 2n+1}{2.4.6 \dots 2n+2} - \frac{n}{1} \frac{1.3.5 \dots 2n+3}{2.4.6 \dots 2n+4} + \frac{n(n-1)}{1.2} \frac{1.3.5 \dots (2n+5)}{2.4.6 \dots (2n+6)} - \&c. \right\}.$$

If we suppose at the same time $m = 1$ and $n = 1$, the equation of the curve becomes $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$, and

$$u = \frac{3}{8} \pi.$$

(391.) It is also essential to remark, that the formula $u = \int_{x_0}^x \phi(x) dx$ is true in the case where the functions $f(x)$, $F(x)$, change their forms, so as to represent the ordinates of several curves, or even of several straight lines drawn one after the other in the plane of x, y . Let us suppose $F(x) = (1 - x^2)$, that is to say, that the first curve is the parabola represented by the equation $y = 1 - x^2$, and that $f(x)$, for negative values of x , coincides with the ordinate of the straight line $y = -\frac{3}{2}x$, and for positive values with the ordinate of the straight line represented by the equation $y = \frac{5}{6}x$. Then, for $x < 0$, we shall have $\phi(x) = 1 - x^2 + \frac{3}{2}x$, and for $x > 0$,

Integral
Calculus.

$\phi(x) = 1 - x^2 - \frac{5}{6}x$. The first value of $\phi(x)$ becomes equal to nothing for $x = -\frac{1}{2}$, and the second for $x = \frac{2}{3}$. Therefore to find the area contained between the parabola and the two straight lines on the positive side of the y , we shall have the formula

$$u = \int_{-\frac{1}{2}}^0 (1x^2 + \frac{3}{2}x) dx + \int_0^{\frac{2}{3}} (1 - x^2 - \frac{5}{6}x) dx = \frac{1}{2} - \frac{1}{24} - \frac{3}{16} + \frac{2}{3} - \frac{8}{81} - \frac{5}{27} = \frac{847}{1296}.$$

It would not be more difficult to find the area limited by the sides of a polygon, these sides being either straight or curve lines.

(392.) Lastly, let us suppose that the surface u is composed of several parts so disposed that the linear section made in this surface by a plane perpendicular to the axis of x , and corresponding to the abscissa x , is composed of several and distinct lengths represented by $f_1(x), f_2(x), f_3(x)$, &c., the first of which is contained between two given curves, the second between two other given curves, &c. If we suppose, moreover, that this surface is limited by the

ordinates corresponding to the abscissæ x_0 and x , we shall have still $u = \int_{x_0}^x \phi(x) dx$, provided we take

$\phi(x) = f_1(x) + f_2(x) + f_3(x) + \&c.$ The same remark applies when the surface u is contained by a curve limited on all sides, and so bent as to be met by the same ordinate in more than two points. Then the equation of the curve resolved with respect to y , would give for each value of x an even number of real values of x ; and after arranging these real values, according to their magnitude, it would be sufficient to subtract successively the first from the second, the third from the fourth, to obtain the quantities which have been designated above by $f_1(x), f_2(x)$, &c., that is to say, the quantities the sum of which would be designated by $\phi(x)$.

(393.) We shall now proceed to prove several propositions which are of great utility in the quadrature of plane surfaces.

1. If the linear sections made in two surfaces by a system of parallel planes are to each other in a constant ratio, the two surfaces are to each other in the same ratio.

Let us suppose that the points of each surface are referred to two rectangular axes contained in the plane of this same surface, the axis of y being moreover contained in one of the parallel planes that are under consideration. If we call $f(x)$ and $\phi(x)$ the linear sections made in these two surfaces by one of these parallel planes, that corresponding to the abscissa x , we shall have $f(x) = a\phi(x)$, a designating the constant ratio between the linear sections. Let x_0 and x_1 be the limits between which the value of x must remain in order that the plane corresponding to this abscissa, and perpendicular to the axis of x should

meet the two surfaces. Then the first surface will be represented by $\int_{x_0}^{x_1} f(x) dx$, and the second by $\int_{x_0}^{x_1} \phi(x) dx$,

but since $f(x) = a\phi(x)$, we shall have $\int_{x_0}^{x_1} f(x) dx = a \int_{x_0}^{x_1} \phi(x) dx$, which is the analytical expression of the proposition to be demonstrated.

Let us consider two curves in the plane of the x, y , the one represented by the equation $f(x, y) = 0$, and the other by the equation $f\left(x, \frac{y}{b}\right) = 0$, b designating a constant quantity. It is clear that, for each value of

the abscissa x , we shall get from the equations of the two curves values of y which will be to each other in the ratio of 1 to b . Consequently, the limits x_0 and x_1 , between which the value of x ought to remain, in order that the value of the ordinate y should be real, will be the same for both curves. If we designate by y_0, y_1, y_2, y_3 , &c., the various real values of y corresponding to an abscissa x given by the equation

$f(x, y) = 0$, those corresponding to the same abscissa given by the equation $f\left(x, \frac{y}{b}\right) = 0$, will be by_0, by_1, by_2, by_3 ,

&c. If, besides, we suppose the quantities y_0, y_1, y_2 , &c. arranged according to their magnitude, the lengths previously designated by $f_1(x), f_2(x)$, &c. will be, respectively, for the curve $f(x, y) = 0$, $y_1 - y_0, y_2 - y_1$, &c., and consequently, the linear section of the surface contained within this curve will be represented by the formula $y_1 - y_0 + y_2 - y_1 + \&c.$ In the same manner, the linear section made in the curve represented

by the equation $f\left(x, \frac{y}{b}\right) = 0$ will be represented by $b(y_1 - y_0 + y_2 - y_1 + \&c.)$ Hence the area limited by

the curve $f(x, y) = 0$ is to the area limited by the curve $f\left(x, \frac{y}{b}\right) = 0$ in the ratio of 1 to b .

By similar reasonings it may be proved, that the area limited by the curve represented by the equation $f(x, y) = 0$ is to the area limited by the curve represented by the equation $f\left(\frac{x}{a}, y\right) = 0$ in the ratio of 1 to a .

But this last area is also to the area of the curve represented by the equation $f\left(\frac{x}{a}, \frac{y}{b}\right) = 0$ in the ratio of 1 to b ,

Integral
Calculus.

Part III.

hence the area limited by the curve represented by the equation $f(x, y) = 0$ is to that limited by the curve represented by the equation $f\left(\frac{x}{a}, \frac{y}{b}\right) = 0$ in the ratio of 1 to $a b$.

Thus, for instance, the area of the ellipse represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ will be equal to $a b$ multiplied by the area of the curve represented by the equation $x^2 + y^2 = 1$, that is to $\pi a b$. In the same manner the expression we have found for the area of the curve represented by the formula $x^{\frac{2m}{2n+1}} + y^{\frac{2m}{2n+1}} = 1$ multiplied by $a b$ will be the area of the curve represented by the equation $\left(\frac{x}{a}\right)^{\frac{2m}{2n+1}} + \left(\frac{y}{b}\right)^{\frac{2m}{2n+1}} = 1$. Thus we have found for the equation of the evolute of the ellipse $\left(\frac{x}{A}\right)^{\frac{2}{3}} + \left(\frac{y}{B}\right)^{\frac{2}{3}} = 1$, in which A and B designate two quantities determined by the formula $A a = B b = \pm (a^2 - b^2)$, therefore the surface limited by this curve will be equal to $A B$ multiplied by the area of the curve represented by the equation $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$, which we have found to be $\frac{3}{8}\pi$. Hence the area of the evolute of the ellipse is equal to $\frac{3}{8}\pi \left(\frac{a^2 - b^2}{a b}\right)^2$.

If we suppose $b = a$, then the equation $f\left(\frac{x}{a}, \frac{y}{a}\right) = 0$ is the equation of a curve similar to the curve represented by the equation $f(x, y) = 0$. Hence the areas limited by two similar curves are to each other as the squares of the linear dimensions of these curves.

Let, as in the preceding paragraph, $f(x)$ and $\phi(x)$ be the linear sections made in two surfaces, situated in the same plane, by a plane perpendicular to the axis of x , drawn at a distance x from the origin of coordinates. Let us suppose that this last plane moves, but so as to be always perpendicular to the axis of x , and that it cannot meet one of the two surfaces without meeting also the other. Let x_0 and x_1 be the limits between which the value of the abscissa x ought to remain, in order that the plane perpendicular to the axis of x should really meet the two surfaces. Then the ratio of the two surfaces contained between the ordi-

nates corresponding to the abscissæ x_0 and x_1 will be $\frac{\int_{x_0}^{x_1} f(x) dx}{\int_{x_0}^{x_1} \phi(x) dx}$. Let the integral general of $f(x) dx$,

beginning with x_0 be represented by $F(x)$, and that of $\phi(x) dx$, beginning also with x_0 , be represented by $\Phi(x)$. Then we have $\int_{x_0}^x f(x) dx = F(x)$, $\int_{x_0}^x \phi(x) dx = \Phi(x)$, and $F(x_0) = 0$, $F'(x) = f(x)$, $\Phi(x_0) = 0$, $\Phi'(x) = \phi(x)$. Consequently,

$$\frac{\int_{x_0}^{x_1} f(x) dx}{\int_{x_0}^{x_1} \phi(x) dx} = \frac{F(x_1)}{\Phi(x_1)} = \frac{F'(x_0 + \theta(x_1 - x_0))}{\Phi'(x_0 + \theta(x_1 - x_0))} = \frac{f(x_0 + \theta(x_1 - x_0))}{\phi(x_0 + \theta(x_1 - x_0))},$$

θ designating a number less than one. But it is obvious that $\frac{f(x_0 + \theta(x_1 - x_0))}{\phi(x_0 + \theta(x_1 - x_0))}$ is one of the values which the fraction $\frac{f(x)}{\phi(x)}$ may obtain for the values of x between the limits x_0 and x_1 , and therefore is a quantity contained

between the greatest and the least of these same values. Hence, the ratio of the two integrals $\int_{x_0}^{x_1} f(x) dx$, $\int_{x_0}^{x_1} \phi(x) dx$, that is of the two surfaces under consideration, is a quantity less than the greatest value of the ratio of the two linear sections $f(x)$ and $\phi(x)$ made in these two surfaces by a plane perpendicular to the axis of x , and greater than the least value of the same ratio.

If, for certain values of the abscissa, the plane perpendicular to the axis of x did not meet one of the surfaces, the last proposition might still be demonstrated, but then x_0 and x_1 ought to designate the values, which values of x ought not to exceed, in order that the plane perpendicular to the axis of x should meet one of the two surfaces, and the value of $f(x)$, or $\phi(x)$, should be considered as becoming equal to nothing for every value of x for which the plane perpendicular to the axis of x should not meet the corresponding surface.

Hence two plane surfaces are equal when a plane remaining in every position parallel to the same plane, cuts,

in every position, each of these two surfaces, so that the linear sections are equal to each other. If we suppose $\phi(x) = 1 \int_{x_0}^{x_1} \phi(x) dx = x_1 - x_0$, and consequently $\int_{x_0}^{x_1} f(x) dx = (x_1 - x_0) f(x_0 + \theta(x_1 - x_0))$. But $x_1 - x_0$

represents the linear projection of the surface on the axis of x , and $f(x_0 + \theta(x_1 - x_0))$ a mean value of $f(x)$, between the limits x_0 and x_1 . Hence the ratio between a plane area, and its projection on the axis of x , is a quantity the value of which is a mean between the various values of the linear sections made in that surface by planes perpendicular to the axis of x .

Quadrature of Curve Surfaces.

(394.) Let $z = f(x, y)$ be the equation of a curve surface referred to three rectangular coordinates. Let p be the point of the surface which has for its coordinates x, y, z , and τ be the acute angle which the tangent plane, at that point, makes with the plane of x, y . Let us suppose, that an element w of the surface, both dimensions of which are very small, and included in the point p , is projected, on the tangent plane, and on the plane of x, y . If through a point of the element w , a third plane is drawn perpendicular to the two first, the intersection of this plane with the element w will be a small arc, and with the two projections of the element two small straight lines which will be the projections of the small arc, or of the chord which joins the extremities of this arc. Now, this chord forms a very small angle with the tangent planes drawn to the surface, first through one of the extremities of the arc, and secondly through the point p , which by hypothesis is very near that extremity. If the last angle is represented by δ , the angle which the same chord will form with the plane of x, y will obviously be $\tau \pm \delta$, and, consequently, the two projections of the chord, or the linear sections made in the two projections of the element w will be to each other in the ratio of the two quantities $\cos \delta, \cos(\tau \pm \delta)$. Therefore, according to the theorem demonstrated in the last number, the projection of the element w on the plane of x, y , will have for its measure the projection of the same element on the tangent plane at the point p multiplied by a quantity less than the greatest value of the ratio $\frac{\cos(\tau \pm \delta)}{\cos \delta}$, and greater than the less value of the same

ratio. But the projection of the element w on the tangent plane may be represented by $w(1 + i)$, i designating a very small quantity, hence the projection on the plane of x, y will be represented by a product of the form $w \frac{\cos(\tau \pm \delta)}{\cos \delta} (1 + i)$. Now, the quantities i and δ being both very small, if we designate by i , a third

quantity, also very small, we shall find $\frac{\cos(\tau \pm \delta)}{\cos \tau} (1 + i) = \cos \tau + i_1$, and since the plane of x, y may be supposed to coincide with any plane whatever, the following proposition is demonstrated.

If w be an element of a curve surface, both dimensions of which are very small, and τ the acute angle formed by the tangent plane drawn through a point p of the element to the surface, and another plane, the projection of w on the last plane will be $w(\cos \tau + i_1)$, i_1 being a very small quantity.

Let now P and p be two points of the surface, the first having for its coordinates x_0 and y_0 , and the second corresponding to the coordinates of the variables x and y . If through each of the points P and p we draw two planes respectively perpendicular to the axis of x and y , the intersections of the surface with these four planes will be four curves, and the area included between these four curves will vary with the position of the point p . The projection of this area on the plane of x, y will be a rectangle, the sides of which will be equal to the numerical values of $x - x_0$ and $y - y_0$. Let $\phi(x, y)$ represent this area, and u its projection on the plane of x, y . We shall have, if we suppose $x > x_0$ and $y > y_0$, $u = (x - x_0)(y - y_0)$. Let also Δx and Δy denote very small increments of the variables x and y , and q the point of the surface corresponding to the coordinates $x + \Delta x, y + \Delta y$. Then $\phi(x + \Delta x, y + \Delta y) - \phi(x, y)$ and $\Delta x \Delta y$ will represent, the first the area of the portion of the surface included between the four planes drawn through the points p and q perpendicularly to the axis x and y , and the second the projection of the same area on the plane of x, y . Hence if τ be the acute angle formed by the tangent plane at p with the plane of x, y , we shall have

$$\Delta x \Delta y = (\cos \tau + i_1) \{ \phi(x + \Delta x, y + \Delta y) - \phi(x, y) \},$$

or

$$\frac{\phi(x + \Delta x, y + \Delta y) - \phi(x, y)}{\Delta x \Delta y} = \frac{1}{\cos \tau + i_1}.$$

If in this formula we suppose that the value of Δx and Δy decreases indefinitely, we shall have

$$\frac{d^2 \phi(x, y)}{dx dy} = \frac{1}{\cos \tau} = \sec \tau.$$

It is now easy to find the value of the area $\phi(x, y)$. This area becomes equal to nothing, at the same time as its projection on the plane of x, y , not only for $y = y_0$ whatever be the value of x , but also for $x = x_0$, whatever be the value of y . We have, therefore, generally,

$$\phi(x, y_0) = 0, \phi(x_0, y) = 0, \text{ and, consequently, } \frac{d\phi(x, y_0)}{dx} = 0, \frac{d\phi(x_0, y)}{dy} = 0.$$

Integral
Calculus.

Part III.

This understood, if we integrate the expression $\frac{d^2 \phi(x, y)}{dx dy} = \sec \tau$, first with respect to y , supposing the integral

to begin with $y = y_0$, and secondly with respect to x , supposing the integral to begin with $x = x_0$, we shall find

successively $\frac{d \phi(x, y)}{dx} = \int_{y_0}^y \sec \tau dy$, and $\phi(x, y) = \int_{x_0}^x \int_{y_0}^y \sec \tau dy dx$. If in this last formula, we substitute

for the coordinates x, y of the variable point p , the coordinates x_1, y_1 of a given point Q , situated on the surface, the value A of the area contained between the four planes drawn through the points P and Q perpendicularly to

the axis of x and y , will be $A = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \sec \tau dy dx$. As to $\sec \tau$, if the differential of the equation of the

surface $z = f(x, y)$, be $dz = p dx + q dy$, we shall have $\sec \tau = \sqrt{1 + p^2 + q^2}$.

It is important to remark that in the formula representing the value of A , the order of the two integrations may be inverted.

(395.) Let us suppose now that the surface represented by the equation $z = f(x, y)$ is cut by two planes perpendicular to the axis of x , the first passing through the given point P corresponding to the abscissa x_0 , the second passing through the variable point p corresponding to the abscissa x . Let the same surface be also cut by two cylindrical surfaces, parallel to the axis of z and represented by the equations $y = \phi(x)$ and $y = \phi_1(x)$. We shall have four curves of intersection, and the area contained between these four curves will vary with the value of x and the position of the corresponding plane. Thus this area will be a function of x , which we shall designate by $\psi(x)$. If Δx be a very small increment of x , the expression $\Delta \psi(x)$ will represent a small surface, the projection of which on the plane of x, y will be contained between two small arcs $p q, r s$, measured on the curves represented by the equations $y = \phi(x)$ and $y = \phi_1(x)$, and between two planes perpendicular to the axis of x , corresponding to the abscissæ x and $x + \Delta x$. This understood, let θ and Θ be two numbers positive and less than unity. We shall find by a process of reasoning similar to that used in (393.) that, among the numerical values of the product $\Delta x (\phi_1(x + \Theta \Delta x) - \phi(x + \theta \Delta x))$ which correspond to the various values of the numbers θ and Θ , the least and greatest represent the surfaces of the inscribed and circumscribed rectangular parallelograms to the projection of the surface $\Delta \psi(x)$, and consequently, the projections of two new surfaces, measured on the surface represented by the equation $z = f(x, y)$. Each of these new surfaces, being included between four planes perpendicular to the axis of x , or the axis of y , and corresponding to abscissæ $x, x + \Delta x$, and to ordinates of the form $\phi(x + \theta \Delta x)$,

$\phi_1(x + \Theta \Delta x)$, will be represented by $\Delta x \int_{\phi(x + \theta \Delta x)}^{\phi_1(x + \Theta \Delta x)} \frac{1}{\cos \tau + i_1} dy$, if we suppose the difference $\phi_1(x) - \phi(x)$

positive. Hence the limit of the ratio $\frac{\Delta \psi(x)}{\Delta x}$, or $\frac{d \psi(x)}{dx} = \int_{\phi(x)}^{\phi_1(x)} \sec \tau dy$. And lastly, as we have obviously

$\psi(x_0) = 0$, we shall have $\psi x = \int_{x_0}^x \int_{\phi(x)}^{\phi_1(x)} \sec \tau dy dx$, and for a particular value x_1 of the abscissa x ,

$A = \int_{x_0}^{x_1} \int_{\phi(x)}^{\phi_1(x)} \sec \tau dy dx$, A being the portion of the curve surface represented by the equation $z = f(x, y)$

limited by two planes perpendicular to the axis of x , the equations of which are $x = x_0$ and $x = x_1$, and by two cylindrical surfaces perpendicular to the plane of x, y , the equations of which are $y = \phi(x)$ and $y = \phi_1(x)$. It is necessary here to observe, that in this last formula, representing the value of A , the order of integration cannot be inverted.

If the bases of the cylindrical surfaces, situated in the plane of x, y , and represented by the equations $y = \phi(x)$ and $y = \phi_1(x)$ meet in two points, and if x_0 and x_1 be the abscissæ of these points, the value A , given by the preceding formula, will represent the surface, the projection of which, on the plane of the x, y , is the area contained between the bases of the two cylindrical surfaces. It would still be the same thing if the two curves $y = \phi(x)$ and $y = \phi_1(x)$ touch each other in two points having for abscissæ x_0 and x_1 . Lastly, $\phi(x)$ and $\phi_1(x)$ might be two values of y derived from the same equation, $\phi(x, y) = 0$ representing a curve limited on all sides, it would then be sufficient to substitute for x_0 and x_1 the least and greatest value of the abscissæ, which correspond to the different points of this curve, in order that the preceding value of A should represent the area, measured on the surface $z = f(x, y)$, and limited by the cylindrical surface represented by the equation $\phi(x, y) = 0$.

(396.) If for the surface $z = f(x, y)$ we substitute a surface limited on all sides, and which cannot be met in more than two points by a parallel to the axis of z , then to derive from the preceding value of A the whole area of this new surface, it will be sufficient to suppose that the cylindrical surface $\phi(x, y) = 0$ of the last paragraph, coincides with the cylindrical surface generated by a line parallel to the axis of z , and tangent, in every position, to the surface $z = f(x, y)$ and to divide the required area in two parts, separated by the curve of contact of the cylindrical surface and of the surface represented by the equation $z = f(x, y)$. In most cases this curve of contact is no other than that formed by the points for which the tangent plane is parallel to the axis of z ; and, consequently, the cylindrical surface abovementioned will be represented by the equation $\cos \tau = 0$, or

2 c 2

$\frac{1}{\sqrt{1+p^2+q^2}} = 0$. The contrary, however, might take place, if the sections made in the surface, represented by the equation $z = f(x, y)$ presented cusps. In the hypothesis we have made, the equation of the surface $z = f(x, y)$ will give for each value of x , two positive values of $\sec \tau$, which ought to be substituted one after the other in the preceding value of A , in order to obtain the two areas, the sum of which will be equal to the required area.

(397.) It is essential to remark, that the formula $A = \int_{x_0}^{x_1} \int_{\phi(x)}^{\phi_1(x)} \sec \tau \, dy \, dx$ may still be employed in the case where the functions $\phi(x)$, $\phi_1(x)$ would change their forms with the value of x so as to represent not the ordinate of one curve, but the ordinates of several curves or straight lines placed one after the other. Thus this formula might be used to find the area, measured on the surface $z = f(x, y)$ and limited by a prism having for its basis a polygon drawn in the plane of x, y , and its lateral edge parallel to the axis of z .

(398.) If the projection of the area A , on the plane of x, y , was composed of several parts so disposed that the linear section made in this surface by a plane perpendicular to the axis of x , and corresponding to the abscissa x , were composed of several lengths, the first contained between two given lines, the second between two other lines, &c., then to find the value of the area A , it would be sufficient to divide it in several parts. Let us suppose that $y_0, y_1; y_2, y_3$ are the functions of x , which represent, between the limits x_0 and x_1 , the ordinates which contain between them the parts of the area. We shall divide the area A , in as many corresponding parts, which will be measured by the integrals

$$\int_{x_0}^{x_1} \int_{y_0}^{y_1} \sec \tau \, dy \, dx, \int_{x_0}^{x_1} \int_{y_2}^{y_3} \sec \tau \, dy \, dx, \&c.$$

and by adding all these integrals we shall have the value of A ,

$$\begin{aligned} A &= \int_{x_0}^{x_1} \int_{y_0}^{y_1} \sec \tau \, dy \, dx + \int_{x_0}^{x_1} \int_{y_2}^{y_3} \sec \tau \, dy \, dx + \&c. \dots \\ &= \int_{x_0}^{x_1} \left\{ \int_{y_0}^{y_1} \sec \tau \, dy \, dx + \int_{y_2}^{y_3} \sec \tau \, dy \, dx + \&c. \dots \right\}. \end{aligned}$$

In the same hypothesis the projection of the area A will be represented by

$$\int_{x_0}^{x_1} \left\{ \int_{y_0}^{y_1} dy + \int_{y_2}^{y_3} dy + \&c. \dots \right\} dx = \int_{x_0}^{x_1} (y_1 - y_0 + y_3 - y_2 + \dots) dx.$$

The formulæ given for the value of A , in the different cases we have examined, would be inexact, if the surface were met in several points by lines parallel to the axis of z . Then it would be necessary to divide the required area in several parts, by means analogous to those we have made use of in the case of a plane area limited by curves.

(399.) We shall now apply those general formulæ to some particular cases. First it may be remarked, that $\int_{y_0}^{y_1} \sec \tau \, dy = (y_1 - y_0) \sec t$, t being a mean value between the various values of τ , corresponding to the various

values of y between the limits y_1 and y_0 . Hence $A = \int_{x_0}^{x_1} (y_1 - y_0) \sec t \, dx = \sec t_1 \int_{x_0}^{x_1} (y_1 - y_0) \, dx$, t_1 being a mean value between the various values of t which correspond to the values of x , between the limits x_1 and x_0 , that is to say, t_1 being a mean value between the acute angles which the plane of x, y forms with the tangent planes at the different points of the surface A . Moreover, since as any plane may be taken for the plane of x, y , the truth of the following proposition becomes obvious. The ratio between a curve surface and its projection upon any plane, is a mean quantity between the values of the secants of the acute angles which the tangent planes at the different points of the surface make with the same plane. Hence, if the angle of the tangent plane with the plane of x, y is the same at every point of the surface, the area measured on this surface is equal to the product of its projection on the plane of x, y by the secant of the angle which the tangent plane makes with the plane of x, y . According to this last proposition, the area of the frustum of a cone is equal to the projection of the surface on the base, which projection is equal to the difference of the two bases, multiplied by the secant of the angle which the side of the frustum makes with the base. And it is easy to verify that this expression of the surface of a frustum of a cone agrees with that given in Geometry.

When the equation of the surface is of the form $z = f(x)$, that is to say, when the surface is a cylindrical surface parallel to the axis of y , we have $\sec \tau = \sqrt{1 + (f'(x))^2}$, and $\sec \tau$ containing only x , the integration relative to y gives immediately $A = \int_{x_0}^{x_1} (y_1 - y_0) \sec \tau \, dx$. By means of this formula, it is easy to find the area measured on a cylindrical surface, parallel to the axis of y , and contained between two straight lines parallel to the axis of y , and two planes perpendicular to it. Then y_1 and y_0 are two constant quantities, and we have $A = (y_1 - y_0) \int_{x_0}^{x_1} \sec \tau \, dx$, $y_1 - y_0$ being the distance between the two planes. Moreover, it is obvious that τ is equal to the angle which the tangent to the base of the cylinder, on the plane of xz , makes with the axis of x ,

Integral
Calculus.

hence $\int_{x_0}^{x_1} \sec \tau dx$ represents the length of the arc measured on this curve, and having for its extremities the two

points corresponding to the abscissæ x_1 and x_0 . Therefore the area measured on the surface of a cylinder, parallel to the axis of y , between two planes parallel to the same line, and two planes perpendicular to it, is equal to the distance of these two planes, multiplied by the length of the arc of the base of the cylinder, contained between the planes parallel to the axis of y .

Let us next consider the surface of the cylinder represented by the equation $x^2 - rx + z^2 = 0$, and let it be proposed to find the portion of this surface limited by the surface of the sphere having the origin for its centre and for radius r . The equation of the sphere is therefore $x^2 + y^2 + z^2 = r^2$, and the curve of intersection of the sphere and cylinder has for projection on the plane of x, y the parabola represented by the equation $y^2 = r(r - x)$, which gives for y the two values $y = -\sqrt{r(r - x)}$, $y = \sqrt{r(r - x)}$. Hence the area measured on the cylindrical surface on the positive side of the axis of z , and limited by the curve of intersection of the

cylinder with the sphere, is equal to $\int_0^r 2 \sqrt{r(r - x)} \sec \tau dx$, τ being the angle made by the tangent plane to the cylindrical surface, at the point corresponding to the abscissa x , with the plane of x, y . If we take $z = +\sqrt{r(r - x)}$, we shall have

$$\sec \tau = \sqrt{1 + \frac{(\frac{1}{2}r - x)^2}{rx - x^2}} = \frac{\frac{1}{2}r}{\sqrt{rx - x^2}},$$

and, consequently,

$$\int_0^r 2 \sqrt{r(r - x)} \sec \tau dx = r^{\frac{3}{2}} \int_0^r \frac{dx}{\sqrt{x}} = 2r^2;$$

therefore A , or the whole surface limited by the sphere, not only on the positive side of z , but also on the negative side, is equal to $4r^2$.

If it were required to find the area of the portion of the surface of the sphere, limited by the surface of the cylinder, on the positive side of y and z , we should make use of the formula of (397.), and assume $x_0 = 0$, $x_1 = r$, $y_0 = \sqrt{r^2 - rx}$, $y_1 = \sqrt{r^2 - x^2}$. Moreover, the equation of the sphere being differentiated successively with respect to x and y , gives

$$x + z \frac{dz}{dx} = 0, \quad y + z \frac{dz}{dy} = 0, \quad \text{and } p = -\frac{x}{z}, \quad q = -\frac{y}{z}, \quad \text{therefore } \sec \tau = \frac{r}{z} = \frac{r}{\sqrt{r^2 - x^2 - y^2}}.$$

This understood, we shall have

$$A = r \int_0^r \int_{\sqrt{r^2 - rx}}^{\sqrt{r^2 - x^2}} \frac{1}{\sqrt{r^2 - x^2 - y^2}} dy dx.$$

We have, generally,

$$\int \frac{dy}{\sqrt{r^2 - x^2 - y^2}} = \sin^{-1} \frac{y}{\sqrt{r^2 - x^2}} + c,$$

c being a quantity independent of y , and, consequently,

$$\int_{\sqrt{r^2 - rx}}^{\sqrt{r^2 - x^2}} \frac{dy}{\sqrt{r^2 - x^2 - y^2}} = \frac{\pi}{2} - \sin^{-1} \sqrt{\frac{r}{r+x}} = \cos^{-1} \sqrt{\frac{r}{r+x}}.$$

Therefore

$$A = \int_0^r \cos^{-1} \left(\sqrt{\frac{r}{r+x}} \right) dx.$$

To find this integral let $s \cos^{-1} \left(\sqrt{\frac{r}{r+x}} \right)^{\frac{1}{2}}$, we shall have $x = r \tan^2 s$, and

$$\int \cos^{-1} \left(\sqrt{\frac{r}{r+x}} \right)^{\frac{1}{2}} dx = \int s dx = sx - \int x ds = sx - r \int \left(\frac{1}{\cos^2 s} - 1 \right) ds = (x + r)s - r \tan s + \text{const},$$

and, finally,

$$\int_0^r \cos^{-1} \left(\sqrt{\frac{r}{r+x}} \right)^{\frac{1}{2}} dx = \left(\frac{\pi}{2} - 1 \right) r^2.$$

This quantity being doubled will give the portion of the surface of the sphere limited by the cylinder on the positive side of the axis of y , and corresponding to both positive and negative values of z . The area of this portion of surface is therefore $(\pi - 2)r^2 = (1.41 \dots) r^2$.

(400.) Let $y = f(x)$ be the equation of a curve drawn in the plane of (x, y) ; and let us suppose that this curve turns round the axis of x , it will generate a surface of revolution represented by the equation $y^2 + z^2 = \{f(x)\}^2$. The portion of this surface, situated on the positive side of the axis of y , between two planes perpendicular to the

axis of x , will be represented by the formula $\int_{x_0}^{x_1} \int_{-f(x)}^{f(x)} \sec \tau dx$, x_1 and x_0 being the abscissæ corresponding to the two given planes, and $f(x)$ being supposed not to change its sign between the limits x_0 and x_1 . As to the

Integral
Calculus.

value of $\sec \tau$, it is easy to find $\sec \tau = \pm \frac{f(x) \sqrt{1 + (f'(x))^2}}{\sqrt{(f(x))^2 - y^2}}$. Moreover, it is easy to see that

$$\int_{-f(x)}^{f(x)} \frac{dy}{\sqrt{(f(x))^2 - y^2}} = \pi,$$

and also that $f(x) \sqrt{1 + (f'(x))^2}$ is precisely the length of the normal of the generating curve of the surface of revolution, corresponding to the abscissa x . Consequently, the integral may be reduced to $\pi \int_{x_0}^{x_1} N dx$. The

double of this formula will be the whole area, measured on the surface of revolution, and contained between two planes perpendicular to the axis of x corresponding to the abscissæ x_0 and x_1 .

This formula being demonstrated for the case in which $f(x)$ does not change sign between the limits x_0 and x_1 , it is sufficient to make use of it in the contrary case, to divide the area A into several parts corresponding to the different portions of the curve situated on different sides of the axis of x .

It is also easy to see, that a portion of the surface A contained between two planes passing through the axis of x , and inclined to each other at an angle ϕ , will be equal to $\phi \int_{x_0}^{x_1} N dx$. Lastly, if $f(x)$ does not change sign between the limits x_0 and x_1 , this last formula and the preceding, or value of A , may be replaced by either of the following expressions

$$\phi \int_{x_0}^{x_1} y \sec \tau dx = \phi \int_{x_0}^{x_1} y \sqrt{1 + y'^2} dx, \text{ and } 2\pi \int_{x_0}^{x_1} y \sec \tau dx = 2\pi \int_{x_0}^{x_1} y \sqrt{1 + y'^2} dx.$$

(401.) We may now apply these general formulæ to a few examples.

Let the generating curve $y = f(x)$ be a straight line parallel to the axis of x , and situated at a distance r from that axis. Then $N = r$, and $A = 2\pi \int_{x_0}^{x_1} r dx = 2\pi r (x_1 - x_0)$, which is the well-known expression of the

lateral surface of the cylinder generated by a parallel to the axis of x , turning round that axis, and limited by two planes perpendicular to the axis of x , and situated at distances x_0 and x_1 from the origin.

Let the generating curve be the circumference having for its radius r , and its centre on the axis of x , then A will represent the surface of a spherical zone, of the sphere generated, having for its height $x_1 - x_0$. In that case, we have still $N = r$, and consequently, $A = 2\pi r (x_1 - x_0)$. Hence the surface of a spherical zone is equal to its altitude multiplied by the circumference of a great circle of the sphere. If the altitude is equal to the diameter of the sphere, the spherical zone becomes the whole area of the sphere, which therefore is equal to four great circles.

Let the generating curve be the ellipse represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. The equation of the surface generated will be $\frac{x^2}{a^2} + \frac{y^2 + z^2}{b^2} = 1$, and we shall have $N = \frac{b}{a} \sqrt{a^2 - \left(1 + \frac{b^2}{a^2}\right) x^2}$. If we suppose $a > b$, we shall have, in designating by e , the eccentricity of the ellipse, $ae = \sqrt{a^2 - b^2}$, and consequently,

$$N = \frac{b}{a} \sqrt{a^2 - e^2 x^2}, \text{ and } A = 2\pi \frac{b}{a} \int_{x_0}^{x_1} \sqrt{\left(\frac{a^2}{e^2} - x^2\right)} dx.$$

But $\frac{2b}{a} \int_{x_0}^{x_1} \sqrt{\left(\frac{a^2}{e^2} - x^2\right)} dx$ is the area of a portion of the ellipse represented by the equation

$\frac{e^2 x^2}{a^2} + \frac{y^2}{b^2} = 1$, limited by the two ordinates corresponding to the abscissæ x_0 and x_1 . Hence, if an ellipse is

supposed to turn round its transverse or greater axis, the surface generated by the revolution of an arc of this ellipse will be equal to the product of the number π by the surface limited by the two planes which contain the bases of the zone, in a second ellipse, which may be derived from the first, in increasing the transverse or greater axis in the inverse ratio of the eccentricity.

When the altitude of the zone becomes equal to the transverse axis of the generating ellipse, the preceding theorem gives the value of the area of the whole ellipsoid, and this may be put under the following form,

$$2\pi b^2 + 2\pi \frac{ab}{e} \tan^{-1} \frac{ae}{b}.$$

If we suppose $a < b$, we shall have by designating still by e the eccentricity of the ellipse, $be = \sqrt{b^2 - a^2}$, and

$$N = \frac{b^2 e}{a} \sqrt{\left(\frac{a^4}{b^2 e^2} + x^2\right)}, \text{ and hence } A = 2\pi \frac{b^2 e}{a^3} \int_{x_0}^{x_1} \sqrt{\left(\frac{a^4}{b^2 e^2} + x^2\right)} dx;$$

but $\frac{2b^2 e}{a^3} \int_{x_0}^{x_1} \sqrt{\left(\frac{a^4}{b^2 e^2} + x^2\right)} dx$ is obviously the area limited by the ordinates corresponding to the abscissæ

Integral
Calculus.

Part III.

z_0 and x_1 , in the hyperbola represented by the equation $\frac{y^2}{b^2} - \frac{e^2 x^2}{a^2} = 1$. Hence if an ellipse is made to turn

round its smaller axis, the surface of the zone generated by the revolution of an arc of this ellipse is equal to the product of the number π by the surface limited by the two planes containing the bases of the zone, in an hyperbola the transverse axis of which coincides with the smaller axis of the ellipse, and the other is equal to the square of the greater axis of the ellipse divided by the square of the smaller axis and the eccentricity.

When the altitude of the zone coincides with the smaller axis $2a$ of the generating ellipse, the preceding theorem gives the area of the whole ellipsoid, and the value may be put under the following form:

$$2\pi b^2 + \frac{\pi a^2}{e} \log \left(\frac{1+e}{1-e} \right).$$

Let the generating curve be the hyperbola represented by one of the equations $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$.

Then the equation of the surface of revolution will be one of the two $\frac{x^2}{a^2} - \frac{y^2 + z^2}{b^2} = 1$, $\frac{y^2 + z^2}{b^2} - \frac{x^2}{a^2} = 1$, and

we shall have $N = \frac{b}{a} \sqrt{\left(1 + \frac{b^2}{a^2}\right)x^2 \mp a^2}$, or if we make $ae = \sqrt{a^2 + b^2}$, $N = \frac{b}{a} \sqrt{e^2 x^2 \mp a^2}$. Therefore

we shall have $A = 2\pi \frac{b}{a} \int_{x_0}^{x_1} \left(x^2 \mp \frac{a^2}{e^2}\right) dx$. But it is obvious that $\frac{2b}{a} \int_{x_0}^{x_1} \left(x^2 \mp \frac{a^2}{e^2}\right) dx$ is the area limited

by the two ordinates corresponding to the abscissæ x_0 and x_1 , in one of the hyperbolas represented by the equations $\frac{e^2 x^2}{a^2} - \frac{y^2}{b^2} = 1$, $\frac{y^2}{b^2} - \frac{e^2 x^2}{a^2} = 1$. Hence, if an hyperbola represented by the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

turns round its real axis, the surface of the zone generated by an arc of this hyperbola is equal to the product of π by the surface limited by the two planes which contain the bases of the zone in a second hyperbola which may be derived from the first, in decreasing the length of the transverse axis of the generating hyperbola, in the inverse ratio of the eccentricity.

Let us next suppose the generating curve to be the parabola represented by the equation $y^2 = 2px$, p being a positive quantity. The equation of the surface generated by the revolution of this parabola round the axis of x

will be $y^2 + z^2 = 2px$, and we shall have $N = \sqrt{2px + p^2}$. Hence $x = N^2 - \frac{p^2}{2}$, $dx = N dN$, and the

value of A will be $\frac{2\pi}{p} \int_{n_0}^{n_1} N^3 dN = \frac{2\pi}{3p} (n_1^3 - n_0^3)$, n_1 and n_0 being the values of the normals at the points, the

abscissæ of which are x_0 and x_1 . If we suppose $x_0 = 0$, then $n_0 = p$, and $A = \frac{2\pi}{3} \left(\frac{n_1^3}{p} - p^2 \right)$.

Let the curve be the hyperbola represented by the equation $xy = \frac{r^2}{2}$. The equation of the surface generated by the revolution of this curve round the axis of x is $x^2(y^2 + z^2) = \frac{1}{4}r^4$, and we shall have

$$A = \pi r^2 \int_{x_0}^{x_1} \sec \tau \frac{dx}{x}. \quad \text{But } \tan \tau = \frac{1}{2} \frac{r^2}{x^2}, \text{ and } x^2 = \frac{r^2}{2 \tan \tau}.$$

Hence

$$lx = l(r) - \frac{1}{2} l(2) - \frac{1}{2} l \tan \tau, \quad \text{or } \frac{dx}{x} = -\frac{1}{2} \frac{d\tau}{\sin \tau \cos \tau}.$$

Consequently, if we call τ_1 and τ_0 , the values of τ corresponding to $x = x_1$ and $x = x_0$, we shall have

$$A = -\frac{1}{2} \pi r^2 \int_{\tau_0}^{\tau_1} \frac{d\tau}{\sin \tau \cos^3 \tau} = \frac{1}{2} \pi r^2 \left\{ \frac{1}{\cos \tau_0} - \frac{1}{\cos \tau_1} + l \tan \frac{\tau_0}{2} - l \tan \frac{\tau_1}{2} \right\}.$$

Let us suppose that the generating curve is the logarithmic represented by the equation $y = e^{\frac{x}{a}}$. We shall have

$$A = 2\pi \int_{x_0}^{x_1} e^{\frac{x}{a}} \sqrt{1 + y'^2} dx, \quad \text{but } y' = \frac{\tau}{a} e^{\frac{x}{a}}, \text{ and } dy' = \frac{A}{a^2} e^{\frac{x}{a}} dx,$$

hence

$$A = 2\pi a^2 \int_{y'_0}^{y'_1} \sqrt{1 + y'^2} dy',$$

y'_1 and y'_0 being the values of y' corresponding to $x = x_1$ and $x = x_0$. Moreover, we have

$$\int \sqrt{1 + y'^2} dy' = \frac{1}{2} y' \sqrt{1 + y'^2} + \frac{1}{4} l \left\{ \frac{\sqrt{1 + y'^2} + y'}{\sqrt{1 + y'^2} - y'} \right\} + \text{const} = \frac{1}{2} \frac{\sin \tau}{\cos^3 \tau} + \frac{1}{2} l \left(\frac{1 + \sin \tau}{1 - \sin \tau} \right) + \text{const}.$$

and we shall have

$$A = \pi a^2 \left\{ \frac{\sin \tau_1}{\cos^2 \tau_1} + l \tan \left(\frac{\pi}{4} + \frac{\tau}{2} \right) - \frac{\sin \tau_0}{\cos^2 \tau_0} - l \tan \left(\frac{\pi}{4} + \frac{\tau_0}{2} \right) \right\}.$$

Let again the curve be the catenary represented by the equation $y = a \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}$. Then $N = \frac{a}{4} (e^{\frac{x}{a}} + e^{-\frac{x}{a}})^2$, and we have, in supposing $x_0 = 0$,

$$A = \pi a \int_{x_0}^{x_1} \left(1 + \frac{e^{2x/a} + e^{-2x/a}}{2} \right) dx, \text{ or } A = \pi a \left\{ x_1 + \frac{a}{4} (e^{\frac{2x_1}{a}} + e^{-\frac{2x_1}{a}}) \right\}.$$

For the last example, we shall suppose that the generating curve is the cycloid represented by the equations $x = r(w - \sin w)$, $y = r(1 - \cos w)$. Then we shall have

$$\sqrt{(1 + y'^2)} dx = \sqrt{(dx^2 + dy^2)} = 2^{\frac{1}{2}} \cdot r(1 - \cos w)^{\frac{1}{2}} dw,$$

and, consequently,

$$A = 2^{\frac{3}{2}} \pi r^2 \int_{w_0}^{w_1} (1 - \cos w)^{\frac{3}{2}} dw = 8 \pi r^2 \int_{w_0}^{w_1} \sin^3 \frac{w}{2} dw. \text{ But } 8 \sin^3 \frac{w}{2} = \left(\frac{e^{\frac{w}{2}} \sqrt{-1} - e^{-\frac{w}{2}} \sqrt{-1}}{\sqrt{1}} \right)^3 = 6 \sin \frac{w}{2} - \sin \frac{3w}{2}.$$

Hence, if we assume $w_0 = 0$, we shall have

$$A = 4 \pi r^2 \left\{ \frac{8}{3} - 3 \cos \frac{w_1}{2} + \frac{1}{3} \cos \frac{3w_1}{2} \right\}.$$

If we wish, in particular, to have the area described by a branch of hyperbola, we must suppose $w_1 = 2\pi$, and then we find $A = \frac{64}{3} \pi r^2 = \frac{\pi}{3} (8r)^2$.

(402.) We shall conclude what we proposed to say with respect to the evaluation of curve surfaces of revolution with two theorems which may be useful.

Let us suppose that the curve which is made to turn round the axis of x is represented by the equation $y = b + f(x)$, b designating a constant and positive quantity. The normal to this curve will be equal to $\{b + f(x)\} \sqrt{1 + (f'(x))^2}$, and we shall have for the area generated by the arc, the extremities of which have for abscissæ x_1 and x_0 ,

$$B = 2\pi \int_{x_0}^{x_1} f(x) \{1 + (f'(x))^2\} dx + 2\pi b \int_{x_0}^{x_1} \sqrt{1 + (f'(x))^2} dx = A + 2\pi b \int_{x_0}^{x_1} \sqrt{1 + (f'(x))^2} dx,$$

A being as before the area generated by the curve represented by the equation $y = f(x)$, revolving round the axis of x . But if s represent the length of the arc of the generating curve,

$$s = \int_{x_0}^{x_1} \sqrt{1 + (f'(x))^2} dx. \text{ Hence } B = A + 2\pi b s.$$

Therefore, if successively an arc of a curve is made to revolve, 1. round an axis arbitrarily chosen, 2. round an axis parallel to the first, and at a distance b from it, the difference between the areas of the two surfaces of revolution generated in the two hypotheses, will be the product of the generating arc by the circumference which a point of the second axis would describe if it turned round the first, provided, however, that the two axes should not meet the generating arc, and should be situated on the same side with respect to this arc.

If for the constant b the constant $-b$ be substituted, and if, moreover, we suppose $fx < b$ for every value of x between the limits x_0 and x_1 , the above formula will become $A - 2\pi b s$, and will represent the area B taken with the sign $-$. Thus we shall have $A + B = 2\pi b s$. Hence, if, in the above supposition the two axes of revolution are situated, with respect to the generating arc, the first on one side, and the other on the other side, the sum of the surfaces generated will be equal to the product of the generating arc, by the circumference which one point of the second axis will describe in turning round the other.

By means of these theorems, which may easily be extended to the case where the generating arc should meet in several points planes perpendicular to the axis of rotation, it will be easy to determine the area of the zone generated by an arc of a circle, represented by s , turning round an axis. Let r be the radius of this arc, b the distance of the centre to the axis, and h the altitude of the zone. Let us imagine that a second axis, drawn through the centre, in a direction parallel to the first, does not meet the arc s . If we call A the surface of the zone generated by the circular arc, revolving round the second axis, we shall have $A = 2\pi r h$, and, consequently, $B = 2\pi (bs + rh)$, or $B = 2\pi (bs - rh)$. If the generating arc is equal to half a circumference, then $h = 2r$, $s = \pi r$, and $B = 2\pi r (\pi b + 2r)$, or $B = 2\pi r (\pi b - 2r)$. These two values added will give the area of the surface, designated by the name of *annular surface*, and which therefore is equal to $4\pi^2 b r$.

In the same manner it may be proved that any curve having a centre, and revolving round an axis which does not meet it, generates a surface which has for its measure the product of its perimeter by the circumference which the centre describes round that axis.

Integral
Calculus.

Part III.

(403.) We shall now proceed to explain the process by which the volume or bulk limited by any given surface or surfaces may be determined. This process is called *cubature*. To arrive more easily at a general solution of the problem of the cubature of bodies, we shall first examine the case where it is required to find the volume of a cylinder, the generating line of which is perpendicular to the base, and which, for that reason, may be called a *right cylinder*.

It is easy to prove that the volume V of such a cylinder, the base of which is represented by U and the altitude by h , is equal to the product of the base by the altitude or to hU .

Let us suppose that all the points of space are referred to three rectangular axes; and let the cylinder be placed in such a manner as to have its generating line parallel to the axis of z , and its base on the plane of x, y . At a distance x from the plane of z, y , let us conceive a plane perpendicular to the axis of x ; the intersection of this plane with the base of the cylinder will be a straight line, the length of which will easily be derived from the equation of the circumference of the base of the cylinder, and will be a certain function of x ; let it be represented by $f(x)$. Let also v be that part of the volume V which is situated with respect to the same plane, on the negative side of the x , and u be the corresponding part of the base U of the cylinder. If x become $x + \Delta x$, Δx being an infinitely small quantity, the volume v will become $v + \Delta v$, and the base u of the volume v , $u + \Delta u$. Now, it is easy to see, 1. that the base Δu of the volume Δv is contained between two rectangular parallelograms the surfaces of which may be represented by the products $(f(x) + i_1) \Delta x$, and $(f(x) + i_2) \Delta x$, i_1 and i_2 designating two infinitely small quantities; 2. that the volume Δv is contained between two rectangular parallelipipeds having the same altitude as the cylinder, and for their bases the two preceding rectangular parallelograms. Hence we shall have $\Delta v = h(f(x) + i) \Delta x$, i being an infinitely small quantity, the value of which is between those of i_1 and i_2 . If we divide both sides of this equation by Δx , we shall have $\frac{\Delta v}{\Delta x} = h(f(x) + i)$, and consequently, if we take the limits of both sides $\frac{dv}{dx} = hf(x)$, or $dv = hf(x) dx$.

This understood, let x_0 and x_1 be the least and the greatest value of x which correspond to the base U , we shall have $v = h \int_{x_0}^x f(x) dx$, and $V = h \int_{x_0}^{x_1} f(x) dx$, but the definite integrals contained in these expressions represent respectively the surfaces u and U , therefore $v = hu$, and $V = hU$.

The preceding demonstration would not apply, if the line, the length of which has been represented by $f(x)$, was composed of several separate lengths $f_1(x)$, $f_2(x)$, $f_3(x)$, &c. contained, the first between two given lines, the second between two other given lines, &c. But then we might easily divide the volume V into several parts, V_1 , V_2 , &c. . . . , and the base U into corresponding parts, U_1 , U_2 , U_3 , &c. . . . , so as to establish by the preceding reasoning the following equations, $V_1 = hU_1$, $V_2 = hU_2$, $V_3 = hU_3$, &c. . . . , and the addition of these would lead to the formula $V = hU$.

Let us suppose now, that the volume V to be determined, is limited by any surface whatever. Let $F(x)$ be the area of the section made in the volume V by a plane perpendicular to the axis of x , and corresponding to the abscissa x , and let again the portion of V , situated with respect to that plane on the negative side of abscissæ, be represented by v . If we consider a second plane perpendicular to the axis of x and corresponding to the abscissa $x + \Delta x$, supposing as before Δx to be an infinitely small quantity, the area of the section corresponding to this new plane may be represented by $F(x) + \Delta F(x)$, and the corresponding volume by $v + \Delta v$. It is obvious that the volume Δv will be contained between two cylinders, the volumes of which may be represented by $(F(x) + i_1) \Delta x$, $(F(x) + i_2) \Delta x$, i_1 and i_2 being two infinitely small quantities; hence we shall have $\Delta v = \{F(x) + i\} \Delta x$, i being also an infinitely small quantity, having a mean value between those of i_1 and i_2 . Therefore, if, after having divided both sides of the last equation by Δx , we take the limits, we shall have $\frac{dv}{dx} = F(x)$, or, which is the same thing, $dv = F(x) dx$. Hence, if we call x_0 and x_1 the least and greatest

value of x corresponding to the points of the volume v , that is to say, the limits between which the value of x ought to remain, in order that a plane perpendicular to the axis of x , and corresponding to that abscissa, should

meet the volume V , we shall have $v = \int_{x_0}^x F(x) dx$, and $V = \int_{x_0}^{x_1} F(x) dx$.

The last formula holds good in the case where the area $F(x)$ of the section made in the volume V by a plane perpendicular to the axis of x is changed into the sum of several surfaces,

$$F_1(x), \quad F_2(x), \quad F_3(x), \quad \&c. \quad \dots$$

Then the volume V is the sum of several volumes represented by the integrals

$$\int_{x_0}^{x_1} F_1(x) dx, \quad \int_{x_0}^{x_1} F_2(x) dx, \quad \int_{x_0}^{x_1} F_3(x) dx, \quad \&c. \dots$$

in each of which the function under the sign $\int$ has constantly a positive value, or is equal to nothing, whilst the abscissa x varies between the limits x_0 and x_1 . This understood, we shall have, at the same time, in the present case,

$$F(x) = F_1(x) + F_2(x) + F_3(x) + \&c. \dots \quad N = \int_{x_0}^{x_1} F_1(x) dx + \int_{x_0}^{x_1} F_2(x) dx + \int_{x_0}^{x_1} F_3(x) dx + \&c. \dots$$

or

$$V = \int_{x_0}^{x_1} F(x) dx.$$

Let us suppose, for instance, that x_0 and x_1 being two constant quantities, y_0 and y_1 two functions of the

Integral Calculus. variable x , and z_0, z_1 two functions of the variables x, y , it is required to find the volume contained between the two curve surfaces represented by the equations $z = z_0, z = z_1$, between the two cylindrical surfaces represented by the two equations $y = y_0, y = y_1$, and finally between the two planes $x = x_0, x = x_1$. Part III.

The section made in this volume by a plane parallel to the plane of y, z , and corresponding to a particular value of the abscissa x , will be the area contained between four lines, namely, two curves, and two lines parallel to the axis of z . Let us add, that in order to obtain the equations of these four lines, it will be sufficient to consider x as a constant quantity in the equations $z = z_0, z = z_1, y = y_0, y = y_1$. Let $F(x)$ represent the area of this section, and $f(x, y)$ the length of the linear section made in the surface $F(x)$ by a plane perpendicular to the axis of y , and corresponding to a particular value of y . We shall obviously have $f(x, y) = z_1 - z_0$, and hence

$$F(x) = \int_{y_0}^{y_1} f(x, y) dy,$$

or which is the same thing,

$$F(x) = \int_{y_0}^{y_1} (z_1 - z_0) dy = \int_{y_0}^{y_1} \int_{z_0}^{z_1} dz dy$$

Consequently, since

$$V = \int_{x_0}^{x_1} F(x) dx, \text{ we shall have } V = \int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} dz dy dx,$$

$$\text{or } V = \int_{x_0}^{x_1} \int_{y_0}^{y_1} (z_1 - z_0) dy dx, \quad \text{or } V = \int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x, y) dy dx.$$

It is necessary to remark, that the function $f(x, y)$ is precisely the length of a linear section made in the volume V by means of two planes perpendicular to the axis of x and y . Moreover it is easy to verify that the preceding expressions for the volume V apply to the case in which the linear section $f(x, y)$ instead of being a single straight line would be composed of several distinct lines, the lengths of which would be

$$f_1(x, y), \quad f_2(x, y), \quad f_3(x, y), \quad \&c. \dots$$

and would be contained, the first between two given surfaces, the second between two other surfaces, &c.

The preceding formulæ might also be established by another train of reasoning. Let us suppose first that the quantities y_0, y_1 are constant, that is to say, independent of x . Then the volume V will be comprised between the surfaces represented by the equations $z = z_0, z = z_1$, and four planes perpendicular to the axis of x and y , passing through two straight lines parallel to the axis of z , the first corresponding to the point the coordinates of which are x_0 and y_0 , and the second to the point the coordinates of which are x_1 and y_1 . Now, if we substitute for this second line another parallel to the axis of z corresponding to a point the coordinates of which are generally x and y , the volume V will be changed into another volume which will be a function of x and y . Let us represent this last volume by $\phi(x, y)$, and let $\Delta x, \Delta y$ be infinitely small increments of x and y . The very small increment of V , which we may designate by $\Delta x \Delta y \phi(x, y)$, will obviously have a mean value between the volumes of two parallelepipeds, the one inscribed and the other circumscribed, that is to say, a mean value between two products of the following form,

$$[f(x, y) + i_1] \Delta x \Delta y, \quad [f(x, y) + i_2] \Delta x \Delta y,$$

i_1 and i_2 being two infinitely small quantities. We shall have, therefore,

$$\Delta_y \Delta_x \phi(x, y) = [f(x, y) + i] \Delta x \Delta y,$$

i being an infinitely small quantity having a mean value between i_1 and i_2 . If, in the last equation, we take the limits, first with respect to Δy and then with respect to Δx , we shall obtain the two following equations,

$$\frac{d \Delta_x \phi(x, y)}{dy} = [f(x, y) + i] \Delta x, \quad \frac{d^2 \phi(x, y)}{dx dy} = f(x, y);$$

and consequently integrating first with respect to y , secondly with respect to x , and observing that the function $\phi(x, y)$ vanishes not only for $x = x_0$, whatever be the value of y , but also for $x = x_0$ whatever be the value of x , we shall find

$$\phi(x, y) = \int_{x_0}^x \int_{y_0}^y f(x, y) dy dx, \quad \text{and } V = \int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x, y) dy dx.$$

This formula is therefore demonstrated for the case in which y_0 and y_1 are constant quantities. To deduce from this particular case the general one, the formula $\{f(x, y) + i\} \Delta x dy$ must first be integrated between the limits y_0 and y_1 . We shall thus find

$$\Delta_x \phi(x, y) = \Delta x \int_{y_0}^{y_1} \{f(x, y) + i\} dy,$$

a formula which will give the value of the volume $\Delta x \phi(x, y)$ contained between the surfaces $z = z_0, z = z_1$, and four planes parallel two and two, namely, the two first being perpendicular to the axis of y , and distant from each other of a quantity $y_1 - y_0$, and the two others perpendicular to the axis of x , at a distance from each other equal to Δx . If, in the equations $y = y_0, y = y_1$, the quantities y_0 and y_1 are no longer supposed to represent constant quantities, but functions of x , and if $v = \psi(x)$ represent that part of the volume V cut off by a plane perpendicular to the axis of x , and situated with respect to this plane on the negative side of x , then it may be easily proved that the volume $\Delta \psi(x)$ has a mean value between those of two other volumes represented by products of the following form,

Integral
Calculus.

$$\Delta x \int_{y_0+i_0}^{y_1+i_1} \{f(x, y) + i\} dy,$$

i_0 or i_1 designating quantities which have a mean value between those of the various increments of y_0 or y_1 , when x receives various infinitely small increments between 0 and Δx . Hence the value $\frac{\Delta \psi(x)}{\Delta x}$ will be a mean between two integrals of the form above, and in each of which i_0 , i_1 and i are infinitely small quantities. Besides, if the numerical value of Δx be supposed to decrease indefinitely, these two integrals will converge towards the same limit, namely, $\int_{y_0}^{y_1} f(x, y) dy$. Therefore the limit of the ratio $\frac{\Delta \psi(x)}{\Delta x}$ or $\frac{d\psi(x)}{dx}$ will be equivalent to $\int_{y_0}^{y_1} f(x, y) dy$. Integrating now this last expression with respect to x , and observing that $\psi(x)$ ought to become equal to nothing when $x = x_0$, we shall conclude

$$\psi(x) = \int_{x_0}^x \int_{y_0}^{y_1} f(x, y) dy dx, \quad \text{and for } x = x_1, V = \int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x, y) dy dx.$$

This last formula leads to the following remarks. If the cylindrical surfaces represented by the equations $y = y_0$, $y = y_1$ should intersect, and the intersection be two generating lines corresponding to the abscissæ x_0 and x_1 , the volume V would be limited on all sides by the surfaces $z = z_0$, $z = z_1$ and the two cylinders. The same remark would hold if the two cylindrical surfaces touched each other along the two generating lines corresponding to $x = x_0$ and $x = x_1$. The two cylindrical surfaces would be reduced to one, if the two functions y_0 and y_1 represented two values of y derived from the same equation between y and x . In the same hypothesis, it might happen, 1. that the surfaces represented by the equations $z = z_0$, $z = z_1$, being separate and distinct surfaces, should have for their intersection a curve situated also on the cylindrical surface; 2. that z_0 and z_1 were the two values of z derived from the same equation $f(x, y, z) = 0$, representing a surface limited on all sides, and to which the cylindrical surface would be circumscribed. In the first case, the volume V would be limited on all sides by the surfaces $z = z_0$ and $z = z_1$, in the second case V would designate the volume contained by the surface $f(x, y, z) = 0$ If $z_0 = 0$, then the volume V would be that limited in the direction of the axis of x , by the two planes $x = x_0$, $x = x_1$; in the direction of the y by the cylindrical surfaces represented by the equations $y = y_0$ and $y = y_1$; finally, in the direction of the z by the surface $z = f(x, y)$.

(404.) We shall now show some applications of the formulæ we have established.

Let us first suppose that after having drawn on a plane perpendicular to the axis of x a curve limiting an area represented by B , a straight line, not parallel to the plane of z, y , moves so as to remain constantly parallel to a given direction, and to meet, in every position, the given curve. This line will generate a cylindrical surface; and if we cut the volume V limited by this surface and the two planes represented by the equations $x = x_0$, $x = x_1$ by another plane perpendicular to the axis of x , the section will be perfectly equal to B . In this case we shall have therefore $F(x) = B$; and consequently $V = B \int_{x_0}^{x_1} dx = B(x_1 - x_0)$, or if we call h the distance of the two planes $V = Bh$. Hence the volume V of an oblique cylinder the base of which is B and the altitude h is equal to the product Bh of the base by the altitude.

(405.) Let us next consider the surface of a cone having its centre at the origin of coordinates. Let us conceive, moreover, that the base of this cone is a plan curve situated in a plane perpendicular to the axis of x , and at a distance from the origin of coordinates equal to 1. Let y_1 and z_1 be the variable coordinates of the curve, and $f(y_1, z_1) = 0$ its equation. The generating line passing through the point y_1 and z_1 of this curve will be represented by the equations $y = y_1 x$, $z = z_1 x$. Now, if y_1 and z_1 be eliminated between this equation and the equation $f(y_1, z_1) = 0$, it is clear that the resulting equation, namely, $f\left(\frac{y}{x}, \frac{z}{x}\right) = 0$, will be the equation of the surface of the cone. This understood, let A represent the area of the base of this cone. The section of this cone by a plane perpendicular to the axis of x , will be a curve similar to the base of the cone, and therefore if x be the distance of this section to the origin of coordinates, the surface of the section will be Ax^2 . We shall have, therefore, in this case $F(x) = Ax^2$, and consequently, $V = A \int_{x_0}^{x_1} x^2 dx = \frac{A(x_1^3 - x_0^3)}{3}$. If the required volume

is that limited by the surface and by the plane represented by the equation $x = x_1$, then $x_0 = 0$, and $V = \frac{1}{3} Ax_1^3$. Besides, if h be the altitude of the cone, and B the surface of the base contained in the plane $x = x_1$, then $h = x_1$ and $B = Ax_1^2$, therefore $V = \frac{1}{3} Bh$. Hence the volume limited by the surface of a cone and a plane not passing through the centre is equal to the third of the product obtained in multiplying the base by the altitude.

With respect to the truncated cone, or volume limited by the surface of the cone, and two parallel planes $x = x_0$, $x = x_1$, it has already been found equal to $\frac{1}{3}(x_1^3 - x_0^3)A$; but if b and B are the surfaces of the two bases, and h the altitude, then $b = Ax_0^2$, $B = Ax_1^2$, and $h = x_1 - x_0$, consequently, $V = \frac{1}{3}h(B + b + \sqrt{Bb})$. Hence, the volume of a truncated cone is equal to the third of the product obtained in multiplying the altitude by the sum of the two bases and a mean proportional between the two bases.

Integral
Calculus.

It is obvious that this last theorem as well as the preceding one relative to the volume of a cylinder would still hold, if the bases instead of being limited by a single continuous line was limited by a system of several lines curve or straight, and consequently, the expressions found for the volumes of cylinders and cones apply also to prisms and pyramids. Part III.

(406.) Let us next suppose, that after having drawn in the plane of x, y a curve represented by the equation $y = f(x)$, this curve is made to revolve round the axis of x , it will generate a surface of revolution, such as the section made by a plane perpendicular to the axis of x will be a circumference having for radius the ordinate of the generating curve. Therefore, if we designate by y this ordinate, the surface of the circle limited by this circumference will be πy^2 . Hence the volume V limited by the surface of revolution and two planes perpendicular to the axis of revolution $x = x_0, x = x_1$, will be $V = \pi \int_{x_0}^{x_1} y^2 dx$.

Let us apply this last formula to a few examples.

Example 1. Let the generating curve be the circle represented by the equation $x^2 + y^2 = r^2$, we shall have in supposing $x_0 = 0$,

$$V = \pi \int_0^x (r^2 - x^2) dx = \pi x \left(r^2 - \frac{1}{3} x^2 \right) = \frac{2}{3} \pi r^2 x + \frac{1}{3} \pi y^2 x,$$

hence the volume of a segment of a sphere, included between two parallel planes, one of which passes through the centre of the sphere, is equal to the volume of the cone having for its base the small base of the segment, and the same altitude, plus the surface of the spherical zone limiting the segment multiplied by one-third of the radius. If the altitude of the segment becomes equal to the radius r , the volume V will be simply reduced to $\frac{2}{3} \pi r^3$, and the double of this product $\frac{4}{3} \pi r^3$ will represent the volume of the sphere.

Example 2. Let the generating curve be the parabola represented by the equation $y^2 = 2px$, we shall find by assuming $x_0 = 0$, $v = \pi p \int_0^x 2x dx = \pi p x^2 = \frac{1}{2} \pi y^2 x$. Hence it follows that the solid generated by the revolution of the parabola produced to the point the abscissa of which is x , has for its measure half the product of its altitude by the surface of the circle πy^2 which may be considered as its base.

Example 3. Let the generating curve be that represented by the equation $y = Ax^a$, A and a being two real and constant quantities. If we suppose also $x_0 = 0$, we shall have

$$V = \pi A^2 \int_0^x x^{2a} dx = \pi A^2 \frac{x^{2a+1}}{2a+1} = \frac{1}{2a+1} \pi y^2 x.$$

If the constant a is positive, the equation $y = Ax^a$ represents a parabola of the degree a , and from the preceding formula we may therefore conclude, that the volume generated by this parabola produced from the origin to the point the abscissa of which is designated by x , is to the volume of the circumscribed cylinder as 1 to $2a+1$. In the particular case where $a = \frac{1}{2}$ we are brought back to the third example.

Example 4. If the generating curve is the logarithmic represented by the equation $y = alx$, we shall have, supposing $x_0 = 0$,

$$V = \pi a^2 \int_0^x (lx)^2 dx = \pi a^2 x \left\{ (lx)^2 - 2(lx) + 2 \right\}.$$

If the equation of the logarithmic was under the form $y = e^{\frac{x}{a}}$, then

$$V = \pi \int_0^x \frac{e^{\frac{2x}{a}}}{a^2} dx = \frac{\pi a}{2} (e^{\frac{2x}{a}} - 1) = \frac{\pi a}{2} (y^2 - 1).$$

Example 5. If the generating curve be the catenary represented by the equation $y = a \frac{e^{\frac{x}{a}} + e^{-\frac{x}{a}}}{2}$, we shall have, supposing $x_0 = 0$,

$$V = \frac{\pi a^2}{4} \int_0^x (e^{\frac{2x}{a}} + 2 + e^{-\frac{2x}{a}}) dx = \frac{\pi a^2}{2} \left\{ x + \frac{a}{4} (e^{\frac{2x}{a}} - e^{-\frac{2x}{a}}) \right\}.$$

(407.) Sometimes the determination of the volume of a body may be made easier by the substitution of another variable for the variable x . Let us suppose that the generating curve is a cycloid, the equation of which has already been given in the preceding pages. Then making use of the same letters, we shall have

$$dx = y dw, \quad y^2 dx = y^3 dw = r^3 (1 - \cos w)^3 dw,$$

and if we designate by w_0 the value of w corresponding to $x = x_0$, we shall have

$$V = \pi r^3 \int_{w_0}^w (1 - \cos w)^3 dw.$$

If we suppose $x_0 = 0$, we shall have necessarily $w_0 = 0$, and consequently,

$$V = \pi r^3 \int_0^w (1 - \cos w)^3 dw.$$

We have, besides,

Integral
Calculus.

Part III.

$$(1 - \cos w)^3 = 2^3 \sin 6 \frac{w}{2} = \frac{1}{(-2)^3} \{ e^{\frac{w}{2}} \sqrt{-1} - e^{-\frac{w}{2}} \sqrt{-1} \}^3 = \frac{1}{4} (10 - 15 \cos w + 6 \cos 2w - \cos 3w).$$

Hence we shall have

$$V = \frac{\pi r^3}{4} \left(10w - 15 \sin w + 3 \sin 2w - \frac{1}{3} \sin 3w \right).$$

In this formulæ w represents the angle described by the radius of the generating circle of the cycloid, therefore to obtain the volume generated by the first branch of the cycloid, we must, in the preceding formula, suppose $w = 2\pi$, and we shall have $V = 5\pi^2 r^3$.

(408.) If we suppose now that two curves situated in the plane of x, y and represented by the equations $y = y_0$ and $y = y_1$, the right sides of these two equations being two functions of x which are positive for every value of x between x_0 and x_1 , are made to turn round the axis of x , the volume limited by the surfaces generated by the two curves, and by the two planes $x = x_0, x = x_1$ will be equal to the difference of the integrals $\pi \int_{x_0}^{x_1} y_1^2 dx$

and $\pi \int_{x_0}^{x_1} y_0^2 dx$, so that if we designate by V this volume we shall have

$$V = \pi \int_{x_0}^{x_1} (y_1^2 - y_0^2) dx = 2\pi \int_{x_0}^{x_1} \int_{y_0}^{y_1} y dy dx.$$

To deduce this formula in a direct manner from the preceding principles, it is sufficient to observe that the volume under consideration being cut by a plane perpendicular to the axis of x , gives for section a zone limited by two circumferences the radii of which coincide with the ordinates y_0 and y_1 , we have therefore, in this case, $F(x) = \pi(y_1^2 - y_0^2)$, and consequently the above expression of the volume V . This volume is precisely that generated by the area U limited, in the plane of x, y , by the curves represented by the equations $y = y_0, y = y_1$ and by the ordinates corresponding to the abscissæ x_0 and x_1 . This understood, let us suppose that the area U is made to slide in the plane of x, y in such a manner as the distance of each point of it to the axis of x increases by the same quantity b , or, which is the same thing, let us suppose that the area B revolves round a line parallel to the axis of x and distant from it by a quantity equal to b . It is obvious that the new solid generated by the revolution of the area U will be a volume equal to the difference

$$\pi \int_{x_0}^{x_1} \{ (y_1 + b)^2 - (y_0 + b)^2 \} dx = \pi \int_{x_0}^{x_1} (y_1^2 - y_0^2) dx + 2\pi \int_{x_0}^{x_1} (y_1 - y_0) dx = V + 2\pi b U.$$

It would be easy to ascertain that the same conclusion would be obtained if the area U were entirely limited by the curves $y = y_0$ and $y = y_1$, or by a single curve. Besides the product $2\pi b$ represents the circumference of a circle which has for its radius the distance b , and the axis of x may coincide with any straight line whatever drawn arbitrarily in the plane of the area U , but entirely out of this area. Hence we may conclude that if a plane surface is made to revolve, 1. round an axis situated in the plane of this area, but entirely out of it, 2. round an axis parallel to the first but more distant from the area, the difference between the volumes generated will be equal to the product of the generating area by the circumference of a circle the radius of which is precisely the distance of the two axes of revolution.

Let us suppose now, that the area U may be divided into two symmetrical parts by a straight line parallel to the axis of x , and distant from it by a quantity equal to b . In this case, the equations of the curves limiting the area U may be written under the following form,

$$y = b - f(x), \quad y = b + f(x),$$

and then

$$V = \pi \int_{x_0}^{x_1} \{ (b + f(x))^2 - (b - f(x))^2 \} dx,$$

or, which is the same thing,

$$V = 4\pi b \int_{x_0}^{x_1} f(x) dx.$$

But we have also

$$U = \int_{x_0}^{x_1} \{ (b + f(x)) - (b - f(x)) \} dx = 2 \int_{x_0}^{x_1} f(x) dx,$$

hence we shall find $V = 2\pi b U$. The same formula subsists when the area U is limited by a single curve, or by several curves joining, provided it may be divided into two symmetrical parts by the straight line $y = b$. Hence, when a plane surface may be divided by a straight line into two symmetrical parts, the solid generated by the revolution of this area round an axis parallel to that line is equal to the product of the area by the circumference of a circle having for radius the distance between the two parallel lines.

Thus, for instance, the solid generated by the revolution of the ellipse represented by the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

round the tangent drawn through one of the extremities of the axis $2b$, is equal to the product obtained in multiplying the surface πab of the ellipse by the circumference $2\pi b$ described by the centre of this curve. Hence the volume V of this body is determined by the equation $V = 2\pi^2 ab^2$.

(409.) Another remarkable theorem results from the formula $V = \int_{x_0}^{x_1} F(x) dx$. Let us consider two parallel

Integral
Calculus.

planes, and parallel to the axis of x , let b be their distance, and let us suppose that a curve is drawn in one of them. If then, from every point of that curve perpendiculars are let fall on the axis, they will generate a surface which together with the two planes will limit a certain volume. Now, this volume is equal to its altitude b multiplied by half the sum of the two bases situated in the two planes. Part III.

To demonstrate this proposition it is sufficient to remark, that if the volume under consideration be cut by a plane perpendicular to the axis of x , the section, the area of which may be as before represented by $F(x)$, will be a trapezium, in which the parallel sides are the linear sections made by this plane in the two bases of the volume. Therefore, if these two linear sections are represented by $f_1(x)$ and $f_2(x)$, we shall have

$$F(x) = \frac{b}{2} (f_1(x) + f_2(x)),$$

and consequently,

$$V = \frac{b}{2} \int_{x_0}^{x_1} \{f_1(x) + f_2(x)\} dx,$$

which equation proves the truth of the theorem stated above.

(410.) To show an application of the formula

$$V = \int_{x_0}^{x_1} \int_{y_0}^{y_1} f(x, y) dy dx,$$

we shall suppose that it is required to find the volume V limited by the plane of x, y , a cylindrical surface of the generating line of which is parallel to the axis of z , and the hyperbolic paraboloid represented by the equation $xy = c^2$. We shall find in this case $f(x, y) = \frac{1}{c} xy$, and consequently,

$$V = \frac{1}{c} \int_{x_0}^{x_1} \int_{y_0}^{y_1} xy dx dy = \frac{1}{2c} \int_{x_0}^{x_1} (y_1^2 - y_0^2) dx dy.$$

If the base of the cylindrical surface is the circumference represented by the equation $(x-a)^2 + (y-b)^2 = r^2$, a and b being two positive quantities greater than r , then

$y_0 = b - \sqrt{r^2 - (x-a)^2}$, $y_1 = b + \sqrt{r^2 - (x-a)^2}$, and $y_1^2 - y_0^2 = 4b\sqrt{r^2 - (x-a)^2}$, $x_0 = a-r$, $x_1 = a+r$, and consequently,

$$V = 2 \frac{b}{c} \int_{a-r}^{a+r} \sqrt{r^2 - (x-a)^2} dx.$$

Then, by assuming $x-a = rt$, we shall have

$$V = 2 \frac{b}{c} r^2 \int_{-1}^{+1} (a+rt) \sqrt{1-t^2} dt.$$

Besides, we have obviously

$$\int_{-1}^{+1} t \sqrt{1-t^2} dt = 0,$$

whilst the integral

$$\int_{-1}^{+1} \sqrt{1-t^2} dt = \frac{\pi}{2}.$$

Hence $V = \pi r^2 \frac{bc}{a}$. Therefore the volume V is equal to the product of the base πr^2 by a fourth proportional to the three lines a, b, c .

If the surface required were to be limited by the same hyperbolic paraboloid, and by four planes perpendicular to the axis of x and y , then the quantities y_0 and y_1 would be constant quantities, and we should find

$$V = \frac{1}{4c} (x_1^2 - x_0^2) (y_1^2 - y_0^2),$$

or which is the same thing,

$$V = (x_1 - x_0) (y_1 - y_0) \frac{x_0 y_0 + x_0 y_1 + x_1 y_0 + x_1 y_1}{4c}.$$

But the product $(x_1 - x_0) (y_1 - y_0)$ represents the surface of the rectangular parallelogram which, in this case, is the base of the volume under consideration. Let

$$\frac{x_0 y_0}{c} = z_1, \quad \frac{x_0 y_1}{c} = z_2, \quad \frac{x_1 y_0}{c} = z_3, \quad \frac{x_1 y_1}{c} = z_4,$$

z_1, z_2, z_3, z_4 , will obviously be the four ordinates of the hyperbolic paraboloid corresponding to the four summits of this rectangular parallelogram, and we shall have

$$V = (x_1 - x_0) (y_1 - y_0) \frac{z_1 + z_2 + z_3 + z_4}{4}.$$

It may be well to remark, that in order to construct the hyperbolic paraboloid which limits this surface, it will be sufficient to join two and two the extremities of the four ordinates z_1, z_2, z_3 , and z_4 , and to suppose that a straight

Integral
Calculus.

line moves in such a manner as to meet two of the opposite sides of the quadrilateral figure so formed, and to be in a plane parallel to the other two sides. Hence the volume limited by the surface so generated, the four planes passing through the sides of the quadrilateral figure, and a fifth plane perpendicular to them, is equal to the product of the rectangular parallelogram which is the base of this volume; multiplied by the fourth part of the sum of the four ordinates corresponding to the four summits of the rectangular base.

(411.) The volume V limited by a given surface, may also be obtained by the following rule.

Let any straight line or axis be drawn, and let us draw in a plane passing through that axis, an auxiliary curve such as the section made in the volume by any plane perpendicular to the axis should always be equal in surface to a rectangular parallelogram having for one of its sides a given line b , and for the other side the linear section made by the same plane in the area drawn on the plane passing through the axis. Then, if we suppose this area limited in the direction of the axis by the two linear sections corresponding to the two extreme positions of the plane supposed to move so as to remain constantly perpendicular to the axis, and to meet the volume under consideration, this volume will be equal to that area multiplied by b .

In order to show the truth of this rule, let us suppose that the axis above mentioned is taken for the axis of x , and if we still call $F(x)$ the area of the section made in this volume by a plane perpendicular to the axis, and

corresponding to the abscissa x , $\frac{F x}{b}$ will be the ordinate of the auxiliary curve; or in other words the linear

section made in the area situated in the plane passing through the axis. Hence this area is equal to

$$\int_{x_0}^{x_1} \frac{F(x)}{b} dx = \frac{1}{b} \int_{x_0}^{x_1} F(x) dx.$$

Now if we multiply this integral by b , we shall find the expression previously found for the volume V .

If the volume V be limited by the surface of a cone or by a pyramid, with any base whatever, and if we take for the axis of the abscissæ a straight line perpendicular to the base, the section $F(x)$ will be proportional to x^2 , and the auxiliary curve will be a parabola the axis of which will be the axis of ordinates. But it has been proved that the area limited by the axis of x , the parabola, and an ordinate, is the third part of the rectangular parallelogram circumscribed to area. This understood, it may be concluded immediately that the volume limited by a cone or a pyramid, is the third part of the cylinder or prism having the same base and same altitude.

If the section $F(x)$ be a trapezium the two parallel sides of which are the linear sections made in two plane surfaces situated in planes parallel to the axis, and if we suppose that b is precisely equal to the distance of the two planes, the ordinate of the auxiliary curve will be precisely half the sum of the linear sections we have just mentioned; hence we shall conclude that the area limited by this auxiliary curve is half the sum of the two plane surfaces, and consequently the theorem of the paragraph (409.)

It would not be more difficult to find by means of the last theorem, the volume limited by two parallel planes, and the surface of a sphere, an ellipsoid, or hyperboloid, &c.

(412.) If the surfaces of the sections of two separate volumes made by a series of parallel planes are in a constant ratio, the two volumes will be to each other in the same ratio.

In order to demonstrate this theorem let us suppose that the different points of space are referred to three rectangular axes, and that the axis of x is perpendicular to the series of parallel planes. If we call $F(x)$ and $f(x)$ the surfaces of the two sections made in the two volumes by one of these planes, namely, by that which corresponds to the abscissa x , we shall have, according to the hypothesis, $F(x) = c f(x)$, c being the constant ratio. Let also x_0 and x_1 be the limits between which the values of the abscissa x must fall, in order that the plane corresponding to that abscissa, and perpendicular to the axis of x , should meet the two volumes. The measure

of the first will be $\int_{x_0}^{x_1} F(x) dx$, and that of the second $\int_{x_0}^{x_1} f(x) dx$, but since $F(x) = c f(x)$, we shall also

have $\int_{x_0}^{x_1} f(x) dx = \frac{1}{c} \int_{x_0}^{x_1} F(x) dx$, which formula agrees with the theorem stated at the beginning of this paragraph.

Hence we may conclude that two volumes are equivalent when the sections made in them by a series of parallel planes are equal in surface.

Let us next consider the sphere represented by the equation $x^2 + y^2 + z^2 = a^2$, and the ellipsoid represented by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. It is obvious that the section made in these two bodies by a plane perpendicular to the

axis of x , and corresponding to the abscissa x , are for the first a circle the radius of which is $(a^2 - x^2)^{\frac{1}{2}}$, and for

the second an ellipse the semi-axes of which are $\frac{b}{a}(a^2 - x^2)^{\frac{1}{2}}$, and $\frac{c}{a}(a^2 - x^2)^{\frac{1}{2}}$; the surface of the first section is

therefore $\pi(a^2 - x^2)$, and that of the second $\frac{\pi b c}{a^2}(a^2 - x^2)$, the ratio of which is that of a^2 to $b c$. Hence we

shall conclude that the volume of the sphere $\frac{4}{3} \pi a^3$ is to that of the ellipsoid as $a^2 : b c$, and therefore that this

last volume is $\frac{4}{3} \pi a b c$. The same ratio subsists clearly between two segments of the sphere and ellipsoid limited by any two parallel planes.

(413.) If we suppose a line to move so as to remain parallel to a given axis, and, if we suppose, besides, that the lengths of this line, intercepted by two separate volumes, in every position, have a constant ratio, then we may conclude that this ratio is also the ratio of the two volumes.

Let, as before, the points of space be referred to three rectangular axes, and let the axis of z be parallel to the line that is supposed to move. This understood, let $f_1(x, y)$, $f_2(x, y)$ be the lengths of that straight line intercepted by the two volumes in any particular position. We shall have, according to the hypothesis $f_1(x, y) = c f_2(x, y)$, c being the constant ratio. Let also x_0, x_1 be constant quantities, and y_0 and y_1 be functions of x such as the equations $x = x_0, x = x_1, y = y_0, y = y_1$, be those of the planes and cylindrical surfaces which limit the space in which the moving line ought to remain in order to meet the two volumes. The first

volume, according to what precedes, will have for its measure $\int_{x_0}^{x_1} \int_{y_0}^{y_1} f_1(x, y) dy dx$, and the second $\int_{x_0}^{x_1} \int_{y_0}^{y_1} f_2(x, y) dy dx$, therefore since $f_1(x, y) = c f_2(x, y)$, we shall have also the first volume equal to the product of the second by c .

The preceding demonstration would not apply if each of the given volumes could be met in several points by a line parallel to the axis of x , or to the axis of y . But then it would be easy to decompose the two volumes into corresponding parts, the ratio of which determined by the preceding reasonings would be equal to c ; and, since when the parts of a sum increase or decrease in the ratio of 1 to c , this sum increases or decreases in the same ratio, it is obvious that the theorem would still be true.

The same theorem would also hold in the case where the lengths $f_1(x, y)$, $f_2(x, y)$ would be composed of several separate lengths, the first included between two given surfaces, the second between two other given surfaces, &c. Let us consider two surfaces each of which completely limits a volume. Let the equation of the

first surface be $f(x, y, z) = 0$, and that of the second $f\left(x, y, \frac{z}{c}\right) = 0$. It is clear that for each system of values

given to x and y , will be derived from these two equations corresponding values of z which will be to each other in the ratio of 1 to c . Hence the limits which the values of x and y ought not to exceed, in order that the values of z should be real, will not change when one of these surfaces is substituted for the other. Moreover, if we designate by $z_0, z_1, z_2, z_3, \dots$ &c. the various real values of z , furnished by the first equation for a given system of values of x and y , those corresponding to the same system given by the second equation will obviously be $c z_0, c z_1, c z_2, c z_3, \dots$ &c. If, besides, we suppose the quantities $z_0, z_1, z_2, z_3, \dots$ &c. to be arranged according to their magnitude, the total length which may be represented by $\phi_1(x, y)$ of the portion of the parallel to the axis of z , corresponding to the system of values of x and y under consideration, contained within the first surface, will be $\phi_1(x, y) = z_1 - z_0 + z_3 - z_2 + \dots$, and for the second the total length corresponding represented by $\phi_2(x, y)$ is equal to $c z_1 - c z_0 + c z_3 - c z_2 + \dots = c(z_1 - z_0 + z_3 - z_2 + \dots)$; we shall have, therefore, $\phi_2(x, y) = c \phi_1(x, y)$, and consequently according to the theorem demonstrated above, we may conclude that the volume limited by the surface represented

by the equation $f\left(x, y, \frac{z}{c}\right) = 0$, is equal to the product of the volume limited by the surface represented by the equation $f(x, y, z) = 0$, multiplied by c . In the same manner it might be proved that the volumes limited by the surfaces $f\left(\frac{x}{a}, y, z\right) = 0$, $f\left(x, \frac{y}{b}, z\right) = 0$ are respectively equal to the volume limited by the surface $f(x, y, z) = 0$ multiplied by a and b .

If now we compare the volumes limited by the surfaces represented by the equations

$$f(x, y, z) = 0, \quad f\left(\frac{x}{a}, y, z\right) = 0, \quad f\left(\frac{x}{a}, \frac{y}{b}, z\right) = 0, \quad f\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) = 0,$$

we shall easily establish that the second is equal to the product of the first by a , the third to the product of the second by b , and the fourth to the product of the third by c . Hence the last is equal to the product of the first by $a b c$.

Therefore, in general, to find the volume limited entirely by a surface represented by the equation $f\left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right) = 0$, it is sufficient to find the volume limited by the surface represented by the equation $f(x, y, z) = 0$, and to multiply it by $a b c$.

From this general theorem it is easy to deduce the volume of the ellipsoid represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ from the volume of the sphere represented by the equation } x^2 + y^2 + z^2 = 1.$$

If we suppose $a = b = c$, then we see that the surface represented by the equation $f\left(\frac{x}{a}, \frac{y}{a}, \frac{z}{a}\right) = 0$ is similar to the surface represented by the equation $f(x, y, z) = 0$, and we conclude from the preceding theorem that the volumes limited by two similar surfaces are to each other as the cubes of their linear dimensions.

CALCULUS OF VARIATIONS.

Calculus of Variations. (414.) IN the problems of Maxima and Minima which have been treated in the preceding pages, the relation between the function and the variable is either immediately known, or is determined by means of some equation of condition arising from the nature of the question; and the problem is easily solved by the ordinary rules. But there is a numerous class of problems, and of vast variety, in which not only is the property of a maximum or minimum to be recognised, but the particular function to be investigated in which this property takes place. Part III.

For example, if it be required to find the shortest distance between two known curves, we must first discover the species of curve to which this property belongs, and then find the least distance.

As a further illustration, let us suppose V to be a function of $x, y, p, q, \&c.$, where $p, q, \&c.$, are the differential coefficients of y with regard to x . The integral $\int V dx$, taken between the same values of x , as a and a_1 , is susceptible of an infinite number of values, dependent upon the relation between y and x ; we may therefore ask what particular equation between these quantities y and x will make $\int V dx$ to become a maximum or minimum. And, here, whether the question be of the least length, between two points, or of the greatest volume, or surface, or any other possible condition, it is obvious that in causing the ordinate y or the abscissa x to vary we must be enabled to pass from one curve or function to another: and the first thing which must be done will be to investigate the change that takes place in the function $\int V dx$, when y and x have changed according to the manner here hinted at.

Problems of this description early occupied the attention of the Mathematicians, who perfected and practised the new Calculus which the genius of Newton had invented, and the notation of Leibnitz had rendered universally acceptable.

But as it would exceed the limits prescribed to us, were we to undertake the task of presenting to the reader even a short account of the methods by which the Bernouillis and Euler solved these problems, or of following them through their laborious efforts to perfect the processes which were at first rude, and sometimes inapplicable, we shall confine ourselves to the method which, under the name of the Calculus of Variations, owes its invention to Lagrange, and of which the notation is not one of the least of its excellencies.

(415.) In this Calculus, the change which the function undergoes in consequence of the alteration in the value of y is called variation, and variation bears to the function the same relation that differentiation does to the original function in the Differential Calculus; in fact, it has been defued to be differentiation under a new symbol. The distinction, however, between the operations may be illustrated by means of a curve.

Suppose we wish to find the shortest distance between two points, of which the coordinates are given.

If C and D be the two given points, (fig. 1.) innumerable curves may be drawn from C to D , and we have to find which of these curves possesses the required property. We must, therefore, in our reasoning pass from one curve to another; as, for instance, from a point P in one curve to a point Q in a curve above it. This change from P to Q is called a variation of the ordinate NP , while a change from P to P_1 , or from Q to Q_1 , P_1 and Q_1 being consecutive points to P and Q respectively, in the curves which are the loci of P and Q , is merely a differentiation of the ordinates NP and NQ .

The symbol (δ) is used to express the variation or change from P to Q , and thus, while dy expresses the alteration of the ordinate y , by passing from one point to another in the same curve, δy represents the alteration of the ordinate when its extremity passes from one curve to another; but δ has also a more extended meaning, as we shall see hereafter: in this case, however,

$$\begin{aligned} \text{If} \quad & NP = y, \quad MP_1 = y + dy \\ \text{and, since} \quad & PQ = \delta y, \quad NQ = y + \delta y. \end{aligned}$$

(416.) From the preceding illustration may be proved a theorem of which great use is made in the Calculus.

$$\begin{aligned} \text{Since} \quad & NQ = y + \delta y \therefore MQ_1 = NQ + d \cdot NQ = (y + \delta y) + d \cdot (y + \delta y) \\ \text{And} \quad & \therefore MP_1 = y + dy \therefore MQ_1 = MP_1 + \delta \cdot MP_1 = y + dy + \delta(y + dy) \\ & \therefore y + \delta y + dy + d\delta y = y + dy + \delta y + \delta dy. \\ & \therefore d\delta y = \delta dy; \end{aligned}$$

whence it appears that the symbols of differentiation and of variation are interchangeable.

Hence, also, if for y we put dy , we have

$$\begin{aligned} d\delta dy &= \delta d^2 y, \quad \text{or, since } d\delta dy = dd\delta y, \\ \therefore d^2 \delta y &= \delta d^2 y, \text{ and similarly } d^n \delta y = \delta d^n y. \end{aligned}$$

These theorems may, however, be proved without the illustration of the curve. For if $y_1 = \text{value of } y + dy$,

$$\delta dy = \delta(y_1 - y) = \delta y_1 - \delta y = d\delta y.$$

(417.) But, to return to the definition of a variation, we instantly perceive that the processes of the Differential Calculus immediately apply, and thus to obtain the variation of a function of y we must write $y + \delta y$ for y , and having expanded the new function according to the powers of δy , subtract from it the original function, and the first term of the difference will be the variation required. Hence, it appears that the first differential coefficient of the function multiplied by δy is the variation of the function.

Thus, if $u = y^n$, the difference of $u = (y + \delta y)^n - y^n$.

$$\therefore \delta u = \text{first term of difference} = ny^{n-1} \cdot \delta y = \frac{du}{dy} \delta y.$$

and
VOL. II.

Also, since if $u = f(x, y, p, q, \&c.)$ where $p, q, \&c.$, are the differential coefficients of y with respect to x ,

$$du = \frac{du}{dx} dx + \frac{du}{dy} dy + \frac{du}{dp} dp + \frac{du}{dq} dq + \&c.$$

or putting

$$M = \frac{du}{dx}, \quad N = \frac{du}{dy}, \quad P = \frac{du}{dp}, \quad Q = \frac{du}{dq}, \quad \&c.$$

$$du = M dx + N dy + P dp + Q dq + \&c.$$

and $\therefore$ by the rule just enunciated, the variation of u or

$$\delta u = M \delta x + N \delta y + P \delta p + Q \delta q + \&c.$$

(418.) In the proof of the theorem $d \delta y = \delta d y$, it has been assumed that the ordinate y alone varied, and that the abscissa x remained constant, but, as we may readily perceive, cases may occur in which both these quantities vary. Such a case will always take place when in consequence of variation the point P is transferred to another point in the curve above it, which is to the left or the right of Q ; the variation may then be conceived to be made up of two variations, one of which is in the direction of the ordinate and the other perpendicular to it. We now proceed to show that whatever be the function u , $\delta d u = d \delta u$.

(419.) Suppose, then, that u be any function whatever of the variables $x, y, dx, dy, d^2x, d^2y, \&c.$, its variation is therefore to be found, by writing instead of $x, dx, d^2x, \&c.$, $y, dy, d^2y, \&c.$, the quantities

$$x + \delta x, \quad dx + \delta dx, \quad d^2x + \delta d^2x, \quad \&c.$$

$$y + \delta y, \quad dy + \delta dy, \quad d^2y + \delta d^2y, \quad \&c.;$$

and then subtracting u from the result, and stopping at the terms which involve the first powers of the variations; but this process will be rendered more easy if, instead of $dx, d^2x, \&c.$, $dy, d^2y, \&c.$, we put $x_1, x_2, \&c.$, $y_1, y_2, \&c.$, and then u will become a function of $x, x_1, x_2, \&c.$ and $y, y_1, y_2, \&c.$; and we shall therefore have

$$du = \frac{du}{dx} dx + \frac{du}{dx_1} dx_1 + \frac{du}{dx_2} dx_2 + \&c. \\ + \frac{du}{dy} dy + \frac{du}{dy_1} dy_1 + \frac{du}{dy_2} dy_2 + \&c.;$$

and from what we have said in our definition of a variation, that it is equal to the product of the differential coefficient by the variation of the variable,

$$\therefore \delta u = \frac{du}{dx} \delta x + \frac{du}{dx_1} \delta x_1 + \frac{du}{dx_2} \delta x_2 + \&c. \\ + \frac{du}{dy} \delta y + \frac{du}{dy_1} \delta y_1 + \frac{du}{dy_2} \delta y_2 + \&c.$$

Keeping this definition in view, we must now find the $\delta \cdot du$ and $d \cdot \delta u$: to do this it will be better to find the variation of each term separately, considering each differential coefficient $\frac{du}{dx}, \frac{du}{dx_1}, \frac{du}{dx_2}, \&c.$, to be functions of $x, x_1, x_2, \&c.$

Now

$$\delta \cdot \frac{du}{dx} dx = \frac{du}{dx} \delta dx + \frac{d^2u}{dx^2} \delta x dx + \frac{d^2u}{dx dx_1} \delta x_1 dx + \&c. \\ = \frac{du}{dx} \delta dx + dx \left\{ \frac{d^2u}{dx^2} \delta x + \frac{d^2u}{dx dx_1} \delta x_1 + \&c. \right\}$$

And

$$\delta \frac{du}{dx_1} dx_1 = \frac{du}{dx_1} \delta dx_1 + dx_1 \left\{ \frac{d^2u}{dx dx_1} \delta x + \frac{d^2u}{dx_1^2} \delta x_1 + \&c. \right\} \\ \&c. = \&c. + \&c. \\ \therefore \delta \cdot du = \frac{du}{dx} \delta dx + dx \left\{ \frac{d^2u}{dx^2} \delta x + \frac{d^2u}{dx dx_1} \delta x_1 + \&c. \right\} \\ + \frac{du}{dx_1} \delta dx_1 + dx_1 \left\{ \frac{d^2u}{dx dx_1} \delta x + \frac{d^2u}{dx_1^2} \delta x_1 + \&c. \right\} \\ + \&c. \dots + \&c.$$

Next, to find the $d \delta u$; differentiating the terms separately.

$$d \cdot \frac{du}{dx} \delta x = \frac{du}{dx} d \cdot \delta x + \frac{d^2u}{dx^2} dx \delta x + \frac{d^2u}{dx dx_1} dx_1 dx + \&c. \\ = \frac{du}{dx} d \delta x + \delta x \left\{ \frac{d^2u}{dx^2} dx + \frac{d^2u}{dx dx_1} dx_1 + \&c. \right\}$$

And

$$d \cdot \frac{du}{dx_1} \delta x_1 = \frac{du}{dx_1} d \delta x_1 + \delta x_1 \left\{ \frac{d^2u}{dx dx_1} dx + \frac{d^2u}{dx_1^2} dx_1 + \&c. \right\} \\ \therefore d \delta u = \frac{du}{dx} d \delta x + \delta x \left\{ \frac{d^2u}{dx^2} dx + \frac{d^2u}{dx dx_1} dx_1 + \&c. \right\}$$

Calculus of Variations.

$$\frac{du}{dx_1} d\delta x_1 + \delta x_1 \left\{ \frac{d^2 u}{dx dx_1} dx + \frac{d^2 u}{dx_1^2} dx_1 + \&c. \right\} \\ + \&c. \quad + \&c.$$

whence

and writing du for u we have also

$$\delta \cdot d^2 u = d \cdot \delta du = \delta^2 du, \text{ and finally } \delta \cdot d^n u = d^n \delta u.$$

Hence, in general, whatever be the value of u , the symbols δ and d are interchangeable, or the variation of the differential gives the same result as the differential of the variation.

Example 1. Let

$$u = y^n \therefore du = n y^{n-1} dy \text{ and } \delta u = n y^{n-1} \delta y$$

$$\therefore \delta du = n y^{n-1} \delta dy + n(n-1) y^{n-2} \delta y dy$$

$$d \cdot \delta u = n y^{n-1} d \delta y + n(n-1) y^{n-2} \delta y dy$$

and

$$\therefore \delta dy = d \delta y \therefore \delta du = d \delta u.$$

Example 2. Let

$$u = x^2 + y^2.$$

$$\therefore du = 2x dx + 2y dy$$

and

$$\delta u = 2x \delta x + 2y \delta y$$

$$\therefore \delta du = 2x \delta dx + 2y \delta dy + 2\delta x dx + 2\delta y dy,$$

and

$$d \cdot \delta u = 2x d \delta x + 2y d \delta y + 2\delta x dx + 2\delta y dy;$$

whence

$$\therefore \delta dx = d \delta x, \text{ and } d \delta y = \delta dy \therefore \delta du = d \delta u.$$

(420.) There is also a similar theorem with respect to the sign of integration, namely, that $\delta \int u = \int \delta u$, or the variation of the integral of a function is equal to the integral of the variation of the function.

To prove this theorem, let $\int u = u_1 \therefore u = du_1$

$$\therefore \delta u = \delta du_1 = d \delta u_1$$

And

the theorem required.

$$\therefore \int \delta u = \int d \delta u_1 = \delta u_1 = \delta \int u,$$

Similarly

$$\int^2 \delta u = \int^2 \delta \int u = \delta \int \int u = \delta \int^2 u,$$

and thus we at length arrive at the equation $\int^n \delta u = \delta \int^n u$.

(421.) Having proved these preliminary theorems, we now proceed to apply them to find the variation of $\int u$, and, in the first instance, let u depend not only on y and its differential coefficients, but upon x and its differential coefficients also. Let, then, u be any function whatever of the variables x and y , and of their differentials, such that

$$du = M dx + N d^2 x + P d^3 x + Q d^4 x + \&c. \\ + m dy + n d^2 y + p d^3 y + q d^4 y + \&c.$$

Then we have, since $d^2 x = d \cdot dx$, $d^3 x = d \cdot d^2 x$, &c.

$$\delta u = M \delta x + N \delta dx + P \delta d^2 x + Q \delta d^3 x + \&c. \\ + m \delta y + n \delta dy + p \delta d^2 y + q \delta d^3 y + \&c.$$

and, consequently,

$$\int \delta u = \int (M \delta x + N \delta dx + P \delta d^2 x + Q \delta d^3 x + \&c.) \\ + \int (m \delta y + n \delta dy + p \delta d^2 y + q \delta d^3 y + \&c.)$$

Now, by transposing the δ and d , i. e. putting δ after d , and integrating by parts, the terms δx and δy may be freed from the sign of differentiation, and the formula for $\int \delta u$ be reduced to a more simple form: thus,

$$\int N \delta dx = \int N d \delta x = N \delta x - \int d N \delta x \\ \int P \delta d^2 x = \int P d^2 \delta x = P d \delta x - \int d P d \delta x \\ = P d \delta x - d P \delta x + \int d^2 P \delta x \\ \int Q \delta d^3 x = \int Q d^3 \delta x = Q d^2 \delta x - \int d Q \cdot d^2 \delta x \\ = Q d^2 \delta x - d Q d \delta x + \int d^2 Q d \delta x \\ = Q d^2 \delta x - d Q d \delta x + d^2 Q \cdot \delta x - \int d^3 Q \delta x \\ \&c. \quad \&c.$$

In the same manner,

$$\int n \delta dy = \int n d \delta y = n \delta y - \int d n \cdot \delta y \\ \int p \delta d^2 y = \int p d^2 \delta y = p d \delta y - d p \delta y + \int d^2 p \delta y \\ \int q \delta d^3 y = \int q d^3 \delta y = q d^2 \delta y - d q d \delta y + d^2 q \cdot \delta y - \int d^3 q \cdot \delta y, \\ \&c. \quad \&c.;$$

and substituting these values in the expression for $\int \delta u$, we have

$$\int \delta u = (N - d P + d^2 Q - \&c.) \delta x + (n - d p + d^2 q - \&c.) \delta y \\ + (P - d Q + \&c.) d \delta x + (p - d q + \&c.) d \delta y \\ + (Q - d R + \&c.) d^2 \delta x + (q - d r + \&c.) d^2 \delta y \\ + \&c. \quad + \&c.$$

$$+ \int (M - dN + d^2P - d^3Q + \&c.) \delta x \\ + \int (m - dn + d^2p - d^3q + \&c.) \delta y.$$

(422.) The result which we have obtained is composed of two similar parts, one of which is due to the variation of x , and the other to the variation of y : hence, we perceive, that if u be a function of x, y, z , or of any number of variables, we may derive from the preceding expression the value of the variation of $\int u$, by adding to it, for each new variable, a series of terms similar to that which the variable x has produced.

Thus, let u be a function of x, y, z , in which case

$$du = Mdx + Nd^2x + Pd^3x + Qd^4x + \&c. \\ + mdy + nd^2y + pd^3y + qd^4y + \&c. \\ + \mu dz + \nu d^2z + \pi d^3z + \rho d^4z + \&c.;$$

then, for the new variable z , we shall have to add the terms

$$(\nu - d\pi + d^2\rho - \&c.) \delta z + (\pi - d\rho + \&c.) d\delta z + \&c. \\ + \int (\mu - d\nu + d^2\pi - d^3\rho + \&c.) \delta z.$$

It must be observed that, in the preceding investigations, y and x , as also y, x , and z , are independent of each other, but may be considered as functions of another variable, as t , which is understood.

Let us consider y and x only, and let y be a function of x , then dx is constant, and

$$du = Mdx \\ + mdy + nd^2y + pd^3y + qd^4y + \&c.$$

and the variation of the $\int u$ is reduced to

$$(n - dp + d^2q - \&c.) \delta y + (p - dq + \&c.) d\delta y + (q - \&c.) d^2\delta y + \&c. \\ + \int M\delta x \\ + \int (m - dn + d^2p - d^3q + \&c.) \delta y,$$

which may be again simplified, since u may be put equal to Vdx ,

and

$$\therefore M = \frac{dV}{dx} dx, \text{ since } \frac{du}{dx} = \frac{dV}{dx} dx;$$

consequently,

$$\int M\delta x = \int \frac{dV}{dx} dx \cdot \delta x = \\ = V\delta x - \int Vd\delta x = V\delta x - \int V\delta dx \\ = V\delta x.$$

Since, if $dx = \text{a constant}$, $\delta dx = 0$, and thus the only terms which remain under the sign of integration are those which are multiplied by δy .

(423.) As an Example of the use of this theorem, let

$$u = \frac{\sqrt{dx^2 + dy^2}}{\sqrt{y}} = \frac{ds}{\sqrt{y}} \\ \therefore \delta u = \sqrt{dx^2 + dy^2} \cdot \delta \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{y}} \delta \sqrt{dx^2 + dy^2};$$

and performing the variations as if we were to obtain differentials

$$\delta \frac{1}{\sqrt{y}} = -\frac{\delta y}{2y^{\frac{3}{2}}}, \delta \sqrt{dx^2 + dy^2} = \frac{dx\delta dx + dy\delta dy}{\sqrt{dx^2 + dy^2}} \\ \therefore \delta u = -\frac{\delta y \cdot ds}{2y^{\frac{3}{2}}} + \frac{dx \cdot \delta dx}{\sqrt{y} \cdot ds} + \frac{dy \delta dy}{\sqrt{y} \cdot ds}$$

Comparing this with the general formula for δu , we have

$$M = 0, m = -\frac{ds}{2y^{\frac{3}{2}}}, N = \frac{dx}{\sqrt{y} \cdot ds}, n = \frac{dy}{\sqrt{y} \cdot ds}$$

and

$$\therefore \delta \int u = N\delta x + n\delta y + \int (-dN)\delta x + \int (m - dn)\delta y \\ = \frac{dx}{\sqrt{y} \cdot ds} \delta x + \frac{dy}{\sqrt{y} \cdot ds} \delta y \\ - \int \left\{ \frac{ds}{2y^{\frac{3}{2}}} \delta y + d \cdot \left(\frac{dy}{\sqrt{y} \cdot ds} \right) \delta y + d \cdot \left(\frac{dx}{\sqrt{y} \cdot ds} \right) \delta x \right\}$$

This method of finding the variation of u is due to Lagrange. Euler, however, has given to it another and more convenient form for the solution of the problems, to which our attention will be directed. Instead of u he put Vdx where V is supposed to be a function of x, y , and the differential coefficients of y with regard to x .

Calculus of Variations. (424.) We proceed, therefore, on this new supposition, to find the variation of $\int V dx$, when y is a function of x , but the reasoning is only applicable when u can be put under the form $V dx$. Part III.

Let

$$dV = M dx + N dy + P dp + Q dq + R dr + \&c.$$

where

$$p = \frac{dy}{dx}, q = \frac{d^2y}{dx^2}, r = \frac{d^3y}{dx^3}, \&c. = \&c.$$

and

$$M = \frac{dV}{dx}, N = \frac{dV}{dy}, P = \frac{dV}{dp}, Q = \frac{dV}{dq}, \&c.$$

Then

$$\delta V = M \delta x + N \delta y + P \delta p + Q \delta q + \&c.,$$

where $M, N, P, Q, \&c.$, are the same as in the equation for dV .

Now

$$\begin{aligned} \delta \int V dx &= \int \delta V dx = \int (V \delta dx + dx \delta V) \\ &= \int (V d\delta x + dx \delta V) \\ &= \int V d\delta x + \int dx \delta V \\ &= V \delta x + \int (dx \delta V - \delta x dV). \end{aligned}$$

and putting instead of δV and dV their values, we have

$$\begin{aligned} \int (dx \delta V - \delta x dV) &= \int dx (M \delta x + N \delta y + P \delta p + Q \delta q + \&c.) \\ &\quad - \int \delta x (M dx + N dy + P dp + Q dq + \&c.) \\ &= \int N dx (\delta y - p \delta x) + \int P dx (\delta p - q \delta x) + \int Q dx (\delta q - r \delta x) + \&c. \\ &\quad + \int R dx (\delta r - s \delta x) + \&c. \end{aligned}$$

We must now endeavour to find the forms of $\delta p - q \delta x$, $\delta q - r \delta x$, $\&c.$, and we shall see that each of these factors depends upon $\delta y - p \delta x$.

Since

$$\frac{dy}{dx} = p, \frac{dp}{dx} = q, \frac{dq}{dx} = r, \&c. = \&c.$$

$$\therefore \delta p = \frac{dx \delta dy - dy \delta dx}{dx^2} = \frac{\delta dy - p \delta dx}{dx} = \frac{d\delta y - p d\delta x}{dx}.$$

$$\therefore \delta p - q \delta x = \frac{\delta dy - p \delta dx - d p \delta x}{dx} = \frac{d\delta y - d.p \delta x}{dx} = \frac{d(\delta y - p \delta x)}{dx}$$

$$\delta q = \delta \frac{dp}{dx} = \frac{dx \delta dp - dp \delta dx}{dx^2} = \frac{\delta dp - q \delta dx}{dx} = \frac{d\delta p - q d\delta x}{dx}$$

$$\therefore \delta q - r \delta x = \delta q - \frac{dq}{dx} \delta x = \frac{d\delta p - q d\delta x - dq \delta x}{dx} = \frac{d(\delta p - q \delta x)}{dx}$$

also

$$\delta r = \delta \frac{dq}{dx} = \frac{dx \delta dq - dq \delta dx}{dx^2} = \frac{d\delta q - r d\delta x}{dx}$$

$$\therefore \delta r - s \delta x = \frac{d\delta q - r d\delta x - dr \delta x}{dx} = \frac{d(\delta q - r \delta x)}{dx}.$$

Now make

$$\delta y - p \delta x = w$$

$$\therefore \delta p - q \delta x = \frac{dw}{dx}.$$

$$\delta q - r \delta x = \frac{d\left(\frac{dw}{dx}\right)}{dx} = \frac{d^2w}{dx^2}.$$

and

$$\begin{aligned} \delta r - s \delta x &= \frac{d\left(\frac{d^2w}{dx^2}\right)}{dx} = \frac{d^3w}{dx^3}. \\ \&c. &= \&c. \end{aligned}$$

$\therefore$ Substituting these values in the expression for $\delta \int (dx \delta V - dV \delta x)$, and adding $V \delta x$, we have

$$\delta \int V dx = V \delta x + \int N w dx + \int P dw + \int Q \frac{d^2w}{dx^2} + \int R \frac{d^3w}{dx^3} + \&c.$$

Then integrating by parts

$$\begin{aligned} \int P dw &= Pw - \int \frac{dP}{dx} w dx \\ \int Q \frac{d^2w}{dx^2} &= Q \frac{dw}{dx} - \int \frac{dQ}{dx} dw \end{aligned}$$

Calculus of
Variations.

$$\begin{aligned}
&= Q \frac{dw}{dx} - w \frac{dQ}{dx} + \int \frac{d^2 Q}{dx^2} w dx \\
\int R \frac{d^2 w}{dx^2} &= R \frac{d^2 w}{dx^2} - \int \frac{dR}{dx} \cdot \frac{dw}{dx} \\
&= R \frac{d^2 w}{dx^2} - \frac{dR}{dx} \cdot \frac{dw}{dx} + \int \frac{d^2 R}{dx^2} \cdot dw \\
&= R \frac{d^2 w}{dx^2} - \frac{dR}{dx} \frac{dw}{dx} + \frac{d^2 R}{dx^2} w - \int \frac{d^3 R}{dx^3} w dx.
\end{aligned}$$

&c. = &c.

$$\begin{aligned}
\therefore \delta \int V dx &= V \delta x + \left(P - \frac{dQ}{dx} + \frac{d^2 R}{dx^2} - \&c. \right) w \\
&\quad + \left(Q - \frac{dR}{dx} + \frac{d^2 S}{dx^2} - \&c. \right) \frac{dw}{dx} \\
&\quad + \left(R - \frac{dS}{dx} + \&c. \right) \frac{d^2 w}{dx^2} \\
&\quad + \&c. \\
&\quad + \int \left(N - \frac{dP}{dx} + \frac{d^2 Q}{dx^2} - \frac{d^3 R}{dx^3} + \&c. \right) w dx.
\end{aligned}$$

(425.) Hence we see that the variation of $\int V dx$ consists of two distinct parts, one of which is under the sign of integration, and the other is not. The latter of which is effected only by the variations of the extreme values of y and x , the former depends upon all the values between these extreme ones. Suppose the extreme values of x and y be x_1, y_1 , and x_2, y_2 ; then the total variation of $\int V dx$.

$$\begin{aligned}
&= V_2 \delta x_2 - V_1 \delta x_1 + \left(P_2 - \frac{dQ_2}{dx_2} + \frac{d^2 R_2}{dx_2^2} - \&c. \right) w_2 \\
&\quad - \left(P_1 - \frac{dQ_1}{dx_1} + \frac{d^2 R_1}{dx_1^2} - \&c. \right) w_1 \\
&\quad + \left(Q_2 - \frac{dR_2}{dx_2} + \frac{d^2 S_2}{dx_2^2} - \&c. \right) \frac{dw_2}{dx_2} \\
&\quad - \left(Q_1 - \frac{dR_1}{dx_1} + \frac{d^2 S_1}{dx_1^2} - \&c. \right) \frac{dw_1}{dx_1} \\
&\quad + \&c. \\
&\quad + \int \left(N - \frac{dP}{dx} + \frac{d^2 Q}{dx^2} - \frac{d^3 R}{dx^3} + \&c. \right) w dx.
\end{aligned}$$

The part under the sign of integration being taken between the same limits.

(426.) If $w = 0$, the formula is reduced to $V \delta x$, but since, if

$$w = 0, \delta y - p \delta x = 0 \therefore \frac{\delta y}{\delta x} = p = \frac{dy}{dx}.$$

there is the same relation between δy and δx as there is between dy and dx , and $\therefore$ the same result ought to be obtained as that arising from ordinary differentiation, and we know that $d \int V dx = V \cdot dx$, the symbols d and $\int$ indicating inverse operations.

(427.) It is obvious, that if u had been a function of three variables x, y, z , instead of the two variables x and y only, and if we still consider x to be the independent variable, then, since the differential of V can be put under the form

$$dV = M dx + N dy + P dp + Q dq + \&c.$$

and

$$\begin{aligned}
&+ N_1 dz + P_1 dp_1 + Q_1 dq_1 + \&c. \\
\therefore \delta V &= M \delta x + N \delta y + P \delta p + Q \delta q + \&c. \\
&+ N_1 \delta z + P_1 \delta p_1 + Q_1 \delta q_1 + \&c.
\end{aligned}$$

we should have for the new variable z a set of terms involving $N_1, P_1, Q_1, \&c.$, precisely similar to those in which $N, P, Q, \&c.$, are introduced; and, making $\delta z - p_1 \delta x = w_1$, the total variation will be found to be expressed by

$$\begin{aligned}
\delta \int V dx &= V \delta x + \left(P - \frac{dQ}{dx} + \frac{d^2 R}{dx^2} - \&c. \right) w \\
&\quad + \left(Q - \frac{dR}{dx} + \frac{d^2 S}{dx^2} - \&c. \right) \frac{dw}{dx} \\
&\quad + \&c. + \left(P_1 - \frac{dQ_1}{dx} + \frac{d^2 R_1}{dx^2} - \&c. \right) w_1
\end{aligned}$$

Calculus of Variations.

$$\begin{aligned}
 & + \left(Q_1 - \frac{d R_1}{d x} + \&c. \right) \frac{d w_1}{d x} + \&c. \\
 & + \int \left(N - \frac{d P}{d x} + \frac{d^2 Q}{d x^2} - \frac{d^3 R}{d x^3} + \&c. \right) w d x \\
 & + \int \left(N_1 - \frac{d P_1}{d x} + \frac{d^2 Q_1}{d x^2} - \frac{d^3 R_1}{d x^3} + \&c. \right) w_1 d x.
 \end{aligned}$$

(428.) In the preceding investigations we have not supposed that the coordinates of the limits enter into the functions which are under the sign $\int$; but there are questions in which these coordinates do vary, as we shall see in the succeeding pages. If, then, the function V does also contain these coordinates and their differentials,

there must be added to the expression for δV a series of terms of the form $\frac{d V}{d x_1} \delta x_1 + \frac{d V}{d y_1} \delta y_1 + \&c.$; and $\therefore$

to the part under the integral sign there will also be added the term $\int \left(\frac{d V}{d x_1} \delta x_1 + \frac{d V}{d y_1} \delta y_1 + \&c. \right) d x$, but as the variations $\delta x_1, \delta y_1, \&c.$, are independent of the indeterminate quantities x and y , they may be put without the sign of integration, and may be written $\delta x_1 \int \frac{d V}{d x_1} d x + \delta y_1 \int \frac{d V}{d y_1} d x + \&c.$, and may be considered to belong to the first part of the expression for the $\delta \int V d x$.

APPLICATION OF THE CALCULUS OF VARIATIONS TO QUESTIONS OF MAXIMA AND MINIMA.

(429.) WE now proceed to the application of the formulæ, which have been deduced in the preceding articles, to those questions of Geometry which by its means find a ready solution. The principles, which, in the Differential Calculus, have guided us to the detection of these tests, by which the existence of a maximum or minimum state of a function may be known, are applicable here.

If there be any function u of the quantities $x, y, dx, dy, \&c.$, the result of the substitution of $x + \delta x, y + \delta y, dx + \delta dx, dy + \delta dy, \&c.$, by stopping at the first terms of the variations, $\delta x, \delta y, \delta dx, \delta dy, \&c.$, will give us the variation of u .

But, as the variation of u only involves the first powers of $\delta x, \delta y, \&c.$, it is clear that the terms which are affected by these variations will change their signs at the same time as the variations change: hence, if u be a maximum or minimum $\delta u = 0$. Now, suppose $u = \int u$, or $\int V d x$; therefore, at a maximum or maximum, $\delta \int u = 0$ or $\delta \int V d x = 0$, according as $u = \int u$ or $u = \int V d x$.

We must not, however, suppose that whenever $\delta \int V d x = 0$, we have a maximum or minimum, for the same restriction applies to the case in question, as to that which has been discussed in the Differential Calculus.

(430.) If, in the first place, we refer to the $\delta \int u$, the result of which is

$$\begin{aligned}
 \delta \int u &= (N - d P + d^2 Q - \&c.) \delta x + (n - d p + d^2 q - \&c.) \delta y \\
 &+ (P - d Q + \&c.) d \delta x + (p - d q + \&c.) d \delta y \\
 &+ \&c. \\
 &+ \int (M - d N + d^2 P - d^3 Q + \&c.) \delta x \\
 &+ \int (m - d n + d^2 p - d^3 q + \&c.) \delta y,
 \end{aligned}$$

we perceive that it is composed of two parts, entirely distinct from each other, one of which is affected with the integral sign, and the other free from it: and since, when $\int u$ is a maximum or minimum, $\delta \int u = 0$, and in the expression to which it is equal must also $= 0$; but as the part out of the integral sign involves the values of δx and δy at the limits only, while the part under the integral sign contains the general values of x and y , we must put these expressions separately equal to nothing.

If the limits be fixed, then $\delta x = 0$ and $\delta y = 0$.

$$\therefore M - d N + d^2 P - d^3 Q + \&c. = 0$$

$$m - d n + d^2 p - d^3 q + \&c. = 0.$$

from which equation the curve which has the required property may be determined.

(431.) But we proceed to examine the variation of $\int V d x$ when $\int V d x$ is a maximum or minimum, and from that expression we shall deduce some theorems which are eminently useful in the solution of the problems to which we shall afterwards refer.

(432.) We have seen that the variation of the $\int V d x$, when taken between the limits of x_1, y_1 , and x_2, y_2 , is expressed by

Calculus of
Variations.

$$\begin{aligned} \delta \int V dx &= V_2 \delta x_2 - V_1 \delta x_1 + \left(P_2 - \frac{d Q_2}{d x_2} + \&c. \right) w_2 \\ &\quad - \left(P_1 - \frac{d Q_1}{d x_1} + \&c. \right) w_1 \\ &\quad + \&c. \\ &\quad + \int \left(N - \frac{d P}{d x} + \frac{d^2 Q}{d x^2} - \frac{d^3 R}{d x^3} + \&c. \right) w dx; \end{aligned}$$

and since, when $\int V dx =$ a maximum or minimum, $\delta \int V dx = 0$, we must have at the same time the expansion $= 0$, and therefore the two parts of which it is composed each $= 0$. Hence we have two equations separately $= 0$, by one of which the variations at the extreme points may be determined, and by the other the nature of the function which possesses the maximum or minimum quality may be found.

Thus, by the equation $N - \frac{d P}{d x} + \frac{d^2 Q}{d x^2} - \&c. = 0$, combined with the equation $d V = M dx + N dy + P dp + \&c.$, may the curve be found, which is the object of our research, and by the equation $V_2 \delta x_2 - V_1 \delta x_1 + P_2 w_2 - P_1 w_1 + \&c. = 0$, may the position of its extreme points be determined, if these points be required.

(433.) But if, as it has been previously remarked, the extreme points be fixed $\delta x_1, \delta x_2, \&c.$ each $= 0$, and the latter equation is of no use.

The two equations may be illustrated by the consideration of the following problem, the solution of which will be first attempted. And, in the first place, had we to find the shortest line between two fixed points, we should

from the equation $N - \frac{d P}{d x} + \frac{d^2 Q}{d x^2} - \frac{d^3 R}{d x^3} + \&c. = 0$ determine the equation to the line: but as the extreme

points are fixed, and the line must pass through them, no other equation is necessary. But if it be required to draw the shortest line between two curves, then since, when drawn, the line must lie between two points, the former equation will give the same result as before, but the equation of the limits is now necessary to determine the precise points from which the shortest line of known species is now to be drawn, and from the equation to the limits which involve the variations of the extreme points, and from the equations to the given curves, may the conditions be found by which the line may be completely determined.

Thus, suppose PQ (fig. 2) to be one of the curves to be drawn between the given curves PP^1 and QQ^1 , and let P^1Q^1 be the curve which arises from making x and y respectively $x + \delta x$ and $y + \delta y$. Then, since P^1

and P are in the same curve PP^1 , and also Q and Q^1 in the same curve QQ^1 , if $\frac{dy}{dx} = m$, and $\frac{dy}{dx} = n$ be the equations

to the given curves: and y_1, x_1, y_2, x_2 be the coordinates of P and Q , the equations $\frac{\delta y_1}{\delta x_1} = m$ and $\frac{\delta y_2}{\delta x_2} = n$ will also be

true, for it is obvious that the ratio of the differentials of AN and PN will be the same as the ratio of the variations of the same quantities.

These values being substituted in the equation of limits, two of the quantities $\delta y_1, \delta y_2, \delta x_1, \delta x_2$, as δy_1 and δy_2 will be eliminated, and two independent variations, as δx_1 and δx_2 , will be left, the coefficients of which may be put $= 0$, and the conditions of the line at the points P and Q determined.

In this illustration it has been tacitly assumed that $\delta dy_1, \delta dx_1, \&c. = 0$, should they not $= 0$, the equations to the curves will give us relations by which some of these quantities may be eliminated, and then the coefficients of the remaining variations may be put equal to zero, and thus the equation of the limits satisfied.

In the succeeding pages we shall find instances illustrating these cases; sometimes the limits will be considered fixed, and at other times unknown, and to be determined by the nature of the problem.

(434.) But as the solution of problems of maxima and minima is much facilitated by deducing from the general expression some particular formulas, we proceed to investigate them.

In general,

$$dV = M dx + N dy + P dp + Q dq + R dr + \&c.,$$

and

$$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3} + \&c. = 0.$$

1. Let all the coefficients, except N and P , $= 0$

$$\therefore dV = N dy + P dp, \text{ and } N - \frac{dP}{dx} = 0 \therefore N = \frac{dP}{dx}.$$

$$\therefore dV = dP \cdot \frac{dy}{dx} + P dp = p dP + P dp = d(Pp + c)$$

$$\therefore V = Pp + c.$$

2. Let all but M, N , and P , be equal to nothing;
then

$$dV = M dx + d(Pp + c)$$

$$\therefore V = \int M dx + Pp + c.$$

3. Let $M = 0, N = 0$, and all the coefficients after $Q = 0$.

Calculus of
Variations.

And

$$\therefore dV = P dp + Q dq.$$

$$\frac{dP}{dx} + \frac{d^2Q}{dx^2} = 0 \therefore dP = \frac{d^2Q}{dx^2} dx$$

$$\therefore P = \int \frac{d^2Q}{dx^2} = \frac{dQ}{dx} + C$$

$$\therefore dV = \frac{dp}{dx} dQ + Q dq + C dp = d(Qq + Cp + C_1)$$

$$\therefore V = Qq + Cp + C'_1.$$

If M also does not = 0, then $V = \int M dx + Qq + Cp + C'_1$.

The following Examples will illustrate the use of these formulas.

(435.) Problem 1. Find the shortest line between two given points. Here

$$\int V dx = \int dx \sqrt{1 + \frac{dy^2}{dx^2}} = \int dx \sqrt{1 + p^2}.$$

$$\therefore V = \sqrt{1 + p^2}.$$

$$\therefore dV = \frac{p}{\sqrt{1 + p^2}} dp = M dx + N dy + P dp + Q dq + \&c.$$

$$\therefore M = 0, N = 0, P = \frac{p}{\sqrt{1 + p^2}}, Q = 0, \&c. = \&c.$$

But

$$N - \frac{dP}{dx} + \&c. = 0 \therefore \frac{dP}{dx} = 0 \therefore P = c$$

$$\therefore \frac{p}{\sqrt{1 + p^2}} = c \therefore p = \frac{c}{\sqrt{1 - c^2}} = a.$$

And $y = ax + b$ the equation to a straight line, and the constants a, b , may be determined by its passing through the given points.

Problem 2. Find the curve of quickest descent from one given point to another given point, the points being in a vertical plane, and gravity the accelerating force.

Make the higher point the origin of coordinates, and let the axis of y be measured downwards.

Then since, by Mechanics, $\frac{ds}{dt} = \sqrt{2gy}$.

$$\therefore t = \int \frac{ds}{\sqrt{2gy}} = \frac{1}{\sqrt{2g}} \cdot \int \frac{dx \sqrt{1 + p^2}}{\sqrt{y}} = \frac{1}{\sqrt{2g}} \cdot \int V dx$$

$$\therefore V = \frac{\sqrt{1 + p^2}}{\sqrt{y}}.$$

$$\therefore dV = -\frac{1}{2} \cdot \frac{\sqrt{1 + p^2}}{y^{\frac{3}{2}}} dy + \frac{p}{\sqrt{y} \sqrt{1 + p^2}} dp = N dy + P dp$$

$$\therefore N = -\frac{\sqrt{1 + p^2}}{2y^{\frac{3}{2}}} \text{ and } P = \frac{p}{\sqrt{y} \sqrt{1 + p^2}}. M = 0, Q = 0.$$

But then

$$V = Pp + c$$

or

$$\frac{\sqrt{1 + p^2}}{\sqrt{y}} = \frac{p^2}{\sqrt{y} \sqrt{1 + p^2}} + c$$

$$\therefore \frac{\sqrt{1 + p^2}}{\sqrt{y}} - \frac{p^2}{\sqrt{y} \sqrt{1 + p^2}} = \frac{1}{\sqrt{y} \sqrt{1 + p^2}} = c = \frac{1}{\sqrt{2a}}$$

by substituting for c

$$\therefore \sqrt{\frac{2a}{y}} = \sqrt{1 + p^2} \therefore p^2 = \frac{2a - y}{y} \therefore p = \frac{\sqrt{2ay - y^2}}{y}$$

$$\therefore \frac{dx}{dy} = \frac{y}{\sqrt{2ay - y^2}} = \frac{-(a - y - a)}{\sqrt{2ay - y^2}} = \frac{a}{\sqrt{2ay - y^2}} - \frac{a - y}{\sqrt{2ay - y^2}}$$

whence

$$x = a V \sin^{-1} \frac{y}{2a} - \sqrt{2ay - y^2}$$

Calculus of the equation to a cycloid, of which the base is coincident with the axis of x , and the curve meets the base in the origin. This is the simplest form of the *Brachystochrone*. Part III.

Problem 3. Required the curve which, by revolution round its axis, shall generate the solid of least resistance.

Resistance

$$= \int \frac{y \cdot p^3}{1 + p^2} dx = \int V dx$$

$$\therefore V = \frac{y p^3}{1 + p^2}$$

$$\begin{aligned} \therefore dV &= \frac{p^3}{1 + p^2} dy + y \cdot \frac{3p^2 \overline{1 + p^2} - 2p^4}{(1 + p^2)^2} dp \\ &= \frac{p^3}{1 + p^2} dy + y \frac{3p^2 + p^4}{(1 + p^2)^2} dp = N dy + P dp \end{aligned}$$

$$\therefore N = \frac{p^3}{1 + p^2}, P = \frac{3p^2 + p^4}{(1 + p^2)^2} y, M = 0, Q = 0, \&c.$$

And since, in this case, $V = Pp + c$

$$\begin{aligned} \therefore \frac{p^3 y}{1 + p^2} &= \frac{3p^3 + p^5}{1 + p^2} y + c \\ \therefore p^3 y + p^5 y &= 3p^3 y + p^5 y + c(1 + p^2) \\ \therefore 2p^3 y + c \overline{1 + p^2} &= 0, \end{aligned}$$

or, writing $-a$ for c ,

$$y = \frac{a \overline{1 + p^2}}{2p^3} = \frac{a}{2p^3} + \frac{a}{p} + \frac{ap}{2} \quad (1)$$

Again, since

$$\frac{dy}{dx} = p \therefore dx = \frac{dy}{p}$$

But from equation (1) by differentiation we have

$$\begin{aligned} dy &= -\frac{3a}{2} \cdot \frac{dp}{p^4} - \frac{a dp}{p^2} + \frac{a}{2} \cdot dp \\ \therefore dx &= -\frac{3a}{2} \cdot \frac{dp}{p^5} - \frac{a dp}{p^3} + \frac{a}{2} \cdot \frac{dp}{p} \\ \therefore x &= \frac{3a}{8} \frac{1}{p^4} + \frac{a}{2p^2} + \frac{a}{2} \cdot \log(p) + b. \quad (2) \end{aligned}$$

And if p could be eliminated between the equations (1) and (2), the relation between y and x , or the equation to the generating curve, would be known.

Problem 4. Required the curve which, within its own arc, its evolute, and radius of curvature, shall contain the least area.

Let R = radius of curvature to any point P at a distance (s) from the origin measured along the curve.

Then, if A = area included between the arc, evolute, and radius,

$$\frac{dA}{dx} = \frac{R}{2} \frac{ds}{dx}$$

But

$$R = \frac{(1 + p^2)^{\frac{3}{2}}}{-q} \quad \text{and} \quad \frac{ds}{dx} = \sqrt{1 + p^2}$$

$$\therefore \frac{dA}{dx} = -\frac{(1 + p^2)^2}{q} \quad \text{and} \quad \therefore V = \frac{(1 + p^2)^2}{q}$$

$$\therefore dV = \frac{2(1 + p^2)p}{q} dp - \frac{(1 + p^2)^2}{q^2} dq \quad \therefore Q = -\frac{(1 + p^2)^2}{q^2}$$

And, as p and q are alone involved,

$$V = Qq + cp + c' \quad \therefore \frac{2(1 + p^2)^2}{q} = cp + c'$$

$$\therefore 2 \frac{dx}{dp} = \frac{cp + c_1}{(1 + p^2)^2}, \text{ but since } dx = \frac{ds}{\sqrt{1 + p^2}}$$

$$\therefore 2 \frac{ds}{dp} = \frac{cp + c_1}{(1 + p^2)^{\frac{3}{2}}} = \frac{cp}{1 + p^2} + \frac{c_1}{(1 + p^2)^{\frac{3}{2}}}$$

$$\therefore 2(s - c') = -\frac{c}{\sqrt{1+p^2}} + \frac{c_1 p}{\sqrt{1+p^2}} = c_1 \frac{dy}{ds} - \frac{c dx}{ds}.$$

$$\therefore (s - c')^2 = c_1 y - c x + c''.$$

We may get rid of the constant c'' by moving the origin along the axis of x , so that x becomes $x + \delta$, and making $-c\delta + c'' = 0$.

$$\therefore s - c' = \sqrt{c_1 y - c x}$$

or

$$s = c' + 2\sqrt{\beta y - a x} \text{ if } c' = 4\beta, \text{ and } c = 4a.$$

which is the same equation as that in Lacroix, but which is there obtained by a more laborious process: the curve is the cycloid, and if, as in Woodhouse's Isoperimetrical Problems, we make $a = 0$ and $c' = 0$, then

$s = 2\sqrt{\beta y}$, a well-known property of the common cycloid.

Problem 5. Find the shortest line that can be drawn between two curves whose equations are known.

Here

$$V = \sqrt{1+p^2}, \quad dV = \frac{p}{\sqrt{1+p^2}} dp \quad \therefore P = \frac{p}{\sqrt{1+p^2}}.$$

and we know from Problem 1 that the line is a straight one, and $p = c$.

Let $dy = m dx$ and $dy = n dx$ be the equations to the two curves, and let y_1, x_1 and y_2, x_2 be the coordinates of the points in which the shortest line intersects them.

Now, since δy_1 and δx_1 represent the variations of y_1 and x_1 , as we pass from one point to another consecutive one in the curve $\frac{dy}{dx} = m$, we shall have

$$\frac{dy_1}{dx_1} = \frac{\delta y_1}{\delta x_1}, \text{ and also } \frac{dy_2}{dx_2} = \frac{\delta y_2}{\delta x_2}.$$

But the equation to the limits is

$$V_2 \delta x_2 - V_1 \delta x_1 + P_2 \delta y_2 - P_1 \delta y_1 = 0.$$

From which we obtain, since the variations of the extreme points are independent of each other,

$$V_1 \delta x_1 + P_1 \delta y_1 = 0, \text{ and } V_2 \delta x_2 + P_2 \delta y_2 = 0;$$

or

$$V_1 \delta x_1 + P_1 \delta y_1 - P_1 p_1 \delta x_1 = 0 \quad (1)$$

and

$$V_2 \delta x_2 + P_2 \delta y_2 - P_2 p_2 \delta x_2 = 0 \quad (2)$$

From (1) we have

$$V_1 + P_1 m - P_1 p_1 = 0,$$

and, therefore,

$$m = p_1 - \frac{V_1}{P_1} = p_1 - \frac{1+p_1^2}{p_1} = -\frac{1}{p_1} = -\frac{1}{c}.$$

From (2) we have

$$V_2 + P_2 n - P_2 p_2 = 0$$

$$\therefore n = p_2 - \frac{V_2}{P_2} = p_2 - \frac{1+p_2^2}{p_2} = -\frac{1}{p_2} = -\frac{1}{c}.$$

$$\therefore 1 + cm = 0 \text{ and } 1 + cn = 0,$$

which equations show that the line must cut both curves at right angles.

And since the equation to a line passing through two given points is $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1} (x - x_1)$, and in this case $c = \frac{y_2 - y_1}{x_2 - x_1}$; therefore, substituting for c , we shall have $1 + m \frac{y_2 - y_1}{x_2 - x_1} = 0$ and $1 + n \frac{y_2 - y_1}{x_2 - x_1} = 0$, and from these two equations, and those of the known curves, the values of y_1, x_1, y_2, x_2 may be found, and the line completely determined.

Problem 6. Find the curve of quickest descent from one given curve to another given curve

Here

$$V = \frac{\sqrt{1+p^2}}{\sqrt{y}}.$$

$$\therefore dV = -\frac{2\sqrt{1+p^2}}{y^{\frac{3}{2}}} dy + \frac{p}{\sqrt{y}\sqrt{1+p^2}} dp = N dy + P dp,$$

whence

$$M = 0 \quad N = -\frac{2\sqrt{1+p^2}}{y^{\frac{3}{2}}} \text{ and } P = \frac{p}{\sqrt{y}\sqrt{1+p^2}}, \text{ \&c.}$$

$$\therefore V = Pp + c;$$

and, as before,

Calculus of
Variations.

$$\frac{1}{\sqrt{y} \sqrt{1+p^2}} = c = \frac{1}{\sqrt{2a}} \dots \sqrt{y} \sqrt{1+p^2} = \sqrt{2a};$$

and the equations of the limits give

$$V_1 \delta x_1 + P_1 \delta y_1 - P_1 p_1 \delta x_1 = 0 \text{ and } V_2 \delta x_2 + P_2 \delta y_2 - P_2 p_2 \delta x_2 = 0.$$

But if $\frac{dy}{dx} = m$ and $\frac{dy}{dx} = n$ be the equations to the curves $\frac{\delta y_1}{\delta x_1} = m$ and $\frac{\delta y_2}{\delta x_2} = n$, whence

$$V_1 - P_1 p_1 + P_1 m = 0 \text{ and } V_2 - P_2 p_2 + P_2 n = 0.$$

But

$$\therefore V - Pp = c = \frac{1}{\sqrt{2a}} \dots = \frac{1}{\sqrt{y} \sqrt{1+p^2}}.$$

and

$$P = \frac{p}{\sqrt{y} \sqrt{1+p^2}} = \frac{p}{\sqrt{2a}}.$$

$$\therefore \frac{1}{\sqrt{2a}} + \frac{p_1 m}{\sqrt{2a}} = 0 \text{ and } \frac{1}{\sqrt{2a}} + \frac{p_2 n}{\sqrt{2a}} = 0$$

$$\therefore 1 + p_1 m = 0 \text{ and } 1 + p_2 n = 0.$$

The last two equations show that the cycloid cuts the two curves at right angles. Also from the equation $\sqrt{y} \sqrt{1+p^2} = \sqrt{2a}$, it appears that the base of the cycloid coincides with the horizontal line from which we have supposed the body to have fallen to acquire the velocity in the cycloid.

(436.) In the problems which have been hitherto solved, the function V does not involve the limits of the integral. There are, however, cases in which this does happen, as, for instance, when the brachystochrone is required between two curves, and the motion is to commence from one of the curves. In such case, which we considered when the general theorems were investigated, we must add to the equation of the limits the terms

$\delta x_1 \int \frac{dV}{dx_1} dx + \delta y_1 \int \frac{dV}{dy_1} dy + \&c.$, and we have the equation

$$\delta \int V dx = V_2 \delta x_2 - V_1 \delta x_1 + \delta x_1 \int \frac{dV}{dx_1} dx + \delta y_1 \int \frac{dV}{dy_1} dy + \&c.$$

$$+ P_2 w_2 - P_1 w_1 + \&c. + \int \left\{ N - \frac{dP}{dx} + \&c. \right\} w dx.$$

Example. Let the curve of quickest descent between two curves be required, the motion commencing at the first.

Here, if y_1 be the vertical coordinate of the point at which the motion commences, and y that of the body at the end of t'' ,

$$t = \frac{1}{\sqrt{2g}} \int \frac{\sqrt{1+p^2}}{\sqrt{y-y_1}} dx$$

$$\therefore V = \frac{\sqrt{1+p^2}}{\sqrt{y-y_1}}$$

$$\therefore dV = \frac{p}{\sqrt{y-y_1} \sqrt{1+p^2}} dp + \frac{\sqrt{1+p^2}}{2(y-y_1)^{\frac{3}{2}}} dy_1 - \frac{\sqrt{1+p^2}}{2(y-y_1)^{\frac{3}{2}}} dy.$$

whence

$$P = \frac{p}{\sqrt{y-y_1} \sqrt{1+p^2}}, \quad N = \frac{-\sqrt{1+p^2}}{2(y-y_1)^{\frac{3}{2}}}, \quad \frac{dV}{dy_1} = \frac{\sqrt{1+p^2}}{2(y-y_1)^{\frac{3}{2}}}$$

we may $\therefore$ use the formula $V = Pp + c$.

$$\therefore \frac{\sqrt{1+p^2}}{\sqrt{y-y_1}} = \frac{p^2}{\sqrt{y-y_1} \sqrt{1+p^2}} + c, \text{ or } \sqrt{y-y_1} \cdot \sqrt{1+p^2} = \frac{1}{c} = \sqrt{2a},$$

an equation to a cycloid, its cusp being at the point at which the motion begins.

Next, referring to the equation of limits, since V does not involve x_1, x_2 , &c., we have merely to find $\frac{dV}{dy_1}$.

But

$$\frac{dV}{dy_1} = \frac{\sqrt{1+p^2}}{2(y-y_1)^{\frac{3}{2}}} = N = -\frac{dP}{dx} \therefore N - \frac{dP}{dx} = 0$$

$$\therefore \int \frac{dV}{dy_1} dy = -P = P_1 - P_2,$$

when taken between the proper values. Hence the equation for the limits becomes

Calculus of
Variations. or

or

$$V_2 \delta x_2 - V_1 \delta x_1 + P_2 w_2 - P_1 w_1 + (P_1 - P_2) \delta y_1 = 0,$$

$$(V_2 - P_2 p_2) \delta x_2 - (V_1 - P_1 p_1) \delta x_1 + P_2 \delta y_2 - P_1 \delta y_1 = 0,$$

$$\frac{\delta x_2}{\sqrt{2a}} - \frac{\delta x_1}{\sqrt{2a}} + \frac{p_2}{\sqrt{2a}} \delta y_2 - \frac{p_1}{\sqrt{2a}} \delta y_1 = 0.$$

and $\therefore$ if $\frac{dy}{dx} = m$ and $\frac{dy}{dx} = n$ be the equations to the limiting curves $\therefore \frac{\delta y_1}{\delta x_1} = m$ and $\frac{\delta y_2}{\delta x_2} = n$. we have

$$(1 + p_2 n) \delta x_2 - (1 + p_1 m) \delta x_1 = 0.$$

whence δx_2 and δx_1 are totally independent of each other,

$$1 + p_2 n = 0 \text{ and } 1 + p_1 m = 0 \therefore m = n,$$

or the tangents at the points where the cycloid meets the curves are parallel, and, since $1 + p_2 n = 0$, it cuts the second curve at right angles.

(437.) The problems which have been the objects of our consideration in the preceding pages have been those of absolute maxima and minima. We proceed to treat of questions of relative maxima and minima. Of this kind is the problem, the length of a curve being given, find its species, that the area included by it may be a maximum. Such is the problem first proposed and solved by James Bernoulli, and to which the name of Isoperimetrical has been given, a name which is extended to all questions of this class.

The problem of relative maxima and minima in its most general point of view may be defined to be this. To find that relation between the variables y and x , that while $\int u_1 dx$ is constant, the $\int u dx$ may be a maximum or minimum. The solution of this question is obtained by multiplying $\int u_1 dx$ by a constant quantity (a), and adding the product to $\int u dx$, and then substituting their sum for $\int V dx$ in the preceding theorems; or, in other words, making the variation of $\int u dx + a \int u_1 dx$, or of $\int dx (u + a u_1) = 0$. and this will appear obvious if we consider that $\therefore \int u_1 dx = c$ and $\int u dx$ is a maximum or minimum; $\therefore \delta \int u_1 dx = 0$ and $\delta \int u dx = 0$, and as the condition of the greatest or least value of $\int u dx$ is dependent upon the equation $\int u_1 dx = c$, every condition of the problem will be included in the variation of $\int (u + a u_1) dx$.

(438.) To put this, however, in a more distinct point of view, the following proof, which is founded upon that given by Sir John Herschell, in a note to the translation of Lacroix's *Elementary Treatise on the Differential and Integral Calculus*, may be here introduced. And, first, we shall show that if there be any number of independent equations, $P = 0$, $Q = 0$, $R = 0$, &c., where P , Q , R , &c., are functions of x , y , p , q , &c., they may be all included in the general equation $P + Qa + Rb + \&c. = 0$, in which the quantities a , b , &c. are entirely independent of x , y , p , q , &c. For the sake of simplicity, let P , Q , and R , be alone taken; then $P + Qa + Rb = 0$ shall include the equations $P = 0$, $Q = 0$, $R = 0$.

For, since P , Q , R , remain unchanged while a , b , may alter, the following equations will hold good simultaneously.

$$P + Qa + Rb = 0$$

$$P + Qa_1 + Rb_1 = 0$$

$$P + Qa_2 + Rb_2 = 0$$

$$\therefore Qa - a_1 + Rb - b_1 = 0 \quad (1)$$

and

$$\therefore Qa - a_2 + Rb - b_2 = 0 \quad (2)$$

$$\therefore \text{from (2)} \quad Q = -R \frac{b - b_2}{a - a_2}.$$

$\therefore$ substituting this value for Q in equation (1).

$$R \cdot \left(\frac{b - b_2 \cdot a - a_1}{a - a_2} - b - b_1 \right) = 0$$

$$\text{i. e. } R (b - b_2 \cdot a - a_1 - b - b_1 \cdot a - a_2) = 0.$$

$$\text{i. e. } R (b a_2 + b_2 a_1 + b_1 a - b_1 a_2 - b a_1 - b_2 a) = 0$$

or

$$R \cdot f(a b a_1 b_1 a_2 b_2) = 0,$$

whence, since $f(a b a_1 b_1 a_2 b_2)$ does not necessarily $= 0$, $\therefore R = 0$. The general equation is now reduced to $P + Qa = 0$; also $\therefore P + Qa_1 = 0$ $\therefore$ by subtraction $Q(a - a_1) = 0$ and $a - a_1$ does not $= 0$. $\therefore Q = 0$ and $\therefore P = 0$. Hence the equation $P + aQ + bR = 0$ does include the equations $P = 0$, $Q = 0$, $R = 0$.

Now, to apply this reasoning to the case before us, let

$$P = \delta \int u dx, \quad Q = \delta \int u_1 dx, \quad R = \delta \int u_2 dx;$$

then will the equation $\delta \int (u + a u_1 + b u_2) dx = 0$ include the separate equations of $\delta \int u dx = 0$, $\delta \int u_1 dx = 0$; $\delta \int u_2 dx = 0$, if a and b be quantities entirely independent of x , y , p , q , &c.

Now, in the particular case of relative maxima and minima, if $\int u dx$ be a maximum or minimum, $\delta \int u dx = 0$, and if $\int u_1 dx$, taken between the same limits, be of a given value, $\delta \int u_1 dx = 0$. Hence every condition of the problem will be included under the equation

$$\delta \int (u + a u_1) dx = 0.$$

Calculus of
Variations.

It is plain that if $\int u_2 dx$ be also constant, we must make $\delta \int (u + au + bu_2) dx = 0$, and so on for any number of functions. Part III.

(439.) Hence, in the variation of $\int V dx$, we must merely write for V , $V + au$, or $V + au + bv$, &c., and proceed as in the cases of absolute maxima and minima. In these examples the total variation will be expressed by $\delta \int V_1 dx$.

(440.) Example 1. To find the curve which under a given perimeter includes the greatest area.

Here $\int u dx = \int \sqrt{1+p^2} dx$ is given, and $\int V dx = \int y dx$,

$$\therefore \int V_1 dx = \int (V + au) dx = \int (y + a\sqrt{1+p^2}) dx$$

$$\therefore V_1 = y + a\sqrt{1+p^2}$$

$$\therefore dV_1 = dy + \frac{ap}{\sqrt{1+p^2}} dp = M dx + N dy + P dp + \&c.$$

$$\therefore M = 0, N = 1, P = \frac{ap}{\sqrt{1+p^2}}, Q = 0,$$

and $\therefore$ from $N - \frac{dP}{dx} = 0$, and $dV = N dy + P dp$; we have

$$V_1 = Pp + c,$$

or

$$y + a\sqrt{1+p^2} = \frac{ap^2}{\sqrt{1+p^2}} + c$$

$$\therefore y - c = a \left\{ \frac{p^2}{\sqrt{1+p^2}} - \sqrt{1+p^2} \right\} = -\frac{a}{\sqrt{1+p^2}}.$$

$$\therefore 1+p^2 = \frac{a^2}{y-c}^2 \quad \therefore p^2 = \frac{a^2 - y - c}{y-c}^2.$$

$$\therefore \frac{dx}{dy} = \frac{y-c}{\sqrt{a^2 - y - c}}.$$

$$\therefore x - c_1 = -\sqrt{a^2 - y - c}.$$

$$\therefore (x - c_1)^2 + y - c = a^2,$$

the equation to a circle.

Example 2. Required the Brachystochrone, when the length of the curve is given.

Here

$$u = \sqrt{1+p^2} \text{ and } V = \frac{\sqrt{1+p^2}}{\sqrt{y}}.$$

$$\therefore V_1 = V + au = \frac{\sqrt{1+p^2}}{\sqrt{y}} + a\sqrt{1+p^2}.$$

$$\therefore dV_1 = \left(\frac{p}{\sqrt{y}\sqrt{1+p^2}} + \frac{ap}{\sqrt{1+p^2}} \right) dp - \frac{\sqrt{1+p^2}}{2y^{\frac{3}{2}}} dy = P dp + N dy$$

$$\therefore N = -\frac{\sqrt{1+p^2}}{2y^{\frac{3}{2}}} \quad P = \frac{p}{\sqrt{y}\sqrt{1+p^2}} + \frac{ap}{\sqrt{1+p^2}}.$$

But

$$V_1 = Pp + c$$

$$\therefore \frac{\sqrt{1+p^2}}{\sqrt{y}} + a\sqrt{1+p^2} = \frac{p^2(1+a\sqrt{y})}{\sqrt{y}\sqrt{1+p^2}} + c$$

$$\therefore \frac{1+p^2}{\sqrt{y}} \cdot (1+a\sqrt{y}) = \frac{p^2(1+a\sqrt{y})}{\sqrt{y}} + c\sqrt{1+p^2}$$

$$\therefore \frac{1+a\sqrt{y}}{\sqrt{y}} = c\sqrt{1+p^2}$$

$$\therefore p^2 = \frac{(1+a\sqrt{y})^2}{c^2 y} - 1 = \frac{1+2a\sqrt{y}+(a^2-c^2)y}{c^2 y}$$

$$\therefore \frac{dx}{dy} = \frac{c\sqrt{y}}{\sqrt{1+2a\sqrt{y}+(a^2-c^2)y}}$$

Calculus of which by putting z^2 for y may be easily integrated, and which will depend upon logarithms or circular arcs, Part III.
 Variations. according as a^2 is greater or less than c^2 .

Example 3. Find the equation to the curve, which, of all those that have the same length and include the same area, shall, by rotation round the axis of x , generate the greatest solid.

Here

$$\int V_1 dx = \int (u + au_1 + bu_2) dx$$

since

$$\text{solid} = \pi \int y^2 dx, \quad \text{length} = \int \sqrt{1+p^2} dx, \quad \text{and area} = \int y dx,$$

$$\therefore V_1 = u + au_1 + bu_2 = y^2 + a\sqrt{1+p^2} + by.$$

$$\therefore dV_1 = (2y + b) dy + \frac{ap}{\sqrt{1+p^2}} dp = N dy + P dp$$

$$\therefore N = 2y + b, \quad P = \frac{ap}{\sqrt{1+p^2}}, \quad M = 0, \quad Q = 0.$$

But, in this case,

$$V = Pp + c$$

$$\therefore y^2 + by + a\sqrt{1+p^2} = \frac{ap^2}{\sqrt{1+p^2}} + c$$

$$\therefore (y^2 + by)\sqrt{1+p^2} + a = c\sqrt{1+p^2}$$

$$\therefore \sqrt{1+p^2} = \frac{a}{c - (y^2 + by)}$$

$$p^2 = 1 - \frac{a^2}{[c - (y^2 + by)]^2} = \frac{(c - y^2 - by)^2 - a^2}{[c - (y^2 + by)]^2}$$

$$\therefore \frac{dx}{dy} = \frac{c - y^2 - by}{\sqrt{(c - y^2 - by)^2 - a^2}}.$$

Example 4. Given the length of a curve to find its form, that its centre of gravity may be the lowest possible

Here

$$\int V dx = \int (u + au_1) dx.$$

Let

$$X = \text{depth of centre of gravity, } \therefore X = \frac{\int x \sqrt{1+p^2} dx}{\int \sqrt{1+p^2} dx} \\ = \frac{\int x \sqrt{1+p^2} dx}{l} \text{ if } l = \text{length of curve}$$

$$\therefore V = \frac{x \sqrt{1+p^2}}{l} + a \sqrt{1+p^2}$$

$$dV = \frac{\sqrt{1+p^2}}{l} dx + \frac{px}{l \sqrt{1+p^2}} dp + \frac{ap}{\sqrt{1+p^2}} dp \\ = M dx + N dy + P dp + \&c.$$

$$\therefore M = \frac{\sqrt{1+p^2}}{l}, \quad N = 0, \quad P = \frac{px}{l \sqrt{1+p^2}} + \frac{ap}{\sqrt{1+p^2}}$$

But

$$N - \frac{dP}{dx} = 0 \therefore P = c.$$

$$\frac{p \cdot x + a}{l \sqrt{1+p^2}} = c$$

$$\frac{\sqrt{1+p^2}}{p} = \frac{x+a}{lc} = \frac{x+a}{b} \text{ if } b = lc$$

$$\frac{1}{p^2} = \frac{(x+a)^2 - b^2}{b^2}$$

$$\therefore p = \frac{b}{\sqrt{(x+a)^2 - b^2}}.$$

$$\therefore y = b \cdot \log(x+a + \sqrt{(x+a)^2 - b^2}) + c_1.$$

the equation to a catenary.

(441.) Find the curve B M, which is the least possible, when the area A B M N is given.

Here, (fig. 3.) area = $\int y dx$, length = $\int \sqrt{1+p^2} \cdot dx$

Calculus of
Variations.

$$\therefore V = \sqrt{1+p^2} + ay$$

$$dV = a dy + \frac{p dp}{\sqrt{1+p^2}} = N dy + P dp + \&c.$$

$$\therefore N = a, P = \frac{p}{\sqrt{1+p^2}}, Q = 0, R = 0, \&c.$$

Therefore, in this case,

$$V = Pp + c,$$

or

$$ay + \sqrt{1+p^2} = \frac{p^2}{\sqrt{1+p^2}} + c$$

$$\therefore ay \sqrt{1+p^2} + 1 = c \sqrt{1+p^2}.$$

and putting

$$a = \frac{1}{a} \text{ and } \beta = \frac{c}{a}, (\beta - y) \sqrt{1+p^2} = (a)$$

$$\therefore 1 + p^2 = \frac{a^2}{(\beta - y)^2} \text{ and } p^2 = \frac{a^2 - (\beta - y)^2}{(\beta - y)^2}$$

$$\therefore \frac{dx}{dy} = \frac{(\beta - y)}{\sqrt{a^2 - (\beta - y)^2}}.$$

whence

$$(x - c_1)^2 + (\beta - y)^2 = a^2.$$

the equation to a circle.

(442.) Find the same when the area included between two lines drawn from a given point, and including an angle, is given.

Here

$$s = \int \sqrt{1+p^2} \cdot dx, \text{ and area} = \frac{1}{2} \cdot \int (x dy - y dx).$$

$$\therefore V = \sqrt{1+p^2} + a \cdot (px - y).$$

$$\therefore dV = \frac{p dp}{\sqrt{1+p^2}} + a \cdot (p dx + x dp - dy) = \frac{p dp}{\sqrt{1+p^2}} + a x dp.$$

$$\therefore M = 0, N = 0, P = \frac{p}{\sqrt{1+p^2}} + ax, \text{ and } \frac{dP}{dx} = 0 \therefore P = c$$

$$\therefore \frac{p}{\sqrt{1+p^2}} = -ax + c = a \cdot \left(\frac{c}{a} - x \right) = \frac{a-x}{\beta} : \text{ if } a = \frac{1}{\beta} \text{ and } \frac{c}{a} = a$$

$$\frac{1}{p^2} = \frac{\beta^2}{(a-x)^2} - 1 = \frac{\beta^2 - (a-x)^2}{(a-x)^2}$$

$$\therefore \frac{dy}{dx} = \frac{a-x}{\sqrt{\beta^2 - (a-x)^2}}$$

whence

$$(y - c_1)^2 + (a - x)^2 = \beta^2,$$

the equation to a circle.

(443.) In the preceding problems we have only considered those cases in which the variable y is a function of x alone; we proceed to show the application of the formulæ to those questions in which a third variable z is introduced. The general expression for the variation of $\int V dx$, when y and z are functions of x , is

$$\begin{aligned} \delta \int V dx &= V \delta x + \left(P - \frac{dQ}{dx} + \frac{d^2R}{dx^2} - \&c. \right) w \\ &+ \left(P_1 - \frac{dQ_1}{dx} + \frac{d^2R_1}{dx^2} - \&c. \right) w_1 \\ &+ \&c. + \&c. + \&c. \\ &+ \int \left(N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3} + \&c. \right) w dx \\ &+ \int \left(N_1 - \frac{dP_1}{dx} + \frac{d^2Q_1}{dx^2} - \frac{d^3R_1}{dx^3} + \&c. \right) w_1 dx. \end{aligned}$$

in which $w_1, N_1, P_1, Q_1, R_1, \&c.$, are of the same form as $w, N, P, Q, R, \&c.$, but z is to be written instead of y .

Calculus of Variations.

Example. Required the shortest distance between two points in space.

Part III.

 Here, if s be the distance, $s = \int \sqrt{1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}} \cdot dx$;

$$\therefore V = \sqrt{1 + \frac{dy^2}{dx^2} + \frac{dz^2}{dx^2}} = \sqrt{1 + p^2 + p_1^2}$$

$$\therefore dV = \frac{p dp}{\sqrt{1 + p^2 + p_1^2}} + \frac{p_1 dp_1}{\sqrt{1 + p^2 + p_1^2}}$$

But, at a maximum or minimum,

$$N - \frac{dP}{dx} + \frac{d^2Q}{dx^2} - \frac{d^3R}{dx^3} + \&c. = 0 \quad (1)$$

and

$$N_1 - \frac{dP_1}{dx} + \frac{d^2Q_1}{dx^2} - \frac{d^3R_1}{dx^3} + \&c. = 0 \quad (2);$$

 whence $\therefore N = 0, Q = 0, R = 0, \&c.$, and $N_1 = 0, Q_1 = 0, R_1 = 0, \&c.$, we have from equations (1) and (2)

$$\frac{dP}{dx} = 0 \text{ and } \frac{dP_1}{dx} = 0, \therefore P = a \text{ and } P_1 = b.$$

But

$$P = \frac{p}{\sqrt{1 + p^2 + p_1^2}} \text{ and } P_1 = \frac{p_1}{\sqrt{1 + p^2 + p_1^2}}$$

$$\therefore \frac{dy}{dx} = a \sqrt{1 + p^2 + p_1^2} \text{ and } \frac{dz}{dx} = b \sqrt{1 + p^2 + p_1^2}$$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{a}{b} \text{ and } \therefore y = \frac{a}{b} z + c.$$

 Whence it appears that the trace of the required distance upon the plane of yz is a straight line; and as, if we had assumed z to be the independent variable, the result would have been similar, *i. e.* of the form $y = \frac{a_1}{b_1} x + c_1$,

we may conclude that the curve line joining the two points lies in one plane, or is a curve of simple curvature.

Hence if the points be given and fixed, and there be no other condition than that simply of shortest distance between them, the case becomes that of the first example, and the line is a straight one, the nature of which may be determined by the given coordinates of the points to which it must be drawn.

(444.) The problem, however, may be solved by the following more general process, and which illustrates the methods given in articles (421 and 422.)

 Thus, supposing $\int ds = \int \sqrt{dx^2 + dy^2 + dz^2}$,

$$\begin{aligned} \therefore \delta \int ds &= \int \delta ds = \int \frac{dx}{ds} \delta dx + \int \frac{dy}{ds} \delta dy + \int \frac{dz}{ds} \delta dz \\ &= \frac{dx}{ds} \delta x + \frac{dy}{ds} \delta y + \frac{dz}{ds} \delta z - \int \left\{ d \frac{dx}{ds} \cdot \delta x + d \frac{dy}{ds} \cdot \delta y + d \frac{dz}{ds} \delta z \right\} \end{aligned}$$

 Now by reference to article (422.), we find $M = 0, m = 0, \mu = 0$,

$$N = \frac{dx}{ds}, n = \frac{dy}{ds}, \nu = \frac{dz}{ds}, \therefore dN = 0, dn = 0, d\nu = 0,$$

$$\therefore d \cdot \frac{dx}{ds} = 0, d \cdot \frac{dy}{ds} = 0, \text{ and } d \cdot \frac{dz}{ds} = 0,$$

$$\therefore \frac{dx}{ds} = a, \frac{dy}{ds} = b, \frac{dz}{ds} = c;$$

whence

$$\frac{dx^2}{ds^2} + \frac{dy^2}{ds^2} + \frac{dz^2}{ds^2} = 1 = a^2 + b^2 + c^2$$

 an equation of condition by which the constants a, b, c are connected. Also, since

$$\frac{dx}{dz} = \frac{a}{c} \text{ and } \frac{dy}{dz} = \frac{b}{c},$$

$$\therefore x = \frac{a}{c} z + c_1 (1) \text{ and } y = \frac{b}{c} z + c_2 (2),$$

the equations to a straight line.

Calculus of Variations. The five constants a, b, c, c_1, c_2 may be found from the four equations which arise from the substitution of the given coordinates of the points for x, y, z , in equations (1) and (2), and from the equation of condition $a^2 + b^2 + c^2 = 1$. Part III.

If one of the points be fixed, but the other situated on a curve surface, let $dz = p dx + q dy$ be the equation to the surface. Then, if the coordinates of the second point be x_2, y_2, z_2 ,

$$\delta z_2 = p_2 \delta x_2 + q_2 \delta y_2,$$

and the equation of the limits, which is

$$\frac{dx_2}{ds_2} \delta x_2 + \frac{dy_2}{ds_2} \delta y_2 + \frac{dz_2}{ds_2} \delta z_2 = 0$$

becomes

$$dx_2 \delta x_2 + dy_2 \delta y_2 + dz_2 \cdot (p_2 \delta x_2 + q_2 \delta y_2) = 0,$$

or

$$(dx_2 + p_2 dz_2) \delta x_2 + (dy_2 + q_2 dz_2) \delta y_2 = 0;$$

and since δx_2 and δy_2 are independent variations, we must put their coefficients separately equal to zero.

$$\therefore dx_2 + p_2 dz_2 = 0 \text{ and } dy_2 + q_2 dz_2 = 0,$$

which show that the required line coincides with the normal of the given surface.

(445.) Next; let it be required to draw the shortest line upon a given surface.

In this case the variations of the coordinates x, y, z , which are under the sign $\int$, must satisfy the differential equation to the surface, since the line in question coincides at every point with it.

Let $\therefore dz = p dx + q dy$ be the equation to the surface, whence we have $\delta z = p \delta x + q \delta y$; and substituting for δz , we have

$$\begin{aligned} \delta \cdot \int ds &= \left(\frac{dx}{ds} + p \frac{dz}{ds} \right) \delta x + \left(\frac{dy}{ds} + q \frac{dz}{ds} \right) \delta y \\ &\quad - \int \left\{ \left(d \cdot \frac{dx}{ds} + p \cdot d \cdot \frac{dz}{ds} \right) \delta x + \left(d \cdot \frac{dy}{ds} + q \cdot d \cdot \frac{dz}{ds} \right) \delta y \right\}; \end{aligned}$$

whence we have from the part under the sign of integration

$$d \cdot \frac{dx}{ds} + p \cdot d \cdot \frac{dz}{ds} = 0 \text{ and } d \cdot \frac{dy}{ds} + q \cdot d \cdot \frac{dz}{ds} = 0,$$

which are the equations of the curve required.

Take for an example the least distance measured along a sphere.

Here

$$z^2 = r^2 - x^2 - y^2;$$

$$\therefore \frac{dz}{dx} = p = -\frac{x}{z} \text{ and } \frac{dz}{dy} = q = -\frac{y}{z},$$

and the equations become, considering ds constant,

$$z d^2 x - x d^2 z = 0 \text{ and } z d^2 y - y d^2 z = 0,$$

from which we also obtain

$$y d^2 x - x d^2 y = 0;$$

whence, by integration,

$$z dx - x dz = a_1 ds \quad (1)$$

$$y dz - z dy = a_2 ds \quad (2)$$

$$x dy - y dx = a_3 ds \quad (3).$$

Multiply (1) by y , (2) by x , and (3) by z , and add; and then

$$a_1 y + a_2 x + a_3 z = 0,$$

the equation to a plane which passes through the centre of the sphere; but the intersection of a plane with a sphere is a circle, and if the plane pass through the centre of the sphere it is a great circle; the required line is, therefore, a portion of a great circle of the sphere.

CALCULUS OF FINITE DIFFERENCES.

Calculus of
Finite
Differences.

(1.) Custom has decided that no Treatise on the Differential Calculus can be reckoned complete, unless it also include the Calculus of Finite Differences: and we adhere to the arrangement which has been made. Indeed we may consider that, by the term Differential Calculus, there is meant that branch of pure Mathematics which solely treats of variable quantities; and this division being allowed, the Calculus of Finite Differences is naturally a subdivision, and rightly occupies the place assigned to it.

Calculus of
Finite
Differences.

The intimate connection between it and the Differential Calculus, properly so called, will soon be seen; and in many instances it may be shown that the formulas of the one can be derived from processes given in the other.

The Calculus of Finite Differences, or of Differences, as Lacroix has proposed to name this branch of Mathematics, may, perhaps, be defined as that which chiefly treats of numbers, and of the change which numerical equations undergo, in consequence of given alterations taking place in the values of the number or numbers of which these are the functions.

Although this statement may at first lead the student to suppose that this Calculus is confined in its views, and limited in its applications, yet will he ultimately find its uses are most extensive: and, as instances, may be informed that the theorems which it demonstrates afford the means and frequently the only means of solution of the interesting Problems to which we are led in our inquiries respecting the Summation of Series, and the questions arising from the Theory of Probabilities, and the construction of Logarithmic Tables.

(2.) We next proceed to explain the terms and the notation made use of in this branch of the Calculus.

If we take the series which is formed by the sum of the squares of the natural numbers, *viz.*

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + \&c. + x^2 + \&c.$$

where x expresses the number of terms, and x^2 is in this case the x th term of the series. It is evident that by giving to x the values 1, 2, 3, &c., the 1st, 2d, 3d, &c. term is produced: on this account x^2 is called the general term of this series; and when the general term is known, the successive terms of the series are also capable of being found.

The general term is therefore a function of x , *i.e.* a function of its place in the series, and mathematicians have, for the most part, agreed to express this particular kind of function by the symbol u_x or v_x : the letter x is called the index, since it marks the place in the series to which any particular term belongs.

And, as in the Differential Calculus, when the form of $f(x)$ is given, that of $f(x+h)$ can be found by means of the differential coefficients of $f(x)$: so also may u_x , as will be shown, be found in terms of the differences of its preceding values.

The nature of these differences may be thus explained. But instead of the series of the squares let us take the following,

$$a + b + c + d + e + \&c.$$

where a, b, c , &c., are numbers connected together by some constant law of derivation, a condition implied by the notion of a series.

If now we subtract a from b , b from c , c from d , &c., we have the results

$$b - a, c - b, d - c, e - d, \&c.,$$

which are called first differences, $b - a$ being the first difference of a ; if the process be continued, we shall have

$$c - 2b + a, d - 2c + b, e - 2d + c, \&c.$$

which are termed second differences; and continuing the process we shall arrive at third, fourth, &c. differences.

(3.) Now if we return to the series of the squares of the natural numbers and write them down according to their actual values, and not symbolically, *viz.*

$$1 + 4 + 9 + 16 + 25 + 36 + 49 + \&c.$$

and if we subtract the first number from the second, the second from the third, the third from the fourth, &c., and afterwards again subtract the first of these differences from the second difference, the second from the third, and so on, we shall have, first writing down the original series, the series of numbers

$$\begin{array}{ll} 1 + 4 + 9 + 16 + 25 + 36 + 49 + \&c. & \text{squares,} \\ 3 + 5 + 7 + 9 + 11 + 13 + \&c. & \text{first differences,} \\ 2 + 2 + 2 + 2 + 2 + \&c. & \text{second differences;} \end{array}$$

whence it appears that the first differences are the odd numbers, and that the second differences are all equal.

Now the use of this investigation is this, that we can by mere addition produce any terms of the series we may wish, if we know three corresponding numbers in each of the three horizontal lines. Thus having given 16, 9, 2, to form the succeeding terms of the series, add 2 to 9 and 9 to 16, and we obtain 11 and 25, the next terms in the second and first lines; and these being found, we may form in a similar manner other terms. In fact, having given the 1, 3, 2, the first numbers of each horizontal line, 2 being assumed constant or the same for every place of the lowest line, all the other terms may be found. Then adding 2 to 3 we have 5, the second number in the middle line, and adding 3 to 1 we have 4, the second figure in the highest line. Again beginning from the constant difference 2, add 2 to 5 and we have 7, add 5 to 4 and we have 9, the next two figures, and similarly may all the terms be found.

Calculus of
Finite
Differences.

(4.) Had we taken the series of the cubes of the natural numbers, or $1 + 8 + 27 + 64 + 125 + 216 + 343 + \&c.$, and taken the first, second, and third differences, the first numbers of the horizontal lines, including the series itself, would be 1, 7, 12, 6, the third difference 6 is constant, and the process indicated in the preceding article would be found to have place here.

Calculus of
Finite
Differences.

And thus if the n th differences between successive logarithms become constant or very nearly so, the logarithms themselves may be found by successive additions only.

The truth of this rule will be afterwards proved, although the formation of the differences makes it almost sufficiently evident.

The term difference which we have used requires a particular definition, and it is understood to express the excess of one term over that which immediately precedes it. The first object of the Direct Calculus of Differences is to find the value of this difference whatever be the function of x ; and we may here add that x is supposed to increase by unity.

(5.) From this definition it appears that the difference of any function is found by writing $x + 1$, for x in it, and subtracting from the result the original function.

This operation is symbolically represented by prefixing the letter Δ before the function under consideration.

Thus Δu_x expresses the difference of u_x , and Δx the difference of x , and thus

$$u_{x+1} - u_x = \Delta u_x \text{ and } x + 1 - x = \Delta x.$$

(6.) But since $u_{x+1} - u_x$ or Δu_x may also be a function of x , and therefore susceptible of change when $x + 1$ is put for x in it: Δu_x will have a difference which is called the second difference of u_x .

And the second difference of Δu_x is from our definition

$$\Delta u_{x+1} - \Delta u_x = \Delta(u_{x+1} - u_x) = \Delta(\Delta u_x) = \Delta^2 u_x.$$

$\Delta^2 u_x$ being the symbol used for the second difference of u_x .

We may stop to remark that the equations

$$u_{x+1} = u_x + \Delta u_x \text{ and } \Delta u_{x+1} = \Delta^2 u_x + \Delta u_x$$

express the method used in determining the places of the numbers in the illustration given at the commencement of the article: thus let $x = 4$,

$$\therefore u_x = 16, \Delta u_x = 9, \Delta^2 u_x = 2, u_{x+1} = u_5 = 16 + 9 = 25 \text{ and } \Delta u_5 = 9 + 2 = 11.$$

(7.) The second difference being written $\Delta^2 u_x$, the third difference is expressed by $\Delta^3 u_x$, the fourth by $\Delta^4 u_x$, and the n th by $\Delta^n u_x$; a system of notation which bears a close analogy to that used in the early part of the Differential Calculus, when we had occasion to find the successive differential coefficients of a function.

(8.) Again referring to the definition of a difference, if $u_0, u_1, u_2, u_3, \&c., u_n$ be the values of u_x when 0, 1, 2, 3, &c. . . . n are put for x , we have the following equations:

$$u_1 - u_0 = \Delta u_0$$

$$u_2 - u_1 = \Delta u_1$$

$$u_3 - u_2 = \Delta u_2$$

$$u_4 - u_3 = \Delta u_3$$

$$\&c. = \&c.$$

$$u_n - u_{n-1} = \Delta u_{n-1},$$

whence by addition we obtain the theorem

$$u_n - u_0 = \Delta u_0 + \Delta u_1 + \Delta u_2 + \Delta u_3 + \&c. + \Delta u_{n-1}$$

Then taking the series $1 + 8 + 27 + 64 + 125 + 216 + \&c.$

$$u_0 = 1, \Delta u_0 = 7, \Delta u_1 = 19, \Delta u_2 = 37, \Delta u_3 = 61$$

$$u_4 - u_0 = \Delta u_0 + \Delta u_1 + \Delta u_2 + \Delta u_3 = 7 + 19 + 37 + 61 = 124,$$

$$\text{or } u_4 = 1 + 124 = 125, \text{ as in the series.}$$

In the same manner, since

$$\Delta u_1 - \Delta u_0 = \Delta(\Delta u_0) = \Delta^2 u_0$$

$$\Delta u_2 - \Delta u_1 = \Delta^2 u_1$$

$$\Delta u_3 - \Delta u_2 = \Delta^2 u_2$$

$$\&c. = \&c.$$

$$\Delta u_n - \Delta u_{n-1} = \Delta^2 u_{n-1}$$

$$\Delta u_n - \Delta u_0 = \Delta^2 u_0 + \Delta^2 u_1 + \Delta^2 u_2 + \&c. + \Delta^2 u_{n-1}.$$

(9.) From the notation which has been used in the preceding articles, we see that the difference of a function is expressed by prefixing the letter Δ ; what has been shown with regard to a monomial is also true for a set of terms connected by the sign $+$ or $-$.

Thus

$$\Delta(u_x + v_x - w_x) = u_{x+1} + v_{x+1} - w_{x+1} - (u_x + v_x - w_x) = u_{x+1} - u_x + v_{x+1} - v_x - w_{x+1} + w_x = \Delta u_x + \Delta v_x - \Delta w_x$$

or the difference of the sum of any number of functions is equal to the sum of the differences of each.

Also since

$$\Delta(a \cdot u_x) = a u_{x+1} - a u_x = a(u_{x+1} - u_x) = a \Delta u_x,$$

it appears that constant quantities may be brought without the sign of the difference.

Calculus of
Finite
Differences.

And since

$$\Delta(u_x \pm c) = \overline{u_{x+1} \pm c} - \overline{u_x \pm c} = u_{x+1} - u_x = \Delta u_x,$$

constant quantities connected to the function by the sign $\pm$ disappear when the difference of the function is taken.

 Calculus of
Finite
Differences.

The constant c need not, however, be independent of x , but must be such a function of it that $c_{x+1} = c_x$.

(10.) When the form of u_x is given, the first and successive differences are easily found, as the following illustrations will show.

Let $u_x = x^2$, or let u_x represent the general term of the series $1 + 4 + 9 + 16 + \&c.$, which is the sum of the squares of the natural numbers, a series to which we have frequently referred.

Then

$$u_{x+1} - u_x = \Delta u_x = (x+1)^2 - x^2 = 2x + 1$$

$$\Delta^2 u_x = 2(x+1) + 1 - (2x+1) = 2. \Delta^3 u_x = 0.$$

Again, if

$$u_x = x^3; \Delta u_x = (x+1)^3 - x^3 = 3x^2 + 3x + 1$$

$$\Delta^2 u_x = 3(2x+1) + 3 = 6x + 6. \Delta^3 u_x = 6, \Delta^4 u_x = 0;$$

this last expression may be thus verified, for $u_x = x^3$ represents the general term of the series $1 + 8 + 27 + 64 + 125 + \&c.$

The first differences of the terms are 7, 19, 37, 61, &c., the second are 12, 18, 24, &c., the third 6, 6, &c., and the 4th differences vanish.

(11.) Again, if $u_x = x^m \therefore \Delta u_x = (x+1)^m - x^m$

$$= m x^{m-1} + m \frac{m-1}{2} x^{m-2} + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} x^{m-3} + \&c.$$

$$= m x^{m-1} + a x^{m-2} + b x^{m-3} + \&c.$$

$$\Delta^2 u_x = m \Delta x^{m-1} + a \Delta x^{m-2} + b \Delta x^{m-3} + \&c.$$

and by writing $m-1, m-2, m-3, \&c.$ for m in the expression for Δx^m , we shall find the value of $\Delta x^{m-1}, \Delta x^{m-2}, \Delta x^{m-3}, \&c.$, and we shall obtain

$$\Delta^2 u_x = m(m-1) x^{m-2} + a_1 x^{m-3} + b_1 x^{m-4} + \&c.$$

Similarly,

$$\Delta^3 u_x = m(m-1)(m-2) x^{m-3} + a_2 x^{m-4} + b_2 x^{m-5} + \&c.$$

and thus proceeding we shall obviously have

$$\Delta^n x^m = m(m-1)(m-2) \dots (m-n+1) x^{m-n} + A x^{m-n-1} + \&c.$$

and if $m = n$, every term but the first will vanish, and we shall finally have

$$\Delta^m x^m = m(m-1)(m-2) \dots (m-3) \dots 3 \cdot 2 \cdot 1,$$

a constant quantity, and, therefore, the higher differences of x^m will vanish.

(12.) Again, if $u_x = A x^m + B x^p + C x^q + \&c. \dots$ where m is the greatest index, then

$$\Delta^m u_x = A \Delta^m x^m + B \Delta^m x^p + C \Delta^m x^q + \&c.$$

But by the preceding article $\Delta^m x^m = m(m-1)(m-2) \dots 3 \cdot 2 \cdot 1$

$$\Delta^m x^p = 0, \Delta^m x^q = 0 \&c.$$

$$\therefore \Delta^m u_x = A m(m-1)(m-2) \dots 3 \cdot 2 \cdot 1.$$

(13.) The following examples contain some useful results, and will entirely explain the process of finding the differences of functions.

Example (1.) Let $u_x = a + bx$, which is the expression for the general term of an arithmetic series, u_0 being the first term.

$$\therefore u_{x+1} = a + b(x+1)$$

and

$$u_{x+1} - u_x = \Delta u_x = (a + b(x+1)) - (a + bx) = b$$

$$\Delta^2 u_x = 0, \text{ results which the formation of arithmetic series fully explains.}$$

Example (2.) Let

$$u_x = a^x \therefore \Delta u_x = a^{x+1} - a^x = (a-1)a^x.$$

$$\therefore \Delta^2 u_x = \Delta a - 1 a^x = a - 1 \Delta a^x = (a-1)^2 \cdot a^x$$

$$\Delta^3 u_x = (a-1)^3 \cdot a^x \text{ and } \therefore \Delta^n u_x = (a-1)^n \cdot a^x.$$

Example (3.) Let

$$u_x = \sin x\theta, \therefore \Delta u_x = \sin x\theta + 1\theta - \sin x\theta$$

$$= 2 \sin \frac{\theta}{2} \cdot \cos \left(x + \frac{1}{2}\right)\theta$$

Since

$$\sin A - \sin B = 2 \sin \frac{A-B}{2} \cdot \cos \frac{A+B}{2}.$$

Again, let

$$u_x = \cos x\theta, \therefore \Delta \cdot \cos x\theta = \cos x\theta + 1\theta - \cos x\theta$$

$$= -2 \sin \frac{\theta}{2} \cdot \sin \left(x + \frac{1}{2}\right)\theta.$$

Since

$$\cos A - \cos B = -2 \sin \frac{A-B}{2} \cdot \sin \frac{A+B}{2}$$

$$\begin{aligned}\Delta^2 \sin x \theta &= 2 \sin \frac{\theta}{2} \cdot \Delta \cdot \cos \left(x + \frac{1}{2} \right) \theta \\ &= 2 \sin \frac{\theta}{2} \cdot \left\{ \cos \left(x + \frac{3}{2} \right) \theta - \cos \left(x + \frac{1}{2} \right) \theta \right\} \\ &= -4 \sin^2 \frac{\theta}{2} \cdot \sin (x + 1) \theta.\end{aligned}$$

$$\begin{aligned}\therefore \Delta^3 \sin x \theta &= -4 \sin^2 \frac{\theta}{2} \cdot 2 \sin \frac{\theta}{2} \cdot \cos \left(x + 1 + \frac{1}{2} \right) \theta \\ &= -8 \sin^3 \frac{\theta}{2} \cdot \cos \left(x + \frac{3}{2} \right) \theta\end{aligned}$$

$$\therefore \Delta^4 \sin x \theta = 2^4 \sin^4 \frac{\theta}{2} \cdot \sin (x + 2) \theta$$

Similarly,

$$\Delta^4 \cdot \cos x \theta = 2^4 \sin^4 \frac{\theta}{2} \cdot \cos (x + 2) \theta.$$

Hence

$$\Delta^{2n} \sin x \theta = \pm \left(2 \sin \frac{\theta}{2} \right)^{2n} \cdot \sin (x + n) \theta$$

$$\Delta^{2n} \cos x \theta = \pm \left(2 \sin \frac{\theta}{2} \right)^{2n} \cdot \cos (x + n) \theta$$

the upper sign to be used when n is of the form $4m$, and the lower when it is of the form $4m + 2$,

and also $\Delta^{2n+1} \sin x \theta = \pm \left(2 \sin \frac{\theta}{2} \right)^{2n+1} \cos \left(x + n + \frac{1}{2} \right) \theta.$

Example (4.) Let

$$u_x = \tan x \theta$$

$$\Delta \tan x \theta = \tan \overline{x+1} \theta - \tan x \theta$$

$$= \frac{\sin \overline{x+1} \theta}{\cos \overline{x+1} \theta} - \frac{\sin x \theta}{\cos x \theta}$$

$$= \frac{\sin \overline{x+1} \theta \cdot \cos x \theta - \sin x \theta \cdot \cos \overline{x+1} \theta}{\cos x \theta \cdot \cos \overline{x+1} \theta} = \frac{\sin \theta}{\cos x \theta \cdot \cos \overline{x+1} \theta}$$

Similarly,

$$\Delta \cotan x \theta = \frac{-\sin \theta}{\sin x \theta \cdot \sin \overline{x+1} \theta}$$

Example (5.) $u_x = \log x$, $\therefore \Delta \log x = \log (x + 1) - \log x$

and $\Delta^2 \log x = \log \overline{x+2} - \log \overline{x+1} - (\log \overline{x+1} - \log x) = \log \overline{x+2} - 2 \log \overline{x+1} + \log x$

$$\Delta^3 \log x = \log \overline{x+3} - 3 \log \overline{x+2} + 3 \log \overline{x+1} - \log x$$

$$\Delta^4 \log x = \log \overline{x+4} - 4 \log \overline{x+3} + 6 \log \overline{x+2} - 4 \log \overline{x+1} + \log x,$$

and similarly it would appear that

$$\begin{aligned}\Delta^n \cdot \log x &= \log (x + n) - n \cdot \log (x + n - 1) + n \cdot \frac{n-1}{2} \log (x + n - 2) \\ &\quad - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} \log (x + n - 3) + \&c. \dots \pm \log x,\end{aligned}$$

a formula which will be shown hereafter to be true for every value of u_x .

(14.) Having found the difference of single terms, we proceed to deduce general expressions for the product and quotient of two or more functions.

Thus

$$\begin{aligned}\Delta (u_x v_x) &= u_{x+1} v_{x+1} - u_x v_x \\ &= (u_x + \Delta u_x) (v_x + \Delta v_x) - u_x v_x \\ &= u_x \Delta v_x + v_x \Delta u_x + \Delta u_x \Delta v_x \\ &= u_x \Delta v_x + v_{x+1} \Delta u_x,\end{aligned}$$

a formula of considerable utility in the inverse method of differences.

Again, $\Delta \left(\frac{u_x}{v_x} \right) = \frac{u_{x+1}}{v_{x+1}} - \frac{u_x}{v_x} = \frac{u_x + \Delta u_x}{v_x + \Delta v_x} - \frac{u_x}{v_x} = \frac{v_x \Delta u_x - u_x \Delta v_x}{v_{x+1} v_x}.$

Thus $\Delta (x^m \cdot a_x) = x^m \cdot \Delta a_x + a_{x+1} \cdot \Delta x^m = a_x \cdot \{ a - 1 x^m + a \Delta x^m \}$

and

$$\Delta \frac{x^m}{a_x} = \frac{x^m \cdot \Delta a_x - a_x \Delta x^m}{a_{x+1} \cdot a_x} = \frac{x^m \cdot (a - 1) - \Delta x^m}{a_{x+1}}.$$

Calculus of
Finite
Differences.

 (15.) Next to find the difference of the product of any number of functions $u_x \cdot u_{x+1} \cdot u_{x+2} \dots u_{x+n}$.

 Calculus of
Finite
Differences.

$$\Delta(u_x \cdot u_{x+1} \cdot u_{x+2} \dots u_{x+n}) = u_{x+1} u_{x+2} u_{x+3} \dots u_{x+n} \{u_{x+n+1} - u_x\}$$

and

$$\Delta \left(\frac{1}{u_x \cdot u_{x+1} \cdot u_{x+2} \dots u_{x+n}} \right) = \frac{1}{u_{x+1} \cdot u_{x+2} \dots u_{x+n}} \left\{ \frac{1}{u_{x+n+1}} - \frac{1}{u_x} \right\} = - \frac{u_{x+n+1} - u_x}{u_x \cdot u_{x+1} \cdot u_{x+2} \dots u_{x+n+1}}$$

 (16.) If $u_x = a + bx$, i. e. if u_x be the general term of an arithmetic series, these latter expressions assume a simple form to which we shall hereafter refer.

For then

$$u_{x+n+1} - u_x = (a + b \cdot x + n + 1) - (a + bx) = b \cdot n + 1.$$

$$\therefore \Delta(u_x \cdot u_{x+1} u_{x+2} \dots u_{x+n}) = n + 1 \cdot b \cdot \{u_{x+1} u_{x+2} \dots u_{x+n}\}$$

and

$$\Delta \left(\frac{1}{u_x \cdot u_{x+1} \dots u_{x+n}} \right) = \frac{-n + 1 \cdot b}{u_x u_{x+1} \dots u_{x+n+1}}.$$

Hence when the function is composed of factorials which are in arithmetic progression, the difference of the product is of the same form as the function itself, except that there is one factor less; and the difference of the fraction is similar to the fraction itself, with the exception of a new factor being added to the denominator.

 (17.) From the preceding expressions, a theorem for u_{x+n} may be deduced, in terms of u_x and its successive differences Δu_x , $\Delta^2 u_x$, $\Delta^3 u_x$, &c.

For since

$$u_{x+1} - u_x = \Delta u_x \text{ or } u_{x+1} = u_x + \Delta u_x, \\ \therefore \Delta u_{x+1} = \Delta u_x + \Delta(\Delta u_x) = \Delta u_x + \Delta^2 u_x.$$

But

$$u_{x+2} = u_{x+1} + \Delta u_{x+1} \\ = u_x + \Delta u_x + \Delta u_x + \Delta^2 u_x \\ = u_x + 2 \Delta u_x + \Delta^2 u_x.$$

And

$$u_{x+3} = u_{x+2} + 2 \Delta u_{x+1} + \Delta^2 u_{x+1} \\ = u_x + \Delta u_x + 2 \Delta u_x + 2 \Delta^2 u_x + \Delta^2 u_x + \Delta^3 u_x \\ = u_x + 3 \Delta u_x + 3 \Delta^2 u_x + \Delta^3 u_x.$$

The form of these expressions, the numerical coefficients of which follow the law of the coefficients of the binomial theorem, would finally lead us to the equation

$$u_{x+n} = u_x + n \Delta u_x + n \frac{n-1}{2} \Delta^2 u_x + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} \Delta^3 u_x + \&c.;$$

the truth, however, of this assumption may be thus shown.

 Let the value for u_{x+n} be supposed to be true:

 Then $u_{x+n+1} = u_{x+n} + \Delta u_{x+n}$

$$= u_x + n \Delta u_x + \frac{n \cdot n-1}{1 \cdot 2} \Delta^2 u_x + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} \Delta^3 u_x + \&c.;$$

$$+ \Delta u_x + n \Delta^2 u_x + \frac{n \cdot n-1}{1 \cdot 2} \Delta^3 u_x + \&c.$$

$$= u_x + (n+1) \Delta u_x + \frac{(n+1) \cdot n}{1 \cdot 2} \Delta^2 u_x + \frac{(n+1) \cdot n \cdot (n-1)}{1 \cdot 2 \cdot 3} \Delta^3 u_x + \&c.;$$

 whence it appears that if the theorem be true for any one term it is true for the next succeeding. Now we have shown that the law of the coefficient does obtain, when $n=2$, and also when $n=3$; the theorem is therefore universally true.

 (18.) The expression for u_{x+n} is very useful in finding the general term of a series when the differences are given.

 For first writing x for n , and then putting $n=0$, we have for the $(x+1)^{\text{th}}$ term, or u_x

$$u_x = u_0 + x \Delta u_0 + x \frac{x-1}{2} \Delta^2 u_0 + \frac{x \cdot x-1 \cdot x-2}{1 \cdot 2 \cdot 3} \Delta^3 u_0 + \&c.$$

$$\therefore x^{\text{th}} \text{ term or } u_{x-1} = u_0 + (x-1) \Delta u_0 + \frac{x-1 \cdot x-2}{1 \cdot 2} \Delta^2 u_0 + \frac{x-1 \cdot x-2 \cdot x-3}{1 \cdot 2 \cdot 3} \Delta^3 u_0 + \&c.$$

Calculus of
Finite
Differences.

Example. Find the general term of the series $19 + 28 + 39 + 52 + \&c.$: arranging the terms,

$$19 + 28 + 39 + 52 + \&c.$$

$$9 + 11 + 13 + \&c. \text{ first differences.}$$

$$2 + 2 + 2 + \&c. \text{ second differences.}$$

$$0 + 0 + 0 + \&c. \text{ third differences.}$$

Here

$$u_0 = 19, \Delta u_0 = 9, \Delta^2 u_0 = 2, \Delta^3 u_0 = 0;$$

$$\therefore u_{x-1} = 19 + (x-1)9 + \frac{(x-1)(x-2)}{1 \cdot 2} 2$$

$$= 10 + 9x + x^2 - 3x + 2 = x^2 + 6x + 12;$$

and by putting 1, 2, 3, &c. for x , the successive terms of the series will be found.

(19.) Having found the value of u_{x+n} in terms of u_x and its differences, we shall now find $\Delta^n u_x$ in terms of u_{x+n} and its preceding values.

For since

$$\begin{aligned} \Delta u_x &= u_{x+1} - u_x \text{ and } \Delta u_{x+1} = u_{x+2} - u_{x+1} \\ \therefore \Delta^2 u_x &= \Delta u_{x+1} - \Delta u_x = (u_{x+2} - u_{x+1}) - (u_{x+1} - u_x) \\ &= u_{x+2} - 2u_{x+1} + u_x, \end{aligned}$$

and

$$\begin{aligned} \Delta^3 u_x &= \Delta u_{x+2} - 2\Delta u_{x+1} + \Delta u_x \\ &= u_{x+3} - u_{x+2} - 2(u_{x+2} - u_{x+1}) + u_{x+1} - u_x \\ &= u_{x+3} - 3u_{x+2} + 3u_{x+1} - u_x, \end{aligned}$$

the law of the coefficients apparently following that of the coefficients of the expanded binomial $(a-b)^n$, and we shall have by analogy

$$\Delta^n u_x = u_{x+n} - n \cdot u_{x+n-1} + n \frac{n-1}{2} u_{x+n-2} - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} u_{x+n-3} + \&c.$$

Assuming this series to express the true value of $\Delta^n u_x$,

$$\therefore \Delta^{n+1} u_x = \Delta^n u_{x+1} - \Delta^n u_x$$

$$= u_{x+n+1} - n u_{x+n} + \frac{n \cdot n-1}{1 \cdot 2} u_{x+n-1} - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} u_{x+n-2} + \&c.$$

$$- u_{x+n} + n \cdot u_{x+n-1} - \frac{n \cdot n-1}{2} u_{x+n-2} + \&c.$$

$$= u_{x+n+1} - (n+1) u_{x+n} + \frac{n+1 \cdot n}{1 \cdot 2} u_{x+n-1} - \frac{n+1 \cdot n \cdot n-1}{1 \cdot 2 \cdot 3} u_{x+n-2} + \&c.$$

Hence if the law be true for the index n , it is true for the index $n+1$, and we have seen that it is true when $n=2$, and when $n=3$; it is therefore true when $n=4$, and thus by successive inductions is always true.

(20.) If $\Delta^n u_x = 0$, or if the differences at length became constant, we have

$$u_{x+n} - n u_{x+n-1} + \frac{n \cdot n-1}{1 \cdot 2} u_{x+n-2} - \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} u_{x+n-3} + \&c. \pm u_x = 0;$$

or, putting $x=1$,

$$u_{n+1} = n u_n - n \frac{n-1}{2} \cdot u_{n-1} + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} u_{n-2} - \&c. \mp u_1;$$

whence it appears that any one term, as u_{n+1} , is expressed by a function of the preceding terms multiplied by constant coefficients, or that a series which has constant differences of any order is a recurring one: the converse of this proposition is not however true.

(21.) We may here make some brief remarks upon a system of notation which has been applied to the formulæ for u_{x+n} , and $\Delta^n u_x$. The letter Δ has hitherto been considered as the symbol of an operation performed upon the function u_x ; so long, therefore, as we remember its primary meaning, we are at liberty to put any expression involving Δ into another form, provided we do not change its value.

Now in the expression

$$u_{x+n} = u_x + n \Delta u_x + n \frac{n-1}{1 \cdot 2} \Delta^2 u_x + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} \Delta^3 u_x + \&c.,$$

if we consider u_x to be separated from Δ , Δ^2 , Δ^3 , &c., we may write

$$u_{x+n} = u_x \left\{ 1 + n \Delta + n \frac{n-1}{2} \Delta^2 + \frac{n \cdot n-1 \cdot n-2}{1 \cdot 2 \cdot 3} \Delta^3 + \&c. \right\};$$

but the expression within the bracket may be expressed by $(1 + \Delta)^n$, for this quantity when expanded will produce precisely the same series in form. Then, finally, we may consider u_{x+n} to be expressed by the equation $u_{x+n} = (1 + \Delta)^n u_x$, the right-hand side of the equation being considered only as the representation of the value of u_{x+n} in terms of u_x and its differences.

Calculus of
Finite
Differences.

Again, since $\Delta^n u_x$ may be written $(1 + \Delta - 1)^n u_x$; for Δ^n and $(1 + \Delta - 1)^n$ have the same meaning.

$$\therefore \Delta^n u_x = u_x \left\{ (1 + \Delta)^n - n(1 + \Delta)^{n-1} + n \frac{n-1}{2} (1 + \Delta)^{n-2} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (1 + \Delta)^{n-3} + \&c. \right\}$$

$$= \overline{1 + \Delta}^n u_x - n(1 + \Delta)^{n-1} u_x + \frac{n(n-1)}{1 \cdot 2} (1 + \Delta)^{n-2} u_x - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (1 + \Delta)^{n-3} u_x + \&c.$$

But from what has preceded $(1 + \Delta)^n u_x$ is the expression for u_{x+n} ; $(1 + \Delta)^{n-1} u_x$ for u_{x+n-1} ; $(1 + \Delta)^{n-2} u_x$ that for u_{x+n-2} , &c.

$$\therefore \Delta^n u_x = u_{x+n} - n u_{x+n-1} + n \frac{n-1}{2} u_{x+n-2} - \&c.;$$

whence it appears how readily the expression for the n th difference of any function u_x may be obtained by means of the notation which has been just used: namely, that notation by which we have expressed the expansion of u_{x+n} under the symbol $(1 + \Delta)^n u_x$.

At present, however, we cannot pursue this subject further, but other applications of this concise system of notation will no doubt occur to the reader.

(22.) The formula $u_{x+2} = u_x + 2 \Delta u_x + \Delta^2 u_x = u_{x+1} + \Delta u_x + \Delta^2 u_x = u_{x+1} + (u_{x+1} - u_x) + \Delta^2 u_x$ conducts us to an expression very useful in the arithmetic of sines.

For suppose $u_x = \sin x \theta$,

$$\therefore u_{x+2} = \sin (x + 2) \theta = \sin (x \theta + 2 \theta);$$

also we have seen that

$$\Delta^2 u_x = -4 \sin^2 \frac{\theta}{2} \cdot \sin (x \theta + \theta),$$

$\therefore$ making the requisite substitutions in the formulæ for u_{x+2} , we have

$$\sin (x \theta + 2 \theta) = \sin (x \theta + \theta) + (\sin (x \theta + \theta) - \sin x \theta) - 4 \sin^2 \frac{\theta}{2} \cdot \sin (x \theta + \theta);$$

and making x successively = 0, 1, 2, 3, 4, &c. and $\theta = 1^\circ$, we have

$$\begin{aligned} \sin 2^\circ &= \sin 1^\circ + (\sin 1^\circ - \sin 0^\circ) - \sin 1^\circ (2 \sin 30')^2 \\ \sin 3^\circ &= \sin 2^\circ + (\sin 2^\circ - \sin 1^\circ) - \sin 2^\circ (2 \sin 30')^2 \\ \sin 4^\circ &= \sin 3^\circ + (\sin 3^\circ - \sin 2^\circ) - \sin 3^\circ (2 \sin 30')^2 \\ &\&c. \quad \&c. \end{aligned}$$

and having computed $\sin 1^\circ$ and $\sin 30'$, by some independent formula, for the sine in terms of the arc, the sines of the arcs will be found by addition and subtraction only.

Again, let $\theta = 1'$, and for $x + 2$ put $x + 1$;

$$\therefore \sin (x' + 1') = \sin x' + (\sin x' - \sin x' - 1') - \sin x' (2 \sin 30'')^2;$$

whence, knowing $\sin 30''$ and $\sin 1'$, the sines of minutes may be found; and by assuming $x = z \cdot 60 + \phi$, a formula may be found by which the sines of arcs consisting of degrees and minutes may be found.

(23.) Since, as we have seen in a preceding article, when n is of the form $4m$,

$$\Delta^n \sin \theta = + \left(2 \sin \frac{\theta}{2} \right)^n \left\{ \sin (x + n) \theta \right\}$$

$$\Delta^{n-1} \sin x \theta = - \left(2 \sin \frac{\theta}{2} \right)^{n-1} \left\{ \cos \left(x + n - \frac{1}{2} \right) \theta \right\}$$

$$\Delta^{n-2} \sin x \theta = - \left(2 \sin \frac{\theta}{2} \right)^{n-2} \left\{ \sin x + n - 1 \theta \right\},$$

$$\therefore \Delta^{n-1} \sin x \theta + \Delta^{n-2} \sin x \theta = - \left(2 \sin \frac{\theta}{2} \right)^{n-2} \left\{ 2 \sin \frac{\theta}{2} \cdot \cos x + n - \frac{1}{2} \theta + \sin x + n - 1 \theta \right\};$$

$$\text{and } \therefore \sin (x + n - 1) \theta = \sin \left(\left(x + n - \frac{1}{2} \right) - \frac{1}{2} \right) \theta$$

$$= - \left(2 \sin \frac{\theta}{2} \right)^{n-2} \left(\sin x + n - \frac{1}{2} \cdot \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \cdot \cos x + n - \frac{1}{2} \theta \right)$$

$$= - \left(2 \sin \frac{\theta}{2} \right)^{n-2} (\sin x + n) \theta.$$

Multiplying both sides by $-\left(2 \sin \frac{\theta}{2} \right)^2$, and transposing,

$$\therefore \left(2 \sin \frac{\theta}{2} \right)^n \sin x + n \theta = - \left(2 \sin \frac{\theta}{2} \right)^2 \left\{ \Delta^{n-1} \sin x \theta + \Delta^{n-2} \sin x \theta \right\}$$

$$\text{or } \Delta^n \sin x \theta = - \left(2 \sin \frac{\theta}{2} \right)^2 \left\{ \Delta^{n-1} \sin x \theta + \Delta^{n-2} \sin x \theta \right\}$$

Calculus of Finite Differences. a formula given by Lagrange; the other cases, when n is an odd number, may be similarly proved; the formula is however general. Calculus of Finite Differences.

Thus having given any two successive differences of the sine of an arc, the rest may be found; and thus the sines may be computed by means of simple additions and subtractions.

(24.) The calculation of logarithms, always laborious, is yet much facilitated by the method of differences. The following example, the numerical results of which are taken from Lacroix, will be an instance of its use.

Let $u_x = \log x$, $\therefore \Delta \log x = \log(x+1) - \log x = \log\left(1 + \frac{1}{x}\right)$

$$= M \left\{ \frac{1}{x} - \frac{1}{2} \frac{1}{x^2} + \frac{1}{3} \cdot \frac{1}{x^3} - \&c. \right\}.$$

$$\Delta^2 \log x = \log(x+2) - 2 \log(x+1) + \log x$$

$$= \log\left(\frac{x+2}{x}\right) - 2 \log\left(\frac{x+1}{x}\right)$$

$$= M \left\{ \frac{2}{x} - \frac{1}{2} \frac{4}{x^2} + \frac{1}{3} \cdot \frac{8}{x^3} - \&c. - 2 \left(\frac{1}{x} - \frac{1}{2} \frac{1}{x^2} + \frac{1}{3} \frac{1}{x^3} - \&c. \right) \right\}$$

$$= -M \left\{ \frac{1}{x^2} - \frac{2}{x^3} + \&c. \right\}.$$

$$\Delta^3 \log x = \log(x+3) - 3 \log(x+2) + 3 \log(x+1) - \log x$$

$$= \log\left(1 + \frac{3}{x}\right) - 3 \log\left(1 + \frac{2}{x}\right) + 3 \log\left(1 + \frac{1}{x}\right)$$

$$= M \cdot \left(\frac{2}{x^3} - \&c. \right).$$

And

$$\Delta^4 \log x = -M \left(\frac{6}{x^4} - \&c. \right).$$

If higher orders of differences be required, we may, by pursuing a similar process, find them; but in general, the values of the third differences are so small that we may stop there, as the omission of the following terms will not sensibly affect the result. The differences being computed, they are substituted in the equation $u_n = u + n \Delta u + \frac{n-1}{2} \Delta^2 u + \frac{n-1}{1 \cdot 2 \cdot 3} \Delta^3 u$, and proper values being given to (n) , the value of u_n , which is the object sought, is discovered.

Thus if

$$x = 10000, \therefore u = \log 10000 = 4.$$

Then

$$\Delta u = 0.00004 \ 34272 \ 76863$$

$$\Delta^2 u = -0.00000 \ 00043 \ 42076$$

$$\Delta^3 u = 0.00000 \ 00000 \ 00868,$$

and let it be required to find the logarithm of 10006.

Here

$$n = 6, \text{ and } u_6 = u + 6 \Delta u + 15 \Delta^2 u + 20 \Delta^3 u;$$

and computing the terms of this equation, we shall find

$$u_6 = \log 10006 = 4.00025 \ 94995 \ 47398.$$

We might, however, have obtained the same result, had we formed a table of differences, and then the operation would have been performed by simple additions and subtractions.

In the common tables of Hutton the logarithm is only given to seven places of decimals; so long, therefore, as the omission of the other terms of the equation does not affect the value of this figure, the differences calculated from the series will be applicable to find the logarithms required; but when this figure is altered, as we can find by trial, we must start from this point, and we compute Δu , $\Delta^2 u$ and $\Delta^3 u$. It is easy to find when this takes place; thus if $n = 50$, the value of the first neglected term $= \frac{50 \times 49 \times 48 \times 47}{1 \cdot 2 \cdot 3 \cdot 4} \times -M \left\{ \frac{6}{x^4} - \&c. \right\}$ is found only to influence the value of the sixteenth figure in the decimal part of the logarithm of 10050.

(25.) Other series more convergent than those given in the last article are readily found, which will render the calculation of Δu , $\Delta^2 u$, and $\Delta^3 u$, much less laborious.

For since $\log(x+1) = \log x + 2M \left\{ \frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \&c. \right\},$

$$\therefore \Delta \log x = \log(x+1) - \log x = 2M \left\{ \frac{1}{2x+1} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} \right\}.$$

Again

$$\therefore \log(z+1) = \log z + M \left\{ \frac{1}{z} - \frac{1}{2z^2} + \frac{1}{3z^3} - \frac{1}{4z^4} + \&c. \right\},$$

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

$$\log(z-1) = \log z - M \left\{ \frac{1}{z} + \frac{1}{2} \frac{1}{z^2} + \frac{1}{3} \frac{1}{z^3} + \frac{1}{4} \frac{1}{z^4} + \&c. \right\},$$

$$\therefore \log(z+1) + \log(z-1) - 2 \log z = -2M \left\{ \frac{1}{2z^2} + \frac{1}{4z^4} + \&c. \right\}.$$

Let

$$z = x + 1,$$

$$\therefore \log(x+2) - 2 \log(x+1) + \log x = \Delta^2 \log x = -2M \left\{ \frac{1}{2(x+1)^2} + \frac{1}{4(x+1)^4} + \&c. \right\};$$

and when x is a large number, it is found to be only necessary to compute the first two terms of the expression for $\Delta \log x$.

"Indeed when $x = 10000$, the second term $2M \cdot \frac{1}{(2x+1)^2}$ is only .00000 00000 00036, and the following term has twenty-two ciphers between the decimal point and the first significant figure, and with respect to $\Delta^2 \log x$, the first term is quite sufficient for the second term, when $x = 10000$ amounts only to .00000 00000 00000 0217."

Hence if N be any number above 10000, we may use without error the formulæ $\Delta \log N = 2M \left\{ \frac{1}{2N+1} + \frac{1}{3} \frac{1}{(2N+1)^3} \right\}$ and $\Delta^2 \log N = -\frac{M}{(N+1)^2}$.

(26.) We next proceed to investigate a formula for the computation of the logarithms of sines.

Since

$$\log(n+z) = \log(n) + 2M \left\{ \frac{z}{2n+z} + \frac{1}{3} \frac{z^3}{(2n+z)^3} + \frac{1}{5} \frac{z^5}{(2n+z)^5} + \&c. \right\};$$

let

$$n = \sin x \theta \text{ and } n+z = \sin x + 1 \cdot \theta \therefore z = \Delta \sin x \theta.$$

$$\therefore \log(\sin x + 1 \cdot \theta) = \log(\sin x \theta) + 2M \left\{ \frac{\Delta \sin x \theta}{2 \sin x \theta + \Delta \sin x \theta} + \frac{1}{3} \left(\frac{\Delta \cdot \sin x \theta}{2 \sin x \theta + \Delta \sin x \theta} \right)^3 + \&c. \right\};$$

a formula given by Delambre, but which was put under another form, to which the above is reducible, of $x\theta = y$ and $\theta = h$;

and

and

$$\therefore \Delta \sin x \theta = \sin(y+h) - \sin y;$$

$$\therefore \log(\sin y + h) = \log(\sin y) + 2M \cdot \left\{ \frac{\sin y + h - \sin y}{\sin y + h + \sin y} + \frac{1}{3} \cdot \left(\frac{\sin y + h - \sin y}{\sin y + h + \sin y} \right)^3 + \&c. \right\}.$$

Now let

and

$$x = 45, \theta = 1^\circ, \therefore \sin x \theta = \sin 45^\circ = \frac{1}{\sqrt{2}}.$$

$$\log \sin x \theta = \frac{1}{2} \log \frac{1}{2};$$

$$\therefore \log \sin 46 = \frac{1}{2} \log \frac{1}{2} + 2M \left\{ \frac{\Delta \sin 45}{2 \sin 45 + \Delta \sin 45} + \frac{1}{3} \left(\frac{\Delta \sin 45}{2 \sin 45 + \Delta \sin 45} \right)^3 + \&c. \right\}.$$

And in a similar manner may the logarithms of the series of other arcs as far as 90 be found; and since as the arc increases the $\Delta \sin x \theta$ decreases, therefore the series becomes the more convergent as we approach 90°, and the sines of arcs greater than 45° being known, the sines of arcs greater than 45° may be computed from the formula

$$\sin \frac{A}{2} = \frac{\sin A}{2 \cos \frac{A}{2}} = \frac{\sin A}{2 \sin \left(90 - \frac{A}{2} \right)};$$

$$\therefore \log \sin \frac{A}{2} = \log \sin A - \log 2 - \log \sin \left(90 - \frac{A}{2} \right).$$

$$\log \sin 31 = \log \sin 62 - \log 2 - \log \sin 59.$$

Thus

A singular formula for the $\log(\cos x)$, also given by Delambre, may have place here.

$$\therefore \cos^2 x = (1 - \sin^2 x) = (1 - y^2) \text{ suppose.}$$

$$\therefore \log \cos^2 x = 2 \log \cos x = \log(1 - y^2) = -M \left\{ y^2 + \frac{y^4}{2} + \frac{y^6}{3} + \frac{y^8}{4} + \&c. \right\},$$

$$\therefore \log \cos x = -M \left\{ \frac{y^2}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \&c. \right\}$$

$$= -M \left\{ \frac{1}{2} (\sin x)^2 + \frac{1}{4} (\sin x)^4 + \frac{1}{6} (\sin x)^6 + \&c. \right\}.$$

(27.) We have already found the expression for $\Delta(u_x v_x)$. We now proceed to find those for $\Delta^2(u_x v_x)$, and $\Delta^3(u_x v_x)$. We have seen that

$$\Delta(u_x v_x) = u_x \Delta v_x + v_x \Delta u_x + \Delta u_x \cdot \Delta v_x (1),$$

and

$$\therefore \Delta^2(u_x v_x) = \Delta(u_x \Delta v_x) + \Delta(v_x \Delta u_x) + \Delta(\Delta u_x \cdot \Delta v_x),$$

each term of which may be found by means of the equation (1).

$$\text{Thus } \Delta(u_x \cdot \Delta v_x) = u_x \cdot \Delta^2 v_x + \Delta v_x \cdot \Delta u_x + \Delta u_x \cdot \Delta^2 v_x$$

$$\Delta(v_x \Delta u_x) = v_x \Delta^2 u_x + \Delta u_x \cdot \Delta v_x + \Delta v_x \cdot \Delta^2 u_x$$

$$\Delta \cdot (\Delta u_x \cdot \Delta v_x) = \Delta u_x \cdot \Delta^2 v_x + \Delta v_x \cdot \Delta^2 u_x + \Delta^2 v_x \cdot \Delta^2 u_x;$$

$$\therefore \Delta(u_x v_x) = u_x \Delta^2 v_x + v_x \Delta^2 u_x + 2 \Delta u_x \cdot \Delta v_x + 2 \Delta u_x \Delta^2 v_x + 2 \Delta v_x \cdot \Delta^2 u_x + \Delta^2 u_x \cdot \Delta^2 v_x.$$

$\Delta^3(u_x v_x)$ may be found in the same manner from equation (1), but the labour would obviously be considerable, and the law of the coefficients is not sufficiently manifest to abridge the labour. The artifice of separating the symbols of operation from those of quantity is here eminently useful.

First, let an accent be placed over the Δ which is prefixed to v_x , to indicate that Δ' applies solely to that function, while Δ refers only to u_x , so that we may write indifferently

$$u_x \Delta' v_x \text{ or } \Delta' u_x v_x, \Delta^2 u_x \Delta'^2 v_x \text{ or } \Delta^2 \Delta'^2 u_x v_x,$$

or in general

$$\Delta^m u_x \cdot \Delta'^n v_x \text{ may be put } \Delta^m \Delta'^n u_x v_x;$$

but we must keep in mind that Δ' does not indicate any different operation from Δ , but that it solely belongs to v_x .

This being premised, resuming the equation for $\Delta u_x v_x$, we have

$$\begin{aligned} \Delta(u_x v_x) &= u_x \Delta' v_x + v_x \Delta u_x + \Delta u_x \Delta' v_x \\ &= (\Delta' + \Delta + \Delta \Delta') u_x v_x; \end{aligned}$$

and

$$\begin{aligned} \therefore \Delta^2(u_x v_x) &= (\Delta + \Delta' + \Delta \Delta') \Delta(u_x v_x) \\ &= (\Delta + \Delta' + \Delta \Delta')^2 \cdot u_x v_x \\ &= (\Delta^2 + \Delta'^2 + \Delta^2 \Delta'^2 + 2 \Delta \Delta' + 2 \Delta^2 \Delta' + 2 \Delta \Delta'^2) u_x v_x \\ &= v_x \Delta^2 u_x + u_x \Delta'^2 v_x + \Delta^2 u_x \Delta' v_x + 2 \Delta u_x \Delta' v_x + 2 \Delta' v_x \Delta^2 u_x + 2 \Delta u_x \Delta'^2 v_x; \end{aligned}$$

which, by obliterating the accent, becomes the same expression that was previously obtained.

(28.) Similarly, since

$$\Delta^3(u_x v_x) = (\Delta + \Delta' + \Delta \Delta')^3 \cdot \Delta(u_x v_x),$$

$$\therefore = (\Delta + \Delta' + \Delta \Delta')^3 u_x v_x,$$

$$\therefore \Delta^n(u_x v_x) = (\Delta + \Delta' + \Delta \Delta')^n u_x v_x.$$

But

$$(\Delta + \Delta' + \Delta \Delta')^n = \{\Delta + \Delta' (1 + \Delta)\}^n,$$

$$\therefore \Delta^n(u_x v_x) = \{\Delta + \Delta' (1 + \Delta)\}^n u_x v_x$$

$$= \left\{ \Delta^n + n \Delta^{n-1} \cdot (1 + \Delta) \cdot \Delta' + n \frac{n-1}{2} \cdot \Delta^{n-2} \cdot \Delta'^2 (1 + \Delta)^2 + \&c. \right\} u_x v_x.$$

$$\text{But } \therefore (1 + \Delta)^m u_x = u_{x+m}, \therefore \Delta^{n-m} (1 + \Delta)^m u_x = \Delta^{n-m} u_{x+m};$$

$$\therefore \Delta^n(u_x v_x) = v_x \Delta^n u_x + n \cdot \Delta^{n-1} u_{x+1} \cdot \Delta v_x + n \frac{n-1}{2} \cdot \Delta^{n-2} u_{x+2} \cdot \Delta^2 v_x + \&c.$$

the law of which is sufficiently manifest.

(29.) By the same system of notation we may arrive at the n th difference of the product of any number of functions. We shall, however, only indicate the process.

Since

$$\Delta^n(u_x v_x) = (\Delta + \Delta' + \Delta \Delta')^n u_x v_x,$$

and that

$$\Delta + \Delta' + \Delta \Delta' = \overline{1 + \Delta} \cdot \overline{1 + \Delta'} - 1,$$

$$\therefore \Delta^n(u_x v_x) = \overline{(1 + \Delta \cdot 1 + \Delta' - 1)^n} u_x v_x.$$

And if we have any number of functions, as $u_x v_x u'_x v'_x$, &c., and suppose Δ'' to refer to u'_x , Δ''' to v'_x , &c., we shall obviously obtain

$$\Delta^n(u_x v_x u'_x v'_x \dots) = \overline{(1 + \Delta \cdot 1 + \Delta' \cdot 1 + \Delta'' \dots - 1)^n} u_x v_x u'_x v'_x \dots;$$

and the right-hand side of the equation being expanded, and the accented letters being prefixed to their respective functions, the true developement of the difference will be found.

(30.) The particular values which in some cases the n th differences of u_x take, demand our attention, and the results which arise from one particular instance are curious and useful.

(31.) Since $\Delta^n u_x = u_{x+n} - n u_{x+n-1} + \frac{n(n-1)}{2} u_{x+n-2} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} u_{x+n-3} + \&c.$, we can, by giving particular values to u_x , readily find values for the n th differences of functions.

Thus, let $u_x = x^m$, then $u_{x+n} = (x+n)^m$, $u_{x+n-1} = (x+n-1)^m$, &c.;

$$\therefore \Delta^n x^m = (x+n)^m - n(x+n-1)^m + \frac{n(n-1)}{2} (x+n-2)^m - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (x+n-3)^m + \&c.$$

But when $n = m$, we have seen, in a preceding article that

$$\Delta^m x^m = 1 \cdot 2 \cdot 3 \dots m - 2 \cdot m - 1 \cdot m;$$

$\therefore$ whatever be the value of x , and first writing m for n ,

$$(x+m)^m - m(x+m-1)^m + \frac{m(m-1)}{2} (x+m-2)^m - \&c. = 1 \cdot 2 \cdot 3 \dots m.$$

If, now, we make $x = 0$, we have the singular result

$$m^m - m(m-1)^m + \frac{m(m-1)}{2} (m-2)^m - \&c. = 1 \cdot 2 \cdot 3 \dots m.$$

Again, if in the general value for $\Delta^n x^m$, we make $x = 0$, we have

$$\Delta^n 0^m = n^m - n(n-1)^m + \frac{n(n-1)}{2} (n-2)^m - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (n-3)^m + \&c.$$

And since $\Delta^m x^m$ is constant, and equal $1 \cdot 2 \cdot 3 \dots m$, therefore $\Delta^n 0^m$ is equal to zero when $n > m$, and equal to the product of the first m integers when $m = n$.

We may remark that, since x^m is the general term of the series, $0^m + 1^m + 2^m + 3^m + \&c.$, $\Delta^n 0^m$ is the symbol for the first term of the n th order of differences of this series.

(32.) Resuming the equation

$$\Delta^n x^m = (x+n)^m - n(x+n-1)^m + \frac{n(n-1)}{1 \cdot 2} (x+n-2)^m - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (x+n-3)^m + \&c.,$$

which, by changing the place of x , may be written

$$= (n+x)^m - n(n-1+x)^m + \frac{n(n-1)}{2} (n-2+x)^m - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (n-3+x)^m + \&c.,$$

and expanding each term according to the powers of x by the binomial theorem, we shall obtain

$$\begin{aligned} \Delta^n x^m &= n^m - n \cdot (n-1)^m + \frac{n(n-1)}{2} (n-2)^m - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (n-3)^m + \&c. \\ &+ m \cdot \left\{ n^{m-1} - n(n-1)^{m-1} + \frac{n(n-1)}{2} (n-2)^{m-1} - \&c. \right\} x \\ &+ m \cdot \frac{m-1}{2} \cdot \left\{ n^{m-2} - n(n-1)^{m-2} + \frac{n(n-1)}{2} (n-2)^{m-2} - \&c. \right\} x^2 \\ &+ \&c. \\ &= \Delta^n 0^m + m \Delta^n 0^{m-1} \cdot x + \frac{m(m-1)}{2} \Delta^n 0^{m-2} \cdot x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \Delta^n 0^{m-3} \cdot x^3 + \&c. \\ &+ \frac{m(m-1)(m-2) \dots (m-n+1)}{1 \cdot 2 \cdot 3 \dots m-n} \Delta^n 0^n \cdot x^{m-n}; \end{aligned}$$

whence, after computing the values of $\Delta^n 0^m$, $\Delta^n 0^{m-1}$, &c., the values of $\Delta^n x^m$ may be found.

(33.) These values, $\Delta^n 0^m$, $\Delta^n 0^{m-1}$, &c. are very important in the theory of series, and in his Collection of Examples on the Calculus of Finite Differences, Sir John Herschell has given the values of $\Delta^n 0^m$ as far as $\Delta^{10} 0^{10}$ in a Table which we have annexed.

These numbers are computed from the formula

$$\Delta^n 0^m = n^m - n(n-1)^m + \frac{n(n-1)}{2} (n-2)^m - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} (n-3)^m + \&c.$$

Thus, let $m = 3$, then, making $n = 1, 2, 3$, successively,

$$\Delta^1 0^3 = 1^3 = 1$$

$$\Delta^2 0^3 = 2^3 - 2 \cdot 1^3 = 8 - 2 = 6$$

$$\Delta^3 0^3 = 3^3 - 3 \cdot 2^3 + 3 \cdot 1^3 = 30 - 24 = 6$$

$$\Delta^4 0^3 = 4^3 - 4 \cdot 3^3 + 6 \cdot 2^3 - 4 = 112 - 112 = 0,$$

the last result we know *à priori*.

Calculus of
Finite
Differences.

Similarly may be found the other values in the annexed Table, and which may be extended to any values of m and n . Also

Calculus of
Finite
Differences.

$$\begin{aligned}\Delta^6 \cdot 0^6 &= 6^6 - 6 \cdot 5^6 + 15 \cdot 4^6 - 20 \cdot 3^6 + 15 \cdot 2^6 - 6 \\ &= 1679616 - 2343750 + 983040 - 131220 + 3840 - 6 \\ &= 2666496 - 2474976 = 191520.\end{aligned}$$

	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6	Δ^7	Δ^8	Δ^9	Δ^{10}
0^1	1									
0^2	1	2								
0^3	1	6	6							
0^4	1	14	36	24						
0^5	1	30	150	240	120					
0^6	1	62	540	1560	1800	720				
0^7	1	126	1806	8400	16800	15120	5040			
0^8	1	254	5796	40824	126000	191520	141120	40320		
0^9	1	510	18150	186480	834120	1905120	2328480	1451520	362880	
0^{10}	1	1022	55980	818520	5103000	16435440	29635200	30240000	16329600	3628800

(34.) Next to find the value of $\Delta \cdot u_x$ and $\Delta^n u_x$ in terms of u_x , and its differential coefficients. Since u_x is a function of x , and u_{x+1} a function of $x+1$, therefore, by Taylor's theorem,

$$\Delta u_x = u_{x+1} - u_x = \frac{d \cdot u_x}{dx} + \frac{d^2 u_x}{dx^2} \cdot \frac{1}{2} + \frac{d^3 u_x}{dx^3} \cdot \frac{1}{2 \cdot 3} + \frac{d^4 u_x}{dx^4} \cdot \frac{1}{2 \cdot 3 \cdot 4} + \&c.$$

or, separating the symbols of operation from those of quantity,

$$\begin{aligned}&= u_x \cdot \left\{ \frac{d}{dx} + \frac{d^2}{dx^2} \cdot \frac{1}{1 \cdot 2} + \frac{d^3}{dx^3} \cdot \frac{1}{2 \cdot 3} + \frac{d^4}{dx^4} \cdot \frac{1}{2 \cdot 3 \cdot 4} + \&c. \right\} \\ &= u_x \cdot \{ e^{\frac{d}{dx}} - 1 \}.\end{aligned}$$

Since $e^{\frac{d}{dx}} - 1$ when expanded will produce a series exactly of the same form; and, similarly, we might infer that $\Delta^n u_x$ may be represented by $(e^{\frac{d}{dx}} - 1)^n u_x$.

But to prove, in a strict manner, this theorem, due to Lagrange; since $u_{x+n} = f(x+n)$, we may use Taylor's theorem, and put

$$\begin{aligned}u_{x+n} &= u_x + \frac{du_x}{dx} n + \frac{d^2 u_x}{dx^2} \frac{n^2}{1 \cdot 2} + \frac{d^3 u_x}{dx^3} \frac{n^3}{2 \cdot 3} + \&c. \\ &= u_x \left\{ 1 + \frac{d}{dx} n + \frac{d^2 n^2}{dx^2} \frac{1}{1 \cdot 2} + \frac{d^3 n^3}{dx^3} \frac{1}{2 \cdot 3} + \&c. \right\} \\ &= u_x e^{n \cdot \frac{d}{dx}} = u_x e^{nt}, \text{ where } t = \frac{d}{dx};\end{aligned}$$

and by writing $n-1$, $n-2$, &c. for n we shall have

$$u_{x+n-1} = u_x e^{(n-1)t}, \quad u_{x+n-2} = u_x e^{(n-2)t}, \quad \&c.$$

Now

$$\Delta^n \cdot u_x = u_{x+n} - n u_{x+n-1} + \frac{n(n-1)}{2} u_{x+n-2} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} u_{x+n-3} + \&c.;$$

therefore writing for u_{x+n} , u_{x+n-1} , &c. the values just obtained,

$$\begin{aligned}\Delta^n . u_x &= u_x \cdot \left\{ e^{nx} - n \cdot e^{(n-1)x} + \frac{n(n-1)}{2} e^{(n-2)x} - \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} e^{(n-3)x} + \&c. \right\} \\ &= u_x \cdot \left\{ (e^x)^n - n(e^x)^{n-1} + \frac{n(n-1)}{2} (e^x)^{n-2} - \frac{n(n-1)(n-2)}{2 \cdot 3} (e^x)^{n-3} + \&c. \right\} \\ &= u_x (e^x - 1)^n = u_x \cdot (e^{\frac{d}{dx}} - 1)^n.\end{aligned}$$

The preceding demonstration is due to Dr. Brinkley, and was given by him in the *Philosophical Transactions* for 1807.

(35.) Again, supposing that $f(\Delta)$ represents any function of Δ capable of being developed according to the powers of Δ , such that

$$\begin{aligned}f(\Delta) &= a + b\Delta + c\Delta^2 + e\Delta^3 + \&c., \\ \therefore f(\Delta) u_x &= a u_x + b\Delta u_x + c\Delta^2 u_x + e\Delta^3 u_x + \&c.\end{aligned}$$

by Lagrange's theorem just demonstrated,

$$\begin{aligned}&= a u_x + b(e^{\frac{d}{dx}} - 1) u_x + c(e^{\frac{d}{dx}} - 1)^2 u_x + e(e^{\frac{d}{dx}} - 1)^3 u_x + \&c., \\ \therefore f(\Delta) u_x &= u_x \cdot \{a + b(e^{\frac{d}{dx}} - 1) + c(e^{\frac{d}{dx}} - 1)^2 + \&c.\} \\ &= u_x \{a + b\Delta_1 + c\Delta_1^2 + \&c.\} \\ &= u_x f(\Delta_1) = u_x f(e^{\frac{d}{dx}} - 1): \end{aligned}$$

for this series having precisely the same form as that which results from the developement of $f(\Delta)$, it must be the same function of $(e^{\frac{d}{dx}} - 1)$ as $a + b\Delta + c\Delta^2 + \&c.$ is of Δ , and therefore may be represented by the same symbol; and

$$\therefore f(\Delta) u_x = f(e^{\frac{d}{dx}} - 1) u_x,$$

a theorem which is due to Arbogast.

(36.) Thus, suppose the form of the function to be such that

$$f(\Delta) = (1 + \Delta)^n;$$

then $\therefore$ for Δ we have written $1 + \Delta$,

$$\therefore f(e^{\frac{d}{dx}} - 1) = (1 + e^{\frac{d}{dx}} - 1)^n = e^{\frac{n}{dx}},$$

and

$$\therefore (1 + \Delta)^n u_x = e^{\frac{n}{dx}} . u_x = u_{x+n},$$

by Article (34.); and this is obviously true, since if $(1 + \Delta)^n$ be expanded, and each term multiplied by u_x , there will arise the series to which u_{x+n} is equal.

(37.) Again, if $f(\Delta) = (\log(1 + \Delta))^n$, we have

$$\{\log(1 + \Delta)\}^n u_x = \{\log(1 + e^{\frac{d}{dx}} - 1)\}^n u_x = \left(\frac{d}{dx}\right)^n u_x = \frac{d^n u_x}{dx^n};$$

whence, if $n = 1$,

$$\frac{d \cdot u_x}{dx} = u_x \log(1 + \Delta) = \Delta u_x - \frac{\Delta^2 u_x}{2} + \frac{\Delta^3 u_x}{3} - \&c.$$

(38.) This last result may be thus verified:

$$u_{x+n} = f(x + n) = u_x + \frac{d \cdot u_x}{dx} \cdot n + \frac{d^2 u_x}{dx^2} \cdot \frac{n^2}{1 \cdot 2} + \&c.$$

But

$$u_{x+n} = u_x + n\Delta u_x + \frac{n(n-1)}{2} \Delta^2 u_x + \frac{n \cdot (n-1)(n-2)}{1 \cdot 2 \cdot 3} \Delta^3 u_x + \&c.;$$

or, expanding according to the powers of n ,

$$\begin{aligned}&= u_x + n \left\{ \Delta \cdot u_x - \frac{1}{2} \Delta^2 \cdot u_x + \frac{1}{3} \Delta^3 \cdot u_x - \frac{1}{4} \Delta^4 \cdot u_x + \&c. \right\} \\ &\quad + \frac{n^2}{1 \cdot 2} \cdot \{ \Delta^2 \cdot u_x - \Delta^3 \cdot u_x + \&c. \} \\ &\quad + \&c.\end{aligned}$$

whence, equating the coefficients of n in both series,

$$\begin{aligned}\frac{d \cdot u_x}{dx} &= \Delta \cdot u_x - \frac{\Delta^2 \cdot u_x}{2} + \frac{\Delta^3 \cdot u_x}{3} - \frac{\Delta^4 \cdot u_x}{4} + \&c. \\ &= u_x \left\{ \frac{\Delta}{1} - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \frac{\Delta^4}{4} + \&c. \right\} \\ &= u_x \log(1 + \Delta),\end{aligned}$$

the theorem in question.

Calculus of
Finite
Differences.

And, by collecting together the coefficients of $n^2, n^3, \&c.$, we may also find the values of $\frac{d^2 u_x}{dx^2}, \frac{d^3 u_x}{dx^3}, \&c.$ in terms of $\Delta \cdot u_x, \Delta^2 \cdot u_x, \&c.$, and thus verify the results which we might at once obtain from the equation $\frac{d^n u_x}{dx^n} = (\log(1+\Delta))^n u_x$, the general form of which will be, since $(\log(1+\Delta))^n = \Delta^n \left\{ 1 - \frac{\Delta}{2} + \frac{\Delta^2}{3} - \frac{\Delta^3}{4} + \&c. \right\}^n$, Calculus of
Finite
Differences.

$$\Delta^n \cdot u_x + A_1 \Delta^{n+1} \cdot u_x + A_2 \Delta^{n+2} \cdot u_x + A_3 \Delta^{n+3} \cdot u_x + \&c.$$

(39.) We have previously found a convenient formula for $\Delta^n \cdot x^m$ in terms of $\Delta^n \cdot 0^m, \Delta^{n+1} \cdot 0^{m-1}$, we can now deduce a general expression for $\Delta^n \cdot u_x$, dependent upon similar values.

Since $\Delta^n \cdot u_x = (e^{\frac{d}{dx}} - 1)^n u_x$, if the right-hand side of the equation be expanded, there will arise a series of the form

$$\begin{aligned} & \left(A_0 + A_1 \frac{d}{dx} + A_2 \frac{d^2}{dx^2} + A_3 \frac{d^3}{dx^3} + \&c. + A_m \frac{d^m}{dx^m} + \&c. \right) u_x \\ &= A_0 u_x + A_1 \frac{d \cdot u_x}{dx} + A_2 \frac{d^2 u_x}{dx^2} + A_3 \frac{d^3 u_x}{dx^3} + \&c. + A_m \frac{d^m \cdot u_x}{dx^m} + \&c. \end{aligned}$$

But if t be written for $\frac{d}{dx}$, it is evident that A_m is the coefficient of t^m , in the expansion of $(e^t - 1)^n$.

$$\text{Now} \quad (e^t - 1)^n = e^{nt} - n e^{(n-1)t} + \frac{n(n-1)}{2} e^{(n-2)t} - \&c.$$

$$\begin{aligned} &= 1 + nt + \frac{n^2 t^2}{1 \cdot 2} + \frac{n^3 t^3}{2 \cdot 3} + \&c. + \frac{n^m t^m}{1 \cdot 2 \cdot 3 \dots m} + \&c. \\ &- n \left\{ 1 + (n-1)t + \frac{(n-1)^2 t^2}{1 \cdot 2} + \&c. + \frac{(n-1)^m t^m}{1 \cdot 2 \cdot 3 \dots m} + \&c. \right\} \\ &+ \frac{n(n-1)}{2} \cdot \left\{ 1 + (n-2)t + \&c. + \frac{(n-2)^m t^m}{1 \cdot 2 \cdot 3 \dots m} + \&c. \right\} \\ &- \&c.; \end{aligned}$$

and $\therefore$ the coefficient of t^m in this expansion, or

$$A_m = \frac{n^m - n(n-1)^m + \frac{n(n-1)}{2}(n-2)^m - \&c.}{1 \cdot 2 \cdot 3 \dots m}.$$

$$\text{But} \quad n^m - n(n-1)^m + \frac{n(n-1)}{2}(n-2)^m - \&c. = \Delta^n \cdot 0^m \quad (\text{Article 31.});$$

$$\therefore A_m = \frac{\Delta^n \cdot 0^m}{1 \cdot 2 \cdot 3 \dots m},$$

which is equal to zero when n is greater than m , and $= 1 \cdot 2 \cdot 3 \dots m$ when $m = n$;

$$\therefore A_n = 1, A_{n+1} = \frac{\Delta^n \cdot 0^{n+1}}{1 \cdot 2 \dots (n+1)}, A_{n+2} = \frac{\Delta^n \cdot 0^{n+2}}{1 \cdot 2 \cdot 3 \dots (n+2)}, \&c.,$$

and therefore

$$\Delta^n \cdot u_x = \frac{d^n u_x}{dx^n} + \frac{\Delta^n \cdot 0^{n+1}}{1 \cdot 2 \dots (n+1)} \cdot \frac{d^{n+1} u_x}{dx^{n+1}} + \frac{\Delta^n \cdot 0^{n+2}}{1 \cdot 2 \dots (n+2)} \cdot \frac{d^{n+2} u_x}{dx^{n+2}} + \&c.$$

(39.) Hence we may find the numerical values of $\Delta^n \cdot 0^{n+1}, \Delta^n \cdot 0^{n+2}, \&c.$

$$\text{For since} \quad \Delta^n = (e^{\frac{d}{dx}} - 1)^n = (e^t - 1)^n = \left(t + \frac{t^2}{1 \cdot 2} + \frac{t^3}{2 \cdot 3} + \frac{t^4}{2 \cdot 3 \cdot 4} + \&c. \right)^n$$

$$= t^n \left\{ 1 + \frac{t}{1 \cdot 2} + \frac{t^2}{2 \cdot 3} + \&c. \right\}^n$$

$$= t^n \cdot \left\{ 1 + \frac{n}{1 \cdot 2} t + \frac{n(3n+1)}{2 \cdot 3 \cdot 4} t^2 + \frac{n^2 \cdot (n+1)}{3 \cdot 4 \cdot 4} t^3 + \&c. \right\}.$$

But from the expression for $\Delta^n \cdot u_x$, obtained in the preceding article, first separating the symbols of quantity u_x from the symbols of operation, and putting $\frac{d}{dx} = t$, we have

$$\Delta^n = (e^t - 1)^n = t^n \left\{ 1 + \frac{\Delta^n \cdot 0^{n+1}}{1 \cdot 2 \cdot 3 \dots (n+1)} t + \frac{\Delta^n \cdot 0^{n+2}}{1 \cdot 2 \dots (n+2)} t^2 + \frac{\Delta^n \cdot 0^{n+3}}{1 \cdot 2 \dots (n+3)} t^3 + \&c. \right\};$$

whence, by equating the coefficients of the same powers of t ,

CALCULUS OF FINITE DIFFERENCES.

241

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

$$\Delta^n \cdot 0^{n+1} = 1 \cdot 2 \cdot 3 \dots (n+1) \cdot \frac{n}{2} = \Delta^n \cdot 0^n \cdot \frac{n^2 + n}{2}.$$

$$\Delta^n \cdot 0^{n+2} = 1 \cdot 2 \cdot 3 \dots (n+2) \cdot \frac{3n^2 + n}{2 \cdot 3 \cdot 4}$$

$$\Delta^n \cdot 0^{n+3} = 1 \cdot 2 \cdot 3 \dots (n+3) \cdot \frac{n^3 + n}{3 \cdot 4^2}$$

&c. &c.;

and, similarly, may other terms be found.

Functions of two Variables.

(40.) Hitherto we have treated of functions of one variable only; we proceed to give a brief account of the direct method of differences when applied to functions of two variables, and the reasoning may be extended to functions of a greater number of variables.

The different values which a function $u_{x,y}$ of two variables takes when different values are given to x and y , x and y being supposed to increase by unity, will form a series of progressions which may be arranged in the following manner.

		Values of x .					
		0	1	2	3	4	x
Values of y .	0	$u_{0,0}$	$u_{1,0}$	$u_{2,0}$	$u_{3,0}$	$u_{4,0}$	$u_{x,0}$
	1	$u_{0,1}$	$u_{1,1}$	$u_{2,1}$	$u_{3,1}$	$u_{4,1}$	$u_{x,1}$
	2	$u_{0,2}$	$u_{1,2}$	$u_{2,2}$	$u_{3,2}$	$u_{4,2}$	$u_{x,2}$
	3	$u_{0,3}$	$u_{1,3}$	$u_{2,3}$	$u_{3,3}$	$u_{4,3}$	$u_{x,3}$
	4	$u_{0,4}$	$u_{1,4}$	$u_{2,4}$	$u_{3,4}$	$u_{4,4}$	$u_{x,4}$
	y	$u_{0,y}$	$u_{1,y}$	$u_{2,y}$	$u_{3,y}$	$u_{4,y}$	$u_{x,y}$

Of which Table each horizontal column is a function of x only, y remaining the same in each term of the series; and each vertical column is a function of y only, y being the only variable, the value of x remaining the same throughout.

The difference between any of the successive terms of these vertical or horizontal series is found by the ordinary rules; but to show which quantity is the independent variable, the suffix x or y is added to the Δ . Thus Δ_x or Δ_y indicate that the difference is taken with regard to x or y .

Thus

$$u_{x+1,y} - u_{x,y} = \Delta_x \cdot u_{x,y} \quad \Delta_x u_{x+1,y} - \Delta_x u_{x,y} = \Delta_x^2 u_{x,y};$$

$$u_{y+1,x} - u_{y,x} = \Delta_y \cdot u_{x,y} \quad \Delta_y \cdot u_{y+1,x} - \Delta_y u_{x,y} = \Delta_y^2 u_{x,y}.$$

$\Delta_x \cdot u_{x,y}$ and $\Delta_y \cdot u_{x,y}$ are called the partial differences of $u_{x,y}$; but if x and y both vary at once the total difference of $u_{x,y}$, which is denoted by $\bar{\Delta} u_{x,y}$, is

$$\bar{\Delta} \cdot u_{x,y} = u_{x+1,y+1} - u_{x,y}.$$

(41.) By means of the partial differences, the general term of any of the columns may be found by the formulæ previously investigated; thus to find $u_{x,0}$, writing $u_{x,0}$ for u_x , and $u_{0,0}$ for u_0 , we have

$$u_{x,0} = u_{0,0} + x \Delta_x \cdot u_{0,0} + \frac{x \cdot x - 1}{1 \cdot 2} \Delta_x^2 u_{0,0} + \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \Delta_x^3 u_{0,0} + \&c.,$$

and

$$\Delta_x^2 u_{0,0} = u_{x,0} - x \cdot u_{x-1,0} + \frac{x \cdot x - 1}{1 \cdot 2} u_{x-2,0} - \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} u_{x-3,0} + \&c.;$$

and to extend these formulæ to any column, as, for instance, to obtain $u_{x,y}$, we must write y for the second 0, and thus

VOL. II.

Calculus of
Finite
Differences.

$$u_{x,y} = u_{0,y} + x \Delta_x \cdot u_{0,y} + \frac{x \cdot x - 1}{1 \cdot 2} \Delta_x^2 u_{0,y} + \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \Delta_x^3 u_{0,y} + \&c.$$

Calculus of
Finite
Differences.

But, obviously, $u_{0,y} = u_{y,0}$; and, therefore, we have

$$u_{0,y} = u_{0,0} + y \cdot \Delta_y u_{0,0} + \frac{y \cdot y - 1}{2} \Delta_y^2 u_{0,0} + \frac{y \cdot y - 1 \cdot y - 2}{1 \cdot 2 \cdot 3} \Delta_y^3 u_{0,0} + \&c.$$

Let this value be substituted in the expression for $u_{x,y}$, so that $u_{x,y}$ may be a function of $u_{0,0}$, and of its partial differences; but in making this substitution we shall evidently meet with the following terms:

$$\Delta_x (\Delta_y \cdot u_{0,0}), \Delta_x^2 (\Delta_y \cdot u_{0,0}), \dots \text{and } \Delta_x^m (\Delta_y^n u_{0,0});$$

the last of which denotes n differentiations with regard to the index y , followed by m differentiations with regard to the index x , and which may be written $\Delta_{x,y}^{m+n} u_{0,0}$; and this notation being used, we have

$$\begin{aligned} u_{x,y} = u_{0,0} &+ \frac{x}{1} \Delta_x \cdot u_{0,0} + \frac{x \cdot x - 1}{2} \Delta_x^2 u_{0,0} + \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \Delta_x^3 u_{0,0} + \&c. \\ &+ \frac{y}{1} \Delta_y u_{0,0} + \frac{xy}{1 \cdot 1} \cdot \Delta_{x,y}^{1+1} u_{0,0} + \frac{y \cdot x \cdot x - 1}{1 \cdot 1 \cdot 2} \Delta_{x,y}^{2+1} u_{0,0} + \&c. \\ &+ \frac{y \cdot y - 1}{1 \cdot 2} \Delta_y^2 \cdot u_{0,0} + \frac{xy \cdot y - 1}{1 \cdot 1 \cdot 2} \Delta_{x,y}^{1+2} u_{0,0} + \&c. \\ &+ \frac{y \cdot y - 1 \cdot y - 2}{1 \cdot 2 \cdot 3} \Delta_y^3 u_{0,0} + \&c.; \end{aligned}$$

a formula which gives the general term of a function of x and y in terms of the first term $u_{0,0}$, and its partial differences.

(42.) It only remains to show how the terms which have the symbols Δ_x , or Δ_y , or $\Delta_{x,y}$ are to be found. Referring to the Table in the article which has just preceded, we shall, by first taking the differences of the series in the first horizontal line, and then in the second, and so on, have

$$\begin{array}{llll} u_{0,0}, & \Delta_x u_{0,0}, & \Delta_x^2 u_{0,0}, & \Delta_x^3 u_{0,0}, \&c. \dots \text{when } y = 0 \\ u_{0,1}, & \Delta_x u_{0,1}, & \Delta_x^2 u_{0,1}, & \Delta_x^3 u_{0,1}, \&c. \dots \text{when } y = 1 \\ u_{0,2}, & \Delta_x u_{0,2}, & \Delta_x^2 u_{0,2}, & \Delta_x^3 u_{0,2}, \&c. \dots \text{when } y = 2 \\ u_{0,3}, & \Delta_x u_{0,3}, & \Delta_x^2 u_{0,3}, & \Delta_x^3 u_{0,3}, \&c. \dots \text{when } y = 3 \\ \&c. & \&c. & \&c. & \&c. \end{array}$$

Now subtract the first line from the second, the second from the third, &c., and the remainders will express the differences with regard to y , and these will be

$$\begin{array}{llll} \Delta_y \cdot u_{0,0}, & \Delta_{x,y}^{1+1} u_{0,0}, & \Delta_{x,y}^{2+1} u_{0,0}, & \Delta_{x,y}^{3+1} u_{0,0}, \&c. \dots \text{when } y = 0 \\ \Delta_y \cdot u_{0,1}, & \Delta_{x,y}^{1+1} u_{0,1}, & \Delta_{x,y}^{2+1} u_{0,1}, & \Delta_{x,y}^{3+1} u_{0,1}, \&c. \dots \text{when } y = 1 \\ \Delta_y \cdot u_{0,2}, & \Delta_{x,y}^{1+1} u_{0,2}, & \Delta_{x,y}^{2+1} u_{0,2}, & \Delta_{x,y}^{3+1} u_{0,2}, \&c. \dots \text{when } y = 2. \end{array}$$

Subtracting again the first line from the second, and the second from the third, &c., the remainders will be the second differences with regard to y ; and thus there will arise

$$\begin{array}{llll} \Delta_y^2 u_{0,0}, & \Delta_{x,y}^{1+2} u_{0,0}, & \Delta_{x,y}^{2+2} u_{0,0}, & \&c. \dots \text{when } y = 0 \\ \Delta_y^2 u_{0,1}, & \Delta_{x,y}^{1+2} u_{0,1}, & \Delta_{x,y}^{2+2} u_{0,1}, & \&c. \dots \text{when } y = 1; \end{array}$$

and similarly we may arrive at any order of differences.

As an example, let the following numbers be placed instead of the values of $u_{0,0}$, $u_{0,1}$, &c.; viz.

5	6	9	14	21	&c.
8	11	16	23	32	&c.
17	22	29	38	49	&c.
32	39	48	59	72	&c.
&c.	&c.				

then, by the application of the preceding reasoning, we shall form the following Table:

First Group.	Second Group.	Third Group.	Fourth Group.
5, 1, 2, 0			
8, 3, 2, 0	3, 2, 0		
17, 5, 2, 0	9, 2, 0	6, 0	
32, 7, 2, 0	15, 2, 0	6, 0	0

$$\begin{aligned} \therefore u_{0,0} &= 5, & \Delta_x u_{0,0} &= 1, & \Delta_x^2 u_{0,0} &= 2, & \Delta_x^3 u_{0,0} &= 0 \\ & & \Delta_y u_{0,0} &= 3, & \Delta_{x,y}^{1+1} u_{0,0} &= 2, & \Delta_{y,x}^{1+2} u_{0,0} &= 0 \\ & & & & \Delta_y^2 u_{0,0} &= 6, & \Delta_{x,y}^{1+2} u_{0,0} &= 0 \\ & & & & & & \Delta_y^3 u_{0,0} &= 0 \end{aligned}$$

$$\begin{aligned} \therefore u_{x,y} &= 5 + x + \frac{x \cdot x - 1}{2} \cdot 2 + 3y + xy \cdot 2 + \frac{y \cdot y - 1}{2} \cdot 6 \\ &= 5 + x^2 + 2xy + 3y^2, \end{aligned}$$

the general value, and by which the table of numbers given may be formed, and their values extended.
(43.) We shall now find the expression for the total differences of a function of any number of variables in terms of its partial differences.

Let $u_{x,y}$ be any function of x and y , and, in the first instance, let x alone vary, then the new state of the function will be represented by $u_{x,y} + \Delta_x \cdot u_{x,y}$; but if now we suppose y also to vary, and if $u'_{x,y}$ represent what $u_{x,y}$ becomes

$$u'_{x,y} = u_{x,y} + \Delta_x u_{x,y} + \Delta_y \cdot (u_{x,y} + \Delta_x \cdot u_{x,y}) = u_{x,y} + \Delta_x u_{x,y} + \Delta_y u_{x,y} + \Delta_y \Delta_x u_{x,y}$$

Again, if u be a function of three variables, as x, y, z , then supposing x and y to vary while z remains constant, the function will become as we have just seen,

$$= u + \Delta_x u + \Delta_y \cdot u + \Delta_y \cdot \Delta_x u;$$

and, therefore, when z varies, we have

$$\begin{aligned} u' &= u + \Delta_x u + \Delta_y u + \Delta_y \cdot \Delta_x u + \Delta_x \cdot (u + \Delta_x u + \Delta_y u + \Delta_y \cdot \Delta_x u) \\ &= u + \Delta_x u + \Delta_y u + \Delta_x \Delta_y u + \Delta_y \cdot \Delta_x u + \Delta_x \Delta_x u + \Delta_x \Delta_y u + \Delta_x \cdot \Delta_y \cdot \Delta_x u; \end{aligned}$$

and the process may be thus extended to any number of variables; but the method of separating the symbols of quantity from those of operation enables us to dispense with any further direct investigations.

For if $u_{x,y}$ be a function of two variables, since

$$1 + \Delta_x + \Delta_y + \Delta_x \Delta_y = (1 + \Delta_x)(1 + \Delta_y);$$

$\therefore u'_{x,y} = (1 + \Delta_x) \cdot (1 + \Delta_y) u_{x,y}$, and $\therefore \Delta u_{x,y} = u'_{x,y} - u_{x,y} = (1 + \Delta_x \cdot 1 + \Delta_y - 1) u_{x,y}$, where $\Delta u_{x,y}$ expresses the total difference.

$$\text{Again, } \therefore 1 + \Delta_x + \Delta_y + \Delta_x \Delta_y + \Delta_x \Delta_x + \Delta_x \Delta_y + \Delta_x \Delta_x \Delta_y = \overline{1 + \Delta_x} \overline{1 + \Delta_y} \overline{1 + \Delta_x};$$

$$\text{and } \therefore \text{if } u = f(x, y, z); \quad u' = \overline{1 + \Delta_x} \cdot \overline{1 + \Delta_y} \cdot \overline{1 + \Delta_z} u;$$

$$\therefore \Delta u = \{ \overline{1 + \Delta_x} \cdot \overline{1 + \Delta_y} \cdot \overline{1 + \Delta_z} - 1 \} u.$$

(44.) Next to find the second difference, change u into Δu .

$$\begin{aligned} \therefore \Delta^2 u_{x,y} &= \Delta_x \Delta u_{x,y} + \Delta_y \cdot \Delta u_{x,y} + \Delta_y \Delta_x \cdot \Delta u_{x,y} \\ &= (\Delta_x + \Delta_y + \Delta_y \Delta_x) \Delta u_{x,y} \\ &= \{ \overline{1 + \Delta_x} \cdot \overline{1 + \Delta_y} - 1 \} \Delta u_{x,y} \\ &= \{ \overline{1 + \Delta_x} \overline{1 + \Delta_y} - 1 \}^2 \cdot u_{x,y}; \end{aligned}$$

and, similarly,

$$\Delta^n u_{x,y} = \{ \overline{1 + \Delta_x} \cdot \overline{1 + \Delta_y} - 1 \}^n u_{x,y}.$$

The expression for $\Delta^2 u_{x,y}$ may be found by substituting for $\Delta u_{x,y}$ its value, and the truth of the above reasoning rendered at once evident.

We may obviously extend the formulæ proved for two variables to functions containing any number, and thus have

$$\Delta^n u = \{ \overline{1 + \Delta_x} \overline{1 + \Delta_y} \overline{1 + \Delta_z} \dots - 1 \}^n u.$$

(45.) The preceding expressions for $\Delta^n u$ may be put under another and remarkable form, by means of the equation deduced in Article 34.

We there see that, by separating the symbols, since $(e^{\frac{d}{dx}} - 1) u_x$ produces the series which is equivalent to Δu_x , it may be written for Δu_x .

Hence for $1 + \Delta_x, 1 + \Delta_y$, we may write $e^{\frac{d}{dx}}, e^{\frac{d}{dy}}$; and thus, if $u_{x,y}$ represent a function of two variables,

$$\Delta^n u_{x,y} = (e^{\frac{d}{dx} + \frac{d}{dy}} - 1)^n u_{x,y};$$

and, similarly,

$$\Delta^n u_{x,y,z,\dots} = (e^{\frac{d}{dx} + \frac{d}{dy} + \frac{d}{dz} + \dots} - 1)^n u_{x,y,z,\dots}$$

(46.) We may observe with regard to these expressions for $\Delta^n u$, and for the sake of illustration referring solely to the more simple case, where

$$\Delta^n u_{x,y} = (e^{\frac{d}{dx} + \frac{d}{dy}} - 1)^n u_{x,y}$$

Calculus of
Finite
Differences.

that no confusion can arise from the powers of $\frac{d}{dx}$ and $\frac{d}{dy}$ when expanded, if we keep in mind their peculiar signification: and observe that any such term as $\frac{d^p}{dx^p} \times \frac{d^q}{dy^q} \cdot u_{x,y}$ is nothing more than $\frac{d^{p+q} u_{x,y}}{dx^p \cdot dy^q}$.

Calculus of
Finite
Differences.

The coefficient of this general term may be thus found. Let t and t_1 be written for $\frac{d}{dx}$ and $\frac{d}{dy}$, then the coefficients of the expansion of $\Delta^n u_{x,y}$ will be the same as those of $(e^{t+t_1} - 1)^n$, and the coefficient of $\frac{d^{p+q} u_{x,y}}{dx^p \cdot dy^q}$ will be the same as that of $t^p \cdot t_1^q$.

But such a term as $t^p \cdot t_1^q$ must arise from the expansion of $(t + t_1)^{p+q}$, and by the binomial theorem the coefficient of $t^p \cdot t_1^q$

$$= \frac{(p+q)(p+q-1)\dots p+1}{1 \cdot 2 \dots q} \cdot \frac{p+1}{1 \cdot 2 \dots p} = \frac{1 \cdot 2 \cdot 3 \dots p+q}{1 \cdot 2 \cdot 3 \dots q \times 1 \cdot 2 \cdot 3 \dots p}.$$

But by Article 39 the coefficient of $(t + t_1)^{p+q}$

$$= \frac{\Delta^n \cdot 0^{p+q}}{1 \cdot 2 \cdot 3 \dots p+q};$$

and, therefore, the coefficient of $t^p \cdot t_1^q$, or of $\frac{d^{p+q} u_{x,y}}{dx^p \cdot dy^q}$, will be

$$\frac{\Delta^n \cdot 0^{p+q}}{1 \cdot 2 \cdot 3 \dots p+q} \times \frac{1 \cdot 2 \cdot 3 \dots p+q}{1 \cdot 2 \cdot 3 \dots p \times 1 \cdot 2 \dots q} = \frac{\Delta^n \cdot 0^{p+q}}{1 \cdot 2 \cdot 3 \dots p \times 1 \cdot 2 \cdot 3 \dots q}.$$

Interpolations.

(47.) One of the most important applications of the theory of Finite Differences is to the subject of interpolations, or the method of inserting between the terms of a series some new term or terms subject to the same law.

The general equation

$$u_x = u + x \cdot \Delta u + \frac{x \cdot x - 1}{2} \Delta^2 u + \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \Delta^3 u + \&c.$$

will always give us any term we may wish of the series of terms, $u, u_1, u_2, u_3, \dots, u_n$, when u and its successive differences, as well as x , are given: but it frequently happens that only a few of these terms are known; and then the result which we obtain, although satisfying the given conditions, cannot be said to be the only one that possesses this property, and thus the problem is an indeterminate one.

(48.) Thus, if we wish to find an algebraical expression which, by giving particular values to x , as 0, 1, 2, shall not only produce the numbers 3, 7, 19, but others in successive order, so as to form a series of which these numbers shall be the first three terms, let us assume that the second differences are constant.

Then

$$\therefore u = 3, \quad \Delta u = 4, \quad \Delta^2 u = 8,$$

$$u_x = 3 + 4x + \frac{x \cdot x - 1}{2} \cdot 8 = 3 + 4x + 4x^2 - 4x = 3 + 4x^2.$$

And now, if we give to x the values 0, 1, 2, 3, 4, &c., a series will arise of which the first three terms will be the given numbers, and of which the second differences are constant. But had we at once assumed $u_x = A + Bx + Cx^2$, and determined the constants A, B, C by the three given conditions of u being successively equal to the numbers 3, 7, 19 when $x = 0, 1, 2$, we shall find that $u_x = 3 + \frac{8}{3}x + \frac{4}{3}x^2$, which will give the very different series 3, 7, 19, 47, 99, &c., and of which the third differences are constant.

(49.) This example thus plainly shows the truth of the assertion, that the problem of interpolations is indeterminate; but it may be further illustrated by the nature of a curve. For considering u_x to be the general expression for the ordinate of an unknown curve, of which $n+1$ values at given intervals are known, it is plainly impossible to find the nature of the curve itself from these conditions, since it is easily to be conceived that innumerable curves may pass through the $n+1$ points, and yet in every other particular differ widely from each other.

But if it be required to find the nature of the curve at some one point lying between two given points, and if we can assume this required point to be very near one of the given ordinates, the chance of error is much lessened.

Now this will come to the same thing, if we expand the expression

$$u_x = u + x \cdot \Delta u + \frac{x \cdot x - 1}{2} \Delta^2 u + \&c.$$

according to the powers of x , when we shall have

$$u_x = u + Ax + Bx^2 + Cx^3 + \&c. \dots + Nx^n;$$

Calculus of Finite Differences. and assume this value of u_x to represent the true ordinate of the curve. The curve thus assumed is of the parabolic species, and if not the one that we are in quest of, is at least of as simple a form, and may have an osculation of the n th order with it.

Calculus of Finite Differences.

When, therefore, we have to interpolate a series within very narrow limits, which is the case alluded to in the preceding illustration, we may assume for the general term an expression developed according to the positive powers of its index, and neglecting all the powers beyond a certain limit, which limit the number of given values will indicate, the general term is known.

(50.) A simple instance of interpolation is given by the question, to place a mean between two terms of the arithmetic series $a, a + b, a + 2b, a + 3b, \&c.$

Then, if it be required to find a term between $a + 2b$ and $a + 3b$, which is also equidistant from either of them,

making

$$u = a + 2b, u_1 = a + 3b; \therefore \Delta \cdot u = b \text{ and } x = \frac{1}{2};$$

$$\therefore u_x = u + x \Delta \cdot u = a + 2b + \frac{b}{2} = a + \frac{5b}{2}.$$

This question may be otherwise solved. For since two terms are given, and the intermediate one required, we may state it, that u and u_2 are given, and u_1 required.

Here we assume

$$\Delta^2 \cdot u = 0; \text{ and } \therefore u_2 - 2u_1 + u = 0; \therefore u_1 = \frac{u + u_2}{2};$$

from which we have the same result as before.

(51.) In general, when it is required to find any one term of the series $u, u_1, u_2, u_3 \dots u_n$, and the rest are given, we may assume that $\Delta^n u = 0$, or that the $(n - 1)$ th differences are constant. For this supposition will obviously give a term which will satisfy the conditions of the problem. In this case we have the simple equation

$$u_n - n u_{n-1} + \frac{n n - 1}{2} u_{n-2} - \frac{n n - 1 n - 2}{1 \cdot 2 \cdot 3} u_{n-3} + \&c. = 0,$$

from which the required term may be found.

Example. Given three values, u, u_1, u_3 , of any function, insert the deficient one u_2 .

Here we make

$$\Delta^3 u = 0; \therefore u_3 - 3u_2 + 3u_1 - u = 0;$$

$$\therefore u_2 = \frac{u_3 + 3u_1 - u}{3};$$

and having found u_2 , we can then find Δu and $\Delta^2 u$, and thus determine $u_x = u + x \Delta u + \frac{x x - 1}{2} \Delta^2 u$, and obtain the general law of the series.

Example 2. Given

$$u = 3, u_1 = 7, u_3 = 39;$$

$$\therefore u_2 = \frac{39 + 21 - 3}{3} = 19;$$

$$\therefore \Delta \cdot u = 4, \Delta^2 \cdot u = 8, \text{ and } u_x = 3 + 4x^2.$$

(52.) Again, when two terms are deficient, let the series of terms be

$$u, u_1, u_2, u_3 \dots u_n, u_{n+1};$$

and here we may make $\Delta^n \cdot u = 0$ and $\Delta^n \cdot u_1 = 0$, whence we have

$$u_n - n u_{n-1} + \frac{n n - 1}{2} u_{n-2} - \frac{n n - 1 n - 2}{1 \cdot 2 \cdot 3} u_{n-3} + \&c. = 0,$$

and

$$u_{n+1} - n u_n + \frac{n n - 1}{2} u_{n-1} - \frac{n n - 1 n - 2}{1 \cdot 2 \cdot 3} u_{n-2} + \&c. = 0;$$

from which two simple equations the unknown term may be found.

Thus, if there be given u, u_1, u_4, u_5 , and u_2, u_3 be required, assume

$$\Delta^4 u = 0 \text{ and } \Delta^4 u_1 = 0;$$

and

$$\therefore u_4 - 4u_3 + 6u_2 - 4u_1 + u = 0,$$

whence

$$u_5 - 4u_4 + 6u_3 - 4u_2 + u_1 = 0;$$

and

$$u_2 = \frac{-3u + 10u_1 + 5u_4 - 2u_5}{10},$$

$$u_3 = \frac{-2u + 5u_1 + 10u_4 - 3u_5}{10};$$

Calculus of and having got these terms we may interpolate any intermediate term whatever by means of the formula

Finite
Differences.

$$u_x = u + x \cdot \Delta \cdot u + \frac{x(x-1)}{2} \cdot \Delta^2 \cdot u + \&c.$$

Calculus of
Finite
Differences.

(53.) Of the use of this formula we will take an example from a table of logarithms, and which with some alterations is the illustration made use of by Lacroix.

Then let it be required to find the logarithm of 314.15926536, having given the logarithm of the equidistant numbers 314, 315, 316, 317, and 318.

In this example the logarithms will be represented by the function u_x , and the numbers by x ; then, calling $u = \log 314$, $u_1 = \log 315$, &c., and omitting the common characteristic, we may form the Table

	Numbers.	Logarithms.	Δu .	$\Delta^2 u$.	$\Delta^3 u$.	$\Delta^4 u$.
u	314	.4969296481				
u_1	315	.4983105538	13809057			
u_2	316	.4996870826	13765288	-43769		
u_3	317	.5010592622	13721796	-43492	+277	
u_4	318	.5024271200	13678578	-43218	+274	-3

after which the differences continue to decrease.

Hence from the Table above we have

$$u_x = u + x \Delta u + \frac{x \cdot x - 1}{2} \Delta^2 u + \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \Delta^3 u + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3}{1 \cdot 2 \cdot 3 \cdot 4} \Delta^4 u,$$

and

$$u = .4969296481$$

$$\Delta u = .0013809057 \quad \Delta^2 u = -.0000043769$$

$$\Delta^3 u = .0000000277 \quad \Delta^4 u = -.0000000003;$$

$$x = .15926536; \frac{x-1}{2} = \frac{x}{2} - \frac{1}{2} = \frac{x}{2} - .5 = -.42036732$$

$$\frac{x-2}{3} = \frac{x}{3} - \frac{2}{3} = -.61357821; \quad \frac{x-3}{4} = \frac{x}{4} - \frac{3}{4} = -.71018366;$$

and these values being substituted in the series for u_x , we find, at length,

$$u_x = \log 314.15926536 = 2.4971498726; \therefore \log \pi = .4971498726.$$

(54.) It is a common problem to insert between two numbers of a given series of numbers any number of equidistant numbers which may follow a law that will comprise not only themselves but also the given numbers.

By such means the tables of logarithms are greatly extended, and the idea of it first appears to have occurred to Briggs.

The following method, remarkable for its simplicity, was given by Mouton in 1670, and by means of it the required terms are computed by addition only.

The principle upon which it is founded is this: first ascertain at what order of the differences of the given numbers the differences become constant; then it is assumed that the series formed by the interpolation of the new terms will have the same order of differences constant.

As an example, let the series be

$$0, 15, 41, 87, 162, 275, \&c.,$$

where $\Delta u_0 = 15$, $\Delta^2 u_0 = 11$, $\Delta^3 u_0 = 9$, $\Delta^4 u_0 = 0$, or the third differences are constant, and it is required to insert two new terms between every two consecutive terms of the given series.

Now making the first term of the series a , the first difference of the series, b , the second difference, c , and the third, d ; let us form the annexed Table according to the principles explained in the introductory articles.

	Numbers.	First Differences.	Second Differences.	Third Differences.
u_0	a			
u_1	$a + b$	b		
u_2	$a + 2b + c$	$b + c$	c	
u_3	$a + 3b + 3c + d$	$b + 2c + d$	$c + d$	d
u_4	$a + 4b + 6c + 4d$	$b + 3c + 3d$	$c + 2d$	d
u_5	$a + 5b + 10c + 10d$	$b + 4c + 6d$	$c + 3d$	d
u_6	$a + 6b + 15c + 20d$	$b + 5c + 10d$	$c + 4d$	d
u_7	$a + 7b + 21c + 35d$	$b + 6c + 15d$	$c + 5d$	d
u_8	$a + 8b + 28c + 56d$	$b + 7c + 21d$	$c + 6d$	d
u_9	$a + 9b + 36c + 84d$	$b + 8c + 28d$	$c + 7d$	d

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

Now, in the example which we have taken, the terms u_0, u_3, u_6 , and u_9 are known, or four equations are given, by which the four unknown quantities a, b, c, d may be determined.

Thus

$$u_0 = a = 0, \quad u_3 = a + 3b + 3c + d = 15,$$

and

$$u_6 = a + 6b + 15c + 20d = 41, \quad \text{and} \quad u_9 = a + 9b + 36c + 84d = 87,$$

and

$$\therefore u_6 - 2u_3 = 11 = 9c + 18d,$$

whence

$$u_9 - 3u_3 = 42 = 27c + 81d;$$

$$d = \frac{1}{3}, \quad c = \frac{5}{9}, \quad \text{and} \quad b = 5 - c - \frac{d}{3} = \frac{13}{3};$$

and from these values the successive terms of the series may be found.

It may be observed that the value of the third difference in the original series is 9, while in the interpolated one it is $\frac{1}{3} = \frac{9}{27} = \frac{9}{3^3}$, or if $D^3 u_0$ and $\Delta^3 u_0$ represent these differences, $D^3 u_0 = 3^3 \cdot \Delta^3 u_0$.

(55.) We proceed to give a general solution of this problem. Let the number of given terms be $n + 1$; and let it be required to insert $m - 1$ terms between each pair of the terms of the original series; as, for instance, between u_x and u_{x+1} .

Now, the general value for u_x is when $\Delta^n u_0$ is constant:

$$u_x = u_0 + x \cdot \Delta u_0 + \frac{x(x-1)}{2} \cdot \Delta^2 u_0 + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} \cdot \Delta^3 u_0 + \&c. \\ + \frac{x(x-1)(x-2) \dots (x-n+1)}{1 \cdot 2 \cdot 3 \dots n} \cdot \Delta^n u_0;$$

and since there are to be $m - 1$ terms between u_x and u_{x+1} , we must suppose unity to be divided into m parts, each $= \frac{1}{m}$, so that x may increase each time by $\frac{1}{m}$; and $\therefore$ after (t) times x will be increased by $\frac{t}{m}$; or, supposing the original value of x to be zero, x may $= \frac{t}{m}$, and $\therefore$

$$u_x = u_0 + \frac{t}{m} \cdot \Delta u_0 + \frac{t \cdot t - m}{1 \cdot 2 m^2} \Delta^2 u_0 + \frac{t \cdot t - m \cdot t - 2m}{1 \cdot 2 \cdot 3 m^3} \Delta^3 u_0 + \&c. \\ + \frac{t \cdot (t - m) (t - 2m) \dots (t - n - 1 m)}{1 \cdot 2 \cdot 3 \dots n \cdot m^n} \cdot \Delta^n u_0.$$

Let the successive differences of this equation be taken; and since, in every algebraical function having positive indices the differences vanish when their order is greater than the index of the variable, we have

$$\Delta^r u_x = \frac{\Delta^r \cdot t \cdot (t - m) (t - 2m) \dots (t - r - 1 m)}{1 \cdot 2 \cdot 3 \dots r \cdot m^r} \Delta^r u_0 \\ + \frac{\Delta^r \cdot t \cdot (t - m) (t - 2m) \dots (t - r m)}{1 \cdot 2 \cdot 3 \dots (r + 1) m^{r+1}} \cdot \Delta^{r+1} u_0 \\ + \&c. \\ + \frac{\Delta^r \cdot t \cdot (t - m) (t - 2m) \dots (t - n - 1 \cdot r)}{1 \cdot 2 \cdot 3 \dots n \cdot m^n} \Delta^n u_0.$$

The terms $\Delta u_0, \Delta^2 u_0$, and $\Delta^n u_0$ are constant, and the symbol Δ' indicates that the differences with regard to x increase by $\frac{1}{m}$.

And now having performed the differentiations, and then made $t = 0$, and having given to r the values 1, 2, 3, &c., we shall find the differences that we are in search of and thus solve the problem which has been proposed, or if we take as an illustration the example in the preceding article, we shall be able to find the differences b, c, d ; and the formulæ for u_x in that particular example will be

$$u_x = u_0 + \frac{t}{3} \Delta u_0 + \frac{t \cdot t - 3}{1 \cdot 2 \cdot 3^2} \Delta^2 u_0 + \frac{t \cdot t - 3 \cdot t - 6}{1 \cdot 2 \cdot 3 \cdot 3^3} \Delta^3 u_0;$$

whence it is obvious, since $\Delta^3 (t \cdot t - 3 \cdot t - 6) = \Delta^3 \cdot t^3 = 1 \cdot 2 \cdot 3$, that $\Delta^3 u_0 = \frac{1}{3^3} \Delta^3 u_0$, an equation which has been previously noticed.

If, however, in the general value for $\Delta^r u_x$, we arrange the terms according to the powers of t , and since

we have $t \cdot (t - m) (t - 2m) (t - 3m) \dots (t - r - 1 \cdot m) = t^r + A m t^{r-1} + B m^2 t^{r-2} + \&c.$,

Calculus of
Finite
Differences.

$$\begin{aligned}\Delta^r \cdot (t^r + A m t^{r-1} + B m^2 t^{r-2} + \&c.) &= \Delta^r \cdot t^r \\ \Delta^r (t^{r+1} + A_1 m t^r + B_1 m^2 t^{r-1} + \&c.) &= \Delta^r \cdot t^{r+1} + A_1 m \Delta^r t^r \\ \Delta^r (t^{r+2} + A_2 m t^{r+1} + B_2 m^2 t^r + \&c.) &= \Delta^r \cdot t^{r+2} + A_2 m \Delta^r t^{r+1} + B_2 m^2 \cdot \Delta^r \cdot t^r \\ &\&c.;\end{aligned}$$

Calculus of
Finite
Differences.

or writing $\alpha, \beta, \gamma, \&c.$ for $\Delta^r \cdot t^r, \Delta^r \cdot t^{r+1}, \Delta^r \cdot t^{r+2}, \&c.$, we have

$$\Delta^r u_x = \frac{1}{1 \cdot 2 \cdot 3 \dots r \cdot m^r} \left\{ \alpha \Delta^r u_0 + \frac{\beta + \alpha A_1 m}{(r+1)m} \Delta^{r+1} u_0 + \frac{\gamma + A_2 m \beta + B_2 m^2 \alpha}{(r+1)(r+2)m^2} \Delta^{r+2} u_0 + \&c. \right\}$$

Let
and

$$r = n, \therefore \alpha = 1 \cdot 2 \cdot 3 \dots n, \Delta^{n+1} u_0 = 0, \Delta^{n+2} u_0 = 0, \&c.,$$

$$\Delta^n u_0 = \frac{\Delta^n u_0}{m^n},$$

which, when $m = n = 3$, is the theorem to which we have been led in the last article by mere inspection.

(56.) Next let it be required to find the index (x), to which, in a series of given numbers, a number included between two terms of this series belongs. Such is the problem when there is given a logarithm not found in the tables, and the corresponding number is to be sought.

The problem is not difficult when the values of $\Delta u_0, \Delta^2 u_0, \Delta^3 u_0, \&c.$ are very convergent; for then, in the equation

$$u_x = u_0 + x \Delta u_0 + \frac{x(x-1)}{2} \Delta^2 u_0 + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} \Delta^3 u_0 + \&c.,$$

u_x is given and x is to be found.

If $\Delta^2 u_0$ be very much smaller than $\Delta \cdot u_0$, we may as a first approximation neglect $\Delta^2 u_0$, and make $u_x = u_0 + x \cdot \Delta u_0$; whence

$$x = \frac{u_x - u_0}{\Delta u_0}.$$

Call this value (α), and then substituting (α) for x , and stopping at the second difference,

$$u_x - u_0 = x \left(\Delta u_0 + \frac{\alpha - 1}{2} \Delta^2 u_0 \right); \therefore x = \frac{u_x - u_0}{\Delta u_0 + \frac{\alpha - 1}{2} \Delta^2 u_0};$$

and calling this value of x, β , and then include the third differences, we shall successively approximate more nearly to the true value of x .

When, however, the differences do not decrease so rapidly as to allow us to neglect the second, third, &c. differences, we must stop at the first very small term, and determine x by the solution of an equation, the dimensions of which will be the last order of differences which we retain; thus if $\Delta^4 u_0$ may be neglected, x must be found by means of the cubic equation

$$u_x - u_0 = \left(\Delta u_0 - \frac{1}{2} \Delta^2 u_0 + \frac{1}{3} \Delta^3 u_0 \right) x + \frac{1}{2} (\Delta^2 u_0 - \Delta^3 u_0) x^2 + \frac{x^3 \cdot \Delta^3 u_0}{6}.$$

(57.) As an illustration, let us take the following question proposed in Herschell's examples.

Three observations of a certain quantity were taken at equal intervals. The values were u_0, u_1, u_2 ; between which its value was a . When did this happen?

Here $u_x = a$ is given, and x is required.

Assuming $\Delta^3 u_0 = 0$,

$$u_0 + x \Delta u_0 + \frac{x(x-1)}{2} \Delta^2 u_0 = a;$$

$$\therefore \Delta^2 u_0 \cdot x^2 + (2 \Delta u_0 - \Delta^2 u_0) x = 2(a - u_0);$$

$$\therefore x = \frac{1}{2 \Delta^2 u_0} \{ \Delta^2 u_0 - 2 \Delta u_0 + \sqrt{(\Delta^2 u_0 - 2 \Delta u_0)^2 + 8(a - u_0) \Delta^2 u_0} \}.$$

If $\Delta^2 u_0 = 0$, or be so small as to be neglected, we have, from the original quadratic,

$$x = \frac{2(a - u_0)}{2 \Delta u_0} = \frac{a - u_0}{u_1 - u_0}.$$

Next to find the value of x , when $\Delta^2 u_0$, although small, must not be neglected, yet its square and higher powers may be omitted: we must first expand the value of x obtained above according to the powers of $\Delta^2 u_0$, and finally retain the first power only.

Now make

$$m = \Delta^2 u_0, n = \Delta u_0, \text{ and } c = a - u_0.$$

Then

$$\begin{aligned}x &= \frac{1}{2m} \cdot (m - 2n + \sqrt{(m - 2n)^2 + 8cm}) \\ &= \frac{1}{2m} \cdot \left(m - 2n + 2n \sqrt{1 - \frac{m}{n} + \frac{2cm}{n^2} + \frac{m^2}{4n^2}} \right)\end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2m} \cdot \left\{ m - 2n + 2n \left(1 - \frac{m}{2n} + \frac{cm}{n^2} + \frac{m^2}{8n^2} - \frac{1}{8} \cdot \frac{m^2}{n^2} \cdot \left(1 - \frac{2c}{n} \right)^2 \right) \right\} \\
 &= \frac{1}{2m} \cdot \left\{ m - 2n + 2n - m + \frac{2cm}{n} + \frac{m^2}{4n} - \frac{m^2}{4n} \left(1 - \frac{4c}{n} + \frac{4c^2}{n^2} \right) \right\} \\
 &= \frac{c}{n} + \frac{cm}{2n^2} - \frac{mc^2}{2n^3} = \frac{c}{n} + \frac{mc \cdot (n-c)}{2n^3} \\
 &= \frac{a - u_0}{\Delta \cdot u_0} + \frac{(a - u_0)(u_1 - a) \cdot \Delta^2 u_0}{2(\Delta \cdot u_0)^3};
 \end{aligned}$$

the latter term may $\therefore$ be considered as a correction to the value of a found on the supposition of $\Delta^2 u_0 = 0$.

(58.) Hitherto it has been supposed that the differences between the values of the variable x are equal; but, as in the application of this theory to practical questions this is not found to be the case, it is necessary to have formulæ which are not thus restricted. We then assume the general term of the series to be represented by a function of x , the powers of which are integral: the reasons for this assumption we have already given. Assuming then

$$u_x = A + Bx + Cx^2 + Dx^3 + \&c. + Nx^n,$$

and supposing that u_x becomes successively $u, u_1, u_2, u_3, \&c.$, when $a, a_1, a_2, a_3, \&c.$ are put for x , there will result the following $(n+1)$ equations to determine the $(n+1)$ quantities $A, B, C, D, \&c. N$:

$$\begin{aligned}
 u &= A + Ba + Ca^2 + Da^3 + \&c. + Na^n \\
 u_1 &= A + Ba_1 + Ca_1^2 + Da_1^3 + \&c. + Na_1^n \\
 u_2 &= A + Ba_2 + Ca_2^2 + Da_2^3 + \&c. + Na_2^n \\
 u_3 &= A + Ba_3 + Ca_3^2 + Da_3^3 + \&c. + Na_3^n \\
 \&c. &= \&c.
 \end{aligned}$$

Subtract the first line from the second, the second from the third, the third from the fourth, and so on in order; and then divide the results respectively by $a_1 - a, a_2 - a_1, a_3 - a_2, a_4 - a_3, \&c.$, and we have

$$\begin{aligned}
 \frac{u_1 - u}{a_1 - a} &= B + C(a_1 + a) + D(a_1^2 + a_1a + a^2) + \&c. \\
 \frac{u_2 - u_1}{a_2 - a_1} &= B + C(a_2 + a_1) + D(a_2^2 + a_2a_1 + a_1^2) + \&c. \\
 \frac{u_3 - u_2}{a_3 - a_2} &= B + C(a_3 + a_2) + D(a_3^2 + a_3a_2 + a_2^2) + \&c. \\
 \&c. &= \&c.;
 \end{aligned}$$

or making, after Laplace,

$$\frac{u_1 - u}{a_1 - a} = \delta u, \quad \frac{u_2 - u_1}{a_2 - a_1} = \delta u_1, \quad \frac{u_3 - u_2}{a_3 - a_2} = \delta u_2, \quad \&c.,$$

we have

$$\begin{aligned}
 \delta u &= B + C(a_1 + a) + D(a_1^2 + a_1a + a^2) + \&c. \\
 \delta u_1 &= B + C(a_2 + a_1) + D(a_2^2 + a_2a_1 + a_1^2) + \&c. \\
 \delta u_2 &= B + C(a_3 + a_2) + D(a_3^2 + a_3a_2 + a_2^2) + \&c.
 \end{aligned}$$

Subtracting these results in the same way as we have done for the values of $\delta u, \delta u_1, \delta u_2, \&c.$, and dividing the respective results by $a_2 - a, a_3 - a_1, \&c.$, and then writing $\delta^2 u, \delta^2 u_1, \&c.$ instead of the fractions $\frac{\delta u_1 - \delta u}{a_2 - a}, \frac{\delta u_2 - \delta u_1}{a_3 - a_1}, \&c.$, in conformity with the notation used in the preceding equations, we have

$$\begin{aligned}
 \delta^2 u &= C + D(a_2 + a_1 + a) + \&c. \\
 \delta^2 u_1 &= C + D(a_3 + a_2 + a_1) + \&c. \\
 \&c. &= \&c.
 \end{aligned}$$

Again subtracting these equations, and dividing the results by $a_3 - a, a_4 - a_1, \&c.$, and making

$$\frac{\delta^2 u_1 - \delta^2 u}{a_3 - a} = \delta^3 u, \quad \frac{\delta^2 u_2 - \delta^2 u_1}{a_4 - a_2} = \delta^3 u_1, \quad \&c. \dots$$

we shall have

$$\delta^3 u = D + \&c.$$

The method of obtaining the equations by which the $n+1$ quantities $A, B, C, \&c. N$ are to be found can now present no difficulty; yet as the calculations would be laborious when the expression for u_x contains many terms, we will stop at the fourth term Dx^3 , or assume that $u_x = A + Bx + Cx^2 + Dx^3$, we shall then have,

$$\begin{aligned}
 D &= \delta^3 u \\
 C &= \delta^2 u - (a_2 + a_1 + a) \delta^3 u \\
 B &= \delta u - (a_1 + a) \delta^2 u + (a_2 a_1 + a_2 a + a_1 a) \delta^3 u \\
 A &= u - a \delta u + a_1 a \delta^2 u - a_2 a_1 a \delta^3 u.
 \end{aligned}$$

Putting these values for A, B, C, and D, in u_x , there arises

$$\begin{aligned} u_x &= u + (x-a)\delta u + (x^2 - (a_1+a)x + a_1a)\delta^2 u \\ &\quad + \{x^3 - (a_2+a_1+a)x^2 + (a_2a_1+a_2a+a_1a)x - a_2a_1a\}\delta^3 u \\ &= u + (x-a)\delta u + (x-a_1)(x-a)\delta^2 u + (x-a_2)(x-a_1)(x-a)\delta^3 u. \end{aligned}$$

The form of this result is remarkable, and we may infer from it the general form of u_x , where there are $n+1$ values of x ; as $a, a_1, a_2, a_3, \dots, a_n$, and we shall have

$$\begin{aligned} u_x &= u + (x-a)\delta u + (x-a)(x-a_1)\delta^2 u + (x-a)(x-a_1)(x-a_2)\delta^3 u + \&c. \\ &\quad + (x-a)(x-a_1)(x-a_2)\dots(x-a_n)\delta^n u. \end{aligned}$$

(59.) When the values a_1, a_2, a_3 , &c. are in arithmetic progression, having a common difference h ; then

$$\begin{aligned} \delta u &= \frac{u_1 - u}{a_1 - a} = \frac{\Delta u}{h} & \delta^2 u &= \frac{\delta u_1 - \delta u}{a_2 - a} = \frac{\Delta^2 u}{2h^2} \\ \delta^3 u &= \frac{\delta^2 u_1 - \delta^2 u}{a_3 - a} = \frac{\Delta^3 u}{2 \cdot 3 h^3}, \&c.; \end{aligned}$$

and making

$$\begin{aligned} x-a &= h_1, \therefore x-a_1 = h_1 - h, x-a_2 = h_1 - 2h, \&c.; \\ u_x &= u + \frac{h_1}{h} \Delta u + \frac{h_1(h_1-h)}{1 \cdot 2 \cdot h^2} \Delta^2 u + \frac{h_1(h_1-h)(h_1-2h)}{1 \cdot 2 \cdot 3 h^3} \Delta^3 u + \&c., \end{aligned}$$

which we have previously obtained, and which may be put under the form

$$\Delta' u = \frac{h_1}{h} \Delta u + \frac{h_1(h_1-h)}{1 \cdot 2 \cdot h^2} \Delta^2 u + \frac{h_1(h_1-h)(h_1-2h)}{1 \cdot 2 \cdot 3 h^3} \Delta^3 u + \&c.,$$

by making $\Delta' u = u_x - u$, where $\Delta' u$ expresses the difference for an interval h_1 .

This formula is used by Lacroix to find the logarithm of 3.141592653, having given the logarithm of 3.14, 3.15, 3.16, 3.17, 3.18, and he assumes

$$a = 3.14, x = 3.141592653; \therefore x-a = h_1 = .0015926536 \text{ and } h = .001;$$

and thus obtains the same result as we have done in a preceding article.

(60.) The coefficients $\delta u, \delta u_1, \delta u_2$, &c. have a remarkable form, which we proceed to show, and thence deduce an expression for u_x by a very elegant method due to Lagrange, and which, as we shall see, possesses the greatest advantages.

For since

$$\begin{aligned} \delta u &= \frac{u_1 - u}{a_1 - a} = \frac{u_1}{a_1 - a} + \frac{u}{a - a_1} \\ \delta^2 u &= \frac{\delta u_1 - \delta u}{a_2 - a} = \frac{1}{a_2 - a} \cdot \left\{ \frac{u_2}{a_2 - a_1} + \frac{u_1}{a_1 - a_2} - \frac{u_1}{a_1 - a} - \frac{u}{a - a_1} \right\} \\ &= \frac{1}{a_2 - a} \left\{ \frac{u_2}{a_2 - a_1} + \frac{u_1(a_2 - a)}{(a_1 - a_2)(a_1 - a)} - \frac{u}{a - a_1} \right\} \\ &= \frac{u_2}{(a_2 - a)(a_2 - a_1)} + \frac{u_1}{(a_1 - a_2)(a_1 - a)} + \frac{u}{(a - a_1)(a - a_2)}; \end{aligned}$$

and, similarly,

$$\begin{aligned} \delta^3 u &= \frac{u_3}{(a_3 - a)(a_3 - a_1)(a_3 - a_2)} + \frac{u_2}{(a_2 - a)(a_2 - a_1)(a_2 - a_3)} \\ &\quad + \frac{u_1}{(a_1 - a)(a_1 - a_2)(a_1 - a_3)} + \frac{u}{(a - a_1)(a - a_2)(a - a_3)} \end{aligned}$$

and putting these values in the place of $\delta u, \delta^2 u$, &c., we shall have for u_x

$$\begin{aligned} u_x &= u \left\{ 1 + \frac{x-a}{a-a_1} + \frac{(x-a)(x-a_1)}{(a-a_1)(a-a_2)} + \&c. \right\} + u_1 \left\{ \frac{x-a}{a_1-a} + \frac{x-a}{(a_1-a_2)(a_1-a)} + \&c. \right\} \\ &\quad + \&c. \\ &= Xu + X_1 u_1 + X_2 u_2 + X_3 u_3 + \&c. + X_n u_n, \end{aligned}$$

by substituting X, X_1, X_2 , &c. X_n for the coefficients of u, u_1, u_2 , &c. u_n .

(61.) The equation to which we have been led by the reasoning in the preceding article might have been inferred from the circumstance that the values of A, B, C, &c., of which we have been in search, could not comprise any other than the first powers of the letters u_1, u_2, u_3 , &c., and that these must be in the numerator only.

Let

$$\therefore u_x = Xu + X_1 u_1 + X_2 u_2 + X_3 u_3 + \&c. + X_n u_n,$$

where X, X_1, X_2 , &c. solely depend upon x and a, a_1, a_2 , &c.

Now this equation must fulfil the condition of u_x becoming successively u, u_1, u_2, u_3 , &c. when a, a_1, a_2, a_3 , &c. are put for x ; and these conditions will be fulfilled if we have, when

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

$$\begin{aligned} x = a, & \quad X = 1, \quad X_1 = 0, \quad X_2 = 0, \quad X_3 = 0, \text{ \&c.}, \quad X_n = 0, \\ x = a_1, & \quad X = 0, \quad X_1 = 1, \quad X_2 = 0, \quad X_3 = 0, \text{ \&c.}, \quad X_n = 0, \\ x = a_2, & \quad X = 0, \quad X_1 = 0, \quad X_2 = 1, \quad X_3 = 0, \text{ \&c.}, \quad X_n = 0, \\ & \vdots \\ x = a_n, & \quad X = 0, \quad X_1 = 0, \quad X_2 = 0, \quad X_3 = 0, \text{ \&c.}, \quad X_n = 1. \end{aligned}$$

If now we read these results in a vertical order, we shall find from the column in which the values of X are tabulated, that $X = 0$, whenever, $x = a_1, a_2, a_3, \text{ \&c.}$, a condition which is satisfied by making

$$x = a \cdot (x - a_1)(x - a_2)(x - a_3) \dots (x - a_n);$$

and that $X = 1$ when $X = a$, a condition which is also satisfied when

$$\alpha = \frac{1}{(a - a_1)(a - a_2)(a - a_3) \dots (a - a_n)};$$

and that all the necessary conditions are fulfilled if

$$X = \frac{(x - a_1)(x - a_2)(x - a_3) \dots (x - a_n)}{(a - a_1)(a - a_2)(a - a_3) \dots (a - a_n)}.$$

In the same manner we may put

$$X_1 = \frac{(x - a)(x - a_2)(x - a_3) \dots (x - a_n)}{(a_1 - a)(a_1 - a_2)(a_1 - a_3) \dots (a_1 - a_n)},$$

$$X_2 = \frac{(x - a)(x - a_1)(x - a_3) \dots (x - a_n)}{(a_2 - a)(a_2 - a_1)(a_2 - a_3) \dots (a_2 - a_n)},$$

$$\vdots$$

$$X_n = \frac{(x - a)(x - a_1)(x - a_2) \dots (x - a_{n-1})}{(a_n - a)(a_n - a_1)(a_n - a_2) \dots (a_n - a_{n-1})}.$$

And thus

$$\begin{aligned} u_x = & \frac{(x - a_1)(x - a_2) \dots (x - a_n)}{(a - a_1)(a - a_2) \dots (a - a_n)} u + \frac{(x - a)(x - a_2) \dots (x - a_n)}{(a_1 - a)(a_1 - a_2) \dots (a_1 - a_n)} u_1 \\ & + \frac{(x - a)(x - a_1) \dots (x - a_n)}{(a_2 - a)(a_2 - a_1) \dots (a_2 - a_n)} \cdot u_2 + \text{\&c.}; \end{aligned}$$

the coefficients of which can be calculated by logarithms.

(62.) The determination of the coefficients $X, X_1, X_2, X_3, \text{ \&c.}$ at once show that the problem of interpolations is an indeterminate one; for any function whatever which would fulfil the requisite conditions might have been assumed as well as that which has been derived from the ordinary theory of equations.

Thus, since $\sin p(x - a_1) \cdot \sin q(x - a_2) \cdot \sin r \cdot (x - a_3) \dots \sin t \cdot (x - a_n)$ vanishes when $x = a_1, a_2, a_3, \text{ \&c.}$, we may assume

$$X = \frac{\sin p(x - a_1) \cdot \sin q \cdot (x - a_2) \dots \sin t(x - a_n)}{\sin p \cdot (a - a_1) \cdot \sin q \cdot (a - a_2) \dots \sin t \cdot (a - a_n)}.$$

And, similarly, for the other terms, and by changing the values of $p, q, r, \text{ \&c.}$, we may have as many values for $X, X_1, \text{ \&c.}$ as we please. Other transformations may also be found; but what is here given is sufficient to show the truth of the proposition, that the problem of interpolations is an indeterminate one.

The Inverse Calculus of Differences.

(63.) In what has preceded our attention has been chiefly directed to the question, To find the difference or successive differences of u_x , when u_x is a given function: we now come to the more important, and, as it so happens to be, the more difficult problem of finding the function from given differences. But, in the first instance, we shall confine ourselves solely to the case in which the difference is an explicit function of the independent variable.

Since $\Delta \cdot (u_x + c)$ and $\Delta \cdot u_x$ give precisely the same results, it will be necessary in reascending from the difference to the function to add to the result a constant quantity, or such a quantity as will not change its value when $x + 1$ is put for x . But it is not necessary that the quantity to be added after integration should be independent of x ; only that its value should not change when $x + 1$ is written for x ; we may, therefore, put c_x to represent the constant, if $c_{x+1} - c_x = 0$, a case which is exemplified by c_x being expressed by $\cos 2\pi x$, for $\cos \cdot 2\pi(x + 1) = \cos(2\pi + 2\pi x) = \cos 2\pi x$; and $\therefore \Delta \cos 2\pi x = 0$.

(64.) The letter Δ prefixed to a function has been used as the sign of the difference of the function; and the letter Σ is also put for the sign of integration, the reverse operation; or Σ and Δ neutralize each other, and

$$\Sigma(\Delta \cdot u_x) = u_x + c.$$

(65.) We have seen in the preceding chapter that

$$\Delta \cdot u_x + \Delta \cdot v_x + \Delta \cdot w_x = \Delta(u_x + v_x + w_x);$$

therefore taking the integral of each side, we have

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

$$\begin{aligned} & \Sigma \{ \Delta \cdot u_x + \Delta \cdot v_x + \Delta \cdot w_x \} \\ & = u_x + v_x + w_x = \Sigma \cdot (\Delta \cdot u_x) + \Sigma (\Delta \cdot v_x) + \Sigma (\Delta \cdot w_x); \end{aligned}$$

or the integral of the sum of any number of differences is equal to the sum of the integrals of each difference taken separately.

Again, since

$$\begin{aligned} a \cdot \Delta \cdot u_x &= \Delta (a \cdot u_x); \\ \therefore \Sigma (a \cdot \Delta u_x) &= a \cdot u_x = a \Sigma (\Delta \cdot u_x), \end{aligned}$$

it appears that a constant coefficient may be brought without the sign of integration.

(66.) The first case which we shall integrate is that in which u_x is a rational function of x .

It has already been shown that the difference of any rational integral function $Ax^m + Bx^{m-1} + Cx^{m-2} + \&c. + Kx + L$ is itself a rational function of the form

$$A_1 x^{m-1} + B_1 x^{m-2} + C_1 x^{m-3} + \&c. + K_1.$$

Conversely, had we to integrate the latter function, in which the highest index is $m-1$, we may assume its integral to be of the form $Ax^m + Bx^{m-1} + Cx^{m-2} + \&c. + Kx + L$, in which the highest index is m , the difference of which expression must equal the given difference; and having found the difference of the assumed series, the coefficients $A, B, C, \&c. K$ may be determined by equating the coefficients of the same powers of x in the two series.

The last coefficient L will not be found by this process, but remains indeterminate, unless it be known from some given relation.

(67.) An example will illustrate our meaning: let the value of $\Sigma (x^3 + 2x^2 + 1)$ be required.

Assume that

$$\begin{aligned} \Sigma (x^3 + 2x^2 + 1) &= ax^4 + bx^3 + cx^2 + ex + f; \\ \therefore x^3 + 2x^2 + 1 &= \Delta (ax^4 + bx^3 + cx^2 + ex + f) \\ &= a \{ 4x^3 + 6x^2 + 4x + 1 \} + b \cdot (3x^2 + 3x + 1) + c \cdot (2x + 1) + e \\ &= 4ax^3 + (6a + 3b)x^2 + (4a + 3b + 2c)x + a + b + c + e; \\ \therefore 4a &= 1; \therefore a = \frac{1}{4}; \quad 6a + 3b = 2; \therefore b = \frac{1}{3} \left\{ 2 - \frac{3}{2} \right\} = \frac{1}{6}; \\ 4a + 3b + 2c &= 0; \therefore c = \frac{1}{2} \cdot \left\{ -1 - \frac{1}{2} \right\} = -\frac{3}{4}; \\ a + b + c + e &= 0; \therefore e = -\frac{1}{4} - \frac{1}{6} + \frac{3}{4} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}; \\ \therefore \Sigma (x^3 + 2x^2 + 1) &= \frac{x^4}{4} + \frac{x^3}{6} - \frac{3x^2}{4} + \frac{x}{3} + f. \end{aligned}$$

(68.) In this manner we may find the integral of any rational function. But without attending to particular examples, we will find the integral of x^n .

Let $\Sigma \cdot x^n = Ax^{n+1} + Bx^n + Cx^{n-1} + Dx^{n-2} + Ex^{n-3} + \&c.$

$$\begin{aligned} \therefore x^n &= A \{ (x+1)^{n+1} - x^{n+1} \} + B \{ (x+1)^n - x^n \} + C \{ (x+1)^{n-1} - x^{n-1} \} \\ &\quad + D \{ (x+1)^{n-2} - x^{n-2} \} + E \{ (x+1)^{n-3} - x^{n-3} \} + \&c. \\ &= A \left\{ \overline{n+1} \cdot x^n + \frac{n \overline{n+1}}{2} x^{n-1} + \frac{\overline{n+1} \cdot n \cdot \overline{n-1}}{1 \cdot 2 \cdot 3} x^{n-2} + \frac{\overline{n+1} \cdot n \cdot \overline{n-1} \cdot \overline{n-2}}{1 \cdot 2 \cdot 3 \cdot 4} x^{n-3} + \&c. \right\} \\ &\quad + B \left\{ \overline{n} \cdot x^{n-1} + \frac{n \overline{n-1}}{1 \cdot 2} x^{n-2} + \frac{n \overline{n-1} \cdot \overline{n-2}}{1 \cdot 2 \cdot 3} x^{n-3} + \&c. \right\} \\ &\quad + C \left\{ \overline{n-1} \cdot x^{n-2} + \frac{\overline{n-1} \cdot \overline{n-2}}{1 \cdot 2} x^{n-3} + \&c. \right\} \\ &\quad + D \left\{ \overline{n-2} \cdot x^{n-3} + \&c. \right\} \\ &\quad + \&c. \end{aligned}$$

Equating coefficients:

$$\begin{aligned} A \cdot (n+1) &= 1; \therefore A = \frac{1}{n+1}. \\ A \cdot \frac{n \overline{n+1}}{2} + nB &= 0; \therefore B = -\frac{1}{2} A (n+1) = -\frac{1}{2} \\ \frac{A \cdot n \cdot \overline{n+1} \cdot \overline{n-2}}{1 \cdot 2 \cdot 3} + \frac{B n \overline{n-1}}{1 \cdot 2} + \overline{n-1} \cdot C &= 0; \end{aligned}$$

or

$$\frac{n \overline{n-1}}{2 \cdot 3} - \frac{n \overline{n-1}}{4} + (n-1) C = 0; \therefore C = \frac{1}{2} \cdot \frac{n}{1 \cdot 2 \cdot 3},$$

and

$$\frac{n \overline{n-1} \cdot \overline{n-2}}{2 \cdot 3 \cdot 4} - \frac{n \overline{n-1} \cdot \overline{n-2}}{3 \cdot 4} + \frac{1}{2} \cdot \frac{n \overline{n-1} \cdot \overline{n-2}}{3 \cdot 4} + D = 0; \therefore D = 0.$$

Similarly,

$$E = -\frac{1}{6} \cdot \frac{n \cdot \overline{n-1} \cdot \overline{n-2}}{2 \cdot 3 \cdot 4 \cdot 5}, F = \frac{1}{6} \cdot \frac{\overline{nn-1} \cdot \overline{n-2} \dots \overline{n-4}}{1 \cdot 2 \cdot 3 \dots 7};$$

and we shall have, by pursuing the investigation,

$$\begin{aligned} \Sigma \cdot x^n &= \frac{x^{n+1}}{n+1} - \frac{x^n}{2} + \frac{1}{2} \cdot \frac{n \cdot x^{n-1}}{2 \cdot 3} - \frac{1}{6} \cdot \frac{\overline{nn-1} \cdot \overline{n-2}}{2 \cdot 3 \cdot 4 \cdot 5} \cdot x^{n-3} \\ &+ \frac{1}{6} \cdot \frac{\overline{nn-1} \cdot \overline{n-2} \cdot \overline{n-3} \cdot \overline{n-4}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} x^{n-5} \\ &- \frac{3}{10} \cdot \frac{\overline{nn-1} \cdot \overline{n-2} \cdot \overline{n-3} \cdot \overline{n-4} \cdot \overline{n-5} \cdot \overline{n-6}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} x^{n-7} \\ &+ \frac{5}{6} \cdot \frac{\overline{nn-1} \cdot \overline{n-2} \cdot \overline{n-3} \cdot \overline{n-4} \dots \overline{n-8}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \dots 11} x^{n-9} \\ &- \frac{691}{210} \cdot \frac{\overline{nn-1} \cdot \overline{n-2} \cdot \overline{n-3} \dots \overline{n-10}}{2 \cdot 3 \cdot 4 \cdot 5 \dots 13} x^{n-11} \\ &+ \frac{35}{2} \cdot \frac{\overline{nn-1} \cdot \overline{n-2} \cdot \overline{n-3} \dots \overline{n-12}}{2 \cdot 3 \cdot 4 \cdot 5 \dots 15} x^{n-13} \\ &- \frac{3617}{30} \cdot \frac{\overline{nn-1} \cdot \overline{n-2} \cdot \overline{n-3} \dots \overline{n-14}}{2 \cdot 3 \cdot 4 \cdot 5 \dots 17} x^{n-15} \\ &+ \frac{43867}{42} \cdot \frac{n(n-1)(n-2)(n-3) \dots (n-16)}{2 \cdot 3 \cdot 4 \cdot 5 \dots 19} x^{n-17} \\ &- \frac{122277}{110} \cdot \frac{n(n-1)(n-2)(n-3) \dots (n-18)}{2 \cdot 3 \cdot 4 \cdot 5 \dots 21} x^{n-19} \\ &+ \&c. \quad + \text{constant}; \end{aligned}$$

a formula in which that part of the coefficient which depends upon the index n follows an obvious law: but not so the other part. These coefficients give rise to some numbers of singular importance in the theory of series, that we shall not hesitate to point them out in this place, although we shall hereafter more particularly allude to them. If we refer to the coefficients we find a fraction independent of (n) , and another of which the numerator consists of the product of (m) factors, viz., $\overline{nn-1} \cdot \overline{n-2} \dots \overline{n-m+1}$; and the denominator is $= 2 \cdot 3 \cdot 4 \dots m+2$, or the product of $(m+1)$ terms. Now the numbers in question are formed by multiplying the first fraction by $\frac{1}{m+2}$. They are called the *numbers of Bernoulli*, since James Bernoulli first remarked them.

The rule just given will show us that the numbers are, taking no notice of the algebraical signs,

$$\frac{1}{2} \cdot \frac{1}{3}, \frac{1}{6} \cdot \frac{1}{5}, \frac{1}{6} \cdot \frac{1}{7}, \frac{3}{10} \cdot \frac{1}{9}, \&c.,$$

or

$$\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \&c.$$

(69.) If we give to n the successive values 0, 1, 2, 3, 4, &c., we shall from the preceding formula find the integrals of 1, x , x^2 , x^3 , &c. Thus

$$\begin{aligned} \Sigma \cdot x^0 &= \Sigma(1) = x + C \\ \Sigma \cdot x &= \frac{x^2}{2} - \frac{x}{2} + C \\ \Sigma \cdot x^2 &= \frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{6} + C \\ \Sigma \cdot x^3 &= \frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4} + C \\ \Sigma \cdot x^4 &= \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} - \frac{x}{30} + C \\ \Sigma \cdot x^5 &= \frac{x^6}{6} - \frac{x^5}{3} + \frac{5}{12}x^4 - \frac{x^2}{12} + C \\ \Sigma \cdot x^6 &= \frac{x^7}{7} - \frac{x^6}{2} + \frac{x^5}{2} - \frac{x^3}{6} + \frac{x}{42} + C \\ &+ \&c. \end{aligned}$$

Calculus of
Finite
Differences.

(70.) The utility of such a Table as the preceding is manifest when we have to integrate an algebraical expression consisting of a series of monomials.

Calculus of
Finite
Differences.

Thus, if

$$\begin{aligned} u_x &= x^3 - 3x^2 + 2x + 1 \\ \Sigma u_x &= \Sigma x^3 - 3\Sigma x^2 + 2\Sigma x + \Sigma 1 \\ &= \frac{x^4}{4} - \frac{x^3}{2} + \frac{x^2}{4} - x^3 + \frac{3x^2}{2} - \frac{x}{2} + x^2 - x + x + C \\ &= \frac{x^4}{4} - \frac{3x^3}{2} + \frac{11x^2}{4} - \frac{x}{2}. \end{aligned}$$

(71.) If the $\Sigma(x)$ and $\Sigma(x^2)$ be both considered to commence when $x = 0$, the constant will disappear,

and
$$\Sigma . x^3 = \frac{x^4 - 2x^3 + x^2}{4} = \left(\frac{x^2 - x}{2}\right)^2 = \left(\frac{x^2}{2} - \frac{x}{2}\right)^2 = (\Sigma(x))^2;$$

a remarkable expression, which when translated shows that the sum of the cubes of the natural numbers is equal to the square of the sum of the numbers themselves. Omitting, however, the consideration of particular results, we return to the remark that the integration of the general term x^n gives us a means by which the integration of any algebraical expression may be effected. Still, in some cases, other and more convenient processes may be used by which the integration may be directly performed, and the rule by which it is done is precise and easily to be remembered.

(72.) Thus, whenever it is possible to cause the function to be made equal to the product of a series of factors of the form $a + bx$, $a + b(x + 1)$, &c., we may dispense with the method of the preceding article

For since
$$\begin{aligned} \Delta \{ (a + bx) \cdot (a + b\overline{x+1}) (a + b\overline{x+2}) \dots (a + b\overline{x+n}) \} \\ &= (a + b\overline{x+1}) (a + b\overline{x+2}) \dots (a + b\overline{x+n}) \{ a + b\overline{x+n+1} - a + bx \} \\ &= (a + b\overline{x+1}) (a + b\overline{x+2}) \dots (a + b\overline{x+n}) b\overline{n+1}; \\ \therefore \Sigma \{ (a + b\overline{x+1}) (a + b\overline{x+2}) (a + b\overline{x+3}) \dots (a + b\overline{x+n}) \} &= \frac{(a + bx) (a + b\overline{x+1}) \dots (a + b\overline{x+n})}{b(n+1)}, \end{aligned}$$

or, writing x instead of $(x + 1)$,

$$\Sigma \{ (a + bx) (a + b\overline{x+1}) \dots (a + b\overline{x+n-1}) \} = \frac{(a + b\overline{x-1}) \cdot (a + bx) \dots (a + b\overline{x+n-1})}{b(n+1)};$$

whence we have this rule: To integrate the product of any number of factors which are in arithmetical progression, multiply the function by the term immediately preceding the first of the given factors, and divide this product by the number of terms so increased, and also by the common difference between each term.

Example (1.) Thus to integrate $u_x = (x - 2)(x - 3)(x - 4)$,
since $b = 1$, and $n = 3$;

$$\therefore \Sigma u_x = \frac{(x - 2)(x - 3)(x - 4)(x - 5)}{4} + C.$$

Example (2.) To integrate x^3 by this method, we must first put x^3 into a proper form.

Now

$$\begin{aligned} x^3 &= x \cdot (x^2 - 1) + x = \overline{x-1} \cdot x \cdot \overline{x+1} + x; \\ \therefore \Sigma x^3 &= \frac{\overline{x-2} \cdot \overline{x-1} \cdot x \cdot \overline{x+1}}{4} + \frac{x \cdot \overline{x-1}}{2} + C \\ &= \frac{x \cdot \overline{x-1}}{2} \cdot \left\{ \frac{\overline{x+1} \cdot \overline{x-2}}{2} + 1 \right\} + C = \left(\frac{x \overline{x-1}}{2} \right)^2 + C. \end{aligned}$$

Example (3.) Let

Here

$$u_x = (2x + 1) \cdot (2x + 3) \cdot (2x + 5).$$

$$b = 2 \text{ and } n = 3;$$

$$\therefore \Sigma (2x + 1)(2x + 3)(2x + 5) = \frac{(2x - 1)(2x + 1)(2x + 3) \cdot (2x + 5)}{8} + C.$$

(73.) The only difficulty in the application of this rule is the resolution of the proposed integral function into factorials. The mode of doing this has been shown in a particular case, when $u_x = x^3$, Example (2). A similar method must be applied to other cases, the facility of which depends only upon practice.

The method of indeterminate coefficients may be used as a means of resolving the function into factorials.

Thus, let

$$\begin{aligned} x^3 &= A \cdot \overline{x-1} \cdot x \cdot \overline{x+1} + B \cdot x \cdot \overline{x-1} + C \overline{x-1} + D \\ &= A \cdot (x^3 - x) + B(x^2 - x) + C \cdot x - C + D \\ &= Ax^3 + Bx^2 - (A + B - C)x + D - C; \\ \therefore A &= 1, B = 0, A + C - B = 0; \therefore C = A = 1, D = C = 1; \\ \therefore x^3 &= \overline{x-1} \cdot x \cdot \overline{x+1} + (x - 1) + 1 = \overline{x-1} \cdot x \cdot \overline{x+1} + x. \end{aligned}$$

Again, let

$$2x^2 - 3x + 1 = Ax + Bx + C$$

$$= Ax^2 + (B + A)x + C;$$

$$\therefore A = 2, B + A = -3; \therefore B = -5, C = 1;$$

$$\therefore 2x^2 - 3x + 1 = 2 \cdot x \cdot x + 1 - 5 \cdot x + 1;$$

which is immediately integrable.

(74.) A general expression for x^n consisting of the sums of the products of x and its preceding values, viz., $x-1, x-2, x-3$, &c., may, however, be deduced from the frequently used theorem for u_{x+n} .

For (n) put y , and we have

$$u_{x+y} = u_y + x \cdot \Delta \cdot u_y + \frac{x \cdot x - 1}{2} \Delta^2 \cdot u_y + \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \Delta^3 \cdot u_y \\ + \frac{x \cdot x - 1 \cdot x - 2 \cdot x - 3}{1 \cdot 2 \cdot 3 \cdot 4} \Delta^4 \cdot u_y + \&c.$$

Let

$$u_y = y^n; \therefore u_{x+y} = (x+y)^n;$$

$$\therefore (x+y)^n = y^n + x \Delta y^n + \frac{x \cdot x - 1}{1 \cdot 2} \Delta^2 \cdot y^n + \frac{x \cdot x - 1 \cdot x - 2}{1 \cdot 2 \cdot 3} \Delta^3 \cdot y^n + \&c.$$

Make $y = 0$; $\therefore y^n = 0$, and we have

$$x^n = \frac{\Delta \cdot 0^n}{1} x + \frac{\Delta^2 \cdot 0^n}{1 \cdot 2} x \cdot x - 1 + \frac{\Delta^3 \cdot 0^n}{1 \cdot 2 \cdot 3} x \cdot x - 1 \cdot x - 2 + \frac{\Delta^4 \cdot 0^n}{2 \cdot 3 \cdot 4} x \cdot x - 1 \cdot x - 2 \cdot x - 3 + \&c.$$

The values of $\frac{\Delta \cdot 0^n}{1}, \frac{\Delta^2 \cdot 0^n}{1 \cdot 2}, \frac{\Delta^3 \cdot 0^n}{2 \cdot 3}$, &c., may be computed by the Table given in Article (33.)

(75.) But if it be required to resolve x^n into products of the successive factors of x , viz., $(x+1), (x+2)$, &c., write $-x$ for x , and let $u_y = y^{n+1}$;

$$\therefore (y-x)^{n+1} = y^{n+1} - x \cdot \Delta y^{n+1} + \frac{x \cdot x + 1}{1 \cdot 2} \Delta^2 y^{n+1} - \frac{x \cdot x + 1 \cdot x + 2}{2 \cdot 3} \Delta^3 y^{n+1} + \&c.$$

Make $y = 0$, and divide both sides by x :

$$(-1)^{n+1} x^n = - \left\{ \Delta \cdot 0^{n+1} - \frac{\Delta^2 \cdot 0^{n+1}}{2} x + 1 + \frac{\Delta^3 \cdot 0^{n+1}}{2 \cdot 3} x + 1 \cdot x + 2 - \&c. \right\};$$

$$\therefore x^n = (-1)^n \left\{ \Delta \cdot 0^{n+1} - \frac{\Delta^2 \cdot 0^{n+1}}{2} x + 1 + \frac{\Delta^3 \cdot 0^{n+1}}{2 \cdot 3} \cdot x + 1 \cdot x + 2 - \&c. \right\};$$

a general formula when the values of $\Delta \cdot 0^{n+1}, \frac{\Delta^2 \cdot 0^{n+1}}{2}$, &c. are known.

(76.) The following method of resolving x^{n+1} into the sum of the products of coefficients which have a constant difference is too remarkable for its elegance to be omitted here.

Let

$$x^2 = v; \therefore x^{n+1} = x \cdot v^n.$$

Now assume that

$$v^n = A_0 + A_1(v-1) + A_2 \cdot (v-1)(v-4) + A_3 \cdot (v-1)(v-4)(v-9) \\ + \&c. + A_n(v-1)(v-4)(v-9) \dots (v-n^2):$$

an assumption which is possible, since if the factors be multiplied out, the highest power of v is v^n , and there will be $n+1$ equations to determine the $n+1$ constants A_0, A_1, A_2, A_3 , &c. A_n .

To determine these, let v be successively made $= 1, 4, 9$, &c., or $1^2, 2^2, 3^2$, &c.;

$$\therefore 1^{2n} = A_0$$

$$2^{2n} = A_0 + A_1(2^2 - 1^2)$$

$$3^{2n} = A_0 + A_1(3^2 - 1^2) + A_2(3^2 - 1^2)(3^2 - 2^2)$$

$$4^{2n} = A_0 + A_1(4^2 - 1^2) + A_2(4^2 - 1^2) \cdot (4^2 - 2^2) + A_3(4^2 - 1^2)(4^2 - 2^2)(4^2 - 3^2) + \&c.;$$

whence

$$A_0 = 1^{2n}$$

$$A_1 = \frac{2^{2n}}{2^2 - 1^2} + \frac{1^{2n}}{1^2 - 2^2}$$

$$A_2 = \frac{3^{2n}}{(3^2 - 1^2)(3^2 - 2^2)} + \frac{2^{2n}}{(2^2 - 1^2) \cdot (2^2 - 3^2)} + \frac{1^{2n}}{(1^2 - 2^2)(1^2 - 3^2)},$$

&c.

and thus knowing A_0, A_1, A_2 , &c., (the law of which terms is evident),

$$x^{n+1} = x \{ A_0 + A_1(x^2 - 1) + A_2(x^2 - 1)(x^2 - 4) + A_3(x^2 - 1)(x^2 - 4)(x^2 - 9) + \&c. \}$$

$$= A_0 x + A_1 x \cdot x - 1 \cdot x + 1 + A_2 x \cdot x - 2 \cdot x - 1 \cdot x \cdot x + 1 \cdot x + 2$$

$$+ A_3 \cdot x \cdot x - 3 \cdot x - 2 \cdot x - 1 \cdot x \cdot x + 1 \cdot x + 2 \cdot x + 3 + \&c.$$

Calculus of
Finite
Differences.

(77.) Again, if any of the factorials in a given product be wanting, the deficient factors may be restored, and the expression rendered fit for integration, as the following examples will show.

Calculus of
Finite
Differences.

Let $u_x = x(x+1)(x+3)$, and let it be required to complete the factorials.

Then

$$u_x = x \cdot \overline{x+1} \cdot (\overline{x+2} + 1) = x \cdot \overline{x+1} \cdot \overline{x+2} + x \cdot \overline{x+1}.$$

Again, let

$$\begin{aligned} u_x &= x \overline{x+1} \overline{x+2} \overline{x+4} \overline{x+5} \\ &= x \overline{x+1} \overline{x+2} \cdot (\overline{x+3} + 1) (\overline{x+5}) \\ &= x \overline{x+1} \overline{x+2} \overline{x+3} \overline{x+5} + x \cdot \overline{x+1} \overline{x+2} \cdot \overline{x+5} \\ &= x \overline{x+1} \cdot \overline{x+2} \overline{x+3} \overline{x+4} + x \overline{x+1} \overline{x+2} \cdot \overline{x+3} + x \overline{x+1} \overline{x+2} \overline{x+3} + 2x \cdot \overline{x+1} \overline{x+2} \\ &= x \overline{x+1} \overline{x+2} \cdot \overline{x+3} \cdot \overline{x+4} + 2x \overline{x+1} \overline{x+2} \overline{x+3} + 2 \cdot x \overline{x+1} \overline{x+2}. \end{aligned}$$

As another example, let

$$\begin{aligned} u_x &= (2x-1) \cdot (2x+1) (2x+5) \\ &= (2x-1) (2x+1) (2+2x+3) \\ &= 2 \cdot (2x-1) (2x+1) + (2x-1) (2x+1) (2x+3). \end{aligned}$$

These examples will be sufficient to show the manner in which the completion of the factorials is to be performed.

(78.) The next case to be considered is that of rational fractions: of these but few can be integrated.

The integral can, however, always be found when the function u_x is of the form $a + bx$, or

$$\frac{1}{u_x \cdot u_{x+1} \cdot u_{x+2} \dots u_{x+n}} = \frac{1}{(a+bx)(a+b\overline{x+1})(a+b\overline{x+2}) \dots (a+b\overline{x+n})}.$$

$$\begin{aligned} \text{For } \Delta \cdot \frac{1}{(a+bx) \dots (a+b\overline{x+n})} &= \frac{(a+bx) - (a+b\overline{x+n+1})}{(a+bx)(a+b\overline{x+1}) \dots (a+b\overline{x+n+1})} \\ &= \frac{-n+1 \cdot b}{(a+bx)(a+b\overline{x+1}) \dots (a+b\overline{x+n+1})}; \end{aligned}$$

$$\therefore \Sigma \cdot \frac{1}{(a+bx)(a+b\overline{x+1}) \dots (a+b\overline{x+n})} = C - \frac{1}{b \cdot n+1} \cdot \frac{1}{(a+bx) \dots (a+b\overline{x+n})}.$$

Whence this rule: To integrate a fraction, of which the denominator is composed of factors in arithmetic progression, and of which the numerator is constant. Change the algebraical sign of the fraction, efface the last factor, divide the result by the number of terms remaining and by the common difference, and add a constant.

Thus

$$\Sigma \cdot \frac{1}{x \cdot \overline{x+1} \overline{x+2} \overline{x+3}} = C - \frac{1}{3 \cdot x \cdot \overline{x+1} \overline{x+2}},$$

and

$$\Sigma \cdot \frac{1}{(2x-1)(2x+1)(2x+3)} = C - \frac{1}{4(2x-1)(2x+1)}.$$

(79.) If any of the terms of the arithmetic progression be wanting, their place must be supplied by multiplying the numerator and denominator by them, and reducing the fraction to others which have constant numerators.

Thus to integrate $\frac{1}{x+1 \overline{x+2} \overline{x+4}}$,

$$\begin{aligned} \frac{1}{x+1 \overline{x+2} \overline{x+4}} &= \frac{x+3}{x+1 \overline{x+2} \dots \overline{x+4}} = \frac{\overline{x+4} - 1}{x+1 \cdot \overline{x+2} \dots \overline{x+4}} \\ &= \frac{1}{x+1 \overline{x+2} \overline{x+3}} - \frac{1}{x+1 \overline{x+2} \dots \overline{x+4}}; \end{aligned}$$

$$\therefore \Sigma \frac{1}{x+1 \overline{x+2} \overline{x+4}} = C - \frac{1}{2 \cdot x+1 \overline{x+2}} + \frac{1}{3} \cdot \frac{1}{x+1 \overline{x+2} \overline{x+3}}.$$

(80.) And, as a general method, had we to integrate

$$\frac{Ax^n + Bx^{n-1} + Cx^{n-2} + \&c. + M}{(a+bx) \cdot (a+b\overline{x+1}) \dots (a+b\overline{x+n+1})},$$

we may assume that

$$Ax^n + Bx^{n-1} + Cx^{n-2} + \&c. = \alpha + \beta(a+bx) + \gamma(a+bx)(a+b\overline{x+1}) + \&c.;$$

and by multiplying out the terms of the equation on the right hand, there will arise, from equating the similar

Calculus of Finite Differences. powers of x , $(n+1)$ equations, by which the $n+1$ quantities α, β, γ , &c. may be found, and the fraction will be resolved into the following fractions, each of which may be integrated:

Calculus of Finite Differences.

$$\frac{\alpha}{(a+bx)(a+bx+1)\dots(a+bx+n+1)} + \frac{\beta}{(a+b \cdot x+1)\dots(a+bx+n+1)} + \frac{\gamma}{(a+bx+2)\dots(a+bx+n+1)} + \&c.$$

(81.) Next to integrate the function $u_x = a^x$.

Since

$$\Delta \cdot a^x = a^{x+1} - a^x = a^x \cdot (a-1);$$

$$\therefore a^x = \frac{\Delta \cdot a^x}{a-1} \text{ and } \Sigma \cdot a^x = \frac{a^x}{a-1} + C.$$

Also if

$$u_x = a^{bx}; \therefore \Delta u_x = a^{bx} \cdot (a^b - 1); \therefore \Sigma \cdot a^{bx} = \frac{a^{bx}}{a^b - 1} + C.$$

(82.) And since

$$\sin x\theta = \frac{e^{x\theta\sqrt{-1}} - e^{-x\theta\sqrt{-1}}}{\sqrt{-1}} \text{ and } \cos x\theta = \frac{e^{x\theta\sqrt{-1}} + e^{-x\theta\sqrt{-1}}}{2},$$

it is manifest we may integrate $\sin x\theta$ and $\cos x\theta$ by the formula for the integral of a^{bx} . The following method is, however, more simple:

$$\therefore \Delta \cdot \cos x\theta = \cos \overline{x+1}\theta - \cos x\theta = -2 \sin \frac{\theta}{2} \cdot \sin \left(\frac{2x+1}{2} \right) \cdot \theta;$$

$$\therefore \Sigma \cdot \sin \left(\frac{2x+1}{2} \right) \theta = C - \frac{\cos x\theta}{2 \sin \frac{\theta}{2}};$$

or, writing x instead of $\frac{2x+1}{2}$,

$$\therefore \Sigma \cdot \sin x\theta = C - \frac{\cos \left(\frac{2x-1}{2} \right) \cdot \theta}{2 \sin \frac{\theta}{2}}.$$

Again

$$\therefore \Delta \cdot \sin x\theta = \sin \overline{x+1}\theta - \sin x\theta = 2 \sin \frac{\theta}{2} \cdot \cos \frac{2x+1}{2} \theta;$$

$$\therefore \Sigma \cdot \cos \left(\frac{2x+1}{2} \right) \theta = C + \frac{\sin x\theta}{2 \sin \frac{\theta}{2}},$$

or

$$\Sigma \cdot \cos x\theta = C + \frac{\sin \left(\frac{2x+1}{2} \right) \theta}{2 \sin \frac{\theta}{2}}.$$

Also, since $\sin^m x\theta$ and $\cos^m x\theta$ may be expressed in terms of the sines and cosines of the multiples of $x\theta$, their integration may be made to depend upon that of $\sin x\theta$ and $\cos x\theta$; but had we to integrate $a^x \cdot \cos x\theta$ or $a^x \sin x\theta$, it would be better to use the exponential values for the $\sin x\theta$ and $\cos x\theta$, as we will show in the next article.

(83.) Integrate $a^x \cdot \cos x\theta$

$$a^x \cos x\theta = a^x \cdot \left(\frac{e^{x\theta\sqrt{-1}} + e^{-x\theta\sqrt{-1}}}{2} \right) = \frac{1}{2} \{ a^x e^{x\phi} + a^x e^{-x\phi} \}, \text{ where } \phi = \theta \sqrt{-1}.$$

Now

$$\Delta (a^x \cdot e^{x\phi}) = a^{x+1} \cdot e^{\overline{x+1}\phi} - a^x e^{x\phi} = a^x e^{x\phi} \cdot \{ a e^{\phi} - 1 \};$$

$$\therefore a^x e^{x\phi} = \frac{\Delta \cdot (a^x e^{x\phi})}{a e^{\phi} - 1}; \therefore \Sigma (a^x e^{x\phi}) = C + \frac{a^x \cdot e^{x\phi}}{a e^{\phi} - 1};$$

and putting $-\phi$ for ϕ ,

$$\Sigma (a^x e^{-x\phi}) = C + \frac{a^x e^{-x\phi}}{a e^{-\phi} - 1};$$

$$\therefore \Sigma (a^x \cdot \cos x\theta) = C + \frac{1}{e} \cdot \left\{ \frac{a^x \cdot e^{x\phi}}{a e^{\phi} - 1} + \frac{a^x e^{-x\phi}}{a e^{-\phi} - 1} \right\}$$

$$= C + \frac{1}{2} \cdot \left\{ \frac{a^{x+1} \cdot (e^{\overline{x-1} \cdot \phi} + e^{\overline{x-1} \cdot \psi}) - a^x \cdot (e^{\overline{x} \cdot \phi} + e^{\overline{x} \cdot \psi})}{a^2 - a(e^{\overline{x} \cdot \phi} + e^{\overline{x} \cdot \psi}) + 1} \right\}$$

$$= C + \frac{a^{x+1} \cdot \cos x - 1 \theta - a^x \cdot \cos x \theta}{a^2 - 2 a \cos \theta + 1}.$$

In a similar manner may $a^x \sin x \theta$ be integrated.

$$(84.) \text{ Again, since } \Delta(u_x u_{x+1} \dots u_{x+n}) = u_{x+1} u_{x+2} \dots u_{x+n} \cdot (u_{x+n+1} - u_x).$$

$$\text{Let } u_x = p \cdot a^x + q; \therefore u_{x+n+1} - u_x = p \cdot (a^{x+n+1} - a^x) = p \cdot a^x (a^{n+1} - 1);$$

$$\therefore \Delta \{ (p \cdot a^x + q) (p \cdot a^{x+1} + q) \dots (p \cdot a^{x+n} + q) \} = (p \cdot a^{x+1} + q) \dots (p \cdot a^{x+n} + q) p \cdot a^x \cdot (a^{n+1} - 1);$$

$$\therefore \Sigma a^x \cdot (p a^{x+1} + q) \cdot (p a^{x+2} + q) \dots (p a^{x+n} + q) = \frac{(p a^x + q) \cdot (p a^{x+1} + q) \dots (p a^{x+n} + q)}{p (a^{n+1} - 1)} + C.$$

For x put $x-1$; and $\therefore a^{x-1} = \frac{a^x}{a}$.

$$\therefore \Sigma a^x \cdot p a^x + q \cdot p a^{x+1} + q \dots p a^{x+n-1} + q = a \cdot \frac{p a^{x-1} + q \dots p a^{x+n-1} + q}{p (a^{n+1} - 1)} + C.$$

$$(85.) \text{ Again, since } \Delta \cdot \frac{1}{u_x \cdot u_{x+1} \dots u_{x+n-1}} = \frac{-(u_{x+n} - u_x)}{u_x u_{x+1} \dots u_{x+n}}.$$

If $u_x = p a^x + q$;

$$\therefore \Delta \cdot \frac{1}{(p \cdot a^x + q) \dots (p a^{x+n-1} + q)} = \frac{-p a^x \cdot (a^n - 1)}{(p \cdot a^x + q) \dots (p a^{x+n} + q)};$$

and then writing $n-1$ for n , we have

$$\Sigma \frac{a^x}{(p a^x + q) \dots (p a^{x+n-1} + q)} = C - \frac{1}{p (a^{n-1} - 1) (p a^x + q) \dots (p a^{x+n-2} + q)}.$$

The utility of these expressions will be apparent when the results of these integrations are applied to the summation of series.

(86.) Since

$$\Delta(u_x v_x) = u_x \Delta v_x + v_{x+1} \Delta u_x;$$

$$\therefore u_x \Delta v_x = \Delta(u_x v_x) - v_{x+1} \Delta u_x;$$

$$\therefore \Sigma(u_x \cdot \Delta v_x) = u_x v_x - \Sigma(v_{x+1} \Delta u_x),$$

a formula which is analogous to $\int y dx = yx - \int x dy$ used in the Integral Calculus.

Now, writing v_x for $\Delta \cdot v_x$, and $\therefore \Sigma v_x$ for v_x ,

$$\Sigma(u_x v_x) = u_x \Sigma v_x - \Sigma(\Delta u_x \Sigma v_{x+1})$$

$$= u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Sigma(\Delta^2 u_x \Sigma^2 v_{x+2})$$

$$= u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Delta^2 u_x \Sigma^3 v_{x+2} - \Delta^3 u_x \Sigma^4 v_{x+3} + \&c.$$

$$\pm \Delta^n u_x \Sigma^{n+1} v_{x+n+1} \mp \Sigma(\Delta^{n+1} u_x \cdot \Sigma^{n+1} v_{x+n+1}),$$

where n may be any integer.

If u_x be a rational function of x , then $\Delta^n u_x$ is constant, and $\Delta^{n+1} u_x = \text{zero}$, and we have

$$\Sigma(u_x v_x) = u_x \Sigma v_x - \Delta u_x \Sigma^2 v_{x+1} + \Delta^2 u_x \Sigma^3 v_{x+2} - \Delta^3 u_x \Sigma^4 v_{x+3} + \&c. \pm \Delta^n \cdot u_x \cdot \Sigma^{n+1} v_{x+n+1}$$

(87.) As an example, let $u_x = x^2$, and $v_x = a^x$.

$$\therefore \Delta u_x = 2x + 1, \Delta^2 u_x = 2, \text{ and } \Delta^3 u_x = 0$$

$$\Sigma v_x = \frac{a^x}{a-1}, \Sigma v_{x+1} = \frac{a^{x+1}}{a-1}, \Sigma^2 v_{x+1} = \frac{a^{x+1}}{(a-1)^2}$$

$$\Sigma^3 v_{x+2} = \frac{a^{x+2}}{(a-1)^3};$$

$$\therefore \Sigma(x^2 \cdot a^x) = \frac{x^2 \cdot a^x}{a-1} - (2x+1) \cdot \frac{a^{x+1}}{(a-1)^2} + 2 \frac{a^{x+2}}{(a-1)^3} + C$$

$$= a^x \left\{ \frac{x^2}{a-1} - \frac{2ax}{(a-1)^2} + \frac{a^2+a}{(a-1)^3} \right\} + C.$$

(88.) Next to obtain general formulæ for $\Sigma^2(u_x v_x)$, $\Sigma^3(u_x v_x)$, &c., and finally $\Sigma^n(u_x v_x)$, we must integrate successively the formula for $\Sigma(u_x v_x)$.

$$\text{And, first, } \Sigma^2(u_x v_x) = \Sigma(u_x \Sigma v_x) - \Sigma(\Delta u_x \cdot \Sigma^2 v_{x+1}) + \&c. \pm \Sigma(\Delta^n u_x \Sigma^{n+1} v_{x+n+1})$$

$$\mp \Sigma^2(\Delta^{n+1} u_x \cdot \Sigma^{n+1} v_{x+n+1}).$$

In this formula we must separately find the terms on the right-hand side of the equation by means of the formula for $\Sigma(u_x v_x)$.

First, for $\Sigma(u_x \Sigma v_x)$; writing Σv_x instead of v_x , we get

$$\Sigma(u_x \Sigma v_x) = u_x \Sigma^2 v_x - \Delta u_x \cdot \Sigma^3 v_{x+1} + \Delta^2 u_x \Sigma^4 v_{x+2} - \&c.$$

$$\pm \Delta^n u_x \Sigma^{n+2} v_{x+n} \mp \Sigma(\Delta^{n+1} u_x \cdot \Sigma^{n+2} v_{x+n+1}).$$

Again, for $\Sigma(\Delta u_x \cdot \Sigma^2 v_{x+1})$; writing Δu_x for u_x , $\Sigma^2 v_{x+1}$ for v_x , and $n-1$ for n ,
 $\Sigma(\Delta u_x \cdot \Sigma^2 v_{x+1}) = \Delta u_x \Sigma^3 v_{x+1} - \&c. \mp \Delta^n u_x \Sigma^{n+2} v_{x+n}$
 $\pm \Sigma(\Delta^{n+1} u_x \cdot \Sigma^{n+2} v_{x+n+1});$

and thus

$$\Sigma(\Delta^n u_x \cdot \Sigma^{n+1} v_{x+n}) = \Delta^n u_x \Sigma^{n+2} v_{x+n} - \Sigma(\Delta^{n+1} u_x \cdot \Sigma^{n+2} v_{x+n+1}).$$

Substituting these values in the expression we have

$$\Sigma^2(u_x v_x) = u_x \cdot \Sigma^2 v_x - 2 \Delta u_x \Sigma^3 v_{x+1} + 3 \Delta^2 u_x \Sigma^4 v_{x+2} - 4 \Delta^3 u_x \Sigma^5 v_{x+3} + \&c.$$

$$\pm (n+1) \Delta^n u_x \cdot \Sigma^{n+2} v_{x+n}$$

$$\mp n+1 \Sigma(\Delta^{n+1} u_x \cdot \Sigma^{n+2} v_{x+n+1}) \mp \Sigma^2(\Delta^{n+1} u_x \Sigma^{n+1} v_{x+n+1}).$$

(89.) If this equation be again integrated by a method similar to that just adopted, we shall get

$$\Sigma^3(u_x v_x) = u_x \Sigma^3 v_x - 3 \Delta u_x \Sigma^4 v_{x+1} + 6 \Delta^2 u_x \Sigma^5 v_{x+2} - \&c.$$

$$\pm \frac{n+1}{1 \cdot 2} \Delta^n u_x \Sigma^{n+3} v_{x+n}$$

$$\mp \frac{n+1}{1 \cdot 2} \Sigma(\Delta^{n+1} u_x \cdot \Sigma^{n+3} v_{x+n+1})$$

$$\mp \frac{n+1}{1} \Sigma^2(\Delta^{n+1} u_x \cdot \Sigma^{n+2} v_{x+n+1}) \mp \Sigma^3(\Delta^{n+1} u_x \cdot \Sigma^{n+1} v_{x+n+1}).$$

And finally, we have

$$\Sigma^m(u_x v_x) = u_x \Sigma^m v_x - m \Delta u_x \Sigma^{m+1} v_{x+1} + \frac{m(m+1)}{1 \cdot 2} \Delta^2 u_x \Sigma^{m+2} v_{x+2} - \&c.$$

$$\pm \frac{n+1}{1 \cdot 2 \dots (m-1)} \Delta^n u_x \Sigma^{n+m} v_{x+n}$$

$$\mp \left\{ \begin{array}{l} \Sigma^m(\Delta^{n+1} u_x \cdot \Sigma^{n+1} v_{x+n+1}) + n+1 \cdot \Sigma^{m-1}(\Delta^{n+1} u_x \cdot \Sigma^{n+2} v_{x+n+1}) \\ + \&c. + \frac{(n+1)(n+2) \dots (n+m-1)}{1 \cdot 2 \dots (m-1)} \cdot \Sigma(\Delta^{n+1} u_x \cdot \Sigma^{n+m} v_{x+n+1}) \end{array} \right\}.$$

It will be observed that the law of the coefficients of this series (first given by Taylor) follows that of the coefficients of the binomial $(1+x)^{-n}$.

For

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2} x^2 - \frac{n(n+1)(n+2)}{1 \cdot 2 \cdot 3} x^3 + \&c.$$

$$\pm \frac{n(n+1)(n+2) \dots (n+m-1)}{1 \cdot 2 \cdot 3 \dots m} x^m \mp \&c.$$

(90.) Returning to the theorem for $\Sigma(u_x v_x)$, and putting instead of $v_{x+1}, v_{x+2} \dots v_{x+n}$ their values in terms of v_x , we get

$$\Sigma(u_x v_x) = u_x \Sigma v_x - \Delta u_x (\Sigma^2 v_x + \Sigma v_x) + \Delta^2 u_x (\Sigma^3 v_x + 2 \Sigma^2 v_x + \Sigma v_x)$$

$$- \Delta^3 u_x (\Sigma^4 v_x + 3 \Sigma^3 v_x + 3 \Sigma^2 v_x + \Sigma v_x) + \&c.;$$

the form under which the theorem was given by Condorcet in his application of analysis to questions of Probabilities.

(91.) We may now classify those functions which it will be possible to integrate.

In the first place, if u_x be a rational function of x , such that $u_x = Ax^m + Bx^{m-1} + Cx^{m-2} + \&c. + L$, then $\Delta^m u_x = m(m-1)(m-2) \dots 2 \cdot 1 \cdot A$, and $\Delta^{m+1} u_x = 0$; and therefore the formula for $\Sigma(u_x v_x)$ will terminate after a given number of terms. And we have seen that if v_x be a fraction whose denominator is a set of factorials in arithmetic progression and numerator constant, that its integral is at once to be found.

Again, if v_x be either an exponential function of x , or a function of the sines and cosines, Σv_x may be found.

Hence we may integrate

1. Every function of the form

$$\frac{Ax^\alpha + Bx^\beta + Cx^\gamma + \&c. + L}{(a+bx)(a+bx+1) \dots (a+bx+n-1)},$$

when the index of x is less than $n-1$; for the fraction

$$\frac{1}{(a+bx) \cdot (a+bx+1) \dots (a+bx+n-1)}$$

can only be integrated $n-1$ times, since the number of the factors of the denominator diminish by unity at each integration, and we cannot integrate when there is but one factor remaining.

Calculus of
Finite
Differences.

And 2. Every function of the form

$$a^{mx} \cdot (A x^\alpha + B x^\beta + C x^\gamma + \&c.)$$

may be integrated, whatever be the value of m ; α , β , &c. being integers.

And 3. Every function of the form

$$(\sin x)^m \cdot (\cos x)^n \{A x^\alpha + B x^\beta + C x^\gamma + \&c.\},$$

m and n being positive, and $\overline{\cos x}^n \cdot \overline{\sin x}^m$ being changed into expressions involving the sines and cosines of the multiple arcs.

(92.) These three cases include almost all the compound functions to which the inverse method of differences can be at once applied, and they merit particular attention both on that account and also since they lead to the summation of a great number of series.

The application of the formula for reduction may be shown in the annexed example.

Integrate $x^2 \cdot \cos x \theta$.

Here

$$u_x = x^2, \Delta u_x = 2x + 1, \Delta^2 u_x = 2, \Delta^3 u_x = 0,$$

$$v_x = \cos x \theta, v_{x+1} = \cos x + 1 \theta, v_{x+2} = \cos x + 2 \theta;$$

$$\therefore \Sigma (x^2 \cos x \theta) = x^2 \cdot \Sigma (\cos x \theta) - (2x + 1) \Sigma^2 \cdot \cos x + 1 \theta + 2 \Sigma^3 \cdot \cos x + 2 \theta;$$

and as $\Sigma \cdot \cos x \theta$ and $\Sigma \cdot \sin x \theta$ have already been found, we can deduce from the results thus known the values of $\Sigma^2 \cos x + 1 \theta$ and $\Sigma^3 \cdot \cos x + 2 \theta$, and thus $\Sigma (x^2 \cos x \theta)$ will be determined; or we may integrate it by

putting for $\cos x \theta$ its value $\frac{e^{x\theta\sqrt{-1}} + e^{-x\theta\sqrt{-1}}}{2}$, or writing ϕ for $\theta \sqrt{-1}$, by putting $\cos x \theta = \frac{1}{2} (e^{x\phi} + e^{-x\phi})$;

$$\therefore \Sigma (x^2 \cos x \theta) = \frac{1}{2} \{ \Sigma \cdot x^2 e^{x\phi} + \Sigma \cdot x^2 \cdot e^{-x\phi} \}$$

$$= \frac{1}{2} \{ x^2 \Sigma (e^{x\phi} + e^{-x\phi}) - (2x + 1) \Sigma^2 \cdot (e^{x\phi} + e^{-x\phi}) + 2 \Sigma^3 (e^{x\phi} + e^{-x\phi}) \},$$

which may be found without difficulty.

(93.) The integration of the general function u_x is of the greatest importance in the summation of series, and necessarily attracted the attention of mathematicians. Euler has shown how it may be made to depend upon the differential coefficients of u_x , and on the integral $\int u_x dx$. We may arrive at the result by the following process.

Since $u_{x+1} = f(x+1)$, where $u_x = f(x)$, we have, by Taylor's theorem,

$$\Delta \cdot u_x = \frac{d \cdot u_x}{dx} + \frac{d^2 u_x}{dx^2} \frac{1}{1 \cdot 2} + \frac{d^3 u_x}{dx^3} \frac{1}{2 \cdot 3} + \frac{d^4 u_x}{dx^4} \frac{1}{2 \cdot 3 \cdot 4} + \&c.;$$

or, writing $\Sigma (u_x)$ for u_x , and consequently u_x for $\Delta \cdot u_x$,

$$u_x = \frac{d \cdot \Sigma u_x}{dx} + \frac{d^2 \Sigma u_x}{dx^2} \frac{1}{1 \cdot 2} + \frac{d^3 \cdot \Sigma u_x}{dx^3} \frac{1}{2 \cdot 3} + \&c.$$

Now assume that

$$\Sigma \cdot (u_x) = A \int u_x dx + B u_x + C \cdot \frac{d u_x}{dx} + D \cdot \frac{d^2 u_x}{dx^2} + E \cdot \frac{d^3 u_x}{dx^3} + \&c.,$$

where A , B , C , &c. are to be determined. A little reflection will show the justness of this assumption.

$$\therefore \frac{d \cdot \Sigma \cdot u_x}{dx} = A u_x + B \cdot \frac{d u_x}{dx} + C \cdot \frac{d^2 u_x}{dx^2} + D \cdot \frac{d^3 u_x}{dx^3} + \&c.$$

$$\frac{d^2 \Sigma u_x}{dx^2} = A \cdot \frac{d u_x}{dx} + B \cdot \frac{d^2 u_x}{dx^2} + C \cdot \frac{d^3 u_x}{dx^3} + \&c.$$

$$\frac{d^3 \Sigma u_x}{dx^3} = A \cdot \frac{d^2 u_x}{dx^2} + B \cdot \frac{d^3 u_x}{dx^3} + \&c.$$

$$\frac{d^4 \cdot \Sigma u_x}{dx^4} = A \cdot \frac{d^3 u_x}{dx^3} + \&c.$$

$$\&c. = + \&c. + \&c.$$

$\therefore$, substituting

$$u_x = A u_x + B \frac{d u_x}{dx} + C \cdot \frac{d^2 u_x}{dx^2} + D \frac{d^3 u_x}{dx^3} + E \cdot \frac{d^4 u_x}{dx^4} + \&c.$$

$$+ \frac{A}{1 \cdot 2} \cdot \frac{d \cdot u_x}{dx} + \frac{B}{1 \cdot 2} \cdot \frac{d^2 u_x}{dx^2} + \frac{C}{1 \cdot 2} \cdot \frac{d^3 u_x}{dx^3} + \frac{D}{1 \cdot 2} \frac{d^4 u_x}{dx^4} + \&c.$$

$$+ \frac{A}{2 \cdot 3} \cdot \frac{d^2 u_x}{dx^2} + \frac{B}{2 \cdot 3} \cdot \frac{d^3 u_x}{dx^3} + \frac{C}{2 \cdot 3} \cdot \frac{d^4 u_x}{dx^4} + \&c.$$

$$+ \frac{A}{2 \cdot 3 \cdot 4} \cdot \frac{d^3 u_x}{dx^3} + \frac{B}{2 \cdot 3 \cdot 4} \frac{d^4 u_x}{dx^4} + \&c.$$

$$+ \frac{A}{2 \cdot 3 \cdot 4 \cdot 5} \frac{d^4 u_x}{dx^4} + \&c.$$

Equating the coefficients of u_x , and putting the coefficients of the other terms equal to zero, we obtain

$$A = 1$$

$$B + \frac{A}{2} = 0$$

$$\therefore B = -\frac{1}{2}.$$

$$C + \frac{B}{2} + \frac{A}{2 \cdot 3} = 0$$

$$\therefore C = \frac{1}{4} - \frac{1}{2 \cdot 3} = \frac{1}{2} \cdot \frac{1}{2 \cdot 3}$$

$$D + \frac{C}{2} + \frac{B}{2 \cdot 3} + \frac{A}{2 \cdot 3 \cdot 4} = 0$$

$$\therefore D = -\frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 4} = 0$$

$$E + \frac{C}{2 \cdot 3} + \frac{B}{2 \cdot 3 \cdot 4} + \frac{A}{2 \cdot 3 \cdot 4 \cdot 5} = 0$$

$$\therefore E = \frac{1}{2 \cdot 3} \cdot \left\{ -\frac{1}{3 \cdot 4} + \frac{1}{2 \cdot 4} - \frac{1}{4 \cdot 5} \right\} = -\frac{1}{2 \cdot 3} \cdot \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}$$

&c.

= &c.;

$$\therefore \Sigma u_x = \int u_x dx - \frac{1}{2} u_x + \frac{1}{2} \cdot \frac{1}{2 \cdot 3} \cdot \frac{d \cdot u_x}{dx} - \frac{1}{6} \cdot \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{d^3 u_x}{dx^3} + \&c.$$

We can immediately from this series arrive at the expression for Σx^m , which has been made the subject of a former article.

For if

$$u_x = x^m; \therefore \frac{d u_x}{dx} = m x^{m-1}, \frac{d^3 u_x}{dx^3} = m \cdot \overline{m-1} \cdot \overline{m-2} \cdot x^{m-3} + \&c.;$$

$$\therefore \Sigma x^m = \frac{x^{m+1}}{m+1} - \frac{1}{2} \cdot x^m + \frac{1}{2} \cdot \frac{m \cdot x^{m-1}}{2 \cdot 3} - \frac{1}{6} \cdot \frac{m \cdot \overline{m-1} \cdot \overline{m-2}}{2 \cdot 3 \cdot 4 \cdot 5} x^{m-3} + \&c.$$

But we hasten, from particular examples of this formula, to explain an extension of our symbolical notation.

(94.) The formula of Lagrange, $\Delta^n u_x = (e^{\frac{d}{dx}} - 1)^n u_x$, which has been proved to be true for positive values of n , may be shown true when n is negative, and thence we may infer the truth of the equation $\Sigma^n u_x = (e^{\frac{d}{dx}} - 1)^{-n} u_x$.

For if we have any expression of differences, as

$$A \Delta^m u_x + B \Delta^n u_x + C \Delta^p u_x + \&c.,$$

and it be integrated, or the symbol Σ be placed before it, the result will be $A \Delta^{m-1} u_x + B \Delta^{n-1} u_x + C \Delta^{p-1} u_x + \&c.$, i. e. the symbols Σ and Δ^{-1} are identical.

Again, if we integrate the expression a second time, or place before the original expression the symbol Σ^2 , the result will be of the form $A \cdot \Delta^{m-2} u_x + B \cdot \Delta^{n-2} u_x + C \cdot \Delta^{p-2} u_x + \&c.$, and thus it appears that Σ^2 and Δ^{-2} are equivalent; and thus Σ^n and Δ^{-n} are equivalent; or, since $\Delta^n = \frac{1}{\Delta^{-n}} = \frac{1}{\Sigma^n}$, it is evident, and Σ^n and Δ^n are inverse expressions, and when applied to the same quantity neutralize each other, and hence $\Sigma^n \Delta^n \cdot u_x = u_x$, the constants which arise from the integrations being omitted.

Hence, also, it appears that whenever in any expansion of the functions $f\left(\frac{d}{dx}\right)$ or $f(\Delta)$ we meet with terms of the form $\left(\frac{d}{dx}\right)^{-n}$ or Δ^{-n} , we may change these symbols into $\int^n$ and Σ^n .

Now assuming the truth of the equation

$$\Delta^n u_x = (e^{\frac{d}{dx}} - 1)^n u_x.$$

Let it be integrated (n) times;

$$\therefore \Sigma^n \cdot \Delta^n \cdot u_x = \Sigma^n \cdot (e^{\frac{d}{dx}} - 1)^n u_x.$$

But

$$\Sigma^n \cdot \Delta^n u_x = u_x; \therefore u_x = \Sigma^n \cdot (e^{\frac{d}{dx}} - 1)^n \cdot u_x.$$

Hence $\Sigma^n \cdot (e^{\frac{d}{dx}} - 1)^n$ should be equal to unity, or Σ^n should make the index of $(e^{\frac{d}{dx}} - 1)$ equal to zero; but this will be the case if $\Sigma^n = (e^{\frac{d}{dx}} - 1)^{-n}$, and hence $\Sigma^n \cdot u_x = (e^{\frac{d}{dx}} - 1)^{-n} \cdot u_x$.

This would be also apparent, if we are at once allowed to separate the symbols Σ^n and $(e^{\frac{d}{dx}} - 1)^n$, treating them as different quantities, for then

$$\Sigma^n = \frac{1}{(e^{\frac{d}{dx}} - 1)^n} = (e^{\frac{d}{dx}} - 1)^{-n}; \text{ and } \therefore \Sigma^n u_x = (e^{\frac{d}{dx}} - 1)^{-n} u_x.$$

(95.) If we make $n = 1$, then $\Sigma(u_x) = \frac{u_x}{e^{\frac{d}{dx}} - 1} = u_x \frac{1}{e^t - 1}$, where t is put for $\frac{d}{dx}$;

$$\therefore \Sigma u_x = u_x \left(\frac{1}{t + \frac{t^2}{1 \cdot 2} + \frac{t^3}{2 \cdot 3} + \frac{t^4}{2 \cdot 3 \cdot 4} + \&c.} \right)$$

And assuming
we find

$$\frac{1}{e^t - 1} = \frac{A}{t} + B + C t + D t^2 + E t^3 + F t^4 + \&c.,$$

$$A = 1, B = -\frac{1}{2}, C = +\frac{1}{2} \cdot \frac{1}{2 \cdot 3}, D = 0, E = -\frac{1}{6} \cdot \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} \&c.;$$

$$\therefore \Sigma(u_x) = \frac{u_x}{t} - \frac{1}{2} u_x + \frac{1}{2} \cdot \frac{t \cdot u_x}{2 \cdot 3} - \frac{1}{6} \cdot \frac{t^3 \cdot u_x}{2 \cdot 3 \cdot 4 \cdot 5} + \&c.;$$

or restoring the values of t , and recalling to mind the signification of t^{-1} ,

$$\Sigma(u_x) = \int u_x dx - \frac{1}{2} u_x - \frac{1}{2} \cdot \frac{1}{2 \cdot 3} \cdot \frac{d \cdot u_x}{dx} - \frac{1}{6} \cdot \frac{1}{2 \cdot 3 \cdot 4 \cdot 5} \cdot \frac{d^3 u_x}{dx^3} + \&c.,$$

the formula we have already found by a direct process.

(96.) We may write the sum of u_x in another form by introducing the representatives of the numbers of Bernoulli, the law of which we shall presently determine: thus

$$\Sigma(u_x) = \int u_x dx - \frac{1}{2} u_x + B_1 \cdot \frac{1}{1 \cdot 2} \frac{d \cdot u_x}{dx} - B_3 \cdot \frac{1}{2 \cdot 3 \cdot 4} \frac{d^3 u_x}{dx^3} + \&c.$$

(97.) We now proceed to find the law of these numbers. We have just seen that if $\frac{1}{e^t - 1}$ or $\frac{t}{e^t - 1}$ be expanded according to the powers of t , that the numerical coefficients taken in order are the same as those of the expansion of Σu_x , or are the same as those of $\int u_x dx, u_x, \frac{d u_x}{dx}, \&c.$; and

$$\therefore \frac{1}{e^t - 1} = \frac{A}{t} + B + C t + E t^3 + \&c.,$$

and

$$\therefore \frac{t}{e^t - 1} = A + B t + C t^2 + E t^4 + \&c.$$

we might infer that there would be no odd powers of t except the first in the expansion for $\frac{t}{e^t - 1}$; but it may be thus shown:

Let

$$f(t) = \frac{t}{e^t - 1};$$

$$\therefore f(-t) = \frac{-t}{e^{-t} - 1} = \frac{t \cdot e^t}{e^t - 1};$$

$$\therefore f(t) - f(-t) = -t \frac{e^t - 1}{e^t - 1} = -t.$$

Now assume

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 + \&c.;$$

$$\therefore f(-t) = a_0 - a_1 t + a_2 t^2 - a_3 t^3 + a_4 t^4 - a_5 t^5 + \&c.;$$

$$\therefore f(t) - f(-t) = t = 2 a_1 t + 2 a_3 t^3 + 2 a_5 t^5 + \&c.;$$

$$\therefore 2 a_1 = 1; \therefore a_1 = \frac{1}{2}, a_3 = 0, a_5 = 0, \&c.;$$

and hence we can only express $\frac{t}{e^t - 1}$ in a series of the form

$$\frac{t}{e^t - 1} = 1 - \frac{t}{2} + B_1 \frac{t^2}{1 \cdot 2} - B_3 \frac{t^4}{2 \cdot 3 \cdot 4} + B_5 \frac{t^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} - \&c.,$$

 $B_1, B_3, B_5, \&c.$ being the numbers of Bernoulli, whose law we proceed to find.(98.) If we attempt to find $B_1, B_3, B_5, \&c.$ by means of Maclaurin's theorem, the differential coefficients of $\frac{t}{e^t - 1}$ become $\frac{0}{0}$ when $t = 0$; and that theorem is therefore inapplicable for the purpose. Laplace avoided the difficulty by an ingenious artifice, which we proceed to give.

Since

$$\frac{a}{x^2 - 1} = \frac{\frac{a}{2}}{x - 1} - \frac{\frac{a}{2}}{x + 1},$$

where x may be $e^{\frac{t}{2}}$;

$$\therefore f(t) = \frac{t}{e^t - 1} = \frac{\frac{t}{2}}{e^{\frac{t}{2}} - 1} - \frac{\frac{t}{2}}{e^{\frac{t}{2}} + 1} = f\left(\frac{t}{2}\right) - \frac{\frac{t}{2}}{e^{\frac{t}{2}} + 1};$$

$$\therefore f\left(\frac{t}{2}\right) - f(t) = \frac{\frac{t}{2}}{e^{\frac{t}{2}} + 1}.$$

And hence the coefficient of t^{2x} in $f\left(\frac{t}{2}\right) - f(t)$ and in $\frac{\frac{t}{2}}{e^{\frac{t}{2}} + 1}$ will be the same.

Now let a_{2x} be the coefficient of t^{2x} in the expansion of $\frac{t}{e^t + 1}$; $\therefore \frac{a_{2x}}{2^{2x}}$ will be the coefficient of t^{2x} in the expansion of $\frac{\frac{t}{2}}{e^{\frac{t}{2}} + 1}$. Also let b_{2x} be the coefficient of t^{2x} in the expansion of $f(t)$, therefore $\frac{b_{2x}}{2^{2x}}$ will be the coefficient of t^{2x} in the expansion of $f\left(\frac{t}{2}\right)$, and we shall have for the coefficient of t^{2x} in $f\left(\frac{t}{2}\right) - f(t)$ the expression $b_{2x} \cdot \left(\frac{1}{2^{2x}} - 1\right)$;

$$\therefore b_{2x} \left\{ \frac{1}{2^{2x}} - 1 \right\} = \frac{a_{2x}}{2^{2x}};$$

$$\therefore b_{2x} = -\frac{a_{2x}}{2^{2x} - 1}.$$

But from the series for $\frac{t}{e^t - 1}$ we have the coefficient of t^{2x} , or

$$(-1)^{x+1} \cdot B_{2x-1} \cdot \frac{1}{1 \cdot 2 \cdot 3 \dots 2x} = b_{2x} = -\frac{a_{2x}}{2^{2x} - 1};$$

$$\therefore B_{2x-1} = (-1)^x \cdot \frac{1 \cdot 2 \cdot 3 \cdot 4 \dots 2x}{2^{2x} - 1} \cdot a_{2x}(1).$$

(99.) We have now to determine the value of a_{2x} , and here the ordinary method will apply, since the differential coefficients of $\frac{t}{e^t + 1}$ do not become vanishing fractions when $t = 0$.

Let $u =$ the function $\frac{1}{e^t + 1}$, then since a_{2x} is the coefficient of t^{2x} in the expansion of $\frac{t}{e^t + 1}$ it must be the coefficient of t^{2x-1} in the expansion of $\frac{1}{e^t + 1}$, or of u ;

$$\therefore a_{2x} = \frac{1}{1 \cdot 2 \cdot 3 \dots (2x-1)} \cdot \frac{d^{2x-1} u}{dt^{2x-1}} (2), \text{ when } t = 0.$$

A few differentiations will show us that $\frac{d^{2x-1} u}{dt^{2x-1}}$ must always be of the form

$$\frac{d^{2x-1} u}{dt^{2x-1}} = \frac{a + b e^t + c e^{2t} + f \cdot e^{3t} + \&c. + k e^{2x-1} t}{(1 + e^t)^{2x}};$$

and, multiplying both sides by $(1 + e^t)^{2x}$, we obtain

$$(1 + e^t)^{2x} \cdot \frac{d^{2x-1} u}{dt^{2x-1}} = a + b e^t + c \cdot e^{2t} + f e^{3t} + \&c. + k \cdot e^{2x-1} t;$$

from which it appears that, however developed, the left-hand side can contain only positive powers of t .

Now make

$$e^t = v; \therefore \frac{1}{e^t + 1} = \frac{1}{v + 1} = (v + 1)^{-1};$$

$$\therefore u = (v + 1)^{-1} = v^{-1} - v^{-2} + v^{-3} - v^{-4} + v^{-5} - v^{-6} + \&c.$$

$$\therefore \frac{du}{dt} = \frac{du}{dv} \frac{dv}{dt} = \frac{du}{dv} v = -v^{-1} + 2v^{-2} - 3v^{-3} + 4v^{-4} - 5v^{-5} + 6v^{-6} - \&c.$$

$$\frac{d^2 u}{dt^2} = 1^2 \cdot v^{-1} - 2^2 v^{-2} + 3^2 v^{-3} - 4^2 v^{-4} + 5^2 v^{-5} - \&c.$$

$$\&c. = \&c.;$$

$$\therefore \frac{d^{2x-1} u}{dt^{2x-1}} = -(1^{2x-1} v^{-1} - 2^{2x-1} v^{-2} + 3^{2x-1} v^{-3} - 4^{2x-1} v^{-4} + \&c.);$$

and

Calculus of
Finite
Differences.Calculus of
Finite
Differences

$$(1 + e^t)^{2x} = (v + 1)^{2x} = v^{2x} + 2x \cdot v^{2x-1} + \frac{2x \cdot 2x-1}{1 \cdot 2} v^{2x-2} + \frac{2x \cdot 2x-1 \cdot 2x-2}{1 \cdot 2 \cdot 3} v^{2x-3} + \&c.$$

Multiplying these equations together, and omitting all the negative powers of v , since they must necessarily destroy each other, we have for $a + b e^t + c e^{2t} + \&c. + k e^{2x-1t}$ the expression

$$\begin{aligned} & -1^{2x-1} \cdot 2^{2x-1} + (2^{2x-1} - 2x \cdot 1^{2x-1}) v^{2x-2} \\ & - (3^{2x-1} - 2x \cdot 2^{2x-1} + 2x \cdot \left(\frac{2x-1}{2}\right) 1^{2x-1}) v^{2x-3} + \&c. \\ & + \&c. \\ & = (1 + e^t)^{2x} \cdot \frac{d^{2x-1} u}{d t^{2x-1}}. \end{aligned}$$

Now make $t = 0$, or $e^t = v = 1$, and putting U_{2x-1} instead of the differential coefficient,

$$\begin{aligned} \therefore 2^{2x} \cdot U_{2x-1} &= -1^{2x-1} + (2^{2x-1} - 2x \cdot 1^{2x-1}) \\ & - \left(3^{2x-1} - 2x \cdot 2^{2x-1} + 2x \cdot \frac{2x-1}{2} 1^{2x-1}\right) \\ & + \&c. \qquad \qquad \qquad + \&c. \end{aligned}$$

$$\therefore a_{2x} = \frac{U_{2x-1}}{1 \cdot 2 \cdot 3 \dots (2x-1)} \cdot \frac{1}{2^{2x}};$$

and, therefore, we have for the x th number of Bernoulli

$$\begin{aligned} B_{2x-1} &= (-1)^x \frac{1 \cdot 2 \cdot 3 \dots 2x}{2^{2x} - 1} \cdot \frac{U_{2x-1}}{1 \cdot 2 \cdot 3 \dots 2x-1} \cdot \frac{1}{2^{2x}} \\ &= \frac{(-1)^{x+1} \cdot 2x}{2^{2x} \cdot (2^{2x} - 1)} \left[\begin{aligned} & 1^{2x-1} - (2^{2x-1} - 2x \cdot 1^{2x-1}) \\ & + \left(3^{2x-1} - 2x \cdot 2^{2x-1} + 2x \cdot \frac{2x-1}{2} \cdot 1^{2x-1}\right) \\ & - \left(4^{2x-1} - 2x \cdot 3^{2x-1} + \frac{2x \cdot 2x-1}{1 \cdot 2} \cdot 2^{2x-1} - \frac{2x \cdot 2x-1 \cdot 2x-2}{1 \cdot 2 \cdot 3} 1^{2x-1}\right) \\ & + \&c. \end{aligned} \right] \end{aligned}$$

(100.) From this formula to deduce $B_1, B_3, B_5, \&c.$; first make $x = 1$;

$$\begin{aligned} \therefore B_1 &= \frac{2}{4 \cdot 3} \cdot \left\{ 1 - (2 - 2) + \left(3 - 2 + 2 \cdot \frac{1}{2}\right) \right. \\ & \qquad \qquad \qquad \left. + (4 - 2 \cdot 3 + 1) - \&c. \right\} \\ &= \frac{1}{6}. \end{aligned}$$

Make $x = 2$;

$$\begin{aligned} \therefore B_3 &= \frac{-4}{16 \cdot 15} \{ 1 - (2^3 - 4) + (3^3 - 4 \cdot 2^3 + 6) \\ & \qquad \qquad \qquad - (4^3 - 4 \cdot 3^3 + 6 \cdot 2^3 - 4) \\ & \qquad \qquad \qquad + (5^3 - 4 \cdot 4^3 + 6 \cdot 3^3 - 4 \cdot 2^3 + 1) \\ & \qquad \qquad \qquad - \&c.; \end{aligned}$$

and, as every term of the form

$$z^n - 4(z-1)^n + 6(z-2)^n - 4(z-3)^n + (z-4)^n$$

is (when z is greater than n) necessarily $= 0$,

$$\therefore B_3 = -\frac{1}{60} \{ 1 - 4 + 1 \} = \frac{2}{60} = \frac{1}{30}.$$

Similarly, $B_5 = \frac{1}{42}.$

(101.) By substitution may other terms of the coefficients $B_1, B_3, B_5, \&c.$ be found; and we here subjoin the first ten numbers of Bernoulli expressed decimally and fractionally.

$$B_1 = 0.166666666666 = \frac{1}{6}$$

$$B_3 = 0.03333333333333 = \frac{1}{30}$$

$$B_5 = 0.0238095238095 = \frac{1}{42}$$

$$B_7 = 0.03333333333333 = \frac{1}{30}$$

$$B_9 = 0.075757575757575 = \frac{5}{66}$$

$$B_{11} = 0.2531135531135 = \frac{691}{2730}$$

$$B_{13} = 1.1666666666666 = \frac{7}{6}$$

$$B_{15} = 7.0921568627451 = \frac{3617}{510}$$

$$B_{17} = 54.9711779448621 = \frac{43867}{798}$$

$$B_{19} = 529.1242424242424 = \frac{1222277}{2310}$$

(102.) The applications of these numbers are very numerous; but we have only room for one or two, and must refer the reader to Lacroix, and to Sir John Herschell's Collection of Examples of the Applications of the Calculus of Finite Differences.

These numbers occur when we wish to find the sum of such series as

$$\frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \frac{1}{4^{2n}} + \&c. \text{ to infinity}$$

$$\frac{1}{1^{2n}} + \frac{1}{3^{2n}} + \frac{1}{5^{2n}} + \frac{1}{7^{2n}} + \&c.$$

It is well known that we may express $\sin \theta$ and $\cos \theta$ by means of the products

$$\sin \theta = \theta \left(1 - \frac{\theta^2}{\pi^2}\right) \left(1 - \frac{\theta^2}{2^2 \pi^2}\right) \left(1 - \frac{\theta^2}{3^2 \pi^2}\right) \left(1 - \frac{\theta^2}{4^2 \pi^2}\right) \dots$$

and

$$\cos \theta = \left(1 - \frac{2^2 \theta^2}{\pi^2}\right) \cdot \left(1 - \frac{2^2 \theta^2}{3^2 \pi^2}\right) \cdot \left(1 - \frac{2^2 \theta^2}{5^2 \pi^2}\right) \left(1 - \frac{2^2 \theta^2}{7^2 \pi^2}\right) \&c.;$$

or, writing $\pi \theta$ and $\frac{\pi}{2} \theta$ instead of θ ,

$$\sin \pi \theta = \pi \cdot \theta \left(1 - \frac{\theta^2}{1^2}\right) \left(1 - \frac{\theta^2}{2^2}\right) \left(1 - \frac{\theta^2}{3^2}\right) \left(1 - \frac{\theta^2}{4^2}\right) \&c.$$

$$\cos \left(\frac{\pi}{2} \theta\right) = \left(1 - \frac{\theta^2}{1^2}\right) \cdot \left(1 - \frac{\theta^2}{3^2}\right) \left(1 - \frac{\theta^2}{5^2}\right) \left(1 - \frac{\theta^2}{7^2}\right) \&c.$$

Taking the logarithm of the first expression, and then its differential coefficient, we have

$$\begin{aligned} \pi \cot \pi \theta &= \frac{1}{\theta} - \left\{ \frac{\frac{2\theta}{1^2}}{1 - \frac{\theta^2}{1^2}} + \frac{\frac{2\theta}{2^2}}{1 - \frac{\theta^2}{2^2}} + \frac{\frac{2\theta}{3^2}}{1 - \frac{\theta^2}{3^2}} + \&c. \right\} \\ &= \frac{1}{\theta} - \frac{2\theta}{1^2} \left\{ 1 + \frac{\theta^2}{1^2} + \frac{\theta^4}{1^4} + \frac{\theta^6}{1^6} + \&c. + \frac{\theta^{2x-2}}{1^{2x-2}} + \&c. \right\} \\ &\quad - \frac{2\theta}{2^2} \left\{ 1 + \frac{\theta^2}{2^2} + \frac{\theta^4}{2^4} + \frac{\theta^6}{2^6} + \&c. + \frac{\theta^{2x-2}}{2^{2x-2}} + \&c. \right\} \\ &\quad - \&c.; \end{aligned}$$

in which it is obvious that the coefficient of θ^{2x-1} will be

$$-2 \left\{ \frac{1}{1^{2x}} + \frac{1}{2^{2x}} + \frac{1}{3^{2x}} + \frac{1}{4^{2x}} + \frac{1}{5^{2x}} + \&c. \right\}.$$

But

$$\begin{aligned} \pi \cot \pi \theta &= \pi \cdot \sqrt{-1} \cdot \left\{ \frac{e^{\pi \theta \sqrt{-1}} + e^{-\pi \theta \sqrt{-1}}}{e^{\pi \theta \sqrt{-1}} - e^{-\pi \theta \sqrt{-1}}} \right\} = \pi \sqrt{-1} \left\{ \frac{e^{2\pi \theta \sqrt{-1}} + 1}{e^{2\pi \theta \sqrt{-1}} - 1} \right\} \\ &= \pi \sqrt{-1} \left\{ 1 + \frac{2}{e^{2\pi \theta \sqrt{-1}} - 1} \right\}; \end{aligned}$$

$\therefore$ the coefficient of θ^{2x-1} will be the same as that of the same power of t in the developement of $\frac{1}{e^t - 1}$.

Calculus of
Finite
Differences.

or of t^{2x} in the developement of $\frac{t}{t^2-1}$, multiplied by $2\pi\sqrt{-1} \cdot (2\pi\sqrt{-1})^{2x-1}$, or $(2\pi\sqrt{-1})^{2x}$, or $(-1)^x \cdot (2\pi)^{2x}$; and is consequently represented by

Calculus of
Finite
Differences.

$$\begin{aligned} (-1)^x \cdot (2\pi)^{2x} b_{2x} &= (-1)^x \cdot (2\pi)^{2x} (-1)^{x+1} \cdot \frac{B_{2x-1}}{1 \cdot 2 \cdot 3 \dots 2x} \\ &= - \frac{2^{2x} \cdot \pi^{2x}}{1 \cdot 2 \cdot 3 \dots (2x)} \cdot B_{2x-1}. \end{aligned}$$

Equating these coefficients of θ^{2x-1} , we obtain

$$\frac{1}{1^{2x}} + \frac{1}{2^{2x}} + \frac{1}{3^{2x}} + \frac{1}{4^{2x}} + \&c. = \frac{2^{2x-1} \cdot \pi^{2x}}{1 \cdot 2 \cdot 3 \dots (2x)} \cdot B_{2x-1};$$

whence, making $x = 1, 2, 3, 4$, we have the values of the following series:

$$\begin{aligned} \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \&c. &= \frac{\pi^2}{6} \\ \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \&c. &= \frac{\pi^4}{90} \\ \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \&c. &= \frac{\pi^6}{945} \\ \frac{1}{1^8} + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \&c. &= \frac{\pi^8}{1350} \end{aligned}$$

(103.) Again, taking the logarithms of $\cos \frac{\pi\theta}{2}$, and of the factors to which it is equal, and then their differential coefficients, we find that

$$-\frac{\pi}{2} \tan \frac{\pi\theta}{2} = \frac{-\frac{2\theta}{1^2}}{1 - \frac{\theta^2}{1^2}} + \frac{-\frac{2\theta}{3^2}}{1 - \frac{\theta^2}{3^2}} + \frac{-\frac{2\theta}{5^2}}{1 - \frac{\theta^2}{5^2}} + \frac{-\frac{2\theta}{7^2}}{1 - \frac{\theta^2}{7^2}} + \&c.;$$

and the coefficient of θ^{2x-1} , by dividing out the fractions, will be found to be

$$-2 \left\{ \frac{1}{1^{2x}} + \frac{1}{3^{2x}} + \frac{1}{5^{2x}} + \frac{1}{7^{2x}} + \&c. \right\}.$$

But

$$\begin{aligned} -\frac{\pi}{2} \tan \frac{\pi\theta}{2} &= \frac{-\pi}{2\sqrt{-1}} \frac{e^{\frac{\pi\theta}{2}\sqrt{-1}} - e^{-\frac{\pi\theta}{2}\sqrt{-1}}}{e^{\frac{\pi\theta}{2}\sqrt{-1}} + e^{-\frac{\pi\theta}{2}\sqrt{-1}}} = \frac{-\pi}{2\sqrt{-1}} \left(\frac{e^{\pi\theta\sqrt{-1}} - 1}{e^{\pi\theta\sqrt{-1}} + 1} \right) \\ &= \frac{-\pi}{2\sqrt{-1}} \left(1 - \frac{2}{e^{\pi\theta\sqrt{-1}} + 1} \right); \end{aligned}$$

and, as we have said before, the coefficient of θ^{2x-1} will be the same as that of the same power of t in the expansion of $\frac{1}{e^t+1}$, or of t^{2x} in the expansion of $\frac{t}{t^2+1}$, but multiplied by $\frac{\pi}{\sqrt{-1}} \cdot (\pi\sqrt{-1})^{2x-1}$, or $(-1)^{x-1} \cdot \pi^{2x}$; and $\therefore$ the coefficient of $\theta^{2x-1} = (-1)^{x-1} \cdot \pi^{2x} \cdot a_{2x}$

$$\begin{aligned} &= (-1)^{x-1} \cdot \pi^{2x} \cdot \frac{(2^{2x} - 1)}{(-1)^x \cdot 1 \cdot 2 \cdot 3 \dots (2x)} \cdot B_{2x-1} \\ &= - \frac{(2^{2x} - 1) \cdot \pi^{2x}}{1 \cdot 2 \cdot 3 \cdot 4 \dots 2x} \cdot B_{2x-1}; \end{aligned}$$

and equating these coefficients of θ^{2x-1} we have

$$\frac{1}{1^{2x}} + \frac{1}{3^{2x}} + \frac{1}{5^{2x}} + \frac{1}{7^{2x}} + \&c. = \frac{1}{2} \cdot \frac{(2^{2x} - 1) \cdot \pi^{2x}}{1 \cdot 2 \cdot 3 \dots 2x} B_{2x-1},$$

whence

$$\begin{aligned} \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c. &= \frac{\pi^2}{8} \\ \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \&c. &= \frac{\pi^4}{96} \\ \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \&c. &= \frac{\pi^6}{960}. \end{aligned}$$

Calculus of
Finite
Differences.

(104.) Since also the series

$$\frac{1}{1^{2x}} - \frac{1}{2^{2x}} + \frac{1}{3^{2x}} - \frac{1}{4^{2x}} + \frac{1}{5^{2x}} - \frac{1}{6^{2x}} + \&c.$$

may be resolved into the two series

$$\frac{1}{1^{2x}} + \frac{1}{3^{2x}} + \frac{1}{5^{2x}} + \&c. - \left(\frac{1}{2^{2x}} + \frac{1}{4^{2x}} + \frac{1}{6^{2x}} + \&c. \right),$$

or into

$$\frac{1}{1^{2x}} + \frac{1}{3^{2x}} + \frac{1}{5^{2x}} + \&c. - \frac{1}{2^{2x}} \left\{ \frac{1}{1^{2x}} + \frac{1}{2^{2x}} + \frac{1}{3^{2x}} + \&c. \right\};$$

therefore the sum of the series will be expressed by

$$\begin{aligned} & \frac{1}{2} \frac{2^{2x} - 1}{1 \cdot 2 \cdot 3 \dots 2x} \cdot B_{2x-1} - \frac{1}{2^{2x}} \cdot \frac{2^{2x-1} \cdot \pi^{2x}}{1 \cdot 2 \cdot 3 \dots 2x} \cdot B_{2x-1} \\ &= \frac{1}{2} \cdot \frac{(2^{2x} - 2) \pi^{2x}}{1 \cdot 2 \cdot 3 \dots 2x} \cdot B_{2x-1} = \frac{(2^{2x-1} - 1) \pi^{2x}}{1 \cdot 2 \cdot 3 \dots 2x} B_{2x-1}; \end{aligned}$$

thence

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \&c. = \frac{\pi^2}{12}$$

and

$$\frac{1}{1^4} - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \&c. = \frac{7 \cdot \pi^4}{720}.$$

We must again refer the reader to Herschell's Examples for many curious applications of these numbers to the summation of series. Without these examples he cannot enter fully into the spirit of the inquiries of this branch of analysis, or form any conception of its extent or power.

 (105.) The expression for the integral of u_x , or

$$\Sigma u_x = \int u_x dx - \frac{u_x}{2} + \frac{B_1}{2} \cdot \frac{d \cdot u_x}{dx} - \frac{B_3}{2 \cdot 3 \cdot 4} \cdot \frac{d^3 u_x}{dx^3} + \frac{B_5}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \frac{d^5 u_x}{dx^5} - \&c.$$

is very useful in determining the approximate values of some numerical expressions which it would be difficult, perhaps impossible, to arrive at by other means.

First, let $u_x = \frac{1}{x}$; then, as we shall hereafter show, if u_x be the general term of a series, Σu_{x+1} corrected is the value of the series itself, $\therefore \Sigma \left(\frac{1}{x} \right)$ is the representative of the series

$$\begin{aligned} & \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c. + \frac{1}{x-1} \\ &= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \&c. + \frac{1}{x} \right) - \frac{1}{x} = S \left(\frac{1}{x} \right) - \frac{1}{x}, \end{aligned}$$

where $S \left(\frac{1}{x} \right)$ represents the sum of the series, of which the last term is $\frac{1}{x}$;

$$\therefore S \left(\frac{1}{x} \right) = C + \log x + \frac{1}{2x} - \frac{B_1}{2x^2} + \frac{B_3}{4x^4} - \frac{B_5}{6x^6} + \&c.;$$

a series which is the more convergent the greater x is: it remains to determine the constant.

We cannot suppose $x = 0$, $\therefore -\log 0 = \infty$ and the other terms are also infinite; but let $x = 1$,

$$\therefore S \left(\frac{1}{1} \right) = 1;$$

$$\therefore 1 = C + \frac{1}{2} - \frac{B_1}{2} + \frac{B_3}{4} - \frac{B_5}{6} + \&c$$

$$\therefore C = \frac{1}{2} + \frac{B_1}{2} - \frac{B_3}{4} + \frac{B_5}{6} - \&c.$$

This series is only convergent for the first few terms for $B_{12} = \frac{7}{6}$, and afterwards the numbers of Bernoulli increase rapidly; but as the terms are alternately $+$ and $-$, the difference between two consecutive terms may decrease, and as we shall have the value of C successively greater or less than its true value, according as we stop at a positive or at a negative term, we may assume the value of C to be very near its real value when the differences diminish and are small.

Then successively putting for B_1, B_3, B_5, B_7 , their values $\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}$, we have

 Calculus of
Finite
Differences.

$$\begin{aligned}
 C &> \frac{1}{2} &< \frac{1}{2} + \frac{1}{12} && \text{or } > .50000 < .58333 \\
 .58333 - \frac{1}{120} &< .58333 - \frac{1}{120} + \frac{1}{252} && \text{or } > .57500 < .57896 \\
 > .57896 - \frac{1}{240} && \text{i.e. } > .57480.
 \end{aligned}$$

Hence, since .57500 and .57480 are decreasing, we may stop at .57896 as an approximate value for C ; or by taking an arithmetic mean between .57500 and .57896, $C = .57698$.

If we make $x = 10$, and let A represent the sum of

$$\begin{aligned}
 \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \&c. + \frac{1}{10} = 2.928968253968 \\
 C = A - \log 10 - \frac{1}{20} + \frac{B_1}{200} - \frac{B_3}{40000} + \&c. = .5772156649015329,
 \end{aligned}$$

by computing as far as the 11th term of the series.

(106.) Let the value of $S\left(\frac{1}{x}\right)$ be required to 1000 terms, in which case it is sufficient to calculate the first four terms of the expansion for $S\left(\frac{1}{x}\right)$, or make

$$S\left(\frac{1}{x}\right) = C + \log 1000 + \frac{1}{2000} - \frac{B_1}{2(1000)^2} + \frac{B_3}{4(1000)^4}$$

Now

$$\begin{aligned}
 C &= .5772156649015 \\
 \log 1000 &= 6.9077552789821 \\
 \frac{1}{2000} &= .0005000000000 \\
 &\quad \underline{7.4854709438836} \\
 \frac{B_1}{2x^2} &= 0.0000000833333 \\
 &\quad \underline{}
 \end{aligned}$$

$$\therefore \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \&c. + \frac{1}{1000} = 7.4854708605503$$

$\frac{B_3}{4(1000)^4}$ is neglected, as there are 13 ciphers after the decimal point.

(107.) Next to find the value of $\sum u_x$, when u_x is the logarithm of x .

$$\begin{aligned}
 \therefore \sum \log(x) &= \int \log \cdot x \cdot dx - \frac{\log x}{2} + \frac{B_1}{1 \cdot 2} \cdot \frac{d \cdot \log(x)}{dx} - \frac{B_3}{2 \cdot 3 \cdot 4} \cdot \frac{d^3 \cdot \log(x)}{dx^3} + \&c. \\
 &= x \log x - x - \frac{\log(x)}{2} + \frac{B_1}{1 \cdot 2} \cdot \frac{1}{x} - \frac{B_3}{3 \cdot 4 \cdot x^3} + \&c. + C \\
 &= \left(x - \frac{1}{2}\right) \log x - x + \frac{B_1}{2x} - \frac{B_3}{3 \cdot 4 \cdot x^3} + \&c. + C \\
 \therefore S(\log x) &= \left(x + \frac{1}{2}\right) \log x - x + \frac{B_1}{2x} - \frac{B_3}{3 \cdot 4 \cdot x^3} + \&c. + C
 \end{aligned}$$

If we make $x = 10$, and calculate the first ten numbers of the series, and then add the Napierian logarithms of first ten numbers together, we shall find that

$$C = .9189385332047.$$

(108.) But the actual value of C may be found in the following manner:

since

$$S(\log \cdot x) = \log 1 + \log 2 + \log 3 \dots + \log x = \log(1 \cdot 2 \cdot 3 \dots x);$$

$$\therefore \log(1 \cdot 2 \cdot 3 \dots x) = \left(x + \frac{1}{2}\right) \log x - x + \frac{B_1}{2x} - \frac{B_3}{3 \cdot 4 \cdot x^3} + \&c. + C.$$

And to find C let x become infinite;

$$\therefore \log(1 \cdot 2 \cdot 3 \dots x) = \left(x + \frac{1}{2}\right) \log x - x + C \quad (1)$$

$$\therefore \log(1 \cdot 2 \cdot 3 \dots 2x) = \left(2x + \frac{1}{2}\right) \log 2x - 2x + C \quad (2).$$

$$\begin{aligned}
 \text{But } \log(2 \cdot 4 \cdot 6 \cdot 8 \dots 2x) &= \log 2^x \cdot (1 \cdot 2 \cdot 3 \cdot 4 \dots x) = \log 2^x + \log(1 \cdot 2 \cdot 3 \dots x) \\
 &= x \log 2 + \log(1 \cdot 2 \cdot 3 \dots x) = x \log 2 + \left(x + \frac{1}{2}\right) \log x - x + C \quad (3).
 \end{aligned}$$

Calculus of Finite Differences. Now subtract equation (3) from equation (2);

$$\therefore \log 1 \cdot 3 \cdot 5 \cdot 7 \dots 2x-1 = x \log x + \left(x + \frac{1}{2}\right) \log 2 - x \quad (4).$$

Then subtract equation (4) from equation (3);

$$\therefore \log \frac{2 \cdot 4 \cdot 6 \cdot 8 \dots 2x}{1 \cdot 3 \cdot 5 \cdot 7 \dots 2x-1} = \frac{1}{2} \log x - \frac{1}{2} \log 2 + C = \frac{1}{2} \log 2x - \log 2 + C;$$

$$\begin{aligned} \therefore 2C - 2 \log 2 &= 2 \log \frac{2 \cdot 4 \cdot 6 \dots 8 \dots 2x}{1 \cdot 3 \cdot 5 \dots 7 \dots 2x-1} - \log 2x \\ &= \log \frac{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \dots (2x)^2}{1 \cdot 3^2 \cdot 5^2 \cdot 7^2 \dots (2x-1)^2} - \log 2x \\ &= \log \frac{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \dots (2x)^2}{1 \cdot 3^2 \cdot 5^2 \cdot 7^2 \dots (2x-1)^2 \cdot 2x} \end{aligned}$$

or since $\frac{1}{2x}$ and $\frac{1}{2x+1}$ are the same when x is very great,

$$\therefore 2C - 2 \log 2 = \log \frac{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2 \dots (2x)^2}{1 \cdot 3^2 \cdot 5^2 \cdot 7^2 \cdot 9^2 (2x-1)^2 (2x+1)^2}.$$

But by Wallis's theorem,

$$\frac{\pi}{2} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdot 8 \cdot 8 \cdot 10 \cdot 10 \dots}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdot 7 \cdot 7 \cdot 9 \cdot 9 \cdot 11 \dots};$$

$$\therefore 2C - 2 \log 2 = \log \frac{\pi}{2} = \log \pi - \log 2;$$

$$\therefore 2C = \log \pi + \log 2 = \log 2\pi; \therefore C = \frac{1}{2} \log 2\pi = \log \sqrt{2\pi};$$

whence

$$S(\log x) = \log \sqrt{2\pi} + \left(x + \frac{1}{2}\right) \log x - x + \frac{B_1}{2x} - \frac{B_3}{3 \cdot 4x^3} + \&c.$$

(109.) If x be very great, so that $\frac{B_1}{2x}$, $\frac{B_3}{3 \cdot 4x^3}$, &c. may be neglected,

$$\begin{aligned} \therefore \log (1 \cdot 2 \cdot 3 \dots x) &= \log \sqrt{2\pi} + \left(x + \frac{1}{2}\right) \log x - x \\ &= \log \sqrt{2\pi} \cdot x^{x+\frac{1}{2}} - x; \end{aligned}$$

$$\begin{aligned} \therefore 1 \cdot 2 \cdot 3 \dots x &= e^{-x+\log \sqrt{2\pi} \cdot x^{x+\frac{1}{2}}} = \sqrt{2\pi} x^{x+\frac{1}{2}} e^{-x} \\ &= \sqrt{2\pi x} \cdot \left(\frac{x}{e}\right)^x. \end{aligned}$$

If x be not so large as to allow all the inverse powers of x to be neglected, we shall have

$$\log \frac{1 \cdot 2 \cdot 3 \dots x}{\sqrt{2\pi} \cdot x^{x+\frac{1}{2}}} = -x + \frac{B_1}{2x} - \frac{B_3}{3 \cdot 4x^3} + \&c.;$$

$$\therefore 1 \cdot 2 \cdot 3 \dots x = \sqrt{2\pi} \cdot x^{x+\frac{1}{2}} \cdot e^{-x} e^{\frac{B_1}{2x} - \frac{B_3}{3 \cdot 4x^3} + \&c.}$$

$$= \sqrt{2\pi x} \left(\frac{x}{e}\right)^x \cdot e^{\frac{1}{12x} - \frac{1}{360x^3} + \&c.}$$

$$= \sqrt{2\pi x} \cdot \left(\frac{x}{e}\right)^x \cdot \left\{1 + \frac{1}{12x} + \frac{1}{288x^2} + \&c.\right\};$$

which reduces itself to the preceding value when by making x so great the terms $\frac{1}{12x}$, $\frac{1}{288x^2}$, &c. become so small as to be inappreciable.

(110.) But when x is considerable, the formula $\sqrt{2\pi x} \cdot \left(\frac{x}{e}\right)^x$ is sufficiently accurate.

Thus if it be required to find the product of the first 1000 numbers;

$$\therefore 1 \cdot 2 \cdot 3 \dots 1000 = \sqrt{2\pi \cdot 1000} \cdot \left(\frac{1000}{e}\right)^{1000}.$$

Calculus of Finite Differences.

Calculus of
Finite
Differences.

Now

$$\begin{aligned}\log \sqrt{2\pi} \cdot 1000 &= 1.8990899 \\ 1000 \cdot \log(1000) &= 3000.000000 \\ &\quad \underline{3001.8990899} \\ 1000 \log e &= 434.2944819\end{aligned}$$

$$\therefore \log(1 \cdot 2 \cdot 3 \dots 1000) = 2567.6046080;$$

and thus the product $1 \cdot 2 \cdot 3 \dots 1000$ will form a number consisting of 2568 figures, the first to the left being 4023537, followed by 2561 ciphers.

If the number be computed by the formula

$$\log 1 \cdot 2 \cdot 3 \dots 1000 = \log \sqrt{2\pi} x + x \log x - Mx + \frac{M B_1}{2x} - \frac{M \cdot B_3}{2 \cdot 4 x^3},$$

where M is the modulus and the logarithms are the ordinary ones, the result will be

$$\log 1 \cdot 2 \cdot 3 \dots 1000 = 2567.6046442;$$

and as .6046442 is the logarithm of 4.02387, the number will have its first six figures expressed by 402387, and these will be succeeded by 2562 ciphers.

The difference between the results obtained by the formula $\sqrt{2\pi} x \left(\frac{x}{e}\right)^x$, and that from the theorem in which the terms involving B_1 and B_3 are not neglected will obviously become smaller as x increases; yet the first formula is sufficiently accurate when the ratios only of high numbers are required.

(111.) The product of the odd numbers is very easily found from the previous investigation; for

$$\begin{aligned}\log 1 \cdot 3 \cdot 5 \dots 2x-1 &= x \log x + \left(x + \frac{1}{2}\right) \log 2 - x \\ &= x \log 2x + \frac{1}{2} \log 2 - x \\ &= \log (2x)^x \cdot 2^{\frac{1}{2}} - x; \\ \therefore \log \frac{1 \cdot 3 \cdot 5 \dots 2x-1}{(2x)^x \cdot \sqrt{2}} &= -x; \\ \therefore 1 \cdot 3 \cdot 5 \dots 2x-1 &= (2x)^x \sqrt{2} \cdot e^{-x} = \sqrt{2} \cdot \left(\frac{2x}{e}\right)^x.\end{aligned}$$

Integration of Equations of Differences.

(112.) Hitherto we have only examined the cases in which the function may be integrated when the difference is given explicitly in terms of x : we proceed to show how the integrations are to be performed when the differences and the functions are involved together.

In such a case the proposed equation will be comprised under the general formula

$$F(x, u_x, u_{x+1}, \&c., \Delta \cdot u_x, \Delta^2 \cdot u_x, \Delta^3 \cdot u_x, \&c.) = 0.$$

But since we can express $\Delta \cdot u_x, \Delta^2 \cdot u_x, \&c.$ in terms of u_x and its successive values, this function will not lose any of its generality if we substitute for $\Delta \cdot u_x, \Delta^2 \cdot u_x, \&c.$ their values in terms of $u_x, u_{x+1}, \&c.$, and write the equation under the form

$$F(x, u_x, u_{x+1}, u_{x+2}, \dots, u_{x+n}) = 0.$$

(113.) We divide equations of differences into orders: those involving $\Delta \cdot u_x$ only are called equations of the first order; and when $\Delta^2 \cdot u_x$ and $\Delta \cdot u_x$ enter into the expression, it is styled an equation of the second order, and so on; or otherwise equations involving u_{x+1} and u_x are equations of the first order, and $f(u_{x+2}, u_{x+1}, u_x) = 0$ is an equation of the second order; and, in general, the equation $f(x, u_x, u_{x+1}, \dots, u_{x+n}) = 0$ is of the n th order; and it is obvious that every equation of finite differences when solved will give the form of a series, and as, when u_x is known, any term is known, hence any particular term will be a function of some of the terms which precede it, and of the index of its place in the series.

(114.) We might infer from what has been previously said respecting the integration of differential equations, that the integration of an equation of finite differences of (n) dimensions would introduce into the solution as many arbitrary constants as there have been integrations, or as many as the order of the equation indicates. But that the general equation $f(x, u_x, u_{x+1}, \dots, u_{x+n}) = 0$ will, under its most complete form, include (n) constants, may be thus shown.

Suppose u_x to contain besides x , (n) constant quantities, $a', a'', a''' \dots a^n$, so that

$$u_x = f(x, a', a'', a''' \dots a^n).$$

Then, writing $x+1, x+2, x+3 \dots x+n$ for x , we may form the equations

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

$$\begin{aligned} u_{x+1} &= f(x+1, a', a'', a''' \dots a^n) \\ u_{x+2} &= f(x+2, a', a'', a''' \dots a^n) \\ u_{x+3} &= f(x+3, a', a'', a''' \dots a^n) \\ &\vdots \\ u_{x+n} &= f(x+n, a', a'', a''' \dots a^n). \end{aligned}$$

 Calculus of
Finite
Differences.

And each of these equations hold good simultaneously, and their number is $n+1$; hence between them the (n) quantities $a', a'', a''' \dots a^n$ may be eliminated, and we shall have at length the equation

$$f(x, u_x, u_{x+1}, u_{x+2}, u_{x+3} \dots u_{x+n}) = 0$$

entirely independent of them. Hence, conversely, as every equation of (n) dimensions in its most general form may be supposed to be derived in the manner just explained, we must, in reascending to the primitive integral, introduce (n) constant quantities.

(115.) And as in differential equations, so in equations of differences will there be particular solutions, but with this difference, that these solutions may in some cases be derived, which will contain as many arbitrary constants as the complete integral itself. This will appear from the circumstance, that the constant need not necessarily be independent of x , and therefore its determination, in order to free the equation from it, may introduce arbitrary constants.

(116.) We now proceed to the integration of equations of differences, beginning with the most simple case, and then ascending to the general theory of such equations. But we shall soon find that the number of cases which can be integrated is very limited, and that the general methods frequently involve the analyst in difficulties greater than those which the original equation presents, and that the solutions of many of the equations depend upon his individual dexterity.

(117.) To integrate the equation of the first degree and of the first order,

$$\Delta u_x + P_x u_x = Q_x,$$

where P and Q involve x only.

This equation is analogous to that of the first degree and first order, $dy + Py dx = Q dx$, and its integral may be found in a similar manner.

Thus, let

$$u_x = v_x \cdot z_x; \therefore \Delta u_x = v_x \Delta z_x + z_x \cdot \Delta v_x + \Delta v_x \cdot \Delta z_x;$$

$\therefore$ substituting in the original equation, we have

$$z_x \cdot (\Delta v_x + P_x \cdot v_x) + v_x \cdot \Delta z_x + \Delta v_x \cdot \Delta z_x = Q_x;$$

and making

$$\Delta \cdot v_x + P_x \cdot v_x = 0, \text{ and } \therefore v_x \Delta z_x + \Delta v_x \cdot \Delta z_x = Q_x,$$

$$\therefore \frac{\Delta v_x}{v_x} = -P_x \text{ and } \Delta z_x = \frac{Q_x}{v_x + \Delta v_x} = \frac{Q_x}{v_{x+1}}$$

But since P_x is a function of x only, make $v_x = e^t$ where t is a function of x only;

$$\therefore \Delta v_x = e^{t+\Delta t} - e^t = e^t (e^{\Delta t} - 1);$$

$$\therefore \frac{\Delta \cdot v_x}{v_x} = -P_x = e^{\Delta t} - 1;$$

$$\therefore e^{\Delta t} = 1 - P_x; \therefore \Delta t = \log(1 - P_x) \text{ and } t = \Sigma \log(1 - P_x).$$

But the sum of the logarithm of $(1 - P_x)$ expresses the sum of all the terms arising from giving to x all the values between the assumed limits, and as the sum of any number of logarithms is equal to the logarithm of their product, we may express this product by $[1 - P_{x-1}]^x$, where $(1 - P_{x-1})$ represents the last term, and the index x expresses the number of the terms $(1 - P_0), (1 - P_1), (1 - P_2), \&c$ in the product, and $\therefore t = \log[1 - P_{x-1}]^x$.

Hence

$$\therefore e^{\log w} = w, \therefore e^t = [1 - P_{x-1}]^x = v_x;$$

$$\therefore z_x = \Sigma \left(\frac{Q_x}{v_{x+1}} \right) + C = \Sigma \left(\frac{Q_x}{[1 - P_x]^{x+1}} \right) + C;$$

$$\therefore u_x = z_x v_x = [1 - P_{x-1}]^x \left\{ \Sigma \frac{Q_x}{[1 - P_x]^{x+1}} + C \right\};$$

whence u_x is known, or the general term of a series is found.

(118.) The method which is generally adopted in the solution of this equation, and which is derived from the preceding investigation, is the following.

Substituting $u_{x+1} - u_x$ for Δu_x , we have

$$u_{x+1} - (1 - P_x) u_x = Q_x;$$

or $u_{x+1} - A_x u_x = Q_x$, where A_x is put for $1 - P_x$.

Assume

$$u_x = A_0 A_1 A_2 A_3 \dots A_{x-1} v_x;$$

$$\therefore u_{x+1} = A_0 A_1 A_2 A_3 \dots A_x v_{x+1};$$

$$\therefore u_{x+1} - A_x u_x = A_0 A_1 A_2 A_3 \dots A_x \{v_{x+1} - v_x\} = Q_x;$$

or

$$A_0 \cdot A_1 \cdot A_2 \cdot A_3 \dots A_x \cdot \Delta v_x = Q_x;$$

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

$$\therefore v_x = \Sigma \left(\frac{Q_x}{A_0 A_1 \dots A_x} \right) + C;$$

$$\therefore u_x = A_0 A_1 \dots A_{x-1} v_x = A_0 A_1 \dots A_{x-1} \left\{ \Sigma \left(\frac{Q_x}{A_1 A_2 \dots A_x} \right) + C \right\};$$

which may be written as in the preceding article

$$u_x = [A_{x-1}]^x \left\{ \Sigma \frac{Q_x}{[A_x]^{x+1}} + C \right\}.$$

Instead, however, of using the notation $[A_{x-1}]^x$ to represent the product of the x factors $A_0 A_1 A_2 \dots A_{x-1}$, Sir John Herschell has proposed to write it $P \cdot A_{x-1}$, P indicating product, and A_{x-1} the last term.

Thus $[A_x]^{x+1}$ will be expressed by $P \cdot A_x$, and thus

$$u_x = P A_{x-1} \left\{ \Sigma \frac{Q_x}{P \cdot A_x} + C \right\}.$$

We shall refer to this expression in the succeeding articles.

Cor. (1.) If the coefficient of u_x be constant and $= A$, then

$$A_0 \cdot A_1 \cdot A_2 \cdot A_3 \dots A_{x-1} = A^x;$$

$$\therefore u_x = A^x \cdot v_x$$

$$u_{x+1} = A^{x+1} v_{x+1};$$

$$\therefore u_{x+1} - A u_x = A^{x+1} \cdot (v_{x+1} - v_x) = A^{x+1} \Delta v_x = Q_x;$$

$$\therefore \Delta(v_x) = \frac{Q_x}{A \cdot A^x}; \therefore v_x = \frac{1}{A} \cdot \left\{ \Sigma \left(\frac{Q_x}{A^x} \right) + C \right\};$$

$$\therefore u_x = A^x v_x = A^{x-1} \cdot \left\{ \Sigma \left(\frac{Q_x}{A^x} \right) + C \right\}.$$

Cor. (2.) We must take care that A_x be not $= 0$, when $x = 0$; but if A_x does vanish when this substitution is made for x , we must omit the factor A_0 , and put $x = 1$.

Cor. (3.) If, when A_x is constant and $= A$, Q_x be also constant $= B$, the expressions for v_x and u_x are much simplified.

For then

$$v_x = \Sigma \frac{Q_x}{A_{x+1}} = \frac{B}{A} \Sigma \frac{1}{A^x} = \frac{B}{A} \cdot \frac{A}{1-A} \cdot \frac{1}{A^x} + C$$

$$= \frac{B}{(1-A) A^x} + C;$$

$$\therefore u_x = A^x v_x = \frac{B}{1-A} + C \cdot A^x.$$

(119.) We shall now give a few simple examples of the preceding formula.

Example (1.) Integrate $u_{x+1} - x + 1 u_x = 1 \cdot 2 \cdot 3 \dots x + 1$.

Here $A_x = x + 1$; $\therefore A_{x-1} = x$ and $Q_x = 1 \cdot 2 \cdot 3 \dots x + 1$.

$\therefore$ let

$$u_x = 1 \cdot 2 \cdot 3 \dots x \cdot v_x;$$

$$\therefore u_{x+1} = 1 \cdot 2 \cdot 3 \dots x + 1 \cdot v_{x+1};$$

$$\therefore u_{x+1} - x + 1 u_x = 1 \cdot 2 \cdot 3 \dots x + 1 \cdot \{v_{x+1} - v_x\} = 1 \cdot 2 \cdot 3 \dots x + 1;$$

$$\therefore v_{x+1} - v_x = \Delta \cdot v_x = 1; \therefore v_x = x + C;$$

$$\therefore u_x = 1 \cdot 2 \cdot 3 \dots x \cdot (x + C).$$

Example (2.) Let

$$u_{x+1} - \frac{x}{x+1} u_x = a \cdot \frac{2x+1}{x+1}.$$

Here

$$A_x = \frac{x}{x+1} \text{ and } A_{x-1} = \frac{x-1}{x}.$$

But $A_0 = 0$; and $\therefore$ we assume $u_x = A_1 A_2 \dots A_{x-1} \cdot v_x$;

$$\therefore u_x = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \dots \frac{x-2}{x-1} \cdot \frac{x-1}{x} v_x = \frac{1}{x} \cdot v_x;$$

$$\therefore u_{x+1} = \frac{1}{x+1} v_{x+1};$$

$$\therefore u_{x+1} - \frac{x}{x+1} u_x = \frac{1}{x+1} \cdot v_{x+1} - \frac{1}{x+1} v_x = \frac{1}{x+1} (v_{x+1} - v_x) = \frac{a \cdot 2x+1}{x+1};$$

$$\therefore \Delta \cdot v_x = 2ax + a; \therefore v_x = a \cdot x \cdot x - 1 + ax + C = ax^2 + C;$$

$$\therefore u_x = \frac{1}{x} v_x = ax + \frac{C}{x}.$$

Calculus of
Finite
Differences.

Example (3.) Integrate the equation $u_{x+1} = \overline{x+1} \cdot u_x$.

Here

$$u_{x+1} - \overline{x+1} \cdot u_x = 0,$$

and

$$A_0 = 1, A_{x-1} = x.$$

Let

$$\therefore u_x = 1 \cdot 2 \cdot 3 \dots x v_x;$$

$$\therefore u_{x+1} = 1 \cdot 2 \cdot 3 \dots \overline{x+1} v_{x+1};$$

$$\therefore u_{x+1} - \overline{x+1} u_x = 1 \cdot 2 \cdot 3 \dots \overline{x+1} \cdot (v_{x+1} - v_x) = 0;$$

$$\therefore \Delta v_x = 0; \therefore v_x = C,$$

and

$$u_x = 1 \cdot 2 \cdot 3 \dots x \cdot C.$$

But the following solution of the question will illustrate the general method:

As

$$u_{x+1} - u_x = x u_x;$$

$$\therefore \Delta u_x = x \cdot u_x.$$

Make

$$u_x = c \cdot e^t; \therefore \Delta u_x = c \cdot e^t \cdot (e^x - 1);$$

$$\therefore \frac{\Delta \cdot u_x}{u_x} = e^x - 1 = x;$$

$$\therefore e^x = 1 + x; \therefore \Delta \cdot t = \log(1 + x);$$

$$\therefore t = \sum \log(1 + x) = \log 1 + \log 2 + \log 3 + \&c. + \log x = \log(1 \cdot 2 \cdot 3 \dots x);$$

$$\therefore e^t = 1 \cdot 2 \cdot 3 \dots x,$$

and

$$u_x = c \cdot 1 \cdot 2 \cdot 3 \dots x.$$

This question is the solution of the problem, "Find the number of permutations which may be made of x things taken together." Here u_x represents the number required, while u_{x+1} will represent the number that can be formed of $x+1$ things; and as the new letter can be written in $x+1$ places, each of the former arrangements can be altered $x+1$ times, and therefore the whole number increased $x+1$ times.

Hence we derive the equation $u_{x+1} = \overline{x+1} u_x$.

And since, when

$$x = 1, u_1 = 1 = C; \therefore C = 1;$$

$$\therefore u_x = 1 \cdot 2 \cdot 3 \cdot 4 \dots \overline{x+1} \cdot x;$$

the same result as is obtained in ordinary algebra.

(120.) Next to integrate the equation of n dimensions,

$$u_{x+n} + A u_{x+n-1} + B u_{x+n-2} + C u_{x+n-3} + \&c. + M u_x + N = 0,$$

where $A, B, C, \&c. M$ and N are constant quantities.

Make

$$u_x = a^x + A_1; \therefore u_{x+n} = a^{x+n} + A_1;$$

and, substituting, the equation becomes

$$a^{x+n} + A a^{x+n-1} + B a^{x+n-2} + C a^{x+n-3} + \&c. + M a^x + A_1 \{1 + A + B + C + \&c. + M\} + N = 0;$$

and to determine A_1 , let

$$A_1 = \frac{-N}{1 + A + B + C + \&c. + M};$$

and $\therefore$ dividing by a^x , we have

$$a^n + A a^{n-1} + B a^{n-2} + C a^{n-3} + \&c. + M = 0,$$

an equation of (n) dimensions. Let its roots be determined, and let them be $\alpha, \beta, \gamma, \&c.$

Then the values of u_x will be $\alpha^x + A_1, \beta^x + A_1, \gamma^x + A_1, \&c.$, and $\alpha^x, \beta^x, \gamma^x, \&c.$ will be the (n) particular integrals of the equation

$$u_{x+n} + A u_{x+n-1} + B u_{x+n-2} + \&c. + M u_x = 0,$$

and $\therefore$ the complete integral of that equation will be

$$C' \alpha^x + C'' \beta^x + C''' \gamma^x + C'''' \delta^x + \&c.;$$

and $\therefore$ the complete integral of the original equation will be

$$C' \alpha^x + C'' \beta^x + C''' \gamma^x + \&c. + \frac{-N}{1 + A + B + \&c. + M}$$

Cor. (1.) If $N = 0, A_1 = 0$; and $\therefore u_x = a^x$ gives the solution of the equation

$$u_{x+n} + A u_{x+n-1} + B u_{x+n-2} + C u_{x+n-3} + \&c. + M u_x = 0.$$

Cor. (2.) If the equation $a^n + A a^{n-1} + B a^{n-2} + \&c. + M = 0$ contain equal roots, then the result would not, under its present form, have a sufficient number of arbitrary constants, and would not therefore be a complete solution.

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

Let us suppose that α and β are equal; and $\therefore C'\alpha^x + C''\beta^x = (C' + C'')\alpha^x$, which may be written $C_1\alpha^x$.
In such a case suppose for an instant that $\beta = (\alpha + \alpha_1)$;

$$\begin{aligned}\therefore C'\alpha^x + C''\beta^x &= C'\alpha^x + C''(\alpha + \alpha_1)^x \\ &= C'\alpha^x + C'' \left(\alpha^x + x \cdot \alpha^{x-1} \cdot \alpha_1 + \frac{x(x-1)}{2} \cdot \alpha^{x-2} \cdot \alpha_1^2 + \&c. \right) \\ &= (C' + C'')\alpha^x + \frac{C''\alpha_1}{\alpha} \cdot \alpha^x \cdot x + \frac{C'' \cdot \alpha^x \cdot x(x-1)}{\alpha^2} \alpha_1^2 + \&c. ;\end{aligned}$$

and putting $C' + C'' = C_1$, and $\frac{C''\alpha_1}{\alpha} = C_2$, and making $\alpha_1 = 0$;

$$\therefore C'\alpha^x + C''\beta^x = (C_1 + C_2x)\alpha^x.$$

And, similarly, if there be three roots, as α, β, γ equal, we must put instead of $C'\alpha^x + C''\beta^x + C'''\gamma^x$ the term $(C_1 + C_2x + C_3x^2)\alpha^x$; and similarly, in the cases where a greater number of roots are equal.

And if any of the roots be impossible, still the expression for u_x may be rendered possible by introducing, instead of the impossible quantities, the cosines and sines of the multiple arc. We shall, however, take an example which will illustrate this remark.

(121.) The general term of a recurring series, the scale of relation of which is given, may now be found. By the scale of relation is meant the constant coefficients which multiply a certain number of the preceding terms of a series, so that by addition of the terms so multiplied a required term may be found.

As a particular instance, let the scale of relation be $p - q + r$: find the general term.

Here

$$u_{x+3} = p \cdot u_{x+2} - q \cdot u_{x+1} + r \cdot u_x.$$

Let

$$u_x = a^x; \therefore u_{x+3} = a^{x+3};$$

$$\therefore a^{x+3} = p a^{x+2} - q a^{x+1} + r a^x;$$

$$\therefore a^3 - p a^2 + q a - r = 0, \text{ dividing by } a^x.$$

Let α, β, γ be the roots of this equation; $\therefore \alpha^x, \beta^x, \gamma^x$ are particular integrals of the original equation, and hence the complete integral is

$$u^x = C'\alpha^x + C''\beta^x + C'''\gamma^x.$$

(122.) It is obvious from the preceding article that if the scale of relation consisted of (n) terms, that the value of u_x would be expressed by the sum of (n) terms also, or that we may write it

$$u_x = C'\alpha^x + C''\beta^x + C'''\gamma^x + \&c. + C^n \nu^x.$$

This result indicates a remarkable circumstance which belongs to the nature of recurring series.

Here put instead of x the numbers 1, 2, 3, 4, &c., and we shall have the successive terms of the series, or u_1, u_2, u_3, u_4 , &c.; and if we express the series which is the sum of u_1, u_2, u_3 , &c., by Σ ,

$$\begin{aligned}\Sigma &= C'(\alpha + \alpha^2 + \alpha^3 + \&c. \dots \alpha^n) \\ &\quad + C''(\beta + \beta^2 + \beta^3 + \&c. + \beta^n) \\ &\quad + C'''(\gamma + \gamma^2 + \gamma^3 + \&c. + \gamma^n) \\ &\quad + \&c. ;\end{aligned}$$

and hence it appears that a recurring series, of which the scale of relation is $p - q + r + \&c.$, is composed of (n) geometric series, the ratios of which are the roots of the equation

$$a^n - p a^{n-1} + q a^{n-2} - r a^{n-3} + \&c. = 0.$$

It is to be observed that the roots are here supposed all possible and different.

Example (1.) Find the general term of the series when the scale of relation is $4 - 3$.

Here

$$u_{x+2} = 4 u_{x+1} - 3 u_x.$$

Make

$$u_x = a^x; \therefore a^2 = 4a - 3, \text{ dividing by } a^x;$$

$$\therefore a^2 - 4a + 3 = 0; \therefore a = 3 \text{ or } 1,$$

and

$$u_x = C' \cdot 3^x + C'' \cdot 1^x = C' 3^x + C''.$$

Example (2.) Let the scale of relation be $6 - 9$: find the general term.

Here

$$u_{x+2} = 6 u_{x+1} - 9 u_x;$$

and

$$\therefore a^2 - 6a + 9 = 0, \text{ or } (a - 3)^2 = 0; \therefore a = 3;$$

and, as two roots are equal, we must assume

$$u_x = (C_1 + C_2x) 3^x.$$

Example (3.) The scale of relation being $2 \cos \theta - 1$, find the general term of the series.

Here

$$u_{x+2} = 2 \cos \theta \cdot u_{x+1} - u_x;$$

$$\therefore \text{making } u_x = a^x, a^2 - 2 \cos \theta \cdot a = -1;$$

$$\therefore a^2 - 2 \cos \theta \cdot a + \cos^2 \theta = \cos^2 \theta - 1 = -\sin^2 \theta;$$

$$\therefore a = \cos \theta \pm \sqrt{-1} \sin \theta;$$

and the roots of the equation are

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

Calculus of
Finite
Differences

and $\therefore$ the particular integrals are

$$(\cos \theta + \sqrt{-1} \sin \theta)^x \text{ and } (\cos \theta - \sqrt{-1} \sin \theta)^x, \text{ or } \cos x\theta + \sqrt{-1} \sin x\theta \text{ and } \cos x\theta - \sqrt{-1} \sin x\theta;$$

and the complete integral is $\therefore$

$$\begin{aligned} u_x &= C' (\cos x\theta + \sqrt{-1} \sin x\theta) + C'' (\cos x\theta - \sqrt{-1} \sin x\theta) \\ &= (C' + C'') \cos x\theta + (C' - C'') \sqrt{-1} \sin x\theta \\ &= B (\sin A \cdot \cos x\theta + \cos A \cdot \sin x\theta) \\ &= B \sin (A + x\theta), \end{aligned}$$

by making

$$C' + C'' = B \sin A \text{ and } (C' - C'') \sqrt{-1} = B \cdot \cos A.$$

If the first term of the series be $\sin A$, then when

$$x = 0 \quad u_0 = \sin A = B \cdot \sin A; \therefore B = 1,$$

and

$$u_x = \sin (A + x\theta),$$

which result may be easily verified.

(123.) Next to integrate the equation $u_{x+2} - A_x \cdot u_{x+1} + B_x \cdot u_x = D_x$, we shall first show that it is integrable whenever $u_{x+2} - A_x u_{x+1} + B_x u_x = 0$ can be integrated.

Let
and

$$u_{x+1} - \alpha_x u_x = v_x, \quad (1)$$

$$v_{x+1} - \beta_x v_x = D_x; \quad (2)$$

$$\therefore u_{x+2} - \alpha_{x+1} u_{x+1} = v_{x+1} = \beta_x \cdot v_x + D_x = \beta_x u_{x+1} - \beta_x \alpha_x u_x + D_x,$$

or

$$u_{x+2} - (\alpha_{x+1} + \beta_x) u_{x+1} + \beta_x \alpha_x u_x = D_x;$$

which being of the same form as the original equation will be identical with it, if

$$\alpha_{x+1} + \beta_x = A_x, \text{ and if } \beta_x \alpha_x = B_x.$$

But the integration of the equations (1) and (2) gives

$$u_x = P \cdot \alpha_{x-1} \left(\sum \frac{v_x}{P \cdot \alpha_x} + C \right)$$

$$v_x = P \beta_{x-1} \left(\sum \frac{D_x}{P \cdot \beta_x} + C' \right);$$

and substituting for v_x , we have

$$\begin{aligned} u_x &= P \cdot \alpha_{x-1} \left(\sum \cdot \frac{P \cdot \beta_{x-1} \cdot \left(\sum \frac{D_x}{P \cdot \beta_x} + C' \right)}{P \cdot \alpha_x} + C \right) \\ &= C \cdot P \cdot \alpha_{x-1} + C' P \alpha_{x-1} \sum \cdot \frac{P \cdot \beta_{x-1}}{P \cdot \alpha_x} + P \cdot \alpha_{x-1} \sum \frac{P \cdot \beta_{x-1}}{P \cdot \alpha_x} \cdot \sum \frac{D_x}{P \cdot \beta_x} \\ &= C U_x + C' U'_x + Q_x \text{ by substitution.} \end{aligned}$$

Hence it appears that when α_x and β_x are known, the equation is integrable; the sole difficulty is therefore to determine these quantities.

Now from the equations $\alpha_{x+1} + \beta_x = A_x$ and $\beta_x \alpha_x = B_x$ it is plain that the values of α_x and β_x are independent of D_x ; and $\therefore$ are the same when $D_x = 0$. But then $u_x = C \cdot U_x + C' U'_x$, which is $\therefore$ the solution of the equation $u_{x+2} - A_x u_{x+1} + B_x u_x = 0$; and if this be integrable the functions, U_x and U'_x , which are its first integrals, will be known; and then α_x, β_x can be found, as we shall now show.

Since

$$U_x = P \cdot \alpha_{x-1}; \therefore U_{x+1} = P \cdot \alpha_x; \therefore \alpha_x = \frac{U_{x+1}}{U_x},$$

and

$$U'_x = P \cdot \alpha_{x-1} \sum \left(\frac{P \cdot \beta_{x-1}}{P \cdot \alpha_x} \right); \therefore U'_{x+1} = P \cdot \alpha_x \cdot \sum \left(\frac{P \cdot \beta_x}{P \cdot \alpha_{x+1}} \right);$$

$$\therefore \frac{U'_x}{U_x} = \sum \cdot \frac{P \cdot \beta_{x-1}}{P \cdot \alpha_x} \text{ and } \frac{U'_{x+1}}{U_{x+1}} = \sum \cdot \frac{P \cdot \beta_x}{P \cdot \alpha_{x+1}};$$

$$\therefore \Delta \left(\frac{U'_x}{U_x} \right) = \frac{P \cdot \beta_{x-1}}{P \cdot \alpha_x} \text{ and } \Delta \cdot \left(\frac{U'_{x+1}}{U_{x+1}} \right) = \frac{P \cdot \beta_x}{P \cdot \alpha_{x+1}};$$

whence

$$\therefore \frac{P \cdot \beta_x}{P \cdot \alpha_{x-1}} = \beta_x \text{ and } \frac{P \cdot \alpha_{x+1}}{P \cdot \alpha_x} = \alpha_{x+1} = \frac{U_{x+2}}{U_{x+1}};$$

$$\therefore \beta_x = \frac{U_{x+2}}{U_{x+1}} \cdot \frac{\Delta\left(\frac{U'_{x+1}}{U_{x+1}}\right)}{\Delta\left(\frac{U'_x}{U_x}\right)};$$

and thus α_x and β_x being found, the complete solution of the equation which is the subject of this article is found.
(124.) In a similar manner the integration of the equation

$$u_{x+3} - A_x u_{x+2} + B_x u_{x+1} - C_x u_x = D_x$$

may be shown to be dependent upon that of

$$u_{x+3} - A_x u_{x+2} + B_x u_{x+1} - C_x u_x = 0.$$

In this case we assume that

$$u_{x+1} - \alpha_x u_x = v_x \quad (1)$$

$$v_{x+1} - \beta_x v_x = w_x \quad (2)$$

$$w_{x+1} - \gamma_x w_x = D_x; \quad (3)$$

whence by eliminating v_x , v_{x+1} , w_x , and w_{x+1} we obtain

$$u_{x+3} - (\alpha_{x+2} + \beta_{x+1} + \gamma_x) u_{x+2} + (\alpha_{x+1} \cdot \beta_{x+1} + \alpha_{x+1} \gamma_x + \beta_x \cdot \gamma_x) u_{x+1} - \alpha_x \cdot \beta_x \cdot \gamma_x \cdot u_x = D_x,$$

which equation will coincide with the original one, if the three following equations be true, viz.

$$\alpha_{x+2} + \beta_{x+1} + \gamma_x = A_x, \quad \alpha_{x+1} \cdot \beta_{x+1} + \alpha_{x+1} \cdot \gamma_x + \beta_x \cdot \gamma_x = B_x,$$

and

$$\alpha_x \beta_x \gamma_x = C_x;$$

and integrating the equations (1), (2), (3), and proceeding as in the last article, we finally obtain

$$U_x = C U_x + C' U'_x + C'' U''_x + Q_x,$$

where

$$Q_x = P \cdot \alpha_{x-1} \Sigma \cdot \frac{P \beta_{x-1}}{P \cdot \alpha_x} \cdot \Sigma \frac{P \cdot \gamma_{x-1}}{P \cdot \beta_x} \cdot \Sigma \frac{D_x}{P \cdot \gamma_x}.$$

Now, as in the last article, the difficulty of integrating the equation will depend upon our finding the values of α_x , β_x , γ_x : but these quantities remain the same when $D_x = 0$, and $\therefore Q_x = 0$, and $\therefore$ when $u_x = C U_x + C' U'_x + C'' U''_x$. But this is the complete integral of $u_{x+3} - A_x u_{x+2} + B_x u_{x+1} - C_x u_x = 0$, and α_x , β_x , γ_x may be found when U_x , U'_x , and U''_x are given. Hence the truth of the proposition.

(125.) The difficulty of the process just detailed is much diminished when A_x , B_x , &c. are constant. As an example, let

$$u_{x+2} + A \cdot u_{x+1} + B \cdot u_x = D_x.$$

Here let
and

$$u_{x+1} - \alpha u_x = v_x \quad (1),$$

$$v_{x+1} - \beta v_x = D_x \quad (2);$$

whence

$$u_{x+2} - (\alpha + \beta) u_{x+1} + \alpha \beta \cdot u_x = D_x;$$

$$\therefore \alpha + \beta = A \text{ and } \alpha \beta = B:$$

i. e. α and β are the roots of $x^2 - A x + B = 0$.

But from (1)

$$u_x = \alpha^{x-1} \cdot \left(C + \Sigma \frac{v_x}{\alpha^x} \right);$$

from (2)

$$v_x = \beta^{x-1} \left(C' + \Sigma \frac{D_x}{\beta^x} \right);$$

$$\therefore u_x = C \alpha^{x-1} + \alpha^{x-1} \cdot \Sigma \cdot \frac{\beta^{x-1} \cdot \left(C' + \Sigma \frac{D_x}{\beta^x} \right)}{\alpha^x}$$

$$= \frac{C}{\alpha} \alpha^x + C' \frac{\alpha^x}{\beta \alpha} \cdot \Sigma \left(\frac{\beta}{\alpha} \right)^{x-1} + \frac{\alpha^x}{\alpha \beta} \cdot \Sigma \left(\frac{\beta}{\alpha} \right)^x \cdot \Sigma \frac{D_x}{\beta^x}$$

$$= C_1 \alpha^x + C_2 \beta^x + \frac{\alpha^x}{B} \Sigma \left(\frac{\beta}{\alpha} \right)^x \cdot \Sigma \cdot \frac{D_x}{\beta^x}.$$

For

$$\Sigma \left(\frac{\beta}{\alpha} \right)^{x-1} = \frac{\alpha}{\beta} \Sigma \left(\frac{\beta}{\alpha} \right)^x = \frac{\alpha}{\beta} \cdot \frac{\left(\frac{\beta}{\alpha} \right)^x}{\frac{\beta}{\alpha} - 1};$$

$$\therefore \frac{C' \alpha^x}{\beta \alpha} \cdot \Sigma \left(\frac{\beta}{\alpha} \right)^{x-1} = \frac{C' \alpha^x}{\beta \alpha} \cdot \frac{\alpha^2}{\beta} \cdot \left(\frac{\beta}{\alpha} \right)^x \cdot \frac{1}{\beta - \alpha} = \frac{C'}{\beta^2 \cdot (\beta - \alpha)} \beta^x = C_2 \beta^x.$$

Calculus of
Finite
Differences.

(126.) The preceding articles lead us to the general proposition, that the integration of the equation

$$u_{x+n} + P_x \cdot u_{x+n-1} + Q_x \cdot u_{x+n-2} + \&c. + U_x \cdot u_x = V_x \quad (1)$$

may be shown to be always dependent upon that of

$$u_{x+n} + P_x u_{x+n-1} + Q_x u_{x+n-2} + \&c. + U_x \cdot u_x = 0 \quad (2).$$

The following proof of this proposition was given by Lagrange, and from its simplicity, as well as the close analogy it bears to the proof of the similar problem in the Differential Calculus, it merits a place here.

 Suppose we have obtained (n) particular integrals of the equation (2), and let these be

$$u'_x, u''_x, u'''_x, \&c.;$$

then the complete integral of the equation (2) will be

$$u_x = C' u'_x + C'' u''_x + C''' u'''_x + \&c.$$

 But if, instead of supposing $C', C'', C''', \&c.$ to be (n) constant quantities, we assume them to be functions of x represented by $C'_x, C''_x, C'''_x, \&c.$ we may then assume

$$u_x = C'_x u'_x + C''_x u''_x + C'''_x u'''_x + \&c.$$

to be the complete solution of equation (1).

 Now writing $x+1$ for x we have

$$\begin{aligned} u_{x+1} &= C'_{x+1} \cdot u'_{x+1} + C''_{x+1} \cdot u''_{x+1} + C'''_{x+1} u'''_{x+1} + \&c. \\ &= C'_x u'_{x+1} + C''_x \cdot u''_{x+1} + C'''_x u'''_{x+1} + \&c. \\ &\quad + u'_{x+1} \Delta C'_x + u''_{x+1} \cdot \Delta C''_x + u'''_{x+1} \cdot \Delta C'''_x + \&c., \end{aligned}$$

by putting

Now make

and we have

whence

$$C'_x + \Delta C'_x, C''_x + \Delta C''_x, \&c. \text{ for } C'_{x+1}, C''_{x+1}, \&c.$$

$$u'_{x+1} \cdot \Delta C'_x + u''_{x+1} \cdot \Delta C''_x + u'''_{x+1} \cdot \Delta C'''_x + \&c. = 0,$$

$$u_{x+1} = C'_x u'_{x+1} + C''_x u''_{x+1} + C'''_x \cdot u'''_{x+1} + \&c.;$$

$$\begin{aligned} u_{x+2} &= C'_{x+1} u'_{x+2} + C''_{x+1} \cdot u''_{x+2} + C'''_{x+1} \cdot u'''_{x+2} + \&c. \\ &= C'_x u'_{x+2} + C''_x u''_{x+2} + C'''_x u'''_{x+2} + \&c. \\ &\quad + u'_{x+2} \cdot \Delta C'_x + u''_{x+2} \cdot \Delta C''_x + u'''_{x+2} \cdot \Delta C'''_x + \&c. \\ &= C'_x u'_{x+2} + C''_x \cdot u''_{x+2} + C'''_x \cdot u'''_{x+2} + \&c., \end{aligned}$$

by making

$$u'_{x+2} \cdot \Delta C'_x + u''_{x+2} \cdot \Delta C''_x + u'''_{x+2} \cdot \Delta C'''_x + \&c. = 0;$$

and thus proceeding, we shall at length have

and

$$u_{x+n-1} = C'_x u'_{x+n-1} + C''_x \cdot u''_{x+n-1} + C'''_x u'''_{x+n-1} + \&c.,$$

$$u'_{x+n-1} \Delta C'_x + u''_{x+n-1} \cdot \Delta C''_x + u'''_{x+n-1} \cdot \Delta C'''_x + \&c. = 0;$$

 and putting x instead of $x-1$ in u_{x+n-1} we finally have

$$\begin{aligned} u_{x+n} &= C'_x u'_{x+n} + C''_x u''_{x+n} + C'''_x u'''_{x+n} + \&c. \\ &\quad + u'_{x+n} \cdot \Delta C'_x + u''_{x+n} \cdot \Delta C''_x + u'''_{x+n} \cdot \Delta C'''_x + \&c.; \end{aligned}$$

 and substituting in equation (1) these values instead of $u_x, u_{x+1},$ and $u_{x+n},$ and bearing in mind that

$$u'_{x+n} + P_x u'_{x+n-1} + Q_x u'_{x+n-2} + \&c. + U_x u'_x = 0$$

$$u''_{x+n} + P_x u''_{x+n-1} + Q_x u''_{x+n-2} + \&c. + U_x u''_x = 0$$

$$\&c.;$$

 and observing that the first line is the coefficient of C'_x , the second of C''_x , $\&c.$, the equation will be reduced to

$$u'_{x+n} \cdot \Delta C'_x + u''_{x+n} \Delta C''_x + u'''_{x+n} \cdot \Delta C'''_x + \&c. = V_x;$$

 and from this and the other $n-1$ equations which involve $\Delta C'_x, \Delta C''_x, \Delta C'''_x, \&c.$, each of these differences may be found in terms of x , and their integration will be that of a single variable only; and thus $C'_x, C''_x, C'''_x, \&c.$ being found, the complete integral of (1) will be determined.

 (127.) The integration of the equation (2) can, however, be seldom effected when there is not some given or obvious relation between the coefficients: it is, however, as we have seen, readily integrated when the coefficients are constant, by the substitution of a^x for the unknown function u_x , and then we can integrate equation (1).

Thus, let the equation to be integrated be

$$u_{x+2} + A u_{x+1} + B u_x = D_x \quad (1);$$

 then the solution of $u_{x+2} + A u_{x+1} + B u_x = 0$ is $u_x = C' \alpha^x + C'' \beta^x$, where α and β are the roots of the equation $u^2 + A u + B = 0$.

 Now putting C'_x and C''_x for C' and C'' , we may assume $u_x = C'_x \alpha^x + C''_x \beta^x$ to be the solution of (1);

 Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

$$\begin{aligned}\therefore u_{x+1} &= C'_{x+1} \alpha^{x+1} + C''_{x+1} \cdot \beta^{x+1} \\ &= C'_x \cdot \alpha^{x+1} + C''_x \beta^{x+1} + \alpha^{x+1} \Delta C'_x + \beta^{x+1} \cdot \Delta \cdot C''_x \\ &= C'_x \alpha^{x+1} + C''_x \beta^{x+1},\end{aligned}$$

Calculus of
Finite
Differences.

by making

$$\alpha^{x+1} \Delta C'_x + \beta^{x+1} \cdot \Delta \cdot C''_x = 0;$$

and

$$\therefore u_{x+2} = C'_x \alpha^{x+2} + C''_x \beta^{x+2} + \alpha^{x+2} \Delta C'_x + \beta^{x+2} \cdot \Delta C''_x$$

whence, by substituting in (1) for u_x , u_{x+1} , and u_{x+2} , we have

$$\alpha^{x+2} \cdot \Delta C'_x + \beta^{x+2} \cdot \Delta C''_x = D_x.$$

Also

$$\alpha^{x+1} \beta \cdot \Delta C'_x + \beta^{x+2} \cdot \Delta C''_x = 0;$$

$$\therefore \Delta C'_x = \frac{D_x}{(\alpha - \beta) \cdot \alpha^{x+1}}; \therefore C'_x = \frac{1}{\alpha - \beta} \cdot \Sigma \left(\frac{D_x}{\alpha^{x+1}} \right) + C_1,$$

and

$$\Delta \cdot C''_x = - \frac{D_x}{(\alpha - \beta) \beta^{x+1}}; \therefore C''_x = - \frac{1}{\alpha - \beta} \Sigma \frac{D_x}{\beta^{x+1}} + C_2;$$

$$\therefore u_x = C_1 \alpha^x + C_2 \beta^x + \frac{\alpha^x}{\alpha - \beta} \Sigma \frac{D_x}{\alpha^{x+1}} - \frac{\beta^x}{\alpha - \beta} \cdot \Sigma \frac{D_x}{\beta^{x+1}}.$$

(128.) The integration of the general equation of differences of the n th order, when the coefficients are functions of x , can seldom be effected.

In the simple case where $u_{x+1} = P_x u_x + Q_x$, we have already shown how the unknown function may be found. Laplace has given a very elegant process by which the equation may be solved; but as it leads only to the same result as we have already obtained, we pass by it and proceed to explain his method of reducing the equation to the one of lower degree.

Let
be the equation.

$$u_{x+n} = P_x u_{x+n-1} + Q_x u_{x+n-2} + \&c. + T_x u_{x+1} + U_x u_x + V_x$$

Suppose

$$\begin{aligned}u_{x+1} &= p_x u_x + q_x; \\ \therefore u_{x+2} &= p_{x+1} \cdot u_{x+1} + q_{x+1} \\ u_{x+3} &= p_{x+2} u_{x+2} + q_{x+2} \\ &\vdots \\ u_{x+n-1} &= p_{x+n-2} u_{x+n-2} + q_{x+n-2} \\ u_{x+n} &= p_{x+n-1} \cdot u_{x+n-1} + q_{x+n-1}.\end{aligned}$$

Multiply the first $n-1$ equations successively by the $n-1$ quantities $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_{n-1}$, and then add the products to the n th equation, and we shall obtain

$$\begin{aligned}u_{x+n} &= (p_{x+n-1} - \alpha_{n-1}) u_{x+n-1} + (\alpha_{n-1} \cdot p_{x+n-2} - \alpha_{n-2}) u_{x+n-2} \\ &\quad + (\alpha_{n-2} p_{x+n-3} - \alpha_{n-3}) u_{x+n-3} + \&c. + \alpha_1 p_x \cdot u_x \\ &\quad + q_{x+n-1} + \alpha_{n-1} \cdot q_{x+n-2} + \alpha_{n-2} \cdot q_{x+n-3} + \&c. + \alpha_1 q_x.\end{aligned}$$

Comparing this with the original equation we have the following results, if the equations be supposed identical:

$$\begin{aligned}P_x &= p_{x+n-1} - \alpha_{n-1} \\ Q_x &= \alpha_{n-1} \cdot p_{x+n-2} - \alpha_{n-2} \\ R_x &= \alpha_{n-2} \cdot p_{x+n-3} - \alpha_{n-3} \\ &\vdots \\ T_x &= \alpha_2 \cdot p_{x+1} - \alpha_1 \\ U_x &= \alpha_1 p_x \\ V_x &= q_{x+n-1} + \alpha_{n-1} \cdot q_{x+n-2} + \alpha_{n-2} \cdot q_{x+n-3} + \&c. + \alpha_1 q_x;\end{aligned}$$

whence we have

$$\begin{aligned}\alpha_{n-1} &= p_{x+n-1} - P_x \\ \alpha_{n-2} &= p_{x+n-1} \cdot p_{x+n-2} - p_{x+n-2} \cdot P_x - Q_x \\ \alpha_{n-3} &= p_{x+n-1} p_{x+n-2} \cdot p_{x+n-3} - p_{x+n-2} \cdot p_{x+n-3} \cdot P_x - p_{x+n-3} Q_x - R_x \\ &\vdots \\ \alpha_1 &= [p_{x+n-1}]^{n-1} - [p_{x+n-2}]^{n-2} \cdot P_x - [p_{x+n-3}]^{n-3} \cdot Q_x - \&c. - T_x;\end{aligned}$$

and

$$\therefore U_x = \alpha_1 p_x = [p_{x+n-1}]^n - [p_{x+n-2}]^{n-1} P_x - [p_{x+n-3}]^{n-2} Q_x - \&c. - T_x p_x;$$

an equation which is of the order $n-1$, since it contains only the values $p_x, p_{x+1} \dots p_{x+n-1}$ but which is not of the first degree. Suppose this to be solved, or that we know some value of p_x which will satisfy it, we shall then obtain the value $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_{n-1}$; and let their values be substituted in the equation

$$V_x = q_{x+n-1} + \alpha_{n-1} \cdot q_{x+n-2} + \alpha_{n-2} \cdot q_{x+n-3} + \&c. + \alpha_1 q_x,$$

Calculus of
Finite
Differences.

which is of $n-1$ dimensions, and this being integrated we shall obtain q_x , and thus finally to find u_x we have to solve the equation $u_{x+1} = p_x u_x + q_x$, of which the solution is known to be

Calculus of
Finite
Differences.

$$u_x = [p_{x-1}]^x \left\{ C + \sum \frac{q_x}{[p_x]^{x+1}} \right\}.$$

(129.) Although this process is difficult of application, yet there are cases in which it may be used with complete success; and, as an illustration, we will take the particular example

$$u_{x+3} = A p^x u_{x+2} + B p^{2x} u_{x+1} + C p^{3x} u_x.$$

Here it is obvious that as $V_x = 0$ $q_x = 0$.

Let

$$\therefore u_{x+1} = v_x u_x; \quad (1)$$

$$\therefore u_{x+2} = v_{x+1} \cdot u_{x+1} \quad (2)$$

$$u_{x+3} = v_{x+2} u_{x+2}; \quad (3)$$

.. multiply (1) and (2) by $\alpha_1 \alpha_2$ respectively, and adding their products to (3),

$$\begin{aligned} u_{x+3} &= (v_{x+2} - \alpha_2) u_{x+2} + (\alpha_2 v_{x+1} - \alpha_1) u_{x+1} + \alpha_1 v_x u_x; \\ \therefore v_{x+2} - \alpha_2 &= A \cdot p^x, \quad \alpha_2 v_{x+1} - \alpha_1 = B p^{2x}, \quad \alpha_1 v_x = C \cdot p^{3x}; \\ \therefore \alpha_2 &= v_{x+2} - A p^x \\ \alpha_1 &= v_{x+2} \cdot v_{x+1} - A \cdot v_{x+1} \cdot p^x - B p^{2x}, \end{aligned}$$

and

$$v_{x+2} \cdot v_{x+1} v_x - A v_{x+1} \cdot v_x \cdot p^x - B v_x p^{2x} - C p^{3x} = 0. \quad (4)$$

The difficulty is to solve this equation.

Now make

$$v_x = m \cdot p^{x+1};$$

$$\therefore v_{x+1} = m p^{x+2} \text{ and } v_{x+2} = m p^{x+3}.$$

Substituting these values in equation (4) we have

$$m^3 p^{3x+6} - A m^2 \cdot p^{3x+3} - B m \cdot p^{3x+1} - C \cdot p^{3x} = 0,$$

or

$$m^3 - \frac{A m^2}{p^3} - \frac{B \cdot m}{p^5} - \frac{C}{p^6} = 0$$

a cubic equation, of which let the roots be a, b, c . Then the values of v_x are $a p^{x+1}, b p^{x+1}, c p^{x+1}$.

Let

To integrate which let

$$v_x = a p^{x+1}; \therefore u_{x+1} = a p^{x+1} u_x.$$

$$u_x = a^{x-1} \cdot (p^1 p^2 \cdot p^3 \dots p^x) u'_x = a^{x-1} \cdot p^{\frac{x \cdot x+1}{2}} u'_x$$

$$\therefore u_{x+1} = a^x \cdot p^{\frac{x+1 \cdot x+2}{2}} u'_{x+1} = \therefore a^x \cdot p^{\frac{x \cdot x+1}{2}} p^{x+1} \cdot u'_x = a^x p^{\frac{x+1 \cdot x+2}{2}} \cdot u'_x;$$

$$\therefore u'_{x+1} = u'_x; \therefore \Delta \cdot u'_x = 0; \therefore u'_x = \text{a constant} = C_1 \cdot a,$$

and

$$\therefore u_x = C_1 a^x \cdot p^{\frac{x \cdot x+1}{2}}.$$

This is only the first integral of u_x , the others are $C_1 b^x \cdot p^{\frac{x \cdot x+1}{2}}$, and $C_2 c^x \cdot p^{\frac{x \cdot x+1}{2}}$; and $\therefore$ the complete integral will be

$$u_x = p^{\frac{x \cdot x+1}{2}} \cdot \{C_1 a^x + C_2 b^x + C_3 c^x\};$$

and by the same process the general equation

$$u_{x+n} = A p^x u_{x+n-1} + B p^{2x} u_{x+n-2} + \&c. L p^{4x} u_x$$

may be solved; and if $a, b, c, \&c. l$ be the roots of the equation

$$m^n - \frac{A m^{n-1}}{p^n} - \frac{B m^{n-1}}{p^{2n-1}} - \frac{C m^{n-2}}{p^{3n-3}} - \&c. - \frac{L}{p^{\frac{n \cdot n+1}{2}}} = 0,$$

then we shall find, as before, that

$$u_x = p^{\frac{x \cdot x+1}{2}} \{C_1 a^x + C_2 b^x + C_3 c^x + \&c. + C l^x\}.$$

(130.) In a similar manner may be integrated the more general equation

$$u_{x+n} = A X_{x+m} \cdot u_{x+n-1} + B \cdot X_{x+m} \cdot X_{x+m-1} \cdot u_{x+n-2} + \&c. + L \cdot X_{x+m} \dots X_{x+m-n+1} u_x$$

where $A, B, C, \&c. L$ are constant quantities.

As in the preceding example, make $u_{x+1} = u_x p_x$, and then to determine p_x we shall have the equation

$$[p_{x+n-1}]^n - A \cdot X_{x+m} \cdot [p_{x+n-1}]^{n-1} - B [X_{x+m}]^2 \cdot [p_{x+n-2}]^{n-2} - \&c. - L [X_{x+m}]^n = 0.$$

Assume

$$p_x = a \cdot X_{x+m-n+1}; \therefore p_{x+1} = a \cdot X_{x+m-n+2} \&c.;$$

$$\therefore [p_{x+n-1}]^n = a^n \cdot [X_{x+m}]^n, \text{ and } [p_{x+n-2}]^{n-1} = a^{n-1} \cdot [X_{x+m-1}]^{n-1} \cdot \&c. = \&c.;$$

Calculus of
Finite
Differences.

and substituting these values, the whole equation will be divisible by $[X_{x+m}]^n$, and we have the equation of (n) dimensions,

Calculus of
Finite
Differences.

$$a^n - A a^{n-1} - B a^{n-2} - C a^{n-3} - \&c. - L = 0.$$

Let the roots of this equation be $\alpha, \beta, \gamma, \delta, \&c.$, then the values of p_x will be, if for simplicity we put $m = n - 1$,

$$\alpha X_x, \beta X_x, \gamma X_x, \delta X_x, \&c.;$$

and $\therefore$ the value of u_x will be found from the integration of the equation

$$u_{x+1} = \alpha X_x \cdot u_x,$$

whence

$$u_x = C_1 \alpha^x \cdot [X_{x-1}]^x,$$

which is one of the first integrals, and $\therefore$ the complete integral will be

$$u_x = [X_{x-1}]^x \cdot \{C_1 \alpha^x + C_2 \beta^x + C_3 \gamma^x + C_4 \delta^x + \&c.\}.$$

Cor. Let
and

$$X_x = p^{x+1}; \therefore X_{x-1} = p^x \text{ and } [X_{x-1}]^x = p^{\frac{x \cdot x+1}{2}},$$

$$u_x = p^{\frac{x \cdot x+1}{2}} \{C_1 \alpha^x + C_2 \beta^x + C_3 \gamma^x + \&c.\},$$

as in the preceding example.

Cor. Hence may be integrated the equation

$$u_{x+n} = A \cdot x \cdot u_{x+n-1} + B x \cdot \overline{x-1} u_{x+n-2} + C \cdot x \cdot \overline{x-1} \overline{x-2} u_{x+n-3} + \&c. \\ + L \cdot x \cdot \overline{x-1} \overline{x-2} \dots 1 \cdot U_x;$$

the integral of which is

$$u_x = (1 \cdot 2 \cdot 3 \dots x) \{C_1 \alpha^x + C_2 \beta^x + C_3 \gamma^x + \&c. + C_n \lambda^x\},$$

where $\alpha, \beta, \gamma, \&c. \lambda$ are the roots of the equation

$$a^n - A a^{n-1} - B a^{n-2} - C a^{n-3} - \&c. - L = 0.$$

(131.) Of equations higher than the first degree little is known, and when solved, their solution is effected by particular artifices.

The equation $u_{x+1} \cdot u_x - a u_{x+1} + b u_x + c = 0$ may be thus solved.

Let

$$u_x = \frac{v_{x+1}}{v_x} + k;$$

and then we have

$$\left(\frac{v_{x+2}}{v_{x+1}} + k\right) \cdot \left(\frac{v_{x+1}}{v_x} + k\right) - a \left(\frac{v_{x+2}}{v_{x+1}} + k\right) + b \left(\frac{v_{x+1}}{v_x} + k\right) + c = 0,$$

or

$$\frac{v_{x+2}}{v_x} + (k-a) \frac{v_{x+2}}{v_{x+1}} + (k+b) \frac{v_{x+1}}{v_x} + k^2 - a k + b k + c = 0.$$

Let $k - a = 0$, or $k = a$, and multiply by v_x ,

$$\therefore v_{x+2} + (a+b) v_{x+1} + (b a + c) v_x = 0;$$

the integral of which may be found by putting $v_x = m^x$; and if α and β be the roots of $m^2 + (a+b)m + b a + c = 0$, the complete integral will be $v_x = p \alpha^x + q \beta^x$, where p and q are constant quantities;

$$\therefore u_x = \frac{v_{x+1}}{v_x} + k = \frac{p \alpha^{x+1} + q \beta^{x+1}}{p \alpha^x + q \beta^x} + \alpha \\ = \frac{\alpha^{x+1} + C \beta^{x+1}}{\alpha^x + C \beta^x} + k, \text{ if } C = \frac{p}{q}.$$

(132.) The following example may be solved by a particular artifice.

Example. Let

$$u_{x+2} \cdot u_{x+1} \cdot u_x = a (u_{x+2} + u_{x+1} + u_x);$$

then, since when

$$A + B + C = 0, \tan A \cdot \tan B \cdot \tan C = \tan A + \tan B + \tan C.$$

Let

$$u_x = \sqrt{a} \tan v_x; \therefore u_{x+1} = \sqrt{a} \cdot \tan v_{x+1}, \text{ and } u_{x+2} = \sqrt{a} \tan v_{x+2};$$

$$\therefore a^{\frac{3}{2}} (\tan v_{x+2} \cdot \tan v_{x+1} \cdot \tan v_x) = a^{\frac{3}{2}} (\tan v_{x+2} + \tan v_{x+1} + \tan v_x);$$

$$\therefore \tan v_{x+2} \cdot \tan v_{x+1} \cdot \tan v_x = \tan v_{x+2} + \tan v_{x+1} + \tan v_x;$$

$$\therefore v_{x+2} + v_{x+1} + v_x = 0.$$

$$x = b^x; \therefore b^2 + b + 1 = 0;$$

Make
whence

$$b = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm \sqrt{-1} \frac{\sqrt{3}}{2} = \cos \frac{2\pi}{3} \pm \sqrt{-1} \sin \frac{2\pi}{3};$$

$$\therefore b^x = \cos \frac{2\pi x}{3} \pm \sqrt{-1} \sin \frac{2\pi x}{3};$$

$$\therefore v_x = C' \left(\cos \frac{2\pi x}{3} + \sqrt{-1} \sin \frac{2\pi x}{3} \right) + C'' \left(\cos \frac{2\pi x}{3} - \sqrt{-1} \sin \frac{2\pi x}{3} \right) \\ = C_1 \cos \frac{2\pi x}{3} + C_2 \sin \frac{2\pi x}{3};$$

whence

$$C_1 = C' + C'' \text{ and } C_2 = (C' - C'')\sqrt{-1}; \\ \therefore u_x = \sqrt{a} \cdot \tan \left\{ C_1 \cos \frac{2\pi x}{3} + C_2 \sin \frac{2\pi x}{3} \right\}.$$

For this and similar examples, see Herschell's Examples on Finite Differences.
(133.) By a similar substitution may be integrated the equation

For

$$u_{x+1} - 2u_x^2 + 1 = 0, \text{ or } u_{x+1} = 2u_x^2 - 1.$$

whence

$$\therefore \cos 2A = 2\cos^2 A - 1, \therefore \text{let } u_x = \cos v_x;$$

whence

$$\therefore u_{x+1} = \cos v_{x+1}; \therefore \cos v_{x+1} = 2(\cos v_x)^2 - 1;$$

$$v_{x+1} = 2v_x; \therefore \text{if } v_x = a^x, \therefore a = 2 \text{ and } v_x = C \cdot 2^x;$$

$$u_x = \cos v_x = \cos (C \cdot 2^x) = \frac{1}{2} (e^{2^x \cdot C\sqrt{-1}} + e^{-2^x \cdot C\sqrt{-1}}) = \frac{1}{2} (C_1^{2^x} + C_1^{-2^x}).$$

(134.) The equation $u_x = V_x \cdot u_{x-1}^{\alpha} \cdot u_{x-2}^{\beta} \cdot u_{x-3}^{\gamma} \dots u_{x-n}^{\nu}$, which expresses that the general term of a series is a function of the product of the preceding terms, may be easily referred to the solution of an equation of the n th order.

Taking the logarithm of each side of the equation,

$$\log u_x = \log V_x + \alpha \log u_{x-1} + \beta \log u_{x-2} + \gamma \log u_{x-3} + \&c. + \nu \log u_{x-n};$$

for $\log V_x$ put U_x , and for $\log u_x$ write v_x ;

$$\therefore v_x = U_x + \alpha v_{x-1} + \beta v_{x-2} + \gamma v_{x-3} + \&c. + \nu v_{x-n}.$$

If U_x be constant, this equation is easily integrable by making

$$v_{x-n} = a^x, \text{ and } \therefore v_x = a^{x+n}.$$

(135.) Equations of mixed differences, when the coefficients are constants, admit of integration by substituting $e^{kx} + H$ for u_x . As an example, we will take the following equation,

$$u_x + A u_{x+1} + B \cdot \frac{d \cdot u_x}{dx} + C \cdot \frac{d \cdot u_{x+1}}{dx} + k = 0;$$

$$\therefore e^{kx} + H + A e^{k(x+1)} + A H + B k e^{kx} + C k e^{k(x+1)} + k = 0.$$

Make

$$H + A H + k = 0, \text{ or } H = \frac{-k}{1+A};$$

$$\therefore e^{kx} + A e^{k(x+1)} + B k e^{kx} + C k e^{k(x+1)} = 0$$

$$1 + A e^k + B k + C k e^k = 0.$$

Let $k_1, k_2, k_3, \&c.$ be the values of k found from this equation, then will

$$v_x = C_1 e^{k_1 x} + C_2 e^{k_2 x} + C_3 e^{k_3 x} + \&c. - \frac{k}{1+A}$$

$$= C_1 \alpha^x + C_2 \beta^x + C_3 \gamma^x + \&c. - \frac{k}{1+A},$$

by making the proper substitutions.

(136.) Want of space alone prevents any mention of those equations of differences which involve more than one variable; the reader is referred to the great work of Lacroix, and to various Papers by Laplace, as well as that invaluable collection of Examples of Sir John Herschell, the almost constant subject of our reference. We shall conclude this portion of our Treatise by an application of the theory of continued fractions to equations of finite differences.

(137.) In the *Mécanique Céleste*, vol. iv., p. 254, Laplace has shown that every linear equation of the second order of finite differences may be reduced to a continued fraction. The following is the method used by him.

Let the equation be expressed by

$$u_x = a_x u_{x+1} + b_x u_{x+2};$$

$$\therefore \frac{u_x}{u_{x+1}} = a_x + b_x \cdot \frac{u_{x+2}}{u_{x+1}};$$

and

$$\therefore \frac{u_{x+1}}{u_x} = \frac{1}{a_x + b_x \cdot \frac{u_{x+2}}{u_{x+1}}};$$

Calculus of
Finite
Differences.

and

Calculus of
Finite
Differences.

$$\frac{u_{x+2}}{u_{x+1}} = \frac{1}{a_{x+1} + b_{x+1} \cdot \frac{u_{x+3}}{u_{x+2}}},$$

$$\frac{u_{x+3}}{u_{x+2}} = \frac{1}{a_{x+2} + b_{x+2} \cdot \frac{u_{x+4}}{u_{x+3}}}, \text{ \&c. ;}$$

and thus substituting we can form the continued fraction

$$\frac{u_{x+1}}{u_x} = \frac{1}{a_x + \frac{b_x}{a_{x+1} + \frac{b_{x+1}}{a_{x+2} + \text{\&c.}}}}$$

And making $a_x = 1$, $b_x = (x+1)q$, or $u_x = u_{x+1} + x+1 \cdot q u_{x+2}$, then

$$\frac{u_{x+1}}{u_x} = \frac{1}{1 + \frac{x+1 \cdot q}{1 + \frac{x+2 \cdot q}{1 + \frac{x+3 \cdot q}{1 + \text{\&c.}}}}}$$

And, finally, making $x = 0$,

$$\frac{u_1}{u_0} = \frac{1}{1 + \frac{2q}{1 + \frac{3q}{1 + \frac{4q}{1 + \text{\&c.}}}}}$$

fractions which express in an elegant manner the relations which exist between two successive terms of a series.

(138.) And, conversely, we may show how the value of a continued fraction can be found by means of an equation of finite differences.

Thus, let $\frac{U}{V}$ be the true value of the fraction

$$= \frac{c}{a + \frac{c}{a + \frac{c}{a + \text{\&c.}}}}$$

and let

$$\frac{u_1}{v_1}, \frac{u_2}{v_2}, \frac{u_3}{v_3} \dots \frac{u_x}{v_x}$$

represent the successive convergent fractions; then it is shown (see ALGEBRA, art. 234.) that

$$u_{x+2} = a u_{x+1} + c u_x,$$

and

$$v_{x+2} = a v_{x+1} + c v_x;$$

whence, by integration,

$$u_x = C_1 \alpha^x + C_2 \beta^x,$$

and

$$v_x = C' \alpha^x + C'' \beta^x,$$

where α and β are the roots of $u^2 = a u + c$;

$$\therefore \frac{u_x}{v_x} = \frac{C_1 \alpha^x + C_2 \beta^x}{C' \alpha^x + C'' \beta^x} = \frac{C_1 + C_2 \left(\frac{\beta}{\alpha}\right)^x}{C' + C'' \left(\frac{\beta}{\alpha}\right)^x},$$

where (α) is supposed to be the greater of the roots α, β ; whence, if F_∞ be the value of the fraction continued to infinity,

$$F_\infty = \frac{C_1}{C'}.$$

(139.) To find the constants C_1, C_2, C', C'' , we have, since

$$\begin{aligned} u_1 &= c, u_2 = c a, \text{ and } v_1 = a \text{ and } v_2 = a^2 + c; \\ \therefore c &= C_1 \alpha + C_2 \beta & a &= C' \alpha + C'' \beta \\ c a &= C_1 \alpha^2 + C_2 \beta^2 & a^2 + c &= C' \alpha^2 + C'' \beta^2; \end{aligned}$$

Calculus of whence
Finite
Differences.

but

$$C_1 = c \frac{a - \beta}{\alpha^2 - \beta \alpha} = \text{and } C' = \frac{a^2 + c - a\beta}{\alpha^2 - \beta \alpha},$$

$$\alpha + \beta = a \text{ and } c = -\alpha\beta; \therefore a - \beta = \alpha \text{ and } a^2 - a\beta = a\alpha;$$

$$\therefore C_1 = \frac{c}{\alpha - \beta}; \therefore C' = \frac{c + a\alpha}{\alpha^2 - \beta \alpha} = \frac{a - \beta}{\alpha - \beta} = \frac{\alpha}{\alpha - \beta};$$

and

$$C_2 \beta = c - C_1 \alpha = c - \frac{c\alpha}{\alpha - \beta} = \frac{-c\beta}{\alpha - \beta}; \therefore C_2 = \frac{-c}{\alpha - \beta} = -C_1$$

$$C'' \beta = a - C' \alpha = a - \frac{\alpha^2}{\alpha - \beta} = \frac{a\alpha - a\beta - \alpha^2}{\alpha - \beta} = \frac{\alpha\beta - a\beta}{\alpha - \beta} = \frac{-\beta^2}{\alpha - \beta};$$

$$\therefore C'' = \frac{-\beta}{\alpha - \beta} = \frac{-\beta}{\alpha} C';$$

$$\therefore \frac{u_x}{v_x} = \frac{C_1 (\alpha^x - \beta^x)}{C' \left(\alpha^x - \frac{\beta}{\alpha} \beta^x \right)} = \frac{c \cdot (\alpha^x - \beta^x)}{\alpha^{x+1} - \beta^{x+1}},$$

a formula of singular neatness.

If we divide each term by α^x , and make $x = \alpha^y$, then, since $\frac{\beta}{\alpha}$ is a proper fraction,

$$F_\infty = \frac{C_1}{C'} = \frac{c}{\alpha}.$$

Example (1.) Find the value of the fraction $\frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \&c.}}}$ to infinity.

Here
whence

$$\alpha = 1, c = 1; \therefore u^2 - u = 1;$$

$$\alpha = \frac{\sqrt{5} + 1}{2};$$

$$\therefore F_\infty = \frac{c}{\alpha} = \frac{2}{\sqrt{5} + 1} = \frac{\sqrt{5} - 1}{2}.$$

Example (2.) Find the value of $\frac{2}{1 + \frac{2}{1 + \frac{2}{1 + \&c.}}}$ to x terms.

Here

$$c = 2, a = 1, u^2 - u = 2; \therefore \alpha = 2 \text{ and } \beta = -1;$$

$$\therefore \frac{u_x}{v_x} = \frac{2 \cdot (2^x - (-1)^x)}{2^{x+1} - (-1)^{x+1}} = \frac{2^{x+1} - 2(-1)^x}{2^{x+1} - (-1)^{x+1}}.$$

The Theory of Generating Functions.

(140.) Let u_x be any function whatever of x ; and let us suppose $\phi(t)$ to be such a function of t , that its developement according to the powers of t shall produce the series

$$\dots + u_{-1} t^{-1} + u_0 + u_1 t + u_2 t^2 + u_3 t^3 + \&c. + u_{x-1} t^{x-1} + u_x \cdot t^x + u_{x+1} \cdot t^{x+1} + \&c.;$$

then $\phi(t)$ is said to be the *generating function* of u_x .

In this expansion we observe that u_x is the coefficient of t^x ; and, in general, the *generating function* of any variable whatever, as u_x , is defined to be such a function of t as, when expanded according to the powers of t , shall have this variable for the coefficient of t^x ; and, conversely, the variable corresponding to a *generating function* is the coefficient of t^x in the expansion of the function according to the powers of t .

It must be observed that the series for $\phi(t)$ is supposed to extend both ways, and thus include negative as well as positive powers of t .

(141.) As examples of these generating functions. Since

$$\frac{t}{(1-t)^2} = 1t + 2t^2 + 3t^3 + 4 \cdot t^4 + \&c. + xt^x + \overline{x+1} t^{x+1} + \&c.,$$

and

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

$$\log \frac{1}{(1-t)} = \frac{t}{1} + \frac{t^2}{2} + \frac{t^3}{3} + \frac{t^4}{4} + \&c. + \frac{t^x}{x} + \frac{t^{x+1}}{x+1} + \&c.;$$

Calculus of
Finite
Differences.

then $\frac{t}{(1-t)^2}$ and $\log \frac{1}{(1-t)}$ are respectively the generating functions of x and $\frac{1}{x}$. For x and $\frac{1}{x}$ are the coefficients of t^x in their respective series.

(142.) Resuming the expansion for $\phi(t)$, we have

$$\phi(t) = \dots + u_0 + u_1 t + u_2 t^2 + u_3 t^3 + \&c. \dots + u_{x-1} t^{x-1} + u_x t^x + u_{x+1} t^{x+1} + \&c.;$$

$$\therefore t \phi(t) = u_0 t + u_1 t^2 + u_2 t^3 + \&c. \dots + u_{x-1} t^x + u_x t^{x+1} + \&c.;$$

in which series, as u_{x-1} is the coefficient of t^x , therefore $t \phi(t)$ is the generating function of u_{x-1} . Again, multiplying by t , we shall have u_{x-2} for the coefficient of t^x in the expansion of $t^2 \phi(t)$; hence $t^2 \phi(t)$ is the generating function of u_{x-2} ; and thus, finally, $t^n \phi(t)$ will be the generating function of u_{x-n} .

Again, if we divide $\phi(t)$ by t , we shall have

$$\frac{1}{t} \phi(t) = \dots + u_1 + u_2 t + u_3 t^2 + \&c. + u_x t^{x-1} + u_{x+1} t^x + u_{x+2} t^{x+1} + \&c.;$$

or $\frac{1}{t} \phi(t)$ is the generating function of u_{x+1} .

Similarly $\frac{1}{t^2} \phi(t)$ is the generating function of u_{x+2} , and $\frac{1}{t^n} \phi(t)$ that of u_{x+n} .

Whence it appears that if we know the generating function of u_x we can immediately obtain those of u_{x-n} or of u_{x+n} by multiplying or dividing by t^n .

(143.) If we take the difference between the expansions of $\frac{1}{t} \phi(t)$ and $\phi(t)$, the coefficient of t^x will be $u_{x+1} - u_x$ or Δu_x , therefore the generating function of Δu_x is $\left(\frac{1}{t} - 1\right) \phi(t)$. Let this be written $\phi'(t)$, then the generating function of $\Delta u_{x+1} - \Delta u_x$ or $\Delta^2 u_x$ will be $\left(\frac{1}{t} - 1\right) \phi' t = \left(\frac{1}{t} - 1\right)^2 \phi t$.

Similarly, the generating function of $\Delta^3 u_x$ is $\left(\frac{1}{t} - 1\right)^3 \phi(t)$, and that of $\Delta^n u_x$ is $\left(\frac{1}{t} - 1\right)^n \phi(t)$.

(144.) We have hitherto considered Δu_x to be equal to $u_{x+1} - u_x$; it might, however, be supposed to be any combination of the quantities $u_x, u_{x+1}, u_{x+2}, \&c.$; suppose it to be $a u_x + b u_{x+1} + c u_{x+2} + \&c. + q u_{x+n}$, and call this quantity ∇u_x .

Then since the coefficient of t^x in the development of

$$\phi(t) \left\{ a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} + \&c. + \frac{q}{t^n} \right\}$$

is

$$a u_x + b u_{x+1} + c u_{x+2} + e u_{x+3} + \&c. + q u_{x+n};$$

$$\therefore \phi(t) \left\{ a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} + \&c. + \frac{q}{t^n} \right\}$$

is the generating function of ∇u_x ; and if, as before, we call this function $\phi'(t)$, and if $\nabla^2 u_x$ be what ∇u_x becomes when ∇u_x is written for u_x , then

$$\phi'(t) \left(a + \frac{b}{t} + \frac{c}{t^2} + \&c. + \frac{q}{t^n} \right) \text{ or } \phi(t) \left\{ a + \frac{b}{t} + \frac{c}{t^2} + \&c. + \frac{q}{t^n} \right\}^2$$

will be the generating function of $\nabla^2 u_x$; and, in general, the generating function of $\nabla^m u_x$ is

$$\phi(t) \left\{ a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} + \&c. + \frac{q}{t^n} \right\}^m.$$

(145.) To extend the reasoning to integrals, let z be the generating function of $\Sigma^n u_x$; then, from what has preceded, $\left(\frac{1}{t} - 1\right)^n z$ will be the generating function of $\Delta^n (\Sigma^n u_x)$ or of u_x ; but as $\phi(t)$ is the generating function of u_x we may put

$$z \cdot \left(\frac{1}{t} - 1\right)^n = \phi(t), \text{ and } \therefore z = \phi(t) \cdot \left(\frac{1}{t} - 1\right)^{-n};$$

whence it appears that we can obtain the generating function of $\Sigma^n u_x$ from that of $\Delta^n u_x$, by merely changing (n) into $(-n)$. We have here neglected the constants and the powers of x which are introduced into the value of the integral by each successive integration; but to make the generating function complete we must assume

$$z = \phi(t) \left(\frac{1}{t} - 1\right)^{-n} + \frac{A_1}{(1-t)^n} + \frac{A_2 t}{(1-t)^n} + \frac{A_3 t^2}{(1-t)^n} + \&c. + \frac{A_n \cdot t^{n-1}}{(1-t)^n};$$

Calculus of Finite Differences. for these fractions being expanded, the coefficients of t^x in the expanded series will give the (n) constants and the powers of x , which are requisite to complete the integral. Calculus of Finite Differences. As an example, let $n = 1$. Then the integration of $\sum u_x$ will introduce but one constant, and we have

$$z = \phi(t) \left(\frac{1}{t} - 1 \right)^{-1} + \frac{A}{1-t}.$$

But the coefficient of t^x in $\frac{A}{1-t}$ is only A , and

$$\therefore \sum(u_x) = f(x) + A.$$

Again, let $n = 2$;

$$\therefore z = \phi(t) \left(\frac{1}{t} - 1 \right)^{-2} + \frac{A_1}{(1-t)^2} + \frac{A_2 t}{(1-t)^2}.$$

But the coefficients of t^x in $\frac{1}{(1-t)^2}$ and $\frac{t}{(1-t)^2}$ are $x+1$ and x ;

$$\therefore \sum^2 u_x = \sum f(x) + (A_1 + A_2)x + A_1 = \sum f(x) + Bx + A_1.$$

(146.) We shall now show the use of the calculus of generating functions, in proving some of the theorems demonstrated in the preceding pages.

The generating function of $\Delta^n u_x$ being $\left(\frac{1}{t} - 1 \right)^n \phi(t)$, expand $\left(\frac{1}{t} - 1 \right)^n$, and multiply each term by $\phi(t)$, and we have

$$\frac{1}{t^n} \phi(t) - n \frac{1}{t^{n-1}} \phi(t) + \frac{n \overline{n-1}}{1 \cdot 2} \frac{1}{t^{n-2}} \phi(t) - \frac{n \overline{n-1} \overline{n-2}}{1 \cdot 2 \cdot 3} \frac{1}{t^{n-3}} \phi(t) + \&c.;$$

but since $\frac{1}{t^n} \phi(t)$ is the generating function of u_{x+n} , and $\frac{1}{t^{n-1}} \phi(t)$ that of u_{x+n-1} , &c. this expansion will be the generating function of

$$u_{x+n} - n u_{x+n-1} + \frac{n \overline{n-1}}{1 \cdot 2} u_{x+n-2} - \frac{n \overline{n-1} \overline{n-2}}{1 \cdot 2 \cdot 3} u_{x+n-3} + \&c.,$$

as well as of $\Delta^n u_x$. Hence, since quantities that have the same generating functions are equal,

$$\Delta^n u_x = u_{x+n} - n u_{x+n-1} + \frac{n \overline{n-1}}{1 \cdot 2} u_{x+n-2} - \frac{n \overline{n-1} \overline{n-2}}{1 \cdot 2 \cdot 3} u_{x+n-3} + \&c.$$

(147.) Again, since the generating function of u_{x+n} is $\frac{1}{t^n} \phi(t)$; and since $\frac{1}{t^n} = \left(1 + \frac{1}{t} - 1 \right)^n$;

$$\begin{aligned} \therefore \frac{1}{t^n} \phi(t) &= \phi(t) + n \cdot \left(\frac{1}{t} - 1 \right) \phi(t) + \frac{n \overline{n-1}}{1 \cdot 2} \left(\frac{1}{t} - 1 \right)^2 \phi(t) \\ &+ \frac{n \overline{n-1} \overline{n-2}}{1 \cdot 2 \cdot 3} \left(\frac{1}{t} - 1 \right)^3 \phi(t) + \&c. \end{aligned}$$

But $\phi(t)$ is the generating function of u_x , and $\left(\frac{1}{t} - 1 \right) \phi(t)$ is that of Δu_x , $\left(\frac{1}{t} - 1 \right)^2 \phi(t)$ that of $\Delta^2 u_x$, &c.

Hence $\frac{1}{t^n} \phi(t)$ is not only the generating function of u_{x+n} , but also of

$$u_x + n \Delta u_x + \frac{n \cdot \overline{n-1}}{1 \cdot 2} \Delta^2 u_x + \frac{n \overline{n-1} \cdot \overline{n-2}}{1 \cdot 2 \cdot 3} \Delta^3 u_x + \&c.;$$

and therefore this last series is equal to u_{x+n} .

(148.) Again, we have seen that if

$$\nabla u_x = a u_x + b u_{x+1} + c u_{x+2} + \&c. + q \cdot u_{x+n},$$

the generating function of $\nabla^m u_x$ will be

$$\left(a + \frac{b}{t} + \frac{c}{t^2} + \frac{e}{t^3} + \&c. + \frac{q}{t^n} \right)^m \phi(t);$$

and if the polynomial be expanded, and each term be multiplied by $\phi(t)$, the generating function of $\nabla^m u_x$ will be of the form

$$a_0 \phi(t) + a_1 \frac{1}{t} \phi(t) + a_2 \frac{1}{t^2} \phi(t) + a_3 \frac{1}{t^3} \phi(t) + a_4 \frac{1}{t^4} \phi(t) + \&c.;$$

whence it is obvious that

$$\nabla^m u_x = a_0 u_x + a_1 u_{x+1} + a_2 u_{x+2} + a_3 u_{x+3} + \&c.$$

(149.) The series obtained in the preceding articles for u_{x+n} are useful in finding the value of that term, by means of other known terms, and as in this consists the theory of interpolation, we shall now obtain such series as are most useful for that purpose.

Calculus of
Finite
Differences.

(1.) Since $\frac{1}{t^n} \phi(t)$, which is the generating function of $u_{x+n} = \frac{1}{\left(1 - t \frac{1}{t} - 1\right)^n} \phi(t)$,

Calculus of
Finite
Differences.

which may be written

$$\begin{aligned} & \left(1 - t \frac{1}{t} - 1\right)^{-n} \phi(t), \\ &= \phi(t) + n \left(\frac{1}{t} - 1\right) \cdot t \phi(t) + \frac{n \overline{n+1}}{1 \cdot 2} \left(\frac{1}{t} - 1\right)^2 \cdot t^2 \phi(t) \\ & \quad + \frac{n \overline{n+1} \overline{n+2}}{1 \cdot 2 \cdot 3} \left(\frac{1}{t} - 1\right)^3 \cdot t^3 \phi(t) + \&c. \dots \end{aligned}$$

But $t \cdot \phi(t)$ is the generating function of u_{x-1} , and $\left(\frac{1}{t} - 1\right) t \cdot \phi(t)$ is that of Δu_{x-1} , and $\left(\frac{1}{t} - 1\right) \cdot t^2 \phi(t)$ that of $\Delta^2 u_{x-2}$, and $\left(\frac{1}{t} - 1\right)^n t^n \phi(t)$ that of $\Delta^n u_{x-n}$;

$$\begin{aligned} \therefore u_{x+n} &= u_x + n \cdot \Delta u_{x-1} + \frac{n \cdot \overline{n+1}}{1 \cdot 2} \Delta^2 u_{x-2} + \frac{n \overline{n+1} \overline{n+2}}{1 \cdot 2 \cdot 3} \Delta^3 u_{x-3} \\ & \quad + \&c.: \text{ see Herschell's Examples.} \end{aligned}$$

(150.) This theorem may, by an extension of the principle of separating the symbols of operation from those of quantity, assume a very remarkable form.

For considering u_x as a factor, and writing $u_{x-n} = u_x \cdot u_{-n}$, and considering u_{-n} as the representative of $\frac{1}{u^n}$.

$$\begin{aligned} \text{Then } u_{x+n} &= u_x \cdot \left\{ 1 + n \cdot \Delta \cdot u_{-1} + \frac{n \overline{n+1}}{2} \Delta^2 u_{-2} + \frac{n \overline{n+1} \overline{n+2}}{1 \cdot 2 \cdot 3} \Delta^3 u_{-3} + \&c. \right\} \\ &= u_x \left\{ 1 + n \frac{\Delta}{u} + \frac{n \overline{n+1}}{2} \frac{\Delta^2}{u^2} + \frac{n \overline{n+1} \overline{n+2}}{1 \cdot 2 \cdot 3} \frac{\Delta^3}{u^3} + \&c. \right\} \\ &= u_x \left(1 - \frac{\Delta}{u} \right)^{-n} = \frac{u_x}{\left(1 - \frac{\Delta}{u} \right)^n}; \end{aligned}$$

and in thus giving to the symbol u_{-n} a new interpretation, no error can arise if, when we expand $\left(1 - \frac{\Delta}{u} \right)^{-n}$, we remember that the powers of u are to be changed into negative suffixes.

(151.) Next let $\frac{1}{t} = 1 + \frac{\alpha}{t^r}$, and let u_{x+n} be required, of which the generating function is $\frac{1}{t^n} \cdot \phi(t)$.

Make $\frac{1}{t} = y$, $\therefore y = 1 + \alpha y^r$; and by Lagrange's theorem for the expansion of functions

$$\phi(t) \cdot y^n = \frac{\phi(t)}{t^n} = \left\{ 1 + n\alpha + \frac{n \overline{n+2r-1}}{1 \cdot 2} \cdot \alpha^2 + \frac{n \cdot \overline{n+3r-1} \cdot \overline{n+3r-2}}{1 \cdot 2 \cdot 3} \alpha^3 \right. \\ \left. + \frac{n \cdot \overline{n+4r-1} \cdot \overline{n+4r-2} \cdot \overline{n+4r-3}}{2 \cdot 3 \cdot 4} \alpha^4 + \&c. \right\} \phi(t).$$

And (α) being equal $t^r \cdot \left(\frac{1}{t} - 1\right)$, $\therefore \alpha \cdot \phi(t)$ is the generating function of $\Delta \cdot u_{x-r}$, and $\alpha^2 \phi(t)$, or $t^{2r} \cdot \left(\frac{1}{t} - 1\right)^2 \cdot \phi(t)$ is the generating function of $\Delta^2 \cdot u_{x-2r}$; $\alpha^3 \phi(t)$ that of $\Delta^3 u_{x-3r}$, &c.; and as $\frac{\phi(t)}{t^n}$ is that of u_{x+n} ,

$$\begin{aligned} \therefore u_{x+n} &= u_x + n \cdot \Delta \cdot u_{x-r} + \frac{n \cdot \overline{n+2r-1}}{1 \cdot 2} \cdot \Delta^2 u_{x-2r} + \frac{n \cdot \overline{n+3r-1} \cdot \overline{n+3r-2}}{2 \cdot 3} \Delta^3 u_{x-3r} \\ & \quad + \&c. \qquad \qquad \qquad + \&c. \end{aligned}$$

(152.) A more general method of interpolation will be found if we suppose $t \cdot \left(\frac{1}{t} - 1\right)^z = z$, and we find an expression for $\frac{1}{t^n}$ in a series arranged according to the powers of z .

It is clear that $\frac{1}{t^n}$ is the same thing as the coefficient of α^n in the developement of the fraction $\frac{1}{1 - \frac{\alpha}{t}}$.

Calculus of
Finite
Differences.

Multiplying numerator and denominator by $1 - \alpha \cdot t$, we have $\frac{1 - \alpha t}{1 - \alpha \left(t + \frac{1}{t}\right) + \alpha^2}$, which becomes,

Calculus of
Finite
Differences.

since

$$t \cdot \left(\frac{1}{t} - 1\right)^2 = z \text{ gives } \frac{1}{t} + t = z + 2,$$

$$\frac{1 - \alpha t}{1 - \alpha z - 2\alpha + \alpha^2} = \frac{1 - \alpha t}{(1 - \alpha)^2 - \alpha^2}.$$

But

$$\frac{1}{1 - \alpha^2 - \alpha z} = \frac{1}{(1 - \alpha)^2} + \frac{\alpha z}{(1 - \alpha)^4} + \frac{\alpha^2 z^2}{(1 - \alpha)^6} + \frac{\alpha^3 z^3}{(1 - \alpha)^8} + \&c.$$

And to find the whole coefficient of α^n , each of these fractions must be developed, and the coefficient of the required power in each separate expansion being found, the sum of these terms will give the object of our inquiry.

Now the coefficient of α^r in the expansion of $\frac{1}{(1 - \alpha)^s}$ or $(1 - \alpha)^{-s} = \frac{s \cdot s + 1 \cdot s + 2 \cdot \dots \cdot s + r - 1}{1 \cdot 2 \cdot 3 \cdot \dots \cdot r}$; whence the coefficient of α^n in the expansion of $\frac{1}{(1 - \alpha)^2}$ is $n + 1$; and in that of $\frac{\alpha}{(1 - \alpha)^4}$ is $\frac{n \cdot n + 1 \cdot n + 2}{1 \cdot 2 \cdot 3}$; in that of $\frac{\alpha^2}{(1 - \alpha)^6}$ is $\frac{n - 1 \cdot n \cdot n + 1 \cdot n + 2 \cdot n + 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}$, &c.; and calling Z the coefficient of α^n , we have

$$\begin{aligned} Z &= n + 1 + \frac{n \cdot n + 1 \cdot n + 2}{1 \cdot 2 \cdot 3} z + \frac{n - 1 \cdot n \cdot n + 1 \cdot n + 2 \cdot n + 3}{2 \cdot 3 \cdot 4 \cdot 5} z^2 \\ &\quad + \frac{n - 2 \cdot n - 1 \cdot n \cdot n + 1 \cdot n + 2 \cdot n + 3 \cdot n + 4}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} z^3 + \&c. \\ &= n + 1 \left\{ 1 + \frac{(n + 1)^2 - 1}{2 \cdot 3} z + \frac{\{(n + 1)^2 - 1\} \{(n + 1)^2 - 4\}}{2 \cdot 3 \cdot 4 \cdot 5} z^2 + \&c. \right\}; \end{aligned}$$

and putting (n) instead of $(n + 1)$, and calling Z' the coefficient of α^n in the developement of $\frac{\alpha}{(1 - \alpha)^2 - \alpha z}$, we shall have

$$Z' = n \left\{ 1 + \frac{n^2 - 1}{2 \cdot 3} z + \frac{(n^2 - 1)(n^2 - 4)}{2 \cdot 3 \cdot 4 \cdot 5} z^2 + \frac{(n^2 - 1)(n^2 - 4)(n^2 - 9)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7} z^3 + \&c. \right\}.$$

But it is obvious that the coefficient of α^n in the developement of $\frac{1 - \alpha t}{(1 - \alpha)^2 - \alpha z}$ or $\frac{1}{(1 - \alpha)^2 - \alpha z} - \frac{\alpha t}{(1 - \alpha)^2 - \alpha z}$ is $Z - Z' t$, and consequently

$$\frac{\phi(t)}{t^n} = \phi(t) \{Z - Z' t\} = Z \cdot \phi(t) - Z' t \cdot \phi(t).$$

But $\frac{\phi(t)}{t^n}$ is the generating function of u_{x+n} ; and $z \phi(t)$, $z^2 \phi(t)$, &c., or $t \cdot \left(\frac{1}{t} - 1\right)^2 \phi(t)$, $t^2 \cdot \left(\frac{1}{t} - 1\right)^4 \phi(t)$, &c. are the generating functions of $\Delta^2 \cdot u_{x-1}$, $\Delta^4 u_{x-2}$, &c., while $t \phi(t)$, $z \cdot t \phi(t)$, $z^2 t \phi(t)$, &c., or $t \phi(t)$, $t^2 \left(\frac{1}{t} - 1\right)^2 \cdot \phi(t)$, $t^3 \cdot \left(\frac{1}{t} - 1\right)^4 \cdot \phi(t)$, &c., are those of u_{x-1} , $\Delta^2 u_{x-2}$, $\Delta^4 u_{x-3}$, &c., and therefore we have

$$\begin{aligned} u_{x+n} &= n + 1 \left\{ u_x + \frac{(n + 1)^2 - 1}{2 \cdot 3} \Delta^2 u_{x-1} + \frac{\{(n + 1)^2 - 1\} \{(n + 1)^2 - 4\}}{2 \cdot 3 \cdot 4 \cdot 5} \Delta^4 u_{x-2} + \&c. \right\} \\ &\quad - n \left\{ u_{x-1} + \frac{n^2 - 1}{2 \cdot 3} \Delta^2 u_{x-2} + \frac{(n^2 - 1)(n^2 - 4)}{2 \cdot 3 \cdot 4 \cdot 5} \Delta^4 u_{x-3} + \&c. \right\}. \end{aligned}$$

(153.) Other expressions for u_{x+n} may be deduced from the preceding value for $\frac{1}{t^n}$.

Let Z'' be the value of Z' when $(n - 1)$ is put for (n) or the value of Z when $(n - 2)$ is put for (n) : in this case we shall change the equation

$$\frac{1}{t^n} = Z - Z' t \text{ into } \frac{1}{t^{n-1}} = Z' - Z'' \cdot t;$$

whence $\frac{1}{t^n} = \frac{Z'}{t} - Z''$; or, adding the two values of $\frac{1}{t^n}$, and then dividing by 2,

$$\frac{1}{t^n} = \frac{1}{2} Z - \frac{1}{2} Z'' + \frac{1}{2} Z' \cdot \frac{1}{1 + t} + \frac{1}{t} - 1.$$

Calculus of
Finite
Differences.

$$\begin{aligned} \text{But } \frac{1}{2}(Z - Z'') &= \frac{1}{2} \cdot \left\{ n+1 + \frac{\overline{n+1} \cdot \overline{n} \cdot \overline{n+2}}{2 \cdot 3} \cdot z + \frac{\overline{n-1} \cdot \overline{n} \cdot \overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3}}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} z^2 + \&c. \right\} \\ &\quad - \frac{1}{2} \left\{ n-1 + \frac{\overline{n-2} \cdot \overline{n-1} \cdot \overline{n}}{2 \cdot 3} z + \frac{\overline{n-3} \cdot \overline{n-2} \cdot \overline{n-1} \cdot \overline{n} \cdot \overline{n+1}}{2 \cdot 3 \cdot 4 \cdot 5} z^2 + \&c. \right\} \\ &= 1 + \frac{n^2}{1 \cdot 2} z + \frac{n^2(n^2-1)}{2 \cdot 3 \cdot 4} z^2 + \frac{n^2 \cdot (n^2-1)(n^2-4)}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} z^3 + \&c.; \end{aligned}$$

Calculus of
Finite
Differences.

and substituting for (z) its value $t \cdot \left(\frac{1}{t} - 1\right)$,

$$\begin{aligned} \therefore \frac{\phi(t)}{t^n} &= \phi(t) \cdot \left\{ 1 + \frac{n^2}{1 \cdot 2} \cdot t \cdot \left(\frac{1}{t} - 1\right) + \frac{n^2 \cdot (n^2-1)}{2 \cdot 3 \cdot 4} \cdot t^2 \cdot \left(\frac{1}{t} - 1\right)^2 + \&c. \right\} \\ &\quad + \frac{n}{2} \cdot \phi(t) \cdot \overline{1+t} \cdot \left\{ \left(\frac{1}{t} - 1\right) + \frac{n^2-1}{2 \cdot 3} \cdot t \cdot \left(\frac{1}{t} - 1\right)^2 + \frac{(n^2-1)(n^2-4)}{2 \cdot 3 \cdot 4 \cdot 5} \cdot t^2 \cdot \left(\frac{1}{t} - 1\right)^3 + \&c. \right\}; \end{aligned}$$

whence by passing from the generating functions to coefficients, and observing that $\phi(t) \cdot (1+t) = \phi(t) + t\phi(t)$ = generating functions of u_x and u_{x+1} ,

$$\begin{aligned} u_{x+n} &= u_x + \frac{n^2}{1 \cdot 2} \cdot \Delta^2 u_{x-1} + \frac{n^2 \cdot \overline{n^2-1}}{2 \cdot 3 \cdot 4} \cdot \Delta^4 u_{x-2} + \frac{n^2 \cdot (n^2-1) \cdot \overline{n^2-4}}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6} \cdot \Delta^6 u_{x-3} + \&c. \\ &\quad + \frac{n}{2} \cdot (\Delta u_x + \Delta u_{x-1}) + \frac{n}{2} \cdot \frac{n^2-1}{2 \cdot 3} \cdot (\Delta^3 u_{x-1} + \Delta^3 u_{x-2}) \\ &\quad + \frac{n}{2} \cdot \frac{(n^2-1)(n^2-4)}{2 \cdot 3 \cdot 4 \cdot 5} (\Delta^5 u_{x-2} + \Delta^5 u_{x-3}) + \&c. \end{aligned}$$

(154.) This formula is useful to interpolate a term between an unequal number $2x+1$ of equidistant quantities, the common interval of which is unity: u_x is the middle term, the series being $u_0, u_1, u_2, \&c. u_x, u_{x+1}, \dots, u_{2x}$, and (n) is the interval between u_x and u_{x+n} .

The values $\Delta^2 u_{x-1}, \Delta u_x + \Delta u_{x-1}, \&c.$, which are the same as $u_{x+1} - 2u_x + u_{x-1}, u_{x+1} - u_{x-1}, \&c.$, are symmetric in their form with regard to u_x , for the terms they involve are arranged at equal intervals before and after u_x .

If, instead of making u_x the middle term of the series, we suppose the terms to be arranged equally about u_0 , we must use negative indices on one side of u_0 , and the series will be $u_{-x}, u_{-x-1}, \dots, u_0, u_1, \dots, u_x$, and u_n will be a function of u_0 and of the differences of the preceding values $u_{-1}, u_{-2}, u_{-3}, \dots, \&c.$

(155.) In the last expression obtained for $\frac{\phi(t)}{t^n}$ write $\overline{n+1}$ for n , and then subtract $\frac{\phi(t)}{t^n}$ from it; the result will then be

$$\begin{aligned} \frac{\phi(t)}{t^{n+1}} - \frac{\phi(t)}{t^n} &= \frac{\phi(t)}{t^n} \cdot \left\{ \frac{1}{t} - 1 \right\} \\ &= \phi(t) \cdot \left\{ \frac{2n+1}{1 \cdot 2} \cdot t \left(\frac{1}{t} - 1\right) + \frac{(4n+2) \cdot \overline{n} \cdot \overline{n+1}}{2 \cdot 3 \cdot 4} \cdot t^2 \cdot \left(\frac{1}{t} - 1\right)^2 + \&c. \right\} \\ &\quad + \frac{\phi(t)}{2} \overline{1+t} \cdot \left\{ \left(\frac{1}{t} - 1\right) + \frac{n \cdot \overline{n+1}}{2} \cdot t \cdot \left(\frac{1}{t} - 1\right)^2 + \frac{\overline{n-1} \cdot \overline{n} \cdot \overline{n+1} \cdot \overline{n+2}}{2 \cdot 3 \cdot 4} \cdot t^2 \cdot \left(\frac{1}{t} - 1\right)^3 + \&c. \right\}; \end{aligned}$$

whence, dividing each side by $\frac{1}{t} - 1$, and transferring the lower line to the upper place,

$$\begin{aligned} \frac{\phi(t)}{t^n} &= \frac{\phi(t)}{2} \cdot (1+t) \left\{ 1 + \frac{n \cdot \overline{n+1}}{2} \cdot t \cdot \left(\frac{1}{t} - 1\right) + \frac{\overline{n-1} \cdot \overline{n} \cdot \overline{n+1} \cdot \overline{n+2}}{2 \cdot 3 \cdot 4} \cdot t^2 \cdot \left(\frac{1}{t} - 1\right)^2 + \&c. \right\} \\ &\quad + \left(n + \frac{1}{2}\right) \phi(t) \cdot \left(\frac{1}{t} - 1\right) \left\{ t + \frac{n \cdot \overline{n+1}}{2} \cdot t^2 \cdot \left(\frac{1}{t} - 1\right) + \frac{\overline{n-1} \cdot \overline{n} \cdot \overline{n+1} \cdot \overline{n+2}}{2 \cdot 3 \cdot 4 \cdot 5} \cdot t^3 \cdot \left(\frac{1}{t} - 1\right)^2 + \&c. \right\} \\ &= \frac{\phi(t)}{2} \overline{1+t} \left\{ 1 + \frac{\left\{ \left(n + \frac{1}{2}\right)^2 - \frac{1}{4} \right\}}{1 \cdot 2} \cdot t \cdot \left(\frac{1}{t} - 1\right) + \frac{\left\{ \left(n + \frac{1}{2}\right)^2 - \frac{1}{4} \right\} \left\{ \left(n + \frac{1}{2}\right)^2 - \frac{9}{4} \right\}}{2 \cdot 3 \cdot 4} \cdot t^2 \cdot \left(\frac{1}{t} - 1\right)^2 + \&c. \right\} \\ &\quad + \left(n + \frac{1}{2}\right) t \cdot \phi(t) \cdot \frac{1}{t} - 1 \left\{ 1 + \frac{\left(n + \frac{1}{2}\right)^2 - \frac{1}{4}}{1 \cdot 2} \cdot t^2 \cdot \left(\frac{1}{t} - 1\right) + \frac{\left\{ \left(n + \frac{1}{2}\right)^2 - \frac{1}{4} \right\} \left\{ \left(n + \frac{1}{2}\right)^2 - \frac{9}{4} \right\}}{2 \cdot 3 \cdot 4} \cdot t^3 \cdot \left(\frac{1}{t} - 1\right)^2 + \&c. \right\} \end{aligned}$$

And now passing to coefficients, since

$$\frac{\phi(t)}{2} (1+t) = \frac{1}{2} \{ \phi(t) + t\phi(t) \}$$

Calculus of and
Finite
Differences.

$$\frac{\phi(t)}{2} \cdot (1+t) \cdot t^r \cdot \left(\frac{1}{t} - 1\right)^{2r} = \frac{1}{2} \left\{ t^r \cdot \phi(t) \cdot \left(\frac{1}{t} - 1\right)^{2r} + t^{r+1} \cdot \phi(t) \left(\frac{1}{t} - 1\right)^{2r} \right\},$$

Calculus of
Finite
Differences.

and remembering that $t^r \cdot \phi(t) \left(\frac{1}{t} - 1\right)^{2r}$ is the generating function of $\Delta^{2r} \cdot u_{x-r}$, and $t^{r+1} \cdot \phi(t) \left(\frac{1}{t} - 1\right)^{2r}$ is that of $\Delta^{2r} \cdot u_{x-r-1}$, we shall have

$$\begin{aligned} u_{x+n} &= \frac{1}{2} (u_x + u_{x-1}) + \frac{\left(n + \frac{1}{2}\right)^2 - \frac{1}{4}}{1 \cdot 2} \cdot \frac{1}{2} \cdot \{\Delta^2 u_{x-1} + \Delta^2 u_{x-2}\} \\ &+ \frac{\left(n + \frac{1}{2}\right)^2 - \frac{1}{4} \cdot \left(n + \frac{1}{2}\right)^2 - \frac{9}{4}}{2 \cdot 3 \cdot 4} \cdot \frac{1}{2} \cdot \{\Delta^4 u_{x-2} + \Delta^4 u_{x-3}\} + \&c. \\ &+ n + \frac{1}{2} \left\{ \begin{aligned} &\Delta u_{x-1} + \frac{\left(n + \frac{1}{2}\right)^2 - \frac{1}{4}}{1 \cdot 2} \cdot \Delta^3 u_{x-2} \\ &+ \frac{\left(n + \frac{1}{2}\right)^2 - \frac{1}{4} \cdot \left(n + \frac{1}{2}\right)^2 - \frac{9}{4}}{2 \cdot 3 \cdot 4} \cdot \Delta^5 u_{x-3} + \&c. \end{aligned} \right\} \end{aligned}$$

(156.) This formula is used for interpolation when there is an equal number $2x$ of equidistant numbers. The two middle terms are y_{x-1} and y_x , (if we suppose y_0 to be the first term) and y_{2x-1} is the last term.

These two formulæ of interpolation were given by Stirling, but without demonstration. He probably arrived at his results by successive inductions. They might have been shown to be true in an earlier part of this Treatise, but we reserved them for a place where they could illustrate the application of the theory of generating functions.

We must refer the reader to the *Théorie Analytique des Probabilités* of Laplace, page 19, for other formulæ of interpolation more general than that which has been just obtained, as our space will not allow us to pursue this part of our subject further.

(157.) The results which we have already obtained show with what facility the theory of generating functions demonstrates the theorems of finite differences. We shall continue to give some important ones as examples of its use.

We readily see that, putting T for $\phi(t)$,

$$\begin{aligned} T \cdot \left(\frac{1}{t^n} - 1\right)^m &= T \cdot \left\{ \left(1 + \frac{1}{t} - 1\right)^n - 1 \right\}^m \\ &= T \left\{ n \cdot \left(\frac{1}{t} - 1\right) + \frac{n(n-1)}{2} \cdot \left(\frac{1}{t} - 1\right)^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \left(\frac{1}{t} - 1\right)^3 + \&c. \right\} \end{aligned}$$

But $T \cdot \left(\frac{1}{t} - 1\right)$ is the generating function of $\Delta \cdot u_x$, $T \cdot \left(\frac{1}{t} - 1\right)^2$ is that of $\Delta^2 u_x$, &c; and thus it appears that $T \cdot \left\{ \left(1 + \frac{1}{t} - 1\right)^n - 1 \right\}^m$ is the generating function of $\{(1 + \Delta \cdot u_x)^n - 1\}^m$, taking care to remember that the powers of Δu_x , which arise from the expansion, are to be considered only as symbols of operation, and that when we meet with the term $\overline{\Delta \cdot u_x}^r$ we must write for it $\Delta^r \cdot u_x$.

But adverting to the left-hand side of the equation, we observe that $T \cdot \left(\frac{1}{t^n} - 1\right) = \frac{T}{t^n} - T$, which is the generating function of $u_{x+n} - u_x$: this may be written $\Delta' u_x$, where Δ' indicates that (n) is the increment of (x) , and therefore $T \cdot \left(\frac{1}{t^n} - 1\right)^2$ is the generating function of $\Delta'^2 u_x$, (x) still increasing by (n) , and hence $T \cdot \left(\frac{1}{t^n} - 1\right)^m$ will be the generating function of $\Delta'^m \cdot u_x$; whence we conclude that

$$\Delta'^m u_x = \{(1 + \Delta \cdot u_x)^n - 1\}^m \quad (1)$$

If the symbol Σ' represent the finite integral, when x increases by the number n , or relative to differences indicated by Δ' , then it is evident that $\Sigma'^m u_x$ will have for its generating function the expression $T \cdot \left(\frac{1}{t^n} - 1\right)^{-m}$, the constants not being taken into account; but we have

$$T \left(\frac{1}{t^n} - 1\right)^{-m} = T \left\{ \left(1 + \frac{1}{t} - 1\right)^n - 1 \right\}^{-m};$$

whence we may conclude that

$$\Sigma'^m \cdot u_x = \{(1 + \Delta \cdot u_x)^n - 1\}^{-m}, \quad (2)$$

Calculus of Finite Differences. observing to change the negative powers of $\Delta \cdot u_x$ into terms of the form $\Sigma u_x, \Sigma^2 u_x, \&c.$: this last formula is implicitly contained in the preceding, since putting $(-m)$ for (m) we may change $\Delta'^{-m} u_x$ into $\Sigma'^m u_x$. (158.) These equations will also be true if we suppose that (x) , instead of increasing by unity in Δu_x , should increase by another quantity, as (k) , provided that the increase of (x) in $\Delta' u_x$ is (nk) . For if (x) be changed

into $\left(\frac{x_1}{k}\right)$, then when (x) increases by unity, $x_1 = kx$ will increase by (k) , and if (x) become $(x+n)$, (x_1) will become $(kx + kn)$, or will increase by (kn) ; or we may consider x to become $x+k$ with regard to the symbol Δ , and $x+nk$ with respect to the symbol Δ' .

If now we consider k to be a very small quantity, or dx , and (n) to be infinitely great, so that nk or ndx may be finite $= \alpha$, and that k is very small, therefore $\Delta u_x = d \cdot u_x$, we shall have

$$\Delta'^m \cdot u_x = \{(1 + d \cdot u_x)^n - 1\}^m$$

$$\Sigma'^m u_x = \{(1 + d \cdot u_x)^n - 1\}^{-m}.$$

But $\log(1 + d \cdot u_x)^n = n \log(1 + d \cdot u_x) = n \cdot d \cdot u_x = \alpha \frac{d \cdot u_x}{dx}$;

omitting the other terms of $\log(1 + d \cdot u_x)$, since k is infinitely small,

$$\therefore (1 + d \cdot u_x)^n = e^{\alpha \frac{d \cdot u_x}{dx}};$$

whence, substituting, we have

$$\Delta'^m \cdot u_x = (e^{\alpha \frac{d \cdot u_x}{dx}} - 1)^m \quad (3)$$

$$\Sigma'^m u_x = \frac{1}{(e^{\alpha \frac{d \cdot u_x}{dx}} - 1)^m}, \quad (4)$$

taking care to transfer to the symbol d the exponents of $d \cdot u_x$, and to change the negative powers into integrals.

If we had considered (n) to be infinitely small and $= dx$, then $\Delta'^m u_x$ will become $d^m u_x$, and $\Sigma'^m u_x$ will become $\frac{1}{d^m} \int^m u_x \cdot dx^m$, and

$$(1 + \Delta \cdot u_x)^n = (1 + \Delta \cdot u_x)^{dx} = e^{dx \cdot \log(1 + \Delta \cdot u_x)} = 1 + dx \cdot \log(1 + \Delta \cdot u_x);$$

and substituting this value for $(1 + \Delta u_x)^n$, equations (1) and (2) become

$$\frac{d^m u_x}{dx^m} = \{\log(1 + \Delta \cdot u_x)\}^m \quad (5)$$

$$\int^m u_x \cdot dx^m = \{\log(1 + \Delta \cdot u_x)\}^{-m}. \quad (6)$$

(159.) At present we have only considered a single function u_x ; the theory may, however, be extended to the case in which the product of many functions is concerned, and then some remarkable theorems may be easily deduced.

Let

T be the function of (t) when u_x is the coefficient of t^x

$T' \dots \dots \dots t' \dots \dots \dots u'_x \dots \dots \dots t'^x$

$T'' \dots \dots \dots t'' \dots \dots \dots u''_x \dots \dots \dots t''^x$

We shall, therefore, have $u_x, u'_x, u''_x, \&c.$ for the coefficient of $t^x, t'^x, t''^x, \&c.$ in the developement of the product $T \cdot T' T'' \dots$: this product will therefore be the generating function of $u_x, u'_x, u''_x \dots$; and, consequently, the

generating function of $u_{x+1} \cdot u'_{x+1} \cdot u''_{x+1}$ will be $\frac{T \cdot T' T'' \dots}{t \cdot t' \cdot t'' \dots}$; whence it follows that

$$\frac{T T' \cdot T'' \dots}{t \cdot t' \cdot t'' \dots} - T T' T'' \dots \left\{ \frac{1}{t \cdot t' \cdot t'' \dots} - 1 \right\}$$

is the generating function of

$$u_{x+1} \cdot u'_{x+1} \cdot u''_{x+1} \dots - u_x u'_x u''_x \dots = \Delta \cdot u_x \cdot u'_x \cdot u''_x \dots;$$

and thus the generating function of

$$\Delta^n \cdot u_x u'_x u''_x \dots \&c. \text{ is } T \cdot T' \cdot T'' \dots \left\{ \frac{1}{t \cdot t' \cdot t'' \dots} - 1 \right\}^n.$$

And if (z) be the generating function of $\Sigma^n \cdot (u_x u'_x u''_x \&c.)$, then that of $\Delta^n \cdot \Sigma^n \cdot (u_x u'_x u''_x \dots)$ or of $u_x u'_x u''_x \&c.$ will be from the preceding $z \cdot \left(\frac{1}{t t' t'' \dots} - 1 \right)^n$, whence

$$T \cdot T' T'' \dots = z \cdot \left(\frac{1}{t \cdot t' \cdot t'' \dots} - 1 \right)^n; \therefore z = T \cdot T' T'' \dots \left\{ \frac{1}{t \cdot t' \cdot t'' \dots} - 1 \right\}^{-n},$$

which is the generating function of $\Sigma^n \cdot (u_x u'_x u''_x \dots)$.

(160.) Applying these results to the product of two functions, u_x and v_x , where v_x is the same as u'_x .

Then the generating function of $\Delta^n u_x v_x$ will be $T \cdot T' \cdot \left(\frac{1}{t t'} - 1 \right)^n$, which may be put under the form

Calculus of
Finite
Differences.

 Calculus of
Finite
Differences.

$$T \cdot T' \left\{ \left(\frac{1}{t} - 1 \right) + \frac{1}{t} \left(\frac{1}{t^1} - 1 \right) \right\}^n$$

$$= T T' \left\{ \left(\frac{1}{t} - 1 \right) + \frac{n}{t} \cdot \left(\frac{1}{t} - 1 \right)^{n-1} \cdot \left(\frac{1}{t^1} - 1 \right) \right.$$

$$\left. + \frac{n \cdot n - 1}{1 \cdot 2 \cdot t^2} \cdot \left(\frac{1}{t} - 1 \right)^{n-2} \cdot \left(\frac{1}{t^1} - 1 \right)^2 + \&c. \right\}$$

But from the preceding article we perceive that the products

$$T \cdot T' \cdot \left(\frac{1}{t} - 1 \right)^n = T' \cdot T \left(\frac{1}{t} - 1 \right)^n$$

$$T \cdot T' \cdot \frac{1}{t} \left(\frac{1}{t} - 1 \right)^{n-1} \cdot \left(\frac{1}{t^1} - 1 \right) = T' \cdot \left(\frac{1}{t^1} - 1 \right) \cdot \frac{T}{t} \cdot \left(\frac{1}{t} - 1 \right)^{n-1}$$

$$T \cdot T' \cdot \frac{1}{t^2} \cdot \left(\frac{1}{t} - 1 \right)^{n-2} \cdot \left(\frac{1}{t^1} - 1 \right)^2 = T' \cdot \left(\frac{1}{t^1} - 1 \right)^2 \cdot \frac{T}{t^2} \cdot \left(\frac{1}{t} - 1 \right)^{n-2}$$

$$\&c. \qquad \qquad \qquad = \&c.$$

are the generating functions respectively of the products $v_x \cdot \Delta^n u_x$, $\Delta v_x \cdot \Delta^{n-1} u_{x+1}$, $\Delta^2 v_x \cdot \Delta^{n-2} u_{x+2}$, &c. Whence passing from generating functions to coefficients,

$$\Delta^n (u_x \cdot v_x) = v_x \cdot \Delta^n u_x + n \cdot \Delta v_x \cdot \Delta^{n-1} u_{x+1} + \frac{n \cdot n - 1}{2} \cdot \Delta^2 v_x \cdot \Delta^{n-2} u_{x+2} + \&c.,$$

and changing (n) into $(-n)$, we shall obtain

$$\Sigma^n (u_x \cdot v_x) = v_x \cdot \Sigma^n u_x - n \Delta v_x \cdot \Sigma^{n+1} u_{x+1} + \frac{n \cdot n + 1}{1 \cdot 2} \cdot \Delta^2 v_x \cdot \Sigma^{n+2} u_{x+2} - \&c.,$$

formulae previously obtained by successive and laborious induction.

(161.) Again, since we may put

$$T \cdot T' T'' \cdot \left\{ \frac{1}{t \cdot t' t'' \dots} - 1 \right\}^n$$

$$= T T' T'' \dots \left\{ 1 + \frac{1}{t} - 1 \cdot 1 + \frac{1}{t'} - 1 \cdot 1 + \frac{1}{t''} - 1 \cdot \dots - 1 \right\}^n$$

we have

$$\Delta^n \cdot u_x u'_x u''_x \dots = \{ 1 + \Delta \cdot 1 + \Delta' \cdot 1 + \Delta'' \cdot \dots - 1 \}^n u_x \cdot u'_x u''_x \dots;$$

taking care that in each term of the development of the right-hand side of the equation we apply the symbol Δ to the variables which are marked with the same accent. Thus if we had but three variables, the term $\Delta^r \cdot u_x u'_x u''_x$ indicates $u'_x u''_x \cdot \Delta^r u_x$, and $\Delta^r \Delta' u_x u'_x u''_x$ is to be written $u''_x \Delta' u_x \cdot \Delta^r u'_x$.

If (n) be negative, the formula will be true for Σ^n .

$$(162.) \text{ Again, since } T T' T'' \cdot \left\{ \frac{1}{t^h \cdot t'^h \cdot t''^h \dots} - 1 \right\}^n$$

$$= T \cdot T' T'' \dots \left\{ \left(1 + \frac{1}{t} - 1 \right)^h \left(1 + \frac{1}{t'} - 1 \right)^h \left(1 + \frac{1}{t''} - 1 \right)^h \dots - 1 \right\}^n,$$

and indicating by $\Delta^h u_x u'_x u''_x \dots$, when x increases by h , we have

$$\Delta'^n \cdot u_x u'_x u''_x \dots = \{ (1 + \Delta)^h (1 + \Delta')^h (1 + \Delta'')^h \dots - 1 \}^n u_x u'_x u''_x \dots;$$

and if we change (n) into $(-n)$, we shall obtain the expression for Σ^n .

For further illustrations of this interesting subject we must refer to the Works of its author, Laplace, and to the third volume of the Differential Calculus of his commentator Lacroix, where the reader will find the theory extended to functions of two or a greater number of variables.

The Summation of Series.

(163.) By the sum of a series is meant, the sum of any given number of the terms $u_1 u_2 u_3 u_4 \dots$ &c. u_x , and we shall show that when the general term is known the summation depends upon the integration of an equation of finite differences.

Let S_x represent the sum of x terms of the series just referred to; or let

$$S_x = u_1 + u_2 + u_3 + u_4 + \&c. + u_x;$$

$$\therefore S_{x+1} = u_1 + u_2 + u_3 + u_4 + \&c. + u_x + u_{x+1};$$

$$\therefore S_{x+1} - S_x = \Delta \cdot S_x = u_{x+1};$$

and

$$\therefore S_x = \Sigma \cdot u_{x+1} + C;$$

Calculus of Finite Differences. and to find C , let $x = 0$ and let $S_0 = 0$, $\therefore \sum u_1 + C = 0$, $\therefore C = -\sum u_1$,
 $\therefore S_x = \sum u_{x+1} - \sum u_1$.

Calculus of Finite Differences.

Hence to find the sum of any series we must find the integral of the $(x+1)$ th term, and this properly corrected will give the sum required.

(164.) If the form of u_{x+1} be not at once evident, we may use the formula

$$u_{x+1} = u_1 + x \Delta u_1 + \frac{x(x-1)}{2} \Delta^2 u_1 + \frac{x(x-1)(x-2)}{1 \cdot 2 \cdot 3} \Delta^3 u_1 + \&c.,$$

which will give us the $x+1$ th term when one of the orders of differences vanishes. Sometimes, however, u_{x+1} can only be found in terms of the preceding terms of the series multiplied by constant quantities. This case will demand a separate investigation: in the following examples, however, the $(x+1)$ th term will be found without difficulty.

(1.) Sum the arithmetic series $a + \overline{a+b} + \overline{a+2b} + \overline{a+3b} + \&c. + \overline{a+x-1b}$.

Here

$$u_x = a + \overline{x-1b} \text{ and } u_{x+1} = a + \overline{xb};$$

$$\therefore S_x = \sum (a + \overline{xb}) = ax + \frac{x \cdot \overline{x-1}}{2} \cdot b + C.$$

$$\text{And if } x = 0, S_0 = 0; \therefore C = 0, \text{ and } S_x = ax + \frac{x \cdot \overline{x-1}}{2} \cdot b = (2a + \overline{x-1b}) \frac{x}{2}.$$

$$\text{Cor. If } a = 1 \text{ and } b = 1, \text{ then } 1 + 2 + 3 + 4 + \&c. + x = \frac{x \cdot \overline{x+1}}{2}.$$

(2.) Sum the geometric series $a + ar + ar^2 + ar^3 + \&c. + ar^{x-1}$.

Here

$$u_x = ar^{x-1}; \therefore u_{x+1} = ar^x;$$

$$\therefore S_x = \sum ar^x = a \sum r^x = a \frac{r^x}{r-1} + C.$$

$$\text{And if } x = 0, S_0 = 0; \therefore 0 = \frac{a}{r-1} + C; \therefore C = -\frac{a}{r-1};$$

$$\therefore S_x = a \frac{r^x}{r-1} - a \frac{1}{r-1} = a \cdot \frac{r^x - 1}{r-1}.$$

(3.) Sum the series $1^2 + 2^2 + 3^2 + 4^2 + \&c. + x^2$.

Here

$$S_x = \sum u_{x+1} = \sum (x+1)^2 = \sum .x \cdot \overline{x+1} + \sum \overline{x+1}$$

$$= \frac{\overline{x-1} \cdot x \cdot \overline{x+1}}{3} + \frac{x \cdot \overline{x+1}}{2} + C.$$

$$\text{And when } x = 0, S_0 = 0; \therefore C = 0;$$

$$\therefore S_x = x \cdot \overline{x+1} \cdot \left\{ \frac{x-1}{3} + \frac{1}{2} \right\} = \frac{2 \cdot \overline{x+1} \cdot x \cdot \overline{x+1}}{2 \cdot 3}.$$

(4.) Sum the series $1^3 + 2^3 + 3^3 + 4^3 + \&c. + x^3$

Here

$$u_{x+1} = (x+1)^3 = (x+1) \{ (x+1)^2 - 1 \} + \overline{x+1} = x \cdot \overline{x+1} \cdot \overline{x+2} + \overline{x+1};$$

$$\therefore S_x = \sum u_{x+1} = \frac{\overline{x-1} \cdot x \cdot \overline{x+1} \cdot \overline{x+2}}{4} + \frac{x \cdot \overline{x+1}}{2} + C \text{ and } C = 0;$$

$$\therefore S_x = \frac{x \cdot \overline{x+1}}{2} \cdot \left(\frac{\overline{x-1} \cdot \overline{x+2}}{2} + 1 \right) = \frac{x \cdot \overline{x+1}}{2} \cdot \frac{x \cdot \overline{x+1}}{2} = \left(\frac{x \cdot \overline{x+1}}{2} \right)^2.$$

Hence

$$1^3 + 2^3 + 3^3 + 4^3 + \&c. + x^3 = (1 + 2 + 3 + 4 + \&c. + x)^2,$$

a result to which we have previously alluded.

(5.) To find the sum of the terms of any order of the figurate numbers.

The series

$$1 + 1 + 1 + 1 + 1 + 1 + \&c.$$

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + \&c.$$

$$1 + 3 + 6 + 10 + 15 + 21 + 28 + \&c.$$

&c.

are called figurate series of the first, second, third, &c. order; it will be easily observed in these three series that the x th term of the second order is the sum of (x) terms of the first order: and the same rule is true for the third order. Hence we have this definition, the x th term of any order of figurate numbers is the sum of x terms of the preceding order.

$$\text{In the first order } u_{x+1} = 1; \therefore \sum .u_{x+1} = x;$$

$$\therefore \text{In the second order } u_{x+1} = x + 1, \text{ and } \sum u_{x+1} = \frac{x \cdot \overline{x+1}}{2}$$

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

$$\therefore \text{In the third order } u_{x+1} = \frac{x+1 \overline{x+2}}{2}; \therefore \Sigma u_{x+1} = \frac{x \overline{x+1} \overline{x+2}}{2 \cdot 3};$$

$$\therefore \text{In the fourth order } u_{x+1} = \frac{x+1 \overline{x+2} \overline{x+3}}{1 \cdot 2 \cdot 3}; \therefore \Sigma u_{x+1} = \frac{x \cdot x+1 \overline{x+2} \overline{x+3}}{2 \cdot 3 \cdot 4}.$$

And thus we might proceed to find the sum of the terms of any order of figurate numbers, and thus it is obvious that the sum of (x) terms of the n th order will be expressed by

$$S_x = \frac{x \cdot x+1 \overline{x+2} \overline{x+3} \dots \overline{x+n-1}}{1 \cdot 2 \cdot 3 \cdot 4 \dots n}.$$

(6.) Sum the series $1 \cdot 2 \cdot 4 + 2 \cdot 3 \cdot 5 + 3 \cdot 4 \cdot 6 + 4 \cdot 5 \cdot 7 + \&c.$

Here

$$\begin{aligned} u_x &= x \cdot x+1 \cdot x+3 = x \overline{x+1} \cdot x+2 + x \cdot x+1; \\ \therefore S_x &= \Sigma u_{x+1} \\ &= \Sigma (x+1 \overline{x+2} \cdot x+3) + \Sigma (x+1 \overline{x+2}) \\ &= \frac{x \overline{x+1} \overline{x+2} \overline{x+3}}{4} + \frac{x \overline{x+1} \overline{x+2}}{3} + C \text{ and } C=0 \\ &= \frac{x \overline{x+1} \overline{x+2} \cdot 3x+13}{3 \cdot 4}. \end{aligned}$$

(7.) Sum the series $3 \cdot 5 \cdot 7 + 5 \cdot 7 \cdot 9 + 7 \cdot 9 \cdot 11 + \&c.$

Here

$$u_x = (2x+1) \cdot (2x+3) (2x+5);$$

$$\therefore S_x = \Sigma u_{x+1} = \Sigma \{2x+3 \cdot 2x+5 \cdot 2x+7\} = \frac{2x+1 \overline{2x+3} \overline{2x+5} \cdot 2x+7}{2 \cdot 4} + C;$$

and

$$x=0 \text{ gives } C = -\frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4};$$

$$\therefore S_x = \frac{2x+1 \overline{2x+3} \overline{2x+5} \overline{2x+7} - 1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4}.$$

(8.) Sum the series $\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \&c.$

Here

$$u_x = \frac{1}{x \cdot x+1 \overline{x+2}}; \therefore u_{x+1} = \frac{1}{x+1 \overline{x+2} \cdot x+3};$$

$$\therefore S_x = \Sigma u_{x+1} = -\frac{1}{2x+1 \overline{x+2}} + C,$$

and

$$0 = \Sigma u_1 = -\frac{1}{4} + C;$$

$$\therefore S_x = \frac{1}{4} - \frac{1}{2 \cdot x+1 \overline{x+2}}.$$

And if S = the sum of an infinite series, we have in this case $S = \frac{1}{4}$.

(9.) Sum the series $\frac{5}{2 \cdot 3 \cdot 4 \cdot 5} + \frac{8}{3 \cdot 4 \cdot 5 \cdot 6} + \frac{11}{4 \cdot 5 \cdot 6 \cdot 7} + \&c.$ The general term of $5+8+11+\&c.$ is $3x+2$;

$$\begin{aligned} \therefore u_x &= \frac{3x+2}{x+1 \cdot x+2 \overline{x+3} \overline{x+4}} = \frac{3(x+4) - 10}{x+1 \overline{x+2} \overline{x+3} \overline{x+4}} \\ &= \frac{3}{x+1 \overline{x+2} \overline{x+3}} - \frac{10}{x+1 \overline{x+2} \overline{x+3} \cdot x+4}; \\ \therefore S_x &= \Sigma u_{x+1} + C = C - \frac{3}{2 \cdot x+2 \overline{x+3}} + \frac{10}{3 \cdot x+2 \overline{x+3} \overline{x+4}} \\ \therefore 0 &= \Sigma u_1 + C = C - \frac{3}{12} + \frac{10}{3 \cdot 24} = C - \frac{1}{4} + \frac{5}{36}; \therefore C = \frac{1}{9}. \\ S_x &= \frac{1}{9} - \frac{3}{2 \cdot x+2 \overline{x+3}} + \frac{10}{3 \cdot x+2 \cdot x+3 \overline{x+4}} \\ &= \frac{1}{9} - \frac{9x+16}{6 \cdot x+2 \overline{x+3} \cdot x+4}, \text{ and } S = \frac{1}{9}. \end{aligned}$$

Calculus of
Finite
Differences.(10.) Sum $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \&c.$

Here

$$u_x = \frac{1}{(2x-1)(2x+1)}; \therefore u_{x+1} = \frac{1}{2x+1 \cdot 2x+3};$$

$$\therefore S_x = \sum u_{x+1} + C = C - \frac{1}{2(2x+1)}, \text{ and } C = +\frac{1}{2};$$

$$\therefore S_x = \frac{1}{2} - \frac{1}{2(2x+1)}; \therefore S = \frac{1}{2}.$$

(11.) Sum the series $1 + 2a + 3a^2 + 4a^3 + \&c. + xa^{x-1}.$

Here

$$u_x = xa^{x-1}; \therefore u_{x+1} = (x+1)a^x = x \cdot a^x + a^x;$$

$$\therefore \sum u_{x+1} = \sum x \cdot a^x + \sum a^x = x \cdot \frac{a^x}{a-1} - \sum \frac{a^{x+1}}{a-1} + \frac{a^x}{a-1} + C$$

$$= \frac{xa^x}{a-1} + \frac{a^x}{a-1} - \frac{a \cdot a^x}{(a-1)^2} + C.$$

And if

$$x=0; \therefore 0 = \frac{1}{a-1} - \frac{a}{(a-1)^2} + C; \therefore C = \frac{1}{(a-1)^2};$$

$$\therefore S_x = \frac{1+a^x(ax-x-1)}{(a-1)^2}; \text{ if } (a) \text{ be a fraction, } S = \frac{1}{(a-1)^2}.$$

(12.) Sum $2 \cdot 5 \cdot 8 + 4 \cdot 8 \cdot 14 + 8 \cdot 14 \cdot 26 + 16 \cdot 26 \cdot 50 + \&c.$

Here

$$u_x = 2^x \cdot (3 \cdot 2^{x-1} + 2)(3 \cdot 2^x + 2);$$

$$\therefore u_{x+1} = 2^{x+1} \cdot (3 \cdot 2^x + 2)(3 \cdot 2^{x+1} + 2)$$

$$= 2 \cdot 2^x \cdot (3 \cdot 2^x + 2)(3 \cdot 2^{x+1} + 2);$$

$$\therefore \sum u_{x+1} = 2 \cdot \sum 2^x (3 \cdot 2^x + 2)(3 \cdot 2^{x+1} + 2).$$

But

$$\therefore \sum a^x \cdot (pa^x + q)(pa^{x+1} + q) = a \cdot \frac{pa^{x-1} + q \cdot pa^x + q \cdot pa^{x+1} + q}{p(a^3 - 1)};$$

$$\therefore S_x = C + 4 \cdot \frac{(3 \cdot 2^{x-1} + 2) \cdot (3 \cdot 2^x + 2) \cdot (3 \cdot 2^{x+1} + 2)}{3 \cdot 7}$$

$$= C + 4 \cdot \frac{27 \cdot 2^{3x} + 63 \cdot 2^{2x} + 42 \cdot 2^x + 8}{3 \cdot 7}, \text{ and } C = -4 \cdot \frac{132}{3 \cdot 7}$$

$$= 4 \cdot \frac{9 \cdot 2^{3x} + 21 \cdot 2^{2x} + 14 \cdot 2^x - 44}{7}$$

$$= \frac{36 \cdot 2^{3x} + 84 \cdot 2^{2x} + 56 \cdot 2^x - 176}{7}.$$

The law of such series as $5 + 8 + 14 + 26 + \&c.$, where the differences are the same, may be found by the following artifice.

Since

$$u_0 = 5, \Delta u_0 = 3, \Delta^2 u_0 = 3, \Delta^3 u_0 = 3, \&c.$$

$$\therefore u_x = 5 + 3 \cdot \overline{x-1} + 3 \cdot \frac{x-1 \cdot x-2}{1 \cdot 2} + \frac{3 \cdot \overline{x-1} \cdot x-2 \cdot x-3}{1 \cdot 2 \cdot 3} + \&c.$$

$$= 5 + 3 \cdot \left(1 + \overline{x-1} + \frac{x-1 \cdot x-1}{1 \cdot 2} + \&c. \dots - 1 \right)$$

$$= 5 + 3 \{ \overline{1+1}^{x-1} - 1 \} = 5 + 3 \cdot (2^{x-1} - 1) = 3 \cdot 2^{x-1} + 2.$$

The summation of many series may be effected by analytical artifices, some of which we subjoin.

(13.) Sum $\frac{1^3 \cdot 4}{2 \cdot 3} + \frac{2^3 \cdot 4^2}{3 \cdot 4} + \frac{3^3 \cdot 4^3}{4 \cdot 5} + \frac{4^3 \cdot 4^4}{5 \cdot 6} + \&c.$

Here

$$u_x = \frac{x^3 \cdot 4^x}{(x+1) \cdot (x+2)}.$$

Now

$$\Delta \cdot \frac{x-2}{x+1} 4^x = \frac{x-1 \cdot 4^{x+1}}{x+2} - \frac{x-2}{x+1} 4^x = 4^x \left(\frac{4 \cdot x-1}{x+2} - \frac{x-2}{x+1} \right)$$

$$= \frac{3x^2 \cdot 4^x}{x+1 \cdot x+2};$$

$$\text{i.e. } u_x = \frac{1}{3} \Delta \cdot \frac{x-2}{x+1} \cdot 4^x; \therefore u_{x+1} = \frac{1}{3} \Delta \cdot \frac{x-1}{x+2} \cdot 4^{x+1}$$

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

$$\therefore S_x = \sum u_{x+1} = \frac{1}{3} \frac{4^{x+1} \cdot x - 1}{x + 2} + C; \therefore C = \frac{2}{3},$$

$$S_x = \frac{2}{3} + \frac{4}{3} 4^x \cdot \frac{x - 1}{x + 2}.$$

$$(14.) \text{ Sum } \frac{5}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2} + \frac{6}{2 \cdot 3 \cdot 4} \cdot \frac{1}{2^2} + \frac{7}{3 \cdot 4 \cdot 5} \cdot \frac{1}{2^3} + \&c.$$

Here

$$u_x = \frac{x + 4}{x \cdot x + 1 \cdot x + 2} \cdot \frac{1}{2^x}.$$

And since

$$\begin{aligned} \Delta \cdot \frac{1}{x \cdot x + 1 \cdot 2^{x-1}} &= \frac{1}{x + 1 \cdot x + 2 \cdot 2^x} - \frac{1}{x \cdot x + 1 \cdot 2^{x-1}} \\ &= \frac{x - 2x - 4}{x \cdot x + 1 \cdot x + 2 \cdot 2^x} = -\frac{x + 4}{x \cdot x + 1 \cdot x + 2 \cdot 2^x}; \end{aligned}$$

$$\therefore u_x = -\Delta \frac{1}{x \cdot x + 1 \cdot 2^{x-1}}; \therefore u_{x+1} = -\Delta \frac{1}{x + 1 \cdot x + 2 \cdot 2^x};$$

$$\therefore S_x = C - \frac{1}{x + 1 \cdot x + 2 \cdot 2^x} = \frac{1}{2} - \frac{1}{x + 1 \cdot x + 2 \cdot 2^x}; \therefore S = \frac{1}{2}.$$

$$(15.) \text{ Sum the series } \frac{3}{5 \cdot 7} - \frac{9}{7 \cdot 29} + \frac{27}{29 \cdot 79} - \frac{81}{79 \cdot 245} + \&c.$$

Here

$$u_x = \frac{(-3)^x}{\{2 - (-3)^x\} \{2 - (-3)^{x+1}\}}.$$

And to integrate this expression we observe that, as

$$\begin{aligned} \Delta \cdot \frac{1}{\beta - \alpha^x} &= \frac{1}{\beta - \alpha^{x+1}} - \frac{1}{\beta - \alpha^x} = \frac{\alpha^x \cdot (\alpha - 1)}{(\beta - \alpha^x) \cdot (\beta - \alpha^{x+1})}, \\ \therefore \frac{\alpha^x}{(\beta - \alpha^x) \cdot (\beta - \alpha^{x+1})} &= \frac{1}{\alpha - 1} \Delta \cdot \frac{1}{\beta - \alpha^x}. \end{aligned}$$

Hence by making $\beta = 2$ and $\alpha = -3$,

$$\therefore u_x = -\frac{1}{4} \Delta \frac{1}{2 - (-3)^x};$$

$$\therefore S_x = \sum u_{x+1} = C - \frac{1}{4} \cdot \frac{1}{2 - (-3)^{x+1}};$$

and making

$$x = 0, \therefore 0 = C - \frac{1}{4 \cdot 5}, \therefore C = \frac{1}{20},$$

$$\therefore S_x = \frac{1}{20} - \frac{1}{4} \cdot \frac{1}{2 - (-3)^{x+1}} = \frac{1}{4} \cdot \left\{ \frac{1}{5} - \frac{1}{2 - (-3)^{x+1}} \right\}, \text{ and } S = \frac{1}{20}.$$

$$(16.) \text{ Sum the series } \frac{1}{1 \cdot 3 \cdot 5 \cdot 7} + \frac{2}{5 \cdot 7 \cdot 9 \cdot 11} + \frac{3}{9 \cdot 11 \cdot 13 \cdot 15} + \&c.$$

Here

$$u_x = \frac{x}{(4x - 3)(4x - 1)(4x + 1)(4x + 3)}.$$

In this example the difference between the successive terms of the denominator being only 2 instead of 4, the ordinary rule will not apply, and we must have recourse to the following method.

Since

$$\Delta \cdot \frac{1}{z_x v_x} = \frac{1}{z_{x+1} \cdot v_{x+1}} - \frac{1}{z_x v_x} = -\frac{z_x \Delta v_x + v_x \Delta z_x + \Delta v_x \cdot \Delta z_x}{z_x v_x \cdot z_{x+1} \cdot v_{x+1}},$$

assume
and

$$z_x = 4x - 3; \therefore z_{x+1} = 4x + 1; \therefore \Delta z_x = 4,$$

$$v_x = 4x - 1; \therefore v_{x+1} = 4x + 3; \therefore \Delta v_x = 4;$$

$$\therefore z_x \Delta v_x + v_x \Delta z_x + \Delta v_x \cdot \Delta z_x = 16x - 12 + 16x - 4 + 16 = 32 \cdot x;$$

$$\therefore \frac{x}{(4x - 3)(4x - 1)(4x + 1)(4x + 3)} = -\frac{1}{32} \cdot \Delta \cdot \frac{1}{(4x - 3) \cdot (4x - 1)};$$

$$\therefore S_x = \sum u_{x+1} + C = C - \frac{1}{32} \cdot \frac{1}{4x + 1 \cdot 4x + 3}.$$

Calculus of
Finite
Differences.

Make

$$x = 0; \therefore C = \frac{1}{96};$$

$$\therefore S_x = \frac{1}{32} \left\{ \frac{1}{3} - \frac{1}{(4x+1)(4x+3)} \right\}, \text{ and } S = \frac{1}{96}.$$

(17.) Sum the series $\sin \theta + \sin(\theta + h) + \sin(\theta + 2h) + \&c. + \sin(\theta + \overline{x-1}h)$.

Here
and

$$u_x = \sin(\theta + \overline{x-1}h); \therefore u_{x+1} = \sin(\theta + xh),$$

$$\Sigma u_{x+1} = \Sigma \sin(\theta + xh).$$

Now

$$\begin{aligned} \Delta \cos(\theta + zh) &= \cos(\theta + \overline{z+1}h) - \cos(\theta + zh) \\ &= -2 \sin \frac{h}{2} \cdot \sin\left(\theta + \frac{2z+1}{2}h\right); \end{aligned}$$

$$\therefore \sin\left(\theta + \frac{2z+1}{2}h\right) = -\frac{\Delta \cos(\theta + zh)}{2 \sin \frac{h}{2}}.$$

Make $\frac{2z+1}{2} = x; \therefore z = \frac{2x-1}{2}$; and integrating,

$$\therefore S_x = \Sigma \sin(\theta + xh) = C - \frac{\cos\left(\theta + \frac{2x-1}{2}h\right)}{2 \sin \frac{h}{2}}.$$

Make

$$x = 0; \therefore C = \frac{\cos\left(\theta - \frac{h}{2}\right)}{2 \sin \frac{h}{2}};$$

$$\therefore S_x = \frac{\cos\left(\theta - \frac{h}{2}\right) - \cos\left(\theta + \frac{2x-1}{2}h\right)}{2 \sin \frac{h}{2}} = \frac{\sin\left(\theta + \frac{x-1}{2}h\right) \cdot \sin \frac{xh}{2}}{\sin \frac{h}{2}}.$$

(18.) Similarly to sum the series $\cos \theta + \cos(\theta + h) + \&c. + \cos(\theta + \overline{x-1}h)$, we have to integrate $\cos(\theta + xh)$.

Now

$$\begin{aligned} \Delta \sin(\theta + zh) &= \sin(\theta + \overline{z+1}h) - \sin(\theta + zh) \\ &= 2 \sin \frac{h}{2} \cdot \cos\left(\theta + \frac{2z+1}{2}h\right); \end{aligned}$$

whence

$$\Sigma \cos(\theta + xh) = C + \frac{\sin\left(\theta + \frac{2x-1}{2}h\right)}{2 \sin \frac{h}{2}};$$

$$\therefore S_x = \frac{\sin\left(\theta + \frac{2x-1}{2}h\right) - \sin\left(\theta - \frac{h}{2}\right)}{2 \sin \frac{h}{2}} = \frac{\cos\left(\theta + \frac{x-1}{2}h\right) \cdot \sin \frac{xh}{2}}{\sin \frac{h}{2}}.$$

Cor. Make $h = 2\theta; \therefore \theta + \frac{x-1}{2}h = \theta + \overline{x-1}\theta = x\theta$, and $\frac{xh}{2} = x\theta$.

$$\therefore \sin \theta + \sin 3\theta + \sin 5\theta + \&c. + \sin \overline{2x-1}\theta = \frac{(\sin x\theta)^3}{\sin \theta},$$

and

$$\cos \theta + \cos 3\theta + \cos 5\theta + \&c. + \cos \overline{2x-1}\theta = \frac{\sin 2x\theta}{2 \sin \theta}.$$

(19.) Sum the series $1 + a \cos \theta + a^2 \cos 2\theta + a^3 \cos 3\theta + \&c. + a^{x-1} \cos \overline{x-1}\theta$.

Here

$$u_x = a^{x-1} \cos \overline{x-1}\theta \text{ and } u_{x+1} = a^x \cos x\theta.$$

Now

$$\Sigma a^x \cos x\theta = C + \frac{a^{x+1} \cos \overline{x+1}\theta - a^x \cos x\theta}{a^2 - 2a \cos \theta + 1}.$$

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

Make

$$x=0; \therefore C + \frac{a \cos \theta - 1}{a^2 - 2a \cos \theta + 1} = 0;$$

$$\therefore S_x = \frac{1 - a \cos \theta + a^{x+1} \cdot \cos x - 1 \theta - a^x \cos x \theta}{a^2 - 2a \cos \theta + 1};$$

and when $x = \infty$, (a) being a proper fraction,

$$S = \frac{1 - a \cos \theta}{a^2 - 2a \cos \theta + 1}.$$

Make

$$\theta = 0; \therefore S_x = \frac{1 - a + a^{x+1} - a^x}{(1 - a)^2} = \frac{1 - a - a^x(1 - a)}{(1 - a)^2} = \frac{1 - a^x}{1 - a},$$

and

$$S = \frac{1 - a}{a^2 - 2a + 1} = \frac{1 - a}{(1 - a)^2} = \frac{1}{1 - a}.$$

(165.) We have already mentioned *recurring series*, and have shown in what manner the general term u_x is to be found, and have seen that the solution of an equation of the first degree is necessary to find u_x : but as this equation has constant coefficients, the difficulty is resolved to the solution of an equation of common algebra.

Example. The scale of relation being $(p + q)$, find the general term and the sum of the series.

Here

Let

$$u_{x+2} = p u_{x+1} + q u_x.$$

$$u_x = a^x, \therefore a^2 = p a + q.$$

Let α and β be the roots of this equation, then α^x, β^x will be prime integrals of the original equation, and $\therefore u_x = C' \alpha^x + C'' \beta^x$ will be the complete solution. As the series is given, C' and C'' may be found by making $x = 1$ and $x = 2$, from the equations

$$u_1 = C' \alpha + C'' \beta \quad \text{Let } C' = A$$

$$u_2 = C' \alpha^2 + C'' \beta^2 \quad C'' = B$$

$$\therefore u_x = A \alpha^x + B \beta^x;$$

$$\therefore S_x = C + \sum u_{x+1} = C + A \alpha \frac{\alpha^x}{\alpha - 1} + B \beta \frac{\beta^x}{\beta - 1};$$

and C may be found by making $x = 0$, when also $S_x = 0$.

(166.) We have supposed the scale of relation to be merely $p + q$; but if it consisted of n terms, the mode of proceeding would be entirely the same, and the general term would be of the form

$$u_x = C_1 \alpha^x + C_2 \beta^x + C_3 \gamma^x + \&c. + C_n \nu^x,$$

and the series would be $\sum u_{x+1} + C$.

Example (1.) The scale of relation being $p + q$, find p and q , and the sum of the series $1 + 5 + 17 + 53 + 161 + \&c$.

Since

$$u_{x+2} = p u_{x+1} + q u_x;$$

$$\therefore 17 = 5p + q, \text{ and } 53 = 17p + 5q; \therefore p = 4 \text{ and } q = -3;$$

$$\therefore u_{x+2} = 4 u_{x+1} - 3 u_x.$$

$$u_x = a^x, \therefore a^2 = 4a - 3, \therefore a = 3 \text{ or } 1;$$

Make

and

$$u_x = C_1 3^x + C_2. \text{ Make } x = 1 \text{ and } 2 \text{ successively;}$$

$$\therefore 1 = 3C_1 + C_2 \text{ and } 5 = 9C_1 + C_2; \therefore C_1 = \frac{2}{3} \text{ and } C_2 = -1;$$

$$\therefore u_x = 2 \cdot 3^{x-1} - 1; \therefore u_{x+1} = 2 \cdot 3^x - 1;$$

$$\therefore S_x = C + 2 \cdot \frac{3^x}{3-1} - x = C + 3^x - x = 3^x - x - 1.$$

Example (2.) Find the scale of relation and the sum of the series

$$1 + 5a + 19a^2 + 65a^3 + 211a^4 + \&c. \dots$$

Here

$$\therefore 19a^2 = p \cdot 5a + q \text{ and } 65a^3 = p \cdot 19a^2 + 5a \cdot q;$$

and

$$\therefore 65a^2 = 19pa + 5q,$$

$$95a^2 = 25pa + 5q;$$

$$\therefore 30a^2 = 6pa; \therefore p = 5a \text{ and } q = 19a^2 - 25a^2 = -6a^2;$$

$$\therefore u_{x+2} = 5a u_{x+1} - 6a^2 u_x.$$

Make

$$u_x = m^x; \therefore m^2 - 5am + 6a^2 = 0; \therefore m = 2a \text{ and } 3a;$$

$$\therefore u_x = C_1 (2a)^x + C_2 (3a)^x;$$

$$\therefore 1 = 2C_1 a + 3C_2 a$$

$$5 = 4C_1 a + 9C_2 a;$$

whence

VOL. II.

Calculus of
Finite
Differences.

$$C_1 = -\frac{1}{a} \text{ and } C_2 = \frac{1}{a}; \therefore u_x = (3^x - 2^x) a^{x-1};$$

$$\begin{aligned} \therefore S_x = \sum u_{x+1} &= \sum 3^{x+1} \cdot a^x - \sum 2^{x+1} \cdot a^x + C \\ &= 3 \cdot \frac{(3a)^x}{3a-1} - 2 \cdot \frac{(2a)^x}{2a-1} + C; \end{aligned}$$

and when

$$\begin{aligned} x=0, C &= \frac{2}{2a-1} - \frac{3}{3a-1} = \frac{1}{1-5a+6a^2}; \\ \therefore S_x &= \frac{6a\{(3a)^x - (2a)^x\} - (3^{x+1} - 2^{x+1})a^x + 1}{1-5a+6a^2} \\ &= \frac{a^{x+1} \cdot (2 \cdot 3^{x+1} - 3 \cdot 2^{x+1}) - a^x \cdot (3^{x+1} - 2^{x+1}) + 1}{1-5a+6a^2}. \end{aligned}$$

Cor. We have here supposed the roots of the equation $a^2 = pa + q$ to be different; if they be equal, and each be $= \alpha$, we must put $u_x = (C' + C''x) \alpha^x$, and proceed as before to determine C' and C'' .

But then

$$\begin{aligned} S_x = \sum u_{x+1} &= \sum (C' + C''x) \alpha^{x+1} \\ &= C + C_1 \sum \alpha^x + C_2 \sum (x \alpha^x) \\ &= C + \frac{C_1 \alpha^x}{\alpha - 1} + C_2 \left(\frac{x \alpha^x}{\alpha - 1} - \frac{\alpha^{x+2}}{(\alpha - 1)^2} \right), \end{aligned}$$

where

$$C_1 = (C' + C'') \alpha, \text{ and } C_2 = C'' \alpha.$$

(167.) But it is not necessary to integrate the expression for u_{x+1} in order to find the sum of the series.

For

$$\therefore u_{x+2} = p u_{x+1} + q \cdot u_x;$$

$$\therefore u_{x+1} + \Delta u_{x+1} = p(u_x + \Delta u_x) + q u_x;$$

$$\therefore \Delta u_{x+1} + (1-p) \Delta u_x + (1-p-q) u_x = 0,$$

by putting $u_{x+1} + \Delta u_{x+1}$ for u_{x+2} and $u_x + \Delta u_x$ for u_{x+1} ;

$$\therefore \Delta \cdot u_{x+2} + (1-p) \Delta u_{x+1} + (1-p-q) u_{x+1} = 0;$$

$$\therefore u_{x+2} + (1-p) u_{x+1} + (1-p-q) \cdot \sum u_{x+1} = 0$$

$$S_x = \sum u_{x+1} = C - \frac{u_{x+2} + (1-p) u_{x+1}}{1-p-q};$$

$$\therefore 0 = C - \frac{u_2 + (1-p) u_1}{1-p-q};$$

$$\therefore S_x = - \frac{(u_{x+2} - u_2) + (1-p)(u_{x+1} - u_1)}{1-p-q}.$$

Application of the Integral Calculus to the Summation of Series.

(168.) As in many cases the value of an integral may be expressed by means of a series, therefore whenever the value of the integral can either be entirely or approximately found the sum of the series can also be found.

Now it is easily seen that

$$\int \frac{dx}{1+x} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \&c.$$

$$\int \frac{dx}{1+x^2} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \&c.$$

$$\int \frac{dx}{\sqrt{1-x^2}} = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \frac{x^7}{7} + \&c.$$

And since $\int \frac{dx}{1+x} = \log(1+x)$; $\int \frac{dx}{1+x^2} = \tan^{-1}x$; and $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x$: we have, by integrating between the limits of $x=0$ and $x=1$,

$$\log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \&c.$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \&c.$$

$$\frac{\pi}{2} = 1 + \frac{1}{2} \cdot \frac{1}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{1}{7} + \&c.$$

Calculus of
Finite
Differences.

and since the approximate values of $\log(2)$ and π are known, the sums of the series of which they are the representatives are also known.

Calculus of
Finite
Differences.

(169.) Also, as we can arrive at the values of certain series, as for instance the sum of the inverse powers of the natural numbers, by other processes than by integration, and as these series can be produced by means of an integral, we can assume the value of the integral to be represented by the sum of these series, and, by a combination of the values of known series and known integrals, the sum of many series may be found which it would be difficult to determine by other means.

(170.) Series of this kind are the following:

$$\frac{1}{1 \cdot 2^2 \cdot 3^2} + \frac{1}{2 \cdot 3^2 \cdot 4^2} + \frac{1}{3 \cdot 4^2 \cdot 5^2} + \&c. \text{ to infinity,}$$

$$\frac{1}{2 \cdot 1^2} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 3^2} + \&c.,$$

which are dependent partly upon the sum of

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \&c. = \frac{\pi^2}{6},$$

and known integrals. We easily see that the last series may be represented by an integral.

For

$$\int \frac{dx}{1-x} = \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \&c.;$$

$$\therefore \int \frac{dx}{x} \cdot \int \frac{dx}{1-x} = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \&c.;$$

and $\therefore \int \frac{dx}{x} \cdot \int \frac{dx}{1-x}$ integrated from $x=0$ to $x=1$, is equal to $\frac{\pi^2}{6}$.

Thus also $\int \frac{dx}{x} \int \frac{dx}{1+x}$, which produces the series $\frac{x}{1} - \frac{x^2}{2^2} + \frac{x^3}{3^2} - \&c.$ is, when taken between the limits of $x=0$ and $x=1$, equal to $\frac{\pi^2}{12}$.

(171.) But before we attend to examples of this kind we will explain a method first given by Euler, by which the sum of many series may be found, and the methods are of very general application.

To take a very simple instance, let

$$s = x + 2x^2 + 3x^3 + 4x^4 + \&c. \dots + nx^n.$$

Then multiplying by $\frac{dx}{x}$, and performing the integrations, to get rid of the coefficients,

$$\int \frac{s dx}{x} = x + x^2 + x^3 + x^4 + x^5 + \&c. + x^n = \frac{x - x^{n+1}}{1-x};$$

whence, by differentiation,

$$\frac{s}{x} = \frac{1 - n + 1 \cdot x^n + n \cdot x^{n+1}}{(1-x)^2}; \therefore s = \frac{x - n + 1 x^{n+1} + n x^{n+2}}{(1-x)^2}.$$

(172.) Again, suppose

$$s = \frac{x}{2} + \frac{2}{3}x^2 + \frac{3}{4}x^3 + \frac{4}{5}x^4 + \&c. \text{ to infinity;}$$

$$\therefore sx = \frac{x^2}{2} + \frac{2x^3}{3} + \frac{3x^4}{4} + \frac{4x^5}{5} + \&c.;$$

$$\therefore \frac{d(sx)}{dx} = x + 2x^2 + 3x^3 + 4x^4 + \&c.$$

And multiplying each side by $\frac{dx}{x}$, and integrating, we shall get rid of the numerical coefficients, and

$$\int \frac{dx}{x} \cdot \frac{d}{dx}(sx) = x + x^2 + x^3 + x^4 + \&c. = \frac{x}{1-x};$$

$$\therefore \frac{1}{x} \cdot \frac{d}{dx}(sx) = \frac{1}{(1-x)^2}, \text{ or } sx = \int \frac{x dx}{(1-x)^2};$$

whence

$$s = \frac{1}{x} \cdot \left\{ C + \frac{1}{1-x} + \log(1-x) \right\};$$

and since, when $x=0$, $s=0$, and $\therefore sx=0$, $\therefore C+1=0$ or $C=-1$;

$$\therefore s = \frac{1}{x} \cdot \left\{ \frac{x}{1-x} + \log(1-x) \right\} = \frac{1}{1-x} + \frac{\log(1-x)}{x}.$$

Calculus of Finite Differences. And the value of the series may be computed whenever x is < 1 ; but if $x = 1$, the series will be infinitely great. Calculus of Finite Differences.

For

$$s = \frac{x + (1-x) \cdot \log(1-x)}{x(1-x)} = \frac{x + \log \sqrt{1-x}}{x(1-x)} = \infty, \text{ if } x = 1.$$

(173.) The method contained in the preceding article will apply to the summation of those series of which the general term is $\frac{A x^n}{B}$, where A and B are rational and integral functions of n resolved into factors of the first degree. The factors of the numerator are made to disappear by successive integrations, and those of the denominator by successive differentiation.

Let $\frac{\alpha + n\beta}{a + nb} x^n$ be the general term of the series

$$s = \frac{\alpha + \beta}{a + b} x + \frac{\alpha + 2\beta}{a + 2b} x^2 + \frac{\alpha + 3\beta}{a + 3b} x^3 + \&c. + \frac{\alpha + n\beta}{a + nb} x^n.$$

Multiply each term by $p x^r$, and differentiate,

$$\therefore p \cdot \frac{d \cdot (s x^r)}{dx} = (\alpha + \beta) \frac{p r + p}{a + b} x^r + \&c. + (\alpha + n\beta) \frac{(p r + p n)}{a + nb} x^{n+r-1};$$

and by making $p = b$, and $p r = a$, or $r = \frac{a}{b}$, the factors of the form $\frac{p r + p n}{a + nb}$ may be made equal to unity, and we shall have

$$b \cdot \frac{d}{dx} (s \cdot x^{\frac{a}{b}}) = (\alpha + \beta) \cdot x^{\frac{a}{b}} + (\alpha + 2\beta) x^{\frac{a}{b}+1} + \&c. + (\alpha + n\beta) \cdot x^{\frac{a}{b}+n-1}.$$

Again, multiplying every term by $p x^r \cdot dx$, and integrating each term, the integral of the general term will be

$$\frac{b p (\alpha + n\beta) \cdot x^{n+r+\frac{a}{b}}}{a + b r + b n};$$

and the factor $\frac{b p (\alpha + n\beta)}{a + b r + b n}$ may be made to disappear by making $a b p = a + b r$, and $b p \beta = b$, or $p = \frac{1}{\beta}$;

$$\therefore r = \frac{a}{\beta} - \frac{a}{b};$$

$$\begin{aligned} \therefore \frac{b}{\beta} \cdot \int \{x^{\frac{a}{\beta}-\frac{a}{b}} \cdot d \cdot (s x^{\frac{a}{b}})\} &= x^{\frac{a}{\beta}+1} + x^{\frac{a}{\beta}+2} + x^{\frac{a}{\beta}+3} + \&c. + x^{\frac{a}{\beta}+n} \\ &= x^{\frac{a}{\beta}+1} \cdot \left(\frac{1-x^n}{1-x} \right) = X \text{ by substitution;} \end{aligned}$$

whence

$$\begin{aligned} d \cdot (s x^{\frac{a}{b}}) &= \frac{\beta}{b} \cdot x^{\frac{a}{b}-\frac{a}{\beta}} \cdot d \cdot X; \\ \therefore s &= \frac{\beta \cdot \int \cdot x^{\frac{a}{b}-\frac{a}{\beta}} \cdot d X}{b x^{\frac{a}{b}}} = \frac{\beta \cdot \int x^{\frac{a}{b}-\frac{a}{\beta}} \cdot d \left\{ \frac{1-x^n}{1-x} \cdot x^{\frac{a}{\beta}+1} \right\}}{b x^{\frac{a}{b}}} \end{aligned}$$

The integral of which must be taken from $x = 0$ to $x = x$.

A similar method will apply whenever the coefficient of x^n contains a greater number of simple factors in the numerator and the denominator; but as the plan is after the previous investigation sufficiently obvious, we pass on to another class of series.

(174.) In the series considered in the preceding articles, the number of the factors either in the numerator or denominator is the same for every term. But there is a class to which the name of hypergeometric has been given by Euler, in which this number increases from one term to the other. The series

$$\frac{\alpha + \beta}{a + b} x + \frac{(\alpha + \beta)(\alpha + 2\beta)}{(a + b)(a + 2b)} x^2 + \&c. + \frac{(\alpha + \beta)(\alpha + 2\beta) \dots (\alpha + n\beta)}{(a + b)(a + 2b) \dots (a + nb)} x^n$$

is of this class, the summation of which (as we shall see) will require the integration of a differential equation.

In the first instance, however, we shall show how the most simple case of this class of series may be reduced to an equation from which (s) is to be deduced.

Let $s = (\alpha + \beta) x + (\alpha + \beta)(\alpha + 2\beta) x^2 + \&c. + (\alpha + \beta) \cdot (\alpha + 2\beta) \dots (\alpha + n\beta) x^n$.

Multiplying each term by $p \cdot x^r \cdot dx$, and integrating, we have

$$p \int s x^r \cdot dx = \frac{p \cdot (\alpha + \beta) \cdot x^{r+2}}{r + 2} + \&c. + \frac{p (\alpha + \beta)(\alpha + 2\beta) \dots (\alpha + n\beta)}{n + r + 1} x^{n+r+1}.$$

Let $p(\alpha + n\beta) = n + r + 1$; $\therefore p = \frac{1}{\beta}$, and $p\alpha = r + 1$; $\therefore r = \frac{\alpha}{\beta} - 1$;

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

$$\therefore \frac{1}{\beta} \cdot \int s x^{\frac{\alpha}{\beta}-1} \cdot dx = x^{\frac{\alpha}{\beta}+1} + (\alpha + \beta) x^{\frac{\alpha}{\beta}+2} + \&c. + (\alpha + \beta) \dots (\alpha + n - 1 \beta) \cdot x^{\frac{\alpha}{\beta}+n};$$

whence, dividing by $x^{\frac{\alpha}{\beta}+1}$, and subtracting unity from each side,

$$\frac{\frac{1}{\beta} \cdot \int s \cdot x^{\frac{\alpha}{\beta}-1} \cdot dx}{x^{\frac{\alpha}{\beta}+1}} - 1 = (\alpha + \beta) x + (\alpha + \beta) (\alpha + 2 \beta) x^2 + \&c. + (\alpha + \beta) \dots (\alpha + n - 1 \beta) x^{n-1}$$

$$= s - (\alpha + \beta) \cdot (\alpha + 2 \beta) \dots (\alpha + n \beta) x^n = s - A x^n,$$

where

$$A = (\alpha + \beta) \cdot (\alpha + 2 \beta) \dots (\alpha + n \beta);$$

whence

$$\int s x^{\frac{\alpha}{\beta}-1} dx = \beta \cdot x^{\frac{\alpha}{\beta}+1} \{1 + s - A \cdot x^n\};$$

whence, by differentiation, we shall, after reduction, find that

$$\beta x^2 \cdot \frac{ds}{dx} + (\alpha + \beta \cdot x - 1) s = (n \beta + \alpha + \beta) A \cdot x^{n+1} - (\alpha + \beta) x.$$

A similar method will apply where the number of factors in each successive coefficient increase by two or by any greater number.

(175.) Next let us take the series in which the factors are in the denominator only; so that

$$s = \frac{x}{\alpha + \beta} + \frac{x^2}{(\alpha + \beta)(\alpha + 2 \beta)} + \&c. + \frac{x^n}{(\alpha + \beta)(\alpha + 2 \beta) \dots (\alpha + n \beta)}.$$

Multiply by $p x^r$, and then differentiate;

$$\therefore p \cdot \frac{d \cdot (s x^r)}{dx} = \frac{p(r+1)x^r}{\alpha + \beta} + \&c. + \frac{p(n+r) \cdot x^{n+r-1}}{(\alpha + \beta)(\alpha + 2 \beta) \dots (\alpha + n \beta)};$$

and making $p n = \beta n$ and $p r = \alpha$, whence $p = \beta$ and $r = \frac{\alpha}{\beta}$;

$$\therefore \frac{\beta \frac{d}{dx} \cdot (s x^{\frac{\alpha}{\beta}})}{x^{\frac{\alpha}{\beta}}} - 1 = \frac{x}{\alpha + \beta} + \&c. + \frac{x^{n-1}}{(\alpha + \beta) \dots (\alpha + n - 1 \beta)} = s - \frac{x^n}{A},$$

whence

$$\frac{ds}{dx} + \frac{\beta - x}{\beta x} s = \frac{A - x^n}{A \beta};$$

a linear equation, the integration of which gives

$$s = \frac{1}{\beta} \cdot e^{\frac{x}{\beta}} x^{-\frac{\alpha}{\beta}} \int e^{-\frac{x}{\beta}} \cdot x^{\frac{\alpha}{\beta}} \left(\frac{A - x^n}{A} \right) dx.$$

(176.) Lastly, if the factors be found both in the numerator and denominator, or if

$$s = \frac{a + b}{\alpha + \beta} x + \frac{(a + b)(a + 2b)}{(\alpha + \beta)(\alpha + 2 \beta)} x^2 + \&c. + \frac{(a + b)(a + 2b) \dots (a + nb)}{(\alpha + \beta)(\alpha + 2 \beta) \dots (\alpha + n \beta)} x^n$$

Multiplying by $p x^r \cdot dx$, and integrating,

$$p \cdot \int s x^r dx = \frac{p(a+b)}{(r+2)(\alpha+\beta)} x^{r+2} + \&c. \dots + \frac{p(a+b)(a+nb)}{(r+n+1)(\alpha+\beta) \dots (\alpha+n\beta)} x^{n+r+1}.$$

Let $p(a+nb) = r+n+1$; $\therefore pb = 1$, and $pa = r+1$; $\therefore p = \frac{1}{b}$ and $r = \frac{a}{b} - 1 = \frac{a-b}{b}$;

$$\therefore \frac{1}{b} \int s x^{\frac{a-b}{b}} dx = \frac{x^{\frac{a-b}{b}+1}}{\alpha + \beta} + \&c. + \frac{(a+b) \dots (a+n-1b)}{(\alpha + \beta) \dots (\alpha + n \beta)} x^{\frac{a-b}{b}+n}.$$

For $\frac{1}{b} \int s x^{\frac{a-b}{b}} dx$ put S , and again multiply by $p x^r$, and differentiate;

$$\therefore p \frac{d \cdot (S x^r)}{dx} = \frac{p \left(\frac{a}{b} + r + 1 \right) x^{\frac{a-b}{b}+r}}{\alpha + \beta} + \dots + \frac{p \cdot \left(\frac{a}{b} + n + r \right) (a+b) \dots (a+n-1b)}{(\alpha + \beta) \dots (\alpha + n \beta)} x^{\frac{a-b}{b}+n+r-1};$$

Let $p \left(\frac{a}{b} + n + r \right) = \alpha + n \beta$; $\therefore p = \beta$, and $p \cdot \left(\frac{a}{b} + r \right) = \alpha$; $\therefore r = \frac{\alpha}{\beta} - \frac{a}{b}$;

$$\therefore \frac{\beta}{b} \cdot \frac{d}{dx} \cdot \left\{ x^{\frac{a-b}{b}} \cdot \int s x^{\frac{a-b}{b}} dx \right\} = \left\{ 1 + s - \frac{(a+b) \dots (a+nb)}{(\alpha + \beta) \dots (\alpha + n \beta)} x^n \right\} x^{\frac{a-b}{b}}.$$

Calculus of
Finite
Differences.

(177.) As an example, find the sum of the series

$$s = \frac{1}{3}x + \frac{1 \cdot 2}{3 \cdot 4}x^2 + \frac{1 \cdot 2 \cdot 3}{3 \cdot 4 \cdot 5}x^3 + \frac{1 \cdot 2 \cdot 3 \cdot 4}{3 \cdot 4 \cdot 5 \cdot 6}x^4 + \&c.$$

Here $a = 0$, $b = 1$; $\alpha + \beta = 3$, $\beta = 1$; $\therefore \alpha = 2$; whence

$$\frac{d}{dx} \left\{ x^2 \cdot \int \frac{s}{x} dx \right\} = \left(1 + s - \frac{x^n}{n+1} \right) x^2.$$

If we suppose the series to consist of an infinite number of terms, and x not greater than unity, the term $\frac{x^n}{n+1}$ may be neglected;

$$\therefore \frac{d}{dx} \left\{ x^2 \cdot \int \frac{s}{x} dx \right\} = x^2 (1 + s);$$

$$\therefore 2x \int \frac{s}{x} dx + sx = x^2 + x^2 s;$$

$$\therefore (x-1) \frac{ds}{dx} = s \left(\frac{2}{x} - 1 \right) - 1;$$

$$\therefore \frac{ds}{dx} + s \cdot \frac{x-2}{x(x-1)} = -\frac{1}{x-1};$$

a linear equation, in which $e^{\int \frac{x-2}{x(x-1)} dx} = \frac{x^2}{x-1}$;

$$\therefore s = \frac{x-1}{x^2} \left\{ C - \int \frac{x^2 dx}{(x-1)^2} \right\} = \frac{x-1}{x^2} \left\{ C + \frac{1+x-x^2}{x-1} - \log(x-1)^2 \right\};$$

and $\therefore$ when $x = 0$, $s = 0$; $\therefore C = 1$;

$$\therefore s = \frac{x-1}{x^2} \left\{ \frac{2x-x^2}{x-1} - \log(x-1)^2 \right\} = \frac{2-x}{x} - \frac{x-1}{x^2} \log(x-1)^2.$$

If $x = 1$, $s = 1$; or $\frac{1}{3} + \frac{1 \cdot 2}{3 \cdot 4} + \frac{1 \cdot 2 \cdot 3}{3 \cdot 4 \cdot 5} + \&c.$ to infinity $= 1$.

(178.) If in the preceding expressions $(-x)$ be written for x , we shall find equations for the sum of series in which the terms are alternately positive and negative; but as the first term will be negative, we must change (s) into $(-s)$, if we deduce the value of the series from the foregoing investigation.

As an example, let us take the infinite series

$$s = 1 \cdot x - 1 \cdot 2x^2 + 1 \cdot 2 \cdot 3x^3 - 2 \cdot 3 \cdot 4x^4 + \&c.,$$

which compared with

$$(\alpha + \beta)x - (\alpha + \beta)(\alpha + 2\beta)x^2 + (\alpha + \beta)(\alpha + 2\beta)(\alpha + 3\beta)x^3 - \&c.$$

gives $\alpha = 0$, $\beta = 1$, and putting $(-x)$ for (x) , and $(-s)$ for (s) , the equation

$$\beta x^2 \cdot \frac{ds}{dx} + (\alpha + \beta \cdot x - 1)s = (n\beta + \alpha + \beta)Ax^{n+1} - (\alpha + \beta)x$$

becomes

$$x^2 \frac{ds}{dx} + (x+1)s = x, \text{ or } \frac{ds}{dx} + \frac{x+1}{x^2}s = \frac{1}{x};$$

the integral of which may be found, since it is a linear equation, by multiplying both sides of the equation by $xe^{-\frac{1}{x}}$;

$$\therefore sxe^{\frac{1}{x}} = \int e^{-\frac{1}{x}} dx; \therefore s = \frac{e^{\frac{1}{x}}}{x} \cdot \int e^{-\frac{1}{x}} dx,$$

in which the integral must be taken from $x = 0$ to $x = x$.

(179.) Similarly we might arrive at the value of (s) when

$$s = x - 1 \cdot x^2 + 1 \cdot 2x^3 - 1 \cdot 2 \cdot 3x^4 + \&c.;$$

but more directly by the following process. Differentiating

$$\frac{ds}{dx} = 1 - 2 \cdot x + 2 \cdot 3x^2 - 2 \cdot 3 \cdot 4x^3 + \&c.$$

$$\therefore x^2 \frac{ds}{dx} = x^2 - 2 \cdot x^3 + 2 \cdot 3x^4 - 2 \cdot 3 \cdot 4x^5 + \&c. = x - s$$

$$\therefore \frac{ds}{dx} + \frac{s}{x^2} = \frac{1}{x}.$$

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

Multiply by $e^{-\frac{1}{x}}$, and integrate,

$$\therefore s e^{-\frac{1}{x}} = \int \frac{e^{-\frac{1}{x}}}{x} dx; \therefore s = e^{\frac{1}{x}} \cdot \int \frac{e^{-\frac{1}{x}}}{x} dx;$$

which, integrated between the limits of $x = 0$ and $x = 1$, gives the sum of the series $1 - 1 + 1 \cdot 2 - 1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 3 \cdot 4 - \&c.$

The value of the integral $\int \frac{e^{-\frac{1}{x}}}{x} dx$ cannot, however, be exactly computed, but Euler has given a series by which its value may be approximately found, the error being scarcely appreciable.

(180.) We proceed to give some applications of the theory laid down in the preceding articles.

Thus, if it be required to sum the series

$$\frac{1}{1 \cdot 3} \pm \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} \pm \frac{1}{7 \cdot 9} + \&c$$

Let

$$s = \frac{x^3}{1 \cdot 3} \pm \frac{x^5}{3 \cdot 5} + \frac{x^7}{5 \cdot 7} \pm \frac{x^9}{7 \cdot 9} + \&c;$$

$$\therefore \frac{ds}{dx} = \frac{x^2}{1} \pm \frac{x^4}{3} + \frac{x^6}{5} \pm \frac{x^8}{7} + \&c;$$

$$\therefore \frac{d}{dx} \cdot \left(\frac{1}{x} \cdot \frac{ds}{dx} \right) = 1 \pm x^2 + x^4 \pm x^6 + \&c$$

$$= \frac{1}{1 \mp x^2};$$

$$\therefore \frac{1}{x} \frac{ds}{dx} = \int \frac{dx}{1 \mp x^2}, \text{ and } s = \int x dx \cdot \int \frac{dx}{1 \mp x^2}.$$

1. Taking the upper sign,

$$s = \frac{x^2}{2} \int \frac{dx}{1-x^2} - \frac{1}{2} \int \frac{x^2 dx}{1-x^2} = \left(\frac{x^2-1}{2} \right) \log \sqrt{\frac{1+x}{1-x}} + \frac{x}{2},$$

which, integrated from $x = 0$ to $x = 1$, gives $s = \frac{1}{2}$.

2. Taking the lower sign,

$$s = \frac{x^2}{2} \int \frac{dx}{1+x^2} - \frac{1}{2} \int \frac{x^2 dx}{1+x^2} = \left(\frac{x^2+1}{2} \right) \tan^{-1} x - \frac{x}{2};$$

whence $s = \frac{\pi}{4} - \frac{1}{2}$, if the integral be taken from $x = 0$ to $x = 1$; or

$$\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \frac{1}{7 \cdot 9} + \&c. \text{ to infinity} = \frac{1}{2}$$

$$\frac{1}{1 \cdot 3} - \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} - \frac{1}{7 \cdot 9} + \&c. = \frac{\pi}{4} - \frac{1}{2}$$

and $\therefore$

$$\frac{1}{1 \cdot 3} + \frac{1}{5 \cdot 7} + \frac{1}{9 \cdot 11} + \&c. = \frac{\pi}{8}.$$

(181.) Next to sum the series $\frac{1}{1 \cdot 2} + \frac{1}{4 \cdot 5} + \frac{1}{7 \cdot 8} + \frac{1}{10 \cdot 11} + \&c. \text{ to infinity.}$

Let

$$s = \frac{x^2}{1 \cdot 2} + \frac{x^5}{4 \cdot 5} + \frac{x^8}{7 \cdot 8} + \frac{x^{11}}{10 \cdot 11} + \&c;$$

$$\therefore \frac{ds}{dx} = \frac{x}{1} + \frac{x^4}{4} + \frac{x^7}{7} + \frac{x^{10}}{10} + \&c;$$

$$\therefore \frac{d}{dx} \cdot \left(\frac{ds}{dx} \right) = 1 + x^3 + x^6 + x^9 + \&c. = \frac{1}{1-x^3};$$

$$\therefore \frac{ds}{dx} = \int \frac{dx}{1-x^3}, \text{ and } s = \int dx \cdot \int \frac{dx}{1-x^3},$$

which, integrated from $x = 0$ to $x = 1$, will give the sum required.

Now

$$\int dx \int \frac{dx}{1-x^3} = x \int \frac{dx}{1-x^3} - \int \frac{x dx}{1-x^3}$$

$$= \frac{1-x}{3} \cdot \log \frac{1-x}{\sqrt{1+x+x^2}} - \frac{1-x}{3\sqrt{3}} \tan^{-1} \frac{x\sqrt{3}}{2+x} + \frac{2}{\sqrt{3}} \tan^{-1} \frac{2x+1}{\sqrt{3}};$$

Calculus of
Finite
Differences.

Calculus of
Finite
Differences.

whence

$$s = \frac{1}{1 \cdot 2} + \frac{1}{4 \cdot 5} + \frac{1}{7 \cdot 8} + \&c. = \frac{2\pi}{3\sqrt{3}}.$$

(182.) Again, sum $\frac{1}{1 \cdot 3} - \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} - \frac{1}{4 \cdot 6} + \&c.$

Let

$$s = \frac{x^3}{1 \cdot 3} - \frac{x^4}{2 \cdot 4} + \frac{x^5}{3 \cdot 5} - \frac{x^6}{4 \cdot 6} + \&c.$$

$$\therefore \frac{ds}{dx} = \frac{x^2}{1} - \frac{x^3}{2} + \frac{x^4}{3} - \frac{x^5}{4} + \&c.;$$

$$\therefore \frac{d}{dx} \cdot \left(\frac{1}{x} \cdot \frac{ds}{dx} \right) = 1 - x + x^2 - x^3 + \&c. = \frac{1}{1+x};$$

$$\therefore \frac{1}{x} \frac{ds}{dx} = \int \frac{dx}{1+x}, \text{ and } s = \int x dx \int \frac{dx}{1+x};$$

$$\therefore s = \frac{x^2}{2} \cdot \log(1+x) - \frac{1}{2} \cdot \int \frac{x^2 dx}{1+x} = \left(\frac{x^2-1}{2} \right) \log(1+x) - \frac{(1-x)^2}{4},$$

which, integrated from $x=0$ to $x=1$, gives $S = \frac{1}{4}$.

(183.) To sum the series $\frac{1}{2 \cdot 1^2} + \frac{1}{3 \cdot 2^2} + \frac{1}{4 \cdot 3^2} + \frac{1}{5 \cdot 4^2} + \&c.$ to infinity.

Let

$$s = \frac{x^3}{2 \cdot 1^2} + \frac{x^4}{3 \cdot 2^2} + \frac{x^5}{4 \cdot 3^2} + \frac{x^6}{5 \cdot 4^2} + \&c.;$$

$$\therefore \frac{ds}{dx} = \frac{x}{1^2} + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \&c.$$

$$\frac{d}{dx} \cdot \left(\frac{ds}{dx} \right) = \frac{1}{1} + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \&c.;$$

$$\therefore \frac{d}{dx} \left\{ x \cdot \frac{d}{dx} \cdot \left(\frac{ds}{dx} \right) \right\} = 1 + x + x^2 + x^3 + x^4 + \&c. = \frac{1}{1-x},$$

$$\therefore x \cdot \frac{d}{dx} \left(\frac{ds}{dx} \right) = \int \frac{dx}{1-x}, \text{ and } \frac{ds}{dx} = \int \frac{dx}{x} \cdot \int \frac{dx}{1-x}$$

Hence, as we have seen in Art. (170.), $\frac{ds}{dx}$, when integrated from $x=0$ to $x=1$, is $= \frac{\pi^2}{6}$.

And

$$s = \int dx \int \frac{dx}{x} \int \frac{dx}{1-x} = x \int \frac{dx}{x} \cdot \int \frac{dx}{1-x} - \int dx \int \frac{dx}{1-x}$$

Now

$$\int dx \int \frac{dx}{1-x} = x \int \frac{dx}{1-x} - \int \frac{x dx}{1-x} = x \int \frac{dx}{1-x} + x - \int \frac{dx}{1-x}$$

$$= x + (1-x) \log(1-x);$$

$$\therefore s = x \int \frac{dx}{x} \int \frac{dx}{1-x} - x - (1-x) \log(1-x),$$

which, integrated from $x=0$ to $x=1$, gives for the sum of the series

$$s = \frac{\pi^2}{6} - 1.$$

These examples will be sufficient to point out the way by which the sum of many other series may be found.

(184.) We here conclude the summation of series, and with it the subject of Finite Differences. Notwithstanding the length to which this Article has run, many things have been but briefly touched on, and some have been omitted.

Among the omissions we must mention the applications of this calculus to geometrical questions. These problems are among the most curious of any that have ever occupied the attention of the analyst, and their perusal will fully reward the labour of the student. But for these and other interesting details connected with Finite Differences we must for the last time refer the reader to the Examples given by Sir J. F. W. Herschell, and finally record our sense of the many obligations we owe to that Work during the preparation of this Article.

The Works too of Laplace and Lacroix, and the Memoirs of Learned Societies, will afford abundant materials for the extension of our knowledge in a subject which appears exhaustless.

Calculus of
Finite
Differences.

CALCULUS OF FUNCTIONS.

CALCULUS OF FUNCTIONS.

Calculus of
Functions.

(1.) To enter into a detail of the applications of the theory of Functions, would be to write a very large Work, as there are few parts of Mathematics into which it has not been carried. This treatise, therefore, confines itself to those matters, which if not considered without reference to application, will not in this period of the Science be considered at all. And by application is meant as well to the ordinary branches of Mathematics as to questions of Physics.

Neither will this Essay embrace all that has been done with the subject, and for the following reasons: those who have invented the little we know on functional equations, little only as compared with the view it gives of the possibility of more, have followed the subject where it led, seizing every relation which presented itself, without inquiring whether the conditions of solution were possible or not. And they were right in so doing, the general interest of the Science being considered; nor do we know whether any benefit would have accrued from deferring to follow out a view which struck them, until any requisite preliminary for complete solution was obtained. But in a work which professes to be so far elementary, that a student who has gone through the usual course of the Differential and Integral Calculus is supposed able to read it, nothing would be less useful than filling an Introduction to the Calculus of Functions with generalities of which particular cases have not been given. That one complicated functional equation has, by a beautiful artifice, been made to depend upon another, of the solution of which there may be more hope, is a matter of great interest, to be recorded for the future inventor, who may thus make two steps in one. But to the student, unless the artifice throw light upon his first principles of notation, or add a general method to his power of combination, such information is secondary. An inventor and an elementary writer stand much in the same relation as an invader and a colonist; the first must seize all he can, the second no more than is useful; the first may go as far as he can make his way, the second no further than he can make a road.

We have, therefore, confined ourselves principally to the elucidation of first principles, and the extension of the ideas which a student must be supposed to have on the subject of notation, together with such methods for the solution of functional equations as have succeeded in giving algebraical results. We have also given such methods of solution as have not yet been reduced to Algebra, in every case wherein we supposed either a useful view might be introduced, or in which we imagined it probable that the next step to be made would be a natural consequence of what has already been done. we have also made such references as will enable the Mathematician to see where may lie the most probable help for any object he may have in view; and this we think enough, considering that there are perhaps in the whole of the Empire not twenty men who are likely to look into a Work of reference with a view to see what has been done in the highest branches of the subject. Even supposing there to be so many, they are already sufficiently familiar with the various Philo-

VOL. II.

sophical Transactions, in which investigators record their results; and to which, as far as we know them, we have referred.

Calculus of
Functions.

Our object will be gained if the student is led to such a knowledge of forms and familiarity with the results of general operations as will render his grasp of ordinary mathematical language more intellectual and less mechanical. The influence of signs on mathematical processes, or, to borrow a term, the etymological branch of Mathematics, is only incidentally a part of the subject-matter of most writings. But, of late years, it has been treated more for its own sake by the following writers,* Babbage, Carnot, Cauchy, Herschel, and Peacock, into whose Works the reader must look if he would judge of the present state of mathematical language.

The very striking differences presented by two functions, which, algebraically speaking, are the same, as will more fully appear in the course of this Essay, sometimes renders the method of drawing general conclusions established in Algebra dangerous, if not altogether unsound. This makes the examination we have given to the manageable cases of functions a matter of some interest to the Mathematician, already sufficiently prepared to find such results by the processes of the Integral Calculus.

We have found it impossible to preserve as much connection as is desirable between the different articles of this Treatise. In a subject hardly yet considered fit for the student, and on which only one *elementary* work exists, of which we have any knowledge, the arrangement which the general methods of the Differential and Integral Calculus enables writers on those subjects to adopt, we soon found to be impossible. More officers must enter the service before functional equations can be drilled into discipline; and while we have, therefore, arranged such materials as we could find or make, in the order in which the student can most easily read, we refer the Mathematician to the general Index at the end.

(2.) By the term *Function* is meant any mathematical expression considered with reference to its form, and not to the value which it derives from giving particular values to the letters contained in it. Thus $a + x$ and $a^2 + x^2$ are both functions of x , but of different forms.

(3.) It is usual to call any expression a function of x only, when it is only with reference to the manner of containing x that it is necessary to consider the function. Thus $a + \log x$ and $b + \log x$ are functions

* See Babbage, in various papers cited in this Work, and his *Elementary Treatise on the Calculus of Functions*; Carnot, *Géométrie de Position*; and *Métaphysique du Calcul infinitésimal*; Cauchy, *Cours d'Analyse*, and *Elémens du Calcul infinitésimal*; Herschel, papers in the *Memoirs of the Analytical Society*, and in the *Philosophical Transactions*, and also his elementary work on the *Calculus of Differences*, and edition of Spence's *Logarithmic Transcendents*; Peacock, *Elements of Algebra*, and *Report on the State of Analysis* in the second volume of the *Memoirs of the British Association*. With the writers of the German School we are not so well acquainted, but if we may judge from such as we have seen, they appear, with the exception of Abel, Jacobi, and Fuss, to have taken a road, which, however useful it may be, does not bring the results of their labours to bear very closely upon the considerations which we have thought it necessary to employ here. The *Calcul des Dérivations* of M. Arbogast, however, has furnished many ideas on the subject.

Calculus of
Functions.

of x of the same form, and so are $\sin px$ and $\sin qx$, (provided a, b, p , and q be independent of x .)

But two functions which are algebraically identical are not therefore of the same form: for instance, $(a+x)(a-x)$ and $a^2 - x^2$, between which algebraical equality always exists, are not of the same form. The processes to which x is subjected in the two are not the same, though the arithmetical result of one set of processes be always the same as that of the other.

(4.) When two expressions contain x and y respectively in such a manner that the substitution of x instead of y changes the first into the second, and *vice versa*, the two are said to be the same functions of x and y . Thus $a - x^2$ is the same function of x which $a - y^2$ is of y . But $a - yx^2$ is not the same function of x which $a - y^3$ is of y ; for though writing y instead of x in the first would give the second, yet writing x instead of y in the second would not give the first.

(5.) Previously to entering upon the symbols peculiar to this branch of Mathematics, it will be useful to look at the nature of the symbols which have preceded. It is usual to say that all symbols are symbols of quantity, and this is true, but only in a sense analogous to that in which $(a+x)(a-x)$ and $a^2 - x^2$ are regarded as identical in Algebra. Considered as representatives of quantity they are the same; not so as representatives of operations. Properly speaking, a symbol of quantity is that in which the connection between the symbol and the quantity is purely arbitrary, without reference to the meaning of any previous symbol, or the result of any defined operation. In Arithmetic, therefore, 1 is the only symbol of quantity: 2 is, in the arithmetic of concrete quantities, the quantity resulting from the operations contained in the symbol $1+1$; in the arithmetic of abstract numbers it is only the representation of the operation $1+1$. For the plainest definition of the science of "Abstract Number" is, the method of reasoning upon the operations which may be performed upon 1, considered independently of the particular quantities which 1 may represent. And Arithmetic itself may be considered as the art of reducing the result of every operation which can be performed upon different functions of 1, to a function of the form

$$a + bx + cx^2 + ex^3 + \&c.$$

where $a, b, c, \&c.$ are severally one or other of the defined symbols 0, 1, 2, 9, x is the operation $9+1$, and generally, pq represents the same function of q which q is of 1. The decimal notation, by which writing is saved, (for in the preceding view every other advantage of it is included,) consists in the substitution of

$$\dots\dots\dots ecba$$

for the preceding series.

(6.) In Algebra (in its most general form) we take as many arbitrary symbols of *quantity* as we please, $a, b, x, \&c.$; understanding also that the symbols may define, as well the magnitude, as any other modification of which concrete quantity admits. But in Algebra, considered as a consequence of *abstract* arithmetic, these are not symbols of quantity, but of undefined, or, *ad libitum*, operations upon 1; having the same indeterminate character with respect to 1, which 1 has with respect to concrete quantity. That is, Algebra is the science which draws conclusions with respect to $a, b, \&c.$, whatever operations they may denote, in the same way as Arithmetic draws conclusions with respect to the definite operations only,

0, 2, &c. whatever concrete magnitudes they may denote. And, strictly speaking, we have in the letters of Algebra our first *arbitrary symbols of operation*.

Calculus of
Functions.

The objects of Algebra are either the determination of the particular characters which $a, b, \&c.$ must have, for the fulfilment of certain conditions, that is, the reduction of problems to equations of condition, and the solutions of these equations; or, the invention of operations which may be substituted instead of given operations, without the alteration of the arithmetical results of any particular case. The latter may be thus stated; *given a function*, and the *form* only of another function, to find the particular case of the form assigned, the results of which shall always be arithmetically* equivalent to the results of the given function.

(7.) But we have yet another case to consider, as follows. Given any number of quantities and any conditions which are to exist between functions of these quantities, required the forms of the functions which will satisfy these conditions. For instance, required that operation which being performed upon x , and again upon the result, shall reproduce x . Of this $a - x$ is a particular case, for if $a - x$ be $p, a - p$, or $a - (a - x)$ is x . Another case is

$$\log(a - \text{No. whose log. is } x)$$

for if this be called p , the repetition of this operation gives

$$\log(a - \text{No. whose log. is } p),$$

$$\text{or, } \log\{a - (a - \text{No. whose log. is } x)\}$$

$$\text{or, } \log(\text{No. whose log. is } x) \quad \text{or } x.$$

We might give more instances, but this will be sufficient to show that as Algebra gives results of greater generality than Arithmetic, so the consideration of the form which will satisfy certain conditions gives results of greater generality than those of common Algebra. The problem enunciated in this article is the general problem of the *Calculus of Functions*.

(8.) Every hint, therefore, which can be gained from the connection of Arithmetic and Algebra, may teach us what to look for, and what to avoid, in the transition from Algebra to the Calculus of Functions. And, first, we must observe, that as the indeterminate character of the algebraical symbols kept out of view the limitations necessary in Arithmetic, so the corresponding character of the symbols to which we shall come, as compared with the definite forms of Algebra, may equally make us lose sight of limitations necessary in the latter science. And as each neglect of the necessary conditions of Arithmetic made algebraical operations more extensive than those of the former Science, the same may happen here.

Though the Calculus of Functions is not yet so far advanced, that we can look back upon such extensions, yet certain anomalies which we shall have occasion to observe oblige us to look forward to them, and we shall therefore now ask, what are these operations of Algebra which have been explained with perfect generality?

The great limitation of pure Arithmetic, namely, that in $a - b$, a must be greater than b , led to the difficulty of negative quantities, since the indeterminate form of a and b prevents the recognition of any such condition. The doctrine of positive and negative quantities, however established, leads to the further difficulty of the square root of

* The word arithmetically is here taken in a very wide sense. It should perhaps be;—Given a function, and the *general* form of another, required (if possible) a *specific* form, such that it and the given function shall be reducible to identity by the same operations.

Calculus of Functions. **Calculus of negative quantity.** The use of $\sqrt{-1}$ (called *impossible*) has of late years pointed out a method of interpretation,* which leaves us finally, we may say, in perfect possession of the following algebraical forms, in which all the letters denote explicit arithmetical quantities, (whole or fractional, rational or irrational.)

$$\pm x, (\pm x)^{\pm y} (+a)^{(\pm y)^n}, \{(-a)^{(\pm y)^n}\}$$

To which we may add the logarithms of the preceding quantities, with their sines, cosines, &c.

But all forms in which the exponent of an exponent has any but a purely numerical form, are not yet, we think, within the dominion of the general theorems of Algebra. For it will appear that such forms may be found to satisfy conditions which common algebraical reasoning would lead us to conclude it impossible to satisfy. And, independently of this subject, it has lately been shown† that the general theorem (or, at least, the observed result) that "all functions which disappear when $x = 0$, can be developed in positive powers of x "

is not true in the case of $e^{-(x)^{-2}}$ or $\frac{1}{x^2}$, which nevertheless vanishes when $x = 0$.

As to definite integrals, the present state of knowledge does not enable us to class them.

We have here no further to do with this subject than to observe at the outset, that, when we prove a case of impossibility by common algebraical reasoning, we mean the impossibility to refer only to those expressions which have been really considered in Algebra, as it now stands. And that, for any thing we know to the contrary, the expressions not yet considered, and *discontinuous* functions in particular, may satisfy an algebraically impossible condition, just as the negative quantity of Algebra satisfies an arithmetically impossible condition.

(9.) The study of Algebra leads to that of variable quantities, in which a letter is considered as assuming all possible values between given limits; and the results of that supposition are traced in the Differential and Integral Calculus. To this we have nothing analogous at present in the Calculus of Functions; we can neither arrange forms in order of magnitude, or according to an order dictated by any other notion which appears necessary to them. It is true that nothing would dispose us to alter the arrangement of the following forms, that is, to change the place of any one in its own set:

a	$\log x.$
$a + bx$	$\log \log x.$
$a + bx + cx^2$	$\log \log \log x.$
&c.	&c.

But suppose we ask, where, in the second series, is the proper place of $a + bx$? We can give no answer, as no place appears more proper than another. But if we write down two sets of algebraical or arithmetical quantities, such as

1	$\frac{3}{2}$	a	$p + q\sqrt{-1}$
2	4	$b + c\sqrt{-1}$	$\sqrt[3]{-1}$
$\frac{1}{2}$	5	$-d$	r
&c.	&c.	&c.	&c.

* See the *Algebra and Report to the British Association* of Mr. Peacock.

† By Professor Hamilton, in a late volume of the *Transactions of the Royal Irish Academy*.

we can, in the first, arrange both sets in an order derived from a notion which we call *comparative magnitude*, which is essential to quantity, meaning that two quantities never present themselves to us unaccompanied by it. And we can immediately remove one out of either set to its proper place in the other. In the algebraical sets, we can do the same in two different ways, answering (as the student acquainted with the theorem

$$a + b\sqrt{-1} = \sqrt{a^2 + b^2} (\sin \theta + \sqrt{-1} \cos \theta)$$

where $\tan \theta = \frac{b}{a}$

will not fail to perceive) to arranging the lines signified by the symbols, in order of inclination or in order of magnitude. But the analogy entirely fails when we take algebraical forms at hazard, or without some specific law of connection; and we therefore propose the following query, which the general results of our future investigations will (in our opinion) incline us to hesitate upon, if not to decide in the affirmative.

May not the Calculus of Functions divide algebraical forms into so many and so distinct varieties, that, judging by the impression remaining on our minds from the differences which have hitherto distinguished *sciences* from one another, we should feel inclined to say that the different classes of functions are rather the objects of different sciences than of different branches of the same science; or, at least, are the objects of branches as different as those which treat of possible and impossible quantities in Algebra?

(10.) But the imperfect considerations connected with functions which physical problems rendered necessary in the Integral Calculus suggested a notion so like what we may conceive to have preceded continuous variation in algebraical quantities, that we do not despair of seeing laws of arrangement established among forms. The notion is that of *discontinuous functions*. When a quantity retains the same value, it is called a *constant*. When a function always retains the same form, no name was ever given to it, no distinction being necessary, because no other case was ever considered. But in the Integral Calculus, it was found that where an arbitrary function, say of x , was introduced in integration, it was allowable to suppose it to have one form for every value of x between a and b , another for every value between b and c , &c. This continuous variation being possible to any extent, we might conceive the limits of invariability diminished without end, so as to present a continuous variation of *form*. We shall afterwards show certain cases in which this is possible, but (remembering that the admissibility of discontinuous functions divided D'Alembert and Euler) we will not undertake to say that a form might be assigned which should satisfy the differential equation; or that if such a form were assigned by one, it would be received as such by another.

(11.) Every new step in the science of symbols has been accompanied by a correlative, usually called the *inverse* step, from whence the first has acquired the name of *direct*. If working with x in a given way produces y , in what way must we work with y to reproduce x ? or, given x , and the way to find y , what is the way to find x when y is given? or, knowing what function y is of x , to find what function x must be of y ? This is properly a question of the Calculus of Functions. Every inverse operation is found to require new modifications of symbols, and, hitherto, every case has pre-

Calculus of
Functions.

sented some difficulties; for it has always been found, that the inverse operation was not generally possible, without some previous extension of the notions necessary to the performance of the direct operation. Thus the first inverse question which presents itself is in Arithmetic, and is the following: if $1 + 1$ be called 2, what is that operation by which 2 may be reconverted into 1? This introduces the notion by which the symbol of quantity itself is considered as a result of operations on certain subordinate symbols of quantity, which would be symbols of pure quantity only, if it were not more consistent with our first ideas to consider the above question as self-evidently answerable, and the inverse operation itself as a new definition resulting from it. But it is to be remarked, that the answer to such questions as the above is not given without the invention of new symbols $\frac{1}{1+1}$, $\frac{1}{1+1+1}$, &c. inverse to $1 + 1$, $1 + 1 + 1$, &c., and an extension of the notion of whole numbers.

The operation of subtraction, the inverse to that of addition, leads to another extension; that of extraction of a root, the inverse of the raising of a power, to another: and so on. And here we observe that every function may have an inverse corresponding to every different quantity which may be considered. If the operation be such as to admit of *interchange*, that is, if the form of a function is the same, whether a or b be considered, the inverses are in both cases of the same form: thus $a + b$ and $a b$, considered with reference to either b or a , give the same inverses. But in a^b , which, considered as a function of a , is not of the same form as when considered as a function of b , there are two inverses, the extraction of a root in the first case, the construction of a logarithm in the second. The following terms are respectively inverse to each other, those which are usually called direct being in the first column:

Addition	Subtraction
Multiplication	Division
Involution	Evolution
Developement	Invelopement*
Exponential	Logarithm
Power	Root
Positive	Negative
Whole number	Fraction
Inversion	Inversion
Reciprocal	Reciprocal
Production	Reproduction
Sine	Inverse sine
Cosine	Inverse cosine
Function	Inverse function
Sum	Difference
Differential	Integral.

(12.) The obvious method of signifying an inverse function would be to invent some abbreviation of the word *inverse*, to be appended to the functional symbol. But it might be equally consistent to take the symbols of any two simple inverse operations we please; and, in whatever sense it may be convenient to connect one of them with the direct functional symbol, in the same

* This term ought to exist; in consequence of the most usual developement being that of a function into the sum of a number of others, the term *summation* usually supplies its place. But in treating of continued products, continued fractions, &c., many awkward circumlocutions are rendered necessary by the want of a general term inverse to developement.

way to connect the other with the inverse functional symbol, so as to make the latter express, both what is the direct operation, and what is the inverse. The following will explain our meaning.

If we consider the series of quantities

$$1, 2, 3, 4, 5, \&c.$$

as being each formed from the preceding by the same general operation, and the same particular case of it, it is evident that the operation is $+1$, or any term being p , the next term is $p + 1$. The inverse of this operation is -1 , and we thus get the series of direct and inverse terms,

$$\dots -2 \quad -1 \quad 0 \quad +1 \quad +2 \quad \dots$$

We may, by extension of our language, say that we here consider the natural numbers as functions of 0, for since it is the definition of an inverse function that the direct operation performed upon it produces the *subject** of the function, and since $-a + a$ is always $= 0$, we must, if we make any such extension at all, when we consider $-a$ as an inverse of a , say that a is considered as a function of 0.

We may also say that the *modus operandi* is absolute, the formation of each from its predecessor having no reference to the place of the latter.

(13.) We may now ask, having got this notion, how are the series of natural numbers to be considered as functions of 1. That is, what direct operation performed upon its inverse always produces 1? The answer is

$\left(\frac{1}{a}\right) \times a = 1$, and the series of direct results being 1, 2, 3, &c. as before, the series of inverse results is $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c., or we get

$$\dots \frac{1}{3}, \frac{1}{2}, 1, 2, 3, \dots$$

We, therefore, see reason to change the imperfect analogies of article (5.) and to consider 0 as there the symbol first operated on, and $+1$ as the *modus operandi* by which x becomes $x + 1$: but when we consider

$x \times \frac{x+1}{x}$ is the *modus operandi*, then 1 is the

symbol first operated on. And here the *modus operandi* is relative, the formation of each number from its predecessor being directly dependent upon the latter.

We may now ask, according to what method of derivation may all numbers be considered as functions of 2. What direct function performed upon its inverse will

always produce 2? Since $(2^x)^{\frac{1}{x}} = 2$, the series of direct quantities 2, 3, 4, may be (λa representing the logarithm of a to the base 2)

$$2^{\lambda 2}, 2^{\lambda 3}, 2^{\lambda 4}, \&c., \text{ or } 2, 3, 4, \&c.;$$

and the inverse set is $2^{\frac{1}{\lambda 2}}, 2^{\frac{1}{\lambda 3}}, 2^{\frac{1}{\lambda 4}}$, &c. the first of which, being $= 2^{\lambda 2}$, (giving 2, which coincides with 0 and 1 of the preceding cases in being its own inverse, and which we shall therefore not distinguish,) may be considered as the first of the previous set and the complete set is

$$\dots 2^{\frac{1}{\lambda 4}}, 2^{\frac{1}{\lambda 3}}, \left(2^{\frac{1}{\lambda 2}} = 2^{\lambda 2}\right), 2^{\lambda 3}, 2^{\lambda 4}, \dots$$

* When a function of x is mentioned, x is the *subject* of the operations.

Calculus of
Functions.

Calculus of
Functions.

We might propose an infinite number of other ways in which the series of natural numbers might be considered as functions of any one among them: but the above will be sufficient to illustrate this point, that each set gives a distinct set of inverses, and that we have therefore an unlimited number of methods by which we may represent inverse functions of different species, choosing of course those methods which present any analogy to the manner of deriving the inverse in question. We shall return to this subject.

(14.) We now proceed to explain the notation of functional symbols. As in Arithmetic the notion of concrete quantity was generalized into that of abstract number, and in Algebra *definite* abstract number was abandoned for symbols of general number, thus enabling us to express any *definite* operation performed with an indefinite number, so in the Calculus of Functions the definite character of the operation is dropped, and indefinite symbols of operation are used, which may be either *unknown*, to be found by means of conditions, or *arbitrary*, to be assigned at pleasure, or *conventionally definite*, (like π and e in Algebra,) to denote particular operations which frequently occur. The symbols usually employed are the letters $F, f, \phi, \psi, a, \beta$: but in the present Treatise *all Greek letters are functional symbols except π and e* , (ω is a functional symbol,) and of the small Roman letters, f only is a functional symbol. And no letter which is a functional symbol in any one place will be used as a symbol of quantity in any other.

The subject of the function is written after or under the functional symbol, as may be most convenient. Thus ϕx or ϕ_x means some operation to be performed upon x : and

$$F x, f x, \phi x, \psi x, \chi x, a x, \beta x,$$

may all denote the results of different operations performed on x , or different functions of x . And when two functional symbols are placed before x , as in $\phi \psi x$, it is meant that the operation ψ having been performed upon x , the operation ϕ is to be performed upon the result. Each functional symbol has a permanence of meaning throughout the same problem.

The symbols of unknown functions will usually be ϕ, ψ , and χ ; those of given functions, a, β , &c., those of arbitrary functions $\theta, \rho, \sigma, \tau$; the symbol λ will always stand for the Napierian logarithm, and $\sin, \cos$, &c. for sine and cosine, &c., as in Trigonometry.

The following examples will illustrate this notation.

1. Let $\phi x = a^x$; then $\phi(x+1) = a^{x+1}$, $\phi \lambda x = a^{x^x}$ $= e^{x \log x}$ $= x^{x^x}$; $\phi(\sqrt{-1}x) = a^{\sqrt{-1}x}$; $\phi \sin x = a^{\sin x}$.

2. Let $\phi x = 1+x$; then $\phi \phi x = 1 + \phi x = 1 + (1+x) = 2+x$; $\phi \phi \phi x = 1 + \phi \phi x = 3+x$,

and so on. Let $\phi x = a^x$; then $\phi \phi x = a^{\phi x} = a^{a^x}$, &c.

3. Let $\phi x = \cos x$, $\psi x = x^2$; then $\phi \psi x = \cos(x^2)$ $\psi \phi x = (\cos x)^2$.

$$\text{Let } \phi x = \frac{1+x}{1-x} \quad \psi x = \frac{1-x}{1+x}$$

$$\phi \psi x = \frac{1 + \frac{1-x}{1+x}}{1 - \frac{1-x}{1+x}} = \frac{1}{x}$$

$$\psi \phi x = \frac{1 - \frac{1+x}{1-x}}{1 + \frac{1+x}{1-x}} = -x.$$

4. What function of x is that which being multiplied by itself gives an equivalent to a second result of the same functional operation? The functional equation from which the form of ϕ is to be found, is evidently $(\phi x)^2 = \phi \phi x$. One solution is evidently $\phi x = x^2$. What function is that which is not changed in value by writing $x+1$ for x ? The functional equation is $\phi(x+1) = \phi x$.

(15.) When an operation is to be performed a number of times in succession upon the same quantity and its results, the symbols $\phi \phi x$, $\phi \phi \phi x$, &c. are abbreviated into $\phi^2 x$, $\phi^3 x$, &c. in which it must be recollected that ϕ being a symbol of operation, the index stands for the number of times the operation is repeated. Thus the question in (7.) was as follows; required solutions of $\phi^2 x = x$.

Thus $\lambda^2 x$ means the logarithm of the logarithm of x , which call the second logarithm of x ; $\sin^2 x$ means the sine of the sine of x ,* the common meaning, the square of the sine of x , being, in this subject, rejected for the preservation of symmetry. The latter is denoted by $(\sin x)^2$, meaning $\sin x \times \sin x$.

(16.) There are two species of inversions to which we shall appropriate distinct symbols. The first, which has more reference to Algebra, (in which the investigation of it in different cases continually occurs,) I shall call the *algebraical* inverse, or simply the inverse, this particular one being understood whenever the single word occurs. But the second, the idea of which will now be met with by the student for the first time, I call the *functional* inverse, never dropping the first word.

If $\phi x = y$, then x is also some function of y , as in the following instance:

$$y = (x-1)^2, \quad x = 1 + \sqrt{y}; \text{ or } x = 1 - \sqrt{y};$$

so that if $x = \psi y$, one form of ψy is $1 + \sqrt{y}$, and one form of ψx is $1 + \sqrt{x}$. This is an *algebraical* inverse of $(x-1)^2$.

Again, suppose we repeat any function any number of times, say that ϕx being $a x$, we find $\phi^2 x$, which is $a(a x)$ or $a^2 x$. The algebraical inverse of this

function is $\frac{x}{a^2}$. In this method we observe that there is no reference to the accumulated operations by which $a^2 x$ was supposed to have been obtained; the inverse is the same, whether we consider $a^2 x$ as our primitive function, or whether we regard it as the third function of $a x$. If we look at the series of direct and inverse quantities

$$-3, -2, -1, 0, +1, +2, +3, \&c.$$

we perceive an analogy. When we here form one from the preceding, the *modus operandi* has no reference to the number of steps by which that predecessor was formed from 0. We shall, therefore, choose these inverses $-1, -2$, &c. as convenient indices of the algebraical inverses of ϕx , $\phi^2 x$, &c. That is, $\phi^{-1} x$ shall be the algebraical inverse of ϕx , (or, by analogy, $\phi^1 x$), $\phi^{-2} x$ of $\phi^2 x$, &c. So that if $\phi x = y$, $x = \phi^{-1} y$; if $\phi^2 x = y$, $x = \phi^{-2} y$, &c. Generally $\phi^{-n} x$ means that

* It must be remembered that the angle of analysis is measured by the abstract number which expresses the ratio of any subtending arc to its radius. Thus the angle 1 (or the angle whose arc = radius) is in degrees $57^\circ 17'$ nearly: its sine is .84135, which, considered as an angle, is equivalent to $48^\circ 18'$, whose sine is .7466 nearly; so that

$$\sin^2 1 = .7466 \text{ nearly.}$$

Calculus of
Functions.

Calculus of Functions. operation upon which if ϕ be repeated n times, the result shall be x , which by analogy from the series is $\phi^0 x$. The operation ϕ^2 performed upon $\phi^{-2} x$ restores x , or $\phi^2 \phi^{-2} x = x$. But we may ask this question; if $\phi^2 x$ be not considered as absolute, but as dependent upon x , or as a function of x , that is, if $\phi^2 x = \psi x$, what is the connection of the operations ϕ and ψ ? To take an instance; let $\phi x = ax$, $\phi^2 x = a^2 x$, or $\psi x = a^2 x$.

Here $a^{\frac{1}{2}} x$ stands to ax and $a^2 x$ in this inverse relation, that whereas $a^2 x$ is the second function of x , when ax

is the first, $a^{\frac{1}{2}} x$ is the first function to which ax is the second. This is what we call the second functional inverse, and it bears relation to the number of operations by which the direct function was obtained. Hence, by a similar analogy, we shall choose the inverses of the set

$$\dots\dots \frac{1}{3}, \frac{1}{2}, 1, 2, 3 \dots\dots$$

to represent these functional inverses, that is, since ψ is ϕ^2 , ϕ shall be $\psi^{\frac{1}{2}}$. And we have the following relation

in the particular case in question, $\psi^{\frac{1}{2}} \psi^{\frac{1}{2}} x = ax$, or $(\psi^{\frac{1}{2}})^2 x = ax = \phi x$.

If we ask what five similar operations performed on x will produce the same result as ϕ performed twice, or what must ψ be that $\psi^5 x = \phi^2 x$, by the same analogy we call $\phi x = \psi^{\frac{2}{5}} x$ and $\psi x = \phi^{\frac{5}{2}} x$, or generally $\psi^{\frac{m}{n}}$ indicates that operation, which being performed n times, shall produce ψ^m .

(17.) We may now show that the extensions of language in (12.) and (13.) may be made to accord with the notions just laid down. If $\phi x = x + 1$, we have $\phi^2 x = \phi x + 1 = x + 2$, and $\phi^n x = x + n$. If we suppose $x = 0$ this gives

$$\left. \begin{array}{l} x \text{ or } \phi^0 x = 0 \\ \phi^1 x = 1 \\ \phi^2 x = 2 \\ \&c. \quad \&c. \end{array} \right\} \begin{array}{l} \text{in the case} \\ \text{where } x = 0, \end{array}$$

and we find the preceding equation to hold good when n is negative. For if we assume

$$\begin{array}{l} \phi^{-2} x = x - 2 \\ \text{then } \phi \phi^{-2} x = \phi^{-2} x + 1 = x - 1 \\ \text{and } \phi^2 \phi^{-2} x = \phi \phi^{-2} x + 1 = x. \end{array}$$

But as it is not the custom to consider functions of particular values of x , we shall say the natural numbers are the successive functions of $x + 1$ when $x = 0$.

The consideration of the successive numbers as functions of 1 is somewhat more refined. If we consider the function ax , we have $\phi^2 x = a(ax) = a^2 x$ and $\phi^n x = a^n x$, giving us

$$\left. \begin{array}{l} \phi^0 x = 1 \\ \phi^1 x = a \\ \phi^2 x = a^2 \\ \phi^3 x = a^3 \\ \&c. \quad \&c. \end{array} \right\} \begin{array}{l} \text{in the particular case} \\ \text{where } x = 1. \end{array}$$

Now we cannot, for any value of a , find the series of natural numbers in the set $1, a, a^2, \&c.$ But by taking a sufficiently near to unity, we may so manage, that all the numbers under a given limit (say a million) shall severally be as near as we please to one or other of the powers of a (say within .000001.) Or, if we express

Calculus of Functions. this theorem in the language of Leibnitz, we should say that if a be infinitely near to unity, some powers of a , $a^{m_1}, a^{m_2}, a^{m_3}$, with infinite exponents, having finite ratios to one another, will be severally equal to 2, 3, 4, &c. In this sense, then, we may say that $(1 + a)x$, where a is infinitely small, is the function which gives the scale of numbers when $x = 1$, but that the successive numbers are not successively produced, there being an infinite number of operations interpolated between each.

When we see quantity in a state of continuous increase, we are so accustomed to consider the increase as absolute, that we admit the notion of a succession of infinitely small and equal increments in infinitely small and equal times, without much repugnance. But because the proportion or the rate of increase is rather harder, we do not so readily admit the notion of quantity increasing at an infinitely small rate per cent. in successive infinitely small times. But we see that the second is the necessary consequence of supposing numbers to be, in our phrase, functions of x , where $x = 1$, instead of functions of x , where $x = 0$.

The preceding view is sufficiently practical, or was, for by it Napier constructed logarithms before exponents were used.* Halley's words in the preceding note, that he considers ratio as a "quantity *sui generis*," correspond to what is meant by saying numbers are functions of 1.

In a similar way, by considering a as infinitely small, we might have supposed any number or fraction to be an n^{th} function of ϕx or $x + a$ in the case where $x = 0$.

(18.) Next to the symbols of quantity in importance come those of the calculus of differences, whether finite or infinitely small, such as $\Delta, d, \Sigma, \int, \&c.$ These are certainly not symbols of quantity, but of operation, and

* This ought to be more prominently stated in elementary Works, as one of the greatest curiosities in the History of Science. We remember discussing the merit of the invention with a friend, when both of us were students at Cambridge, the one for, and the other against, both being under the impression that the system flowed naturally from the extended exponential symbol, and neither (as might be supposed) being Historian enough to know the contrary. Napier published his *Mirifici Logarithmorum Canonis* &c., in 1614, and even in the time of Halley the symbols of Algebra were insufficient to furnish a theory. The ideas of the latter we quote: "Logarithms may be properly said to be *numeri rationum exponentes*; wherein we consider ratio as a *quantitas sui generis*, beginning from the ratio of equality, or 1 to 1 = 0; and these ratios we suppose to be measured by the number of *rationunculae* contained in each. Now these *rationunculae* are so to be understood as in a continued scale of proportionals infinite in number between the two terms of the ratio, which infinite number of mean proportionals is to that infinite number of the like and equal *rationunculae*, between any other two terms, as the logarithm of the one ratio is to the logarithm of the other. Wherefore, if the root of any infinite power be extracted out of any number, the *differentiola* of the said root from unity shall be as the logarithm of that number." *Misc. Cur.* vol. ii. p. 38, 39. Halley, of course, had whole exponents, and even fractional ones, and thus was within (one would think) a hair's breadth of the simple definition, "if $y = a^x$, x is the

logarithm of y ." In this very paper he used $(1 + q)^{\frac{1}{m}}$ not expressly as the m^{th} root, but as an abbreviation of Newton's development. On such small matters of symbolic notation do the simplifications of analysis depend, that the Calculus of Functions assumes a character of importance. And, unfortunately, improvements in symbols travel more slowly than any others. Perhaps the abolition of the fluxional notation in England is not in point, as national prejudice interfered: but when was the decimal point (certainly used by Napier) finally introduced into schools and colleges in France? We have evidence before us that it was after 1710, and perhaps long after, and also that the Ptolemaic system was taught as the most probable at the same time.

yet are not strictly *functional* symbols, in the sense in which we have to consider functions. As nothing is of more consequence in this Science than to avoid letting the reader fall, for want of warning, into a notion that the definitions of ϕ , ψ , &c., and their consequences,

have any necessary affinity with those of Δ , $\frac{d}{dx}$, &c.

and their consequences, we shall examine the degree of connection which exists between the two.

It is plain that Δu_x or $u_{x+h} - u_x$ may always be considered as a function of u_x ; that is, * *dependent upon x through u_x only*. For if $u_x = p$, from which x may be found in terms of p , and substituted in $u_{x+h} - u_x$, the result is a function of p only. To take two instances,

$$u_x = \sin x \quad \Delta u_x = u_x \cos h + \sqrt{1 - u_x^2} \sin h - u_x,$$

$$u_x = x^2 \quad \Delta u_x = 2h\sqrt{u_x} + h^2.$$

Consequently, $\Delta \phi x$ is a function in which the functional nature of Δ depends upon that of ϕ . But in $\psi \phi x$, ψ is considered merely as the index of certain operations to be performed, not on the form of ϕ , but on ϕx considered as a result. Thus if $\psi x = x + x^2$, we have

$$\psi \phi x = \phi x + (\phi x)^2 \quad \psi e^x = e^x + (e^x)^2$$

$$\psi(x + \chi x) = x + \chi x + (x + \chi x)^2$$

$$\psi a \beta \gamma x = a \beta \gamma x + (a \beta \gamma x)^2;$$

or $\psi \phi x$ performs the same operations on ϕx , which $\psi a \beta \gamma x$ does on $a \beta \gamma x$. But $\Delta \sin x$ does not perform the same operation on $\sin x$ which Δx^2 does on x^2 . In fact, we may see that though the very simple nature of the connection between Δu_x and u_x , expressed by

$$\Delta u_x = u_{x+h} - u_x$$

renders the operations easy, yet that in principle we have, by the definition of Δ , ascended one step higher than the Calculus of Functions in the scale of generalization, by which concrete Arithmetic became abstract, abstract Arithmetic Algebra, and Algebra the Calculus of Functions. For we have thus proposed operations upon the form of ϕx , that is, Δ (if $\Delta \phi x$ be considered as a function of ϕx) stands to ϕ in the same relation as ϕ to x , or as x to abstract, or as abstract to concrete quantity.

But there is another point of view (and that, it may be, the more natural one) in which $\Delta \phi x$ is not considered with respect to ϕx , but with respect to x . If we arrange quantities in an order dictated by any law whatever

$$\dots a_{-3} a_{-2} a_{-1} a_0 a_1 a_2 a_3 \dots$$

and consider $+ \Delta u_x$ as the operation by which we pass from u_x to u_{x+1} , it is clear that we have a perfect analogy with the contents of (12.) If we consider that in the symbol $+ \Delta u_x$, the u_x is merely inserted to recall to the mind at what point of the series we suppose ourselves to be, we shall see that $u_x + \Delta u_x$ has a connection with $(1 + \Delta)u_x$, of the same kind as $a + ba$ with $(1 + b)a$; a being considered as the result of an operation on 1,

* This phrase we have often found necessary: its meaning is as follows. If y be a function of x , then ϕy is also a function of x , but only because y is so. Therefore, we say, ϕy is a function of x , *through y* . Thus $x^2 + 2xz + z^2$ is a function of x and of z through $x + z$; if the latter equal y , the function is y^2 , not containing x or z . But $x^2 + 2xz$, though it may be considered as a function of either x or z , through $x + z$, cannot be thus considered as a function of both; for if $x + z = y$, $x^2 + 2xz$ must be either a function of y and x , or of y and z . One of the two, x and z , being eliminated, the other remains.

and $(1 + b)a$ being the same operation on a which $1 + b$ is on 1. Still more direct is the analogy of $u_x + \Delta u_x$ and $(1 + \Delta)u_x$ with 7 pebbles + 3 pebbles and $(7 + 3)$ pebbles. Considered in this light, the Calculus of Differences is (whichever word may be deemed most correct) a refinement, extension, or generalization of the method of figures, "the only true demonstrable root of human knowledge."

(19.) The preceding view may perhaps remove some of the feeling of unmixed astonishment with which the student always regards (or ought to regard) the now common method of the *separation of the symbols of operation from those of quantity* in the higher Mathematics. Lagrange's well known theorem

$$(1 + \Delta)^n u_x = u_x + n \Delta u_x + n \frac{n-1}{2} \Delta^2 u_x + \dots \text{&c. (A),}$$

the truth of which may be easily established, *treats* (not considers) Δ as a quantity on the first side, and as a symbol of operation on the second. And so, in an enlarged view of Algebra, does

$$(1 + a)^n b = b + n a b + n \frac{n-1}{2} a^2 b + \dots \text{&c. (B)}$$

as explained in (5.) The connection was necessary to Halley, in explaining logarithms without* the direct aid of exponents.

In his version of the preceding theorem, b was omitted, and $(1 + a)^n$ was a *quantitas sui generis*; even the meaning of numerical symbols had been altered, and 0 merely meant "ratio of one to one."

(20.) It would not be difficult so to generalize the notion of successive counting, that Algebra should become a Science of operations, even in the higher sense in which Δ , an operation, is distinguished from u_x , a quantity. By that method, (B) would become a particular case of (A) in which u_x is derived from the equation $\Delta u_x = b u_x$. And Laplace's theory of generating functions might easily be connected with such a system.

(21.) The Calculus of Ratios, for so it should be called, really exists, though for convenience logarithmically connected with that of differences. From analogy with (13.), we might express the method of passing from term to term of the series in (18.) by $\times \nabla u_x$, in which case $u_{x+1} = u_x \nabla u_x$. But as, if $v_x = \lambda u_x$, we have

$$\Delta v_x = \lambda \nabla u_x, \text{ or } \nabla u_x = \epsilon^{\Delta v_x},$$

the theorems of the second Science are virtually deducible from certain cognate theorems of the first. But the study of the consequences of defining

$$\nabla u_x \text{ to be } \phi(u_x, u_{x+1}, \dots, u_{x+n})$$

is yet in the merest infancy.

(22.) If we proceed to apply the preceding notions to the Calculus of Functions, we are stopped by a warning from the method by which Arithmetic was made Algebra. The nature of the extension by which we passed from $u_x + \Delta u_x$ to $(1 + \Delta)u_x$ required no apparent extension of the usage of the sign $+$, which continued to signify algebraical addition.

But, in the same manner as in Algebra, the sign $+$ acquired an extended meaning in the formula $a + \sqrt{-1}b$, or $a + (-b)$, so it may happen that the most

* We mean that his exponents were simply abbreviations of subsequent algebraical operations, not employed in his definition or theory of logarithms.

Calculus of Functions.
 general algebraical meaning of $+$ will not render those theorems true of $(\phi + \psi)x$, considered as meaning $\phi x + \psi x$, which are true of $(a + b)x$ where a and b are quantities.

The first enquiry in Algebra appears to be, how to invent a simple and consistent notation, and deduce rules from it. But this soon gives place to an inverse enquiry, namely, how to invent notation, so that the rules which follow from it shall be simple and analogous to those which have preceded. We might, if we pleased, say, let $(\phi + \psi)x$ mean $\phi x + \psi x$, in which case $(\phi + \psi)^2 x$ must stand for the same function of $\phi x + \psi x$, which the latter is of x , or for

$$\phi(\phi x + \psi x) + \psi(\phi x + \psi x).$$

But this definition would be of no use, so far as its connection with previous results is concerned, unless we

$$\begin{aligned}\phi^2 x &= a \{a(x+m) - m + m\} - m = a^2(x+m) - m, \\ \phi \psi x &= a \{b(x+m) - m + m\} - m = ab(x+m) - m = \psi \phi x, \\ \phi x + \psi x + m &= (a+b)(x+m) - m = (\phi + \psi)x, \text{ by definition,} \\ (\phi + \psi)^2 x &= (a+b)^2(x+m) - m, \\ &= \phi^2 x + 2\phi \psi x + \psi^2 x + 3m.\end{aligned}$$

But, by the same rule, which laid down that

$$(\phi + \psi)x \text{ means } \phi x + \psi x + m,$$

we have

$$\begin{aligned}(\phi^2 + \phi \psi)x &= \phi^2 x + \phi \psi x + m, \\ (\phi^2 + \phi \psi + \phi \psi)x &= \phi^2 x + 2\phi \psi x + 2m, \\ (\phi^2 + 2\phi \psi + \psi^2)x &= \phi^2 x + 2\phi \psi x + \psi^2 x + 3m.\end{aligned}$$

Hence it appears that the separation of the symbols of operation and quantity can (for these functions) be carried to the same extent at least as in the Calculus of Differences, with this extension of meaning of the sign $+$; that $+$ placed between two isolated signs of operation, each taken once, requires the addition of m when the symbols of operation are reconnected with their symbols of quantity; or, that if

$$\phi_1 x = a_1(x+m) - m,$$

$$\phi_2 x = a_2(x+m) - m,$$

$$\dots \dots \dots$$

$$\phi_n x = a_n(x+m) - m,$$

$$(p_1 \phi_1 + p_2 \phi_2 + \dots)x \text{ means } p_1 \phi_1 x + p_2 \phi_2 x + \dots + (p_1 + p_2 + \dots - 1)m.$$

Hence $(\phi - \psi)x$ means $\phi x - \psi x + (1 - 1 - 1)m$, or $\phi x - \psi x - m$.

The inverse of $a(x+m) - m$ is $\frac{1}{a}(x+m) - m$;

or if $\phi x = a(x+m) - m$,

$$\phi^{-1}x, \text{ by analogy } \frac{1}{\phi}x, \text{ is } \frac{1}{a}(x+m) - m;$$

whence $\psi \frac{1}{\phi}x$, by analogy $\frac{\psi}{\phi}x$, is $\frac{b}{a}(x+m) - m$,

and $\left(\frac{\psi}{\phi}\right)^2 x$, considered merely as a repetition of the

operation $\frac{\psi}{\phi}$, is $\frac{b^2}{a^2}(x+m) - m$: but considered as $\psi^2 \phi^{-2}$,

or as the operation ψ twice repeated upon the second

algebraical inverse of ϕ , which is $\frac{1}{a^2}(x+m) - m$,

it gives, as might be expected, the same result.

Similarly, $(1 + \psi)x$ will be $x + \psi x + m$, or

could deduce relations between ϕx , ψx , and $(\phi + \psi)^2 x$, similar to those which exist between ax , bx , and $(a+b)^2 x$: for instance, our definition will be useless unless $\phi \psi x = \psi \phi x$, and

$$(\phi + \psi)^2 x = (\phi^2 + 2\phi \psi + \psi^2)x.$$

If $\phi x = ax$ and $\psi x = bx$, the above equation is true, but this is nothing more than calling a coefficient an operation instead of a quantity. But there may be more complicated cases, in which the indifference of the symbols of operation and quantity may be established. For instance, let

$$\phi x = a(x+m) - m \quad \psi x = b(x+m) - m.$$

Here, if $(\phi + \psi)x$ stand for $\phi x + \psi x + m$, we have the following:

$(a+1)(x+m) - m$ and x must be considered as $\phi_1 x$, when $a_1 = 1$;

$$\begin{aligned}(\epsilon^{\psi})x &= \left(1 + y\psi + \frac{y^2}{2}\psi^2 + \dots\right)x \\ &= x + y\psi x + \frac{y^2}{2}\psi^2 x + \dots + (\epsilon^y - 1)m\end{aligned}$$

$$\begin{aligned}\lambda \overline{1 + y\psi}x &= \left(y\psi - \frac{y^2}{2}\psi^2 + \frac{y^3}{3}\psi^3 - \dots\right)x \\ &= y\psi x - \frac{y^2}{2}\psi^2 x + \dots + \{\lambda(1+y) - 1\}m;\end{aligned}$$

or rather the second sides must analogically be considered as operations on x , represented by ϵ^{ψ} and $\lambda \overline{1 + y\psi}$, considered as functional symbols. And thus a meaning may be given to ϕ^y considered as a functional symbol.

(23.) But in the greater number of cases which might be proposed, analysis does not yet furnish the means of assigning how $+$ and $-$ between symbols of operation may be defined, so that $(\phi + \psi)^2 x$ and $(\phi^2 + 2\phi \psi + \psi^2)x$ may have the same meaning, &c. And this even in the case of functions which appear altogether congruous, such as $ax + b$ and $px + q$. One case more, however, is virtually contained in the theory of logarithms. If $\phi x = x^m$ and $\psi x = x^n$, the functional symbols may be treated as quantities, provided $(\phi + \psi)x$ mean $\phi x \times \psi x$, &c. For in that case $(\phi + \psi)x$ means x^{m+n} , and $(\phi + \psi)^2 x$ means $x^{(m+n)^2}$, and $(\phi^2 + 2\phi \psi + \psi^2)x$ means $\phi^2 x \times 2\phi \psi x \times \psi^2 x$, or $x^{m^2 + 2mn + n^2}$, which agrees with the former. And from the case in the last article, and the present, we may use the symbols of operation as quantities, provided the

functions are all of the form $\frac{(mx)^a}{m}$, where m is the same

in every case, and where $(\phi + \psi)x$ denotes $m \times \phi x \times \psi x$, &c.

But suppose we ask, what relation exists between the logarithm and the sine, so as to establish any connection between the second logarithm and sine, the sine of the logarithm and the logarithm of the sine, and to simplify operations upon these functions by constituting

Calculus of Functions. $(\log + \sin) x$ a defined operation. What meaning will it here be convenient to give to $+$? This question we have not the most remote means of answering.

(24.) The last two articles open a wide field of speculation upon the possible limits of analysis. Looking at common Algebra merely to pick out the leading features of its notation, we observe two sets of inverse operations; the first $+$ and $-$, the second $\times$ and $\div$, so connected with the first, that from them follows a primary branch of Mathematics, called the theory of logarithms, which is properly the theory of the connection between the first set of operations and the second. But suppose it had so happened that the construction of our reasoning faculties had made some one of the thousand lights in which Multiplication and Division might be viewed more simple than that of their derivation from Addition and Subtraction. We think there is ground to argue that this might have been in some degree the case with the Greek geometers and their successors; at any rate the general supposition is possible, for any thing we know to the contrary. Addition and Subtraction would then have been performed by an inverse method of logarithms; the numbers would have been arranged in geometrical instead of arithmetical series; the ratios which are incommensurable (that is, not expressible in the way we think most simple) might have been commensurable, and *vice versa*. Suppose, for instance, that the series of powers of 2 had been pointed out as the scale of *natural*

numbers. Then $2^{\frac{1}{2}}$, $2^{\frac{1}{3}}$, &c. would have stood (as to simplicity) in the light of fractions, 3 would have been an incommensurable fraction, and $2 + 3$ (if ever it were discovered) a very difficult operation.

Similarly, if it had so happened that $a(x + m) - m$ had been more easily tabulated, either on paper or in memory, than what we call a product, it would itself have been denoted by ax , or, as we have called it, ϕx ; and the product which we now call ax would have been considered under another form, perhaps as $\phi(x + 1) - \phi(1)$.

(25.) The Calculus of Functions, therefore, may be looked at in the following light, when considered in its most general form. Granting any given forms of operation, ϕx , x , &c., as fundamental, required the translation into our notions of the most simple relations, and *vice versa*: also, what must the fundamental operations be, in order that certain modes of connection (according to our views) may exist between them. The use of such questions must be obvious to those who consider how apt we are to take the conventions derived from our own* first view of numbers as the limits of our method of considering symbols. "Addition and Subtraction must be the most simple mode of considering magnitudes, it is in the nature of things," says many a mental philosopher. It is worth while to see how far another nature can be given to things, (if the words of an unintelligible phrase are to be considered as having sepa-

* Many other points of speculation might be added, all tending to disembarass the mind of false connections between symbols and reasoning, or phenomena and reasoning. If all mankind had spoken one language, in what light would the possibility of different languages have been viewed? If the rays of light had not moved in straight lines, what would have been the effect on our notions of figure? Would the straight line in that case have been the one to which all other lines would have been referred? &c. &c.

To the student of mathematical symbols we should decidedly recommend attention to the methods by which the deaf and dumb are taught to read and write.

rate meaning,) and to recollect that, as every extension of notation has, by laying down new operations as fundamental, assumed the possession of certain new powers, and that the mathematical sciences seem to have slackened their progress during the present century, the relations of symbols are worth looking at as modes of invention. It is not a bold surmise, that any great step in the Mathematics can only be made by observing some new relations existing between forms, and assuming the value of these forms to be known in all cases, an assumption which is rendered a truth by placing them in tables. Thus sines, logarithms, elliptic integrals, values of $\int e^{-t^2} dt$, &c., have been successively tabulated, and put in the light of assumed operations.

(26.) Two more warnings may be obtained from algebraical recollections. The first, not to be hasty in proposing new symbols of notation, but to wait until theorems point out consequences to which it will be convenient to adapt notation. And here we mean particularly compound notation: that is, it must be immaterial whether ξ or ρ stand for an arbitrary function; but it may be dangerous to the progress of the Science to assume that ξ_2 and ρ_2 stand for operations which have some specific connection with ξ and ρ , unless the convenience of that connection appear in the methods which made ξ and ρ necessary. The student who endeavours to apply principles must immediately become an inventor of notation, and much of his success must depend upon whether he does this well or not. He must abstain from generality in symbols, unless where there is generality in the theorems which can be immediately connected with them. By inventing a shorter method of writing, he is almost sure to lose sight of some relation, unless his new notation* *only abbreviates, and does not abolish*. So far as it ceases to suggest, it becomes no notation at all. This remark is particularly necessary in treating the Calculus of Functions, because much notation is used which is not abbreviation. But what is rejected is not done for the sake of abbreviation, but purposely to throw away all that the rejected notation would suggest. If the student carries this process much further, he will never find his mistake, simply because he will never have clear ideas on the subject.

Symmetry of notation is indispensable in all complicated processes. There is a limit at which the mind requires a dictionary, if the similar relations between different sets of symbols be not suggested by some similarity of character in the notation. Symmetry begets symmetry, and an unsymmetrical result from symmetrical data is almost a proof of error, while the contrary is a presumption of truth. For it is very unlikely that error should not affect this point.

(27.) The second caution is as follows. When, in Algebra, a particular case requires that some term or terms of a general theorem become evanescent, nothing or zero, whichever it may be called, some peculiarity of form is generally observed. As the zero of quantity is that which is conceived neither to increase nor diminish quantity, so we shall term the zero of operation that which is conceived to make no alteration in the subject of a function, and which is denoted by ϕ^0 . Thus $\phi^0 x = x$,

* Some writers carry this so far, that they hardly leave more than one form of equation, $U = 0$, in the whole of Mathematics

Calculus of
Functions.

meaning that if we ask what function of x will fulfil certain conditions, and the answer is $\phi^0 x$, we know that x itself is the answer desired, and x , considered as a function of itself, we may say is operated upon by the zero of operation, that is, not changed at all. Observe, then, that wherever we make a supposition which destroys a real operation, we substitute ϕ^0 for ϕ , and may find among our symbols effects analogous to those produced by making $x = 0$ in Algebra. For instance, if $c \phi x$ be a term of an equation, of which we have the general solution, we must not hastily conclude that the general solution will apply when $c = 1$; for by this supposition we substitute an operation which does not change value for one which does. We are more likely to find the general solution apply to the case where $c = 0$, because $0 \times \phi x$ is an operation which does change ϕx ; though in the latter case we must remember that we have all the possibilities of common Algebra to encounter. Again, in a general solution, with respect to $\phi x + c$, we may safely, functionally speaking, suppose $c = 1$, though we are now not sure what may result from $c = 0$.

We have no reason to assume that expressions which resemble each other algebraically, are the same in their functional properties. Thus $\frac{2+2x}{6-2x}$ and $\frac{2+2x}{2-6x}$ appear very analogous particular cases of $\frac{a+ax}{b-cx}$, but we shall see that, considered as functions of x , they are as strikingly different as x^a and a^x .

(28.) In all that has preceded, we have considered functions with reference only to one subject, or as functions of one magnitude. We shall denote a function in which two or more magnitudes are considered, in the usual way, by such symbols as

$$\phi(x, y) \quad \phi(x, y, z) \quad \phi(x, \psi x, \chi x.)$$

Thus $\psi x + x^2 \chi x$ is a case of $\phi(x, \psi x, \chi x.)$ and is a function of x only, but directly, as well as through ψx and χx . Similarly, $\psi(x, \chi(y+z))$ is a function of x, y , and z , but of the two latter through $y+z$; so that, if y and z vary, and $y+z$ remain constant, $\psi(x, \chi(y+z))$ does not vary. As we are principally concerned with functions of one subject, we shall defer further definition.

(29.) The direct question of the formation of a functional equation, which shall have for one solution $\phi x = ax$, may be answered in an infinite number of ways. Suppose we wish to have an equation which shall give $\phi x = x^2$. As all our operations are independent of the magnitude or form of the subjects, we have (a representing a known function) $\phi ax = (ax)^2$, between which and the former equation we may eliminate x wholly or partially: that is, if we call the first equation $U = 0$, and the second $V = 0$, and if βx be another known function of x , we have $V + \beta x U = 0$ when $\phi x = x^2$, or

$$\phi ax + \beta x \phi x - \{(ax)^2 + x^2 \beta x\} = 0,$$

of which, give a and β what form we may, $\phi x = x^2$ is one solution: and, as there are two arbitrary functions, we may determine them algebraically from any conditions. But we are not to assume that $\phi x = x^2$ is the only solution of the preceding. Suppose, for instance, $ax = ax$ and $\beta x = -a^2$, (or $-a^2 x^0$), we have

$$\phi(ax) = a^2 \phi x,$$

of which one solution is $\phi x = x^2$, but of which $x^2 \xi x$ is also a solution, if there can be a function ξ which shall not be altered by changing x into ax , or if a solution can be found to $\xi ax = \xi x$. For in that case

$$\phi(ax) = (ax)^2 \xi(ax) = a^2 x^2 \xi x = a^2 \phi x.$$

We have thus made an indefinite solution of $\phi(ax) = a^2 \phi x$ depend upon finding any number of functions of x which shall not be altered by changing x into ax . A general solution to which we shall afterwards come is,

$$\text{If } \phi(ax) = a^2 \phi x, \phi x = x^2 \times \text{any } f^0 \text{ of } \left(\cos 2\pi \frac{\lambda x}{\lambda a} \right)$$

with certain limitations on the word *any* derived from our future consideration of inverse functions.

(30.) To find a functional equation, admitting of two particular solutions, $\omega_1 x$ and $\omega_2 x$, we may evidently take $F\{\phi ax - \omega_1 ax, \phi \beta x - \omega_2 \beta x, \dots, \phi x - \omega_1 x, \phi x - \omega_2 x\} = 0$;

where a, β , &c. are known functions, and every term contains none but positive powers of $\phi ax - \omega_1 ax$, &c. But this case may be reduced to the last, if a function ξx can be found which will give the same results whether $\omega_1 x$, or $\omega_2 x$ be substituted for x ; or if a form of ξ can be found to $\xi \omega_1 x = \xi \omega_2 x$. For, if the value of the two last be called θx , and if an equation be formed from the last Article by means of $\xi \phi x - \theta x = 0$, $\xi \phi ax - \theta ax = 0$, &c., then that equation will be satisfied either by $\phi x = \omega_1 x$, or by $\phi x = \omega_2 x$.

(31.) Before proceeding further we must investigate the properties of several functional relations. And first, as to the algebraical inverse of ϕx . We have seen that ϕx may have more inverse functions than one; for example, the inverses of x^2 are the two values of $x^{\frac{1}{2}}$; those of $ax^2 + bx$ are

$$\frac{-b + \sqrt{b^2 + 4ax}}{2a} \text{ and } \frac{-b - \sqrt{b^2 + 4ax}}{2a}.$$

Again, $\cos 2\pi x$ has an infinite number of inverses, all contained in the formula

$$\frac{1}{2\pi} (\cos^{-1} x) \pm m,$$

where m is any whole number. Here $\cos^{-1} x$ placed in brackets signifies an angle less than π , and $\cos^{-1} x$ is the inverse of $\cos x$, or the angle whose cosine is x .

The inverses of e^x are infinite in number; for since e^x in its most general form (m being a whole number) is $e^{x \pm 2m\pi \sqrt{-1}}$, if we put $\psi x = e^{x \pm 2m\pi \sqrt{-1}}$, and λx for the numerical logarithm of x , we find $x \pm 2m\pi \sqrt{-1} = \lambda \psi x$, in which, if the inverse $\psi^{-1} x$ be substituted for x , which, by definition, gives $\psi \psi^{-1} x = x$, we have $\psi^{-1} x = \lambda x \pm 2m\pi \sqrt{-1}$.

The following examples of direct and inverse functions may be verified by the student.

1. Examples of direct functions which are their own inverses:

$$x, \frac{1}{x}, a - x, \frac{x-2}{x-1} (a^n - x^n)^{\frac{1}{n}}, \log(a - e^x)$$

$$\log\left(\frac{e^x - 2}{e^x - 1}\right), \log \log \frac{e^x + 1}{e^x - 1}.$$

Calculus of
Functions.

Calculus of Functions. The method by which these may be derived will presently be explained. They are here given to be verified.

2. The functions, in the second of the following columns are severally inverse to the opposite functions in the first.

$x + a$	$x - a$
bx	$\frac{x}{b}$
$a + bx$	$\frac{x - a}{b}$
$\frac{a + bx}{a' + b'x}$	$-\frac{a - a'x}{b - b'x}$
$\log x$	e^x , or $\log^{-1} x$
$\sin x$	$\sin^{-1} x$, angle whose sine is x .
$\cos x$	$\cos^{-1} x$, angle whose cosine is x .
x^n	$\frac{1}{x^n}$
ax^b	$\left(\frac{x}{a}\right)^{\frac{1}{b}}$
$ax^2 + bx + c$	$\frac{-b \pm \sqrt{b^2 - 4a(c-x)}}{2a}$
a^{e^x}	$\frac{\lambda^2 x - \lambda^2 a}{\lambda b}$ (λ^2 is log of log)
$\log(a + b e^x)$	$\log\left(\frac{e^x - a}{b}\right)$.

(32.) From comparison of various algebraical cases it will appear that the inverses of the equation $y = \psi x$ may be all reduced to any one inverse of the equation $y e^{2\pi m \sqrt{-1}} = \psi x$, the different forms of which, arising from different whole values given to m , will present the various inverses of $y = \psi x$. This is the case whenever the operation of inversion is direct, not depending upon any subsidiary inverses.* But, when subsidiary inversions take place, then each quantity inverted should be previously multiplied by the form $e^{2\pi n \sqrt{-1}}$, where n is a whole number. And for the multiplier we may

write $(1)^m$, if we recollect that $(1)^{\frac{m}{n}}$ has n different values. For instance, we shall invert

$$(x^2 - 1)^3 = y, \text{ or } (x^2 - 1)^3 = y(1)^m.$$

Let us choose† the direct arithmetical form of inversion:

$$x^2 - 1 = \sqrt[3]{y(1)^{\frac{m}{3}}} \quad x^2 = \{1 + \sqrt[3]{y(1)^{\frac{m}{3}}}\} (1)^n$$

$$x = \sqrt{1 + \sqrt[3]{y(1)^{\frac{m}{3}}}} (1)^{\frac{n}{2}}$$

or if $\psi x = (x^2 - 1)^3$ $\psi^{-1} x = \sqrt{1 + \sqrt[3]{x(1)^{\frac{m}{3}}}} (1)^{\frac{n}{2}}$ where m and n are any whole numbers. The cases contained reduce themselves to six, as follows, (p and q being whole numbers, and $1, r, r^2$ being the cube roots of unity.)

* For example, the usual inversion of $x^3 + ax$ by Cardan's formula requires the inversion of $z^6 + bz^3$, and all that can be said is that the method fails to present one form which contains inverses required, and inverses required only. But another form is frequently given, in which this is not the case.

† Let $\sqrt[3]{y}$ signify the arithmetical cube root of y , in opposition to $(y)^{\frac{1}{3}}$ of which $\sqrt[3]{y}$ is one particular case.

	Form of m .	Form of n .	Corresponding Inverse.
1.	$3p$	$2q$	$\sqrt{1 + \sqrt[3]{x}}$
2.	$3p$	$2q + 1$	$-\sqrt{1 + \sqrt[3]{x}}$
3.	$3p + 1$	$2q$	$\sqrt{1 + r\sqrt[3]{x}}$
4.	$3p + 1$	$2q + 1$	$-\sqrt{1 + r\sqrt[3]{x}}$
5.	$3p + 2$	$2q$	$\sqrt{1 + r^2\sqrt[3]{x}}$
6.	$3p + 2$	$2q + 1$	$-\sqrt{1 + r^2\sqrt[3]{x}}$.

Calculus of Functions.

The first of these inverses we shall call the arithmetical inverse, or, where distinction between it and the functional inverses is necessary, the arithmetico-algebraical inverse, being that algebraical inverse in which the processes of Algebra are supposed to take no wider form than those of Arithmetic. All the six satisfy the equation of definition $\psi \psi^{-1} x = x$, from which it would seem that we might argue as follows. In the preceding equation write ψx instead of x , which gives $\psi(\psi^{-1} \psi x) = \psi x$; but when $\psi v = \psi x$, $v = x$, therefore $\psi^{-1} \psi x = x$, or if ψ^{-1} be inverse to ψ , ψ is also inverse to ψ^{-1} . But the clause in Italics contains a *petitio consequentiæ*, as the algebraist will readily perceive; for from $\psi v = \psi x$, can only be inferred that one or more values of v are severally $= x$. And it will be found, upon trial, that while the arithmetical inverse does satisfy $\psi^{-1} \psi x = x$, the others do not. For instance, putting ψx instead of x , in all the six inverses already found, we shall have the following results for $\psi^{-1} \psi x$.

1. $\psi^{-1} \psi x = x$
2. $\psi^{-1} \psi x = -x$
3. $\psi^{-1} \psi x = \sqrt{1 - r + rx^2}$
4. $\psi^{-1} \psi x = -\sqrt{1 - r + rx^2}$
5. $\psi^{-1} \psi x = \sqrt{1 - r^2 + r^2 x^2}$
6. $\psi^{-1} \psi x = -\sqrt{1 - r^2 + r^2 x^2}$.

(33.) Even in the results of $\psi \psi^{-1} x$, we shall find a difference of form, if we agree not to suppress r^2, r^6 , &c. because they are severally equal to 1, &c. Thus the six forms (they are identical in value) are

1. x
2. $\{(-1)^2 - 1 + (-1)^2 \sqrt[3]{x}\}^3$
3. $r^3 x$
4. $\{(-1)^2 - 1 + (-1)^2 r \sqrt[3]{x}\}^3$
5. $r^6 x$
6. $\{(-1)^2 - 1 + (-1)^2 r^2 \sqrt[3]{x}\}^3$;

and these, writing ψx instead of x , give the six forms of ψx , from which the six inverses follow arithmetically; thus, from

$$(x^2 - 1)^3 = r^3 \psi x, \text{ or } r^3 y,$$

we find arithmetically

$$x = \sqrt{1 + r \sqrt[3]{y}};$$

whence value (3.) of $\psi^{-1} x$. From

$$(x^2 - 1)^3 = \{(-1)^2 - 1 + (-1)^2 r \sqrt[3]{y}\}^3,$$

we find

$$x = (-1) \sqrt{1 + r \sqrt[3]{y}};$$

whence value (4.) of $\psi^{-1} x$, and so on. Whence, and from similar processes in the case of other functions, we lay down the following definition, and deduce the following theorem.

Definition.—Two operations are said to be convertible when the value of the result is independent of the order in which they are performed. Thus ϕ and ψ are convertible when $\phi \psi x = \psi \phi x$.

Theorem.—Every inverse is convertible with one or other of the forms of the direct function. This theorem assumes the convertibility of the direct function with the arithmetical inverse which we have seen; and we

Calculus of Functions. have shown how to give the direct function the form by &c., so that the recurrence has already begun with Calculus of Functions. which any inverse may be arithmetically derived from it. $\phi^m x = x$.

Thus $(x^2 - 1)^3 = r^3 \psi x$, giving ψx the form $\frac{(x^2 - 1)^3}{r^3}$;

this form is convertible with the corresponding inverse form $\sqrt{1 + r\sqrt[3]{x}}$.

For example, the inverses $1 + \sqrt{x}$ and $1 - \sqrt{x}$ are not both convertible with $(x - 1)^{\frac{1}{2}}$; but the first is convertible with $(x - 1)^{\frac{1}{2}}$, and the second with the equivalent form $(1 - x)^{\frac{1}{2}}$. Again, $\sin x$ is convertible with $\sin^{-1} x$, but not with $\sin^{-1} x - 2\pi$; the latter being convertible with $\sin(x + 2\pi)$, another form of $\sin x$.

(34.) When we drop the supposition that ψx represents any form of a function, and confine it to one form, we shall let $\psi^{-1}x$ stand for its arithmetical or *convertible* inverse, and (in order to remove the notation of exponents where the analogy does not exist) we shall signify any *inconvertible* inverse by $\psi_{-1}x$. Then, from definition, $\psi^{-1}\psi x = x$, but $\psi_{-1}\psi x$ is not always $= x$.

Those factors, or functional symbols, which are introduced to change the form of a function, and which are symbolical representations of changes of form only, we call *formal* factors or functions, in opposition to *real* factors or functions, which produce a change of value.

*Theorem.** If ψ_{-1} be any invertible inverse of the form ψ , and if $\psi_{-1} \psi x = \phi x$, and if the successive functions of ϕx be taken, namely $\phi^2 x$, $\phi^3 x$, &c., then shall $\psi \phi^n x$ always be a form of ψx : and all the forms of ψx shall be contained among them.

Let us first consider $\psi \phi x$, or $\psi (\psi_{-1} \psi x)$. Let ρ be the formal symbol, which makes $\psi_{-1} x$ the arithmetical inverse to $\rho \psi x =$ the given form. Then, as we have seen, $\psi \psi_{-1} x$ gives x in the form ρx , and $\psi \psi_{-1} (\psi x) = \rho \psi x$ another form of ψx . Again, in $\psi \psi_{-1} x = \rho x$ repeat the operations, which gives $\psi \psi_{-1} \psi \psi_{-1} x = \rho^2 x$, the same or another form of x ; for if $\rho x = x$ in value, $\rho^2 x = \rho x = x$ in value. Write ψx instead of x in the last equation, then $\psi \psi_{-1} \psi \psi_{-1} \psi x = \rho^2 \psi x$, the same or another form of ψx : or $\psi (\psi_{-1} \psi \psi_{-1} \psi x) = \rho^2 \psi x$, that is, $\psi \phi^2 x = \rho^2 \psi x$, and so on.

That all the forms of ψx may be thus found, follows from all the inverses having forms of the function from which they are arithmetically deducible.

Hence, if we represent the various inconvertible inverses of ψ by ψ_{-1} , ψ_{n-1} , &c., and if $\psi_{-1} \psi x = \phi x$, $\psi_{n-1} \psi x = \phi_n x$, &c., the several results

$\psi x, \psi \phi, x, \psi \phi^2 x, \psi \phi^3 x, \&c.$
 $\psi x, \psi \phi, x, \psi \phi' x, \psi \phi'' x, \&c.$
 $\&c. \quad \&c. \quad \&c. \quad \&c.$

are all forms of ψx . If we consider one horizontal series (the first) we see that, if ψx have only a finite number of inverses, there can only be a finite number of corresponding forms, answering to a finite number of values of the succession

$$x, \phi, x, \phi^2 x, \phi^3 x, \&c.;$$

after the first value of these, which is equal to a preceding value, the series recommences, and x will be the first which reappears. For (we are now speaking of values only) let $\phi^{n+m}x = \phi^n x$, or $\phi^m(\phi^n x) = \phi^n x$. Let $\phi^n x = z$; then $\phi^m z = z$, or† $\phi^m(x) = x$, $\phi^{m+1}x = \phi x$,

* We consider the demonstration of this theorem incomplete in several points: it is worth further consideration.

Remember, that we only employ general functional equations,

&c., so that the recurrence has already begun with Calculus of
 $\phi^m x = x$. Functions.

(35.) When $\phi^m x = x$, ϕx is called a *periodic function* of the m th order. The above is an *a priori* proof of the possibility of such functions.

(36.) To show a case of the preceding theorem, let $\psi x = (x^2 - 1)^3$, $\psi_{-1} x = \sqrt{1 + r\sqrt[3]{x}}$, whence

$$\begin{aligned}\psi_{-1}\psi x &= \phi x = \sqrt{1-r+rx^2} \\ \phi^2 x &= \sqrt{1-r+r\{\sqrt{1-r+rx^2}\}^2} \\ &= \sqrt{1-r^2+r^2x^2} \\ \phi^3 x &= \sqrt{1-r^3+r^3x^2} = x, \text{ since } r^3 = 1.\end{aligned}$$

$$\left. \begin{aligned} \psi \phi x &= (-r + r x^2)^3 = r^3 (x^2 - 1)^3 \\ \psi \phi^2 x &= (-r^2 + r^2 x^2)^3 = r^6 (x^2 - 1)^3 \end{aligned} \right\} \begin{array}{l} \text{forms of} \\ (x^2 - 1)^3. \end{array}$$

Again, let $\psi x = \sin x$, $\psi_{-1} x = \sin^{-1} x + 2\pi$

$$\psi_{-1} \psi x = \phi x = x + 2\pi, \quad \phi^n x = x + 2n\pi$$

this function is *not* periodic; but $\sin x$ has an infinite number of inverses; and $\psi \phi^n x = \sin(x + 2n\pi)$, which includes all one set of forms of ψx . Similarly, $\psi_{-1} x = \sin^{-1} x - 2\pi$ gives all the forms included in $\sin(x - 2n\pi)$. But $\psi_{-1} x = \pi - \sin^{-1} x$ gives $\phi x = \pi - x$ a periodic function; $\psi_{-1} x = 3\pi - \sin^{-1} x$ gives $\phi x = 3\pi - x$ another periodic function, &c.

(37.) We shall now proceed to the consideration of the successive functions ϕx , $\phi^2 x$, $\phi^3 x$, &c. ; and, firstly, as to their values or the limit of their values, for a particular value of x .

Theorem. The limit (if there be one) of $\phi^n x$, made by increasing n without limit, is a solution of the equation $\phi x \equiv x$.

For let us suppose that n can be made so great that $\phi^n x$ and $\phi^{n+1} x$ shall not differ from a limit L by so much as l and l_1 , which two latter may be made as small as we please. That is, let

$$\begin{aligned}\phi^n x &= L + \text{less than } l, \\ \phi^{n+1} x &= L + \text{less than } l_1.\end{aligned}$$

But $\phi^{n+1}x = \phi(\phi^n x)$, that is, $L +$ less than $l_1 = \phi(L + \text{less than } l)$; or, since l and l_1 may be made as small as we please, $L = \phi(L)$. If this equation have only one possible root, then the limit is independent of the value of x first chosen; if it have more than one possible root, some values of x may cause $\phi x, \phi^2 x, \&c.$ to approach one limit, and some another.

(38.) It may happen, that the values of $\phi^n x$ may approach in a regulated succession to different limits. Suppose three limits, L , L' , and L'' , to which $\phi^n x$, $\phi^{n+1} x$, and $\phi^{n+2} x$ approach in succession; then, by preceding reasoning, $\phi^{n+3} x = \phi^n x$, or $\phi^3 x = x$ is the equation of which L , L' , and L'' are different roots.

(39.) The preceding theorem is sometimes used explicitly in different parts of Mathematics. As when it is inferred that the value of

$\frac{1}{2 + \frac{1}{2} + \text{&c. ad. inf.}}$ is x where $x = \frac{1}{2 + x}$;

let $\phi x = \frac{1}{2+x}$ then $\phi^2 x = \frac{1}{2+\frac{1}{2+x}}$, &c., the limit of

which is the solution of $\phi x = x$.

independent of the actual values of the subject of the symbols. Thus $\phi z = z$ is the same as $\phi x = x$.

Calculus of Functions.

(40.) But this theorem is used in a less defined form throughout every part of the practice of Algebra, from Division and Involution to Newton's rule for approximating to the roots of equations, and the similar processes of the Differential Calculus; all of which depend upon the following: that if a be an assumed value for x (practically, it must be rather near) in the equation

$$\phi x = 0; \text{ and if } \psi a = a - \frac{\phi a}{\phi' a}, \text{ where } \phi' a \text{ is the value}$$

of the differential coefficient of ϕx for $x = a$, the results of the succession $\psi a, \psi^2 a, \&c.$, will approach a limit* which is a solution of $\phi x = 0$; or, as we should now

say, of $x - \frac{\phi x}{\phi' x} = x$, that is, either of $\phi x = 0$, or of

$\frac{1}{\phi' x} = 0$. The first is the practical theorem, the second illustrates the difficulty sometimes found in applying it.

(41.) Thus, in any commercial transaction, if its funds be acted upon nearly according to the same law every year, and in such a manner that they do not tend to diminish or accumulate without limit, the effect of successive years will render it comparatively immaterial how the funds stood at the commencement, and will tend to reduce the business to a state of permanency, in which the receipts and expenditure are equal.† And various mechanical problems, in which the disappearance of the effects of the primitive state of the system is owing to a succession of resistances, partake of the same analogy.

(42.) We may illustrate the value of $\phi^n x$ geometrically as follows. Take two rectangular axes $O X, O Y$, (fig. 1,) cut by a line $O A$, equally inclined to both. Let $O 12, \&c.$ be the curve whose equation is $y = \phi x$, and take $O M$ any value of x , the ordinate of which is $M 1$, then $M 1 = \phi(O M)$. Take $O N = M 1$, the ordinate of which is $N 2$, whence $N 2 = \phi(M 1) = \phi^2(O M)$, and so on. By constructing different curves, the reader will see that the finite limit (if there be one) of $\phi^n(O M)$ must be that position of the ordinate at which $O A$ cuts the curve, or at which $\phi x = x$.

(43.) Or take any table in which the tabulated quantity is at some part or other equal to its argument; a table of the *mean duration* (commonly called the *expectation*) of human life is convenient for the purpose. For example, in the Northampton Table,‡ opposite to 28 we find 29.30, and opposite to 29 we find 28.79, so that, if ϕx represent the mean duration of a life aged x , the root of $\phi x = x$ lies between 28 and 29. If, going into this Table with any age, we take the nearest whole year of mean duration, and repeat the process, the result will finally be a repetition of 29, in whatever part of the Table we commence. For instance,

* In the Appendix to the second edition of Legendre's *Théorie des Nombres*, the principle is applied to the solution of all classes of algebraical equations.

† Mutual insurance offices would be such when they reached their stationary points, if they did not guarantee a fixed sum at least to their insurers. And if Dr. Price had well considered that the rate of interest depends on the quantity of money, he might have seen reason to suppose that his farthing put out to compound interest at the beginning of the world would never have exceeded a fixed sum. The theory of the unlimited tendency of population to increase might, for any thing its advocate could know to the contrary, have required a similar modification independent of the price of provisions.

‡ *Morgan on Assurances*, London, 1821, p. 236.

Value of	Case 1.	Case 2.	Case 3.
x	9	0	63
ϕx	40	25	12
$\phi^2 x$	23	31	38
$\phi^3 x$	32	28	24
$\phi^4 x$	27	29	31
$\phi^5 x$	30	28	28
$\phi^6 x$	28	29	29
$\phi^7 x$	29	29	29
$\phi^8 x$	29	29	29
&c.	&c.	&c.	&c.

Calculus of Functions.

(44.) It is evident that $\phi^n x$ may be considered as a function of n , and its second function might be taken

with respect to n , which would be denoted by $\phi^{\phi^n x}$, &c. But, to avoid this complication of notation, let us signify $\phi^n x$, treated as a function of n and x , by the capital letter Φ , so that $\phi^n x = \Phi(n, x)$. That this will also be true in the case of fractions, is evident from the following reasoning. Since $(\phi^m)^n x$ will become $\Phi(m n, x)$ generally, if the operations be those of ordinary Algebra, (in which there is no distinction between the results of whole numbers, *generally expressed*, and fractions) it

follows that if $m = \frac{p}{n}$, and if $\phi^{\frac{p}{n}}$, (meaning as yet un-

known, $\phi^{\frac{p}{n}} x$ being merely considered as a representation

of $\Phi\left(\frac{p}{n}, x\right)$) be called ψ , then $\psi^n x$ will be $\Phi(p, x)$, or

$(\phi^{\frac{p}{n}})^n x = \Phi(p, x) = \phi^p x$; so that $\Phi\left(\frac{p}{n}, x\right)$ is one of

the n th functional inverses of ϕx , or may be called $\phi^{\frac{p}{n}} x$ in the sense explained in (16). We shall proceed to find some general forms of $\phi^n x$.

(45.) Since we shall afterwards see that expressions of the same form, algebraically speaking, (such as $a + b x$ and $c + e x$), may present important functional differences, we shall not call $a + b x$ and $c + e x$ particular cases of the same general form, but *cognate* forms. And all the functions hitherto imagined or treated of will fall under one or other of the following cases, when considered with respect to the properties of their successive functions.

1. Cases in which the successive functions all assume *different* forms, algebraically speaking, *ad infinitum*. Such is $1 + x^2$, the second function of which is $2 + 2x^2 + x^4$, and so on. Such also is $\log x$, the second function of which is $\log \log x$, &c.

2. Cases in which the successive functions of x assume only a finite number of recurring *different* forms. In this case we have seen that $\phi^n x$ will for some particular value of n become equal to x , $\phi^{n+1} x$ to ϕx , &c. See (7.) and (36.)

3. Cases in which all the successive functions are cognate forms, either with or without recurrence. The only cases of this kind known in common Algebra are

$$a + b x, \frac{a + b x}{a' + b' x}, \text{ and } a x^b.$$

(46.) A general method has been given by Laplace,* which reduces the determination of successive functions, and indeed the solution of functional equations of all kinds, to the solution of equations of differences.

* *Mém. des Savans Etrangers*, 1773.

Calculus of Applied to the determination of successive functions, it Functions. is as follows:

Let $x, \phi x, \phi^2 x, \&c.$, be considered as the successive terms of $u_n = \Phi(n, x) = \phi^n x$. Then, $u_{n+1} = \phi^{n+1} x = \phi \phi^n x = \phi u_n$, or u_{n+1} is a given function of u_n . But, unfortunately, we know so little of general methods for integrating $u_{n+1} = \phi u_n$, that this is of little use in finding general solutions.*

But this method has the following use. It points out that we are not to consider the successive functions of ϕx as absolutely irreducible to the same form, because they present differences which are in common Algebra considered as essential. For instance,† let

$$\phi x = \frac{2x}{1-x^2},$$

whence

$$u_{n+1} = \frac{2u_n}{1-u_n^2},$$

which equation can be integrated as follows: assume $u_n = \tan v_n$, then $u_{n+1} = \tan v_{n+1} = \tan 2v_n$, from the equation; consequently, $v_{n+1} = 2v_n$, or $v_n = c \cdot 2^n$, and $u_n = \tan(c \cdot 2^n)$, which may, by the exponential formula, be reduced to the following, as to form:

$$u_n = \phi^n x = \sqrt{-1} \frac{1-c^2}{1+c^{2^n}},$$

where c must be determined from the condition that $u_0 = x$, which gives

$$x = \sqrt{-1} \frac{1-c}{1+c}, \text{ or } c = \frac{\sqrt{-1}-x}{\sqrt{-1}+x};$$

so that, if $\phi x = \frac{2x}{1-x^2}$,

$$\phi^n x = \sqrt{-1} \frac{(\sqrt{-1}+x)^{2^n} - (\sqrt{-1}-x)^{2^n}}{(\sqrt{-1}+x)^{2^n} + (\sqrt{-1}-x)^{2^n}};$$

in the reduction of which for any particular value of n , all impossibility will vanish, as is sufficiently obvious to the practised algebraist.

Another example given in the same Memoir is the following: If $\phi x = 2x^2 - 1$, or $u_{n+1} = 2u_n^2 - 1$, the integral of which is (assuming $u_n = \cos v_n$, and proceeding as before)

$$u_n = \frac{1}{2} \{c^{2^n} + c^{-2^n}\}$$

and

$$\phi^n x = \frac{1}{2} \left\{ (x + \sqrt{x^2-1})^{2^n} + (x - \sqrt{x^2-1})^{2^n} \right\}$$

(47.) The above will be found to succeed when the values of n are fractional or negative. If, for instance, we make $n = -1$, we find in the second case

$$\phi^{-1} x = \frac{1}{2} \left\{ (x + \sqrt{x^2-1})^{\frac{1}{2}} + (x - \sqrt{x^2-1})^{\frac{1}{2}} \right\}$$

which is an equivalent form to $\left(\frac{1+x}{2}\right)^{\frac{1}{2}}$, the real inverse of $2x^2 - 1$. But it may be suspected that this is

* Whether indeed this method is to be called reduction of functional to differential, or of differential to functional, equations, must depend upon which class has had most substantial progress made in its solution.

† *Memoirs of the Analytical Society*, p. 50. The Paper in which this is found is attributed, we believe correctly, to Sir John Herschel, whose treatise on the Calculus of Differences is one of the richest analytical Works of its size extant.

not the most general form of $\phi^n x$, when n is fractional, for the following reasons.

(48.) The arbitrary constant introduced in the integration of an equation need not be considered as never changing; it is sufficient that it does not change under any changes which the verification of the equation may render necessary. In the Differential Calculus, in which every possible change of value of the independent variable is contemplated, the constant must under no circumstances change with a change in the variable, and therefore cannot be a function of it. But in the Calculus of Differences, in which, generally speaking, the variable is considered as changing only from one whole number to another, the arbitrary constant is any function of the variable which is the same for all whole numbers, the most general form of which is $\psi(\cos 2n\pi)$. Thus the solution of $u_{n+1} = u_n + 1$ is not $u_n = n + C$, but $u_n = n + \psi(\cos 2n\pi)$; where, however, ψ must not contain the inversion of the cosine. Again, in a functional equation, the arbitrary constant may be any function of x , which does not change upon substituting for x all the known operations which are performed upon it in the equation. Thus, one solution of $\phi(x^2) = x\phi x$ is $\phi x = x$; but if a function ξx can be found which shall not change when x^2 is substituted for x , that is, if the equation $\xi x^2 = \xi x$ admit of solution, then $x\theta\xi x$ is a solution of $\phi(x^2) = x\phi x$, where θ is any function we please, which does not invert ξ ; for

$$\xi x^2 = \xi x, \theta\xi x^2 = \theta\xi x, \text{ and } x^2\theta\xi x^2 = x \times x\theta\xi x.$$

Taking the second instance in (47.), in which $\phi x = 2x^2 - 1$ and $u_{n+1} = 2u_n^2 - 1$, it appears, then, that if we can so find ξx as that $\xi(2x^2 - 1) = \xi x$, the arbitrary function $\theta\xi x$ may be substituted for the constant c ; for the change of n into $n+1$ changes x into ϕx , or $2x^2 - 1$, and ξx remains unaltered. Hence the integral is (see Remark at the end of this Treatise)

$$u_n = \frac{1}{2} \{(\theta\xi x)^{2^n} + (\theta\xi x)^{-2^n}\}$$

Now there is no objection, consistently with the conditions, to supposing $\theta\xi x$ to contain n among the constants which are introduced, provided the functions of n thus considered do not change in changing n from one whole number to another. For instance, $\theta\xi x$ might be $\xi x \cdot \cos 2\pi n$. And though, in this case, all the whole values of n would not give $\phi^n x$ different in form from the value in (47.), the fractional values of n would.

When we talk of the most general form of a function, we must be considered as speaking conjecturally, for we have no test by which to try the generality of any solution whatsoever. Though n is not an assignable function of x , yet there does exist between them a species of functional relation; namely, that a change of n into $n+1$ is equivalent to a change of x into $2x^2 - 1$. Either, therefore, more general forms of the functional inverses may be given than are explicitly contained in the expression for $\phi^n x$, or there is some connection between ξx and $\cos 2\pi n$, which must oblige us to consider one as a function of the other, or at least as having this sort of relation, that $\phi(\xi x, \cos 2\pi n)$ is not more general than $\phi(\xi x)$.

(49.) Much of what has been done in the Calculus of Functions depends upon a class* of forms to which we

* Mr. Babbage was the first who made this form known; but he requests us to state that the method of expression, $\phi \alpha \phi^{-1} x$, was suggested to him by W. H. Maule, Esq., of Trinity College, Cambridge, and of Lincolns Inn.

Calculus of now proceed. By observing a few processes of Algebra Functions. functionally, we shall see that we are continually using such forms; though, where value or developement is the prominent idea, we lose sight of mere form.

In the formula $\sqrt{x^2 + c}$, we see the direct and inverse operations $\sqrt{\quad}$ and $(\quad)^2$; but the inverse not performed upon the direct function, but upon a function of it.

The steps are $x, x^2, x^2 + c, \sqrt{x^2 + c}$. Similarly, in expanding $\sin mx$ in terms of $\sin x$, we are seeking for $\sin m \sin^{-1} p$ in terms of p ; and the steps are $p, \sin^{-1} p, m \sin^{-1} p, \sin m \sin^{-1} p$. Again, x^n , when represented by $e^{n \log x}$, is another instance.

And the algebraist cannot but have remarked, that triple operations of this sort give expressions of more utility as well as simplicity of relations than any others in which the first and third are not inverse. Thus $\sqrt{x^2 + c}$ is of a much higher degree of importance than $\sqrt{x^2 + c}$ in all the practice of Algebra.

The general form of such functions is either $\phi \alpha \phi^{-1} x$, $\phi \alpha \phi^{-1} x$, $\phi^{-1} \alpha \phi x$, or $\phi^{-1} \alpha \phi x$; of which we shall reject the second and fourth, and not consider them at present. Let us call $\phi \alpha \phi^{-1} x$ a derivative* of αx , (a derivative of αx by ϕ , if it be necessary to distinguish ϕ .) We shall then have the following theorems.

(A.) Derivatives of derivatives of αx are derivatives of αx .

Let $\phi \alpha \phi^{-1} x$ be a derivative of αx ; and $\psi (\phi \alpha \phi^{-1} x)$ $\psi^{-1} x$, or $\psi \phi \alpha \phi^{-1} \psi^{-1} x$ a derivative of $\phi \alpha \phi^{-1} x$. Then, if $\psi \phi x = \chi x$, we have $\psi x = \chi \phi^{-1} x$, $x = \chi \phi^{-1} \psi^{-1} x$, or $\chi^{-1} x = \phi^{-1} \psi^{-1} x$, whence $\psi \phi \alpha \phi^{-1} \psi^{-1} x$ is $\chi \alpha \chi^{-1} x$,

another derivative of αx . Thus $\log (e^x)$, or $n x$, is one derivative of x^n , therefore $\sin (n \sin^{-1} x)$ is another.

(B.) If μ and ν be corresponding derivative forms of α and β , then $\mu \nu$ is a derivative of $\alpha \beta$. For let $\mu x = \phi \alpha \phi^{-1} x$, $\nu x = \phi \beta \phi^{-1} x$. Then $\mu \nu x = \phi \alpha \phi^{-1} \phi \beta \phi^{-1} x$. But $\phi^{-1} \phi x = x$; therefore, $\phi^{-1} \phi \beta \phi^{-1} x = \beta \phi^{-1} x$, or $\mu \nu x = \phi \alpha \beta \phi^{-1} x$, whence $\mu \nu$ is a derivative of $\alpha \beta$.

(C.) If two functions α and β be inverse to one another, or convertible each with the other, then their corresponding derivatives are inverse or convertible. Thus, if $\alpha \beta x = x$, $\phi \alpha \phi^{-1} \phi \beta \phi^{-1} x = \phi \alpha \beta \phi^{-1} x = \phi \phi^{-1} x = x$. Or, if $\alpha \beta x = \beta \alpha x$, $\phi \alpha \phi^{-1} \phi \beta \phi^{-1} x = \phi \alpha \beta \phi^{-1} x = \phi \beta \alpha \phi^{-1} x = \phi \beta \phi^{-1} \phi \alpha \phi^{-1} x$.

(D.)† The successive functions of the derivatives of αx are the corresponding derivatives of the successive functions of αx . Thus $(\phi \alpha \phi^{-1})^2 x = \phi \alpha \phi^{-1} \phi \alpha \phi^{-1} x = \phi \alpha^2 \phi^{-1} x$. Again, if $(\phi \alpha \phi^{-1})^n x = \phi \alpha^n \phi^{-1} x$, then $(\phi \alpha \phi^{-1})^{n+1} x = \phi \alpha \phi^{-1} (\phi \alpha \phi^{-1})^n x = \phi \alpha \phi^{-1} \phi \alpha^n \phi^{-1} x = \phi \alpha^{n+1} \phi^{-1} x$; so that this theorem, being true for the second function, is true for all which succeed.

(E.) The derivatives of a periodic function are periodic functions of the same order. Thus, if $\alpha^n x = x$, $(\phi \alpha \phi^{-1})^n x = \phi \alpha^n \phi^{-1} x = \phi \phi^{-1} x = x$.

(F.) If two functions be algebraically cognate in form, so will be all their corresponding derivatives. For if we take $2 + 3x$ and $3 + 4x$, their derivatives $\phi (2 + 3 \phi^{-1} x)$ and $\phi (3 + 4 \phi^{-1} x)$ will be particular cases of $\phi (m + n \phi^{-1} x)$.

* We do not like to suggest any other than a generic name. Any person may specify to himself what derivative he will call it.

† This and the next theorem are due to Mr. Babbage. See *Philosophical Transactions*, 1815.

(50.)* If functions αx and βx be given, neither of which are periodic, required to find whether β be a derivative of α , and, if so, what derivative. Let $\beta x = \phi \alpha \phi^{-1} x$: for x write ϕx , and the preceding becomes $\beta \phi x = \phi \alpha x$, which equation is to be solved. Assume

$$\alpha^n x = A(n, x) \quad \beta^n y = B(n, y),$$

and from these two equations eliminate n ; giving

$$F(x, y, \alpha^n x, \beta^n y) = 0.$$

Between y and x suppose that relation to exist which would arise from putting arbitrary constants C and C' in place of $\alpha^n x$ and $\beta^n y$; that is, define $y = \phi x$ by the equation

$$F(x, y, C, C') = 0.$$

Then, since

$$\alpha^{n+1} x = A(n, \alpha x) \quad \beta^{n+1} y = B(n, \beta y),$$

elimination of n gives

$$F(\alpha x, \beta y, \alpha^{n+1} x, \beta^{n+1} y) = 0;$$

or $\beta y = \phi \alpha x$ satisfies the equation

$$F(\alpha x, \beta y, C, C') = 0;$$

consequently, the system of equations,

$$y = \phi x, \quad \beta y = \phi \alpha x,$$

are both true when $F(x, y, C, C') = 0$ is true: but these two, by elimination of y , give $\beta \phi x = \phi \alpha x$, or $\beta x = \phi \alpha \phi^{-1} x$. Therefore, ϕx must satisfy the condition $F(x, \phi x, C, C') = 0$.

(51.) As an example of the last theorem, let us ask what derivative is $a' + b'x$ of $a + bx$.

Here (22.) let us put $a + bx$ and $a' + b'x$ in the form $p(x + q) - q$, which gives

$$\beta y = b' \left(y + \frac{a'}{b' - 1} \right) - \frac{a'}{b' - 1}$$

$$\alpha x = b \left(x + \frac{a}{b - 1} \right) - \frac{a}{b - 1}$$

whence (22.)

$$\beta^n y = b'^n \left(y + \frac{a'}{b' - 1} \right) - \frac{a'}{b' - 1}, \text{ call this } C' - \frac{a'}{b' - 1}$$

$$\alpha^n x = b^n \left(x + \frac{a}{b - 1} \right) - \frac{a}{b - 1} \dots \dots \dots C - \frac{a}{b - 1}$$

eliminate n , which gives (writing C' and C for $\lambda C'$ and λC)

$$\lambda \cdot \left(y + \frac{a'}{b' - 1} \right)^{\lambda b'} - \lambda \cdot \left(x + \frac{a}{b - 1} \right)^{\lambda b} = C' \lambda b - C \lambda b';$$

or, writing C for $\log^{-1} (C' \lambda b - C \lambda b')$, which is merely an arbitrary constant, (we shall frequently make a similar change,)

$$\frac{\left(y + \frac{a'}{b' - 1} \right)^{\lambda b'}}{\left(x + \frac{a}{b - 1} \right)^{\lambda b}} = C$$

$$\phi x = C^{\frac{1}{\lambda b}} \left(x + \frac{a}{b - 1} \right)^{\frac{\lambda b'}{\lambda b}} - \frac{a'}{b' - 1}$$

$$\phi^{-1} x = C^{-\frac{1}{\lambda b'}} \left(x + \frac{a'}{b' - 1} \right)^{\frac{\lambda b}{\lambda b'}} - \frac{a}{b - 1}$$

* This theorem is due to M. Collins, who refers to it in the *Petersburg Transactions*, 6th series, vol. i. p. 350. We have not seen his demonstration.

Calculus of
Functions.

$$\begin{aligned}\alpha\phi^{-1}x &= bC^{-\frac{1}{\lambda b'}}\left(x + \frac{a'}{b'-1}\right)^{\frac{\lambda b'}{\lambda b}} - \frac{a}{b-1} \\ \phi\alpha\phi^{-1}x &= C^{\frac{1}{\lambda b}}\left(\alpha\phi^{-1}x + \frac{a}{b-1}\right)^{\frac{\lambda b'}{\lambda b}} - \frac{a'}{b'-1} \\ &= C^{\frac{1}{\lambda b}}b^{\frac{\lambda b'}{\lambda b}}C^{-\frac{1}{\lambda b}}\left(x + \frac{a'}{b'-1}\right)^{\frac{\lambda b'}{\lambda b}} - \frac{a'}{b'-1} \\ \left\{b^{\frac{\lambda b'}{\lambda b}} = b'\right\} &= b'\left(x + \frac{a'}{b'-1}\right) - \frac{a'}{b'-1} = \beta x.\end{aligned}$$

(52.) We can now illustrate what has been said in (27.) about functional distinctions. If, in the preceding case, $b' = 1$, then the form of ϕx , as derived from the preceding, is

$$C^{\frac{1}{\lambda b}}\left(x + \frac{a}{b-1}\right)^{-\frac{1}{\lambda b}} - \alpha,$$

which is useless. If a value could be derived, or a limit of ϕx , when $b' - 1$ diminishes without limit, the function $a' + x$ might be considered as a case of $a' + b'x$ upon principles adopted in Algebra.* But the pre-

ceding expression is simply $-C^{\frac{1}{\lambda b}} - \alpha$, which can in no sense be called a representation of the function to which we shall come, so long as C is a constant. But we have treated C as a constant of the Differential Calculus; whereas all that is necessary is, that it should be a function of x , which does not change when either αx , βx , ϕx , or $\phi^{-1}x$ is substituted for x . If we could give the general form, the preceding expression might become of the form $\alpha - \alpha$ when $b' = 1$; in which case its value might be determined by differentiation. For if P and Q be functions of b' , which become infinite when $b' - 1 = 0$, we have

$$P - Q = \lambda \frac{e^P}{e^Q} = \lambda \frac{e^P P'}{e^Q Q'},$$

where P' and Q' are the values $\frac{dP}{db'}$ and $\frac{dQ}{db'}$ when $b' = 1$; and the value of $P - Q$ may be finite if $P' = Q'$. But we proceed to find how $a' + x$ may be derived from $a + bx$.

(53.) Let

$$\beta y = a' + y \quad \alpha x = b\left(x + \frac{a}{b-1}\right) - \frac{a}{b-1}$$

$$\beta^n y = na' + y \quad \alpha^n x = b^n\left(x + \frac{a}{b-1}\right) - \frac{a}{b-1}$$

Let

$$\begin{aligned}na' + y &= C \\ b^n\left(x + \frac{a}{b-1}\right) - \frac{a}{b-1} &= C' - \frac{a}{b-1}\end{aligned}$$

from the last

$$n = \frac{\lambda C' - \lambda\left(x + \frac{a}{b-1}\right)}{\lambda b}$$

* By Algebra, we mean also the theory of limits: we consider that common Arithmetic introduces the theory of limits in such expressions as $\sqrt{2}$; nor can we find a meaning for the equation $\sqrt{8} = 2\sqrt{2}$, without either the theory of limits or Geometry. We assume, therefore, the former as a part of Algebra.

or

$$\phi x \text{ or } y = C - a' \frac{\lambda C' - \lambda\left(x + \frac{a}{b-1}\right)}{\lambda b}$$

and has the form

$$\begin{aligned}C + \frac{a}{\lambda b} \cdot \lambda\left(x + \frac{a}{b-1}\right) \\ \phi^{-1}x &= \varepsilon^{\frac{(x-C)\lambda b}{a}} - \frac{a}{b-1} \\ \alpha\phi^{-1}x &= b \cdot \varepsilon^{\frac{(x-C)\lambda b}{a}} - \frac{a}{b-1} \\ \phi\alpha\phi^{-1}x &= C + \frac{a'}{\lambda b}\left(\lambda b + (x - C)\frac{\lambda b}{a'}\right) \\ &= a' + x = \beta x.\end{aligned}$$

Calculus of
Functions.

(54.) We return to the subject of periodic functions. By (49.) it appears, that if αx be a periodic function of the n th order, then so is $\phi\alpha\phi^{-1}x$. So many different periodic functions, therefore, as we are able to find, *apparently*, so many different classes of periodic functions can we ascertain, each class being infinite in the number of its individual cases. For instance, $(1)^{\frac{1}{n}}x$ is a periodic

function of the n th order; so therefore is $\phi(1)^{\frac{1}{n}}\phi^{-1}x$, giving the following theorem, a particular case of which we have already arrived at in a different manner.

If y be any non-arithmetical root of $y^n - \phi^{-1}x = 0$, ϕy is a periodic function of x . Thus $\phi(-\phi^{-1}x)$

is a periodic function of the second order. Again, $\frac{1}{x}$ and

$1 - x$ are periodic functions of the second order; so also must be $\phi\left(\frac{1}{\phi^{-1}x}\right)$ and $\phi(1 - \phi^{-1}x)$; for instance,

$$\begin{aligned}\phi x = 2 + 3x \quad \phi\left(\frac{1}{\phi^{-1}x}\right) &= 2 + \frac{9}{x-2} = \frac{2x+5}{x-2} \\ \phi x = x^3 \quad \phi(1 - \phi^{-1}x) &= (1 - \sqrt[3]{x})^3\end{aligned}$$

We have seen (36.) that if r be one of the n th roots of unity, then $\sqrt[n]{1 - r + rx^n}$ is a periodic function of the n th order. For we shall find

$$\alpha^n x = \sqrt[n]{1 - r^n + r^n x^n},$$

which is $= x$ when $w = n$, since r^n then $= 1$. Consequently,

$$\phi\left(\sqrt[n]{1 - r + r(\phi^{-1}x)^n}\right)$$

is a periodic function of the n th order.

If we take $\alpha x = \sqrt[n]{a + bx^n}$, we shall find

$$\alpha^n x = \sqrt[n]{a \cdot \frac{b^n - 1}{b - 1} + b^n x^n},$$

which is periodic of the n th order, if $b^n = 1$, independently of a .

The preceding cases arise from bx being periodic if b be a root of unity.

(55.) We shall now proceed to the consideration of the third class in (45.), namely, those functions of which the successive functions are algebraical cognates, those which immediately strike us, and from which indeed all others must be derived, being

$$a + bx, \quad \frac{a + bx}{a' + b'x}, \quad ax^b.$$

Calculus of Functions. (56.) It will be convenient to investigate the conditions of convertibility of these several classes. We first examine $a + bx$. In order that $a + bx$ and $p + qx$ may be convertible, we must have

$$a + b(p + qx) = p + q(a + bx),$$

$$\text{or } a + bp = p + aq, \text{ that is, } \frac{a}{b-1} = \frac{p}{q-1}.$$

Let $\frac{a}{b-1} = m$, then $a + bx = b(x + m) - m$, a form

we have already used, all the cognates in which b only is changed being convertible with it. It will always be most convenient to write functions in the form in which they are convertible with their cognates; thus $2x + 3$ should always be written $2(x + 3) - 3$, and $1 - 2x$ should be written $-2\left(x - \frac{1}{2}\right) + \frac{1}{2}$.

(57.) The conditions of convertibility of $\frac{a + bx}{a' + b'x}$ are derived from the equation

$$\frac{a + b \frac{p + qx}{p' + q'x}}{a' + b' \frac{p + qx}{p' + q'x}} = \frac{p + q \frac{a + bx}{a' + b'x}}{p' + q' \frac{a + bx}{a' + b'x}}$$

or

$$ap' + bp = a'p + aq$$

$$aq' = b'p$$

$$a'q' + b'q = b'p' + bq',$$

the third of which is true if the first and second be true; giving, as the conditions of convertibility,

$$\frac{a' - b}{a} = \frac{p' - q}{p} \quad \frac{b'}{a} = \frac{q'}{p}.$$

Let

$$\frac{a' - b}{a} = m \quad \frac{q'}{a} = k;$$

whence

$$\frac{a + bx}{b + am + akx}$$

is convertible with every cognate in which a and b only are changed.

Suppose $a = 1$ (to which it may always be reduced) and we have the simple expression

$$\frac{1 + bx}{b + m + kx}.$$

In which form we shall generally use this expression.

(58.) Similarly, the condition of convertibility of two functions of the form ax^b is that $a^{\frac{1}{b-1}}$ shall be the same in both; and the most convenient form of the function is

$$a^{\frac{1}{b-1}} = m \quad ax^b = \frac{mx^b}{m}$$

(59.) Theorem. When $\phi^n x$ is a cognate of ϕx , and ϕx is put in the form in which some of the constants do not enter into the criterion of convertibility, then the performance of the successive operations alters only the independent constants just mentioned.

This follows from the convertibility of the successive functions; for $\phi^n x$ being convertible with ϕx , or $\phi^n \phi x$ being the same as $\phi \phi^n x$, it follows that every condition of convertibility must exist between $\phi^n x$ and ϕx . Now

VOL. II.

Calculus of Functions. ϕx being ax^b or $\frac{mx^b}{m}$, and $\phi^n x$ being of the form $a'x^{b'}$, or $\frac{m'x^{b'}}{m'}$ and $m = \text{const.}$, being the condition of convertibility, it follows that $m' = m$.

(60.) The successive functions of $a + bx$ and ax^b or $b(x + m) - m$ and $\frac{mx^b}{m}$ are contained in the equations

$$\phi^n x = b^n(x + m) - m$$

$$\phi^n x = \frac{(mx)^b}{m}.$$

(61.) In order to find the successive functions of $\frac{a + bx}{a' + b'x}$ or $\frac{1 + bx}{b + m + kx}$, assume

$$\phi^n x = \frac{1 + b_n x}{b_n + m + kx}$$

Then

$$\begin{aligned} \phi^{n+1} x &= \frac{b_n + b + m + (b_n b + k)x}{b_n b + k + (b_n + b + m)m + (b_n + b + m)kx} \\ &= \frac{1 + \frac{b_n b + k}{b_n + b + m}x}{\frac{b_n b + k}{b_n + b + m} + m + kx} \end{aligned}$$

$$\text{whence } b_{n+1} = \frac{b_n b + k}{b_n + b + m}$$

From this equation the successive functions may be readily formed; but a general integral may be given of the equation between b_n and b_{n+1} .

Assume $b_n = pB_n + q$; and substitute the values of b_n and b_{n+1} in

$$b_{n+1}(b_n + b + m) = b_n b + k,$$

which will give the following equation:

$$B_n B_{n+1} + \frac{q + m + b}{p} B_{n+1} + \frac{q - b}{p} B_n + \frac{q^2 + m q - k}{p^2} = 0$$

$$\text{Assume } \frac{q + m + b}{p} + \frac{q - b}{p} = 0$$

$$\text{and } \frac{q^2 + m q - k}{p^2} = 1$$

whence

$$q = -\frac{m}{2} \quad p^2 = -\left(\frac{m^2}{4} + k\right)$$

and the equation may be reduced to

$$\frac{B_{n+1} - B_n}{1 + B_n B_{n+1}} = -\frac{2p}{2b + m}$$

or

$$\tan^{-1} B_{n+1} - \tan^{-1} B_n = \tan^{-1} \left(-\frac{2p}{2b + m} \right)$$

or

$$\Delta \cdot \tan^{-1} B_n = \tan^{-1} \left(-\frac{2p}{2b + m} \right)$$

Therefore

$$\tan^{-1} B_n = n \tan \left(-\frac{2p}{2b + m} \right) + C$$

and

$$b_n = p \tan \left(n \tan^{-1} \left\{ -\frac{2p}{2b + m} \right\} + C \right) - \frac{m}{2}.$$

2 T

Calculus of
Functions.

The value of C may be determined from $b_1 = b$, which gives $C = \frac{\pi}{2}$, or, all reductions made,

$$b_n = p \cot \left(n \tan^{-1} \frac{2p}{2b+m} \right) - \frac{m}{2}.$$

There are three distinct forms to consider, namely, where p^2 is positive, nothing, and negative. This depends upon whether $m^2 + 4k$ is $-$, 0 , or $+$.

$$\begin{array}{ll} m^2 > -4k & p^2 \text{ is } - \\ m^2 = -4k & p^2 \text{ is } 0 \\ m^2 < -4k & p^2 \text{ is } + \end{array}$$

In the last of the three cases, the expression above given for b_n is the rational form, and can always be reduced to a simple algebraical form; in the first it becomes of an impossible form, trigonometrically speaking, and must be reduced to an exponential form.

By means of the expressions

$$\sqrt{-1} \tan(x\sqrt{-1}) = \frac{1 - e^{2x}}{1 + e^{2x}}$$

$$2\sqrt{-1} \tan^{-1}(x\sqrt{-1}) = \lambda \left(\frac{1-x}{1+x} \right)$$

we find the following transformation for b_n .

Let
$$p^2 = -p^2 = \frac{m^2}{4} + k$$

$$b_n = p' \frac{(2b+m+2p')^n + (2b+m-2p')^n}{(2b+m+2p')^n - (2b+m-2p')^n} - \frac{m}{2}$$

when p^2 or $\frac{m^2}{4} + k = 0$, the term

$$p \cot n \tan^{-1} \frac{2p}{2b+m} \text{ or } \frac{p}{\tan n \tan^{-1} \frac{2p}{2b+m}}$$

becomes

$$\frac{2b+m}{2n},$$

or

$$b_n = \frac{2b - (n-1)m}{2n};$$

or if

$$\phi x = \frac{1+bx}{b+m-\frac{m^2}{4}x}$$

$$\phi^n x = \frac{1 + \frac{2b - (n-1)m}{2n}x}{\frac{2b - (n-1)m}{2n} + m - \frac{m^2}{4}x}$$

(62.) The function $\frac{1+b_n x}{b_n+m+kx}$ is equal to x when

b_n is infinite: this cannot happen in the last-mentioned case except when $n=0$ or $\phi^0 x = x$, which fulfils the condition universally. But in the other two cases, the conditions of a periodic function of the n th order are

$$n \tan^{-1} \frac{2p}{2b+m} = w\pi,$$

where w is a whole number, positive or negative. This gives

$$\frac{2p}{2b+m} = \tan \frac{w\pi}{n} = \sqrt{\frac{1 - \cos \frac{2w\pi}{n}}{1 + \cos \frac{2w\pi}{n}}}$$

from which, by substitution for p , and reduction, is found*

$$k = - \frac{b^2 - 2b(b+m) \cos \frac{2w\pi}{n} + (b+m)^2}{2 \left(1 + \cos \frac{2w\pi}{n} \right)}$$

The relation analogous to this, derived from

$$(2b+m+2p')^n - (2b+m-2p')^n = 0,$$

presents k in an impossible form. It may also be obtained from the above.

(63.) If when $\phi x = \frac{1+bx}{b+m+kx}$, we wish to find $\phi^n x$

more directly, we must solve the equation of differences

$$u_{n+1} = \frac{1+bu_n}{b+m+ku_n} \dots \dots (1)$$

which has been virtually done in integrating

$$u_{n+1} = \frac{k+bu_n}{b+m+u_n} \dots \dots (2)$$

for, write the first in the form

$$u_{n+1} = \frac{\frac{1}{k} + \frac{b}{k}u_n}{\frac{b}{k} + \frac{m}{k} + u_n} \dots \dots (3)$$

the solution of (2), with such a value of C as will make $u_0 = x$, is

$$p \tan \left(\tan^{-1} \frac{2x+m}{2p} - n \tan^{-1} \frac{2p}{2b+m} \right) - \frac{m}{2}.$$

To make this the solution of (3), write $\frac{1}{k}$ for k , $\frac{b}{k}$ for b , and $\frac{m}{k}$ for m , and therefore $\frac{p}{k}$ for p , which will give the following;

$$\phi^n x = \frac{p}{k} \tan \left(\tan^{-1} \frac{2kx+m}{2p} - n \tan^{-1} \frac{2p}{2b+m} \right) - \frac{m}{2}$$

which may be easily reduced to the form

$$\frac{1+b_n x}{b_n+m+kx}.$$

(64.) But the most direct algebraical method is as follows. Let the n th function of

$$\frac{1+bx}{b+m+kx} \text{ be } \frac{A_n+B_n x}{A'_n+B'_n x},$$

substitute the first for x in the second, and after reduction equate the result identically with

$$\frac{A_{n+1}+B_{n+1}x}{A'_{n+1}+B'_{n+1}x},$$

which gives two sets of equations of the forms

$$\begin{array}{l} A_n(b+m) + B_n = A_{n+1} \\ A_n k + B_n b = B_{n+1} \end{array}$$

which give

$$\begin{array}{l} A_{n+1} - (2b+m)A_n + (b^2+bm-k)A_{n-1} = 0 \\ B_n = bA_n - (b^2+bm-k)A_{n-1}. \end{array}$$

* This is the formula cited by Mr. Babbage (*Elem. Tr. Calc. Functions*, p. 5) from a paper by Mr. Horner, in the *Annals of Philosophy*, November, 1817, which we have not seen. We presume the third method we have given is that of Mr. Horner.

Calculus of
Functions.

Calculus of Functions. The solution by the usual method is obvious, and the result is as follows. Let r and r' be the roots of

$$R^2 - (2b + m)R + (b^2 + bm - k) = 0,$$

if those roots be possible. Then

$$\phi^n x = \frac{r^n - r'^n + \{b(r^n - r'^n) - r r' (r^{n-1} - r'^{n-1})\} x}{(b+m)(r^n - r'^n) - r r' (r^{n-1} - r'^{n-1}) + k(r^n - r'^n) x}.$$

If the roots be impossible, assume

$$(b+m)^2 - 2(b+m)b \cos 2\theta + b^2 + 2k(1 + \cos 2\theta) = 0$$

$$h^2 = b^2 + mb - k;$$

then

$$\phi^n x = \frac{\sin n\theta + \{b \sin n\theta - h \sin (n-1)\theta\} x}{(b+m) \sin n\theta - h \sin (n-1)\theta + k \sin n\theta \cdot x}$$

from which the condition of periodicity may be derived as before, and it is only in the latter case that a possible cognate of ϕx can be periodic.

(65.) All the successive functions which have here been treated of may be made to give algebraical and functional inverses, or functions with negative and fractional indices; but, as before observed, we have no reason to imagine that we have thus obtained the most general form of $\phi^n x$ in those cases. Thus, if

$$\phi x = b(x+m) - m \quad \phi^n x = b^n(x+m) - m,$$

we find that the operation $b^{-n}(x+m) - m$ performed

upon $b^n(x+m) + m$ will give x , and that $b^p(x+m) - m$ performed on x , p times, will give $b^n(x+m) - m$.

(66.) The periodic functions are the only cases which at present elude this method, but the difficulty may sometimes be conquered as follows.*

Let $r_1, r_2, r_3, \dots, r_n$ be the n th roots of 1; then

$$r_1^w + r_2^w + r_3^w + \dots + r_n^w$$

= 0 for every value of w , except $w =$ multiple of n , when it is $= n$. If we call the preceding $n R_w$, then only one of the following series

$$R_w, R_{w-1}, R_{w-2}, \dots, R_{w-n+1}$$

is equal to 1, and all the rest to 0. Consequently, if ϕx be a periodic function of the n th order, then

$$\phi^m x = x R_w + \phi x R_{w-1} + \dots + \phi^{n-1} x R_{w-n+1},$$

where $w = (\text{mult. of } n) + m$.

For example, let us try a periodic function of the second order. The roots are 1 and -1 , which gives

$$2 R_w = 1 + (-1)^w \quad w = 2p + m,$$

or the m th function of ϕx is

$$\phi^m x = \phi x \frac{1 + (-1)^{m-1}}{2} + x \frac{1 + (-1)^m}{2}$$

$$= \phi x \frac{1 - (-1)^m}{2} + x \frac{1 + (-1)^m}{2}.$$

If for m we substitute a fraction, we shall find that the preceding equation may or may not be true, according to the form of ϕ . If $\phi x = 1 - x$ and $m = \frac{1}{2}$; then

$$\phi^{\frac{1}{2}} x = \frac{1 - \sqrt{-1}}{2} + \sqrt{-1} x$$

$$\left(\phi^{\frac{1}{2}}\right)^2 x = 1 - x = \phi x.$$

But if $\phi x = \frac{1}{x}$, and $m = \frac{1}{2}$, the preceding is not true.

A reason to suspect such a result we may see as follows. In deducing the various cases hitherto arrived at of

* Sir J. Herschel, *Examp. Calc. Diff.*, sect. xi., p. 137.

$$\phi^n x = \Phi(n, x),$$

we have employed processes which, though only demonstrable, or at least demonstrated, when n is a whole number, are yet general in form, not introducing the condition that n is whole, as part of the evidence of the form of the result. Thus

$$(\phi^m)^n x = \phi^m \phi^m \phi^m \dots \phi^m x$$

is a result of the operations, without reference to the value of m or n ; or, though m and n are understood to be whole numbers, yet no consequence of this understanding affects, or can affect, the passage from any one step to another. In the case immediately preceding, however, the contrary happens; the general expression for $\phi^n x$ cannot be changed into that for $\phi^{n+1} x$ by the substitution of ϕx for x , without direct reference to the whole value of m and its consequences, namely, that $1 - (-1)^m = 0$, or 1 and $1 + (-1)^m = 1$ or 0.

Again, in the preceding cases we arrived at a function of n ; in the present we have introduced an arbitrary function of n . We say arbitrary, because there is an infinite number of methods of accomplishing the same object. Let

$$p = \frac{1 - (-1)^m}{2} \quad q = \frac{1 + (-1)^m}{2}$$

then

$$x^p (\phi x)^q \quad \frac{x^{2p} (\phi x)^{2q}}{p x + q \phi x}, \text{ \&c.}$$

are all forms which become x or ϕx , according as m is even or odd. And, generally,

$$\psi^{-1}(p \psi x + q \psi \phi x)$$

is a representation of $\phi^m x$ ($\phi^2 x = x$), in which, by varying the form of ψ , m itself may be introduced as a constant in an infinite number of different ways.

In the same way we may see that if $\phi^n x = x$,

$$\phi^m x = \psi^{-1}(R_{m-1} \psi x + R_{m-2} \psi \phi x + \dots + R_{m-n} \psi \phi^{n-1} x).$$

The student must not, however, confound this expression with that which has been called the derivative of

$$R_{m-1} x + R_{m-2} \phi x + \dots + R_{m-n} \phi^{n-1} x,$$

which would be

$$\psi^{-1}(R_{m-1} \psi x + R_{m-2} \phi \psi x + \dots + R_{m-n} \phi^{n-1} \psi x).$$

(67.) The equations by means of which the functional inverses of $\phi^m x$ ($\phi^n x = x$) are to be generally found, might be determined as follows. Let $n = 2$, and say we

wish to determine $\phi^{\frac{1}{2}} x$, or to find χx , so that $\phi x = \chi^2 x$. We have, therefore, to find ψ and χ from the following two equations:

$$\psi^{-1}(p \psi x + q \psi \phi x) = \chi x$$

or

$$p \psi x + q \psi \phi x = \psi \chi x,$$

and

$$\chi^2 x = \phi x.$$

What we shall afterwards see of such equations will lead us to suppose that periodic functions have no functional inverses which contain arbitrary functions within the limits of Algebra as defined in (8.)

(68.) But there is a certain class of functional inverses which are identical with the various successive functions of ϕx itself ($\phi^n x = x$). When p is a whole number, we have $\phi^p x = x$, which gives $\phi^p \phi^m x = \phi^{p+m} x = \phi^m x$.

Consequently, if $\phi^{\frac{a}{b}} x$ be $\phi^m x$, where m is a whole num-

Calculus of Functions. ber, or if $\phi^a x$ be $\phi^{mb} x$, it follows that $mb = np + a$, which equation can always be satisfied by whole values of m and p , except only where b and n have a common

measure which a has not. If, then, we suppose $\frac{a}{b}$ to be reduced to its lowest terms, we have the following theorem.

If $\phi^n x = x$, some value or other of $\phi^m x$ is a value of $\phi^{\frac{a}{b}} x$, except only where b and n have a common measure greater than unity.

Thus, if $\phi^6 x = x$, we have a particular case of $\phi^{\frac{1}{3}}$ in ϕ^5 , for $(\phi^5)^3 x = \phi^{15} x = \phi x$. If $\phi^7 x = x$, we have a case of $\phi^{\frac{1}{3}}$ in ϕ^4 , for $(\phi^4)^3 x = \phi^{12} x = \phi^3 x$, &c.

(69.) We shall now stop to consider a most important part of the subject, for which the preceding articles furnish materials, namely, the criterion of impossibility, so far as the same may be established or suspected. We have as yet regarded the functional symbol as one of continued quantity, but the preceding method of representing the successive functions of ϕx ($\phi^n x = x$) has led us to a function discontinuous in value, namely, one which is impossible in value for any fractional value of n , but possible for all whole values of n . And we must either say, that a general expression for $\phi^n x$ is either impossible, or (if we extend our notion of the meaning of the functional form) discontinuous.

We are here in the same situation as in Algebra, when we say that $a - b$ ($a < b$) is either impossible, or (if we extend our notion of quantity) negative. The symbol of the latter term is obtained by performing the operation indicated, as if it were rational; and in this as in other cases, we find that though the symbols of analysis can in no case be made to express the limitations under which operations are possible, or to give warning at the moment of overstepping them, they leave (each in its own way) a mark of impossibility in the result; or, if the impossibility be evaded by extensions of meaning, a mark of negative, imaginary, &c., quantity, as the case may be. We shall, therefore, perform an operation of which we know the result to be impossible, (or, if extension be necessary, discontinuous,) meaning impossible, if we stick to our primitive idea of the functional symbol; discontinuous, if we allow it to include discontinuous functions.

(70.) We have proved (49.) that the derivatives of a periodic function are periodic, and of the same order: we must, therefore, not expect any discriminating symbol if we attempt to express one periodic function as a derivative of another of the same order. In the proposition of (50.) we excluded periodic functions, only because we had then no mode of expressing $\phi^n x$ in general terms. But let us now suppose ϕx and ψx to be two periodic functions of the second order, of which ψx is to be made a derivative of ϕx .

$$\phi^n x = \phi x \frac{1 - (-1)^n}{2} + x \frac{1 + (-1)^n}{2}$$

$$\psi^n y = \psi y \frac{1 - (-1)^n}{2} + y \frac{1 + (-1)^n}{2}.$$

Eliminate n between these two, and remark that at the same time we eliminate altogether $(-1)^n$, the symbol of discontinuity. Write c and c' for $\phi^n x$ and $\psi^n y$, as in (50.) The result is

$$y \phi x - x \psi y = c(y - \psi y) - c'(x - \phi x);$$

Calculus of Functions.

from which y is to be determined in terms of x , and vice versa.

For instance, let $\psi y = \frac{1}{y}$, $\phi x = 1 - x$: which gives

$$y = \frac{-c'(2x-1) + \sqrt{c'(2x-1)^2 - 4(c-x)(1-x-c)}}{2(1-x-c)}$$

$$= \Theta x$$

$$x = \frac{y^2(1-c) - c'y + c}{y^2 - 2c'y + 1} = \Theta^{-1} y$$

$$\frac{1}{x} = \Theta(1 - \Theta^{-1} x)$$

(71.) Suppose we now try (what we know to be impossible) to represent a function which is not periodic in the form of a derivative from one which is periodic. For instance, to determine Θ so that

$$x + 1 = \Theta(1 - \Theta^{-1} x).$$

As before, eliminate n from

$$\phi x \frac{1 - (-1)^n}{2} + x \frac{1 + (-1)^n}{2} = c$$

$$y + n = c',$$

in which the sign of discontinuity is not eliminated with its exponent. This gives

$$\frac{\phi x + x}{2} - \frac{\phi x - x}{2} (-1)^{c'-y} = c;$$

or

$$y = c' - \lambda \left(\frac{\phi x + x - 2c}{\phi x - x} \right)^{\frac{1}{\lambda(-1)}}$$

$$= \Theta x$$

$$y + 1 = \Theta \phi x, \text{ or } x + 1 = \Theta \phi \Theta^{-1} x.$$

In verifying the last equation, remember that logarithms of different signs must be chosen for -1 and

$$\frac{1}{-1}. \text{ The equation } \lambda \left(\frac{1}{x} \right) = -\lambda x, \text{ where}$$

$$\lambda x = \log x + 2m\pi\sqrt{-1}$$

gives relatively to itself

$$\lambda \frac{1}{x} = -\log x - 2m\pi\sqrt{-1}$$

It therefore appears, and will appear, it is easy to see how, that when such an exponent as $\frac{1}{\lambda(-1)}$ occurs, there

has been discontinuity expressed or implied in the previous operations or conditions. And those things which are impossible of continuous functions are not therefore impossible of discontinuous functions: all ordinary reasoning upon functions is based upon principles which are only (as far as has been shown) necessarily true of continuous functions; consequently, conditions which such reasoning proves to be impossible are not necessarily impossible, either of discontinuous functions, or of those which retain the test of their appear-

ance in the process, namely, $\frac{1}{\lambda(-1)}$ as an exponent, or any other.

(72.) This I apprehend to be the answer to the dif-

Calculus of Functions. Calculus of Functions. one sort of discontinuity (at least *not* continuity) as the means of its removal; namely, the employment of $\phi_{-1}x$ in one part of a verification, and of $\phi^{-1}x$ in the other.

It is as follows: if $\phi\left(\frac{1}{x}\right) = c\phi(x)$, it follows that $\phi x = c\phi\left(\frac{1}{x}\right)$, by substituting $\frac{1}{x}$ for x ; whence by common multiplication $1 = c^2$ and $c = +1$ or -1 .

Therefore (and as to functions unaffected with $(\)^{\frac{1}{\lambda(-1)}}$ or an equivalent there can be no doubt) no function can satisfy this condition. Against this it is urged, that

$\left(\frac{1-x}{1+x}\right)^{\frac{\lambda c}{\lambda(-1)}}$ does satisfy the condition: for

$$\frac{1 - \frac{1}{x}}{1 + \frac{1}{x}} = \frac{1-x}{1+x} \times -1$$

$$\left(\frac{1 - \frac{1}{x}}{1 + \frac{1}{x}}\right)^{\frac{\lambda c}{\lambda(-1)}} = \left(\frac{1-x}{1+x}\right)^{\frac{\lambda c}{\lambda(-1)}} \times (-1)^{\frac{\lambda c}{\lambda(-1)}}$$

$$= \left(\frac{1-x}{1+x}\right)^{\frac{\lambda c}{\lambda(-1)}} \times e^{\lambda c} = c \left(\frac{1-x}{1+x}\right)^{\frac{\lambda c}{\lambda(-1)}}$$

Now, though the preceding solution is not obviously discontinuous, (it must be remembered that we are speaking of discontinuity of value, as distinguished from discontinuity of law,) yet its inverse is so: for if

$$ax = \left(\frac{1-x}{1+x}\right)^{\frac{\lambda c}{\lambda(-1)}} \quad a^{-1}x = \frac{1 - (-1)^{\frac{\lambda c}{\lambda(-1)}}}{1 + (-1)^{\frac{\lambda c}{\lambda(-1)}}},$$

which presents rational value only when x is a whole power of c , giving $a^{-1}x = 0$ for all odd powers, and $a^{-1}x = \infty$ for all even powers. And the same may be proved

generally of $(\phi x)^{\frac{1}{\lambda(-1)}}$, the inverse of which is $\phi^{-1}\{(-1)^{\lambda c}\}$,

which will generally present rational values only when x is a whole power of c .

(73.) Our definition of a discontinuous function being that which has possible values for definite possible values of its subject, and impossible for all between, and implying no more than is meant in this definition, we may now show that a *discontinuous* function of a *discontinuous* function may be a *continuous* function.† For instance, the second function of

$$x^{\frac{1}{\lambda(-1)}} \text{ is } x^{\frac{1}{\lambda(-1)^2}} = x^{\frac{1}{\pi^2} \times \text{sq. of any odd No.}}$$

This theorem will show that problems relating to con-

* Observations on the analogy between the Calculus of Functions and other branches of analysis.—Phil. Trans. Read April 17, 1817, p. 18.

† An argument might be drawn from this which would be as rational as, and would set in rather a ludicrous light, a practice which formerly prevailed in the explanation of algebraical terms. If $-(-a) = +a$, because the negation of a negation is an affirmation, surely the *discontinuance* of *discontinuity* has a right to be considered as the commencement of *continuity*.

tinuous functions may find rational solutions among discontinuous functions, and the continuity presumed in the problem may be satisfied.

(74.) What we have here called discontinuity is discontinuity of value, as distinguished from discontinuity of law, and the second may be expressed by means of the first. And the theorem of the last article is, that a discontinuous function (in value) of a discontinuous function (in value) may be continuous in value. Of functions discontinuous in law the same thing is sufficiently obvious. For if ϕx be a function which is ax from $x = a$ to $x = b$, and βx from $x = b$ to $x = c$; then if ψx be another discontinuous function, which is $\gamma a^{-1}x$ from $x = a$ to $x = b$, and $\gamma \beta^{-1}x$ from $x = b$ to $x = c$, we shall have from a to c , $\psi \phi x = \gamma x$. And hence it is a necessary condition that the breach of continuity should take place at the same values of x in both. The latter may also be shown with respect to functions discontinuous in value, it being known that possible functions of possible functions are possible, and that of an impossible function only an impossible function can be possible. For if ϕx be possible only when $x =$ one of $a, b, c, \&c.$, then if ψx be discontinuous, and $\psi \phi x$ continuous, ψ must be impossible where ϕ is impossible.

(75.) We shall now proceed to the expression of any one periodic function as a derivative of any other of the same order. Let ϕx and ψx be the two functions in question, or let $\phi^n x = x$, $\psi^n x = x$. We have then to eliminate m between the equations

$$0 = c + x R_m + \phi x R_{m-1} + \dots + \phi^{n-1} x R_{m-n+1}$$

$$0 = c' + y R_m + \psi y R_{m-1} + \dots + \psi^{n-1} y R_{m-n+1}$$

where

$$n R_m = r_1^m + r_2^m + \dots + r_n^m,$$

and $r_1, r_2, \dots, r_n$ are the roots of $r^n - 1 = 0$.

If r be one of these impossible roots, such that the rest shall be $r^2, r^3, \dots, r^n$, we shall have

$$n R_m = r^m + r^{2m} + \dots + r^{nm};$$

and (writing c and c' for $\frac{c}{n}$ and $\frac{c'}{n}$), the substitution of this in the above equations gives two equations of the form

$$P_0 + P_1 r^m + P_2 r^{2m} + \dots + P_{n-1} r^{(n-1)m} = 0,$$

where

$$P_0 = c + x + \phi x + \dots + \phi^{n-1} x,$$

or

$$c + y + \psi y + \dots + \psi^{n-1} y$$

$$P_1 = x + r^{-1} \phi x + r^{-2} \phi^2 x + \dots + r^{-(n-1)} \phi^{n-1} x$$

$$P_2 = x + r^{-2} \phi x + r^{-4} \phi^2 x + \dots + r^{-2(n-1)} \phi^{n-1} x.$$

&c.

&c.

&c.

If we eliminate r^m from these two equations, in the usual way, remembering that $r^{n+p} = r^p$, we shall have an equation of the form

$$Q_0 + Q_1 r + Q_2 r^2 + \dots + Q_{n-1} r^{n-1} = 0.$$

If we multiply this by r^{n-v} it becomes

$$Q_0 r^{n-v} + Q_1 r^{n-v+1} + \dots + Q_v + Q_{v+1} r + \dots + Q_{n-1} r^{n-v-1} = 0,$$

or

$$Q_v + Q_{v+1} r + \dots + Q_{n-1} r^{n-1} = 0,$$

or these functions $Q_0, \&c.$, must be such that the same change which makes Q_0 become Q_v makes Q_1 become Q_{v+1} , &c. Again, the only admissible change is that which arises from changing x and y into $\phi^m x$ and $\psi^m y$. Hence the form of the final equation is

Calculus of Functions. $\chi(x, y) + r\chi(\phi x, \psi y) + r^2\chi(\phi^2 x, \psi^2 y) + \dots + r^{n-1}\chi(\phi^{n-1} x, \psi^{n-1} y) = 0.$

It is unnecessary to find more than the form; for the preceding process immediately suggests that *any form whatever* of χ will satisfy the conditions. For the object being to obtain $y = \Theta x$ in such a manner that $\psi y = \Theta \phi x$, $\psi^2 y = \Theta \phi^2 x$, &c., the preceding process makes it appear that $\psi y = \Theta \phi x$, &c., might be deduced by the same steps which give $y = \Theta x$. Nor would the particular form derived from the preceding methods be of any use, as it is obtained (66.) from a very limited method of expressing $\phi^m x$ and $\psi^m x$.

The equation in (70.) is a particular case of the preceding result; where $n = 2$ or $r = -1$; being $y\phi x - cy + c'x + (-1)(x\psi y - c\psi y + c'\phi x) = 0$, for which we are therefore at liberty to substitute

$$\chi(x, y) = \chi(\phi x, \psi y).$$

Thus, in the case of (70.), where we make $\frac{1}{x}$ a derivative of $1 - x$, we may, instead of the equation there given, determine y from

$$x + y = 1 - x + \frac{1}{y},$$

or from
$$x\sqrt{y} = \frac{1-x}{\sqrt{y}}, \text{ \&c.,}$$

the last of which gives the most simple value of y , namely, $\frac{1-x}{x}$.

(76.) We now come to the subject of convertible functions generally: that is, of those which satisfy the equation $\phi\psi x = \psi\phi x$. And, in the first place, ϕ^n and ϕ^m are convertible for all values of m and n . When m and n are whole numbers, this follows from the nature of the operation, for $\phi^2\phi^3 x$ signifies $\phi\phi(\phi\phi\phi x)$ or $\phi^5 x$, and $\phi^3\phi^2 x$ signifies $\phi\phi\phi(\phi\phi x)$, which is also $\phi^5 x$. When m and n are negative, we have, by hypothesis, restricted the meaning of $\phi^{-1}x$ to the inverse which is convertible with ϕx , &c. Hence ϕ^{-2} , for example, is convertible with ϕ^{-3} , both being ϕ^{-5} . We shall also restrict the meaning of $\phi^{\frac{1}{n}}$ to those operations of the kind which are convertible with ϕ , that is, if any others shall be found. Hence $\phi^{\frac{1}{3}}\phi^{\frac{1}{2}}x$ is the same as $\phi^{\frac{1}{2}}\phi^{\frac{1}{3}}x$, both being $\phi^{\frac{5}{6}}x$.

We have proved that if ϕ and ψ be convertible functions, all their corresponding derivatives are convertible. The converse of this is true, since the primitive function is itself a derivative of its derivative ($\Theta^{-1}(\Theta\phi\Theta^{-1})\Theta x = \phi x$). But here it must be remembered that if

$$\Theta\phi\Theta^{-1}x = ax, \quad \Theta\psi\Theta^{-1}x = \beta x,$$

and if a and β be convertible, we do not know that more than one of the values of ϕ which satisfy the first equation is convertible with any one of those which satisfy the second. We are not to assume, for instance, if

$$\Theta\phi\Theta^{-1}x = \Theta\psi\Theta^{-1}x,$$

that therefore $\phi x = \psi x$; though this is evidently one of the values which satisfy that equation.

(77.) To find functions which are convertible with a given function ax , take any other function which has another convertible with it, and such that the equation

$$\Theta\beta\Theta^{-1}x = ax$$

Calculus of Functions

can be easily solved. Let γx be the function which is convertible with βx ; then $\Theta\beta\Theta^{-1}x$ and $\Theta\gamma\Theta^{-1}x$, or ax and $\Theta\gamma\Theta^{-1}x$ are convertible. If Θ contain an arbitrary function, an infinite number of solutions may be thus proved, and sometimes obtained.

For instance, $1+x$ and $x+\phi(\cos 2\pi x)$ are evidently convertible. Let

$$\alpha\Theta x = \Theta(1+x),$$

that is, let

$$\alpha^n x = A(n, x),$$

and eliminate n between

$$c' = y + n \text{ and } c = A(n, x),$$

or find

$$y = \Theta x \text{ from } c = A(c' - y, x).$$

Then

$$\Theta(\Theta^{-1}x + \phi \cos(2\pi\Theta^{-1}x))$$

will be convertible with αx . Generally, let ξx be a solution of $\xi ax = \xi x$, and let

$$\beta^n x = B(x, n, c, c' \dots).$$

Let $\Theta\beta\Theta^{-1}x = ax$, as before, then a and β are convertible functions. But if, instead of n, c, c' , we write $\eta\xi x, \zeta\xi x, \kappa\xi x$, &c., η, ζ , and κ being any functions, we find that

$$ax \text{ and } \Theta B(\Theta^{-1}x, \eta\xi x, \zeta\xi x, \kappa\xi x \dots)$$

are convertible. For $\eta\xi x$, &c., are constants with respect to the only change x undergoes in the function ΘB , namely, a change into ax .

(78.) We have already considered the principal convertible functions of Algebra: we shall only observe further that all derivatives of ϕx with respect to Θ are equal to ϕx , if Θ and ϕ be convertible: for in that case $\Theta\phi\Theta^{-1}x = \Theta\Theta^{-1}\phi x = \phi x$.

(79.) If two periodic functions be convertible, and if they be of the m th and n th order, the function compounded of them is periodic, and not of a higher order than the (mn) th. For if

$$\phi^m x = x, \quad \psi^n x = x, \text{ and } \phi\psi x = \psi\phi x,$$

then

$$\begin{aligned} \phi\psi \dots \phi\psi \dots (\text{repeated } mn \text{ times}) &= \phi^{mn}\psi^{mn}x \\ &= \overline{\phi^n}^m \overline{\psi^m}^n x = \overline{\phi^n}^m x = x. \end{aligned}$$

If r and r' be m th and n th roots of 1, then rx and $r'x$ are convertible periodic functions of those orders, and so are $\Theta r\Theta^{-1}x$ and $\Theta r'\Theta^{-1}x$. The function compounded of them is $\Theta r\Theta^{-1}\Theta r'\Theta^{-1}x$, or $\Theta rr'\Theta^{-1}x$, which may easily be shown to be as above.

(80.) There is one class of functions remaining to be noticed, namely, that of functions which are *symmetrical* with respect to two or more letters. A function of x and y is symmetrical with respect to those letters when both enter under precisely the same circumstances, so that an interchange of the two does not affect the value of the function. Such functions are

$$x+y, \quad xy, \quad \frac{x^2+y^2}{x+y}, \quad \sin^{-1}x + \sin^{-1}y.$$

The following are symmetrical functions of x, y , and z ; that is, an interchange of any two of the preceding three does not affect the value of the function.

$$\begin{aligned} x+y+z, \quad xy+yz+zx \\ \sin^{-1}(x+y) + \sin^{-1}(y+z) + \sin^{-1}(z+x). \end{aligned}$$

If a function be symmetrical with respect to x and y , and to y and z , it will be so with respect to x and z ;

Calculus of Functions. for if x may be put in place of y , and z the same, and the corresponding interchanges being made, if no new terms be introduced thereby, then, since every term or factor which contains y is only removed in either case, there are the same terms and factors with respect to x as to z . This will, however, appear more clearly by the following reasoning.

(81.) If we call $x + y = 2p$ and $xy = q^2$, then any function of x and y is a function of p and q . But all unsymmetrical functions are twofold functions of p and q , that is, present double values, according to whether we take one sign or the other for the values of x and y derived from the above equation. For instance, $x^2 + y$ is either

$$(p + \sqrt{p^2 - q^2})^2 + p - \sqrt{p^2 - q^2},$$

or

$$(p - \sqrt{p^2 - q^2})^2 + p - \sqrt{p^2 - q^2}.$$

But in symmetrical functions this cannot be the case; for, since x and y can be interchanged, so can any values we may write for them. The characteristic of a symmetrical function is therefore that it has no more values when considered as a function of p and q than it has as a function of x and y .

(82.) Let there be a function which is symmetrical with respect to x and y , and is also a function of z . Consequently, it can be represented by

$$\phi(x + y, xy, z).$$

Let it also be symmetrical with respect to x and z , so that x and z may be interchanged in the preceding, which, as before observed, is unambiguous in value. That is,

$$\phi(x + y, xy, z) = \phi(z + y, zy, x),$$

which last is the form of a function symmetrical with respect to z and y .

(83.) A symmetrical function of x , y , and z can be represented by an unambiguous form of

$$\phi(x + y + z, xy + yz + zx, xyz),$$

and so on.

(84.) A function is symmetrical with respect to any two of n quantities, when it is symmetrical with respect to $(n - 1)$ pairs out of the $\frac{1}{2}n(n - 1)$ which can be

taken, provided every letter be contained once. Take five quantities, a, b, c, d , and e , and let $(ab, bc; ac)$ be the abbreviation of the deduction of the symmetry of the function with respect to a and c from that with respect to a and b and b and c . Let ab, bc, cd, de , be the pairs which enter into the hypothesis

$$\begin{array}{ll} ab, bc; ac & bc, cd; bd \\ ac, cd; ad & bd, de; be \\ ad, de; ae & cd, de; ce \end{array}$$

and there are but ten pairs from the five quantities a, b, c, d, e . If the pairs chosen had been ab, ac, bd, ce , we should have had

$$\begin{array}{ll} ab, bd; ad & ba, ae; be \\ ba, ac; bc & cb, bd; cd \\ ac, ce; ae & da, ae; da \end{array}$$

(85.) If $\phi^n x = x$, any symmetrical function of $x, \phi x, \phi^2 x, \dots, \phi^{n-1} x$ is constant when $\phi^n x$ is changed into $\phi^{n-1} x$, $\phi^{n-1} x$ into $\phi^{n-2} x$, &c. For, on account of the recurrence of $x, \phi x$, &c., in $\phi^n x, \phi^{n+1} x$, &c., this change amounts to a mere interchange of the subjects of the function. For instance, if $\phi^3 x = x$, then writing $\phi^2 x$ for x , in

$$x + \phi x + \phi^2 x + x \cdot \phi x \cdot \phi^2 x,$$

gives

$$\phi x + \phi^2 x + x + \phi x \cdot \phi^2 x \cdot x.$$

(86.) If we investigate the conditions under which

symm. function $(a_1 x, a_2 x, a_3 x, \dots, a_n x)$

may remain constant when x is changed into ϕx , we shall find

$$a_2 \phi x = a_1 x \quad a_3 \phi x = a_2 x \dots a_1 \phi x = a_n x,$$

or

$$a_n x = a_1 \phi x = a_2 \phi^2 x = \dots = a \phi^n x,$$

or

$$a_n x = a_n \phi^n x.$$

Consequently, either $\phi^n x = x$, in which case the function may be reduced to

symm. function of $(a_1 x, a_1 \phi x, a_1 \phi^2 x, \dots, a_1 \phi^{n-1} x)$,

or ϕx must be the n th functional inverse of one of the invertible algebraical inverses of $a_n x$.

(87.) Any function of the form

$$\psi x \times \text{symm. function of } (x, \phi x, \dots, \phi^{n-1} x)$$

is periodical in its values when x is changed into $\phi x, \phi^2 x$, &c., in succession, if $\phi^n x = x$.

(88.) But when $\phi^n x = x$, a more general species of symmetry will satisfy the equation $\psi \phi x = \psi x$. It is sufficient to take such a function of $x, \phi x, \phi^2 x, \dots, \phi^{n-1} x$, as will remain the same when x is changed into $\phi x, \phi x$ into $\phi^2 x, \dots, \phi^{n-1} x$ into $\phi^n x$ or x , all at once. For instance, if $\phi^3 x = x$,

$$\frac{x}{\phi x} + \frac{\phi x}{\phi^2 x} + \frac{\phi^2 x}{\phi^3 x}$$

$$x^2 \phi x + (\phi x)^2 \phi^2 x + (\phi^2 x)^2 x$$

do not change when ϕx is substituted throughout instead of x . But a simple interchange of ϕx and x would alter either of the preceding functions.

(89.) Hence it appears that a solution of $\psi \phi x = \psi x$ is not necessarily a function of the coefficients of the equation whose roots are $x, \phi x, \phi^2 x, \dots, \phi^{n-1} x$.

If we denote by $\chi(a, b, c)$ any function of a, b , and c , and by $\chi(b, c, a)$ a similar function, in which a has become b , b has become c , and c has become a , the general form of the solution of $\psi \phi x = \psi x$ where $\phi^3 x = x$ is

$$\psi x = \chi(x, \phi x, \phi^2 x) + \chi(\phi x, \phi^2 x, x) + \chi(\phi^2 x, x, \phi x),$$

or any other properly symmetrical function of the three forms of χ .

(90.) Here it is to be observed, that when ψx represents a function of $x, \phi x, \phi^2 x, \dots, \phi^{n-1} x$, which satisfies the equation $\psi \phi x = \psi x$, whence we have $\psi \phi^2 x = \psi \phi x$, &c., and

$$\psi x = \frac{1}{n} \{ \psi x + \psi \phi x + \dots + \psi \phi^{n-1} x \} \dots (A)$$

in which all interchanges would seem possible: this is nevertheless not necessarily the case. If ψx were a function of x only, and not identical with $\psi \phi x$, then no interchange would alter the form of the second side of (A). But this is not always the case where ψx is also a function of ϕx , &c., and never when $\psi \phi x = \psi x$, unless where the interchanges in question can already be made in ψx by itself, without altering its form.*

* The late M. Abel, whose talents have most deservedly secured a reputation almost unequalled, considering the age at which he died, has (Crelle's Journal, vol. iv.) overlooked this distinction, and in consequence has inferred, that whenever two roots of an irreducible equation are connected by a known and rational relation,

Calculus of
Functions.

(91.) The two species of symmetry described in (80.) and (88.) might be distinguished by the names of *disjunctive* and *periodical*, which, for want of better, we shall adopt. Thus

$$ab + bc + ca$$

is disjunctively symmetrical with respect to a , b , and c ; while

$$ab^2 + bc^2 + ca^2$$

is periodically symmetrical. All disjunctively symmetrical functions are periodically so, but the converse is not true.

(92.) In all cases, a particular solution of $\phi a^2 x = \phi x$ may be made to give a general solution of $\phi a x = \phi x$. Let ωx be the particular solution, say of $\phi a^2 x = \phi x$, then any periodically symmetrical function of ωx , $\omega a x$, and $\omega a^2 x$ being taken, for instance,

$$\frac{\omega x}{\omega a x} + \frac{\omega a x}{\omega a^2 x} + \frac{\omega a^2 x}{\omega x},$$

any arbitrary function of this is a solution of $\phi a x = \phi x$.

(93.) Mr. Babbage has proposed to let $\bar{\phi}(x, y)$ stand for a function which is symmetrical with respect to x and y ; $\phi(\bar{x}, \bar{y}, z)$ to stand for a function of x , y , and z , which is symmetrical with respect to x and y only. But as we shall not in this Treatise have sufficient occasion to use symmetrical functions in very complicated operations, we prefer some circumlocution to the substitution of a purely arbitrary symbol for a *derived* notion. The future progress of the Science will doubtless, as in all cases which have hitherto occurred, suggest the proper method of representing a symmetrical function.

We shall be continually obliged to mention functions which are constant under ax for x , (as we may call it,) that is, which do not change form when ax is written for x . These we shall always denote by ξ ; that is, ξax means ξx , a function which satisfies the condition $\xi x = \xi ax$. Thus, $\cos 2\pi x$ is a particular case of $\xi(x+1)$.

Most of the symmetrical functions which we shall have occasion to use are those of $x, \phi x, \dots, \phi^{n-1}x$, where $\phi^n x = x$; and the symmetry need be only periodical (91.) This is then a particular case of $\xi \phi x$ (88.); and though there may be other values of ξ (possibly) which may satisfy $\xi \phi x = \xi x$, and which are not symmetrical, yet as our assumption of a symmetrical function will always arise from our not knowing what these other solutions may be, and as the other solutions would do as well, if they could be found, we shall use $\xi \phi x$ as the representation of a symmetrical function of $x, \phi x, \dots, \phi^{n-1}x$, when $\phi^n x = x$.

(94.) We now come to the various classes of functional equations which it may be necessary to consider. If there be any number of quantities, x, y, z , &c., between which a functional relation exists, the questions which may be asked are as follows:

1. Where all the functional forms are known, what functional relations must exist between x, y, z , &c., in order that the equation may be satisfied? This embraces the whole of common Algebra: for instance, the solution of

the equation admits of solution by means of equations of a lower degree. His demonstration, however, from what has been here said, cannot stand, unless any solution of $\psi \phi x = \phi x$ can be expressed in terms of the coefficients of the equation whose roots are $x, \phi x, \dots, \phi^{n-1}x$, without solving the equation itself. This, we believe, has never been done.

$$x^2 - xy + ay^2 = 0$$

is a particular case of the following,

$$F(x, y, \phi y, \psi x, \&c.) = 0,$$

where F, ϕ, ψ , &c. are known forms; what relation does the preceding require to exist between y and x ?

2. Where the forms which enter into the equation are unknown, and it is required to satisfy the equation independently of any relations between x, y, z , &c. For instance,

$$\phi x + \phi y = \phi(x + y)$$

occurs in various branches of Mathematics, where x and y are to be independent. The solution (it is the most general) is $\phi x = mx$, where m is independent of x and y .

3. Where x, y, z , &c. are related in a given manner, and one or more of the functional forms in the equation are to be found so as to satisfy it. For instance, if in the last equation, $y = x + 1$, and we therefore require to solve

$$\phi(x) + \phi(x + 1) = \phi(2x + 1).$$

Here cx is a solution as before: but it is a more general solution in one sense than is necessary, though not the most general in another respect. Properly speaking, this is the solution of

$$\phi x + \phi y = \phi(x + y),$$

where y is independent of x , and c retains its value, whatever y may be when x is changed into y or into $x + y$. But the last equation but one only requires that c should remain unchanged when x is changed into $x + 1$ or $2x + 1$; and if ξx be such a function, $x \times \xi x$ satisfies the equation: for

$$\begin{aligned} x \xi x + (x + 1) \xi(x + 1) &= x \xi x + (x + 1) \xi x \\ &= (2x + 1) \xi x = (2x + 1) \xi(2x + 1), \end{aligned}$$

because

$$\xi x = \xi(x + 1) = \xi(2x + 1).$$

(95.) The third case is the one to which this Treatise is devoted: but the second is a particular branch of the subject which deserves some consideration.

If we take an equation, such as

$$\phi x - \phi(y + 1) = \phi x \times \phi y,$$

in which x and y do not enter symmetrically, it is often easy to prove that no continuous function of x or y will satisfy it. In the preceding instance, let us, for the present, write m for ϕx , which gives

$$\phi(y + 1) = m(1 - \phi y).$$

Of this equation it can be shown, that if ωy be any solution, and $\omega y + \psi y$ the most general solution, ψy can be found from another equation. For we have

$$\omega(y + 1) = m(1 - \omega y),$$

which, subtracted from the preceding, gives

$$\phi(y + 1) - \omega(y + 1) = -m(\phi y - \omega y)$$

or

$$\psi(y + 1) = -m\psi y.$$

Now $\frac{m}{1+m}$ is a particular solution of the equation;

therefore $\frac{m}{1+m} + \psi y$ is the general solution, which

must, if it is to be independent of x , be independent of

m . That is, ψy must have the form $-\frac{m}{1+m} + \chi y$,

where χy is independent of m . In this form, ψy cannot

Calculus of
Functions.

Calculus of
Functions.

be found, and hence $\frac{m}{1+m}$ or $\frac{\phi x}{1+\phi x}$ independent of

x , is the most general solution of the first-mentioned equation. Hence ϕx must be a constant, or the first equation can only be solved by $\phi x = c$, which gives

$$c - c = c \times c, \text{ or } c = 0.$$

(96.) The only remaining case is where x and y enter symmetrically, which may be subdivided again into cases reducible to $Fx = Fy$, and those which are not so reducible. An instance of the first is

$$\phi x + \phi y = \phi(2x) + \phi(2y).$$

Here the change of y into x does not alter the equation, which is moreover immediately reducible to

$$\phi x - \phi(2x) = \phi y - \phi(2y),$$

the only solution of which is

$$\phi x - \phi 2x = \text{a constant},$$

for any other solution gives a relation between x and y . This case then is reducible to the third mentioned in (94.)

But if we take such a relation as

$$\phi x + \phi y = \phi(2x + 2y),$$

which is not reducible to the form $Fy = Fx$, we know that the solutions of this equation must be contained among those of

$$\phi x + p = \phi(2x + q),$$

being such among the solutions of the latter as satisfy the additional conditions

$$q = 2y \quad p = \phi(2y) \quad \frac{d}{dy} \phi x = 0.$$

But all that precedes relates to continuous functions only. We may not (69.) assume any of the conclusions drawn from continuous forms to be true of those which are discontinuous.

(97.) Several cases, however, of the last-mentioned forms admit of solutions by common methods. Such are

$$\phi x + \phi y = \phi(x + y),$$

from which are derived

$$\phi x + \phi y = \phi x \times \phi y$$

$$\phi x + \phi y = \phi(xy)$$

$$\phi x \times \phi y = \phi(x + y)$$

$$\phi x \times \phi y = \phi(xy)$$

and various others.

The use of the Calculus of Functions in regard to these is to show that the obvious algebraical form which satisfies them is also the *most general of continuous functions*. The following process may be applied to all the preceding.

Let $\phi x_1 + \phi x_2 = \phi(x_1 + x_2)$; for x_2 write $x_2 + x_2$; then

$$\phi x_1 + \phi(x_2 + x_2) = \phi(x_1 + x_2 + x_2),$$

or

$$\phi x_1 + \phi x_2 + \phi x_2 = \phi(x_1 + x_2 + x_2),$$

or, generally,

$$\Sigma \phi x = \phi(\Sigma x).$$

Hence

$$n \phi x = \phi n x,$$

where n is any whole number. Let $my = nx$, n being a whole number, then

$$\phi my = \phi nx, \text{ or } m \phi y = n \phi x;$$

VOL. II.

Calculus of
Functions.

or $\phi y = \frac{n}{m} \phi x$, that is $\phi\left(\frac{n}{m}x\right) = \frac{n}{m} \phi x$;

whence $\phi nx = n \phi x$ is true when n is fractional. Let $x_1 + x_2 = 0$, whence by the original equation

$$\phi(-x_1) = -\phi x_1,$$

or

$$\phi(-nx_1) = -\phi(nx_1) = -n \phi x,$$

or $\phi nx = n \phi x$ is true when n is negative. Whence in the equation $\phi nx = n \phi x$ the value of n is universal. Let $nx = y$, then

$$\phi y = \frac{y}{x} \phi x, \text{ or } \frac{\phi y}{y} = \frac{\phi x}{x};$$

whence $\phi y = cy$.

The following theorems, which may be easily proved, contain the preceding demonstration.

1. If α be such a function that its inversion with respect to ϕx substitutes $\frac{1}{n}$ for n , then the equation $\phi nx = \alpha(\phi x, n)$ is true for fractional values of n , if it be true for whole values. Such are $n \phi x, \overline{\phi x}^n$.

2. If α be such a function that its inversion with respect to ϕx changes n into $-n$, then $\phi(x+n) = \alpha(\phi x, n)$ is true for negative values of n , if it be true for positive values. Such is $\phi x + n$.

(98.) The general functional equation containing only one unknown function of one subject may be put under the following form :

$$F(x, \phi x, \phi \alpha x, \phi \beta x, \dots) = 0,$$

where F, α, β , &c., are known forms, and ϕ is an unknown form to be so determined as to satisfy the equation. In almost every case, it will be necessary to suppose a particular solution found, either of the equation itself, or of one connected with it, in such a way that $\psi \alpha x, \psi \beta x$, &c. all enter into it. This, at first sight, appears hardly to be a real solution, but only the reduction of the solution to that of some particular case. But a moment's recollection will show that precisely the same thing takes place in Algebra. For instance, $x^2 + ax = b$ is not solved, except by reduction to the more simple form $y^2 = b$, for which there are no methods of further reduction in Algebra, independent of these which Arithmetic applies to every particular value of b .

For $x = (b)^{\frac{1}{2}}$ is not a solution, but a conventional representation of a solution, being, by definition, nothing more than the solution of $y^2 = b$. Similarly, every equation of the first degree is reduced to $ax = b$, of

which $\frac{b}{a}$ is not a solution, but a representation of a solution.

Let us suppose a particular solution found, and let it be

$$\psi x = \varpi(x, a, b, c, \dots),$$

where a, b, c , &c. are arbitrary constants. A general solution* may then be found, if a function of x can be found which does not change upon the substitution of $\alpha x, \beta x, \dots$ respectively for x . For in that case, if the function be called $\xi \alpha, \xi \beta, \dots, x$, the following values may be substituted for $a, b, c, \dots$:

$$a = \theta \xi x \quad b = \theta' \xi x \quad c = \theta'' \xi x \quad \&c.,$$

* Whether the most general or no is not known. By a general solution we always mean one which admits an indefinite number of forms.

Calculus of Functions. in which $\theta, \theta', \theta'', \dots$ may be any functions whatever, with a limitation presently to be noticed. For in that case, $a, b, c, \dots$ are constant under every change which can take place in verifying the equation $F = 0$ by means of $\psi = \omega$. For instance,

$$\phi(x) + \phi(-x) = x^2$$

is evidently satisfied by $\phi x = \frac{x^2}{2} + cx$; but in place of c may be written any function which contains only even powers of x , or $\xi(-x)$, giving the solution $\phi x = \frac{x^2}{2} + x\xi(-x)$.

(99.) The limitation alluded to above is the following: $\theta \xi x, \theta' \xi x, \&c.$ must preserve the property of unchangeability under x into $\alpha x, x$ into $\beta x, \&c.$, that is, $\theta, \theta', \&c.$ must not involve any operation inverse to ξ . For instance, θx must not be $x + \xi^{-1}x$; for in that case $\theta \xi x$ is $\xi x + x$, which becomes $\xi x + \alpha x$ when x is changed into αx . Thus, in the integration of the equation of differences, $u_{n+1} = v_n$ or $u_n = f(\cos 2\pi n)$, $f x$ may not be $x + \alpha \cos^{-1}x$, for $f \cos 2\pi n$ then becomes $\cos 2\pi n + \alpha(2\pi n)$, and may change when n is changed into $(n+1)$.

(100.) The method of Laplace corresponding to the method given in (46.) is as follows. Let the equation be

$$F(x, \phi \alpha x, \phi \beta x) = 0,$$

where F, α , and β are known forms, and ϕ is to be found. Assume

$$u_x = \alpha x \quad u_{x+1} = \beta x.$$

By elimination from which we obtain

$$u_{x+1} = \beta \alpha^{-1} u_x,$$

or u_{x+1} is a known function of u_x , giving a common equation of differences. If this can be integrated, suppose it done, either in the common form

$$u_x = f(z, C),$$

or in the more general form

$$u_x = f(z, \cos 2\pi z).$$

Next, assume

$$\phi(u_x) = v_x \quad \phi(u_{x+1}) = v_{x+1},$$

which makes the original equation become

$$F(\alpha^{-1} u_x, v_x, v_{x+1}),$$

another equation of differences, which, if it can be integrated, gives

$$v_x = F_1(u_x, \cos 2\pi z),$$

or

$$\phi \alpha x = F_1 x,$$

whence

$$\phi x = F_1 \alpha^{-1} x.$$

For further extensions of this method, see the *Memoirs of the Analytical Society*, p. 98, &c.

(101.) We may exemplify this method hereafter; at present we return to our first method, and observe that we have reduced the solution of a functional equation,

1. To finding a particular solution which contains at least one arbitrary constant.

2. To finding the function which is constant with respect to the equation; that is, which is not changed by changing x into αx , into βx , &c., $\alpha x, \beta x, \&c.$ being all the forms which occur.

For the first, no general process can be given; we

proceed to ascertain how far we can fulfil the second condition. Calculus of Functions.

(102.) And first we propose the equation

$$\phi \alpha x = \phi x,$$

the solution of which, when found, we have agreed to call $\xi \alpha x$. This is evidently a particular case of

$$\phi \alpha x = \beta \phi x,$$

already solved in (50.) Here $\beta x = x$, and the two equations between which n must be eliminated appear to be

$$\alpha^n x = c \quad y = c',$$

from which n cannot be eliminated, because it does not appear in the second.

But, instead of c and c' , we are at liberty to write any functions of n which do not change on changing n into $n+1$, which gives

$$\alpha^n x = \mu \cos 2\pi n \quad y = \nu \cos 2\pi n.$$

Let us suppose these equations give

$$n = Ax \quad n = By;$$

then, by substituting αx for x in the original equations, we obtain

$$n = A \alpha x \quad n = B y,$$

and the fundamental relation is preserved, namely, that a change of n into $n+1$ is equivalent to a change of x into αx . Hence we have

$$y = B^{-1} A x \quad y = B^{-1} A \alpha x,$$

or

$$\xi \alpha x = B^{-1} A x.$$

It might appear that Ax itself satisfies the conditions: but the reasoning of (50.) is based upon the theorem, that in the result of elimination it matters not whether the quantity eliminated were constant or variable; it must be remembered that $n = Ax$ and $n = A \alpha x$ are not both true equations; but that the second set of equations should be

$$n - 1 = A \alpha x \quad n - 1 = B y$$

for

$$\alpha^{n-1}(\alpha x) = \mu \cos 2\pi(n-1).$$

Let $\mu \cos 2\pi n = c$; then, since $y = \nu \cos 2\pi n$, we have the following theorems.

1. If $\alpha^n x = c$, the value of n (or Ax) is a function of x , which satisfies the equation

$$A \alpha x = Ax - 1.$$

2. A very general value of $\xi \alpha x$ is contained in

$$\xi \alpha x = \nu \cos 2\pi A x,$$

where ν is any function which does not invert the cosine: but much more general values, containing, it may be supposed, many transcendents which Ax cannot now express, may be obtained by using

$$\alpha^n x = \mu \cos 2\pi n \quad \text{instead of} \quad \alpha^n x = c.$$

(103.) The function c^{-Ax} is a particular solution of

$$\psi \alpha x = c \psi x,$$

and a particular solution of

$$\psi \alpha x = c' \psi x$$

can be deduced from it, in the form

$$\psi x = c^{-\frac{\lambda \alpha' Ax}{\lambda c}}.$$

Also the function $-cAx$ is a particular solution of

$$\psi \alpha x = \psi x + c,$$

Calculus of Functions.

α^{-Ax} of $\psi \alpha x = (\psi x)^c$.

And in this way a large number of forms may be found with known particular solutions. But we proceed with the investigation of the properties of A .

(104.) Where a name is necessary, we shall call Ax the *exponential inverse* of αx .^{*} If $\alpha_1 x$ be a derivative of αx with respect to Θ , and $A_1 x$ the value of n derived from $\alpha_1 x = c$; then $A_1 x = A \Theta^{-1} x$. For if $\alpha_1 x = \Theta \alpha \Theta^{-1} x$, we have

$$\alpha_1 x = \Theta \alpha \Theta^{-1} x = c;$$

or, writing c for $\Theta^{-1} c$ we find

$$\alpha^n \Theta^{-1} x = c \quad n = A \Theta^{-1} x = A_1 x,$$

from which new classes of particular solutions may be deduced.

(105.) If, instead of αx , we take $\alpha^m x$, we find from the same process $(\alpha^m)^n x = c$, or $m n = A x$; that is,

$$n = \frac{1}{m} A x.$$

Thus of

$$\psi \alpha^m x = \psi x - 1,$$

a particular solution is $\frac{1}{m} A x$. This may be verified, for since

$$A \alpha x = A x - 1,$$

we have

$$A \alpha^2 x = A \alpha x - 1 = A x - 2, \text{ \&c.,}$$

whence

$$A \alpha^m x = A x - m,$$

or

$$A_m \alpha^m x = A_m x - 1, \text{ where } A_m x = \frac{A x}{m}.$$

This holds also where m is a negative whole number: for

$$A \alpha x = A x - 1,$$

we have

$$A \alpha^{-1} x = A x + 1$$

$$A \alpha^{-m} x = A x + m,$$

or

$$A_m \alpha^{-m} x = A_m x - 1, \text{ where } A_m x = -\frac{A x}{m}.$$

But for $m = 0$ we find $A x = \infty$, as might be expected for a solution of

$$A x = A x - 1.$$

That the preceding will also hold when m is fractional may be inferred in all those cases in which the form of $\alpha^n x$ remains true when n is fractional.

(106.) To apply the preceding method, let us take the function $a + b x$, reduced to the form $b(x + m) - m$, in which case it is a derivative of $b x$, being $\Theta b \Theta^{-1} x$ where $\Theta x = x - m$.[†] Let $\alpha x = b x$, which gives $\alpha^n x = b^n x = c$, or

$$n = A x = \frac{\lambda c - \lambda x}{\lambda b} = \frac{c - \lambda x}{\lambda b},$$

changing the form of the constant. Hence

$$\xi b x = \theta \cos 2\pi \frac{c - \lambda x}{\lambda b},$$

* The reader may, as in the case of the derivative, supply any names he likes better. It is important to observe that, in finding ξx , $A x$ may stand for the solution of $A \alpha x = A x \pm p$, where p is any whole number.

† The analogy between functional symbols and products is such that a factor may occupy the place of, and be considered as, a functional symbol.

or, which is the same form, (if the cosine be expanded, and the specific form of θ changed,) Calculus of Functions.

$$\xi b x = \theta \cos 2\pi \frac{\lambda x}{\lambda b},$$

where θ is any form which does not invert the cosine. Again, the exponential inverse of $b(x + m) - m$ is

$$A \Theta^{-1} x, \text{ or } \frac{c - \lambda(x + m)}{\lambda b}; \text{ whence, as before,}$$

$$\xi(b(x + m) - m) = \theta \cos 2\pi \frac{\lambda(x + m)}{\lambda b},$$

or

$$\xi(a + b x) = \theta \cos 2\pi \frac{\lambda(a + b x - x)}{\lambda b}.$$

When $b = 1$, there is an indication of a change of form. In fact, the preceding process fails by the reduction of $\Theta b \Theta^{-1} x$ to x . But if $\alpha x = a + x$, we have $\alpha^n x =$

$$x + n a = c \text{ and } A x = \frac{c - x}{a}. \text{ We have then}$$

$$\xi(a + x) = \theta \cos 2\pi \frac{x}{a}.$$

(107.) Applying the same process to αx^b in the form $\frac{m x^b}{m}$, which is a derivative of $x^b \left(\Theta x = \frac{x}{m} \right)$, we find, if

$$\alpha x = x^b \quad A x = \frac{c - \lambda^2 x}{\lambda b}$$

$$\xi(x^b) = \theta \cos 2\pi \frac{\lambda^2 x}{\lambda b},$$

where $\lambda^2 x$ denotes $\lambda \lambda x$, the logarithm of the logarithm of x .

$$\xi\left(\frac{(m x)^b}{m}\right) = \theta \cos 2\pi \frac{\lambda^2 m x}{\lambda b}$$

$$\xi(a x^b) = \theta \cos 2\pi \frac{\lambda^2 a^{\frac{1}{b}-1} x}{\lambda b}.$$

This fails when $b = 1$, in which case a similar process will give

$$\xi(a x) = \theta \cos 2\pi \frac{\lambda x}{\lambda a}.$$

(108.) The function $\frac{a + b x}{a' + b' x}$, when put in the form $\frac{1 + b x}{b + m + k x}$ (57.), does not readily appear as the derivative of a more simple function of the same kind; but its exponential inverse may be directly found from the expression given in (64.) for its n th function. The most convenient value of c is zero, for the mere purpose of finding a value of $A x$; as appears from the preceding instances. We have, however, retained it hitherto, as solutions with an arbitrary constant are frequently necessary. But in the present case, let us make

$$\alpha x = \frac{a + b x}{a' + b' x} \quad \alpha^n x = 0,$$

which gives (64.)

$$a(r^n - r'^n) + \{b(r^n - r'^n) - r r' (r^{n-1} - r'^{n-1})\} x = 0,$$

from which we obtain, by dividing by r'^n and deducing the value of $\left(\frac{r}{r'}\right)^n$,

Calculus of
Functions.

$$n = \frac{\lambda(a+bx-rx) - \lambda(a+bx-r'x)}{\lambda r - \lambda r'} = Ax,$$

whence we have (r and r' being the roots of

$$R^2 - (a'+b)R + (a'b - ab') = 0)$$

$$\xi \left\{ \frac{a+bx}{a'+b'x} \right\} = \theta \cos 2\pi \lambda \left\{ \frac{a+bx-rx}{a+bx-r'x} \right\}^{\frac{1}{\lambda r - \lambda r'}}.$$

This expression fails in the case already noticed, in which the two roots of the preceding equation are equal, or in which

$$\frac{a+bx}{a'+b'x} \text{ takes the form } \frac{1+bx}{b+m-\frac{1}{4}m^2x}.$$

We might obtain Ax in this case from the expression in (61.); but by differentiating the numerator and denominator of the first expression for Ax in this article with respect to r , in the usual manner, we find, since r must be $\frac{1}{2}(a'+b)$,

$$Ax = -\frac{rx}{a+bx-rx} = \frac{(a'+b)x}{(a'-b)x-2a'},$$

whence

$$\xi \left(\frac{a+bx}{a'+b'x} \right) = \theta \cos 2\pi \frac{(a'+b)x}{(a'-b)x-2a'}$$

$$\xi \left(\frac{1+bx}{b+m-\frac{1}{4}m^2x} \right) = \theta \cos 2\pi \frac{(2b+m)x}{mx-2}.$$

(109.) In (104.) is proved the following theorem. If Ax be a solution of

$$A\alpha x = Ax \pm \text{wh. no.}$$

then

$$\xi(f\alpha f^{-1}x) = \theta \cos 2\pi A f^{-1}x.$$

Hence the determination of $\xi\psi x$ is reduced in all cases to the finding of a particular solution of

$$\psi x = f\alpha f^{-1}x, \text{ or } \psi fx = f\alpha x.$$

Where αx is one of the fundamental forms already noticed, or any other in which n can be found from the equation

$$\alpha^n x = \text{function}(n, x) = 0.$$

This differs from the theorem in (102.), inasmuch as it is not here necessary to determine $\psi^n x$.

(110.) We may moreover observe that the finding of the n th function by means of equations of differences may be more easily done by a similar method, and that, generally speaking, the circumstances which facilitate the integration of $u_{n+1} = \phi(u_n)$ are derived from the expressibility of ϕx in the form $f\alpha f^{-1}x$. For instance, in the case of $\phi x = 2x^2 - 1$ we have evidently $\phi x = \cos 2 \cos^{-1}x$, whence $\phi^n x = \cos 2^n \cos^{-1}x$, from which the expression for $\phi^n x$ in a preceding article may easily be deduced.

(111.) We shall now apply what has preceded to

$$a+bx+cx^2 = f\alpha f^{-1}x.$$

If we assume

$$\alpha x = g+hx^2 \quad fx = p+qx,$$

whence

$$f^{-1}x = \frac{x-p}{q},$$

we find that

$$a+bfx+cf(fx)^2 = f\alpha x$$

is satisfied by one relation only between g and h , namely,

$$gh = \frac{4ac - b^2 + 2b}{4}$$

if at the same time

$$p = -\frac{b}{2c} \quad q = \frac{h}{c}.$$

It therefore remains to ascertain $\alpha^n x$. If, as before, we assume $\alpha_n x = u_n$, we find

$$u_{n+1} = g + hu_n^2,$$

an equation which we are not aware has been or can be integrated, except in the case of

$$gh = 0, \text{ or } -2.$$

If we assume

$$g = 0 \quad h = 1,$$

we find

$$u_n = c^{2^n},$$

or

$$\alpha x = x^2 \quad \alpha^n x = x^{2^n};$$

from which we find that if

$$\phi x = a+bx+cx^2 \quad c = \frac{b^2-2b}{4a}$$

$$\alpha x = x^2 \quad fx = -\frac{b}{2c} + \frac{x}{c} \quad f^{-1}x = \frac{2cx+b}{2}$$

$$\phi x = f\alpha f^{-1}x$$

$$\phi^n x = f\alpha^n f^{-1}x = -\frac{b}{2c} + \frac{1}{c} \left\{ \frac{2cx+b}{2} \right\}^{2^n}$$

$$\xi \phi x = \theta \cos 2\pi \frac{\lambda^2 \{cx + \frac{1}{2}b\}}{\lambda 2}$$

If we assume

$$g = -2 \quad h = 1,$$

or

$$\alpha x = x^2 - 2 \quad u_{n+1} = u_n^2 - 2,$$

or

$$u_n = P^{2^n} + P^{-2^n},$$

where

$$P + \frac{1}{P} = x,$$

we find that if

$$b^2 - 2b + 8 - 4ac = 0$$

$$\phi x = a+bx+cx^2$$

$$\phi^n x = f \left\{ \left(\frac{f^{-1}x + \sqrt{(f^{-1}x)^2 - 4}}{2} \right)^{2^n} + \left(\frac{f^{-1}x - \sqrt{(f^{-1}x)^2 - 4}}{2} \right)^{2^n} \right\}$$

where

$$fx = -\frac{b}{2c} + \frac{x}{c}$$

$$\xi \phi x = \theta \cos 2\pi \frac{\lambda^2 (f^{-1}x + \sqrt{(f^{-1}x)^2 - 4})}{\lambda 2}$$

(112.) We can thus find either the n th function or the exponential inverse of any of the following forms:

$$\Theta(a+b\Theta^{-1}x)$$

$$\Theta(a(\Theta^{-1}x)^b)$$

$$\Theta\left(\frac{a+b\Theta^{-1}x}{a'+b'\Theta^{-1}x}\right),$$

where Θ is any function whatever. All periodic functions may be directly treated, as in (75.)

(113.) When the exponential inverse is given, the primitive function may be found from the equation

Calculus of
Functions.

Calculus of
Functions.

from which we may also find the expression for $\alpha^n x$, or

$$\alpha x = A^{-1}(Ax - 1),$$

$$\alpha^n x = A^{-1}(Ax - n).$$

(114.) The most general solution of the equation $\phi \psi x = \psi \phi x$ hitherto given is $\psi x = \phi^n x$, where n may be either whole or fractional, positive or negative. Thus $\alpha^m x$ and $\alpha^n x$ are necessarily convertible; and by this method a large class of convertible functions may be found. But in the case where αx is periodic, this method furnishes at first only a finite number of forms convertible with αx .

(115.) The preceding method is directly applicable to the investigation of convertible functions, since

$$\phi \alpha x = \alpha \phi x$$

is solved by eliminating n from the two equations

$$\alpha^n x = \mu \cos 2\pi n \quad \alpha^n y = \nu \cos 2\pi n,$$

in which, however, the present state of analysis will oblige us to assume either μ or ν constant forms.

Assuming $\mu \cos 2\pi n = \text{constant}$, we find

$$n = A(x, c),$$

where the arbitrary constant is explicitly mentioned for convenience. In the same way we find

$$A(y, \cos 2\pi n) = A(x, c),$$

or

$$A(y, \nu \cos 2\pi A x) = A(x, c),$$

whence y is what $A^{-1}Ax$ becomes, when in the second operation A^{-1} a different value of c is used, which is replaced by $\nu \cos 2\pi Ax$ or $\xi \alpha x$ after the completion of the operation. Thus to find those functions which

are convertible with x^p , where $Ax = \frac{\lambda^2 x}{\lambda p} + c$, if we invert this, writing c' for c , we have

$$A^{-1}x = \varepsilon^{p^{-c'-c}},$$

in which, substituting $\frac{\lambda^2 x}{\lambda p} + c$ for x , we get

$$x^{p^{-c'-c}},$$

in which, writing ξx^p for $c - c'$, we have

$$x^{p^{-\xi x^p}}$$

as the most general form which can be obtained of functions which are convertible with x^p . We may reduce this to the more simple form $x^{\xi x^p}$ or $(\xi x^p)^{\lambda^2}$.

(116.) A large number of convertible functions may be deduced as follows. Since $m x$ and $m' x$ are convertible, it follows (49.) that the following pairs are convertible,

$$\begin{array}{ll} \cos m \cos^{-1} x & \text{and} \quad \cos m' \cos^{-1} x \\ \sin m \sin^{-1} x & \text{and} \quad \sin m' \sin^{-1} x \\ \tan m \tan^{-1} x & \text{and} \quad \tan m' \tan^{-1} x, \\ & \&c. \qquad \qquad \&c. \end{array}$$

all of which are common algebraical expressions in a transcendental form. For since, for instance, $\cos m \phi x$ can be expanded where m is a whole number in the form (we take that which is best known)

$$\begin{aligned} & (\cos \phi x)^m - m \cdot \frac{m-1}{2} (\cos \phi x)^{m-2} (1 - \cos^2 \phi x) + \\ & m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} (\cos \phi x)^{m-4} (1 - \cos^2 \phi x)^2 - \&c. \end{aligned}$$

If for ϕx we write $\cos^{-1} x$, we have the expression

$$\begin{aligned} & x^m - m \cdot \frac{m-1}{2} x^{m-2} (1 - x^2) + \\ & m \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} x^{m-4} (1 - x^2)^2 - \&c., \end{aligned}$$

any two forms of which, for different values of m , are convertible. For instance, let $m = 2$ and $m = 3$ be two values of m ; then we have

$$2x^2 - 1 \text{ and } 4x^3 - 3x,$$

which are convertible functions, that is,

$$4(2x^2 - 1)^3 - 3(2x^2 - 1) = 2(4x^3 - 3x)^2 - 1.$$

(117.) We can also find, by an extension of the preceding method, an indefinite* number of solutions of the following:—What function of $x_1, x_2, x_3, \&c.$ is that which is changed in a given manner by changing x_1 into $\alpha_1 x_1, x_2$ into $\alpha_2 x_2, \&c.$? Find $\alpha_1^n x_1, \alpha_2^n x_2, \&c.$, and let $F_1, F_2, \&c.$ be given functions, and $\mu_1, \mu_2, \&c.$ arbitrary functions. Let y be the function required, and βy the change of form which is to take place. Eliminate $n_1, n_2, \&c.$, and m from as many of the following as may be necessary for the purpose,

$$\begin{aligned} F_1(\alpha_1^n x_1, \alpha_2^n x_2, \dots, \beta^n y) &= \mu_1 \{ \cos 2\pi n_1, \cos 2\pi n_2, \dots \} \\ F_2(\alpha_1^n x_1, \alpha_2^n x_2, \dots, \beta^n y) &= \mu_2 \{ \cos 2\pi n_1, \cos 2\pi n_2, \dots \} \\ &\&c. \qquad \qquad \&c. \end{aligned}$$

The result will be a relation between $y, x_1, x_2, \&c.$, which equally holds between $\beta y, \alpha_1 x_1, \alpha_2 x_2, \&c.$, and y , obtained in terms of $x_1, x_2, \&c.$, will be the function required.

(118.) But cases of the simplest nature will not give a result of any generality of form, considering the wide terms of the question. Suppose, for instance, we ask what is that function of x_1 and x_2 which is not changed by changing x_1 into $x_1 + 1$, when x_2 is changed into $2x_2$. We can eliminate n (in finite terms) from the equations

$$\begin{aligned} 2^{x_1+n} + 2^n x_2 &= \alpha y \\ 2^{x_1+n} &= 2^n x_2 + c, \end{aligned}$$

and we obtain

$$y = c \alpha^{-1} \left(\frac{2^{x_1} + x_2}{2^{x_1} - x_2} \right),$$

which satisfies the conditions: but though it admits of an infinite number of instances, it is not sufficiently general to give the smallest idea of what even, in the present state of analysis, must be considered as the most general solution of the problem.

(119.) And we may give the forms hitherto obtained for ξx an indefinite extension of generality. For if we consider the equation

$$\phi \alpha x = \phi x - 1 \dots (1.)$$

of which Ax is a particular solution, we find

$$\phi \alpha x - A \alpha x = \phi x - Ax,$$

or

$$\phi x - Ax = \text{any form of } \xi \alpha x$$

is also a solution of (1.) Hence for Ax we may write

$$Ax + \xi \alpha x,$$

or

$$\theta \cos 2\pi (Ax + \xi \alpha x)$$

* That is, supposing the n th functions of all the given functions can be expressed.

 Calculus of
Functions.

Calculus of Functions. is also a form of $\xi \alpha x$. Generalizing this process, we find that u_{Ax} is a form of $\xi \alpha x$ if it satisfies the following equation,

$$u_{n+1} = \theta \cos 2\pi (n + u_n),$$

where θ is any function which does not invert the cosine, and where $u_0 = \theta \cos 2\pi A x$.

(120.) The next step in the process of solving functional equations is to find a function which shall not change when either of αx , βx , γx , &c. is substituted for x . We must content ourselves here with showing that we want a process analogous to one of ordinary Algebra, which, if it had not been found, would have left that Science as full of difficulties as is the Calculus of Functions at present.

Let us consider the very simple case of finding a function which shall not be changed either by substitution of $x + 1$ or of x^2 , instead of x . Since then we have

$$\phi(x + 1) = \phi x \quad \phi(x^2) = \phi(x);$$

we have also

$$\phi(x^2 + 1) = \phi x^2 = \phi x$$

$$\phi(x + 1^2 + 1) = \phi x + 1^2 = \phi(x + 1) = \phi x$$

$$\phi(x^2 + 1^2 + 1) = \phi x, \text{ \&c. \&c. \&c.};$$

or generally, if

$$\alpha x = x + 1 \quad \beta x = x^2,$$

$$\phi(\alpha^n \beta^n \alpha^n \beta^n \dots) x = \phi x.$$

If a particular value of ϕ could then be found, say ϕ_1 , in which case $\theta \phi_1$ would be a general value, we should have in one expression a general value of $\xi \alpha x$ for an infinite number of different forms of α , for no one of which can the most limited particular solution be found by our most general previous methods. If this case could then be solved, it would become the most unfailing instrument of removing the difficulties of the preceding; but the converse is not true. For instance, the finding of $\xi(a + bx + cx^2)$ in all cases would be reduced to finding that function which is not changed when

x is changed into $a + bx$ and into $1 + \frac{c}{b}x$.

(121.) In Algebra, the difficulty of resolving such an equation as $\alpha \beta x = \gamma x$ is overcome by developing $\alpha \beta$ and γ in powers of x , that is, changing all functions into functions of one general form. But in the Calculus of Functions there is as yet no theorem by which any function may be expressed in terms of the successive functions of another function, or of functions in any other way related to it.

(122.) The only cases in which the preceding question can now be solved are where the given functions are among the successive functions of some other function. For if $\xi \alpha x = \xi x$, we have $\xi \alpha^2 x = \xi \alpha x = \xi x$, &c., and the same may be extended to algebraical and, under limitation, to functional inverses.

(123.) We shall take this opportunity to lay before the student some idea of the only route by which (if experience be a guide) any of the difficulties of this subject can be removed. The mathematical term *transcendental* is often used, but is seldom specifically defined. The following will be a convenient definition, and will include all the various senses in which the word has been used.

One function is said to be transcendental with respect to another when the second cannot be expressed by any finite number of the operations which are presumed

as granted in the formation of the first. Thus, taking Calculus of Functions. algebraical operations only, if we begin with the addition and subtraction of whole numbers, then multiplications and the raising of powers, in finite numbers, are not transcendental, but divisions, generally speaking, are transcendental, though particular cases may not be so. Thus the 7th part of 1 is a transcendental with respect to addition, subtraction, and multiplication. When we grant a new fundamental operation, we use a new

symbol in all cases; here it is $\frac{1}{7}$. Observe, also, that

this is a transcendental of a peculiar species, which cannot be expressed, even approximately, in terms of additions, subtractions, and multiplications. But if one transcendental of the kind be given, say $\frac{1}{10}$, and the operations of

addition, subtraction, and multiplication be applied to it, we can then express all other transcendentals of the same nature, either directly or approximately. Still

$\left(\frac{1}{10} \text{ and its powers only being granted}\right) \frac{1}{7}$ is a transcendental with respect to whole numbers and $\frac{1}{10}$, the

operations granted being addition, subtraction, multiplication by any whole numbers, and division by 10.

Granting any division, we next come to the operation $\sqrt[n]{a}$, which is generally transcendental,* but not always so. Similarly, the logarithm is a new transcendental, the sine another, if we consider only possible functions, and so on.

(124.) That all transcendentals arise out of inverse operations, springs simply from this, that in selecting operations we are naturally led to choose those combinations of preceding operations which are evidently always possible. But though this may generally be the method, it by no means follows that every subject will admit of it. And it is matter of convenience, sometimes of accident, whether we shall first take that operation which it will ultimately be proper to call direct,† or its inverse. When Napier invented his logarithms, he found $\psi^{-1}x$ where $\psi x = a^x$; when Archimedes found the area of the parabola, he solved a case of integration, which is transcendental with respect to all the curves he considered, instead of differentiation, which, it so happened, was transcendental to none. The ingenuity by which the difficulties of an inverse operation were thus overcome, independently of the direct one, is a principal part of the merit of the two great men alluded to.

(125.) The Calculus of Functions is essentially an inverse calculus, or rather we have presented first the inverse part. We have avoided mentioning the direct calculus in the statement of the conditions under which

* In the following succession, each operation after the third is transcendental with respect to the aggregate of those which precede. Addition, subtraction, multiplication of whole numbers, division, extraction of n th root, formation of logarithm, sine, &c.

† This, however, is true only of particular inverses, absolutely considered. For if the operation $\sqrt[n]{a}$ had preceded a^x it would have been afterwards found convenient to consider the latter as direct, and the former as one of its inverses. According to the received ideas of these terms, the following question is direct: "To what town does this road lead?" and the following inverse: "In how many and what ways can I go to that town?"

Calculus of Functions. it proposes to solve questions. This was done to avoid formally introducing a branch of the subject which is sufficiently considered in ordinary Algebra. But to show the inverse nature of the functional calculus, and thence the high probability of its requiring new transcendentials, we shall now take one or two instances of the direct calculus.

(126.) The general question is, given a number of functions, required the functional relations which exist between any of them and the rest. To take a simple case, suppose we ask what functional relations exist between $x + c$ and $c x^2$. If we call the first αx and the second βx , we ask, what equations of the form

$$F(x, \alpha x, \beta x) = 0$$

must be identically true when $\alpha x = x + c$ and $\beta x = c x^2$. This, it is evident, can be solved in an infinite number of ways; we may, therefore, add some condition or conditions to the general question.

1. Suppose the constant is to be eliminated; we have then

$$\beta x = (\alpha x - x) x^2.$$

2. Suppose x is to be eliminated, except as the subject of a function; then

$$\beta x = c (\alpha x - c)^2.$$

3. Suppose both the former cases are to be included in one. We may do this in an infinite number of ways, for instance,

$$\alpha \beta x = c x^2 + c \quad \beta x = c x^2$$

$$c = \alpha \beta x - \beta x.$$

Let γx be any function of x . The same reasoning gives

$$c = \alpha \beta \gamma x - \beta \gamma x;$$

therefore

$$\alpha \beta \gamma x - \alpha \beta x = \beta \gamma x - \beta x.$$

Or we might differentiate the first value of c with respect to x , which would give

$$\alpha' \beta x = 1 \text{ where } \alpha' x = \frac{d \alpha x}{d x}.$$

4. We may express one function by expressing its n th function in terms of the n th function of the other, and then eliminating n , except where it appears as an index. For instance,

$$\beta^n x = c^{n-1} x^{2^n} \quad \alpha^n x = x + n,$$

whence

$$\alpha^n x - x = \frac{1}{\lambda 2} \{ \lambda^2 c \beta^n x - \lambda^2 c x \}.$$

(127.) Every question of the Calculus of Functions is an inverse to some such question as one of the preceding, and, by experience, we must look for new transcendentials, which is confirmed by our having already found new modifications of the old ones, such as $\cos \lambda x$, &c., or $\frac{1}{2} (x^{\sqrt{-1}} + x^{-\sqrt{-1}})$, and these only in very limited solutions. We have seen that no question, however simple, can be completely solved without the entrance of a transcendental, which is of the following form:

$$f(x, \cos x, \cos \cos x, \cos \cos \cos x, \dots).$$

That is, considering $\cos^n x$ as a separate transcendental for every different value of n , we have an infinite number of transcendentials thus introduced. And even these are only one particular set; for (102.) every different

form of μ in the equation $\alpha^n x = \mu \cos 2 n \pi$ may introduce a different set.

(128.) The difficulties thus introduced by new transcendentials it is useless to attempt to meet by reducing them to the old and well-known forms. The mathematical reader is aware how the attempts to integrate elliptic functions, exponentials of the form $e^{-t^2} dt$, &c., have settled down, so to speak, into the consideration of these quantities as primary functions, in the same manner as λx , $\cos x$, &c., to be calculated and tabulated as the starting points of new branches of analysis. It is most desirable that a similar course should be pursued with respect to other functions.

(129.) The periodic functions have hitherto appeared to stand in some sort of relation to others, similar to that of rational and irrational expressions in Algebra. But we may also find a class of solutions with regard to periodic functions, similar to those which we have attained for others; and we may then show that the new solutions so attained are not really different in form from the results of (87.), obtained by reference to the fundamental property of periodicity. Let ϕx be a periodic function of the second order, then we have

$$\phi^n x = x \cdot \frac{1 + (-1)^n}{2} + \phi x \frac{1 - (-1)^n}{2}.$$

Hence, by equating this to zero, and finding n , we have an exponential inverse of ϕx , namely,

$$\lambda \cdot \left(\frac{\phi x + x}{\phi x - x} \right)^{\frac{1}{\lambda(-1)}},$$

and

$$\xi \phi x = \theta \cdot \cos 2 \pi \lambda \cdot \left(\frac{\phi x + x}{\phi x - x} \right)^{\frac{1}{\pi \sqrt{-1}}},$$

which, by the equation

$$\cos 2 \pi \lambda \psi x = (\psi x)^{2 \pi \sqrt{-1}} + (\psi x)^{-2 \pi \sqrt{-1}},$$

gives

$$\xi \phi x = \theta \left\{ \left(\frac{\phi x + x}{\phi x - x} \right)^2 + \left(\frac{\phi x - x}{\phi x + x} \right)^2 \right\},$$

which, since

$$(\phi x - x)^2 \text{ or } \overline{\phi x + x}^2 - 4 x \phi x,$$

is an unambiguous* function of $\phi x + x$ and $x \phi x$; that is, the preceding is simply equivalent to any function of $\{x + \phi x, x \phi x\}$, or (81.) a symmetrical function of $(x, \phi x)$.

(130.) We may exhibit a form analogous to that already obtained for periodic functions of higher orders, as follows. The function

$$\chi x = x + r \phi x + r^2 \phi^2 x + \dots + r^{n-1} \phi^{n-1} x,$$

where $\phi^n x = x$ and $r^n = 1$, is such that the substitution of ϕx for x is the same thing as multiplication by r^{n-1} , or

$$\chi \phi x = r^{n-1} \chi x,$$

whence an exponential inverse of ϕx is

$$\frac{\lambda \chi x}{(n-1) \lambda r} \text{ or } \frac{\lambda (x + r \phi x + r^2 \phi^2 x + \dots + r^{n-1} \phi^{n-1} x)}{(n-1) \lambda r},$$

or

$$\xi \phi x = \theta \cos 2 \pi \lambda \{ x + r \phi x + \dots + r^{n-1} \phi^{n-1} x \}^{\frac{1}{(n-1) \lambda r}},$$

where $p^{\frac{1}{n-1}}$ means the arithmetical $(n-1)$ th root only.

* θ may be what we please, and therefore is unambiguous. See (81.)

Calculus of Functions. The subject of $\theta \cos 2\pi\lambda$ in the preceding may be reduced as before, in such a way that the whole shall merely imply an arbitrary function of

$$(x + r\phi x + r^2\phi^2 x + \dots + r^{n-1}\phi^{n-1}x)^n,$$

which we might therefore suppose must be a symmetrical function of $x, \phi x, \&c.$ And it is so, because, being of the form $(\chi x)^n$, where an interchange of x and $\phi^m x$ gives

$$\chi \phi^m x = \chi x \times r^{\text{wh. no.}},$$

we have

$$(\chi \phi^m x)^n = \chi x \times r^{\text{mult. of } n} = \chi x.$$

As it is indifferent what root of 1 is taken, we may, when n is an even number, take -1 , which shows that

$$(x - \phi x + \phi^2 x - \dots - \phi^{2n-1}x)^{2n}$$

is a symmetrical function of $x, \phi x, \&c.$

(131.) There is no general method of finding a particular solution of a functional equation, which (101.) is necessary as yet to its complete solution, and (98.) may be suspected to be essential to it. But there is an indefinite number of ways in which the solution of one equation may be made to depend upon that of another, either with or without a particular solution. We shall first take cases in which no particular solution has been found.

(132.) Any function of x may be substituted instead of x throughout the equation. If, for instance, we have

$$\phi \alpha x = \beta x \phi x + \gamma x,$$

where α, β , and γ are known functions, and ϕ is to be found, write κx instead of x , which gives

$$\phi \alpha \kappa x = \beta \kappa x \phi \kappa x + \gamma \kappa x:$$

every solution of this equation is a solution of the original.

1. If $\kappa x = \gamma^{-1}x$, the equation is reduced (writing α for $\alpha\gamma^{-1}$, &c.) to the form

$$\phi \alpha x = \beta x \phi \gamma^{-1}x + x;$$

or, writing ψ for $\phi \gamma^{-1}$, to

$$\psi \gamma \alpha x = \beta x \cdot \psi x + x,$$

which is of the form

$$\psi \alpha x = \beta x \psi x + x.$$

2. If for $\phi \kappa$ we write ψ , we have

$$\psi (\kappa^{-1} \alpha \kappa) x = \beta \kappa x \psi x + \gamma \kappa x.$$

In which, by taking a proper form for κ , we may make either $\beta \kappa x$ or $\gamma \kappa x$ what we please; so that the equation

$$\phi \alpha x = \beta x \phi x + \gamma x$$

can be solved independently of γx , if it can be solved independently of βx , or *vice versa*.

3. Let $\phi x = \mu x \psi x + \nu x$, in which case the equation becomes

$$\mu \alpha x \psi \alpha x + \nu \alpha x = \beta x \mu x \psi x + \beta x \nu x + \gamma x,$$

or

$$\psi \alpha x = \frac{\beta x \mu x}{\mu \alpha x} \psi x + \frac{\beta x \nu x + \gamma x - \nu \alpha x}{\mu \alpha x}.$$

If then νx be a particular solution of the equation itself, and μx a particular solution of

$$\chi \alpha x = \beta x \chi x,$$

the equation is reduced to $\psi \alpha x = \psi x$, or ψx is $\xi \alpha x$; whence, in order to solve

$$\phi \alpha x = \beta x \phi x + \gamma x,$$

we must have particular solutions: 1. of the equation itself; 2. of the equation deprived of its known term; 3. the function which does not change when αx is substituted for x .

(133.) There cannot be two particular solutions of the preceding equation, independent of the form of $\xi \alpha x$. For let $\phi_1 x$ and $\phi_2 x$ be two particular values of ϕ ; then we have

$$\phi_1 \alpha x = \beta x \phi_1 x + \gamma x$$

$$\phi_2 \alpha x = \beta x \phi_2 x + \gamma x$$

$$\phi_1 \alpha x - \phi_2 \alpha x = \beta x (\phi_1 x - \phi_2 x).$$

Consequently, $\phi_1 x - \phi_2 x$ is a particular solution of

$$\chi \alpha x = \beta x \chi x.$$

Let

$$\phi_1 x - \phi_2 x = \chi_1 x,$$

then

$$\frac{\chi \alpha x}{\chi_1 \alpha x} = \frac{\chi x}{\chi_1 x},$$

or $\chi x \div \chi_1 x$ is a particular solution of

$$\xi \alpha x = \xi x;$$

or χx must be of the form

$$(\phi_1 x - \phi_2 x) \xi \alpha x.$$

Take another particular solution, $\phi_3 x$; hence another particular value of χx is $(\phi_3 x - \phi_1 x) \xi_1 \alpha x$, and since

$\frac{\chi_1 x}{\chi_2 x}$ must be a form of $\xi \alpha x$, it follows that

$$\frac{\phi_3 x - \phi_1 x}{\phi_2 x - \phi_1 x}$$

must always be of the form $\xi \alpha x$. Consequently, all the forms $\phi_1, \phi_2, \&c.$ must be included in

$$M \xi \alpha x + N,$$

and obtained by giving different forms to ξ , without changing M and N .

(134.) This remark is important, because it seems to show (as we shall hereafter do more fully) that *some certain* solution of an algebraical character is the *only* such, independent of the transcendentals peculiar to this subject; except only in the case of periodic functions, in which the transcendentals alluded to take algebraical forms. We say *some certain* solutions, because it does not follow that any one we may happen to light upon is the most general of the algebraical forms. This may be tried by following the preceding process, and introducing an arbitrary constant in place of $\xi \alpha x$, which, it is plain, satisfies the conditions. For instance, let

$$\phi(x^2) = \left(x + \frac{1}{x}\right) \phi x - 1,$$

which is evidently satisfied by $\phi_1 x = x$. Consequently, $\phi x - x$ must be a particular solution of

$$\chi(x^2) = \left(x + \frac{1}{x}\right) \chi x,$$

of which $c\left(x - \frac{1}{x}\right)$ is a solution, and every other solution must be of the form $\left(x - \frac{1}{x}\right) \xi(x^2)$. Consequently

$$\phi x = x + \left(x - \frac{1}{x}\right) \xi(x^2),$$

and the most general algebraical form is

Calculus of
Functions.

$$x + c \left(x - \frac{1}{x} \right).$$

(135.) In the same way we can reduce the solution of

$$a_x \phi \alpha x + b_x \phi \beta x + \dots + m_x = 0,$$

where $a_x, b_x, \&c.$ are given functions of x , to the finding a particular solution, and the solution of the equation deprived of its last term. For if ϕ, x be a particular solution, then $\phi x - \phi, x$ is a solution of the equation deprived of its last term: consequently, so many distinct algebraical solutions as we can find of the curtailed equation, so many terms can we give the general value of ϕx . If $p_x, q_x, r_x, \&c.$ be particular algebraical solutions, then

$$\phi x = m_1 p_x + m_2 q_x + \dots$$

where $m_1, m_2, \&c.$ may be constants, which gives the general algebraical form, or forms* of

$$\xi \alpha x \beta x \dots,$$

which gives the most general form.

(136.) In the case where $\alpha, \beta, \&c.$ are successive functions of a periodic function, a particular solution may generally be obtained. Suppose, for instance,

$$\alpha x = \frac{1}{x} \quad \alpha^2 x = x$$

$$\phi \alpha x = (x+1) \phi x + x^2;$$

for x write αx , which gives

$$\phi x = \frac{x+1}{x} \phi \alpha x + \frac{1}{x^2},$$

or

$$\phi x = \frac{(x+1)^2}{x} \phi x + (x+1)x + \frac{1}{x^2}$$

$$\phi x = \frac{x^4 + x^3 + 1}{x^2 - x(x+1)^2}.$$

This is a particular solution only, and we must now look for a particular solution of

$$\phi \alpha x = (x+1) \phi x.$$

This equation admits of no continuous solution, for by substituting αx for x , we find

$$\phi x = \left(\frac{1}{x} + 1 \right) \phi \alpha x,$$

or

$$\left(\frac{1}{x} + 1 \right) (x+1) = 1,$$

which is not true. But from (70.) it may admit of discontinuous solutions.

(137.) To examine this further, let us take

$$\phi \alpha x = \beta x \phi x + \gamma x \quad \alpha^2 x = x$$

$$\phi x = \beta \alpha x \phi \alpha x + \gamma \alpha x$$

$$\phi x = \frac{\gamma \alpha x + \gamma x \beta \alpha x}{1 - \beta x \beta \alpha x}.$$

But

$$\phi \alpha x = \beta x \phi x$$

cannot, as before, be solved, unless

$$\beta x \beta \alpha x = 1,$$

in which case the preceding value of ϕx becomes infinite, unless also

$$\gamma \alpha x + \gamma x \beta \alpha x = 0.$$

* Meaning a function which does not change when $\alpha x, \beta x, \&c.$ are substituted for x .

(138.) First, let

$$\beta \alpha x \beta x = 1,$$

then

$$\phi \alpha x - \beta x \phi x = \gamma x;$$

which, writing αx for x , becomes

$$\phi x - \beta \alpha x \phi \alpha x = \gamma \alpha x;$$

by which, and $\beta \alpha x = \frac{1}{\beta x}$, we find

$$\gamma x + \gamma \alpha x \beta x = 0$$

which is the same as

$$\gamma \alpha x + \gamma x \beta \alpha x = 0.$$

Consequently, the proposed equation can have no continuous solution, unless this second condition be also satisfied. But from the condition $\beta x \beta \alpha x = 1$, the preceding may be thus written,

$$\frac{\gamma \alpha x}{\sqrt{\beta \alpha x}} + \frac{\gamma x}{\sqrt{\beta x}} =$$

of which a solution is

$$\gamma x = \sqrt{\beta x} \theta(x - \alpha x) \times \text{symm. func.}(x, \alpha x),$$

where θx is any function which changes sign when x is changed into $-x$. Substituting this in the original equation, we obtain

$$\phi \alpha x = \beta x \phi x + \sqrt{\beta x} \theta(x - \alpha x) \xi \alpha x,$$

or

$$\sqrt{\beta \alpha x} \phi \alpha x = \sqrt{\beta x} \phi x + \theta(x - \alpha x) \xi \alpha x,$$

which gives

$$\phi x = - \frac{\xi \alpha x \theta(x - \alpha x)}{\sqrt{\beta x}} \frac{\theta_1 x}{\theta_1 x + \theta \alpha x},$$

where $\theta(-x) = -\theta x$ and θ_1 is any function whatever. The form of β must be $\beta x = \theta_2 \alpha x \div \theta_2 x$, where θ_2 is any function whatever. We shall hereafter return to this subject in a more general form. See the Index at the end of this Treatise.

(139.) If we proceed in exactly the same way with

$$\phi \alpha x + \beta x \phi x = \gamma x \quad \alpha^2 x = x,$$

we find the necessary conditions of continuous solution (with an arbitrary function) to be

$$\beta x \beta \alpha x = 1 \quad \gamma \alpha x - \beta \alpha x \gamma x = 0,$$

and the solution is

$$\phi x = \frac{\xi \alpha x}{\sqrt{\beta x}} \frac{\theta_1 x}{\theta_1 x + \theta_1 \alpha x}.$$

In which the student must remember that changing βx into $-\beta x$ changes the form of γx , which will admit of continuous solution, and therefore he is not to attempt to deduce one solution from the other. Take the two equations

$$\left. \begin{aligned} \psi \alpha x + \beta x \psi x &= \gamma x \\ \phi \alpha x - \beta x \phi x &= \gamma x \end{aligned} \right\} \alpha^2 x = x \dots (A)$$

Eliminate γx , determine βx from the result, and then write αx for x ; the equation $\beta x \beta \alpha x = 1$ will then give

$$\frac{\phi \alpha x}{\psi \alpha x} = - \frac{\phi x}{\psi x},$$

which answers to the difference of form already obtained. This is independent of γx , but applying the last condition to the equations, we find

$$\gamma x (\phi \alpha x - \psi \alpha x) = 0,$$

 Calculus of
Functions.

Calculus of Functions. which can only be satisfied by $\gamma x = 0$. So that the two equations (A) cannot both have continuous solutions, unless $\gamma x = 0$.

(140.) The discontinuous solutions of the preceding equation would arise from the application of a general method (if one existed) for the solution of the case where αx is not periodic. We shall now proceed to some simple cases of the latter. Let

$$\phi \alpha x = c \phi x + c',$$

where c and c' are given constants. A particular solu-

tion of this is $\frac{c'}{1-c}$, and a particular solution of $\phi \alpha x =$

$c \phi x$ is $c^{-\Lambda x}$ where Λx is the exponential inverse of αx , which satisfies $\Lambda \alpha x = \Lambda x - 1$; hence a general solution of the given equation is

$$\phi x = \frac{c'}{1-c} + c^{-\Lambda x} \xi \alpha x.$$

Whence the following instances in which the second function given is $c^{-\Lambda x}$, or the multiplier of $\xi \alpha x$ in the general solution of $\phi \alpha x = c \phi x$, that of the original equation being made by addition of $\frac{c'}{1-c}$.

αx	Particular solution of $\phi \alpha x = c \phi x$
$a + bx$	$(a + bx - x)^{\frac{\lambda c}{\lambda b}}$
$a + x$	$c^{\frac{x}{a}}$
bx	$x^{\frac{\lambda c}{\lambda b}}$
ax^b	$\{\lambda (ax^{b-1})\}^{\frac{\lambda c}{\lambda b}}$
x^b	$(\lambda x)^{\frac{\lambda c}{\lambda b}}$
$\frac{a+bx}{a'+b'x}$	$\left(\frac{a+bx-r'x}{a+bx-rx}\right)^{\frac{\lambda c}{\lambda r-\lambda r'}}$ (108.) r and r' unequal
$\frac{a+bx}{a'+b'x}$	$c^{\frac{rx}{a+b x-rx}}$ r and r' equal.

Solutions might also be obtained for all other cases in which αx is a known derivative of one of the preceding.

(141.) Let $\phi \alpha x = \phi x + c'$. The general solution is $\phi x = -c' \Lambda x + \xi \alpha x$.

(142.) Some additional forms may be derived from the preceding by differentiation. It is evident that the solution of $\phi \alpha x = c \phi x$ gives a solution of

$$\alpha' x \psi \alpha x = c \psi x \quad \alpha' x = \frac{d \alpha x}{d x},$$

in which $\psi x = \phi' x$. From this it follows, that if ϕx be a solution of $\phi \alpha x = c \phi x$, $(\phi' x)^m$ is a solution of

$$\psi \alpha x = \left(\frac{c}{\alpha' x}\right)^m \psi x.$$

Hence we find the solution of

$$\phi(x^b) = p x^m \phi x$$

to be

$$\phi x = x^{\frac{m}{b-1}} (\lambda x)^{\frac{\lambda p}{\lambda b}} \xi(x^b);$$

and thus also may be solved the equation

$$\phi\left(\frac{a+bx}{a'+b'x}\right) = (a'+b'x)^m \phi x.$$

(143.) The cases which we cannot succeed in solving generally are very often marked by this peculiarity, that several particular solutions can be obtained, but none containing an arbitrary constant. Thus $\phi^n x = x^p$ has

the solution $\phi x = x^{\frac{1}{p^n}}$, where, r being a primitive n th root of unity, we may write for $p^{\frac{1}{n}}$ either of $\sqrt[n]{p}$, $r \sqrt[n]{p}$, $r^2 \sqrt[n]{p}$... up to $r^{n-1} \sqrt[n]{p}$. The following equation,

$$\phi^n x = \frac{d^n \phi x}{d x^n},$$

admits of a solution of the form $c' x^c$, in which c is a root of the equation

$$c^n - c + m = 0,$$

and

$$c' = \{c \cdot c - 1 \cdot c - 2 \cdot \dots \cdot c - m + 1\}^{\frac{1-c}{m}},$$

which gives $m n$ particular solutions, but contains no arbitrary constant. And, in a similar way, particular solutions may be found to

$$\frac{d^n \phi^b x}{d x^n} = \frac{d^{a'} \phi^{b'} x}{d x^{a'}},$$

all of the form $c' x^c$.

(144.) Many cases occur in which a particular solution may be obtained; but we shall rather proceed to consider such as more immediately depend upon the preceding part of this work. Unless we can combine our various cases by some method of connection, there is little use in considering them at present.

The nature of compound functions is of the first importance. What connection exists between $\phi \psi x$ and $\psi \phi x$? Till something more than we shall be able to exhibit is made known upon this point, the Calculus of Functions will be in much the same position as Arithmetic without the knowledge of the relation which exists between b times a and a times b .

We may call the preceding operations (namely, such as $\phi \psi x$, $\phi \psi \chi x$, &c.) *compound functions* of the second, third, &c. orders, and ϕ , ψ , &c. their *functional factors*. And $\phi \psi$ and $\psi \phi$ may be distinguished by calling the first the *converse* of the second, a term which must be carefully distinguished from the *inverse*. The latter term it might be more logical to call the *contrary*, but custom has settled its present meaning.

(145.) Given the results of a compound operation of the second order, and of its converse, required the operations themselves. That is, to solve the equations

$$\phi \psi x = \alpha x \quad \psi \phi x = \beta x,$$

which give

$$\alpha x = \phi \beta \phi^{-1} x \quad \text{and} \quad \beta x = \psi \alpha \psi^{-1} x;$$

and therefore ϕ and ψ are respectively the functions by which α is a derivative of β , and β of α . From the method of determining these functions, given in (50.), it appears that ϕ and ψ are in form inverse to each other. But as they cannot be real inverses of each other, since then $\phi \psi x$ and $\psi \phi x$ would be respectively $= x$, the forms only must be preserved, and different arbitrary constants or functions taken. But this will require that one relation should exist between those two arbitrary constants or functions. For instance, let us propose

$$\phi \psi x = \cos a \cos^{-1} x \quad \psi \phi x = \cos b \cos^{-1} x;$$

the method requires that we should eliminate n between $\cos a^n \cos^{-1} x = c$ and $\cos b^n \cos^{-1} y = c'$,

Calculus of which gives
Functions.

$$\begin{aligned} y &= \cos \left\{ C (\cos^{-1} x)^{\frac{\lambda b}{\lambda a}} \right\} \\ \psi x &= \cos \left\{ C (\cos^{-1} x)^{\frac{\lambda b}{\lambda a}} \right\} \quad \phi x = \cos \left(C' (\cos^{-1} x)^{\frac{\lambda a}{\lambda b}} \right) \\ \psi \phi x &= \cos (C C'^{\frac{\lambda b}{\lambda a}} \cos^{-1} x) \\ \phi \psi x &= \cos (C' C^{\frac{\lambda a}{\lambda b}} \cos^{-1} x) \end{aligned}$$

which satisfy the original conditions, if

$$C C'^{\frac{\lambda b}{\lambda a}} = b, \text{ which gives } C' C^{\frac{\lambda a}{\lambda b}} = a.$$

(146.) But if α and β be corresponding derivatives of two other functions, as in the preceding case, the question may be reduced as follows. Let it be required to solve

$$\phi \psi x = \Theta \alpha \Theta^{-1} x \quad \psi \phi x = \Theta \beta \Theta^{-1} x,$$

and let $\phi_1 x$ and $\psi_1 x$ be solutions of

$$\phi_1 \psi_1 x = \alpha x \quad \psi_1 \phi_1 x = \beta x;$$

then assume

$$\phi x = \Theta \phi_1 \Theta^{-1} x \quad \psi x = \Theta \psi_1 \Theta^{-1} x.$$

(147.) If we apply the preceding conditions to the solution of

$$\phi \psi x = a x \quad \psi \phi x = x^b,$$

we find

$$\phi x = a c^{-\frac{\lambda a}{\lambda b}} (\lambda x)^{\frac{\lambda a}{\lambda b}} \quad \psi x = e^{c x^{\frac{\lambda b}{\lambda a}}},$$

consequently to solve

$$\phi \psi x = a(x+m) - m \quad \psi \phi x = (x+m)^b - m,$$

we have

$$\phi x = a c^{-\frac{\lambda a}{\lambda b}} (\lambda x + m)^{\frac{\lambda a}{\lambda b}} - m \quad \psi x = e^{c(x+m)^{\frac{\lambda b}{\lambda a}}} - m.$$

(148.) The following question is a particular case of the last, in the same manner as that in (102.) is a particular case of that in (50.) Required a function, of which one of the inconvertible inverses shall be such that $\phi_{-1} \phi x$ is a given function of x . For instance, suppose

$$\phi \psi x = x \quad \psi \phi x = x^b,$$

in which, by the first equation, ψ is an inverse of ϕ ; and, by the second equation, not a convertible inverse. If we eliminate n between the equations

$$y = \cos 2\pi n \quad x^n = c,$$

and pursue the preceding method, we find

$$\phi x = \cos 2\pi \left(\frac{\lambda^2 c' - \lambda^2 x}{\lambda b} \right)$$

$$\psi x = c^{\frac{1}{2\pi} \cos^{-1} x},$$

and the condition $c = c^b$ makes

$$\phi \psi x = x \text{ and } \psi \phi x = x^{\frac{\lambda c}{\lambda c'}},$$

which is a very limited particular case of the problem in question.

(149.) By the same method convertible functions may be found, but not in a more general form than that already given in (115.)

(150.) Required a function which shall be inverted with respect to one variable, by an interchange of two others. Of this a particular case is $x + y = z$, the inverse of which with respect to x is $x + z = y$.

Take any two similar functions of z and y , but which

have the same form with respect to x . Calling $\alpha_x x$ and $\alpha_y x$ these* functions, solve

$$\phi \psi x = \alpha_x x \quad \phi \psi x = \alpha_y x;$$

then while by the method of solution ψx is (the constant excepted) an inverse of ϕx , it is evident that, in the equations themselves, an interchange of z and y , accompanied by an interchange of ϕ and ψ , makes no change in the conditions of the problem. But the relation between the constants must be determined according to the particular forms of each case.

For instance, if we assume

$$\alpha_x x = \cos z \cos^{-1} x \quad \alpha_y x = \cos y \cos^{-1} x,$$

we find the function in question to be

$$\cos \left\{ (\cos^{-1} x)^{\frac{\lambda y}{\lambda z}} \right\},$$

and it may easily be shown that the derivatives of functions which possess this property also have the same; so that

$$\phi(\phi^{-1} x + y - z) \quad \phi\left(\frac{y}{z} \phi^{-1} x\right) \quad \phi\left((\phi^{-1} x)^{\frac{y}{z}}\right)$$

are all functions in which y and z are interchanged by inversion with respect to x .

(151.) Converse functions are simple derivatives of each other. Thus $\alpha \beta$ is $\alpha \beta \alpha^{-1}$ or $\alpha(\beta \alpha) \alpha^{-1}$. Hence converse functions are periodical together, and the n th function of either is directly ascertained from that of the other. And since any inconvertible inverse α_{-1} gives $\alpha \alpha_{-1} x = x$, we have $\alpha \alpha_{-1} \alpha \alpha^{-1} x = x$, or $\alpha_{-1} \alpha x$ being βx , we find that αx is a value of $\xi \beta x$. This is evident in the case chosen in (148.)

(152.) We now turn our attention to the general equation

$$\phi^n x = \alpha x \dots (1),$$

the solution of which is the general expression for

$\alpha^n x$, or the n th functional inverse of x . It would appear at first sight that $\phi^n x = x$ ought to be considered as a particular case of this, and undoubtedly it will be so considered when the solutions of functional equations are as general as the ideas implied in the equations themselves. But in the mean while the following argument will show that a peculiarity of x as distinguished from αx makes every solution which we can obtain of (1) appear much less extensive than those of $\phi^n x = x$.

From the preceding equation it follows that

$$\phi^n(\phi x) \text{ being } = \phi(\phi^n x) \quad \therefore \phi \alpha x = \alpha \phi x,$$

or ϕ and α must be convertible. Now when $\alpha x = x$, it is convertible with every function of x ; but when αx is any other function of x , however simple, it is convertible only with a very inconsiderable variety of forms, and those not all assignable at present. Now though $\phi^n x = x$ is a case of $\phi^n x = \alpha x$, we cannot for any useful purpose consider them together, until we have some idea how to fill up the void, if we may so speak, between αx and x , so as to assign functions continually approaching nearer and nearer to x in the extensiveness of the forms with which they are convertible. See (9.) and (27.)

(153.) One solution of $\phi^n x = \alpha x$ arises from the substitution of $\frac{1}{n}$ instead of n in the expression for $\alpha^n x$. But

* We here adopt a combination of the notation α_x and αx mentioned in (14.), to signify that while $\alpha_x x$ and $\alpha_y x$ are cognate functions of x , they are similar functions of z and y . Here x is the subject to which the stated property is referred.

Calculus of
Functions.in like manner as we found various forms of $\phi^{-1}x$ in (33.),by writing $\phi x = \alpha x$ in the form $\phi x = \alpha x \cdot e^{2k\pi\sqrt{-1}}$, k being any positive or negative whole number, so we may find many forms (whether all or not we cannot say)of $\frac{1}{\alpha^n}x$, by substituting for αx the form $\alpha x \times e^{2k\pi\sqrt{-1}}$. For instance, let us suppose $\alpha x = x^a$, obvious n th functional inverses of which are the n forms of $x^{\frac{1}{a^n}}$. But suppose

$$\phi^n x = x^a e^{2k\pi\sqrt{-1}}.$$

Then, remembering that if

$$\beta x = p x^q, \quad \beta^n x = p^{\frac{n}{q-1}} x^{q^n},$$

we find

$$\phi x = e^{\frac{1}{a^n-1} 2k\pi\sqrt{-1}} x^{\frac{1}{a^n}}$$

$$= \left\{ \cos\left(\frac{1}{a^n-1} 2k\pi\right) + \sqrt{-1} \sin\left(\frac{1}{a^n-1} 2k\pi\right) \right\} x^{\frac{1}{a^n}},$$

which gives an infinite number of values, corresponding to the positive and negative whole values of k , in n different series corresponding to the n forms of $\frac{1}{a^n}$; one of these series containing only a finite number when $\frac{1}{a^n}$ and a are commensurable with the unit. The convertibility of the preceding expression with x^a is easily established, if we remember that

$$\frac{a^{\frac{1}{a^n}-1}}{a-1} \times a = \frac{a^{\frac{1}{a^n}+1}-1}{a-1} - 1;$$

and thus we have an extension of the forms of the functions which are convertible with x^a .

We shall here take the opportunity of observing that, if n be increased without limit, the limit of the preceding is x , or $\phi^n x$, as we might suppose it would be.

(154.) A similar extension of form may be given to the values of $\phi^n x$ deduced in (46.), though without prejudice to the truth of the remarks in (48.). If in the first example we assume $v_{n+1} = 2v_n + k\pi$, k being any positive or negative whole number, we have, if

$$\phi x = \tan 2 \tan^{-1} x = \frac{2x}{1-x^2}$$

$$\phi^n x = \tan \{2^n \tan^{-1} x + (2^n - 1)k\pi\};$$

which is equally true when n is fractional, but which in the latter case gives as many different values as there may be values of k .

(155.) But we have not any reason to suppose that we can thus, even in these simple cases, give the most remote idea of the general nature of the function which, repeated n times, will give αx . Nor can we, from analogy, suppose the complete solution of any inverse operation which does not contain an arbitrary function or functions. And if we are to treat forms as different, which we cannot prove to be the same, the following consideration will show that we are not even to suppose

that $\alpha^n x$ is limited in the number or character of its arbitrary functions, but may require an infinite number of arbitrary functions, with different subjects.

Let us again take $\phi^n x = \alpha x$; and suppose α to be a derivative of βx ; thus

$$\alpha x = \mu \beta \mu^{-1} x.$$

Calculus of
Functions.

In the general determination of μ from β , an arbitrary function must be supposed to appear, from (50.) If we now assume $\phi x = \mu \psi \mu^{-1} x$, we find that the original equation is satisfied by $\psi^n x = \beta x$, in the general solution of which an arbitrary function also enters. Consequently, in determining $\mu \psi \mu^{-1} x$, we produce a result with two arbitrary functions, having no means of proving that these two reduce themselves to one, and knowing that very different forms can be given to the two functions, because the first depends upon β and α , and the second on β only; while β may be any given form. And being thus able to vary the forms of the arbitrary function, the general solution must be one which, explicitly or implicitly, contains every possible form. We may, therefore, conceive it possible that the general solution contains an infinite number of arbitrary functions, the subjects of which must satisfy one or more relations derived from the nature of the function αx .

On the other hand, perhaps the preceding reasoning may lead some to suspect that there should be no arbitrary function in the solution of $\phi^n x = \alpha x$. And the following reasoning may appear to be in favour of such a suspicion. If we could find $\varpi(x, c)$ a particular solution of $\psi^n x = \beta x$, containing an arbitrary constant, the latter might be immediately converted into an arbitrary function by substituting θx , the solution of

$$\theta \varpi(x, \theta x) = \theta x \dots (A).$$

Now, as yet we have not the means of introducing even an arbitrary constant (other than an integer) into any single case of $\alpha^n x$.

(156.) The following method of solving $\psi^n x = \alpha x$ is given by Mr. Babbage.* Assume

$$\psi x = \phi f \phi^{-1} x \quad \alpha x = \phi f^n \phi^{-1} x,$$

where f is a known function. The solution of the question is then reduced to the determination of ϕ from

$$\phi f^n x = \alpha \phi x.$$

Let us proceed to solve this as in (50.) If we call F and A the exponential inverses of f and α , and eliminate m from the two equations

$$f^m x = c \quad \alpha^m \phi x = c',$$

we find

$$m n = F x \quad m = A \phi x$$

$$\phi x = A^{-1} \frac{1}{n} F x \quad \phi^{-1} x = F^{-1} n A x$$

$$\psi x = A^{-1} \frac{1}{n} F f F^{-1} n A x = A^{-1} \left(A x - \frac{1}{n} \right),$$

because

$$F f x = F x - 1 \quad F f F^{-1} x = x - 1,$$

a form in which the n th functional inverse has already been obtained. With the forms of exponential inverses, which are actually attainable, the preceding method is therefore equivalent to the actual expression of $\alpha^n x$, by giving n a fractional value. But for any thing which appears to the contrary, the solution of

$$f^m x = \mu \cos 2\pi m \quad \alpha^m \phi x = \nu \cos 2\pi m$$

may contain more general solutions. But some attempts which we have made lead to an ambiguity which will occur in a less complicated form hereafter.

(157.) The whole question relative to the solution of $\psi^n x = \alpha x$ is thus reduced to the solution of $\psi^n x = x - 1$;

* Essay, &c. Phil. Trans., 1815.

Calculus of Functions. since any particular solution of this, ϖx for example,

gives $A^{-1} \varpi A x$ for a value of $\alpha^{\frac{1}{n}} x$, where A is the exponential inverse of α . But we have not any direct means of determining any other values for ψx , except $x - \frac{1}{n}$. But by referring all functions to x^a , we may

arrive at a value of $\alpha^{\frac{1}{n}} x$ for every value there given of the n th functional inverse of x^a ; as follows:

Since we have generally (50.)

$$\alpha x = A^{-1} B \beta B^{-1} A x,$$

(where α and β are any functions whatsoever, and A and B their exponential inverses) let $\beta x = x^a$, in which case

$$B x = -\frac{\lambda^2 x}{\lambda a} \quad B^{-1} x = e^{x \lambda a}.$$

Let ϖx be a value of $\beta^{\frac{1}{n}} x$: whence we find by successive operation, that

$$A^{-1} B \varpi B^{-1} A x$$

is a value of $\alpha^{\frac{1}{n}} x$. Take for ϖx its value already obtained in (153.), namely,

$$\lambda^{-1} \left(\frac{a^{\frac{1}{n}} - 1}{a - 1} 2 k \pi \sqrt{-1} \right) \times x^{\frac{1}{a}},$$

using $\lambda^{-1} x$ for e^x . We have then

$$\varpi B^{-1} A x = \lambda^{-1} \left(\frac{a^{\frac{1}{n}} - 1}{a - 1} 2 k \pi \sqrt{-1} \right) \times (\lambda^{-2} - \lambda a A x)^{\frac{1}{a}}.$$

and therefore $\alpha^{\frac{1}{n}} x$ or $A^{-1} B \varpi B^{-1} A x$ is the function A^{-1} of

$$-\frac{1}{\lambda a} \lambda^2 \left\{ \lambda^{-1} \left(\frac{a^{\frac{1}{n}} - 1}{a - 1} 2 k \pi \sqrt{-1} \right) \times (\lambda^{-2} - \lambda a A x)^{\frac{1}{a}} \right\}.$$

But here we must observe, that the absolute manner in which the n th inverse of x^a was obtained depended upon the successive operations not inverting $e^{2k\pi\sqrt{-1}}$, so that the result appeared in the form $e^{2k\pi\sqrt{-1}} \cdot x^a$, or x^a . But in any other case we must remember that the use of the general form $\phi x \times e^{2k\pi\sqrt{-1}}$ requires an equally general form of inversion: and without it, as we shall see, we cannot make the preceding give an n th inverse of αx , but only of a particular class of its derivatives. If then we assume

$$\lambda \lambda^{-1} \left(\frac{a^{\frac{1}{n}} - 1}{a - 1} 2 k \pi \sqrt{-1} \right) = \frac{a^{\frac{1}{n}} - 1}{a - 1} 2 k \pi \sqrt{-1} + 2 k' \pi \sqrt{-1}$$

$$\lambda \lambda^{-2} (-\lambda a A x) = \lambda^{-1} (-\lambda a A x) + 2 k'' \pi \sqrt{-1},$$

and substitute, writing

$$K \text{ for } \frac{a^{\frac{1}{n}} - 1}{a - 1} k + k' + a^{\frac{1}{n}} k'',$$

we have for the value of $\alpha^{\frac{1}{n}} x$

$$A^{-1} \left\{ -\frac{\lambda (2 \pi K \sqrt{-1} + a^{\frac{1}{n}} \Theta^{-1} x)}{\lambda a} \right\}.$$

(158.) We may easily discover the conditions of verification of this expression by observing that it is a derivative of

$$2 \pi K \sqrt{-1} + a^{\frac{1}{n}} x,$$

or that it is

$$\Theta (2 \pi K \sqrt{-1} + a^{\frac{1}{n}} \Theta^{-1} x)$$

where

$$\Theta x = A^{-1} \left(-\frac{\lambda x}{\lambda a} \right) \quad \Theta^{-1} x = a^{\frac{1}{n}} x,$$

whence, the n th function of

$$2 \pi K \sqrt{-1} + a^{\frac{1}{n}} x \text{ being } \frac{a - 1}{a^{\frac{1}{n}} - 1} 2 \pi K \sqrt{-1} + a x,$$

the n th function of the preceding is

$$\Theta \left(\frac{a - 1}{a^{\frac{1}{n}} - 1} 2 \pi K \sqrt{-1} + a \Theta^{-1} x \right),$$

the only apparent case in which this gives the value αx or $A^{-1} (A x - 1)$ is when $K = 0$, which explicitly reduces the n th function to the form $A^{-1} (A x - 1)$; so that, returning to the general form, we find the follow-

ing as an expression for $\alpha^{\frac{1}{n}} x$; namely,

$$A^{-1} \left[-\frac{1}{\lambda a} \lambda^2 \left\{ \lambda^{-1} \left(\frac{a^{\frac{1}{n}} - 1}{a - 1} 2 \pi k \sqrt{-1} \right) \times (\lambda^{-2} a^{-\lambda x})^{\frac{1}{a}} \right\} \right]$$

subject to the following equation between the arbitrary integers implied in its most general meaning;

$$\frac{a^{\frac{1}{n}} - 1}{a - 1} k + k' + a^{\frac{1}{n}} k'' = 0.$$

Various functions will arise in which the equation of condition becomes unnecessary; but, generally speaking, it must not be neglected.

Unsatisfactory as the preceding must necessarily be, we have thought it might be useful, in order to direct the attention of our readers to this curious and by no means finally settled subject of the inversion of the exponential function of Algebra.* The difficulty is of the same character as that contained in $\cos^{-1} \cos x$, or any other function which has different inverses, and is comparatively little felt in problems whose scope and meaning have been more attended to than the present one. We see, however, that we have not yet succeeded in obtaining a solution with an arbitrary constant: for though a is such at the outset, the equation of condition deprives it of that character, by making it a function of arbitrary integers only.

The solution of $\psi^n x = \alpha x$, in all its generality, is a fundamental inquiry in the subject of the present Treatise: nor can more essential progress be looked for without it, than in common Algebra without the complete solution of $x^n = 1$. Our own suspicion is, that the solutions are infinite in number, but embraced under a finite number of forms; but we hope the subject will be further considered.

(159.) The method of representing the n th functions

* The extension lately asserted of the logarithms of unity and of numbers in general has not been adopted by us, because we cannot agree with those who think it established. The fundamental assumptions have not been explained with sufficient clearness for us to undertake to specify our objections. If it be really meant to be asserted that

e to the $(2k \div 2k' - \sqrt{-1})$ th power = 1,

in the usual sense of such a formula, we must entirely dissent from the conclusion.

Calculus of Functions.

Calculus of Functions. of a periodic function in (66.) fails, as we have seen, to give a form of their functional inverses. And it is necessary that every such inverse of a periodic function should be a periodic function: for if $\phi^n x = \alpha x$, where $\alpha^n x = x$, we must have $\phi^n x = x$. And since these conditions

can be fulfilled in periodic functions of the form $(1)^{\frac{1}{n}}x$, $x^{(1)^{\frac{1}{n}}}$, &c., and since any periodic function may be made a derivative of any other of the same kind, we can express any periodic function of the p th order in the form $\phi(1)^{\frac{1}{p}}\phi^{-1}x$, and the n th functional inverse of this will be $\phi(1)^{\frac{1}{np}}\phi^{-1}x$. The arbitrary function contained in the general method of determining ϕ , given in (75.), shows that in this case, at least, there is an arbitrary function in $\alpha^n x$, unless, which may happen, it disappears in forming $\phi(1)^{\frac{1}{np}}\phi^{-1}x$ as well as $\phi(1)^{\frac{1}{n}}\phi^{-1}x$.

(160.) We now take the equation $\phi^n x = \alpha^n x$, of which one class of solutions is contained in the various n th inverses of $\alpha^n x$. But another class is found as follows. Let β be any periodic function of the order n , or a submultiple of n , and assume $\phi x = \beta \alpha x$. Then, if β and α be convertible, it is easily shown that $\phi^n x$ is the same as $\beta^n \alpha^n x$ or $\alpha^n x$. Thus $\frac{1}{x}$ being a periodic function of

the second order, convertible with x^2 , $\frac{1}{x^2}$ is a solution of $\phi^2 x = x^4$. But we are not in possession of any method by which it can be found what periodic functions are convertible with any given function.

(161.) If it ever be necessary to consider the successive functions of αx as derivatives of each other, it may be done by the method of (50.), and the result will be as follows: if

$$\alpha^p x = f \alpha^q f^{-1} x,$$

then $f x$ must be deduced from the equation

$$\frac{A f x}{A x} = \frac{p}{q},$$

where $A x$ is the exponential inverse of αx , and different values of the arbitrary constant may be employed in the numerator and denominator.

The following theorem may also be deduced. Let

$$\alpha^p x = f_p \alpha f_p^{-1} x,$$

and let

$$\alpha^{p+1} x = \phi_p \alpha^p \phi_p^{-1} x,$$

then among all the functions which may be found to satisfy the conditions imposed on f_p and ϕ_p , a set may be found between which the following relations exist:

$$\begin{aligned} f_2 x &= \phi_1 x \\ f_3 x &= \phi_2 \phi_1 x \\ f_4 x &= \phi_3 \phi_2 \phi_1 x, \text{ \&c.,} \end{aligned}$$

or

$$f_{n+1} x = \phi_n \phi_{n-1} \dots \phi_{n-p} f_{n-p} x.$$

(162.) Mr. Babbage* has given the following method of ascertaining the value of a function which does not change when x is changed into $\alpha_1 x, \alpha_2 x, \dots, \alpha_n x$. Find a particular solution of $\xi \alpha_1 x$, which call $f_1 x$; then find a particular solution of $\xi \alpha_2 x$, and find what function the latter is of $f_1 x$, or exhibit it in the form $f_2 f_1 x$: so that

Calculus of Functions. $f_2 f_1 \alpha_2 x = f_2 f_1 x$: similarly, let $f_3 f_2 f_1 x$ be a particular solution of $\xi \alpha_3 x$, and so on. Then will $\theta(f_n f_{n-1} \dots f_2 f_1 x)$ be a general solution of the requisite conditions: for though, for instance, $f_1 \alpha_2 x$ is different from $f_1 x$, yet similarity of form is restored by the operation f_2 , and so on for the rest. A similar method may be applied to the solution of the simultaneous equations $\phi \alpha_1 x = \phi \beta_1 x$, $\phi \alpha_2 x = \phi \beta_2 x$, which indeed amounts to substituting in the preceding $\alpha_1 \beta_1^{-1} x$ for $\alpha_1 x$, and so on.

(163.) But the preceding process must be subject to the condition that f_n does not invert f_{n-1} , f_{n-1} does not invert f_{n-2} , &c. according to the remark in (99.) We shall proceed to some instances.

Let $m x$ and $n x$ be the functions in question. A particular solution of $\phi(m x) = \phi x$ is $\cos 2\pi \frac{\lambda x}{\lambda m}$: we have therefore to solve the equation

$$\phi \cos 2\pi \frac{\lambda n x}{\lambda m} = \phi \cos 2\pi \frac{\lambda x}{\lambda m}.$$

Let

$$\cos 2\pi \frac{\lambda x}{\lambda m} = \Theta x \quad \phi \Theta n x = \phi \Theta x,$$

or

$$\phi \Theta n \Theta^{-1} x = \phi x,$$

where $\phi \Theta x$ is the function now to be found, and ϕx is itself, by preceding methods, the same as $\psi \Theta^{-1} x$ where $\psi n x = \psi x$. Hence $\psi \Theta^{-1} \Theta x$ is the function in question, which is, for convertible inverses, the same as $\psi n x$, which by hypothesis (unless the solution is assumed in the solution) satisfies $\psi n x = \psi x$ only. We have then, owing to the inversion above alluded to, and contrary to the limitation mentioned in (99.) abandoned the condition already satisfied, in trying to secure the other. And it must be observed that the theorem of (104.) is not true, if an inconvertible inverse of Θ be made to take the place of Θ^{-1} .

(164.) If, however, taking inconvertible inverses, we carry on the process, we may arrive at this sort of solution, namely, an ambiguous function of x , which shall be absolutely unchanged when x is changed into $n x$, and changed from one of its values to another when x is changed into $m x$. For we find $\psi x = \cos 2\pi \frac{\lambda x}{\lambda n}$.

$$\Theta_{-1} x = \varepsilon^{\frac{\lambda m}{2\pi} \cos^{-1} x}$$

$$\begin{aligned} \psi \Theta_{-1} \Theta x &= \cos 2\pi \frac{1}{\lambda n} \lambda \varepsilon^{\frac{\lambda m}{2\pi} \cos^{-1} \cos 2\pi \frac{\lambda x}{\lambda m}} \\ &= \cos \frac{\lambda m}{\lambda n} \left(2\pi \frac{\lambda x}{\lambda m} + 2w\pi \right). \end{aligned}$$

where w is any whole number, positive or negative. This is

$$\cos 2\pi \left(\frac{\lambda x}{\lambda n} + w \frac{\lambda m}{\lambda n} \right),$$

in which a change of x into $n x$ causes no change, and x into $m x$ adds 1 to w , or gives another value of the function. The conditions are indeed satisfied when m is a whole power of n , but this gives only a case of $\phi \alpha^p x = \phi x$, following from $\phi \alpha x = \phi x$.

If possible, let there be a function of x which does not change when x is changed into $m x$ or $n x$. It is easily proved that it will not change when $m^a n^b x$ is substituted for x , a and b being any two whole numbers, positive or negative. But these whole numbers may be

* Philosophical Transactions, 1815.

Calculus of
Functions.

so taken, that m^n may, if λm and λn be incommensurable, be made as near as we please to any given quantity c . Hence the *continuous* solution, in a form independent of the specific values of m and n , is a solution of $\phi cx = \phi x$, for all values of c . Such a solution is then impossible, in any sense in which we consider equations as solved in Algebra.

(165.) The preceding solution is an accident arising from our having functions for which $\Theta_{-1} \Theta x$ is of the form $x + c$. If we take x^m and $n x$, and extend the theorem in (104.) to the case where Θ_{-1} is written for Θ^{-1} , for trial only, according as we take one function or the other first, we arrive at the results

$$\cos 2\pi \frac{m^n \lambda x}{\lambda n} \text{ and } \cos 2\pi \frac{\lambda (\lambda x + w \lambda n)}{\lambda m};$$

the first of which again, by a peculiarity of this case, is such a solution as that in the last article for all *whole* values of m , and *positive whole* values of w . The second is not in any case such a solution, but it leads us to the following remark.

If we may suppose w to have any value whatsoever, then the second formula becomes a solution of the question; for the substitution of x^m instead of x gives

$$\cos 2\pi \frac{\lambda \left(\lambda x + w \frac{\lambda n}{\lambda m} \right) + \lambda m}{\lambda m},$$

equivalent to changing w into $\frac{w}{\lambda m}$: while the substitution of $n x$ for x gives

$$\cos 2\pi \frac{\lambda (\lambda x + (w + 1) \lambda n)}{\lambda n},$$

which is equivalent to changing w into $w + 1$. We can, by this method, generally obtain a solution of the following problem: Given $\alpha_1 x, \alpha_2 x, \&c.$, required such a function ϕx , that $\phi \alpha_1 x, \phi \alpha_2 x, \&c.$ shall only differ from ϕx in the values of constants.

(166.) Let us take any two functions, αx and βx , the exponential inverses of which are $A x$ and $B x$, and $\cos 2\pi A x$ is a value of $\xi \alpha x$. We have, therefore, to solve the equation

$$\phi \cos 2\pi A \beta x = \phi \cos 2\pi A x;$$

so that $\phi \cos 2\pi A x$ may be a value both of $\xi \alpha x$ and $\xi \beta x$. A value of ϕx (102.) is $\cos 2\pi B A^{-1} \frac{1}{2\pi} \cos^{-1} x$,

and therefore the equation derived from the preceding method is

$$\phi \cos 2\pi A x = \cos 2\pi B A^{-1} (A x + w) \text{ or } \cos 2\pi B \alpha^n x$$

as appears from (105.), since w may be indifferently a positive or negative whole number. Here it is evident that in all cases the substitution of αx for x is the alteration of a constant, and that if αx and βx be convertible, the substitution of βx for x changes $B \alpha^n x$ into $B \beta \alpha^n x$ or $B \alpha^n x - 1$ giving no change in the cosine.

Thus the ambiguous solution of the first example may be obtained in all cases of convertible functions.

(167.) If the two functions in question be periodic and convertible, the condition may be generally satisfied. If, for instance, αx and βx be periodic convertible functions of the second order, then

$$x + \alpha x + \beta x + \alpha \beta x$$

does not change when x is changed into αx , or into βx , becoming in these two cases

$$\alpha x + x + \beta \alpha x + \beta x$$

and

$$\beta x + \alpha \beta x + x + \alpha x,$$

which are equal, since $\alpha \beta x = \beta \alpha x$. Hence

$$\theta (x + \alpha x + \beta x + \alpha \beta x) = \phi x$$

fulfils the conditions $\phi \alpha x = \phi x, \phi \beta x = \phi x$.

Similarly, periodic and convertible functions of the third order, αx and βx , being given, the function

$$\theta \left(\begin{array}{l} x + \alpha x + \beta x \\ + \alpha^2 x + \alpha \beta x + \beta^2 x \\ + \alpha^2 \beta x + \alpha \beta^2 x + \alpha^2 \beta^2 x \end{array} \right)$$

satisfies the conditions $\phi \alpha x = \phi \beta x = \phi x$.

And generally, if $\alpha_1 x, \alpha_2 x, \dots, \alpha_m x$ be m periodic functions of the n th order, which are mutually convertible, two and two, it follows that

$$\theta (\Sigma \alpha_1^{a_1} \alpha_2^{a_2} \alpha_3^{a_3} \dots \alpha_m^{a_m} x),$$

for every value of each of the quantities $a_1, a_2, \dots$, between 0 and $n - 1$ inclusive, satisfies the conditions

$$\phi \alpha_1 x = \phi x \quad \phi \alpha_2 x = \phi x, \&c.$$

(168.) When periodic functions are not convertible, a similar formula may be constructed, if their compound functions are periodic: but it must be observed, that in the case of periodic functions of the second order, if the compound function be periodic, and of the second order, the functions must be convertible: and the converse. For if

$$\alpha^2 x = x \quad \beta^2 x = x \quad \alpha \beta \alpha \beta x = x,$$

we have

$$\alpha^2 \beta \alpha \beta x = \alpha x \text{ or } \beta \alpha \beta x = \alpha x \\ \beta \alpha \beta^2 x = \alpha \beta x \text{ or } \beta \alpha x = \alpha \beta x.$$

(169.) When the compound function is not periodic we have no method of solving this question. In fact, if $\phi \alpha x = \phi x, \phi \beta x = \phi x, \alpha^n x = x, \beta^n x = x$, we have also $\phi \alpha \beta x = \phi x$, and if $\alpha \beta$ be not periodic, we cannot solve this equation directly or indirectly, except only in the case where $\alpha \beta$ is one of the forms noticed in (112.)

(170.) We shall now give some methods of reduction of functional equations to more simple forms. Several are derived from Mr. Babbage's Papers in the *Philosophical Transactions*. Indeed, without these papers, and the elementary treatise of examples published with the examples to the Cambridge translation of Lacroix, the third branch of the subject mentioned in (94.) could not have been said to exist in a defined form.

Let

$$(\psi \alpha)^n \psi \beta x = \gamma x,$$

where by $(\psi \alpha)^2$ is meant $\psi \alpha \psi \alpha, \&c.$, and the reader must not confound $(\psi \alpha)^n$ with $\psi^n \alpha^n$, the two being the same only when ψ and α are convertible. Let

$$\psi \alpha x = \phi f \phi^{-1} x,$$

which gives

$$\phi f^n \phi^{-1} \phi f \phi^{-1} \alpha^{-1} \beta x = \gamma x,$$

or

$$\phi f^{n+1} \phi^{-1} \alpha^{-1} \beta x = \gamma x,$$

or

$$\phi f^{n+1} x = \gamma \beta^{-1} \alpha \phi x:$$

this equation may also be reduced to

$$\chi^{n+1} x = \gamma \beta^{-1} \alpha x \text{ where } \chi x = \psi \alpha x.$$

Calculus of
Functions.

Calculus of
Functions.

(171.) Though the reduction of any periodic function to a derivative of any other of the same order makes it obvious that no apparent generality of solution of $\psi^n x = x$ does in reality contain more than can be comprised under the form $\phi(1)^{\frac{1}{n}} \phi^{-1} x$, yet the following solution may give an idea of the variety of forms which the last-mentioned expression may contain. Let

$$\psi^n x = x \text{ and } \psi x = \phi f \phi^{-1} x,$$

or

$$\phi f^n \phi^{-1} x = x \text{ and } \phi f^n x = \phi x.$$

Let $f x$ be a periodic function of the k th order, and take ϕx any symmetrical function of

$$x, f^n x, f^{2n} x, \dots, f^{(k-1)n} x,$$

then ϕx is a solution of the preceding equation.

(172.) Questions might be proposed which occupy the same position in the Calculus of Functions with those on indeterminate equations in common Algebra. One example of such questions is contained in the following.* Let αx and βx be two rational and integral functions of the degrees m and n , having the roots

$$\begin{matrix} a_1 & a_2 & a_3 & \dots & a_m \\ b_1 & b_2 & b_3 & \dots & b_n. \end{matrix}$$

Let γx be any third function whatsoever; required solutions of the equation

$$\alpha x \phi x + \beta x \psi x = \gamma x.$$

If we signify by $\alpha' x$ and $\beta' x$ the differential coefficients of αx and βx , it follows from the theory of equations that of the quantities

$$\frac{\alpha x}{x - a_1} \cdot \frac{1}{\alpha' a_1}, \quad \frac{\alpha x}{x - a_2} \cdot \frac{1}{\alpha' a_2}, \quad \&c.$$

all disappear when x is equal to either of the series $a_1, a_2, \&c.$, except only one which becomes equal to unity; while

$$\frac{\beta x}{\beta a_1} \cdot \frac{\beta x}{\beta a_2} \quad \&c.$$

are in all those cases severally equal to unity, and vanish when x is any root of βx . Hence the expression

$$\begin{aligned} & \frac{\alpha x}{x - a_1} \cdot \frac{\beta x}{\alpha' a_1 \cdot \beta a_1} + \frac{\alpha x}{x - a_2} \cdot \frac{\beta x}{\alpha' a_2 \cdot \beta a_2} + \dots \\ & = 0 \text{ when } x \text{ is any root of } \beta x, \\ & = 1 \text{ when } x \text{ is any root of } \alpha x. \end{aligned}$$

Similarly,

$$\begin{aligned} & \frac{\beta x}{x - b_1} \cdot \frac{\alpha x}{\beta' b_1 \cdot \alpha b_1} + \frac{\beta x}{x - b_2} \cdot \frac{\alpha x}{\beta' b_2 \cdot \alpha b_2} + \dots \\ & = 1 \text{ when } x \text{ is any root of } \beta x, \\ & = 0 \text{ when } x \text{ is any root of } \alpha x; \end{aligned}$$

consequently the expression

$$\beta x \Sigma \left(\frac{\alpha x}{x - a} \cdot \frac{1}{\alpha' a \cdot \beta a} \right) + \alpha x \Sigma \left(\frac{\beta x}{x - b} \cdot \frac{1}{\beta' b \cdot \alpha b} \right)$$

is equal to unity when x is any root either of αx or βx , that is, for $m + n$ values of x . But its degree is only $m + n - 1$, therefore it cannot contain x , and must be identically equal to unity. From which it is evident, that γx multiplied by the preceding coefficients of αx and βx give ϕx and ψx the solutions of the equation.

The same process may be extended to any number of terms, thus enabling us always to give a solution of the equation

* Cauchy, *Exercices de Mathématiques*, vol. i. p. 160.

$$\Sigma(\alpha x \cdot \phi x) = \gamma x,$$

Calculus of
Functions.

where $\alpha_1 x, \alpha_2 x, \&c.$ are rational and integral functions. The distinction between this solution and any other of the infinite number which might evidently be taken consists in its always giving a rational and integral solution whenever γx is rational and integral.

(173.) Required a curve of such a nature, that any ordinate y being taken to the abscissa x , and also to an abscissa which is a given function of the first abscissa and ordinate, the second ordinate may be equal to the first abscissa.

Let $y = \psi x$ be the equation of the curve required, and let $\alpha(x, y)$ be the given function of the abscissa and ordinate. Then by the conditions, the ordinate, which has $\alpha(x, y)$ for its abscissa, is equal to x ; or

$$\psi \alpha(x, \psi x) = x.$$

Let $\psi x = \phi f \phi^{-1} x$, which gives

$$\phi f \phi^{-1} \alpha(x, \phi f \phi^{-1} x) = x,$$

or

$$\begin{aligned} \phi f \phi^{-1} \alpha(\phi x, \phi f x) &= \phi x \\ \alpha(\phi x, \phi f x) &= \phi f^{-1} x, \end{aligned}$$

which, taking any given form for f , is a common functional equation with respect to ϕ .

If, for instance, we take for $f x$ any periodic function of the second order, in which case $f^{-1} x = f x$, we can from the preceding equation deduce one of the following form,

$$\phi f x = \beta \phi x;$$

the solutions of which are discontinuous, unless β be also a periodic function of the second order; in which case we have the means of solution from (50.)

(174.) Let

$$F(x, \psi x, \psi^2 \alpha x, \psi^3 \beta x, \dots) = 0,$$

$F, \alpha, \beta, \dots$ being given forms.

Find a function ξx which does not change when $\alpha x, \beta x, \dots$ are severally substituted for x , and write $\xi^{-1} \phi \xi x$ for ψx . The equation then becomes

$$F(x, \xi^{-1} \phi \xi x, \xi^{-1} \phi^2 \xi \alpha x, \xi^{-1} \phi^3 \xi \beta x, \dots) = 0.$$

Let $y = \xi x$, which gives

$$F(\xi^{-1} y, \xi^{-1} \phi y, \xi^{-1} \phi^2 y, \dots) = 0,$$

or, ξ being known, the equation is reduced to the form

$$F_1(y, \phi y, \phi^2 y, \dots) = 0.$$

Let $\phi y = \chi f \chi^{-1} y$, f being a known form; which gives

$$F_1(y, \chi f \chi^{-1} y, \chi f^2 \chi^{-1} y, \dots) = 0,$$

or

$$F_1(\chi y, \chi f y, \chi f^2 y, \dots) = 0.$$

The solutions of this equation may be divided into two kinds: those which are periodic and those which are not periodic. If we assume for f a periodic form, then ϕ and consequently ψ must be periodic, if χ and ξ be continuous. In that case, the non-periodic values of ψ will answer to discontinuous forms of χ : conversely, if we assume f to be non-periodic, the periodic forms of ψ , if any, will answer to discontinuous values of χ . In the preceding solution, however, a point must be held in mind which considerably limits its generality. In the first substitution $\xi^{-1} \phi \xi x$, and the subsequent assumption of $\xi^{-1} \phi \xi \alpha x = \xi^{-1} \phi \xi x$, because $\xi \alpha x = \xi x$, we may be led into the mistake pointed out in (163.) Generally speaking, it must be taken as a rule that the operation ξ^{-1} performed upon ξ , whatever may have been the intermediate operations, produces a result which does

Calculus of Functions.

not possess the properties of χ performed upon ξ with intermediate operations, where χ and ξ are wholly independent. If, for instance, we were to apply the preceding reasoning to the equation

$$F(x, \psi x, \psi \alpha x \dots) = 0,$$

say, for a particular example, to

$$\psi x + \psi \alpha x = \beta x,$$

the first result of the substitution would be

$$\xi^{-1} \phi \xi x + \xi^{-1} \phi \xi \alpha x = \beta x,$$

or

$$2 \xi^{-1} \phi \xi x = \beta x,$$

from which it would appear that we could immediately determine ϕx . But the truth is, that though $\xi \alpha x = \xi x$ yet, firstly, the consequence $\phi \xi \alpha x = \phi \xi x$ is only true of those forms of ϕ which do not invert any periodic terms of ξ , and we thus perhaps throw away some forms of the solution; secondly, it does not follow that $\xi^{-1} \phi \xi \alpha x = \xi^{-1} \phi \xi x$, and indeed will not be true of most of the forms which we have found. Take, for instance, $\alpha x = a x$, and let

$$\xi x = x^{\frac{2\pi\sqrt{-1}}{\lambda a}} \quad \xi^{-1} x = x^{\frac{\lambda a}{2\pi\sqrt{-1}}}$$

Let

$$\phi x = x^2 \quad \xi^{-1} \phi \xi x = x^2 \quad \xi^{-1} \phi \xi \alpha x = \overline{a x}^2;$$

the preceding is the construction of $\xi^{-1} \phi \xi x$, &c., in the ordinary and usual meaning of the symbols of compound functions; that is, assuming $\phi \psi \chi x$ to be the same thing as $(\phi \psi) \chi x$, and $(\xi^{-1} \phi \xi) \alpha x$ to mean the same thing as $\xi^{-1} \phi (\xi \alpha x)$. It is true that, if we perform successively the operations αx , $\xi \alpha x$ (which is ξx) &c., we do arrive at the same result as $\xi^{-1} \phi \xi x$; but owing to the ambiguity of the process of inversion, we cannot say that $(\xi^{-1} \phi \xi) \alpha x = (\xi^{-1} \phi \xi) x$, where $\xi^{-1} \phi \xi$ is considered as one defined operation. But this is what is meant in the substitution of $\xi^{-1} \phi \xi x$ instead of ψx in the solution of

$$\psi x + \alpha x \psi (\alpha x) = \beta x,$$

and the introduction of the substitution without attention to this circumstance would be equivalent to confounding the preceding equation with

$$\psi x (1 + \alpha x) = \beta x.$$

(175.) The reduction of an equation to an equation of differences can be more simply done by means of the exponential inverse: and the method we now give is the same in effect as that of Laplace, the finding of the exponential inverse required answering to the first of the integrations in article (100.) Let

$$F(x, \phi \alpha x, \phi \beta x) = 0:$$

write $\alpha^{-1} x$ for x , and this becomes

$$F(\alpha^{-1} x, \phi x, \phi \beta \alpha^{-1} x) = 0.$$

Find $E x$ the exponential inverse of $\beta \alpha^{-1} x$, and assume $\phi x = \psi E x$. Then $\phi \beta \alpha^{-1} x = \psi E \beta \alpha^{-1} x = \psi (E x - 1)$. Substitute in the equation, and write $E^{-1} x$ for x , which gives

$$F\{\alpha^{-1} E^{-1} x, \psi x, \psi (x - 1)\} = 0,$$

or, writing u_x for $\psi (x - 1)$,

$$F(\alpha^{-1} E^{-1} x, u_{x+1}, u_x) = 0,$$

a common equation of differences. Similarly, if

$$F(x, \phi x, \phi \alpha x, \phi \alpha^2 x \dots \phi \alpha^n x) = 0$$

let $A x$ be the exponential inverse of αx , and let $\psi A x$ be the solution required. Then

VOL. II.

Calculus of Functions.

$$\phi \alpha x = \psi A \alpha x = \psi (A x - 1)$$

$$\phi \alpha^2 x = \psi (A \alpha x - 1) = \psi (A x - 2), \text{ \&c.}$$

Substitute these values, write $A^{-1} x$ for x , and u_x for ψx , which gives

$$F(A^{-1} x, u_x, u_{x+1}, \dots, u_{x+n}) = 0.$$

Let

$$F(x, \psi x, \psi^2 x \dots \psi^n x) = 0$$

$$F(\alpha x, \phi \alpha x, \phi \alpha^2 x \dots \phi \alpha^n x) = 0$$

where

$$\phi \alpha \phi^{-1} x = \psi x.$$

This, as before, may be reduced to

$$F(A^{-1} x, u_x, u_{x+1}, \dots, u_{x+n-1}) = 0.$$

(176.) The question of the actual solution of any given functional equation is yet far from capable of being solved, unless in a few particular cases. By observing, however, one general form, we shall be able to see to what species of processes we must look as the means of attaining the object. Let us take the general equation of the first degree

$$\phi x + \beta x \phi \alpha x = \gamma x \dots (1).$$

Firstly, two solutions of this equation can differ only by a solution of

$$\phi x + \beta x \phi \alpha x = 0 \dots (2);$$

for if ϖx and $\varpi_1 x$ both satisfy equation (1), we have

$$(\varpi x - \varpi_1 x) + \beta x (\varpi \alpha x - \varpi_1 \alpha x) = 0,$$

or $\varpi x - \varpi_1 x$ satisfies equation (2).

Again, it must be evident that if ϖx and $\varpi_1 x$ are particular solutions of (2), then $\frac{\varpi x}{\varpi_1 x}$ is also a solution of

$$\phi x - \phi \alpha x = 0 \dots (3),$$

which can readily be obtained when $A x$ is known.

The solution of equation (1) either contains one arbitrary constant only, (that is, more than is contained in αx , &c.) or an infinite number. For if it contain a finite number, let ϕx be the solution with the greatest possible number of constants. Let one of these constants be c , and differentiate both sides of (1) with respect to c , giving

$$\frac{d \phi x}{d c} + \frac{d \phi \alpha x}{d c} \cdot \beta x = 0.$$

But $\frac{d \phi \alpha x}{d c}$ is the same function of αx which $\frac{d \phi x}{d c}$ is of x ; so that $\frac{d \phi x}{d c}$ is a solution of (2), and thence

$\phi x + c' \frac{d \phi x}{d c}$ is also a solution of (1); that is, there

may be one more than the number of constants assumed, which is against the supposition. By the same reasoning it may be shown that the most general solution of (2) as to constants can have one only or an infinite number. And since the difference of any two solutions of (1) is a solution of (2), &c., the most general algebraical form of the solution of (1) is

$$\mu x + c \cdot \nu x,$$

where μx is a solution of (1), not containing a constant, and νx a solution of (2). If in this we write $\xi \alpha x$ for c , we have the most general solution, unless a solution can be given for (1) containing an infinite number of arbitrary constants.

2 Y

Calculus of
Functions.

An infinite number of arbitrary constants corresponds to an infinite number of algebraical forms different from each other, which are solutions of the equation. The same does any one constant entering in a manner which admits of the result being algebraically expanded in a series of powers of the constant. For if

$$c_1 \varpi_1 x + c_2 \varpi_2 x + \dots \text{ad inf.},$$

or

$$\varpi_0 x + \varpi_1 x \cdot c + \varpi_2 x \cdot c^2 + \dots \text{ad inf.}$$

be solutions of equation (1), in which c does not enter into αx , βx , or γx , it follows that $\varpi_0 x$, $\varpi_1 x$, &c. are severally solutions of (1). But this cannot be if $\varpi_0 x$, $\varpi_1 x$, &c. contain only the ordinary transcendents of Algebra, still less if they are common algebraical functions, as in that case $\varpi_m x$, $\varpi_n x$, and $\varpi_p x$ being solutions of (1), $\varpi_m x - \varpi_n x$ and $\varpi_n x - \varpi_p x$ are solutions of (2), and

$$\frac{\varpi_m x - \varpi_n x}{\varpi_n x - \varpi_p x} \text{ of (3);}$$

that is, the equation $\phi \alpha x = \phi x$ is solved by a function containing only the common expressions of Algebra, which, unless in the case of a periodic function, we think preceding evidence entitles us to say, cannot be; at least, never has been found so to be. This finally reduces the most general form of the algebraic solution of (1) to

$$\mu x + c \cdot \nu x,$$

where μx is the most general algebraic solution independent of new constants, and νx the same of (2). Consequently

$$\phi x = \mu x + \xi \alpha x \cdot \nu x$$

is the most general solution.

(177.) The particular solutions of (1), as found by trial or otherwise, may be of any number of forms, since, if any *specific* value be given to c , $\mu x + c \nu x$ does not contain an arbitrary constant. But the particular solutions of (2) will be all of one form, the difference being only in a constant factor. Thus in the case of

$$\phi x = \frac{x^5 - x^4}{x^3} + \frac{1}{x^3} \phi(x^3),$$

of which $x^2 + x^3$ and $x^2 - x^3$ are particular solutions, $c x^3$ is found to be the only algebraical solution of $\phi x = \frac{1}{x^3} \phi(x^3)$.

(178.) We have before seen that the theory of discontinuous functions must form the link between those of periodic and non-periodic functions. The former are, if we may so express it, the *rational quantities* of this branch of the subject, when considered in all their extent: the *whole numbers*, when the successive functions only, and not functional inverses, are considered. The discontinuous function is the *radical sign*, and is necessary to the expressing of periodic by means of non-periodic functions, and *vice versa*.

To any functional equation whatsoever, an infinite number of discontinuous solutions may be found, subject only to the condition that the solutions of continuity must take place at the values of x contained in the series

$$\dots \alpha^{-1} c, c, \alpha c, \alpha^2 c, \alpha^3 c, \dots$$

where c is any given quantity, and αx one of the functions which enters under an unknown functional symbol. For example, take the equation (1) now under consideration, and suppose, for instance, that αx increases

from $x = -\infty$ to $x = +\infty$, and let the values of αx when $x = 0$ be denoted by c_1, c_2 , &c.

Let there be a discontinuous function, which, from $x = 0$ to $x = c_1$, is ϖx , from $x = c_1$ to $x = c_2$, is $\varpi_1 x$, and so on. Assuming, therefore, A_n to represent a function of x which is unity for all values of x between c_n and c_{n+1} , and nothing for all others, let us assume

$$\phi x = A_0 \varpi x + A_1 \varpi_1 x + A_2 \varpi_2 x + \dots$$

If for x we take a value lying between c_n and c_{n+1} , we have

$$\phi x = A_n \varpi_n x \quad \phi \alpha x = A_{n+1} \varpi_{n+1} \alpha x,$$

or

$$\phi x = \varpi_n x \quad \phi \alpha x = \varpi_{n+1} \alpha x;$$

let the following conditions exist between ϖx , $\varpi_1 x$, &c.

$$\varpi x + \beta x \varpi_1 \alpha x = \gamma x$$

$$\varpi_1 x + \beta x \varpi_2 \alpha x = \gamma x, \text{ \&c.};$$

and the equation (1) is evidently satisfied for all values of x from nothing to infinity. An additional series would make the solution true for all negative values, and the generality of the solution might be further increased by supposing ϖx itself discontinuous, $\varpi_1 x$, &c. being then discontinuous in a corresponding manner.

(179.) Thus, to solve the equation $\phi(x+a) = \phi x$, let a straight line carrying points equidistant from each other by a move parallel to itself, so that one of the points moves from its first position to that of the next point, over an arc of any curve, continuous or discontinuous. Then the succession of branches so obtained is a discontinuous solution of the equation.

In the same manner, any other case of $\phi \alpha x = \phi x$ may be shown to have an infinite number of discontinuous solutions; and therefore the solution of (1) or

$$\phi x = \mu x + \xi \alpha x \cdot \nu x$$

may be subject to discontinuity in the term $\xi \alpha x$, even where the functions μx and νx are continuous.

(180.) Wherever an arbitrary constant is a part of a solution, that is to say, a constant which does not enter expressly into the conditions of the problem, it must be determined by the particular circumstances of the problem. The constant of integration is only one particular case, though the appearance of a constant is usually held almost peculiar to integration. But a wider induction from observation of solutions leads us to infer that though the solution of the problem of giving a *value* to a *form* does not introduce a constant, yet the latter must always be looked for where the object is to assign *forms* which stand in specified relations to other *forms*.

By the term constant we see that we now mean any function whatsoever which does not change under any changes which are supposed in the equation, or which become necessary to be considered in the actual verification of the conditions of the problem. In the Differential Calculus only is this constant absolutely constant in the primitive sense of the term, the reason being, that every possible continuous change is undergone by the variable, and the constant must therefore satisfy $\Delta c = 0$ for every value of the difference of the variable. In the Calculus of *Finite Differences*, (which perhaps analogy may at a future time denominate *discontinuous differences*), the condition is $\Delta c = 0$ for given *finite* values of the difference of the variable. In the Calculus of Functions the condition is $\Delta c = 0$ for given *changes of form* of the variable. And this, it must be observed, may lead us,

Calculus of
Functions.

Calculus of Functions. since any manner of satisfying the conditions is a solution, to either of the following species of arbitrary constants.

1. A continuous function of x , unchanged when x is changed into αx , βx , &c.
2. A discontinuous function of the same kind.
3. A continuous constant, that is to say, a constant in the primitive notion of the word.
4. A discontinuous constant, that is, a discontinuous function of x , which only so far depends upon x , as certain values of the latter mark the epochs of the solutions of continuity.*

(181.) The last-mentioned sort of constant, being of a somewhat unusual character, requires some discussion. The question is, can the equation $\phi \alpha x = \phi x$ be satisfied by a function of x , which is

$$\begin{aligned} & \dots \dots \dots \\ c_1 & \text{ from } x = -a_2 \text{ to } x = -a_1 \\ c_0 & \text{ from } x = -a_1 \text{ to } x = 0 \\ c_2 & \text{ from } x = 0 \text{ to } x = a_2 \\ & \dots \dots \dots \end{aligned}$$

or the like? It is obvious that this will be the case for such limits as contain αx and x between them at the same time. For, calling c_x the constant in question, if a_1 and a_2 be values of x between which c_x remains the same, then $c_{\alpha x} = c_x$ will be true if αx be between the limits when x is so. Or rather this should be extended as follows. If there be any number of values $a_1, a_2, a_3, \dots$, ranged in order of magnitude, and having this property, that when x lies any where between any two consecutive values, αx lies either in the same interval, or entirely in some other, then the constant may be made discontinuous, provided it always have the same value in the intervals in which x and αx are simultaneously found. And if this can be partially done, the arbitrary function may be made discontinuous, being constant for the intervals just mentioned, and otherwise continuous or discontinuous in all other intervals.

The condition may be simply represented geometrically (fig. 2) as follows. Take any origin O , and draw a line OA inclined at an angle of 45° to rectangular axes of coordinates. Take any points in this line, and through them draw intersecting parallels to the axes. Then if the curve whose ordinate is αx cut these lines in intersections of the parallels, the preceding conditions are fulfilled wholly or partially, except where the diagonal curve and diagonal right line are in immediately contiguous rectangles. For the higher curve in figure 2, the dotted lines on the same level denote intervals in which the constants must be the same. For the lower curve the constant must be the same in the interval

* A distinguished analyst calls Algebra the science of pure time, as Geometry is that of pure space. The notions of time, succession, and number are so nearly related that we have no doubt very admissible conventions would make this palpably true. In the mean while we are very glad to avail ourselves of such an authority to introduce a most convenient term. We have always regretted the loss of the metaphysics of fluxions, which there was no necessity for discarding with the notation. Admit the following definition—Time is the succession of values of the independent variable; so that a condition is said to be satisfied at an earlier or a later time, according as it is satisfied for a greater or less (or less or greater) value of the independent variable—and the language of Algebra will be materially facilitated. If all the circumlocutions by which English mathematicians have of late years endeavoured to prove themselves free from the notion of successive values being given in successive times were collected together, we should write in the title-page "*corrigenda aut potius vitanda*."

between the two intersections with the diagonal line, and may take another value for all greater values of x , positive or negative. In figure 3 there may be discontinuity in the constant in the manner indicated by the dotted lines. It must be observed, however, that it is not in this arbitrary way that a discontinuous function can be made constant in all the several intervals of continuity, throughout the complete scale of magnitude, from $-\infty$ to $+\infty$. The general determination of this question is a problem of considerable difficulty, amounting to the following. Required a scale of magnitudes

$$\dots a_{-1} \ a_0 \ a_1 \ a_2 \dots$$

proceeding from $-\infty$ to $+\infty$, and such that

$$(x - a_n)(x - a_{n+1}) \text{ and } (\alpha x - a_{n+m})(\alpha x - a_{n+m+1})$$

may always be negative together, continuous or discontinuous relations existing between n and m , but such that no interval may be missing.

(182.) We shall now take some algebraical examples of these discontinuous constants. And first we ask what discontinuous constant satisfies $\phi \log x = \phi x$. If x be less than 1, so is $\log x$; but if x be greater than 1, $\log x$ is not always so. In fact, the condition cannot in any way be completely satisfied, except where the constant is altogether continuous.

Secondly, let $\phi x^m = \phi x$, m being a positive integer. Here, when x lies between $+1$ and -1 , x^m has the same limits, while from $x = -1$ to $x = -\infty$, and from $x = +1$ to $x = +\infty$, x^m has the same limits when m is odd, and the latter limits when m is even. Hence, when m is even, the discontinuous constant is

$$C_{-\infty, -1} + C'_{-1, +1} + C_{+1, +\infty},$$

meaning by $C_{-\infty, -1}$ a constant which is C for all values between $-\infty$ and -1 , and zero for all others. When m is odd, the most general solution is

$$C_{-\infty, -1} + C'_{-1, 0} + C''_{0, +1} + C'''_{+1, \infty}.$$

Thirdly, the solution of $\phi(a + bx) = \phi x$ is

$$C_{-\infty, \frac{a}{1-b}} + C'_{\frac{a}{1-b}, +\infty}.$$

Fourthly, both $\sin x$ and x lie between 0 and ∞ , when x lies between 0 and π , 2π and 3π , &c., and between 0 and $-\infty$ when x lies between 0 and $-\pi$, -2π and -3π , &c. A solution of $\phi \sin x = \phi x$ is

$$\begin{aligned} & C_{-2m\pi, -(2m+1)\pi} + C_{(2m+1)\pi, (2m+2)\pi} \\ & + C'_{2m\pi, (2m+1)\pi} + C'_{-(2m+1)\pi, -(2m+2)\pi}. \end{aligned}$$

(183.) For those values of x which satisfy the equation $\alpha x = x$, the constant may possibly assume what we may call a single value belonging to that one value of x only, and totally independent of the preceding and following values. Thus, when m is an even whole number, the values of the constant which satisfies $\phi x^m = \phi x$ may be single, for $x = 0$ and $x = 1$, and the discontinuous constant may be written thus,

$$C_{-\infty, -1} + C'_{-1, 0} + C''_0 + C'_{0, +1} + C'''_{+1} + C_{+1, +\infty}.$$

(184.) We shall not here attempt application of the preceding theory to the case of impossible values of x , but shall proceed to show that what we have done is as necessary to the complete understanding of a functional equation as the arbitrary constant to that of a differential equation, and for the same reasons.

The developement of a given function into any finite series of terms is a direct operation, not admitting more ambiguity of expression than is contained in the function itself which is to be developed. The same may be said

Calculus of Functions. of the envelopement of a finite series of terms; these being direct and inverse operations of which either may be assumed as direct, the question being the same in both. But the developement of a function in an infinite series of terms involves the inverse operation of envelopement, with the consideration of a remainder, unknown both in form and magnitude, except by its developement. If this remainder be to be determined by a functional equation, and directly or indirectly many are so determined, an arbitrary constant may enter the solution, and the process may be vitiated if its existence be unknown, as has repeatedly happened. If the remainder be that of a convergent series, and diminishing without limit as the number of preceding terms is increased without limit, the constant being neglected, this does not make the last-mentioned constant, if entering by itself, diminish without limit. On the contrary, it is rather in a divergent series that an independent constant term may be neglected; and it happens, in fact, to be the case, that series which are or may be divergent present fewer difficulties in this respect than those which are always convergent. But in the case of a divergent series the form of the remainder is independent of the magnitude of the quantity, and the arbitrary constant must not be neglected.

Some writers have rejected divergent series, on account of the difficulties to which they are subject. But we shall endeavour to make it appear likely that series which are *never* divergent should be much more cautiously used. Or rather we should say, that this has been already taught in the case of series arising from the inversion of periodic series, such as those for the sine, cosine, &c., and that it may be useful to make the same more apparent in the case of other series.

The difficulties which lie in the way of the solution of functional equations will render it impossible to apply the preceding method in its full extent. But whereas it has been usual to search both for the form of the constant, (that is, to determine the intervals of continuity, where discontinuity has been proved or suspected,) and its values from the circumstances of each particular case, the preceding method may be made useful in pointing out the former, in cases where the determination of the required envelopement is not possible. But we shall give an instance, to show that the method does explain some of the circumstances which have occasioned perplexity in treating of series. The first example will show that the constant of the calculus of finite differences is a particular case.

(185.) Let

$$\psi x = \beta x (1 + \psi \alpha x),$$

which is evidently the functional equation from which the envelopement of the following series may be obtained,

$$\beta x + \beta x \beta \alpha x + \beta x \beta \alpha x \beta \alpha^2 x + \dots$$

Similarly

$$\psi x = \beta x + \psi \alpha x$$

gives

$$\psi x = \beta x + \beta \alpha x + \beta \alpha^2 x + \dots,$$

and

$$\psi x = \beta x + \gamma x \psi \alpha x$$

gives

$$\psi x = \beta x + \gamma x \beta \alpha x + \gamma x \gamma \alpha x \beta \alpha^2 x + \dots$$

Let $\alpha x = x + 1$. It will be found that $\phi(x+1) = \phi x$ cannot be satisfied by any but a *periodically* discontinuous constant. Let h be any given value of x , and m being a whole number, positive or negative, let the value of the constant be C between the limits $h + m$

and $h + m + \frac{1}{2}$, and C' between $h + m + \frac{1}{2}$ and $h + m + 1$. This is expressed by

$$C_{h+m, h+m+\frac{1}{2}} + C'_{h+m+\frac{1}{2}, h+m+1}$$

Consequently the solution of every equation of finite differences of the usual form may contain a discontinuous constant of this species, in which it is worthy of remark that the determination of the epochs of discontinuity contains an arbitrary constant.

(186.) Next, let $\alpha x = x^2$. The equation $\phi x^2 = \phi x$ may be solved by a discontinuous constant of the form

$$C_{-\infty, -1} + C'_{-1, 0} + C''_0 + C'_{0, +1} + C'''_{+1} + C_{+1, \infty},$$

these being what we have called single values when $x = 0$, and when $x = 1$.

Let us consider the equation

$$\phi x = \frac{x}{1+x^2} (1 + \phi x^2)$$

a particular solution of which is x . Another particular solution is $\frac{1}{x}$ (177.); but we find a solution of

$$\phi x = \frac{x}{1+x^2} \phi(x^2)$$

$$\phi x = x - \frac{1}{x},$$

and hence the general solution of the equation (as to continuous constant solutions of $\phi x^2 = \phi x$) is

$$\phi x = x + c \left(x - \frac{1}{x} \right);$$

the developement of ϕx in a series is as follows,

$$\phi x = \frac{x}{1+x^2} + \frac{x}{1+x^2} \cdot \frac{x^2}{1+x^4} + \frac{x}{1+x^2} \cdot \frac{x^2}{1+x^4} \cdot \frac{x^4}{1+x^8} + \dots$$

which presents several curious circumstances. It is always convergent, is not periodic for possible values of x , is a symmetrical function (apparently) of x and $\frac{1}{x}$, while the envelopement of it cannot be so, even if we take the most general of the solutions which we have found, namely,

$$x + \left(x - \frac{1}{x} \right) \theta \cos 2\pi \frac{\lambda^2 x}{\lambda^2}.$$

The value of the series may to all appearance be determined by common Algebra as follows. Divide ϕx by $1 - x^2$, which gives

$$\frac{\phi x}{1-x^2} = \frac{x}{1-x^4} + \frac{x^3}{1-x^8} + \dots = \sum \frac{x^{2^m-1}}{1-x^{2^{m+1}}},$$

which will be found by developing each term to be

$$x + x^3 + x^5 + \dots \text{ad infinitum,}$$

for

$$2^{m+1} \times 2n + 2^m - 1$$

can be easily shown to be equal to a given odd number for only one whole value of m and n . That is, we have

$$\frac{\phi x}{1-x^2} = \frac{x}{1-x^2} \quad \phi x = x:$$

but proceeding in precisely the same manner to develop each term in powers of $\frac{1}{x}$ we find that

Calculus of Functions.

$$\frac{\phi x}{1-x^2} = x^{-1} + x^{-3} + \dots \text{ or } \phi x = \frac{1}{x}.$$

But at the same time, the series being always convergent, we may easily find by summation that it fulfils the following conditions:

$$\begin{aligned} \text{when } x \text{ lies between } \pm 1 \quad \phi x &= x \\ \text{when } x = 0 \quad \phi x &= 0 \\ \text{when } x > 1 \text{ or } < -1 \quad \phi x &= \frac{1}{x}. \end{aligned}$$

If we now proceed to determine the discontinuous constant, we may do it, knowing the epochs of discontinuity, by observing that, as x diminishes or increases

without limit, $\frac{\phi x}{x}$ and $x \times \phi x$ have the limit 1. Assuming, therefore, the expression

$$x + c \left(x - \frac{1}{x} \right),$$

we see that, since the only changes of the constant (the single value excepted) take place at $x = +1$ and $x = -1$, the value of the constant must be expressed by

$$(-1)_{-\infty, -1} + (0)_{-1, +1} + (-1)_{+1, +\infty} + (0)_0,$$

in which, as it happens, it is unnecessary to express the single value. And this confirms what precedes.

(187.) The principle in which the preceding process differs from common Algebra is in the adoption of an axiom which may be thus stated: *discontinuous functions are to be admitted as solutions of an equation, when they satisfy its conditions.* All the rest is a necessary consequence of ordinary reasoning. This axiom is already admitted and used in various parts of Mathematics; and is, in fact, tacitly allowed in the explanation which is usually given of the above. Let

$$V^2 - a_n V + b_n = 0$$

be the equation whose roots are x^n and y^n . Then

$$\begin{aligned} x &= \frac{b_1}{a_1} + \frac{1}{a_1} x^2 = \frac{b_1}{a_1} + \frac{1}{a_1} \frac{b_2}{a_2} + \frac{1}{a_1 a_2} x^4, \text{ \&c.} \\ &= \frac{b_1}{a_1} + \frac{b_2}{a_1 a_2} + \frac{b_4}{a_1 a_2 a_4} + \frac{b_8}{a_1 a_2 a_4 a_8} + \text{\&c.} \end{aligned}$$

By similar reasoning, and neglecting the remainder, y may be expanded into the same series; which is, therefore, an analytical representation of either root. If

$y = \frac{1}{x}$, this series becomes the preceding; and, as in all

other cases yet observed, though it represents either root, gives the *least* root by arithmetical approximation. This theorem, namely, that the development of an ambiguous function gives, arithmetically considered, the least of the values of the envelopment, has never, so far as we know, been generally proved. Generally speaking, divergency has been the consequence of an attempt to expand any other value, but in the preceding instance this is not the case. And the same occurs in cases of infinitely continued fractions, the envelopment of which is generally ambiguous, the arithmetical value of the continued fraction always giving the least of the values.

(188.) As another example, take the case of

$$\phi x = \frac{1}{1+x} \phi x^2.$$

The solution of which is $C(1-x)$, where C may be subject to the same discontinuity as before. If we consider the continued product

$$\frac{1}{1+x} \cdot \frac{1}{1+x^2} \cdot \frac{1}{1+x^4} \dots \text{ ad inf.,}$$

or

$$\frac{1-x}{1-x^2} \frac{1}{1+x^2} \dots \text{ or } \frac{1-x}{1-x^\infty},$$

we see that it is $1-x$ for all values of x lying between -1 and $+1$, and 0 for all others. This confirms the solution derived from the functional equation, which is

$$\{(1)_{-1, +1} + (0)_{-\infty, -1} + (0)_{+1, +\infty}\} (1-x).$$

(189.) From what precedes we may gather that the discontinuous constant (for all discontinuously variable functions may be reduced to functions involving discontinuous constants) supplies the place of the symbols of ambiguity in Algebra, in cases where the generality of any unknown form prevents the ambiguity from being explicitly considered: and also that the epochs of discontinuity may be determined from the known functions of which unknown functions enter.

The discontinuity of the arbitrary function which enters the solution of a partial differential equation has long been allowed to be possible; but we are not aware that the possible discontinuity of the constant of integration has received any attention. But the fundamental difference of a common functional symbol and a differential coefficient renders it unlikely that the last-mentioned branch of the subject will present very perceptible analogies with that just considered.

(190.) We shall now proceed to consider the difficulties which stand in the way of actually finding particular solutions of the equations

$$\psi x = \beta x + \gamma x \psi \alpha x \dots (1)$$

$$\psi x = \gamma x \psi \alpha x \dots (2),$$

corresponding to the equations of differences

$$u_x = \beta A^{-1} x + \gamma A^{-1} x u_{x-1}$$

$$u_x = \gamma A^{-1} x u_{x-1}.$$

As all that has hitherto preceded seems to point out the difficulty, perhaps impossibility, of making any progress, except in cases which admit of $\alpha^n x$ being expressed as an explicit function of n , we shall assume this operation as fundamental in the case of all positive and negative whole values of n . That is, we shall assume the exponential inverse Ax to be given when the form of αx is given.

(191.) All functions may be divided into three classes.

1. Periodic functions of a finite order, which satisfy the equation $\alpha^n x = x$ for some whole value of n .

2. What we will call periodic functions of an infinite order, being those in which $\alpha^n x$ has a finite limit when n increases without limit. For, as we have shown in (37.), the limit in question is a solution of $\alpha x = x$, and gives $\alpha \alpha^n x = x$ when n is infinite.

3. Functions in which $\alpha^n x$ increases or decreases without limit, $x = \infty$ or $x = 0$ not being a root of $\alpha x = x$, and which therefore have no periodic character whatsoever.

(192.) In the first of these cases, equation (1) admits of absolute solution. We have already had an instance (137.): the general method (which is due to Mr. Babbage) being, if $\alpha^n x = x$, eliminate $\psi \alpha x, \psi \alpha^2 x, \dots \psi \alpha^{n-1} x$, from the equations

Calculus of
Functions.

$$\begin{aligned}\psi x &= \beta x + \gamma x \psi \alpha x \\ \psi \alpha x &= \beta \alpha x + \gamma \alpha x \psi \alpha^2 x \\ &\vdots \\ \psi \alpha^{n-1} x &= \beta \alpha^{n-1} x + \gamma \alpha^{n-1} x \psi x,\end{aligned}$$

giving

$$\psi x = \frac{\beta x + \gamma x \beta \alpha x + \gamma x \gamma \alpha x \beta \alpha^2 x + \dots + \gamma x \dots \gamma \alpha^{n-2} x \beta \alpha^{n-1} x}{1 - \gamma x \gamma \alpha x \gamma \alpha^2 x \dots \gamma \alpha^{n-1} x}$$

When the numerator and denominator are both finite, no arbitrary constant can enter; for a solution of $\phi x = \gamma x \phi \alpha x$ gives

$$1 = \gamma x \gamma \alpha x \dots \gamma \alpha^{n-1} x,$$

which makes the denominator of the preceding infinite. Excepting discontinuous solutions, we have here then the complete and only solution. But it must be observed, that the discontinuous function is of the species noticed in (72.), and the method might be extended to this case.

(193.) In what follows, we mean by P_0, α_0 , &c., functions of x ; by P_1, α_1 , &c. the corresponding functions of αx , and so on. So that the solution of

$$P_0 = B_0 + c_0 P_1 \quad (\alpha^n x = x) \dots (1)$$

is

$$P_0 = \frac{B_0 + c_0 B_1 + c_0 c_1 B_2 + \dots + c_0 c_1 \dots c_{n-2} B_{n-1}}{1 - c_0 c_1 c_2 \dots c_{n-1}}.$$

The denominator of this solution must vanish whenever the numerator vanishes, except only when $B_0 = 0$. For if the numerator vanish without the denominator, $P_0 = 0, P_1 = 0$, and therefore $B_0 = 0$. But this appears more clearly by observing that the first of the following equations gives all the rest, and that they give an equation of condition among c_0, c_1 , &c. For simplicity, we suppose the periodic function to be of the fourth order, ($c_4 = c_0, B_4 = B_0$, &c.)

$$\begin{aligned}B_0 + c_0 B_1 + c_0 c_1 B_2 + c_0 c_1 c_2 B_3 &= 0 \\ B_1 + c_1 B_2 + c_1 c_2 B_3 + c_1 c_2 c_3 B_0 &= 0 \\ B_2 + c_2 B_3 + c_2 c_3 B_0 + c_2 c_3 c_0 B_1 &= 0 \\ B_3 + c_3 B_0 + c_3 c_0 B_1 + c_3 c_0 c_1 B_2 &= 0.\end{aligned}$$

Now there cannot be two perfectly independent relations between c_0, c_1 , &c., which reconcile the preceding, for if there be two such, namely $V_0 = 0$ and $W_0 = 0$, then $V_1 = 0$ and $W_1 = 0$ fulfil the same conditions, and so on; that is, there are more than $n-1$, namely $(2n-2)$ conditions among c_0, c_1 , &c., and they cease to be indeterminate. Except only when both V_0 or W_0 are symmetrical functions of c_0, c_1 , &c., or else when the two are connected by an equation of the form

$$f(V_0, W_0) = \xi \alpha x.$$

Therefore if one of the two be a symmetrical function of c_0, c_1 , &c. (or a form of $\xi \alpha x$), both are so. Now

$$1 - c_0 c_1 c_2 c_3 = 0$$

makes the preceding equations identical, as appears by substituting in either the reciprocal of the one omitted in the last term for the product of all the rest. It is therefore the sole condition under which the preceding equations can be true, which may be verified by actual solution.

When the denominator of P_0 vanishes without the numerator, it is a sign that there is no continuous solution whatsoever; in fact, the last-mentioned condition combined with the equation (1) will be found to give an equation of condition, namely, that the function in the

numerator of P_0 shall be nothing. It may also be easily shown that the condition $1 - c_0 c_1 \dots c_{n-1} = 0$ is required in order that $P_0 = c_0 P_1$ may be possible.

(194.) Given αx , required values of c_0, c_1 , &c., which will satisfy the condition

$$1 - c_0 c_1 \dots c_{n-1} = 0.$$

It is evident that

$$c_0 = \frac{m_0}{m_0 + m_1 + m_2 + \dots + m_{n-1}}$$

satisfies the condition

$$c_0 + c_1 + c_2 + \dots + c_{n-1} = 1:$$

that

$$\frac{m_0}{\sum m_0} - \frac{p_0}{\sum p_0}$$

therefore makes the same sum = 0; or that

$$\frac{m_0}{\sum m_0} - \frac{p_0}{\sum p_0} = c_0$$

satisfies the condition in question; and the exponent may be multiplied by any form of $\xi \alpha x$, or any symmetrical function of $x, \alpha x \dots \alpha^{n-1} x$.

(195.) We may integrate the equation

$$P_0 = c_0 P_1,$$

the preceding condition being fulfilled, as follows. Let the degree of the function be the fourth, and assume

$$\begin{aligned}P_0 &= 1 + c_0 + c_0 c_1 + c_0 c_1 c_2 \\ P_1 &= 1 + c_1 + c_1 c_2 + c_1 c_2 c_3 \\ c_0 P_1 &= c_0 + c_0 c_1 + c_0 c_1 c_2 + 1 = P_0,\end{aligned}$$

whence

$$P_0 = (1 + c_0 + c_0 c_1 + c_0 c_1 c_2) \xi \alpha x$$

is the general solution of $P_0 = c_0 P_1$.

(196.) Given c_0 , so that $c_0 c_1 c_2 c_3 = 1$ required solutions of

$$B_0 + c_0 B_1 + c_0 c_1 B_2 + c_0 c_1 c_2 B_3 = 0 \quad (\alpha^4 x = x).$$

If we assume the first of the succeeding equations, we have those which follow:

$$\begin{aligned}B_0 &= V + V' c_0 + V'' c_0 c_1 + V''' c_0 c_1 c_2 \\ c_0 B_1 &= V c_0 + V' c_0 c_1 + V'' c_0 c_1 c_2 + V''' \\ c_0 c_1 B_2 &= V c_0 c_1 + V' c_0 c_1 c_2 + V'' + V''' c_0 \\ c_0 c_1 c_2 B_3 &= V c_0 c_1 c_2 + V' + V'' c_0 + V''' c_0 c_1,\end{aligned}$$

whence the equation in question is satisfied if

$$V + V' + V'' + V''' = 0,$$

these being severally symmetrical functions of $x, \alpha x, \alpha^2 x$, and $\alpha^3 x$.

The preceding process fails when $c_0 = 1$, by giving only the solution $B_0 = 0$; but in that case we have, as before,

$$B_0 = \frac{p_0}{\sum p_0} - \frac{q_0}{\sum q_0}.$$

(197.) Let us now take the equation

$$\phi x = \beta x + \gamma x \phi \alpha x \quad (\alpha^4 x = x) \dots (\phi),$$

or

$$P_0 = B_0 + c_0 P_1,$$

where

$$\begin{aligned}B_0 + c_0 B_1 + c_0 c_1 B_2 + c_0 c_1 c_2 B_3 &= 0 \dots (B) \\ 1 - c_0 c_1 c_2 c_3 &= 0 \dots (c)\end{aligned}$$

Assume

$$P_0 = Q_0 (1 + c_0 + c_0 c_1 + c_0 c_1 c_2) = Q_0 C_0, \quad B_0 = B'_0 C_0,$$

which makes (ϕ) become

$$Q_0 C_0 = B'_0 C_0 + C_0 Q_1,$$

or

$$Q_0 = B'_0 + Q_1.$$

Calculus of
Functions.

Calculus of Functions. And because C_0 is a solution of $P_0 = c_0 P_1$, the condition (B) gives

$$B'_0 + B'_1 + B'_2 + B'_3 = 0.$$

Assume $Q_0 = m B'_0 + m' B'_1 + m'' B'_2$,
giving $\left. \begin{matrix} m B'_0 \\ + m' B'_1 \\ + m'' B'_2 \end{matrix} \right\} = B'_0 + \left\{ \begin{matrix} m B'_1 \\ + m' B'_2 \\ + m'' (-B'_0 - B'_1 - B'_2) \end{matrix} \right\}$,
which is satisfied by making

$$m = 1 - m'' \quad m = \frac{3}{4}$$

$$m' = m - m'' \quad m' = \frac{2}{4}$$

$$m'' = m' - m'' \quad m'' = \frac{1}{4},$$

or

$$Q_0 = \frac{1}{4} (3 B'_0 + 2 B'_1 + B'_2)$$

$$P_0 = Q_0 C_0 = \frac{C_0}{4} \left(3 \frac{B_0}{C_0} + 2 \frac{B_1}{C_1} + \frac{B_2}{C_2} \right).$$

This is only a particular solution, but it may be made general by adding $C_0 \xi \alpha x$, the general solution of $P_0 = c_0 P_1$, which gives

$$P_0 = \frac{C_0}{4} \left(3 \frac{B_0}{C_0} + 2 \frac{B_1}{C_1} + \frac{B_2}{C_2} \right) + C_0 \xi \alpha x \\ = \frac{1}{4} (3 B_0 + 2 c_0 B_1 + c_0 c_1 B_2) + C_0 \xi \alpha x.$$

And the solution of the same equation, when αx is of the n th order, and the conditions are altered accordingly, is

$$P_0 = \frac{1}{n} \{ \overline{n-1} B_0 + \overline{n-2} c_0 B_1 + \dots + c_0 c_1 \dots c_{n-2} B_{n-2} \} \\ + (1 + c_0 + c_0 c_1 + \dots + c_0 c_1 \dots c_{n-2}) \xi \alpha x.$$

In the case of a periodic function of the second order the preceding assumes the form

$$P_0 = \frac{1}{2} \beta x + (1 + \gamma x) \xi \alpha x,$$

and B_0 must be of the form $V - V \gamma x$, where V is a symmetrical function of x and αx , or it may be of γx

and $\gamma \alpha x$, that is, of γx and $\frac{1}{\gamma x}$. If we assume

$$V = \frac{\sqrt{-\gamma x}}{1 - \gamma x} \xi \alpha x$$

we may get the form of solution given by Mr. Babbage, (*Examples, &c.*, p. 21.)

The preceding solution, in an equivalent form, might be obtained by finding $V, V', &c.$ from the equations

$$B_0 = V + V' c_0 + \dots$$

$$B_1 = V + V' c_1 + \dots$$

$$\dots$$

and solving the equation

$$P_0 = V + V' c_0 + \dots + c_0 P_1$$

by such an assumption as

$$P_0 = W + W' c_0 + \dots$$

(198.) We now come to the second of the cases mentioned in (191.), namely, where αx is a periodic function of an infinite order, as we have called it by

extension, namely, where $\alpha^n x$ has a finite limit when n increases without limit. The analogy is not very perfect, because the infinitely periodic function, instead of approximating to the value of x first set out with, approximates to a determinate value of x , namely, a root of $\alpha x = x$. Nor have we any method of solution in this case which cannot also be applied to the third species of functions in (191.) We shall, therefore, confine what we have to say on infinitely periodic functions to this article, and then drop the name.

Let p be a root of $\alpha x = x$. Then, since some values of x always make $\alpha^n x$ approximate to p (we have found this by observation), and since the equation

$$\phi x = \beta x + \gamma x \phi \alpha x$$

gives

$$\phi \alpha^n x = \beta \alpha^n x + \gamma \alpha^n x \phi \alpha^{n+1} x,$$

the reasoning of (37.) shows that we must have

$$\phi p = \beta p + \gamma p \phi p,$$

or

$$\phi p = \frac{\beta p}{1 - \gamma p},$$

and if $\gamma p = 1$, ϕp is infinite or $\beta p = 0$. Again, if there be an arbitrary function in the solution, then $\phi x = \gamma x \phi \alpha x$ must admit of solution, or $\phi p = \gamma p \phi p$, and either ϕp is nothing or infinite, or $\gamma p = 1$. Such reasoning as the above is worth little, except as an index to the sort of methods which are probably good, in attempting to solve an equation of the kind specified. We have found it confirmed by instances in which the equation was formed from a given algebraical solution, as follows: the equation of the form in question, which has the solution $\mu x + \nu x \cdot \xi \alpha x$ is

$$\phi x = \frac{\mu x \cdot \nu \alpha x - \mu \alpha x \nu x}{\nu \alpha x} + \frac{\nu x}{\nu \alpha x} \cdot \phi \alpha x,$$

in all the cases in which νp is finite, we have $\gamma p = 1$ and $\beta p = 0$; while $\nu p = 0$ gives $\frac{1}{\alpha' p}$ for the value of γp , and $\mu p \left(1 - \frac{1}{\alpha' p} \right)$ for that of βp .

(199.) A process similar to that employed in the case of periodic functions gives (using the same denominations as before, namely, $P_0 = \phi x \quad B_0 = \beta x \quad c_0 = \gamma x \quad P_1 = \phi \alpha x$, &c.)

$$P_0 - c_0 P_1 = B_0$$

$$P_0 - c_0 c_1 P_2 = B_0 + c_0 B_1$$

$$P_0 - c_0 c_1 c_2 P_3 = B_0 + c_0 B_1 + c_0 c_1 B_2, \&c.$$

Or if

$$\varpi_n = c_0 c_1 c_2 \dots c_n$$

$$P_0 - \varpi_n P_{n+1} = B_0 + \varpi_0 B_1 + \varpi_1 B_2 \dots + \varpi_{n-1} B_n.$$

Consequently, in forming the series from a given involution, the true arithmetical value is

$$\phi x - \varpi_\infty \phi p \dots (\varpi),$$

and is only ϕx where $\varpi_\infty \phi p = 0$. But in obtaining the involution from the solution of the functional equation, we need not attend to any correction of the general value of ϕx ; for ϖ_∞ is a solution of $P_0 = c_0 P_1$, and the second term of (ϖ) may be considered as a part of $\nu x \xi \alpha x$ in the complete solution $\mu x + \nu x \xi \alpha x$. But we here see reason to confirm the entry of a *discontinuous constant* rather than a *discontinuous variable function*, into the form of the solution of the general equation, which gives the involution of the above-

Calculus of
Functions.

mentioned series. For different values of x may cause the limits of $\alpha^n x$ to be different roots of $\alpha x = x$, or may make it increase without limit. And since there can be but one solution (really algebraical) of $\phi x = \gamma x \phi \alpha x$, in all cases where the continued product ϖ_∞ is algebraical, it is the solution in question: whence if p_1, p_2, p_3 , &c. be the several roots of p , the preceding term must have the form

$$(C_1 \phi p_1 + C_2 \phi p_2 + \dots) \varpi_\infty,$$

where C_1, C_2 severally disappear for all values of x , except those which lie between certain limits. This also tends to establish what we have hitherto observed, namely, that among the epochs of discontinuity are the arithmetical values which satisfy $\phi x = x$.

(200.) The same considerations apply to the integration of equations of differences of the form

$$u_x = B_x + G_x u_{x-1};$$

but the case which has been completely resolved, namely, where αx is a periodic function, gives only equations of differences which involves discontinuous functions (in value), as appears from (64.) and the reduction of

$$\phi x = \beta x + \gamma x \phi \alpha x \text{ in (175.)},$$

which is

$$u_x = \beta A^{-1} x + \gamma A^{-1} x u_{x-1}.$$

(201.) We proceed to point out some cases in which the general functional equation of the first degree, containing one unknown function of a known function, is integrable on the supposition that Ax is given when αx is given. The equation

$$\phi x = \alpha' x \phi \alpha x$$

is integrable, where $\alpha' x$ is the differential coefficient of αx .

The general solution is $A'x \xi \alpha x$. For the differentiation of both sides of the equation

$$A \alpha x = A x - 1$$

gives

$$A' \alpha x \cdot \alpha' x = A' x,$$

or $A'x$ is a particular solution of the equation in question. Thus, to integrate

$$\phi x = 2 x \phi x^2,$$

we find that if

$$\alpha x = x^2, A x = -\frac{\lambda^2 x}{\lambda 2}, A' x = -\frac{1}{\lambda 2 \cdot x \lambda x},$$

or

$$\phi x = -\frac{1}{\lambda 2 \cdot x \lambda x} \cdot \theta \cos 2\pi \frac{\lambda^2 x}{\lambda 2}.$$

Denoting the differential coefficient of ϕx by $D \phi x$, we find in the same way, that

$$\phi x = \frac{D \alpha^m x}{D \alpha^n x} \phi \alpha^{m-n} x$$

gives

$$\phi x = A' \alpha^n x \cdot \xi \alpha x,$$

because

$$A \alpha^n x = A \alpha^m x + m - n.$$

We have also

$$D \alpha^m x = D \alpha \cdot \alpha^{m-1} x \cdot D \alpha \cdot \alpha^{m-2} x \dots D \alpha x,$$

where $D \alpha$ is considered as one functional symbol.

(202.) The equation $\phi x = A x \phi \alpha x$ can be solved in the form of a definite integral only. Let $\phi x = \chi A x$, giving

$$\chi A x = A x \chi A \alpha x = A x \chi (A x - 1)$$

or

$$\chi x = x \chi (x - 1),$$

whence

$$\phi x = \int e^{-x} a^{Ax} da, \text{ from } a = 0 \text{ to } a = \infty;$$

here χx is what is usually denoted by Γx .

(203.) If $\varpi_1 x$ and $\varpi_2 x$ be particular solutions of the equations

$$\phi x = \gamma_1 x \phi \alpha x \quad \phi x = \gamma_2 x \phi \alpha x,$$

then $\varpi_1 x \cdot \varpi_2 x$ and $\frac{\varpi_1 x}{\varpi_2 x}$ are particular solutions of

$$\phi x = \gamma_1 x \gamma_2 x \phi \alpha x \quad \phi x = \frac{\gamma_1 x}{\gamma_2 x} \phi \alpha x.$$

From this we find that

$$\phi x = (D \alpha x)^n \phi \alpha x \text{ gives } \phi x = (D A x)^n \xi \alpha x,$$

and since $\phi x = c \phi \alpha x$ gives $\phi x = c^{Ax}$

$$\phi x = c (D \alpha x)^n \phi \alpha x \text{ gives } \phi x = c^{Ax} (D A x)^n \cdot \xi \alpha x.$$

(204.) From the solution of $\phi x = \gamma x \phi \alpha x$ immediately follows that of $\phi x = \gamma \alpha_1 x \phi \alpha x$, where $\alpha_1 x$ is any function which is convertible with αx . For if ϖx solve the first equation, then $\varpi \alpha_1 x = \gamma \alpha_1 x \varpi \alpha_1 x = \gamma \alpha_1 x \varpi \alpha_1 \alpha x$ or $\varpi \alpha_1 x$ solves the second equation. Thus, having got a particular solution of

$$\phi x = \frac{1}{1+x} \phi (x^2),$$

namely, $\phi x = 1 - x$, we can, by this article and the last, find particular solutions of the equation

$$\phi x = \left(\frac{c x^m}{1+x^n} \right)^p \phi x^2$$

$$1. \phi x = \frac{c^m}{2} \phi x^2 \quad \text{gives } \phi x = \left(\frac{c^m}{2} \right)^{-\frac{\lambda^2 x}{\lambda 2}} = (\lambda x)^{-\frac{\lambda c}{m \lambda 2} + 1}$$

$$2. \phi x = 2 x \phi x^2 \quad \dots \phi x = \frac{d}{dx} \left(\frac{\lambda^2 x}{\lambda 2} \right) = \frac{1}{\lambda 2 \cdot x \lambda x}$$

$$3. \phi x = c^m x \phi x^2 \quad \dots \phi x = \frac{(\lambda x)^{-\frac{1}{m} \frac{\lambda c}{\lambda 2}}}{\lambda 2 \cdot x}$$

$$4. \phi x = c x^m \phi x^2 \quad \dots \phi x = \frac{(\lambda x)^{-\frac{\lambda c}{2}}}{(\lambda 2)^m \cdot x^m}$$

$$5. \phi x = \frac{1}{1+x} \phi x^2 \quad \dots \phi x = 1 - x$$

$$6. \phi x = \frac{1}{1+x^n} \phi x^2 \quad \dots \phi x = 1 - x^n$$

$$7. \phi x = \left(\frac{c x^m}{1+x^n} \right) \phi x^2 \quad \dots \phi x = \frac{(\lambda x)^{-\frac{c}{\lambda 2}} (1-x^n)}{(\lambda 2)^m x^m}$$

$$8. \phi x = \left(\frac{c x^m}{1+x^n} \right)^p \phi x^2 \quad \dots \phi x = \frac{(\lambda x)^{-\frac{\lambda c}{2}} (1-x^n)^p}{(\lambda 2)^{mp} x^{mp}}.$$

(205.) The last is a particular case of the following. The solutions of

$$\phi x = \gamma \theta x \cdot \phi \alpha x$$

and

$$\phi x = \gamma x \phi \theta \alpha \theta^{-1} x$$

depend upon each other. If ϖx solve the first, $\varpi \theta^{-1} x$ solves the second: if ϖx solve the second, $\varpi \theta x$ solves the first. Hence a simple general expression of the problem may be given as follows. The solution of

Calculus of
Functions.

Calculus of Functions. $\phi x = \gamma x \phi \alpha x$ may be reduced to that of $\phi x = x \phi \gamma \alpha \gamma^{-1} x$, or the question is:—to express a given

function in the form $\phi^{-1} \left(\frac{\phi x}{x} \right)$. And it will be essential,

in future researches, to remark that the mode of solution of the equation does not depend on γx or αx , except in so far as they affect the form of $\gamma \alpha \gamma^{-1} x$. Given this last, the solution is given; or whatever γx may be, the method of finding the algebraical solution of

$$\phi x = \gamma x \phi \gamma^{-1} \kappa \gamma x$$

is independent of γx , or may be reduced to the solution of an equation not containing γx . For example, we find λx to be the value of ϕx in the equation

$$\phi x = \gamma x \phi (x^{\frac{1}{\gamma x}}).$$

(206.) When $\phi x = \gamma x \phi \alpha x$ can be solved, the solution of

$$\phi x = \beta x + \gamma x \phi \alpha x$$

can always be made to depend upon a form in which $\phi \alpha x$ has no coefficient except unity. For if νx be the solution of the former equation, we have

$$\frac{\phi x - \beta x}{\nu x} = \frac{\phi \alpha x}{\nu \alpha x}$$

or

$$\psi x = \frac{\phi x}{\nu x}, \quad \psi x = \frac{\beta x}{\nu x} + \psi \alpha x.$$

If a particular solution can be found to this, namely μx , then $\mu x \cdot \nu x$ is a particular solution of the preceding, and $\nu x (\mu x + \xi \alpha x)$ is the general solution. The corresponding equation of differences can evidently be solved by direct integration; being

$$\psi A^{-1} x = \frac{\beta A^{-1} x}{\nu A^{-1} x} + \psi A^{-1} (x - 1)$$

or

$$\Delta u_{x-1} = \frac{\beta A^{-1} x}{\nu A^{-1} x} \text{ a given function,}$$

where $u_x = \psi A^{-1} x$. And since the solution of

$$\phi x = \gamma x \phi \alpha x$$

also involves that of

$$\phi x = \lambda \gamma x + \phi \alpha x,$$

one general form of soluble equations is contained in the following conditions:

1. Let $\phi x = \gamma x \phi \alpha x$ be soluble by νx .
2. Let $\beta x = \nu x \lambda \gamma_1 x$, where $\phi x = \gamma_1 x \phi \alpha x$ is soluble.
3. Let the process of finite integration be granted, or the solution of $\Delta \psi x =$ a given function of x .

Thus we can solve

$$\phi x = (A'x)^m (n \lambda \alpha'x + c) + (\alpha'x)^m \phi \alpha x$$

as follows. A particular solution of the equation deprived of its known term is $\phi x = (A'x)^m$, and therefore we have to solve

$$\phi x = \lambda (\epsilon^x (\alpha'x)^m) + \phi \alpha x$$

or

$$\epsilon^{\phi x} = \epsilon^x (\alpha'x)^m \cdot \epsilon^{\phi \alpha x},$$

which gives

$$\epsilon^{\phi x} = \epsilon^{\epsilon \alpha x} \cdot (A'x)^m$$

or

$$\phi x = c A x + n \lambda A'x,$$

whence, in the original equation,

$$\phi x = (A'x)^m \{c A x + n \lambda A'x + \xi \alpha x\}.$$

VOL. II.

(207.) The two equations

$$\phi x = \beta x + \gamma x \phi \alpha x$$

$$\psi x = \beta_1 x + \gamma x \psi \alpha x$$

give the solution of

$$\chi x = m \beta x + m_1 \beta_1 x + \gamma x \chi \alpha x,$$

the sole condition between them being, that

$$\chi x - m \phi x - m_1 \psi x$$

must be a solution of

$$\phi x = (1 - m - m_1) \gamma x \phi \alpha x.$$

(208.) The instances in which we are able to solve this simple form of the functional equation of the first degree are of the most partial and isolated character. The instance $\phi x = x \phi (x - 1)$ gives a sufficient insight into the difficulty. Though the resolving function of this equation has long been used under the notation of Γx , it has never been expressed in any general form except as a definite integral. The well-known difficulties which attend the determination of that integral for all values of x are a sufficient indication of the transcendental which may be expected in the solution of the general equation.

(209.) If one of the functions $\beta x, \gamma x, \alpha x$, contain an arbitrary constant which it is required should itself be a particular value of the resolving function, a new condition is introduced which considerably limits the generality of the solution.* Let us suppose, for instance,

$$\phi x = \phi (x \phi a)$$

the solution of this equation, by preceding methods, is

$$\phi x = \theta \cos 2\pi \frac{\lambda x}{\lambda \phi a};$$

but θ is no longer indeterminate, since, upon making $x = a$, we have

$$\phi a = \theta \cos 2\pi \frac{\lambda a}{\lambda \phi a},$$

which condition θ must satisfy. At the same time, however, though the preceding stipulation limits the number of forms of the solutions yet obtained, we must remember (119.) that these solutions are, to all appearance, a very small proportion of those which might be found. And we also observe, that the appearance of another letter, with conditions as to the manner in which it must enter, entirely changes the class of problems to which the question must be referred. Since the solution of the preceding must be a function of x and a , let us represent it by $\phi(x, a)$, then what we have called ϕa means $\phi(a, a)$, in which a enters in a twofold manner; first, as it entered before, secondly, as x entered in the first. The substitution of $x \phi a$ for x is now that of $x \phi(a, a)$ for x , and the equation becomes

$$\phi(x, a) = \phi(x \phi(a, a), a),$$

in which a is to have a given value. But this condition can in no way affect the general question proposed in the problem, except so far as this: that though there may be a class of solutions which is true for the specific value of a only, and not for others, yet every solution of the general equation

$$\phi(x, y) = \phi(\tau \phi(y, y), y)$$

becomes a solution of the proposed equation, when y is made equal to a . Unless, therefore, we limit the pro-

* *Memoirs of the Analytical Society*, p. 107.

Calculus of
Functions.

blem at the beginning of this article, by adding, that the solutions presented are to be such as are true for one specific value of a only, we rather extend than limit the problem. But the preceding solution is not subject to any such limitation, the value of a being general.

To take a case of less difficulty, let us examine that given in the excellent Memoir already cited, namely,

$$\phi x^2 = \phi x + \phi a:$$

this we proceed to distinguish from the equation

$$\phi x^2 = \phi x + c.$$

The solution of the latter is

$$\phi x = c \frac{\lambda^2 x}{\lambda^2} + \xi x^2.$$

Assign any value whatever, $C \lambda^2$ to ϕa , and let ξx^2 be an arbitrary constant C' . Then if

$$C \lambda^2 = C \lambda^2 a + C',$$

or if

$$\phi x = C (\lambda^2 x - \lambda^2 a + \sqrt{2}),$$

the first equation is evidently satisfied. But this solution is as general as the corresponding solution of the second equation.

(210.) We had at first intended to exemplify at more length the method of Laplace, already described in (100.) But, on a closer examination, we find that the methods already given apply to every case in which such a reduction is useful with greater facility. Among the instances integrated by Laplace are*

$$(\phi x)^2 = \phi 2x + 2$$

$$\phi x^q = \phi m x + p.$$

With regard to the first, the equation

$$\phi x = c + c' (\phi \alpha x)^m$$

is easily solved with one particular solution, containing an arbitrary constant; and the independent artifice by which the equation of differences, corresponding to the first of the preceding, is integrated, is a step precisely corresponding to finding by observation the particular solution

$$\phi x = c^x + c^{-x}.$$

Of the second equation it is unnecessary to speak (103.) An equation solved by Monge, amounting to the following,

$$\phi (a u^m) - \phi (b u^m) = c u,$$

may be immediately reduced, by substituting x for $a u^m$, to the form

$$\phi x = p x^q + \phi (r x),$$

of which a particular solution may be immediately found by assuming $C x^m = \phi x$. And another equation in the same Memoir, of the form

$$\phi a x - \phi b x = p x^m - q x^n,$$

evidently admits of a particular solution of the form

$$C x^m - C' x^n + C'',$$

in which C and C' are determinate and C'' arbitrary

* We take these from the Memoir last cited, p. 111. It is necessary, however, that we should inform the reader (as otherwise he might, if he consulted that Memoir, dispute our assertion) that there is a material misprint in the equation cited from Monge. The two exponents n and m should be the same; the equation of Monge being of the form

$$\phi a u^m - \phi b u^m = c u,$$

not

$$\phi a u^m - \phi b u^n = c u.$$

See Monge's Memoir in the *Savans Etrangers*, 1773, p. 320.

(211.) We shall now proceed to some questions in Calculus of
Functions. which the Differential Calculus is combined with that of Functions.

In order to illustrate the distinction between the functional and differential symbols, let us propose the equations

$$\phi \left(\frac{dy}{dx} \right) = \phi y \quad \phi \alpha y = \phi y$$

for comparison. If ϕ be given, to find y or αy , then the first is a question of the inverse differential calculus, the second of inverse algebra. If y be a specific function of x in the first, and α a specific form in the second, then the two questions are of the same kind, since we can always express $\frac{dy}{dx}$ as a function of y , in any given case.

But if we leave y indeterminate, the first question can no more be solved than the second, when α is left indeterminate.

From the above we may, by a process precisely similar to those in (50.) and (102.), solve the first of these equations for a specific form of y , as follows. Express the n th differential coefficient of y as a function of n and y ; thus

$$D^n y = F(n, y),$$

in which a simultaneous substitution of $n-1$ for n , and $D y$ for y makes no change. Consider the first side as an arbitrary constant, and on this supposition determine (what we may by analogy call the differential inverse) the value of n . This value (δy) will be a function of y , which satisfies the equation

$$\phi \left(\frac{dy}{dx} \right) = \phi y - 1,$$

whence

$$\theta \cos 2\pi \delta y \text{ satisfies } \phi \left(\frac{dy}{dx} \right) = \phi y$$

$$c^{-\delta y} \text{ satisfies } \phi \left(\frac{dy}{dx} \right) = c \phi y,$$

$$\&c. \quad \&c. \quad \&c.$$

If y be called αx , and if we eliminate n between

$$D^n \alpha x = F(n, \alpha x) \text{ and } D^n \beta z = F_1(n, \beta z),$$

and then replace $D^n \alpha x$ and $D^n \beta z$ by arbitrary constants, or by any functions of $\cos 2\pi n$, we obtain a relation between βz and αx which is not altered by substituting $D \beta z$ and $D \alpha x$ for βz and αx , or if

$$\beta z = \phi(\alpha x) \quad D \beta z = \phi(D \alpha x).$$

But here we cannot, as in (50.), substitute for βz its value $\phi \alpha x$, and the remark in (18.) will show the reason.

For $D \beta z$ is not the same as $D \phi \alpha x$, the symbol D having a meaning dependent on the form of the functional symbol which follows it. Thus we find that

$$e^{bx} = (e^{ax})^{\frac{\lambda b}{\lambda a}}$$

is a relation between e^{bx} and e^{ax} , which is not altered by substituting $e^{bx} \cdot b$ for e^{bx} and $e^{ax} \cdot a$ for e^{ax} ; but the equation

$$D(e^{ax})^{\frac{\lambda b}{\lambda a}} = (D e^{ax})^{\frac{\lambda b}{\lambda a}}$$

is not true.

(212.) But we may make the preceding process verify a result we have before seen reason to anticipate, namely, that the transcendentals which satisfy $\phi \alpha x = \phi x$

Calculus of Functions. are of a much wider form than can be obtained from the most general use we have made of the function Ax .

It appears that we can solve $\phi\left(\frac{dy}{dx}\right) = \phi y$ by means of

the differential inverse of y , in the same manner as the preceding equation by the exponential inverse of αx . If then we assume

$$\frac{dy}{dx} = \alpha y,$$

and find the differential inverse of the resulting value of y expressed in terms of x , we obtain the following solution of $\phi\left(\frac{dy}{dx}\right) = \phi y$; namely, the expression $u_{\delta y}$,

where

$$u_{n+1} = \theta \cos 2\pi(n + u_n) \quad u_0 = \theta \cos 2\pi \delta y.$$

But this gives also a solution of $\phi \alpha y = \phi y$, and the least consideration will show that the two functions δy and Ay are of the most distinct characters: the first requiring only algebraical inversion, while the second requires the inversion of Γx , or of the solution of $\phi x = x\phi(x-1)$. If there be then one general expression which embraces every possible case of $\phi \alpha x = \phi x$, it must be one which indifferently contains all the solutions derived from the preceding, as well as from the equation in (119.)

But the preceding only serves to show how to deduce still more general notions of the possible extent of the solutions which $\phi \alpha x = \phi x$ may obtain. Let ∇y be a function derived from y , in any manner whatsoever, which will admit of successive derivations $\nabla^2 y$, $\nabla^3 y$, &c., subject to the condition

$$\nabla^m(\nabla^n y) = \nabla^{m+n}(y).$$

This includes ordinary functions, finite and infinitely small differences, as well as the more general sense in which this symbol ∇ has been used by Laplace. Indeed it is evident that any sense in which ∇ may be taken is an opening for a new and more general definition. Let

$$\nabla^n y = F(n, y),$$

an equation in which contemporaneous changes of n into $n-1$, and of y into ∇y does not affect the result. Putting an arbitrary constant for the first side, the value of n deduced from the preceding is an exponential inverse to ∇y , which satisfies the equation

$$\psi \nabla y = \psi y - 1,$$

and if

$$u_{n+1} = \theta \cdot \cos 2\pi(n + u_n) \quad u_0 = \theta \cos 2\pi \psi y,$$

then $u_{\psi y}$ is a solution of $\phi \nabla y = \phi y$.

In so many ways, therefore, as we can make $\nabla y = \alpha y$, in so many forms can the conditions of solution of $\phi \alpha y = \phi y$ be expressed.

The various problems in which the exponential inverse has been used have their obvious counterparts in cases

where $\frac{dy}{dx}$ is substituted for αy ; but the difficulties in

the way of expressing the most simple differential inverses will render the process useless for the present. The inversion of Γx , directly considered, amounts to the solution of the equation

$$\psi\left(\frac{x}{\psi x}\right) = \psi x - 1,$$

a very limited solution of which would be the solution

of $x = Ax \alpha x$, where Ax is the exponential inverse of αx . But this we have no means of solving. Merely as an expression of what sort of difficulty is in question we may write the equation $x = Ax \alpha x$ as follows,

$$\frac{\alpha x}{\alpha^2 x} = \frac{x}{\alpha x} - 1.$$

On this we will make one remark for students (and there are many such) who have acquired familiarity with the various artifices of analysis, without paying much attention to their connection. No process which any one method of solution requires was ever dispensed with in any other method. However different the various ways of solution of a question may appear to be, if method A were abandoned on account of a certain difficulty which did not explicitly appear in method B, then it is certain that the method B does contain within itself the direct way of overcoming the difficulty of method A. If, for instance, a problem of Mechanics may be shown to depend upon a certain integral which has not been found, and also upon a process of geometrical Algebra which presents no direct integration, and which solves the mechanical problem; then the method of Algebra must contain the method of finding the integral alluded to. Thus the question of finding x in terms of y in the equation

$$y = \int_0^\infty \alpha^{-x} x^a da$$

really contains that of finding a particular solution of

$$\alpha^{\frac{x}{\alpha^2 x}} = \alpha^{\frac{x}{\alpha^2 x} - 1} x,$$

where the functional exponents are themselves functions of x .

In the preceding, we do not speak of facility of solution, but only of possibility. And one method may be much more valuable than another as a method of suggestion of the artifices proper to be employed. For instance, the integration of equations of differences really contains the solution of all functional equations. But the employment of what we have called derivative functions frequently suggests a method of solution, in cases where the equation of differences would baffle all our efforts. But the solution of the functional equation once obtained, we find the equation of differences to be integrable, and also that greater power of penetration would have suggested an independent method for the latter.

(213.) But clearly to admit the assertion, that mathematical difficulties must always be conquered, and cannot be avoided or evaded, and that any successful artifice, which appears not to introduce the difficulty, does, in fact, introduce and surmount it, will require close consideration of the various cases which occur. It was immediately suggested to us by a friend, on our stating this principle to him, that the following was a case in point against us: an equation of n dimensions cannot be directly deprived of its n th term, or first power of x , without the solution of an equation of the $(n-1)$ th degree; whereas, by changing the roots into their reciprocals, the thing can be done by an equation of the first degree. But here, on further consideration, we see that we have, in fact, changed the problem, solved it as thus changed, and the whole question now lies in this: can the first problem be reduced to the second without supposing the solution of an equation of the $(n-1)$ th degree? The first problem is: given an equation with roots $a, a', \&c.$, to find another with roots $a+h, a'+h,$

Calculus of Functions &c., which shall be without the n th term. We find readily an equation whose roots are $\frac{1}{a} + k, \frac{1}{a'} + k, \&c.$,

which shall be without the second term. But what we want is h , and it will be found that we cannot determine h in terms of k without the solution of an equation of the $(n-1)$ th degree at least. If we know a , we can determine h from k ; but this supposes the solution of an equation of the n th degree: if we do not know a , we must solve an equation of the $n-1$ th degree.

(214.) The equation

$$\psi \frac{dy}{dx} = \frac{d}{dx} \psi y$$

is satisfied, without reference to the form of y , by

$$\psi y = a_0 y + a_1 \frac{dy}{dx} + a_2 \frac{d^2 y}{dx^2} + \&c.;$$

this belongs to a class of questions bearing no very perceptible analogy with those in (115.)

The same applies when ∇y is substituted for y , where ∇y is a linear function of successive values of y , when x increases by equal differences.

(215.) The equation

$$\psi \alpha x = \frac{d\psi x}{dx} \dots (1)$$

can be integrated whenever the following equation

$$\psi(x+h) - \psi x = h \psi \alpha x \dots (2)$$

can be solved. The following remark will illustrate the necessity of caution in applying conclusions drawn from one form to other forms. Equation (2) being solved with an arbitrary constant, the latter may be replaced by any function which does not change when x becomes either $x+h$, or αx . (We suppose αx independent of h .) It might seem, then, that h being diminished without limit in the solution of (2), which gives the solution of (1), the only condition for the arbitrary constant of (1) is, that it should not change when x is changed into αx . But this is certainly not true; for if ϖx satisfy (2), then $c\varpi x$ also satisfies it, but $\xi \alpha x \cdot \varpi x$ does not satisfy (1), unless $\xi \alpha x$ be constant. At the same time the following alternative may be easily proved. Either the function which does not change under x into $x+h$ and x into αx becomes a constant when $h=0$, or there is no such function.

(216.) The equation

$$\psi \alpha x = \frac{d^m \psi x}{dx^m}$$

has been reduced to a differential equation of the ordinary kind by Mr. Babbage, as follows. If for x we write $\alpha^p x$, we have

$$\psi \alpha^{p+1} x = \frac{d^m}{(d\alpha^p x)^m} \cdot \psi \alpha^p x,$$

where the second side denotes the performance of the following operation m times upon $\psi \alpha^p x$, namely, differentiation and division by $\frac{d\alpha^p x}{dx}$. Consequently, if the function αx be periodic of the n th order, we have, making $\psi x = \psi \alpha^n x = y$,

$$y = \frac{d^m}{(d\alpha^{n-1} x)^m} \cdot \frac{d^m}{(d\alpha^{n-2} x)^m} \dots \frac{d^m}{(dx)^m} \cdot y.$$

In the case where $m=1$ and $n=2$ we have for the solution of

$$\psi \alpha x = \frac{d\psi x}{dx}, \quad \alpha^2 x = x$$

$$y = \frac{d}{d\alpha x} \cdot \frac{dy}{dx}, \quad y = \psi x,$$

or

$$\frac{d^2 y}{dx^2} = \alpha' x \cdot y.$$

If $\alpha x = a - x$, this gives

$$y = c \cos x + c' \sin x = \psi x,$$

which satisfies the equation, when

$$c \cos a + c' \sin a = c'$$

$$c \sin a - c' \cos a = -c,$$

which two equations are the same in different forms.

(217.) Either the equation (1) preceding has an infinite number of solutions, or the equation

$$\phi \alpha x - \phi x = \frac{\varpi x}{\varpi \alpha x} \cdot \frac{d\phi x}{dx}$$

has no solution except $\phi x = \text{constant}$. For if ϖx be a particular solution of (1), then if there be another particular solution not of the form $c\varpi x$, let it be $\phi x \cdot \varpi x$. Then the preceding equation is the condition which determines ϕx . If the preceding have a particular solution $\varpi_1 x$, then a similar equation can be constructed which shall determine ϕx , so that $\phi x \varpi_1 x \varpi x$ is a particular solution. Proceeding in this way, and observing that the sum of two particular solutions of (1), or a particular solution multiplied by a constant, is also a particular solution, we have as a general solution of (1)

$$\psi x = c \cdot \varpi x + c_1 \varpi_1 x \varpi x + \dots,$$

whence follows the proposition stated.

(218.) The general expression for $\alpha^n x$ being

$$\Lambda^{-1}(\Lambda x - n),$$

we find the following differential equation,

$$\frac{d\alpha^n x}{dx} + \frac{d\Lambda x}{dx} \cdot \frac{d\alpha^n x}{dn} = 0,$$

consequently the equation

$$\frac{du}{dx} \cdot \frac{d^2 u}{dn^2} - \frac{du}{dn} \cdot \frac{d^2 u}{dx dn} = 0$$

contains among its solutions the n th functions of all functions of x whatsoever, and its integral may be written $\psi \phi^n x$, where ψ and ϕ are any forms whatsoever. Hence, by interchanging n and x , follows this theorem: that if $f(n, x)$ be such a function that its x th function with respect to n is the same as its n th function with respect to x , then $f(x, n)$ is one ordinate of a developable surface, where x and n are the other two.

We may also note that $\frac{d\alpha^n x}{dn}$ must be a function of $\alpha^n x$

only, or must be the solution of an equation of the form $\frac{du}{dn} = \phi u$.

(219.) Between the differential coefficients of $\alpha x, \alpha^2 x, \&c.$, where $\alpha^n x = x$, we find the following relations by differentiating both sides of the preceding equation with respect to x .

$$1 = \alpha' \alpha^{n-1} x \cdot \alpha' \alpha^{n-2} x \dots \alpha' \alpha x \cdot \alpha' x \dots (1)$$

where $\alpha' x$ means $\frac{d\alpha x}{dx}$, and consequently $\alpha' \alpha x$ means

$$\frac{d\alpha^2 x}{d\alpha x}. \quad \text{Hence } \alpha' x \text{ is a solution of the equation}$$

Calculus of
Functions.

$$\psi x \cdot \psi \alpha x \cdot \psi \alpha^2 x \dots \psi \alpha^{n-1} x = 1,$$

of which the general solution is $\psi x = e^{\theta x - \theta \alpha x}$, and by what has been proved (194.), the most general solution, exclusive of discontinuous functions or constants. And we have before found (201.) that $\alpha'x$ may be expressed in the preceding form.* Differentiating the logarithm of both sides of (1), we find

$$\frac{\alpha'x}{\alpha'x} dx + \frac{\alpha''\alpha x}{\alpha'\alpha x} d\alpha x + \dots + \frac{\alpha''\alpha^{n-1}x}{\alpha'\alpha^{n-1}x} d\alpha^{n-1}x = 0,$$

whence $\int \xi \alpha x \frac{\alpha''x}{\alpha'x} \cdot dx$ is a solution of the equation

$$\psi x + \psi \alpha x + \dots + \psi \alpha^{n-1} x = \text{constant} \dots (2),$$

or the preceding integral is a form of $C + \theta x - \theta \alpha x$. The integrals

$$\int \xi \alpha x \frac{\alpha''\alpha x}{\alpha'\alpha x} d\alpha x \quad \int \xi \alpha x \frac{\alpha''\alpha^2 x}{\alpha'\alpha^2 x} d\alpha^2 x, \text{ \&c.}$$

all possess the same property, and therefore the same form, though apparently very different, and considered as such when first given.†

(220.) The general solution of (2) in the last article has a particular case in which the function has been said to have the *complementary* property, but which might better be thus described. The mean of the successive values is always *one given value* of the function itself. Such a relation we see in

$$\alpha x = \frac{\pi}{2} - x \quad \alpha^2 x = x$$

$$\cos^2 x + \cos^2 \left(\frac{\pi}{2} - x \right) = 2 \cos^2 \frac{\pi}{4};$$

and in

$$\alpha x = \frac{1}{x} \quad \alpha^2 x = x$$

$$\tan^{-1} x + \tan^{-1} \frac{1}{x} = 2 \tan^{-1} 1.$$

If we assume

$$\psi x + \psi \alpha x + \dots + \psi \alpha^{n-1} x = n \psi k,$$

or $\psi x = \psi k + \theta x - \theta \alpha x$, we must determine k from the equation $\theta k - \theta \alpha k = 0$, one set of values of which corresponds to $k - \alpha k = 0$.

If the function αx be not periodic, the only solution of the preceding will be found to be

$$\psi x = \psi k + r^{-\Delta x} \xi \alpha x \quad r^n = 1.$$

(221.) A general method‡ has been given for reducing to a common differential equation

$$F \left(\begin{array}{cccc} \psi x, & \psi \alpha x, & \psi \alpha^2 x, & \dots & \psi \alpha^{n-1} x \\ \frac{d\psi x}{dx}, & \frac{d\psi \alpha x}{dx}, & \frac{d\psi \alpha^2 x}{dx}, & \dots & \frac{d\psi \alpha^{n-1} x}{dx} \end{array} \right) = 0;$$

but as it leads to very complicated details, we shall only here describe it generally.

In the preceding equation, substitute $\alpha x, \alpha^2 x, \dots, \alpha^{n-1} x$ instead of x , and for

* In the simple case of $\frac{1}{x}$, we have

$$\lambda^{-1} \left(\lambda \left(\frac{1}{x} \cdot \frac{1}{x} \right) - \lambda (\lambda x \cdot x) \right) = -\frac{1}{x^2}.$$

† Sir J. Herschël, *Edition of Spence*, p. 166.

‡ Babbage, *Philosophical Transactions*, 1816.

 Calculus of
Functions.

$\frac{d^n \psi \alpha^p x}{(d \alpha^p x)^n}$ write $\frac{d}{dx} \frac{1}{(\alpha^p)' x} \frac{d}{dx} \dots \psi \alpha^p x$ expanded.

We commence with $n(r+1)$ quantities

$$\begin{array}{cccc} \psi x & \psi' x & \psi'' x & \dots \psi^{(r)} x \\ \psi \alpha x & \psi' \alpha x & \psi'' \alpha x & \dots \psi^{(r)} \alpha x \\ \vdots & \vdots & \vdots & \vdots \\ \psi \alpha^{n-1} x & \psi' \alpha^{n-1} x & \psi'' \alpha^{n-1} x & \dots \psi^{(r)} \alpha^{n-1} x, \end{array}$$

of which all except the first line are to be eliminated; that is, $(n-1)(r+1)$ are to be eliminated. By successive substitutions we set out with n distinct equations; and every differentiation adds n to the number of equations, and only $n-1$ to the number of quantities to be eliminated, for those in the continuation of the first line of the above, namely,

$$\psi^{(r+1)} x \quad \psi^{(r+2)} x \dots$$

need not be eliminated. Consequently, after V differentiations, we have $n(1+V)$ equations with $(n-1)(r+1+V)$ quantities to be eliminated. If we make the first of these greater by 1 than the second, or

$$n(1+V) = (n-1)(r+1+V) + 1,$$

giving $V = (n-1)r$, we shall, after $(n-1)r$ differentiations, have one more equation than there are quantities to be eliminated. If, for instance, we take

$$\psi x + a \psi(1-x) = \frac{d\psi x}{dx} + b \frac{d\psi(1-x)}{dx}.$$

Here $n=2, r=1$, and one differentiation will be sufficient. The corresponding equation will be

$$\psi(1-x) + a \psi x = -\frac{d\psi(1-x)}{dx} - b \frac{d\psi x}{dx};$$

by differentiating these two equations, we obtain four equations between which to eliminate $\psi(1-x)$ and its first two differential coefficients. In this particular case we may easily see that the resulting equation will be linear, and of the second degree, that its solution will be of the form

$$C e^{mx} + C' e^{ux},$$

and finally, that $\psi x = 0$ is the only solution.

(222.) The preceding method will fail if the n equations of successive substitution are reducible to a smaller number, say to $n-p$. But it must be recollected, that we have as yet only *differentiated after substitution*, which will not, generally speaking, give the same result as if we *substituted after differentiation*. For instance,

if we take $\frac{1}{x}$, we find that

substitution and differentiation gives 1

differentiation and substitution gives $-x^2$.

If, then, we suppose that the first substitution gives $n-p$ independent equations, the number after substitution and V differentiations will be

$$n-p + (n-p)V.$$

Generally speaking, the dependence of the equations on each other will be destroyed in differentiation, so that from each of the V differential equations which arise from the primitive equation $n-1$ new equations can be produced by successive substitutions, or the total number will be

$$n-p + (n-p)V + (n-1)V;$$

this will be greater than

Calculus of
Functions. when

$$(n-1)(r+1+V)$$

$$1+V > \frac{n-1}{n-p}(r+1).$$

But cases may arise, in which, for any thing appearing to the contrary, both methods together will not produce more equations than new differential coefficients; and to these the method will not apply.

Again, the process may fail on account of a differential coefficient being introduced in one equation only, and in no other; that is, the number of differential equations specified in the formula may fail. In this case more differentiations must be made, and such equations must be chosen as will answer the purpose.

(223.) The above method is general in form, but any combinations of the equations which will eliminate all the results of successive operation is equally good. For instance, take the equation

$$\psi x \frac{d\psi x}{dx} = \psi \alpha x, \quad \alpha^2 x = x,$$

$$\text{substitution gives } \psi \alpha x \frac{d\psi \alpha x}{dx} = \alpha' x \psi x,$$

$$\text{multiplication gives } \frac{d\psi x}{dx} \cdot \frac{d\psi \alpha x}{dx} = \alpha' x.$$

Differentiate the first, and substitute from the third, which gives, making $y = \psi x$,

$$\left(\frac{dy}{dx}\right)^2 + y \frac{dy}{dx} \frac{d^2 y}{dx^2} = \alpha' x.$$

If $\alpha x = 1 - x$, a particular solution of the preceding is $y = C - x$, and it will be found that $\frac{1}{2} - x$ satisfies the conditions.

The preceding equation belongs to a class no general method of integrating which has been found. We here see a great distinction between the case in which $\alpha^2 x = x$ and all others. In the first, $\frac{dy}{dx}$ is a function which, for

certain values of the constants, is of the form $\frac{\psi x}{\psi \alpha x}$,

that is, satisfies the equation $\chi x \chi \alpha x = 1$. This leads us to the following observation. It cannot have escaped the notice of any one who has looked at the history of the Differential Calculus, how soon, after attention was first turned to the subject in a general point of view, the great forms of differential equations which are now integrable were discovered; and how little progress has been made during the present century. We do not speak of particular equations, involving no general functional symbol, but of those in which some form is left indeterminate. The preceding equation adds some additional force to the presumption, that the reason why no more general forms are discovered lies in the incapability of analysis, as it stands at present, to express the primitive functions of such equations. For it draws a remarkable distinction between the case of $\alpha^2 x = x$ and all others. We need not allude to the equation known by the name of Riccati, and to others, in which the ordinary processes of analysis succeed only with very limited cases.

It is true that infinite series may be made to express many otherwise unattainable results, but these themselves are open to the same difficulties. When their envelopes are not reducible either to the integration of a

differential equation, or the solution of a functional equation, they are, analytically considered, quantities of an entirely unknown form. Generally speaking, either a differential or functional equation may be chosen, but as this may not appear with regard to many series, we shall show in the following article one method of proceeding.

(224.) Suppose we wish to reduce the series

$$a - \frac{a^2}{2 \cdot 3} + \frac{a^3}{2 \cdot 3 \cdot 4 \cdot 5} - \&c.$$

to a particular case of the solution of a functional equation. In the denominators, instead of 1, 2, &c. write $x+1$, $x+2$, &c. Call the result ψx , which gives

$$\begin{aligned} \psi x &= \frac{a}{x+1} - \frac{a^2}{(x+1)(x+2)(x+3)} + \&c. \\ &= \frac{a}{x+1} - \frac{a^2}{(x+1)(x+2)} \psi(x+2). \end{aligned}$$

The general solution of which, if it could be found, would give $\psi(0) = \sin a$. By a process of this kind Legendre* reduced the series for the tangent of x to a continued fraction, which he employed to demonstrate the incommensurability of the circle and its diameter.

(225.) We cannot quit the subject of the connection of the functional and differential calculus without saying something on the very remarkable theorem of Fourier,† by which one value of a function is expressed in terms of a definite integral which contains the general value. If we write the two series

$$F x = A_1 \cos x + A_2 \cos 2x + A_3 \cos 3x + \dots$$

$$f x = a_1 \sin x + a_2 \sin 2x + a_3 \sin 3x + \dots$$

we see in them the conditions

$$F(x) = F(-x) \quad f(x) = -f(-x)$$

by the known properties of the sine and cosine. Hence an attempt to represent a function of the form $F x$ by a series of sines, or $f x$ by a series of cosines, must introduce some symbol of impossibility, or at least of discontinuity. And the same may be said of the series

$$F x = Q_1 \cos q_1 x + Q_2 \cos q_2 x + \dots$$

$$f x = P_1 \sin p_1 x + P_2 \sin p_2 x + \dots$$

where $Q_1 \dots P_1 \dots q_1 \dots p_1 \dots$ are not functions of x .

The integral $\int \phi x dx$, between the limits $x = a$ and $x = b$, may be conceived to be the limit of the following process. Divide the interval $b - a$ into n parts, each equal to k . Then the sum of the several terms $\phi a \cdot k + \phi(a+k) \times k + \phi(a+2k) \times k + \dots + \phi(a+nk) \times k$ is an expression, the limit of which, made by increasing n or diminishing k without limit, is the definite integral in question.

If, then, in the preceding series, we assume

$$q_1 = k \quad q_2 = 2k \dots q_n = nk,$$

and suppose $Q_1, Q_2, Q_3 \dots$ to be the corresponding values of a certain function of q or mk , each multiplied by k , namely,

$$Q_1 = k \psi k \quad Q_2 = k \psi 2k \dots Q_n = k \psi nk,$$

and if we then suppose k to diminish without limit, we have

$$F x = \int_0^x \psi q \cdot \cos qx \cdot dq,$$

in which $F x$ now represents the limit of the series.

* See notes to Sir D. Brewster's translation of his *Geometry*.

† *Théorie de la Chaleur*, p. 449.

Calculus of Functions. But let the expanded equation, as it stands, be multiplied by $\cos mkx \times dx$, and then integrated with respect to x , from $kx = 0$ to $kx = \pi$. By the common processes of the integral calculus, we know that (m and r being whole numbers)

$$\int_0^\pi \cos mkx \cdot \cos rkx \, dx = 0 \text{ or } \frac{1}{2} \frac{\pi}{k}$$

the first when m and r are different, the second when they are the same. Hence it follows, that the preceding integration destroys every term, except only that in which $\cos mx$ is found, which it reduces to

$$k \psi mk \times \frac{\pi}{2k} \text{ or } \frac{\pi}{2} \psi mk.$$

Let k diminish without limit, and let m at the same time increase without limit, in such a way that mk is always equal to the general value q ; then the superior

limit of x in the integration, which is $\frac{\pi}{k}$, (for kx then $= \pi$), increases without limit, and the result is, that

$$\frac{\pi}{2} \psi q = \int_0^\infty Fx \cdot \cos qx \cdot dx$$

if

$$Fx = \int_0^\infty \psi q \cos qx \cdot dq.$$

Exactly in the same way, using the sines instead of cosines, and remembering that

$$\int_0^\pi \sin mkx \cdot \sin rkx \cdot dx = 0 \text{ or } \frac{1}{2} \frac{\pi}{k},$$

according as m and r are different or equal, we deduce that

$$\frac{\pi}{2} \chi p = \int_0^\infty fx \cdot \sin px \cdot dx$$

if

$$fx = \int_0^\infty \chi q \sin px \, dp.$$

From which it appears that $\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \psi q$ and Fx , when ψq and Fx satisfy either equation, have this property, that an interchange of x and q is accompanied by an interchange of F and ψ ; and also, that $\left(\frac{\pi}{2}\right)^{\frac{1}{2}} \chi p$ and fx have the same property. M. Cauchy* has proposed to call these pairs *reciprocal functions* of the first and second kind; but the name reciprocal has been given to other functions, and is besides too valuable as a general term to be thus appropriated. We should propose to call them *Fourier's reciprocals*, after their discoverer.

(226.) Substitution in the second equation of each set gives the following theorems:

$$Fx = \frac{2}{\pi} \int_0^\infty \cos qx \, dq \int_0^\infty Fx \cdot \cos qx \cdot dx$$

$$fx = \frac{2}{\pi} \int_0^\infty \sin px \, dp \int_0^\infty fx \cdot \sin px \, dx;$$

and in the first integration we may write v for x , the general symbol of the variable being indifferent in a definite integral: we may also make $q = p$, since the limits of the two are the same; and we then have, observing that we have now a separation of variables in the successive integrations,

$$Fx = \frac{2}{\pi} \int_0^\infty \int_0^\infty Fv \cdot \cos qv \cdot \cos qx \, dq \, dv$$

$$fx = \frac{2}{\pi} \int_0^\infty \int_0^\infty fv \sin qv \sin qx \, dq \, dv.$$

If the integration with respect to v be taken from $-\infty$ to $+\infty$, then each integral is merely doubled; for since

$$Fv = F(-v) \text{ and } \cos qv = \cos(-qv),$$

we have

$$Fv \cdot \cos qv = F(-v) \cos(-qv).$$

Similarly, since

$$fv = -f(-v) \text{ and } \sin qv = -\sin(-qv),$$

we have also

$$fv \sin qv = f(-v) \sin(-qv),$$

whence the proposition stated. But for a contrary reason, namely, because

$$Fv \sin qv = -F(-v) \sin(-qv)$$

$$fv \cdot \cos qv = -f(-v) \cos(-qv),$$

we have

$$\int_{-\infty}^\infty Fv \cdot \sin qv \, dv = 0 \quad \int_{-\infty}^\infty fv \cdot \cos qv \, dv = 0;$$

consequently,

$$\int_{-\infty}^\infty (Fv + fv) \sin qv \, dv = \int_{-\infty}^\infty fv \sin qv \, dv$$

$$\int_{-\infty}^\infty (Fv + fv) \cos qv \, dv = \int_{-\infty}^\infty Fv \cos qv \, dv.$$

Substituting these in the preceding results, taking the first integration from $-\infty$ to $+\infty$, and writing

$$\frac{1}{2} \int_{-\infty}^\infty \text{ instead of } \int_0^\infty,$$

we find

$$Fx = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty (Fv + fv) \cos qv \cos qx \, dq \, dv$$

$$fx = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty (Fv + fv) \sin qv \sin qx \, dq \, dv;$$

whence, by addition,

$$Fx + fx = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty (Fv + fv) \cos q(x - v) \, dq \, dv.$$

(227.) Any function whatsoever may be reduced to the form $Fx + fx$, supposing constants and zero to be included in these forms Fx and fx . For any function ϕx may be written

$$\phi x = \frac{\phi x + \phi(-x)}{2} + \frac{\phi x - \phi(-x)}{2},$$

the first term of which falls under Fx , and the second under fx .

Consequently, we have this general theorem, which we may write in two different ways. If ϕx be any function of x whatsoever,

$$\phi x = \int_0^\infty \int_{-\infty}^\infty \phi v \cos q(x - v) \, dq \, dv$$

$$= \int_{-\infty}^\infty \phi v \, dv \int_0^\infty \cos q(x - v) \, dq;$$

between which, however, there is a distinction of some importance to be drawn. In the case of $\iint P Q \, dx \, dy$, where P and Q are functions of x only and of y only, and the limits are definite and independent of x and y , it is absolutely indifferent whether we proceed in the order denoted by $\int P \, dx \int Q \, dy$, or in that of $\int Q \, dy \int P \, dx$. The same thing may be said as to definite and independent limits of the general form $\iint \psi x \phi(x, y) \, dx \, dy$. But where we have to integrate such a function as the preceding, and when the integration with respect to y ,

* Exercices de Mathématiques, vol. ii, p. 141.

Calculus of Functions. owing to the particular values of the limits, causes any limitation of form, such as the loss of terms which appear when the limits are general, &c., it is obvious that the subsequent integration with respect to x will be altered in form: in fact, the operation is not performed on the same function as when the limits are kept general.* This order, then, is *absolutely necessary* when the deduction of the integral produced it; and *absolutely false* when the deduction of the integral produced the other. The order is a necessary condition of the result. If, for instance, the result of an investigation be

$$\int y^x dx dy, \text{ from } y = 1 \text{ to } y = a, \\ \text{from } x = -\infty \text{ to } x = -1.$$

According as we integrate, first with respect to x , or to y , we have the results

$$\int_1^a \frac{dy}{y \log y} \text{ or } \int_{-\infty}^{-1} \frac{a^{x+1} - 1}{x+1} dx,$$

which are not the same.

The preceding theorem of Fourier was deduced upon the supposition of the first integration being with respect to v . It remains to see what will be the consequence of supposing it to be with respect to q . Generally, we have ($c = x - v$)

$$\int \cos cq \cdot dq = \frac{\sin cq}{c} + \text{constant},$$

which we cannot attempt to integrate within the required limits, because the cosine of ∞ (an undefined constant) enters. Also the expression of

$$\int_0^\infty \cos cx dx = \int_0^\pi \cos cx \cdot dx + \int_\pi^{2\pi} \cos cx dx + \dots$$

gives the result zero; as does every other similar method. This result not making the theorem true, we must either integrate between general limits, and make these limits infinite when the final result has been obtained, or keep to the order of the investigation.

(228.) By the first of the two methods M. Defflers has established the theorem synthetically.† Taking $\int \cos q(x-v) dq$ from $q=0$, to an undetermined limit, we have

$$\int_0^\infty \cos q(x-v) dq = \frac{\sin q(x-v)}{x-v} \\ \phi x = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\sin q(x-v)}{x-v} \phi v dv,$$

in the result of which q is to be made infinite. Assume

$$v = x + \frac{z}{q}, \text{ which gives}$$

$$\phi x = \frac{1}{\pi} \int_{-\infty}^\infty \frac{\sin z}{z} \phi \left(x + \frac{z}{q} \right) dz:$$

expand the function of the binomial, which gives

$$\phi x = \frac{1}{\pi} \int_{-\infty}^\infty \phi x \frac{\sin z}{z} dz,$$

the other terms vanishing when q is made infinite, unless $\int_{-\infty}^\infty \sin z z^m dz$ be infinite for finite values of m , which

* A general value of a constant is one which does not produce any reduction in the result. Integrate $2x+a$ from $x=0$ to $x=c$, and we have c^2+ac , which is general; the general form can be restored from it. Integrate from $x=0$ to $x=a$, and we have $2a^2$, from which the general form cannot be recovered.

† Formerly a pupil of the *Ecole Normale*, and deceased some years ago. This demonstration is given by M. Poisson, *Théorie de la Chaleur*, p. 209.

is not the case. The preceding equation is true, for it is known that $\int_{-\infty}^\infty \frac{\sin z}{z} dz = \pi$ Calculus of Functions.

(229.) The very singular nature of Fourier's theorem, and its not having appeared in English, to the best of our knowledge, in any work yet published, must be our excuse for presenting yet one more demonstration, given by M. Poisson,* and which contains several analytical considerations worthy of attentive study. We know that so long as a constant is perfectly general in value, it is indifferent whether it is made to vanish before or after integration; consequently, Fourier's theorem may be presented in the following form:

$$\phi x = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \phi v \cos q(x-v) e^{-kq} dq dv,$$

provided that in the result $k=0$. And that this theorem is true may be thus proved. If we integrate the following by parts in the usual way, we find

$$\int_0^\infty \cos q(x-v) e^{-kq} dq = \frac{k}{k^2 + (x-v)^2},$$

whence

$$\phi x = \frac{1}{\pi} \int_{-\infty}^\infty \frac{k \phi v dv}{k^2 + (x-v)^2} \dots (A)$$

Let the reader remark that the indefinite character of the first integration has now disappeared, or rather is deferred to the end of the process, when it will be reduced to making $k=0$. We now give a literal translation of the process of M. Poisson, to which we shall add one which is more accessible to the common reader. It is also to be observed that the former process applies only to those functions in which ϕv is not infinite for any value of v , while the latter is general.

"Now, ϕv never becoming infinite, it is evident that this simple integral (A) will be infinitely small at the same time as k , except for the extent, in which the values of v differ infinitely little from those of x ; it will suffice, then, to integrate from $v=x-u$ to $v=x+u$, u being a positive and infinitely small quantity; between these limits ϕv may be considered constant, and equal to ϕx ; in consequence we have

$$\int_{-\infty}^\infty \frac{k \phi v dv}{k^2 + (x-v)^2} = \phi x \int_{-\infty}^\infty \frac{k dv}{k^2 + (x-v)^2} = \phi x \tan^{-1} \frac{v-x}{k}$$

which integral, between the assigned limits, is

$$2 \phi x \tan^{-1} \frac{u}{k},$$

which is $\pi \phi x$ when $k=0$."

Or we may proceed as follows. In the left-hand side of (A) write $x+h$ for v ; we have then $dh=dv$, and expanding $\phi(x+h)$ by Taylor's theorem, we have, for the integral required,

$$\frac{1}{\pi} \left(\phi x \int \frac{k dh}{k^2 + h^2} + \sum \frac{d^n \phi x}{d x^n} \frac{1}{2 \cdot 3 \dots n} \int \frac{k h^n dh}{k^2 + h^2} \right).$$

If, now, we consider separately (n being a whole number)

$$k \int \frac{h^{2n+1} dh}{k^2 + h^2} \text{ and } k \int \frac{h^{2n} dh}{k^2 + h^2},$$

we see from the form of the first that it vanishes when taken between $h=-\infty$ and $h=\infty$, which correspond

* In a Memoir entitled *Sur la Théorie des Ondes*, in the Memoirs of the Institute, Année 1816.

Calculus of Functions. $v = -\infty$ and $v = +\infty$. Now the second, by algebraical reduction, may be brought to the form

$$k \int (h^{2n-2} + Ah^{2n-4} + \dots + G) dh + k^{2n+1} \int \frac{dh}{k^2 + h^2}.$$

All the independent terms of which disappear, even independently of k , when integrated between the given limits; and the last term, being generally

$$k^{2n} \tan^{-1} \frac{h}{k}, \text{ or, between the limits, } \pi k^{2n},$$

vanishes when $k = 0$. We have, therefore, only to consider

$$\frac{1}{\pi} \phi x \int \frac{k dh}{h^2 + k^2}, \text{ or } \frac{\phi x}{\pi} \tan^{-1} \frac{h}{k} \text{ (from } h = -\infty \text{ to } h = +\infty),$$

which is ϕx , the same as before.

(230.) We shall now take some miscellaneous examples of simple functional equations. Some of them are from Mr. Babbage's collection of examples in the supplementary volume to the Cambridge translation of Lacroix.

$$\text{Let } \beta x = \frac{\psi x + \psi \alpha x}{1 - \psi x \cdot \psi \alpha x}.$$

Assume

$$\psi x = \tan \phi x,$$

which gives

$$\beta x = \tan(\phi x + \phi \alpha x) \text{ or } \phi x + \phi \alpha x = \tan^{-1} \beta x.$$

If ωx be a particular solution of the last, the general solution of the first is

$$\tan(\omega x + \theta \cos \pi A x) \text{ if } \theta(-x) = -\theta x.$$

If $\alpha^2 x = x$, the first equation is incongruous, unless βx be a symmetrical function of x and αx , in which case a solution of the third equation is

$$\frac{\theta x}{\theta x + \theta \alpha x} \tan^{-1} \beta x,$$

whence

$$\psi x = \tan\left(\frac{\theta x}{\theta x + \theta \alpha x} \tan^{-1} \beta x\right),$$

or

$$\psi x = \frac{1}{\sqrt{-1}} \frac{(1 + \sqrt{-1} \beta x)^{\theta_1 x} - (1 - \sqrt{-1} \beta x)^{\theta_1 x}}{(1 + \sqrt{-1} \beta x)^{\theta_1 x} + (1 - \sqrt{-1} \beta x)^{\theta_1 x}}$$

where

$$\theta_1 x = \frac{\theta x}{\theta x + \theta \alpha x}.$$

The preceding is an instance suggested by the formulæ of Trigonometry; but if for βx we write $\sqrt{-1} \beta x$, and $\sqrt{-1} \psi x$ for ψx , we have for the solution of

$$\begin{aligned} \beta x &= \frac{\psi x + \psi \alpha x}{1 + \psi x \cdot \psi \alpha x} \\ \psi x &= \frac{(1 + \beta x)^{\theta_1 x} - (1 - \beta x)^{\theta_1 x}}{(1 + \beta x)^{\theta_1 x} + (1 - \beta x)^{\theta_1 x}} \\ &= \frac{1}{\sqrt{-1}} \tan(\theta_1 x \tan^{-1}(\sqrt{-1} \beta x)). \end{aligned}$$

(231.) Let

$$\psi x + \psi \alpha x = 0 \quad \alpha^2 x = 0.$$

The method given by Mr. Babbage for the determination of the form of ψ is as follows. Solve the equation

$$\psi x + c \psi \alpha x = c' \beta x,$$

giving

VOL. II.

or

$$\psi \alpha x + c \psi x = c' \beta \alpha x,$$

$$\psi x = \frac{c'}{1 - c^2} (\beta x - c \beta \alpha x).$$

Now suppose c' a function of c , which vanishes when $c = 1$, which may be satisfied so as to make the value of $\frac{c'}{1 - c^2}$ any whatever: let it be C . Hence

$$\psi x = C (\beta x - \beta \alpha x) = \xi \alpha x (\beta x - \beta \alpha x).$$

It might here appear as if there were two arbitrary functions ξ and β ; but this is not in reality the case since $\beta x - \beta \alpha x$ is itself of the form

$$(x - \alpha x) \xi \alpha x \text{ because } \frac{\beta x - \beta \alpha x}{x - \alpha x}$$

is unchanged by writing αx for x , that is, by interchanging x and αx .

(232.) Further to illustrate this appearance of more arbitrary functions than one, in a case where it has been proved that such a thing cannot be, let us take

$$\psi x + \psi \alpha x = \beta x \quad \alpha^2 x = x.$$

Now βx must itself be a case of $\xi \alpha x$, and we find

$$\psi x = \frac{\theta x}{\theta x + \theta \alpha x} \beta x + \xi \alpha x (x - \alpha x),$$

using the reduced form of the last solution. But

$$\frac{\theta x}{\theta x + \theta \alpha x} = \frac{1}{2} + \frac{1}{2} \cdot \frac{\theta x - \theta \alpha x}{\theta x + \theta \alpha x},$$

the second term of which (as well before as after multiplication by βx) is a form of $\xi \alpha x (x - \alpha x)$, whence

$$\psi x = \frac{\beta x}{2} + \xi \alpha x (x - \alpha x)$$

is really as general as any other solution.

The solution of

$$\psi x + \psi \alpha x + \psi \alpha^2 x = 0 \quad \alpha^3 x = x$$

is

$$\psi x = \left(\frac{\theta x}{\theta x + \theta \alpha x + \theta \alpha^2 x} - \frac{1}{3} \right) \xi \alpha x,$$

which is the same general form as

$$(2\theta x - \theta \alpha x - \theta \alpha^2 x) \xi \alpha x,$$

and appears to have two arbitrary functions, and not more. For if we add two such forms as the preceding together, which is a solution apparently containing four arbitrary functions, it amounts to no more than writing

$$\theta x + \theta_1 x \frac{\xi_1 \alpha x}{\xi \alpha x} \text{ instead of } \theta x$$

in the preceding; that is, changing the form of an arbitrary function, by making it a function of other arbitrary functions. And the same may be proved of

$$(a \cdot \theta x + a_1 \theta \alpha x + \dots + a_{n-1} \theta \alpha^{n-1} x) \xi \alpha x \quad (\alpha^n x = x).$$

But these two arbitrary functions are only one in reality; for if we write the preceding under the form

$$(\theta x - \theta \alpha^2 x) \xi \alpha x - (\theta \alpha x - \theta x) \xi \alpha x,$$

we see a form of $\gamma x - \gamma \alpha x$.

To this form every solution of the equation

$$\psi x + \psi \alpha x + \dots + \psi \alpha^{n-1} x = 0 \quad \alpha^n x = x$$

may be reduced. A particular solution of the equation is

Calculus of
Functions.

$C\theta x + C'\theta\alpha x + \dots + C^{(n-1)}\theta\alpha^{n-1}x$,
where $C, C', \&c.$ are forms of $\xi\alpha x$, and

$$C + C' + \dots + C^{(n-1)} = 0.$$

The application of the condition last named gives

$$C(\theta x - \theta\alpha^{n-1}x) + C_1(\theta\alpha x - \theta\alpha^{n-1}x) + \dots;$$

each term of which is of the form $\gamma x - \gamma\alpha x$; for if we denote $\theta\alpha^m x$ by P_m , we find that

$$P_m - P_{m+1} = P_m + P_{m+1} + \dots + P_{m+n-1} \left\{ \begin{array}{l} \gamma x \\ - (P_{m+1} + P_{m+2} + \dots + P_{m+n}) \end{array} \right\} \text{ or } \left\{ \begin{array}{l} \gamma x \\ - \gamma\alpha x. \end{array} \right.$$

Hence it follows that the general solution of

$$\psi x + \psi\alpha x + \dots + \psi\alpha^{n-1}x = \beta x \quad (\alpha^n x = x)$$

is

$$\psi x = \frac{\beta x}{n} + \theta x - \theta\alpha x,$$

and βx must be a form of $\xi\alpha x$, or when this is not the case, the solutions are all discontinuous.

(233.) The equation $\psi x = b + c\psi\alpha x$, $\alpha^n x = x$ gives $\psi x = \frac{b}{1-c}$ for the only continuous solution; but

taking this as a particular solution, we find the general solution to be

$$\psi x = \frac{b}{1-c} + c^{\Lambda x} \xi\alpha x;$$

where, however, Λx is discontinuous in value. In a similar way, we find for the solution of

$$\begin{aligned} \psi\alpha &= \beta x + \gamma x\psi\alpha x \quad (\alpha^n x = x) \\ \psi x &= \mu x + \nu x\xi\alpha x, \end{aligned}$$

where μx is to be obtained from (192.), and νx is a discontinuous solution of $\psi x = \gamma x\psi\alpha x$. It appears in this way that the equation has in all cases an arbitrary function; but that where only a single solution is found, as in (192.), the arbitrary function has a coefficient of discontinuous value, while the appearance of the result in the indeterminate form considered in (197.) is the sign of the conditions being satisfied, under which a continuous arbitrary function is possible.

(234.) When one or more of the terms is missing in the equation

$$\psi x + \psi\alpha x + \dots = 0 \quad \alpha^n x = x$$

there is no continuous solution, except $\psi x = 0$. For suppose, for instance, that $\psi\alpha^m x = P_m$, and that

$$P_{m_1} + P_{m_2} + \dots = 0,$$

where $m_1, m_2, \&c.$ do not contain the complete series from 0 to $n-1$, both inclusive. Let ΣP denote this complete series; then, by successive substitution of αx for x , and addition, the equation

$$\Sigma P + \Sigma P + \dots = 0$$

is obtained, or the solutions of the first, if any, must be solutions of $\Sigma P = 0$, that is, must be of the form $\phi x - \phi\alpha x$, from which we find, by means of the first equation, that

$$\phi_{m_1} + \phi_{m_2} + \dots = \phi_{m_1+1} + \phi_{m_2+1} + \&c.,$$

or each side is a symmetrical function of $x, \alpha x, \dots$ up to $\alpha^{n-1}x$. This cannot be the case, unless each term be such. Suppose, for instance, that $(\alpha^2 x = x)$, the following incomplete sum is a form of $\xi\alpha x$:

$$\phi x + \phi\alpha x + \phi\alpha^2 x + \phi\alpha^3 x.$$

Then, since the remainder of the complete sum is the

same, and since successive substitution does not alter this property, we find that each of the following is a form of $\xi\alpha x$, Calculus of
Functions.

$$\begin{aligned} & (P_2 + P_3 + P_4) (P_0 + P_1 + P_2) (P_2 + P_4 + P_5 + P_6) \\ & (P_0 + P_2 + P_3 + P_4) (P_1 + P_3 + P_5) (P_0 + P_4 + P_5) \&c., \end{aligned}$$

whence the sum of any four, or of any three, is a form of $\xi\alpha x$, whence the difference of any two is the same. Hence

$$P_0 - P_1 = P_1 - P_2 \dots = P_{n-1} - P_0,$$

but $\Sigma (P_0 - P_1) = 0$; therefore $P_0 - P_1 = 0$ or P_0 is itself a form of $\xi\alpha x$, and $P_0 - P_1$, the solution of the given equation, is $= 0$.

The same thing might be proved by observing, that the particular solution, derived as before,

$$C P_0 + C' P_1 + \dots,$$

when the series is incomplete, does not give one equation of condition $\Sigma C = 0$, but $n-1$ partial sums, severally equal to nothing, from which follows $C = 0$, $C' = 0$, &c.

In a similar manner it may be shown, that the equation

$$\pm \psi x \pm \psi\alpha x \pm \dots = 0$$

is impossible when any of the terms differ in sign. For in that case, one substitution and addition will either destroy or double any given term, and the last case results.

We must except from the preceding the case where the number of negative and positive terms are equal, in which case the general solution is evidently $\xi\alpha x$; and, secondly, the case in which the terms are alternately positive and negative, in which substitution reproduces the equation. In the latter case, we have either an even number of terms, as in the case of

$$P_0 - P_1 + P_2 - P_3 = 0 \quad \alpha^4 x = x,$$

equivalent to $Q_0 + Q_2 = 0$, or $Q_0 = 0$, or $P_0 = P_1$, that is, to $P_0 = \xi\alpha x$: or we have an odd number of terms, as in

$$P_0 - P_1 + P_2 - P_3 + P_4 = 0 \quad \alpha^5 x = x,$$

in which successive substitution and addition gives

$$P_0 + P_1 + P_2 + P_3 + P_4 = 0,$$

whence

$$P_0 + P_2 + P_4 = 0 \quad P_0 = 0.$$

(235.) The general solution of the equation

$$a_0 P_0 + b_0 P_1 + c_0 P_2 + \dots + k_0 P_{n-1} = l_0,$$

where P_1 is the same function of αx which P_0 is of x , and where $\alpha^n x = x$, offers no other difficulty except the very complicated character of the terms which must be employed. But it may be easily reduced in the case when $n = 3$, to the one already solved in (197.), by assuming $P_0 = Q_0 + v_0 Q_1$; and the following is the general process for reducing the preceding equation to the form

$$m_0 Q_0 + n_0 Q_1 = l_0.$$

Assume

$$P_0 = Q_0 + b_0 Q_1 + c_0 Q_2 + \dots + i_0 Q_{n-2}$$

$$P_1 = Q_1 + b_1 Q_2 + c_1 Q_3 + \dots + i_1 Q_{n-1}$$

$$P_2 = Q_2 + b_2 Q_3 + c_2 Q_4 + \dots + i_2 Q_0$$

$$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$P_{n-1} = Q_{n-1} + b_{n-1} Q_0 + c_{n-1} Q_1 + \dots + i_{n-1} Q_{n-2}$$

Determine the $n-2$ quantities $b_0, c_0, \dots, i_0$ by following equations:

Calculus of Functions.

$$\begin{aligned} b_0 i_1 + c_0 h_2 + e_0 g_3 + \dots + i_0 b_{n-2} + k_0 &= 0 \\ a_0 i_0 + b_0 h_1 + c_0 g_2 + \dots + g_0 b_{n-3} + i_0 &= 0 \\ k_0 i_{n-1} + a_0 h_0 + b_0 g_1 + \dots + f_0 b_{n-4} + g_0 &= 0 \\ \dots &\dots \end{aligned}$$

and so on until there are $n - 2$ equations. For a particular instance, suppose $n = 5$, and that

$$a_0 P_0 + b_0 P_1 + c_0 P_2 + e_0 P_3 + f_0 P_4 = l_0.$$

Assume

$$P_0 = Q_0 + b_0 Q_1 + c_0 Q_2 + e_0 Q_3,$$

and

$$\begin{aligned} b_0 e_1 + c_0 c_2 + e_0 b_3 + f_0 &= 0 \\ a_0 e_0 + b_0 c_1 + c_0 b_2 + e_0 &= 0 \\ f_0 e_4 + a_0 c_0 + b_0 b_1 + c_0 &= 0, \end{aligned}$$

which equations (in this and in all other cases) are changeable by successive substitution into the following:

$$\begin{aligned} b_0 e_1 + c_0 c_2 + e_0 b_3 + f_0 &= 0 \\ a_1 e_1 + b_1 c_2 + c_1 b_3 + e_1 &= 0 \\ f_2 e_1 + a_2 c_2 + b_2 b_3 + c_2 &= 0, \end{aligned}$$

from which equations e_1 , c_2 , and b_3 can be determined, and thence, by restitution, e_0 , c_0 , and b_0 . The original equation is now reduced to

$$(c_0 e_2 + e_0 c_3 + f_0 b_4 + a_0) Q_0 + (e_0 e_3 + f_0 c_4 + a_0 b_0 + b_0) Q_1 = l_0,$$

which has been solved. With regard to the notation of such questions as the preceding, observe

1. That it will very often be found convenient to write such equations as the preceding in the form

$$a_0 P_0 + b_1 P_1 + c_2 P_2 + e_3 P_3 + f_4 P_4 = 0,$$

expressing the coefficient of P_m by the function which it is, not of x , but of $\alpha^m x$.

2. That in carrying on investigations connected with periodic equations, it will often be useful to write them round a circle drawn upon the paper, instead of in the usual way.

3. That in questions of common Algebra, which have reference to elimination, &c., from equations of the first degree, a form of notation derived from analogy will frequently furnish help to the memory: for instance, the following

$$\begin{aligned} ax + b'x' + c''x'' + e'''x''' &= h \\ a'x' + b''x'' + c'''x''' + ex &= h' \\ a''x'' + b'''x''' + cx + e'x' &= h'' \\ a'''x''' + bx + c'x' + e''x'' &= h''' \end{aligned}$$

will be the best mode of writing such equations.

The preceding solution is, of course, to be adopted only in cases where the disappearance of the numerator and denominator of the result obtained from the successive equations shows that a continuous arbitrary function enters the solution.

When the equations which determine b_0 , c_0 , &c. give indeterminate results, one of them may be assumed at pleasure (say $b_0 = 1$), and c_0 and e_0 must be determined from any two of the equations. If the result be still indeterminate, let $c_0 = 1$, and determine e_0 from one of the equations.

(236.) The equation*

$$a_0 \phi m_0 x + a_1 \phi m_1 x + \dots = k_0 + k_1 x + k_2 x^2 + \dots$$

admits of an algebraical resolution, whether we suppose

* A particular case of this equation is cited by Sir J. Herschel (Edition of *Spence*, &c. p. xxviii.) as having been found among the papers of Mr. Spence.

both sides finite, or one or both infinite in the number of their terms. If we assume

$$\phi x = K_0 + K_1 x + K_2 x^2 + \dots,$$

make the substitutions on the first side of the equation, and equate corresponding terms, we find

$$K_0 \Sigma a = k_0 \quad K_1 \Sigma a m = k_1 \quad K_2 \Sigma a m^2 = k_2, \text{ \&c.,}$$

where

$$\Sigma a m^m \text{ means } a_0 m_0^m + a_1 m_1^m + \dots$$

Hence

$$\phi x = \frac{k_0}{\Sigma a} + \frac{k_1}{\Sigma a m} x + \frac{k_2}{\Sigma a m^2} x^2 + \dots$$

(237.) The equation

$$\phi x \cdot \phi \alpha x \cdot \dots \cdot \phi \alpha^{n-1} x = \beta x,$$

where βx is a form of $\xi \alpha x$, has for its general solution

$$\phi x = (\beta x)^{\frac{1}{n}} \frac{\theta x}{\theta \alpha x}.$$

But it must be observed, that the ambiguity of the second side sometimes leads to results of this kind, that some of the quantities $\phi x, \phi \alpha x, \dots$ must be considered as belonging to one of the forms, and some to others. This is always the case when the n th root can actually be extracted, and when, as may happen, the root itself, absolutely considered, is not a form of $\xi \alpha x$. For instance, suppose it required to find the curve whose ordinates equidistant from the axis of x on either side form a rectangle equal to the square of their common abscissa. The equation of the curve being $y = \phi x$, we have

$$\phi(x) \times \phi(-x) = x^2 \quad \phi x = \sqrt{x^2} \frac{\theta x}{\theta(-x)},$$

which, in its present form, appears to satisfy the conditions; but the form in question is, if θx be rational, the equation of two distinct curves denoted by

$$\phi x = x \frac{\theta x}{\theta(-x)} \quad \phi x = -x \frac{\theta x}{\theta(-x)},$$

which can only be considered as the same curve under an extension of analysis, by which any number of curves may be contained under one equation. Taking one ordinate off each curve, the condition is satisfied; but taking both from the same, we have $\phi x \times \phi(-x) = -x^2$ instead of x^2 . To get a curve which satisfies the condition within the common meaning of the phrases, we

must let $\frac{\theta x}{\theta(-x)}$ be an irreducible square root; that is, let

$$\phi x = \sqrt{x^2} \sqrt{\frac{\theta x}{\theta(-x)}}, \text{ or } y^2 = x^2 \frac{\theta x}{\theta(-x)}.$$

Similarly, we find for the curve the rectangle of whose ordinates is constant and equal to b , when that of the abscissæ is constant and equal to a , the equation

$$y^2 = b \theta x \div \theta \left(\frac{a}{x} \right).$$

(238.) We shall now examine further the method of derivation of periodic functions. The form $\phi_{-1} \phi x$, which has been shown (33.) to contain periodic functions, does in reality contain every periodic function. For every such function of the n th order can be exhibited in the form $\phi^{-1} r \phi x$, where $r^n = 1$. But since the different inverses of a function ϕx may be obtained by solving the equation $\phi x = r y$ arithmetically as to form, we find that $\phi_{-1} \phi x$ and $\phi^{-1} r \phi x$ are the same.

Calculus of Functions. It has appeared, therefore, that the periodic functions are in all cases the results of performing an inverse operation upon any other form of the direct function, except that with which the inverse is convertible. But the preceding inversion is only one particular case of those which are possible; and it will appear from considerations similar to those employed in (31.), that if a direct (that is integral) algebraical expression be given, its complete inversion cannot be found without applying to every term in which there is a power of x , the factor unity in the form r^n , where n remains unknown until the inversion shows what root must be extracted.

We found the forms of periodic functions hitherto obtained by supposing a particular solution; and the derivation of a non-periodic from a periodic function, or *vice versa*, give a certain limited symbol of discontinuity. We may show how to extend that solution in the following way. Instead of supposing αx a particular solution of $\psi^n x = x$, let it not be so, and let $\phi \alpha \phi^{-1} x = \psi x$. Then we have

$$\phi \alpha^n x = \phi x,$$

or

$$\phi x = \theta \cos 2\pi \frac{\Lambda x}{n},$$

which being supposed, $\phi \alpha \phi^{-1} x$ is periodic. But, by actually performing the operation, all traces of αx disappear as follows.*

$$\begin{aligned} \phi^{-1} x &= \Lambda^{-1} \left(\frac{n}{2\pi} \cos^{-1} \theta^{-1} x \right) \\ \phi \alpha \phi^{-1} x &= \theta \cos 2\pi \frac{1}{n} \Lambda \alpha \Lambda^{-1} \frac{n}{2\pi} \cos^{-1} \theta^{-1} x \\ &= \theta \cos 2\pi \frac{1}{n} \left(\frac{n}{2\pi} \cos^{-1} \theta^{-1} x - 1 \right) \\ &= \theta \cos \left(\cos^{-1} \theta^{-1} x - \frac{2\pi}{n} \right), \end{aligned}$$

which is evidently a periodic function of the n th order, but is derived from $x - \frac{2\pi}{n}$, which is not a periodic function. At the same time it must be observed, that the preceding is equally a derivative of $\frac{2\pi}{n} - x$, and we should therefore state, as a theorem of some curiosity, that *periodic functions of all orders can be derived from periodic functions of the second order*, if the method above given did not lead to a more remarkable extension of our preceding views.

The equation $\psi^n x = x$ is solved when $\alpha^n x = x$, ψx being $\phi \alpha \phi^{-1} x$. That is, when $\alpha^n x = \phi^{-1} \phi x$; but if $\alpha^n x = \phi_{-1} \phi x$, (where ϕ_{-1} , as before, stands for an inconvertible inverse of ϕ) we have still

$$\psi^n x = \phi \phi_{-1} \phi \phi^{-1} x = x;$$

the above is an instance, for $\cos^{-1} x - 2\pi$ is one of the inconvertible inverses of $\cos x$. The form of periodic functions thus obtained is

$$\phi (\phi_{-1} \phi)^{\frac{1}{n}} \phi^{-1} x.$$

Thus, a periodic function of the n th order is

$$\theta e^{\left(\Lambda \theta^{-1} x + \frac{2m\pi\sqrt{-1}}{n} \right)} \text{ or } \theta \left(e^{\frac{2m\pi\sqrt{-1}}{n}} \theta^{-1} x \right),$$

* We do not use brackets; every functional operation is relative to the total result of all which precede in the order of operation.

which does not, however, differ from forms already obtained. Calculus of Functions.

(239.) The preceding attempt to derive a periodic from a non-periodic function terminated in the disappearance of the first mentioned; the following method gives the same result.

In the equation $\phi \alpha^n x = \phi x$, let $\phi = \psi \Lambda$, which gives $\psi (x - n) = \psi x$, of which r^n is a particular solution when $r^n = 1$. And since all the values of r are powers of some one or more amongst them, it follows that the general theorem deduced from the preceding is:

$$\text{if } \phi x = \theta (r^{\Lambda x}) \quad r^n = 1,$$

and if

$$\psi x = \phi \alpha \phi^{-1} x, \text{ then } \psi^n x = x,$$

αx being any function whatever, and Λx its exponential

inverse. The resulting value of ψx is $\theta \frac{1}{r} \theta^{-1} x$.

(240.) We shall now consider the equation

$$\psi x + \psi \alpha x + \dots + \psi \alpha^n x = 0,$$

where αx is not periodic. If we assume $k^{-\Lambda x}$ as a particular solution, we find

$$a_0 + a_1 k + a_2 k^2 + \dots + a_n k^n = 0;$$

and if $k_1, k_2, \&c.$ be the roots of this equation, the general solution of the preceding is

$$\xi_1 \alpha x \cdot k_1^{-\Lambda x} + \xi_2 \alpha x \cdot k_2^{-\Lambda x} + \dots$$

in which there are n arbitrary functions, unless the equation have one or more sets of equal roots.

In the case of a couplet of equal roots, the particular solution corresponding to them is

$$(\xi_1 \alpha x + \Lambda x \xi_2 \alpha x) k_1^{-\Lambda x}.$$

In the case of a triplet of equal roots, the solution is

$$(\xi_1 \alpha x + \Lambda x \xi_2 \alpha x + (\Lambda x)^2 \xi_3 \alpha x) k_1^{-\Lambda x},$$

and so on.

The preceding equation shows the connection between functional equations and equations of differences very completely. In fact, if we denote any function ϕx by $\psi \Lambda x$, &c., the equation

$$\beta x \phi x + \beta_1 x \phi \alpha x + \beta_2 x \phi \alpha^2 x + \dots = \nu x$$

may be immediately reduced to

$$B x \psi x + B_1 x \psi (x-1) + B_2 x \psi (x-2) + \dots = N x,$$

which is a common equation of differences. There is no occasion, therefore, to dwell further upon this case, as we may refer the reader to all works on equations of differences.

(241.) Let

$$\beta x + \beta_1 x \psi x + \beta_2 x \psi^2 x + \dots + \beta_n x \psi^n x = 0.$$

Assume $\psi x = \phi \alpha \phi^{-1} x$, where α is any given function. Then, by substitution, and writing ϕx for x , we have

$$\beta \phi x + \beta_1 \phi x \cdot \phi \alpha x + \dots + \beta_n \phi x \phi \alpha^n x = 0,$$

which may in various cases be reduced to the last form; and it would appear at first as if we might get as many distinct solutions as we can assign forms of αx . But this does not follow; for since every function can be made a derivative of any given function, it may happen that any two solutions, $\phi_1 \alpha_1 \phi_1^{-1}$ and $\phi_2 \alpha_2 \phi_2^{-1}$, may be the same. But the arbitrary function (which depends only on αx) may vary with the form which we assign to α .

We have already considered a particular case of the preceding process, in the method of finding periodic functions. In order to show generally the disappearance of αx , remember that the several steps are, 1. $\psi x =$

Calculus of $\phi \alpha \phi^{-1}x$. 2. $\phi x = \chi A x$. 3. An equation of differences, giving $\chi x = F x$, $\phi x = F A x$, (it must be remembered that, as far as we can assign them, the arbitrary functions are also functions of $A x$.) $\phi^{-1}x = A^{-1}F^{-1}x$, $\psi x = F A \alpha A^{-1}F^{-1}x = F(F^{-1}x - 1)$, or the function of which $F^{-1}x$ is the exponential inverse. Hence ψx appears to be altogether independent of αx . It is really so if the coefficients be constant, or if $F x$ is expressed, as will frequently happen, in the form $F_1 A x$, not containing x except through $A x$.

(242.) As an instance of the last, let us suppose

$$\psi^2 x - 2a\psi x + x = 0,$$

in which we immediately find

$$\begin{aligned}\phi x &= c r^{Ax} + c' r^{-Ax} \quad r = a + \sqrt{a^2 - 1} \\ \psi x &= \phi \alpha \phi^{-1}x = a x + \sqrt{a^2 - 1} \sqrt{x^2 - C} \\ &= \sqrt{C} \cos\left(\cos^{-1}a + \cos^{-1}\frac{x}{\sqrt{C}}\right),\end{aligned}$$

which fulfils the conditions. But the general solution of

$$\psi^2 x + a\psi x + b x = 0$$

depends upon the inversion of

$$\chi x = c r^x + c' r'^x, \text{ where } r r' = b;$$

and, generally, $\sum a_n \psi^n x = 0$ depends upon the inversion of $\sum c_n r^n = y$. It must be observed, however, 1. That the preceding equation always has one particular solution of the form $c x$, derived from taking one particular solution only of the form r^x , to the equation which gives χx . 2. That if the product of any two roots of $\sum a_n r^n = 0$ be equal to unity, a more general solution may be found, as in the particular case above, namely

$$\frac{1}{2}\left(r + \frac{1}{r}\right)x + \frac{1}{2}\left(r - \frac{1}{r}\right)\sqrt{x^2 - C},$$

where r is one of the roots in question.

(243.) The equation

$$x^0(\psi x)^{a_1}(\psi^2 x)^{a_2} \dots (\psi^n x)^{a_n} = 1$$

may be solved as follows. Let $\psi x = \phi \alpha \phi^{-1}x$, and take the logarithm of the result, which gives

$$a_0 \lambda \phi x + a_1 \lambda \phi \alpha x + \dots + a_n \lambda \phi \alpha^n x = 0 \dots (\lambda \phi),$$

which gives

$$\lambda \phi x = \sum c_m r_m^{Ax},$$

where r_m is a root of

$$a_0 + a_1 r + \dots + a_n r^n = 0;$$

the form of the solution is

$$\lambda^{-1} F(F^{-1} \lambda x - 1)$$

in all cases in which the equation $(\lambda \phi)$ gives $\lambda \phi x$ as a function of $A x$; that is, in all cases at present soluble.

(244.) We shall here proceed to consider the manner in which the inconvertible inverses of a function must be used in connection with the direct function, when we take into consideration direct functions with more values than one.

1. If $f(a, b)$ and $\phi(a, b)$ be two different forms of the same function, so that when a and b are unambiguous,

$$f(a, b) = \phi(a, b),$$

it does not therefore follow that when we substitute instead of a and b functions with more values than one, that any value of a combined with any other value of b will make the preceding equation true, unless we use the

same values of a and b on both sides of the equation. Hence, in

$$a b = c, \text{ which gives } a^{\frac{1}{m}} b^{\frac{1}{m}} = c^{\frac{1}{m}},$$

it is not true that any value of $a^{\frac{1}{m}}$ multiplied by any value of $b^{\frac{1}{m}}$ may be equated with any value of $(a b)^{\frac{1}{m}}$, but only with that value which arises from using the same forms of a and b as those from which the $a^{\frac{1}{m}}$ and $b^{\frac{1}{m}}$ in question were derived in an arithmetical form. If one of the primitive m th roots of unity be called r , and if a be considered as $a r^{pm}$ and b as $b r^{qm}$, then $a b$ on the second side must be considered as $a b r^{(p+q)m}$.

2. When $y = \phi x$ gives $x = \phi_{-1} y$, this equation is not identically true, unless we give y one particular form for each particular form of ϕ_{-1} . In fact, as we have seen, $\phi_{-1} \rho \phi x$ is the representation of x , as derived from inversion and reinversion, where ρ is a symbol of a change of form without change of value.

3. When we take the function $\psi \phi_{-1} x$, we have different results according as we use one or another form of ϕ_{-1} , except when ψ admits of unambiguous representation in the form $\chi \phi$, in which case $\psi \phi_{-1} x$ is $\chi \phi \phi_{-1} x$ or χx .

4. If we assume $\chi x = \phi \psi x$, then $\chi_{-1} x = \psi_{-1} \phi_{-1} x$, and any form of ψ_{-1} applied to any form of ϕ_{-1} gives one of the forms of χ_{-1} . And if the forms of ϕ_{-1} , ψ_{-1} , and χ_{-1} can be found by giving different values to m , n , and p , as in

$$\phi_{-1} x = \Phi(m, x) \quad \psi_{-1} x = \Psi(n, x) \quad \chi_{-1} x = X(p, x),$$

then the equation $\chi_{-1} x = \psi_{-1} \phi_{-1} x$ is an equation of condition between m , n , and p .

5. If there be a constant c in ϕx which admits of several values, then we can only consider the introduction of different values as forming entirely different functions of x . For instance, one of the forms of

$1 + (1)^{\frac{1}{2}} x$ is a periodic function, and the other is not.

6. Let there be a function of x which has more values than one, and let us suppose it to be ϕx . If we make $\phi x = y$, we must remember that y is now either of the values of ϕx , and in every alteration and substitution which we make, we must remember that the general value of y must be used, and not the particular value, in every case where the general value of ϕx is used. For instance, let us suppose $a^x = y$, where the first side has as many values as there are units in the denominator of x . If we call m this denominator, then y being the arithmetical value of a^x , we have $a^x = (1)^{\frac{1}{m}} y$, or, which will do equally well, $a^x = (1)^x \cdot y$; the true expansion of a^x being

$$a^x = (1)^x \left\{ 1 + x + \frac{x^2}{2} + \dots \right\},$$

derived from the general binomial theorem

$$(1 + a)^x = (1)^x \left\{ 1 + x a + x \frac{x-1}{2} a^2 + \dots \right\}.$$

If then $l a$ and $l x$ be the arithmetical logarithms of a and x , we find

$$e^{(l a + 2\pi m \sqrt{-1})x + 2n\pi \sqrt{-1}} = (1)^x e^{l y + 2\pi p \sqrt{-1}},$$

an equation which is true for all values of m , n , and p , provided that value of $(1)^x$ be taken which corresponds

Calculus of Functions. to the given value of m ; provided, in fact, we take $\varepsilon^{2\pi m x \sqrt{-1}}$ for $(1)^x$; that is, we have

$$\varepsilon^{(la+2\pi m \sqrt{-1})x+2\pi n \sqrt{-1}} = \varepsilon^{ly+2\pi m x \sqrt{-1}+2\pi p \sqrt{-1}},$$

an equation which is certainly true, and might be verified by actual expansion. But we are not at liberty to assert the equality of the exponents, unless we make $n=p$.

The theorems which have been lately proposed* with respect to the logarithms of unity, amount, in fact, to equating the preceding exponents without the restriction upon the value of $(1)^x$ above alluded to, and the assertion that x , as thus derived, is the real solution of $a^x = y$. For that process gives

$$(la + 2\pi m \sqrt{-1})x + 2\pi n \sqrt{-1} = ly + 2\pi m'x \sqrt{-1} + 2\pi p \sqrt{-1},$$

or

$$x = \frac{ly + 2\pi v \sqrt{-1}}{la + 2\pi w \sqrt{-1}},$$

where v and w are any whole numbers. If $a = \varepsilon$ and $y = 1$, this gives for the logarithms of unity

$$\frac{2\pi v \sqrt{-1}}{1 + 2\pi w \sqrt{-1}} \text{ or } \frac{2\pi v}{2\pi w - \sqrt{-1}},$$

(245.) The examples in (240, &c.) are closely connected with the history of the methods employed in this branch of Mathematics, on which we shall now introduce some remarks. Every expression of Algebra must necessarily give an idea rather of a set of operations to be performed upon quantities than of the quantity itself

* With great respect for the talents of Mr. Graves and of Sir William Hamilton, as well as of some distinguished foreigners, whom they state to have produced coincident results, we must suppose them to have fallen into this error, directly or indirectly, until some one among them produces a more explicit statement of their fundamental conventions, pointing out whether they employ those in common use or not. After an attentive perusal of both papers by Mr. Graves, that in the *Philosophical Transactions* of 1829, and that in the *Fourth Report of the British Association*, we do not know whether the assertion, that the logarithm of 1 is

$\frac{2\pi v}{2\pi w - \sqrt{-1}}$ means, that

$$\left(1 + 1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \dots\right)^{\frac{2\pi v}{2\pi w - \sqrt{-1}}} = 1 \dots (A)$$

or not.

To find a relation between x and y in the equation $a^x = y$, Mr. Graves supposes $a = \cos \theta + \sqrt{-1} \sin \theta$, and hence concludes that $a^x = f(xf^{-1}a)$ where $f\theta = \cos \theta + \sqrt{-1} \sin \theta$. But this is equally true where $f\theta = e^{c\theta}$, c having any value whatever, as appears sufficiently if the common processes of Algebra be true. If they be not true, we cannot see how the theorem

$$f\theta = \text{the common expansion of } \varepsilon^{\theta \sqrt{-1}}$$

(which is admitted on all sides) is true.

If we were to hear an objection to the equation $\varepsilon^{2\pi v \sqrt{-1}} = 1$, grounded on an objection to impossible quantities, we might appeal to the actual expansion

$$1 - \frac{\pi^2 v^2}{2} + \dots + (2\pi v - \dots) \sqrt{-1} = 1,$$

or $1 + 0 \times \sqrt{-1} = 1$. Is it really asserted that the expansion of the left-hand side of A would give the right-hand side? If so, let it be shown, and our objections of course fall to the ground. But in the mean while we hold it safer, when two methods give discordant results, to make the better known of the two throw suspicion upon the other.

which results from performing these operations. Nothing is more difficult to a beginner in Algebra than the necessity of considering a complicated literal expression as the value of an unknown quantity, instead of the exhibition of the processes by which the value is to be obtained, so soon as the subjects of operation shall have received specific values. Before the introduction of specific symbols for indeterminate known quantities by Vieta, the language and methods of algebraists were fast approaching to a *calculus of operations*, distinguished from a *calculus of functions* by the absence of symbols, but on this very account approximating more perhaps to the consideration of the *functional* than of the ordinary algebraic character of expressions. "*Solve quamvis questionem . . . serva operationes . . . et habebis regulam de modo pro omni consimili questione*" are the words of Cardan, which comprise his method of making one arithmetical example point out the solution of another. When the literal expressions of Vieta were introduced, the object seems to have been to forget as much as possible the method of formation which each expression carried with it, unless at the moment when a new operation was to be performed. And thus it became necessary to describe operations in words when it was desired to point the attention upon a function, as such; and the rule of Cardan, above cited, became as necessary in the Algebra of Vieta as in pure Arithmetic, simply because a confined mode of viewing the subject had conventionally invested the results of the former with the character of the latter. Hence rules and processes took the place of expressions; and our readers must remember that it was, and still is, usual to deduce the roots of an equation of the second degree *de novo* on every separate occasion. This anxiety to distinguish the operations of Algebra from the symbols which conventionally represented the operations themselves, is strikingly manifest in the older algebraists. Take the following example of a process from Bartholinus.*

Exempla additionis simplicium.

$$\text{Add. } \begin{cases} a \\ b \end{cases}$$

$$\text{Summa } a + b.$$

The Differential Calculus, or rather the Integral Calculus, first turned attention expressly on the forms which will satisfy given conditions, considered independently of values. The celebrated isoperimetrical problems of the Bernoullis are among the earliest of the kind; but before this the *Principia* of Newton had appeared, full of questions in which *law* and not *quantity* was the subject of investigation. The considerations entering into physical questions made a more functional algebra absolutely necessary: the pure mathematician was rapidly becoming (out of Geometry) a machine for solving numerical equations, summing arbitrarily formed series, and trying to square the circle, when the call to follow up the researches of Newton threw him upon difficulties which were not of his own making.

(246.) We cannot find it stated precisely who first used the indeterminate functional symbol. But this is a matter of little consequence, since the time when real questions of the inverse functional calculus began to be agitated must date from the researches of Euler, Lambert, D'Alembert, Lagrange, Monge, and Laplace. In

Calculus of the year 1760 Lagrange* integrated the equation which Functions. we should represent by

$$\Sigma p_n \phi(a + b \alpha^n x) = \xi \alpha x,$$

where $p_0, p_1, \&c.$ are forms of $\xi \alpha x$, and $\alpha x = a' + b'x$. The process consisted in a developement of $\phi(a + b \alpha^n x)$ by Taylor's theorem, and had no close connection with the difficulties of the subject. The next step was made by Monge, as we shall proceed to state.

(247.) The solution of an ordinary differential equation is shown by the direct method of forming such equations to amount to the inquiry what function of x and y must be constant in order that a given relation

between $x, y, \frac{dy}{dx}$, &c. may exist. The existence of a

functional relation between two quantities is necessary and sufficient to the fulfilment of the condition, that the one shall be constant when the other is so. A partial differential equation, besides being known to arise from the elimination of an arbitrary function, is shown to depend upon two simultaneous differential equations of the common kind, giving two functions of the variables, of which it is sufficient for the conditions that they should be constant together; that is, that one should be any function of the other. The determination of the form of the arbitrary function from given conditions leads of course to questions in the Inverse Calculus of Functions. To take a very simple case, suppose it is required to describe a surface of revolution with a given axis, which shall pass through a given curve. This amounts geometrically to describing through every point of the curve a circle perpendicular to the axis, and having its centre in the axis. This, it is evident, can be done, whether the given curve be continuous or discontinuous. If the axis of revolution be made that of z , and axes of x and y be taken perpendicular to it and to each other, the differential equation of the surface in question is

$$y \frac{dz}{dx} = x \frac{dz}{dy},$$

which is common to all surfaces of revolution having the axis in question, and its complete integral is

$$z = \theta(x^2 + y^2).$$

If αx and βx be the values of y and z , corresponding to the ordinate x of the curve in question, the condition that the surface must pass through the curve requires that

$$\beta x = \theta(x^2 + (\alpha x)^2)$$

and the determination of the form of θ depends on the inversion of $x^2 + (\alpha x)^2$.

The preceding method will frequently give the answer required, because the importance of simple inversion has led to the consideration of many cases, and inversion has been invented (the process being practically supplied by tabulation) where the ordinary processes of Arithmetic have not been found sufficient. But in a great number of cases the integral itself cannot be otherwise determined than by eliminating the subject of an arbitrary

function from between two equations: for instance, Calculus of take the manner in which Monge expressed the equation Functions. of a *developable* surface, where z is a function of x and y , obtained by eliminating V between the two following,

$$z - \psi V = (x - V) \frac{d\psi V}{dV}$$

$$y - \phi V = (x - V) \frac{d\phi V}{dV},$$

where ϕ and ψ may be any forms whatsoever.

(248.) Let there be a function z defined as follows:

$$z = \phi \alpha(x, y) + \psi \beta(x, y),$$

where α and β are known forms, and ϕ and ψ are to be so determined, that

$$z = f x \text{ when } y = F x \text{ } \{ f, F, f_1, F_1$$

$$z = f_1 x \text{ when } y = F_1 x \text{ } \}$$
 being known.

Hence we have the following equations,

$$f x = \phi \alpha(x, F x) + \psi \beta(x, F x)$$

$$f_1 x = \phi \alpha(x, F_1 x) + \psi \beta(x, F_1 x).$$

Let $\beta(x, F x) = u$ give* $x = \beta u$, which gives

$$f \beta u = \phi \alpha(\beta u, F \beta u) + \psi u,$$

from which the form of ψ is known, when that of ϕ is known; and by following a process exactly similar with the second equation, making $\beta(x, F_1 x) = u$, give

$x = \beta_1 u_1$, we find

$$f_1 \beta_1 u_1 = \phi \alpha(\beta_1 u_1, F_1 \beta_1 u_1) + \psi u_1,$$

and as these equations must be identically true, whatever we may write for u_1 or u , write u for u_1 in the second, and subtract it so altered from the first, which gives

$$f \beta u - f_1 \beta_1 u = \phi \alpha(\beta u, F \beta u) - \phi \alpha(\beta_1 u, F_1 \beta_1 u),$$

an equation of the form

$$\phi \mu x - \phi \nu x = \omega x,$$

which may be reduced to the form

$$\phi x = \beta x + \gamma x \phi \alpha x,$$

already considered. The preceding process will apply equally well to

$$z = a_{x,y} + b_{x,y} \phi \alpha(x, y) + c_{x,y} \psi \beta(x, y),$$

and will give a result of similar form; as also will

$$z = a_{x,y} + \phi \{ b_{x,y} + c_{x,y} \psi \beta(x, y) \},$$

which must be previously reduced to the form

$$\phi^{-1}(z - a_{x,y}) = b_{x,y} + c_{x,y} \psi \beta(x, y).$$

(249.) The equation

$$z = \Sigma \gamma_n(x, y) \psi_n \alpha_n(x, y),$$

in which there are n undetermined functions, $\psi_1, \psi_2, \&c.$ to be determined by n conditions of the form

$$z = f_n x \text{ when } y = F_n x,$$

may be reduced by the preceding process to n equations, containing each a term of the form $\omega u \psi_1 u$: elimination reduces these to $n - 1$ equations, wanting $\psi_1 u$. A repetition of the process eliminates $\psi_2 u$, and so, till we have two equations only, from which the elimination of $\psi_{n-1} u$ leaves an equation of the form

* βx was the notation proposed by Monge for what we call $\beta^{-1} x$ (a notation introduced, we believe, by Sir John Herschel.) It will be convenient to reserve the former for particular inversions required by problems, as in the present case.

* *Misc. Taur.*, vol. iii. We cite here from the Memoir of Monge already cited. Several writers have used expressions which would lead us to infer they thought Lagrange had solved a more general case. If it be so, we cannot see it from Monge's account, and we have not had access to the original. In the *Memoirs of the Analytical Society*, p. 96, it is stated that Lagrange had reduced the question to the solution of a transcendental equation, such as those in (242.), which might be done.

Calculus of
Functions.

$$\psi_n \mu u + \omega u \psi_n \nu u + \omega_1 u = 0.$$

(250.) The process by which Monge reduced the equations he thus obtained to equations of differences, or rather generalized the conditions implied in equations of differences so as to raise them to functional equations, is as follows. Taking the difference of x to be that by which it increases when it becomes αx , or

$$\Delta x = \alpha x - x,$$

and y a function of x to be determined, he defines Δy by $\psi \alpha x - \psi x$,

$$\Delta^2 y \text{ by } \psi \alpha^2 x - \psi \alpha x - (\psi \alpha x - \psi x),$$

and so on. Thus the equation

$$\Delta y + A y + B = 0$$

answers to what, in our notation, would be

$$\psi \alpha x = (1 - A) \psi x - B.$$

We shall give, in the words of Monge, his integration (as we must now call it) of the preceding equation, for the cases where $\alpha x = x + a$, $\alpha x = a + (b + 1)x$, and $\alpha x = a x^n$. It is extracted from the ninth volume of the *Memoirs of the Academy of Sciences*, page 357. It must be observed, that what we call 2π , Monge calls π .

"PROBLÈME V.

"Intégrer complètement l'équation $\Delta y + A y + B = 0$, dans le cas où la différence finie de la variable principale seroit 1°. = constante = a ; 2°. = $a + bx$; 3°. = $a x^n - x$.

"SOLUTION.

"Soit fait $A y + B = e^u$; ce qui donnera

$$\frac{e^{u+\Delta u} - e^u}{A} + e^u = 0,$$

et par conséquent $\frac{\Delta u}{L \cdot (1 - A)} = 1$; soit x la variable

principale: cela posé, pour le premier cas, on a $\Delta x = a$ ou $\frac{\Delta x}{a} = 1$ et par conséquent $\frac{\Delta u}{L \cdot (1 - A)} =$

$\frac{\Delta x}{a}$, dont l'intégrale sera $u = \frac{x L \cdot (1 - A)}{a}$; et, mettant

pour u sa valeur, $L \cdot (A y + B) = \frac{x}{a} L \cdot (1 - A)$, ou

enfin $A y + B = (1 - A)^{\frac{x}{a}}$; or nous avons vu que, pour compléter cette intégrale, il falloit ajouter la fonction arbitraire $\phi\left(\sin \frac{\pi x}{a} \text{ et } \cos \frac{\pi x}{a}\right)$; donc l'intégrale complète, dans le premier cas, sera

$$A y + B = (1 - A)^{\frac{x}{a}} \times \phi\left(\sin \frac{\pi x}{a} \text{ et } \cos \frac{\pi x}{a}\right).$$

"Pour le second cas, on fera $a + bx = e^w$, d'où l'on tirera $\Delta w = L \cdot (b + 1)$, ou $\frac{\Delta w}{L \cdot (b + 1)} = 1$; ainsi,

ayant d'ailleurs $\frac{\Delta u}{L \cdot (1 - A)} = 1$, on aura $\frac{\Delta u}{L \cdot (1 - A)} =$

$\frac{\Delta w}{L \cdot (b + 1)}$, dont l'intégrale sera $u = w \frac{L \cdot (1 - A)}{L \cdot (b + 1)}$,

et, mettant pour u et w leurs valeurs,

$$L \cdot (A y + B) = L \cdot (a + bx) \frac{L \cdot (1 - A)}{L \cdot (b + 1)};$$

or nous avons vu (Prob. III.) que la fonction arbitraire qu'il faut ajouter pour compléter l'intégrale, est

$$\phi\left(\sin \frac{\pi L \cdot (a + bx)}{L \cdot (b + 1)} \text{ et } \cos \frac{\pi L \cdot (a + bx)}{L \cdot (b + 1)}\right);$$

donc on aura, pour intégrale complète,

$$L \cdot (A y + B) = L \cdot (a + bx) \frac{L \cdot (1 - A)}{L \cdot (b + 1)}$$

$$+ L \cdot \phi\left(\sin \frac{\pi L \cdot (a + bx)}{L \cdot (b + 1)} \text{ et } \cos \frac{\pi L \cdot (a + bx)}{L \cdot (b + 1)}\right),$$

et, passant aux nombres,

$$A y + B = (a + bx)^{\frac{L \cdot (1 - A)}{L \cdot (b + 1)}}$$

$$\times \phi\left(\sin \frac{\pi L \cdot (a + bx)}{L \cdot (b + 1)} \text{ et } \cos \frac{\pi L \cdot (a + bx)}{L \cdot (b + 1)}\right).$$

"Pour le troisième cas; c'est à dire, lorsque l'on a $\Delta x = a x^n - x$, on fera $x = e^w$, d'où l'on tirera $\Delta w =$

$(n - 1)w + L a$: faisant ensuite $(n - 1)w + L a = e^u$

on aura $\Delta w' = L \cdot n$, ou $\frac{\Delta w'}{L \cdot n} = 1$; or on a $\frac{\Delta u}{L \cdot (1 - A)}$

$= 1$; par conséquent on aura $\frac{\Delta u}{L \cdot (1 - A)} = \frac{\Delta w'}{L \cdot n}$, dont

l'intégrale est $u = w' \frac{L \cdot (1 - A)}{L \cdot n}$; et, mettant pour u

et w' leurs valeurs,

$$L \cdot (A y + B) = L \cdot L \cdot a x^{n-1} \times \frac{L \cdot (1 - A)}{L \cdot n};$$

ajoutant enfin la fonction arbitraire du Problème IV., on aura, pour intégrale complète,

$$L \cdot (A y + B) = L \cdot L \cdot (a x^{n-1}) \frac{L \cdot (1 - A)}{L \cdot n}$$

$$+ L \cdot \phi\left(\sin \text{ et } \cos \frac{\pi L \cdot L \cdot a x^{n-1}}{L \cdot n}\right)$$

enfin, passant aux nombres,

$$A y + B = (L \cdot (a x^{n-1}))^{\frac{L \cdot (1 - A)}{L \cdot n}}$$

$$\times \phi\left(\sin \text{ et } \cos \frac{\pi L \cdot L \cdot a x^{n-1}}{L \cdot n}\right).$$

The method by which Monge discovers the nature of the arbitrary function is, in fact, repeated again in the above, though not directly alluded to, by finding x and ω in such a manner that a given increase of ω shall increase x by $\alpha x - x$, or *vice versa*; he then makes the sine and cosine of ω multiplied by 2π , and divided by the given increase, the subjects of the arbitrary function, having first expressed ω in terms of x . In fact,

a being the increase of ω , $-\frac{\omega}{a}$ is the exponential inverse

of αx , as we have termed it. The Memoir of Monge cited is certainly worth the attention of the advanced student; but after all, it is an effort of genius striving to force a mathematical method beyond its powers.

(251.) While Monge was employed in generalizing the Calculus of variables with constant differences into a Calculus of Functions, Laplace, as he informed Monge at the time, was reducing functional equations to equations of constant differences, in the manner which has been already described. A little before the same time,*

* Lambert's first ideas were published in the *Acta Helvetica* in 1758; Lagrange's theorem in the *Memoirs of the Academy of Sciences* for 1768; Laplace's method for functional equations in the *Memoirs of the Savans Etrangers* for 1773.

Calculus of
Functions.

Calculus of Functions. Lambert communicated to Euler and Lagrange some expansions of the root of a trinomial equation in series, which the latter generalized into his celebrated theorem, the greatest step yet made in the calculus of inversion. It is not, however, of any service in our present pursuit, owing to the unlimited number of terms which always occurs. And, in the method of Laplace, many of the integrations of equations of differences which can be effected are unserviceable, owing to the appearance of a sum or product, the number of terms of which is a function of x .

(252.) The method of Laplace can only apply, with perfect generality, to cases in which the unknown function has not more than two subjects; for if $\alpha x = u_x$, and $\beta x = u_{x+1}$, which is always possible, subject to the integration of an equation of differences, it does not follow that γx will be u_{x+n} , &c.; but if this happen, the method applies.

In the *Memoirs of the Analytical Society* already cited, it is shown how, by means of equations of partial differences, to reduce more extensive classes of functional equations to depend upon finite integration. The following is a sketch of the method.

Let the proposed equation be

$$F(x, \phi \alpha x, \phi \beta x, \phi \gamma x) = 0 \dots (1),$$

where F , α , β , and γ are known forms, and ϕx is to be determined. Assume a function of v and z , namely, $P_{v,z}$, to be determined by the following conditions:

$$\alpha x = P_{v,z}, \quad \beta x = P_{v-1,z}, \quad \gamma x = P_{v,z-1},$$

or

$$\beta^{-1} P_{v-1,z} = \alpha^{-1} P_{v,z} \dots (2) \quad \gamma^{-1} P_{v,z-1} = \alpha^{-1} P_{v,z} \dots (3),$$

giving

$$P_{v,z} = \Pi(v, z, \theta(\cos 2\pi v, z)) \dots (2')$$

$$P_{v,z} = \Pi_1(v, z, \theta_1(v, \cos 2\pi z)) \dots (3').$$

That Π and Π_1 are known forms is the assumed integration; and θ , θ_1 are arbitrary functions *subject only to the conditions $\Pi = \Pi_1$, and that of not inverting the cosine, provided always these two conditions agree.* But we have

$$x = \alpha^{-1} P_{v,z},$$

and substitutions in (1) give (if $\phi P_{v,z} = \Phi_{v,z}$)

$$F(\alpha^{-1} P_{v,z}, \Phi_{v,z}, \Phi_{v-1,z}, \Phi_{v,z-1}) = 0 \dots (4),$$

another equation of differences, which we will suppose integrated in the form

$$\Phi_{v,z} = F_1(v, z, \Theta \cos 2\pi v, \Theta_1 \cos 2\pi z) \dots (5),$$

F_1 being known by integration, and Θ and Θ_1 being arbitrary functions, *which do not invert the cosine.* From (2') and (3') v and z can be expressed in the form

$$v = \Pi P_{v,z} \quad z = \Pi_1 P_{v,z},$$

giving

$$\phi P_{v,z} = F_1(\Pi P_{v,z}, \Pi_1 P_{v,z}, \Theta \cos 2\pi \Pi P_{v,z}, \Theta_1 \cos 2\pi \Pi_1 P_{v,z})$$

$$\phi x = F_1(\Pi x, \Pi_1 x, \Theta \cos 2\pi \Pi x, \Theta_1 \cos 2\pi \Pi_1 x).$$

(253.) The words which we have put in italics relate to points which have not been adverted to by the distinguished writer from whom we cite, (indeed, it hardly lay in his road to do so); and in the perfect impossibility of treating even the most simple case in all its generality, we are compelled to resort to analogy, and even to conjecture. But as our object is to create doubt, not to establish certainty, such a method is more than sufficient, since the simple word of denial is a sufficiently

VOL. II.

conclusive argument against whatsoever is not established by demonstration.

The simultaneous existence of any number of equations may be of two kinds, very distinct in their character, though not always distinguished. The first, to which the name more properly belongs, (but which is not the common species,) and which we shall call *dependent coexistence*, is where we require that one or more of the equations should be true *only* when the rest are true, but not without them, *necessarily*. Instead of separate equations,

$$V_1 = 0 \quad V_2 = 0 \dots W_1 = 0 \quad W_2 = 0 \dots,$$

where the existence of the first set does not necessarily make that of the second follow, the dependent coexistence only requires that

$$c_1 V_1 + c_2 V_2 + \dots + h_1 W_1 = 0$$

$$c'_1 V_1 + c'_2 V_2 + \dots + h'_1 W_2 = 0$$

$$\dots \dots \dots$$

where the constants may be, and generally will be, determinate. On the other hand, independent coexistence requires the absolute existence of all the first sets of equations together. The want of sufficient nominal distinction between these two different things creates a difficulty in the Calculus of Variations well known to those who remember the early obstacles to their progress. But we proceed to the bearing of this distinction upon our subject.

(254.) A question may arise as to the equations (2) and (3), whether their coexistence is dependent or independent. If the first, the two equations are equivalent only to one; and the solution, though correct, (for the dependent includes the independent case,) is very limited. That the solutions of the independent case are much more limited than those of the other is sufficiently evident; but we shall apply the distinction to the common theory of differential equations; that is, we shall digress to show that the arbitrary function appears in the solution of *dependently* coexistent differential equations, even where no partial differentiations occur. That the introduction of the arbitrary function into *partial* differential equations may be referred to this circumstance follows from the well-known method of deducing their integration.

Let $\phi(M, N) = P$ be the result of an integration, where M , N , and P are respectively functions of x , y , and z . Common differentiation shows that

$$\frac{d\phi}{dM} dM + \frac{d\phi}{dN} dN = dP,$$

or that the preceding condition is sufficient for the dependent coexistence of

$$dM = 0, \quad dN = 0, \quad dP = 0,$$

or

$$\frac{dM}{dx} dx + \frac{dM}{dy} dy + \frac{dM}{dz} dz = 0, \text{ \&c., \&c.,}$$

in which the appearance of the complete differentia. form may be lost by suppression of factors. But let us consider independently the coexisting equations

$$\frac{dy}{dt} - x = 0, \quad \frac{dx}{dt} - y = 0,$$

the independent coexistence of which implies that

$$x = ce^t + c'e^{-t}, \quad y = ce^t - c'e^{-t},$$

which latter equations will be dependently coexistent when

Calculus of
Functions.

$$\frac{dc}{dt} \epsilon^t + \frac{dc}{dt} \epsilon^{-t} = \theta t \left(\frac{dc}{dt} \epsilon^t - \frac{dc}{dt} \epsilon^{-t} \right),$$

where θt is any function whatsoever. Here c may be assumed as any function of t , and c' determined accordingly. Let $c = \psi t$, then the elimination of ψt between the two equations which give x and y , leaves one relation between x , y , and t , containing the arbitrary function θt ; which is sufficient to satisfy the conditions. Or we may multiply the second of the given equations by θt , and add it to the first, which gives

$$\frac{dy}{dt} + \theta t \frac{dx}{dt} = x + y \theta t.$$

Assume $y = \psi t$, which gives by the common process

$$x = \epsilon^{\int \frac{dt}{\theta t}} \int \epsilon^{-\int \frac{dt}{\theta t}} \left(\psi t - \frac{\psi' t}{\theta t} \right) dt,$$

in which it is easily verified that $\psi' t = x$ gives $\frac{dx}{dt} = \psi t$. The elimination of ψt then gives one relation between x , y , and t , with the arbitrary function θt . The above method shows that we have implicitly proceeded step by step in the formation of a partial differential equation more complicated than the circumstances require, and it is known that a given set of dependently coexistent equations may be made to arise from an infinite number of partial equations. But if we take the most simple partial differential equation which depends upon the two given equations, which is

$$y \left(\frac{dy}{dx} \right) + \left(\frac{dy}{dt} \right) = x,$$

we have the most simple form of the relation required, namely,

$$x^2 - y^2 = \theta (x + y \epsilon^{-t}),$$

subject to which it is generally true that y and x being functions of t , $\frac{dy}{dt} = x$, when $\frac{dx}{dt} = y$. There is, how-

ever, what we may call a *singular* solution for every particular form of θ , which deserves the attention of mathematicians, but is foreign to our present subject.

(255.) If we grant the equations (2) and (3) in (252.) to be dependently coexistent, the arbitrary function in the solution of the single equation which then connects the two must be neither of the form $\xi(v-1)$ or $\xi(z-1)$, but of the form $\xi(v-1, z-1)$. Supposing them independently coexistent, we come to the consideration of the two conditions:

1. $\Pi = \Pi_1$, which connects θ and θ_1 ;
2. Neither θx nor $\theta_1 x$ must invert $\cos x$.

Under what conditions these may be made to agree must remain a question; but the generality of the result will not be altogether destroyed by supposing arbitrary constants in the place of θ and θ_1 , owing to the final arbitrary functions Θ and Θ_1 . But the form given to these certainly appears too limited.

(256.) The following considerations will show that the preceding method must be further considered before it can be used with perfect safety. The coexistence of (2) and (3), if *independent*, makes v a function of z (for all assigned forms of θ and θ_1), determined by eliminating $P_{v,z}$ from (2') and (3'); that is, (4) is not an equation of partial differences, as supposed; if *dependent*, then only one value of $P_{v,z}$ results, whether we suppose it obtained from the single equation

$$(2) + c(3) = 0,$$

or by assigning such a relation between θ and θ_1 as shall make (2') and (3') identically the same equations. In either case, both v and z cannot be separately assigned in terms of $P_{v,z}$, which the subsequent solution requires. In either case, however, solutions may be obtained without difficulty when the arbitrary functions are supposed constant; but when they are not supposed constant, an equation of condition results between arbitrary functions.

When (2) and (3) are considered as independently coexistent, let the condition which must be assigned between θ and θ_1 be deferred, and the process continued to the end, as described. If the expression of v and z , by means of two equations, which are to be made only equivalent to one, leads to any indeterminate forms, the solution fails; if not, a relation may be assigned between the constants or arbitrary functions in the result, which will make the final value of ϕx a solution. But it must be observed that the elucidation of this point cannot be given in the present state of the theory of simultaneous equations of partial differences.

(257.) It would take more consideration than we can here give to decide *a priori* upon the preceding method; and the only examples in which the solution of the *dependent* equations (2) and (3) is possible are not conclusive, owing to the limited character of the functions which must be employed, and of the solutions obtained. We shall show, however, by an example, that the method cannot be employed, as given, without the assignment of a relation between the arbitrary quantities obtained.

Let the equation be

$$\phi x + \phi 2x = \phi 3x$$

$$x = P_{v,z}, \quad 2x = P_{v-1,z}, \quad 3x = P_{v,z-1},$$

$$P_{v-1,z} = 2 P_{v,z} \text{ or } P_{v,z} = \left(\frac{1}{2} \right)^v \theta = \Pi$$

$$P_{v,z-1} = 3 P_{v,z} \text{ or } P_{v,z} = \left(\frac{1}{3} \right)^z \theta_1 = \Pi_1,$$

and

$$\Phi_{v,z} + \Phi_{v-1,z} = \Phi_{v,z-1},$$

or

$$\Phi_{v,z} = \Theta^v \Theta_1^z \times \Theta_2$$

subject to the condition $1 + \frac{1}{\Theta} = \frac{1}{\Theta_1}$.

Assume θ and θ_1 arbitrary constants, and we have

$$\Pi x = \frac{\lambda \theta - \lambda x}{\lambda 2}, \quad \Pi_1 x = \frac{\lambda \theta_1 - \lambda x}{\lambda 3},$$

and assuming Θ , &c. to be constant, we have

$$\phi x = \Theta^{\frac{\lambda \theta - \lambda x}{\lambda 2}} \times \left(\frac{\Theta}{1 + \Theta} \right)^{\frac{\lambda \theta_1 - \lambda x}{\lambda 3}} \times \Theta_2 = V V_1 \Theta,$$

which gives

$$\phi 3x = V V_1 \Theta^{-\frac{\lambda 3}{\lambda 2}} \left(\frac{\Theta}{1 + \Theta} \right)^{-1} \Theta_2$$

$$\phi x + \phi 2x = \left\{ 1 + \Theta^{-1} \left(\frac{\Theta}{1 + \Theta} \right)^{-\frac{\lambda 2}{\lambda 3}} \right\} V V_1 \Theta_2,$$

which are not equal except when

$$1 + \Theta^{-1} \left(\frac{\Theta}{1 + \Theta} \right)^{-\frac{\lambda 2}{\lambda 3}} = \Theta^{-\frac{\lambda 3}{\lambda 2}} \left(\frac{\Theta}{1 + \Theta} \right)^{-1}.$$

(258.) The only method which we can perceive of

Calculus of
Functions.

Calculus of deducing a certain solution from the method in (252.) Functions. is as follows. Integrate (2) and (3), and assume θ and θ_1 , so as to make the two Π and Π_1 identically the same. Then find $\Phi_{v,x}$, as before; eliminate z between $P_{v,x}$ and $\phi P_{v,x}$, giving ϕx in terms of x and v . Then v is an arbitrary function of x , which becomes $v - 1$ when αx becomes βx , and remains the same when αx becomes γx . If we apply this method to the preceding equation, we find

$$P_{v,x} = \left(\frac{1}{2}\right)^v \left(\frac{1}{3}\right)^x = x$$

$$\phi P_{v,x} = \Theta^v \Theta_1^x \times \Theta_2 \quad 1 + \frac{1}{\Theta} = \frac{1}{\Theta_1}$$

$$\phi x = \Theta^v (2^v x)^{-\frac{\lambda \Theta_1}{\lambda^2}} \times \Theta_2,$$

which will be found to satisfy the conditions.

Two species of arbitrary functions are thus introduced; firstly, Θ_1 , which must not change when αx is changed, either into βx , or γx , and v , which is changed into $v - 1$, and not changed at all, in the first and second of these cases respectively.

(259.) We shall briefly observe, that the same sort of proof which shows that differential equations of the n th order are not completely integrated without the introduction of n arbitrary constants, has been applied to functional equations, as follows. If we suppose

$$\psi x = F_1(x, a_1, a_2, \dots, a_{n-1}), \quad \psi \alpha x = F_1(\alpha x, a_1, \dots)$$

$$\psi^2 x = F_2(x, a_1, a_2, \dots, a_{n-1}), \quad \psi \alpha x = F_2(\alpha x, a_1, \dots)$$

$$\dots \dots \dots$$

$$\psi^n x = F_n(x, a_1, a_2, \dots, a_{n-1}), \quad \psi \nu x = F_1(\nu x, \dots);$$

then the elimination of the $n - 1$ constants between each set of n equations gives an equation of the form $\Phi(x, \psi x, \psi^2 x, \dots, \psi^n x) = 0$, $\Phi(x, \alpha x, \dots, \nu x) = 0$, from which it is inferred that the complete solution of an equation of this form has at least $n - 1$ arbitrary constants. In the differential calculus the inference is so far just; but we have never been satisfied with any of the inverse inferences which have been made, to the effect that not more than $n - 1$ arbitrary constants could exist in the integral, though of course sufficiently sure by experience that the result, whether correctly inferred or not, is correct. Mr. Babbage* seems to have had a doubt on this subject on his mind, when he expresses himself upon "the train of reasoning usually made use of to prove that a differential equation of the n th order requires in its complete integral n arbitrary constants." And M. Lacroix, who has done as much for the differential calculus as Delambre did for astronomy, has not, in his researches, found even an attempt at a rigid demonstration. In speaking of what is, in fact, the same subject, namely, the various differential equations of the first order, which one of the n th implies, he uses these remarkable words: "*En s'abandonnant à l'analogie, on en conclura qu'une équation différentielle de l'ordre n a un nombre n d'intégrales premières.*"† We shall, therefore, not consider it an impertinent digression if we attempt to give what may be generalized into a proof of the proposition in question, in illustration of a remarkable point connected with functions.

(260.) In considering a function of x , y , and z , we must remark two distinct species. 1. Functions of x , y , and z , which cannot be reduced to the form $\phi(x, p)$

where p is a function of y and z . 2. Functions of x , and (y and z), which can be so reduced. Thus $y x^3 + z$ is an instance of the first sort; $x^3 + y z$ is an instance of the second. See (18.) and note. This precisely amounts to the distinction of the conditions under which, x only being variable, y and z are equivalent to two arbitrary constants, or to one only. The conditions of the second case are as follows. If $u = \phi(x, y, z)$, which can also be reduced to the form $\psi(x, p)$, then

$$\frac{du}{dz} \cdot \frac{dp}{dy} - \frac{du}{dy} \cdot \frac{dp}{dz} = 0,$$

which can easily be shown to be the necessary and sufficient condition, that u shall only contain x and y , because it is a function of p , or of some function of p . Now, if possible, let

$$\frac{du}{dx} = \chi(x, u) \text{ have the primitive } u = \phi(x, c, c_1);$$

the differentiation of the second gives

$$\frac{du}{dx} = \phi'(x, c, c_1) \quad c = \psi\left(x, \frac{du}{dx}, c_1\right).$$

The substitution of the form ψ for c in ϕ must eliminate c_1 ; for if not, we can express $\frac{du}{dx}$ in the form $\varpi(x, u, c_1)$, and therefore

$$\varpi(u, x, c_1) = \chi(x, u),$$

or the complete value of u can only contain c_1 , which is against the supposition. So that $\varpi(u, x, c_1)$ must be identical with $\chi(x, u)$, or otherwise a relation results which is not consistent with the supposition of u , containing two arbitrary constants. Consequently, $\phi(x, \psi, c_1)$ is independent of c_1 , or

$$\frac{d\phi}{d\psi} \cdot \frac{d\psi}{dc_1} + \frac{d\phi}{dc_1} = 0.$$

But from the identical equation

$$\phi'(x, \psi, c_1) - \frac{du}{dx} = 0 \text{ we get } \frac{d\phi'}{d\psi} \cdot \frac{d\psi}{dc_1} + \frac{d\phi'}{dc_1} = 0,$$

and from the two last we obtain

$$\left(\frac{d\phi}{d\psi} \cdot \frac{d\phi'}{dc_1} - \frac{d\phi}{dc_1} \cdot \frac{d\phi'}{d\psi}\right) \frac{d\psi}{dc_1} = 0.$$

Now, if $\frac{d\psi}{dc_1} = 0$, we have either $\frac{d\phi}{dc_1} = 0$, or $\frac{d\phi}{d\psi}$ is infinite for all values of ψ (a form of c); which last cannot be, and the first shows that ϕ is not a function of c_1 . But if

$$\frac{d\phi}{d\psi} \cdot \frac{d\phi'}{dc_1} - \frac{d\phi}{dc_1} \cdot \frac{d\phi'}{d\psi} = 0,$$

we see that ϕ can only be a function of ψ and c_1 , or of c and c_1 , in the form

$$u = \phi(x, \theta(c, c_1));$$

that is, the two arbitrary constants are equivalent only to one.*

* The (to us) apparent laxity of demonstration with which this subject is usually treated continues up to the present moment. M. Poisson, in the *Théorie de la Chaleur* just published, has a chapter on partial differential equations, in which the subject is treated with his usual happy combination of clearness and power. But he begins with a proof that the integral of a differential equation of the n th degree "*doit contenir, pour être complète, un nombre n de constantes arbitraires.*" This he does prove, with the exception of the proposition implied in the words *pour être complète*; for he does not show, or attempt to show, that no more than n arbitrary constants can be contained in the general integral.

* Essay, &c. part i. *Phil. Trans.* 1815.

† *Calc. Diff.* vol. ii. p. 294.

Calculus of Functions. (261.) The preceding process cannot be applied to equations of differences, and it is important to notice why this most simple case of functional equations differs so much, in general, from what may, in one point of view, be considered a particular case. In $\frac{dy}{dx}$ we are considering the properties of one term of a series; in $\frac{\Delta y}{\Delta x}$ we must consider the aggregate of all the terms.

If we let $\frac{\Delta a}{\Delta b} \Delta b$ stand for the difference of a , upon the supposition that b only varies, we do not find that

$$\frac{\Delta u}{\Delta y} \frac{\Delta \psi}{\Delta z} - \frac{\Delta u}{\Delta z} \cdot \frac{\Delta \psi}{\Delta y} = 0$$

is necessary when u is a function of x and $\theta(z, y)$. And it may be made evident that we can satisfy a functional equation by a solution containing as many arbitrary constants as we please. For instance,

$$\psi^2 x = x \text{ gives } \phi(1)^{\frac{1}{2}} \phi^{-1} x,$$

where ϕ may be a function which contains any number of arbitrary constants. To take a single instance, $(a-x)^{\frac{1}{2}}$, containing two arbitrary constants, satisfies $\psi^2 x = x$, the complete integral of which has been surmised to contain only one. But at the same time, remembering what we know relative to periodic functions, as opposed to all others, we cannot allow even a suspicion derived from them to affect our views of other functions.

The fact is, that in the most limited sense of the term, every functional equation admits of an infinite number of arbitrary constants, in admitting an arbitrary function. What we should ask in reference to the process in (259.) is this:—How many arbitrary constants can be found in such a manner as to admit of being converted into *separate arbitrary constants of the equation*, that is, arbitrary functions which remain constant during the verification of the equation? And, in this point of view, we cannot assign even so many arbitrary functions as might seem possible to the solution of $\psi^n x = x$, since we have found only one, and even given a method which shows all assignable solutions, at least, to be particular cases of that one.

(262.) We shall now proceed to the consideration of functions of more than one independent subject. If we take such a function as

$$\psi(x_1, x_2, x_3, \dots, x_{n-1}, x_n),$$

containing n independent subjects, we have n different ways of proceeding to what we might denote by ψ^2 ; or in a wider sense, we have

$$1 + n + n \cdot \frac{n-1}{2} + \dots + n \cdot \frac{n-1}{2} + n \text{ or } 2^n - 1$$

such methods. Thus, in the case of $\psi(x, y, z)$, which we shorten (where it is an ultimate form) into ψ , we have the following results of one repetition:

$$\begin{array}{lll} \psi(x, y, \psi) & \psi(x, \psi, \psi) & \\ \psi(x, \psi, z) & \psi(\psi, y, \psi) & \psi(\psi, \psi, \psi) \\ \psi(\psi, y, z) & \psi(\psi, \psi, z) & \end{array}$$

(263.) Before we proceed with this subject, it is necessary to remark, that the preceding operations contain more *indeterminateness*, if we may use the phrase, in proportion to those already considered, than we should

suppose by comparing $x+y$ and x , which is such an analogy as would naturally present itself. In fact, we are not proceeding from the consideration of x to that of $x+y$ where $+$ is defined, but to $x+y$ where $+$ may represent any operation whatever. But we should first consider cases in which the method in which x and y enter the function is of imperfect generality; for instance, such as

$$\psi\{\alpha(x, y)\}, \quad \psi(x+y), \quad \psi(axy-y^2), \text{ \&c.,}$$

in which x and y enter through a given function.

Again, the successive functions arising out of the preceding method may subject x and y to new operations at every step, that is, to operations not performed upon them in the original function. This cannot happen in functions of a single subject. If we see $\sin x$ in the tenth function, it must have been in the first. We may avoid this by allowing only what we will call *lineal* substitution, that is, not regarding any letter as a variable, except it come from itself in the original function. This may be best done as follows: instead of substituting $\psi(x, y)$ for x , and thus introducing y under all the signs of operation to which x is subject, give a specific value b to y before substitution. If, for the present, we denote by $\psi_y \psi$ and $\psi_x \psi$, the results of one substitution of this kind, in the first of which x becomes a before substitution for y , and in the second of which y becomes b before substitution for x , we find

$$\psi_y \psi_x \psi = \psi_x \psi_y \psi = \psi(\phi \overline{x}, b, \psi \overline{a}, y),$$

whence it follows that $\psi_x \psi_y \psi_x \psi = \psi_x^2 \psi_y \psi = \psi_y \psi_x^2 \psi$; or, generally, $\psi_x^m \psi_y^n \psi = \psi_y^n \psi_x^m \psi$.

To avoid proposing different classes of notation for different species of substitution, which we shall not have much occasion to employ together, we shall denote lineal substitutions simply by affixing the equations of condition which determine the specific values of x and y . Thus, by

$$\psi_{\substack{x=a \\ y=b}}^{m, n}(x, y) \text{ or } \psi^{m, n}(x, y) \left[\substack{x=a \\ y=b} \right],$$

we mean that m lineal substitutions of $\psi(x, b)$ for x and n substitutions of $\psi(a, y)$ for y have taken place. If, when the process is finished, we restore x and y instead of a and b , we may write

$$\psi_{\substack{x=a \\ y=b}}^{m, n}(x, y) \text{ or } \psi^{m, n}(x, y) \left[\substack{x=a \\ y=b} \right];$$

and if we finally write a and b instead of x and y , we may signify the result by

$$\psi_{\substack{x=a \\ y=b}}^{m, n}(a, b) \text{ or } \psi^{m, n}(a, b) \left(\substack{x=a \\ y=b} \right).$$

(264.) General substitution is that explained in (262.), and the immediate consequence of the distinction between it and lineal substitution is, that the order of the operations is not indifferent. Mr. Babbage has proposed the following notation,*

$$\psi_{1 \ 2 \ 3 \ 4 \ 5}^{m_1 \ m_2 \ m_3 \ m_4 \ m_5}(x, y, z),$$

to denote that 1. the m th function of the preceding is taken relatively to the letter written first, namely x ; 2. the result is considered as a function of z , (the third letter,) and its n th function taken on that supposition; 3. the p th function of the result is taken relatively to y ; 4. the q th function of the result relative to x ; 5. the r th function of the result relative to y . By the symbol

* About the same time M. Cauchy proposed an analogous method of representing interchanges. *Journ. Ec. Polytech. Cah.* 17, 1815.

Calculus of
Functions.

$$\psi^{m,m,m}(x,y,z)$$

we understand m general simultaneous substitutions, that is, we write $\psi(x,y,z)$ at once instead of both x,y , and z . Thus

$$\psi^{3,3}(x,y) = \psi(\psi(\psi(x,y), \psi(x,y)), \psi(\psi(x,y), \psi(x,y))).$$

(265.) It is much more easy to invent notation for cases which it may be conceived will hereafter arise, than to enter into the general solution of any one instance. The following extract* will show both the excessive complication of the questions which might be proposed, and the difficulty of taking in at one view the conditions of the problem.

$$\psi^{2,3,2,2,2,2,2,2,2,2}(x,y,z,v) = f x \frac{z}{\beta x} z = \gamma x \}.$$

The equation contains the analytical enunciation of the following problem. What must be the form of a function of four quantities $\psi(x,y,z,v)$, such that taking the second function relative to z , the third relative to x , and the second simultaneous one relative to z and v ; if, in the result, αx be put for y and βz for v , and the whole be then considered as a function of x and z ; and if, on this hypothesis, the third function be taken relative to z , and the second relative to x ; and, if γx be now put for z , and the third function of the expression considered merely as a function of x be taken, then it is required that the final result shall be equal to $f x$, a given function of x ?

We shall now proceed to the consideration of some cases of functions of more subjects than one.

(266.) So long as we consider substitution relative to one subject only, it is evident that we do not take a more general case than those which have been considered in the former part of this Treatise; thus we have

$$\psi^{n,1}(x,y) = \alpha^n x \text{ where } \psi(x,y) = \alpha x$$

$$\psi^{1,n}(x,y) = \beta^n y \text{ where } \psi(x,y) = \beta y$$

$$\psi^{0,1}(x,y) = x \quad \psi^{1,0}(x,y) = y.$$

In these cases one subject only is considered as variable, and the other may be treated as an arbitrary constant. In a similar manner, we have

$$\psi^{0,m}(x,y) = x \quad \psi^{m,0}(x,y) = y;$$

for since $\psi^{0,1}(x,y)$ is not a function of y , the successive functions of it with respect to y are the same in value as the first function. We must remember, that though by definition $\psi^{1,1}(x,y)$ and $\psi^{1,1}(x,y)$ are the same, being, in fact, only the same function considered as a basis for two different kinds of substitutions; yet, since $\psi^{m,n}(x,y)$ has two different values according as the substitutions are made first relative to x and then to y , or *vice versa*, the function $\psi^{0,0}(x,y)$ may be either $\psi_{1,2}^{0,0}(x,y)$ or $\psi_{2,1}^{0,0}(x,y)$. The first, as a particular case of $\psi^{0,m}(x,y)$, is x ; the second, for a similar reason, is y .

(267.) By the definition of simultaneous substitution, we have the following equation,

$$\psi^{m,m}(\psi x, y, \psi x, y) = \psi^{m+1, m+1}(x, y) \dots (A),$$

for all whole values of m from 1 upwards. We shall also readily find that the preceding order is invertible, as follows:

$$\psi\{\psi^{m,m}(x,y), \psi^{m,m}(x,y)\} = \psi^{m+1, m+1}(x,y) \dots (B).$$

* Babbage, *Essay, &c.*, part ii. *Phil. Trans.* 1816.

For if we consider the various manners in which ψ enters into $\psi^{3,3}$, for instance, as follows:

$\overline{1, 1}$	$\overline{2, 2}$	$\overline{3, 3}$
ψ	ψ	ψ
	ψ	ψ
		ψ

And if we next consider $\psi^{4,4}$ as derived from substituting ψ for both x and y in the preceding, we have the following expression, without brackets:

$$\left\{ \begin{array}{ccc} & \psi & \psi \\ \psi & & \psi \\ & \psi & \psi \\ \psi & & \psi \\ & \psi & \psi \\ \psi & & \psi \end{array} \right\}$$

which, after the brackets are written, appears exactly the same as substituting $\psi^{3,3}$ for x and y in $\psi x, y$. If, then, we make $m = 0$ in (A) and (B), in like manner as we hitherto have applied extended notation only to cases which satisfy such conditions of convertibility as are satisfied by the general notation, we must make the following equations determine $\psi^{0,0}$, namely,

$$\psi^{0,0}\{\psi(x,y), \psi(x,y)\} = \psi(x,y) \dots (A)$$

$$\psi\{\psi^{0,0}(x,y), \psi^{0,0}(x,y)\} = \psi(x,y) \dots (B)$$

and we are thus led to a condition which we shall consider as analogous to convertibility, and shall denominate *simultaneous convertibility*, namely,

$$\beta(\alpha x, y, \alpha x, y) = \alpha(\beta x, y, \beta x, y).$$

(268.) But there may be functions which satisfy one of the preceding equations, but not the other, analogous in species to the inconvertible inverses already considered.

For example, if $\psi(x,y) = x + y$, we find that

$$\psi^{0,0}(x,y) = \frac{1}{2}x + \frac{1}{2}y \text{ satisfies both equations, while}$$

$\psi^{0,0}(x,y) = x$ satisfies the first only, as does $\psi^{0,0}(x,y) = y$. In analogy with $\phi^0 \phi x = \phi \phi^0 x = x$, which do not involve a condition, because $\phi^0 x$ or x is convertible with all

functions, we shall let $\psi^{0,0}$ signify those functions only in which both equations are satisfied; while those which satisfy the first only shall be denoted by $\psi_{0,0}^{0,0}$; those which satisfy the second only belong to an infinite undefined class, presenting analogies with the functions we have denoted by the letter ξ , and point out an analogy which has not struck us till this moment. The equation $\phi \phi^0 x = \phi x$ being satisfied by $\phi^0 x = x$ is also satisfied by every form of $\xi \phi^0 x$. Agreeably to the character of the notation we have hitherto employed, $\phi^0 x$ should be the symbol of αx in the equation $\phi \phi^0 x = \phi x$, or in $\xi \alpha x = \xi x$ αx is $\xi_0 x$.

(269.) The functions which satisfy the first equation only are infinite in number, and (which is a further analogy with $\phi^0 x$ in $\phi \phi^0 x = \phi x$) independent of the form of $\psi(x,y)$. For equation (A) simply gives

$$\psi^{0,0}(a,a) = a.$$

Thus $m x - n y$ or $x^m y^{-n}$, (where $m - n = 1$); $x \alpha \left(\frac{y}{x}\right)$.

Calculus of
Functions.

where $\alpha(1) = 1$; $\cos \left\{ \frac{y}{x} \cos^{-1} x \alpha \left(\frac{y}{x} \right) \right\}$, &c. &c. are all general forms of $\psi_{0,0}(x, y)$.

And it must be observed, that whenever the second equation is satisfied, the first is satisfied by one or other of the values of $\psi_{0,0}$ derived from it. For, if we determine a from the equation

$$\psi(a, a) = \psi(x, y),$$

and if we afterwards make $x = y$, it is evident that one value of a reduces itself to x .

For example, given

$$\psi(x, y) = x^2 + y, \text{ required } \psi_{0,0}(x, y);$$

$$a^2 + a = x^2 + y \text{ gives } a = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + x^2 + y},$$

and the positive sign gives the function required, for

$$\sqrt{x^2 + x + \frac{1}{4}} - \frac{1}{2} = x.$$

Given $\psi(x, y) = \sin(xy)$, required $\psi_{0,0}(x, y)$; this gives $\psi_{0,0}(x, y) = \sqrt{xy}$.

(270.) If we call $\psi_{0,0}$ the zero-function* of ψ , we have the following theorems.

1. The zero-function of $\alpha(x, y)$ is also a zero-function of $\psi\alpha(x, y)$, ψ being any form whatever of a single subject. For the equation $\alpha(a, a) = \alpha(x, y)$ gives $\psi\alpha(a, a) = \psi\alpha(x, y)$, and $\alpha_{0,0}(a, a) = a$ gives

$$\alpha_{0,0}(\psi\alpha x, y, \psi\alpha x, y) = \psi\alpha x, y;$$

so that

$$(\psi\alpha)_{0,0}(x, y) = \alpha_{0,0}(x, y).$$

2. The zero-function of a zero-function is the zero-function itself; for if $\chi(a, a) = a$, which is the sufficient characteristic of χ being the zero-function of some function or other, we see that $\chi(a, a) = \chi(x, y)$ gives $a = \chi(x, y)$.

We are thus led back again to the remark already made relative to the step from ψx to $\psi(x, y)$ being too great a generalization. The consideration of $\psi\alpha(x, y)$, for the various given forms of α , divides all functions of two subjects into a number of classes, the zero-function of all the individuals of a class being the same. And we shall denote by the name of *primitive zero-functions* all those which have no other zero-function except themselves; that is, all functions of the form $\chi(x, y)$ which satisfy the equation $\chi(x, x) = x$.

The same notions may be extended to functions of three, or any number of subjects. The zero-functions of sums and products are sufficiently known under the names of *means* and *mean proportionals*.

Every function $\phi(x, y)$ can be represented under the form $\Phi\phi_0(x, y)$, where ϕ_0 means the zero-function of ϕ , and is one of the primitive zero-functions contained in the general solution of $\phi(x, x) = x$.

(271.) No zero-function can be a function of any other zero-function as a single subject, that is, *all zero-functions are primitives*, and we may therefore drop the latter term. For if $\phi(x, x) = x$, and $\psi(x, x) = x$, and if $\phi(x, y) = \alpha\psi(x, y)$, we have, making $y = x$,

$$x = \alpha x, \text{ or } \phi(x, y) = \psi(x, y).$$

But any zero-function of two or more zero-functions

* The simultaneous zero-function, if distinction be required.

is also a zero-function, as is sufficiently evident: and any zero-function may be expressed in terms of any other, together with one of the functions presently to be discussed, in which $f(x, x) = 0$.

If, for example,

$$\alpha^0 = \beta^0 + P,$$

P must be reduced to zero by $y = x$.

(272.) Let $\psi(x, y)$ be a function such that $\psi(x, a) = x$; then if $\chi(x, y)$ be any zero-function, the following contains an extensive class of the same kind,

$$\chi\left(x, \psi\left\{x, a \alpha\left(\frac{y}{x}\right)\right\}\right), \alpha(1) = 1;$$

and thus, by successive substitution, we may show that no form exists under which all zero-functions can be shown with any finite number of arbitrary functions.

But the most obvious form of all zero-functions is

$$x + \theta(x - y)\psi(x, y),$$

where $\theta(0) = 0$ and $\psi(x, x)$ is not infinite.

(273.) Homogeneous functions are reduced by making $x = y$ to the form ax^n , where n is the dimension of the function. The form being

$$\psi(x, y) = ax^n \alpha\left(\frac{y}{x}\right), \alpha(1) = 1;$$

we have

$$\psi_{0,0}(x, y) = x \sqrt[n]{\alpha\left(\frac{y}{x}\right)}.$$

(274.) If the function $\psi(x, y)$ be of such a kind that $\psi(x, x) = \text{constant}$, including 0 or ∞ under the general term constant, the preceding method fails, since we cannot discover a real function of a to equate with $\psi(x, y)$. But in this case it must be observed, that we have the equation

$$\psi_{m,m}(x, y) = \text{constant},$$

independently of m . We shall presently see that we shall be obliged to consider the corresponding zero-functions

as *infinite*, or at least of the form $\omega\left(\frac{1}{0}\right)$. Thus in $\alpha x - \alpha y$

we see a class of functions which satisfy the equation $\psi_{0,0}(x, y) = \omega\left(\frac{1}{0}\right)$. But the original function cannot

be derived from the zero-function in this case, by either of the equations (A) or (B), nor can two functions which have the same zero-functions be made functions of each other. Thus we cannot make $x^2 - y^2$ a function of $x - y$, though both have the zero-function ∞ . Those functions which have infinite zero-functions present analogies in some respect with the constant of the differential calculus, and also with the absolute zero of Algebra. For instance, if $\chi(x, y)$ represent a function which has an infinite zero-function (that is, which gives $\chi(x, x) = 0$); then $\psi(x, y) + \chi(x, y)$ has the same zero-function as $\psi(x, y)$, after $\psi + \chi$ has been substituted instead of ψ : thus the zero-function of xy being $\sqrt{xy}$, that of $xy + x - y$ is $\sqrt{xy + x - y}$. Again, $\chi(x, y) \times \psi(x, y)$ is another form of $\chi(x, y)$, if $\psi(x, x)$ be finite. It is, moreover, evident that

$$\alpha(\phi(x, y), \phi_0(x, y))$$

has the zero-function $\phi_0(x, y)$ the same as that of $\phi(x, y)$. And many other theorems might be deduced, all conveying certain ideas of classification among functions of two

Calculus of
Functions.

Calculus of Functions.

or more variables, and by means of which the every-day algebraical properties of simple functions may be made the guides to generalization.

"Any zero-function" is the answer to the following question:—What is the value of z in terms of x and y , which shall give

$$1 = \frac{dz}{dx} + \frac{dz}{dy} \text{ (when } y = x \text{)}$$

$$\text{or } 0 = \frac{d^2 z}{dx^2} + 2 \frac{d^2 z}{dx \cdot dy} + \frac{d^2 z}{dy^2} \text{ \&c. \&c. ?}$$

For a given value of $\frac{dz}{dy}$, there is one zero function, namely,

$$\psi(x, y) - \psi(x, x) + x,$$

where

$$\psi(x, y) = \int \frac{dz}{dy} \cdot dy.$$

(275.) By the *modulus* of $\alpha(x, y)$ we mean the function $\alpha(x, x)$ considered as of a single subject. The moduli of the sum, difference, product, &c. of two functions are the sum, difference, &c. of the moduli. And the modulus of $\frac{dz}{dx} + \frac{dz}{dy}$, where z is a function of x and y , is the differential coefficient of the modulus of z . For if we put the zero-function in the form

$$\alpha^{0,0}(x, y) = \frac{x+y}{2} + \psi(x-y)\chi(x, y), \quad \psi(0) = 0,$$

and if we denote the modulus of $\alpha(x, y)$ by $\bar{\alpha}x$, we have

$$z = \bar{\alpha} \alpha^{0,0}(x, y),$$

$$\frac{dz}{dx} = (\bar{\alpha}' \alpha^{0,0}) \left(\frac{1}{2} + \psi'(x-y)\chi(x, y) - \psi(x-y) \frac{d\chi}{dx} \right),$$

or

$$\left(\frac{dz}{dx} \right) = \bar{\alpha}' x \left(\frac{1}{2} + \psi' 0 \cdot \chi x \right);$$

similarly,

$$\left(\frac{dz}{dy} \right) = \bar{\alpha}' x \left(\frac{1}{2} - \psi' 0 \bar{\chi} x \right),$$

whence

$$\left(\frac{dz}{dx} + \frac{dz}{dy} \right) = \bar{\alpha}' x.$$

Similarly we find the modulus of

$$\beta(x, y) \frac{dz}{dx} + \beta(y, x) \frac{dz}{dy} \text{ to be } \bar{\beta} x \bar{\alpha}' x,$$

of which the well-known theorem relative to the value of $x \frac{dz}{dx} + y \frac{dz}{dy}$ in homogeneous functions will give a particular case.

(276.) If we consider a function which has a numerator and denominator with zero for the moduli of both, having the form

$$\frac{\psi(x-y)\chi(x, y)}{\psi_1(x-y)\chi_1(x, y)} \quad \psi 0 = 0, \chi 0 = 0,$$

we find its modulus to be $\frac{0}{0}$, which is in this case

$$\frac{\psi' 0 \chi x}{\psi'_1 0 \chi_1 x}$$

(277.) Sir J. Herschel* has applied equations of differences to the determination of functions, which satisfy certain conditions when y is made a given function of x . We give one instance of this method, as follows.

Suppose it required to solve

$$\psi(x, x) - \psi(x, 0) = a.$$

Let there be a function of x and y , which becomes $f(x, h)$ when $y = 0$ and $f(x, h+1)$ when $y = x$.

Such a function is $f(x, h+P)$, where $x = 0$ makes $P = 0$ and $x = y$ makes $P = 1$. Assume $\psi(x, y) = f(x, h+P)$, which gives

$$f(x, h+1) - f(x, h) = Q;$$

and this gives

$$f(x, h) = ah + Q,$$

where Q is any function which is the same whether 0 or x be substituted for y . This equation being identically satisfied for all values of h , write $h+P$ for h , and we have

$$\psi(x, y) = a(h+P) + Q.$$

The general form of Q is

$$\alpha x + \psi x \cdot \psi_1(y-x)\chi(x, y),$$

where $\psi 0 = 0, \psi_1 0 = 0$. And the form of P is

$$\beta \frac{x}{y} + \omega x \omega_1(y-x)\chi_1(x, y),$$

where $\beta 0 = 0, \beta 1 = 1, \omega 0 = 0, \omega_1 0 = 0$. Whence the general form, supposing ah included in αx , is

$$\alpha x + a\beta \frac{x}{y} + \theta x \theta_1(x-y)\Theta(x, y),$$

where

$$\theta 0 = 0 \quad \theta_1 0 = 0.$$

(278.) *Simultaneous derivation*† is the formation of such functions as $\phi \alpha(\phi^{-1}x, \phi^{-1}y)$ from $\alpha(x, y)$, and it will be found that these derivatives possess the properties already treated in speaking of functions of one subject. When we wish to consider a derivative or other compound function as a simple function of x and y , we shall write it in [] brackets, which shall be reserved for this purpose only, and the exponents of operation relatively to the whole may be written on the outside.

(A) Derivatives of derivatives of $\alpha(x, y)$ are derivatives of $\alpha(x, y)$.

(B) If $\mu(x, y)$ and $\nu(x, y)$ be corresponding derivatives (derived by means of the same function ϕ) of $\alpha(x, y)$ and $\beta(x, y)$, then $\mu\{\nu(x, y), \nu(x, y)\}$ is the corresponding derivative of $\alpha\{\beta(x, y), \beta(x, y)\}$.

(C) If α and β have the corresponding derivatives μ and ν , and if $\alpha(\beta, \beta) = \beta(\alpha, \alpha)$, then $\mu(\nu, \nu) = \nu(\mu, \mu)$.

(D)† The successive functions, however taken, of any derivative of $\alpha(x, y)$, are the corresponding derivative of the same successive functions of α . Suppose it true for any one derivative of $\alpha(x, y)$, and any one value of $\frac{m}{1}, \frac{n}{2}$, that

$$[\phi \alpha(\phi^{-1}x, \phi^{-1}y)]^{\frac{m}{1}, \frac{n}{2}} = \phi \alpha^{\frac{m}{1}, \frac{n}{2}}(\phi^{-1}x, \phi^{-1}y),$$

then a substitution such as changes n into $n+1$ on the first side, actually made in the second side, gives

$$\phi \alpha^{\frac{m}{1}, \frac{n}{2}}(\phi^{-1}x, \alpha(\phi^{-1}x, \phi^{-1}y));$$

but if we now take $\alpha^{\frac{m}{1}, \frac{n}{2}}$ and change n into $n+1$ before derivation, we find

* Camb. Phil. Trans. 1820.

† Babbage, Essay, &c. part ii.

‡ Id. lb.

Calculus of
Functions.

$$\alpha^{m, n+1}(x, y) = \alpha^{m, n}(x, \alpha(x, y)),$$

whence the theorem readily follows by the usual method of successive induction. Let P be the symbol of

$$m, m', m'', \dots$$

$$n, n', n'', \dots$$

where $n, n', \dots$ are severally either 1 or 2, and we have

$$[\phi \alpha(\phi^{-1}x, \phi^{-1}y)]^P = \phi \alpha^P(\phi^{-1}x, \phi^{-1}y).$$

(279.) All the derivatives of zero-functions are zero-functions. For if $\alpha(x, x) = x$, we have $\phi \alpha(\phi^{-1}x, \phi^{-1}x) = \phi \phi^{-1}x = x$, or $\phi \alpha(\phi^{-1}x, \phi^{-1}y)$ is a zero-function. And if $\psi_0(x, y)$ be the zero-function of $\psi(x, y)$, then $\phi \psi_0(\phi^{-1}x, \phi^{-1}y)$ is the zero-function of $\chi \phi \psi(\phi^{-1}x, \phi^{-1}y)$. Similarly if $\alpha(x, x) = \text{constant}$, the derivatives of $\alpha(x, y)$ have the same property.

(280.) If any function be *simultaneously periodic*, then all its derivatives are the same. Or if

$$\alpha^{m, m}(x, y) = \alpha^{0, 0}(x, y),$$

then

$$\begin{aligned} \phi \alpha^{m, m}(\phi^{-1}x, \phi^{-1}y) &= \phi \alpha^{0, 0}(\phi^{-1}x, \phi^{-1}y) \\ &= [\phi \alpha(\phi^{-1}x, \phi^{-1}y)]^{0, 0} \end{aligned}$$

It remains to inquire what are the various classes of simultaneously periodic functions.

Firstly, all zero-functions themselves are what we may call permanent; for if $\chi(x, x) = x$, we have

$$\chi^{2, 2}(x, y) = \chi(\chi(x, y), \chi(x, y)) = \chi(x, y),$$

and, generally,

$$\chi^{n, n}(x, y) = \chi(x, y) = \chi^{0, 0}(x, y).$$

To obtain functions which are not permanent, but periodic, and of one certain degree, take any periodic function of a single subject, of the order required, and substitute instead of the subject a zero-function of x and y . That is, let $\phi^n x = x$, and $\alpha(x, x) = x$, then $\phi \alpha(x, y)$ is simultaneously periodic. For we have

$$(\phi \alpha)^{2, 2}(x, y) = \phi \alpha(\phi \alpha x, \phi \alpha y) = \phi^2 \alpha(x, y),$$

or, generally,

$$\begin{aligned} \phi \alpha^{m, m}(x, y) &= \phi^m \alpha(x, y) \\ &= \alpha(x, y) = \alpha^{0, 0}(x, y) = (\phi \alpha)^{0, 0}(x, y). \end{aligned}$$

Thus, if we take the zero-function $\frac{1}{2}(x + y)$, and the periodic function of it of the second degree $\frac{2}{x + y}$, we find

$$\alpha^{2, 2}(x, y) = \frac{2(x + y)}{2 + 2} = \frac{x + y}{2} = \alpha^{0, 0}(x, y).$$

(281.) The easiest mode of actually exhibiting the simultaneous functions of $\alpha(x, y)$ consists in writing it in the form $A \alpha^{0, 0}(x, y)$, where $\alpha^{0, 0}$ is z in the equation $\alpha(z, z) = \alpha(x, y)$. For we then have

$$\begin{aligned} \alpha^{2, 2}(x, y) &= A \alpha^{0, 0} \{A \alpha^{0, 0}(x, y), A \alpha^{0, 0}(x, y)\} \\ &= A A \alpha^{0, 0}(x, y) = A^2 \alpha^{0, 0}(x, y), \end{aligned}$$

and, generally,

$$\alpha^{m, m}(x, y) = A^m \alpha^{0, 0}(x, y).$$

In perfect analogy with functions of one subject, we should call $\alpha^{0, 0}$ the subject, and A the functional symbol: but the necessity of considering $\alpha^{0, 0}$ separately as a function of two subjects will oblige us to relinquish this

method of expression. We have called A the *modulus** of the function $\alpha(x, y)$. Consequently, we can immediately assign the simultaneous functions of a very large class of complicated functional forms, the essentials being that $\alpha(z, z) = \alpha(x, y)$ shall be capable of solution, and that $\alpha(z, z)$ shall also be of one of the forms whose n th functions (considered as functions of a single subject) shall be attainable in finite terms. Thus in the case of

$$\alpha(x, y) = \frac{x + y \log \left(\frac{x}{y} \right)}{x^2 + xy + y^2}$$

$$\alpha^{0, 0}(x, y) = 3\alpha(x, y), \quad Az = \frac{z}{3}$$

$$\alpha^{n, n}(x, y) = \frac{1}{3^n} \cdot \frac{x^3 + y^3 \log \left(\frac{x}{y} \right)}{x^2 + xy + y^2}.$$

(282.) Since every function can be reduced to the form $A \alpha^{0, 0}(x, y)$, a change of a constant or other change which reduces $\alpha(x, x)$ to zero, or Ax to zero, will at the same time reduce $\alpha^{0, 0}$ in $A \alpha^{0, 0}$ to a function of α . Thus, if we take $\omega x - m \omega y$, the zero-function of which is

$$\omega^{-1} \left(\frac{\omega x - m \omega y}{1 - m} \right), \quad \text{and } Az = \omega z(1 - m),$$

the supposition of $m = 1$ reduces Az to zero, and $\alpha^{0, 0}(x, y)$ to $\omega^{-1} \left(\frac{\omega x}{0} \right)$, which is constant, nothing, or

infinite. In the case of $x - ay$, we have $\alpha^{0, 0} = \alpha$

The addition of a constant makes no difference in the zero-function, but only in the modulus, to which it adds a constant.

(283.) Since $\phi(\alpha x, \alpha y) = \bar{\phi} \alpha \alpha^{-1} \phi^{0, 0}(\alpha x, \alpha y)$, and the modulus of this function is $\bar{\phi} \alpha$, its zero-function is $\alpha^{-1} \phi^{0, 0}(\alpha x, \alpha y)$, or we see that $\alpha^{-1} \phi^{0, 0}(\alpha x, \alpha y)$ is the zero-function of $\phi(\alpha x, \alpha y)$.

It is almost needless to say, that if two functions be identically the same, their zero-functions and moduli must be the same.

(284.) If β be a derivative of α , then both the modulus and zero-function of β are the corresponding derivatives of those of α . For let

$$\beta(x, y) = B \beta^{0, 0}(x, y) \quad \alpha(x, y) = A \alpha^{0, 0}(x, y),$$

and

$$B \beta^{0, 0}(x, y) = \phi A \alpha^{0, 0}(\phi^{-1}x, \phi^{-1}y),$$

if we make $x = y$, we find $Bx = \phi A \phi^{-1}x$; and, therefore,

$$\phi A \phi^{-1} \beta^{0, 0}(x, y) = \phi A \alpha^{0, 0}(\phi^{-1}x, \phi^{-1}y),$$

which is ratified by

$$\beta^{0, 0}(x, y) = \phi \alpha^{0, 0}(\phi^{-1}x, \phi^{-1}y).$$

It is evident, therefore, that the problem of making any function β a derivative of any other function α cannot be generally solved. First make the modulus of one a derivative of the modulus of the other; then see

* Such terms as *modulus*, *derivative*, &c. have been purposely used, because, from the various senses in which they have been used, they are *movable* terms. They must not be considered as of permanent application.

Calculus of
Functions.

Calculus of Functions. whether the deriving function so obtained will make the zero-function of one the derivative of that of the other.

We shall apply the process in (50.) to this problem.

If we consider $\alpha^{n,n}$ and $\beta^{n,n}$ as functions of n , and $\alpha^{0,0}$ and $\beta^{0,0}$ respectively, and eliminate n between the two equations,

$$A^n \alpha^{0,0}(x, y) = c \quad B^n \beta^{0,0}(x', y') = c',$$

we find an equation of the following for

$$\beta^{0,0}(x', y') = \varpi \alpha^{0,0}(x, y),$$

where ϖ is evidently the deriving function of B from A . This equation, by the method of obtaining it already explained, remains true if we substitute $\alpha(x, y)$ for x and y , and $\beta(x', y')$ for x' and y' : that is,

$$\beta(x', y') = \varpi \alpha(x, y) \dots \beta^{m,m}(x', y') = \varpi \alpha^{m,m}(x, y)$$

are all consequences of the first. The condition under which β may then be a derivative of α is the permanence of the first of these equations, when ϖx and ϖy are substituted for x' and y' . Let us suppose

$$\alpha(x, y) = ax + by \quad \beta(x', y') = \sqrt{ax'^2 + by'^2}$$

$$\alpha^{n,n}(x, y) = (a + b)^{n-1} (ax + by)$$

$$\beta^{n,n}(x', y') = (a + b)^{\frac{n-1}{2}} \sqrt{ax'^2 + by'^2},$$

and

$$ax'^2 + by'^2 = ax + by$$

is the equation which results from equating the preceding to unity, and eliminating n . This gives between the zero-functions the equation

$$\beta^{0,0}(x, y) \text{ or } \sqrt{\frac{ax'^2 + by'^2}{a+b}} = \sqrt{\frac{ax + by}{a+b}} \text{ or } \sqrt{\alpha^{0,0}(x, y)};$$

so that $\varpi z = \sqrt{z}$. Substitute $\sqrt{x}$ for x' and $\sqrt{y}$ for y' , and the equation becomes identically true. Hence the first is a derivative of the second, as is indeed sufficiently evident.

(285.) It remains to inquire whether any given zero-function can be made a derivative of any other. It does not appear that this can be generally done, as in the case of functions of a single subject. For instance, all the derivatives of symmetrical functions are symmetrical. And on comparing the question in the following manner with the corresponding question as to a single subject (which will often be found useful in tracing analogies) we see a material limitation introduced. The equation

$$\beta(x, y) = \phi \alpha(\phi^{-1}x, \phi^{-1}y)$$

is evidently a particular case of the derivation of $\beta(x, y)$ from $\alpha(x, y')$, considered as functions of one subject only, and y and y' being independent arbitrary constants. The deriving function ϕx , being thus obtained, must now be further subjected to the condition $y' = \phi^{-1}y$, which it may be impossible to satisfy. But the following method will show that (continuously or discontinuously) a zero-function may, in some cases, be made a derivative of another. Let $\alpha(x, y)$ and $\beta(x, y)$ be two zero-functions, and assume the equations

$$\beta(x', y') = c' \quad \alpha(x, y) = c,$$

which, by the permanence of zero-functions under simultaneous substitutions, admit of $\beta(x', y')$ and $\alpha(x, y)$ being substituted for x' and y' and for x and y . From these deduce

$$y' = Bx' \quad y = Ax$$

and assume $y' = \gamma y$. We have thus, by substitutions shown to be legitimate,

$$\beta(x', y') = B\beta(x', y'), \quad \alpha(x, y) = A\alpha(x, y)$$

$$Bx' = \gamma Ax, \quad B\beta(x', y') = \gamma A\alpha(x, y);$$

or

$$\beta(B^{-1}\gamma Ax, \gamma y) = B^{-1}\gamma A\alpha(x, y),$$

which are necessary consequences of the first two equations and true for all forms of γ . To obtain β as a derivative of α , we have only to assume

$$B^{-1}\gamma Ax = \gamma x, \text{ or } Bx = \gamma A\gamma^{-1}x,$$

or the function by which one zero-function β is derived from another α , is the same as that by which the value of y' in $\beta(x', y') = c'$ is derived from the value of y in $\alpha(x, y) = c$. It thus appears why symmetrical functions can only be derived continuously from each other: for if $\beta(x, y)$ be a symmetrical function of x and y , the equation $\beta(x, y) = c$ shows y to be a periodic function of x of the second degree; thus, if $\beta(x, \phi x) = c$, and if $\beta(\phi x, x) = \beta(x, \phi x)$, the substitution of $\phi^{-1}x$ for x in the first gives $\beta(\phi^{-1}x, x) = c = \beta(\phi x, x)$, one solution of which is $\phi x = \phi^{-1}x$ or $\phi^2 x = x$.

But in the preceding solution it must be remembered that the initial suppositions

$$\beta(x' y') = c' \quad \alpha(x, y) = c$$

remain in force, restricting the truth of the derivation to such values of x and y as simultaneously satisfy these relations. The derivation can only be general in such cases as give values of γx , which, by the process, are independent of c and c' , or in which c and c' are otherwise removed from the final equation. In the case where neither zero-function is symmetrical, two new arbitrary constants are introduced in the derivation of B from A , by means of particular values of which it may happen that the constants c and c' do not appear in γx . In the case where both are symmetrical, there is an infinite number of ways in which B may be derived from A (75.), and any number of arbitrary constants may be introduced.

For instance, suppose it required to derive $\sqrt{xy}$ from $\frac{x+y}{2}$. Here we have

$$\alpha(x, y) = \frac{x+y}{2} \quad \beta(x', y') = \sqrt{x'y'} \quad Ax = 2c - x \quad Bx' = \frac{c'^2}{x'}.$$

One value of γx is

$$\frac{c'(2c - x)}{x} \text{ or } \gamma^{-1}x = \frac{2cc'}{x + c'}$$

giving

$$\gamma(2c - \gamma^{-1}x) = \frac{c'^2}{x}.$$

But

$$\gamma \alpha(\gamma^{-1}x, \gamma^{-1}y) = \frac{2xy + (x+y)c'}{2c' + x + y},$$

which is $= \sqrt{xy}$, when $xy = c'^2$ and $x + y = 2c$

But if we choose $\gamma x = e^{mx}$, giving

$$\gamma(2c - \gamma^{-1}x) = \frac{e^{2cm}}{x},$$

we find that $\frac{c'^2}{x}$ is thus made a derivative of x , if

$e^m = c'^{\frac{1}{c}}$, and we find that in this case $\sqrt{xy}$ is a derivative of $\frac{x+y}{2}$; for if

Calculus of
Functions.

$$\gamma x = c^{\frac{x}{c}} \quad \gamma^{-1} x = c^{\frac{\lambda x}{\lambda c'}}$$

$$\sqrt{xy} = \varepsilon^{\frac{\lambda x + \lambda y}{2}} = c^{\frac{1}{2c}} \left(c^{\frac{\lambda x}{\lambda c'}} + c^{\frac{\lambda y}{\lambda c'}} \right).$$

The question, therefore, whether any one zero-function can be made a derivative of any other remains unsettled; for in the preceding process there are two places certainly, and two more possibly, in which the present state of analysis requires the substitution of constants instead of arbitrary functions, the former in deriving B from A, the latter in the constants c and c' . For it is sufficient that c and c' should be functions which do not change when $\alpha(x, y)$ and $\beta(x', y')$ are severally substituted for x and y and for x' and y' . The only forms of such functions known to us are $\theta \alpha(x, y)$ and $\theta_1 \beta(x', y')$, the substitution of which for c and c' is in effect the putting the initial equations in a more complicated form, and nothing more. But there may be more general solutions containing arbitrary functions.

(286.) The theory of the simultaneous inverses is reduced to that of inverses of functions of one subject only, by the equations

$$\phi(x, y) = \Phi \phi^{0,0}(x, y) \quad \phi^{n,n}(x, y) = \Phi^n \phi^{0,0}(x, y);$$

thus Φ^{-1} is the modulus of the convertible inverses, giving

$$\Phi \alpha^{0,0}(\Phi^{-1} \alpha^{0,0}, \Phi^{-1} \alpha^{0,0}) = \Phi \Phi^{-1} \alpha^{0,0} = \alpha^{0,0}(x, y)$$

$$\Phi^{-1} \alpha^{0,0}(\Phi \alpha^{0,0}, \Phi \alpha^{0,0}) = \Phi^{-1} \Phi \alpha^{0,0} = \alpha^{0,0}(x, y).$$

Similarly, Φ_{-1} is the modulus of any inconvertible simultaneous inverse of $\phi(x, y)$, or of $\phi_{-1,-1}(x, y)$, and we have the equation

$$\phi_{-1,-1}(\phi(x, y), \phi(x, y)) = \Phi_{-1} \phi^{0,0}(x, y).$$

Thus, if we suppose $\phi(x, y) = x + y$, we have

$$\phi^{0,0}(x, y) = \frac{x+y}{2} \quad \Phi z = 2z \quad \Phi^{-1} z = \frac{z}{2}$$

$$\phi^{0,-1}(x, y) = \frac{x+y}{4}.$$

If $\phi(x, y) = x^2 - y$, we have

$$\phi^{0,0}(x, y) = \frac{1}{2} + \sqrt{\frac{1}{4} + (x^2 - y)} \quad \Phi z = z^2 - z$$

$$\Phi^{-1} z = \frac{1}{2} + \sqrt{\frac{1}{4} + z} \quad \Phi_{-1} z = \frac{1}{2} - \sqrt{\frac{1}{4} + z}$$

$$\phi^{0,-1}(x, y) = \frac{1}{2} + \sqrt{\frac{3}{4} + \sqrt{\frac{1}{4} + (x^2 - y)}}$$

$$\phi_{-1,-1}(x, y) = \frac{1}{2} - \sqrt{\frac{3}{4} + \sqrt{\frac{1}{4} + (x^2 - y)}}.$$

(287.) Let

$$[\phi_{-1,-1}(\phi(x, y), \phi(x, y))]^{n,n} = \psi^{n,n}(x, y);$$

then $\phi(\psi^{n,n}, \psi^{n,n})$ must be a form of $\phi(x, y)$. This is an extension of the theorem in (34.), and follows from

$$\Phi(\Phi_{-1} \Phi)^n z \text{ being always a form of } \phi z.$$

(288.) The generality of the symbol $\phi(x, y)$ allows us to imagine another mode of substitution, which we may call *internal* substitution. If in ψx we substitute αx for x , and repeat this operation, we have a result capable of being expressed, consistently with the notions contained in ψ and α , by $\psi \alpha^n x$. But if in $\phi(x, y)$ we

suppose $\alpha(x, y)$ to be substituted for x , and $\beta(x, y)$ for y , and continue this any number of times in any specified order of substitution, we cannot express the result by what has preceded. But if we agree to denote

$$\psi(x, y) \text{ by } \psi(x, y)_{\alpha, \beta}^{0,0}$$

$$\psi\{\alpha(x, y), \beta(x, y)\} \text{ by } \psi(x, y)_{\alpha, \beta}^{1,1}$$

we have in

$$\psi(x, y)_{\alpha, \beta}^{m,n} \quad \psi(x, y)_{\beta, \alpha}^{n,m} \quad \psi(x, y)_{\alpha, \beta}^{\overline{m}, \overline{n}}$$

symbols analogous to

$$\psi_{1,2}^{m,n}(x, y) \quad \psi_{2,1}^{n,m}(x, y) \quad \psi^{\overline{m}, \overline{n}}(x, y),$$

and differing from them only in this, that $\alpha(x, y)$ and not $\psi(x, y)$ is substituted for x , and $\beta(x, y)$ not $\psi(x, y)$ for y . Here we may see, 1. That $\alpha = \psi$ and $\beta = \psi$ reduce this to common successive substitution. 2. That if α be a function of x only, and β of y only, the order of substitution is altogether indifferent. Of this last lineal substitution may be regarded as a particular case (262.)

(289.) We shall now proceed to some functional equations containing functions with two independent subjects. We may remark, as we have done before, that the transcendentials which may enter are more and higher than can be ascertained by any methods of solution which we can give. And, as an instance analogous to $\phi x = x \phi(x-1)$ we may assign the equation

$$f'(x, y) \times f(x+y, z) = f(x, z) \times f(x+z, y),$$

which was shown by Euler* to be a property of

$$f(x, y) = \int_0^1 \frac{a^{x-1} da}{(1-a^n)^{\frac{n-y}{n}}}.$$

(290.) To simplify the notation, let it be understood that a functional symbol unaccompanied by any subject means a function of the two subjects x and y , the order of operations being first with respect to x , and next with respect to y , unless the contrary be specified. Let simultaneous substitutions be denoted by the single exponent with a bar over it, instead of the exponent twice repeated; thus, $\bar{n}$ instead of n, n . And let the modulus of any function be denoted by a bar placed over the functional symbol: thus

$$\phi(x, y) = \bar{\phi} \phi^{0,0}(x, y) \text{ or } \phi = \bar{\phi} \phi^{0,0},$$

$$\phi^n(x, y) = \bar{\phi}^n \phi^{0,0}(x, y) \text{ or } \phi^n = \bar{\phi}^n \phi^{0,0}.$$

(291.) Given $\phi(\alpha, \beta) = \phi$, required forms of $\phi(x, y)$.

Firstly, if two particular solutions ω_1 and ω_2 be known, then $\theta(\omega_1, \omega_2)$ is a solution. Or if one particular solution be known, then $\theta \omega_1$ is a solution.

Secondly, we may suppose ϕ to be a zero-function, for if we can solve the equation $\bar{\phi}(\alpha, \beta) = \bar{\phi}^{0,0}$, we have a solution of $\bar{\phi} \phi^{0,0}(\alpha, \beta) = \bar{\phi} \phi^{0,0}$, or of $\phi(\alpha, \beta) = \phi$.

Thirdly, if α and β be the same, and be also zero-functions, then $\theta \alpha(x, y)$ is a solution, for by the property of a zero-function $\alpha(\alpha, \alpha) = \alpha$.

The general solution of the preceding equation depends upon the determination of

$$\alpha(x, y)_{\alpha, \beta}^{\overline{n}, \overline{n}} \text{ and } \beta(x, y)_{\alpha, \beta}^{\overline{n}, \overline{n}}$$

as explicit functions of n, x , and y . Let

$$\alpha(x, y)_{\alpha, \beta}^{\overline{n}, \overline{n}} = V(x, y, n) \quad \beta(x, y)_{\alpha, \beta}^{\overline{n}, \overline{n}} = W(x, y, n),$$

* Legendre, *Mém. de l'Institut*. 1809. *Fonct. Ellipt.* vol. ii. p. 370.

Calculus of Functions and f being any convenient function, obtain

$$n = F(x, y) \text{ from } f(V, W) = 0,$$

it is clear, as in (50.), that

$$V(x, y, n) = V(\alpha, \beta, n-1)$$

$$W(x, y, n) = W(\alpha, \beta, n-1),$$

so that the function F must satisfy the equation

$$n-1 = F(\alpha, \beta),$$

or

$$F(\alpha, \beta) = F(x, y) - 1.$$

From this we may either derive transcendental solutions, such as

$$\phi(x, y) = \theta \cos 2\pi F(x, y);$$

or, by taking another form, f_1 instead of f , and thus deriving a new form F_1 , we see that

$$\phi(x, y) = F(x, y) - F_1(x, y).$$

In this way we may obtain any number of particular solutions, and may thus form a general solution.

(292.) Required, for example, the solution of

$$\phi(x+y, 2y) = \phi(x, y).$$

Let

$$\alpha(x, y) = x + y \quad \beta(x, y) = 2y$$

$$\alpha(x, y)_{\alpha, \beta}^{n, n} = x + (2^n - 1)y \quad \beta(x, y)_{\alpha, \beta}^{n, n} = 2^n y.$$

We have now to determine n by making $x - y + 2^n y$ any function whatsoever of $2^n y$; or, which is the same thing, making

$$2^n = \frac{1}{y} \psi(x-y), \text{ or } n = \psi(x-y) - \frac{\lambda y}{\lambda 2}.$$

Taking two such solutions, the difference of which merely gives an arbitrary function of $x - y$, we find $\theta(x - y)$ as a general form of solution. But we also find a more general solution in

$$\theta\left(\theta_1(x-y), \theta_2 \cos 2\pi \left\{\theta_3(x-y) - \frac{\lambda y}{2}\right\}\right).$$

(293.) Let

$$\phi(ax + by, a'x + b'y) = \phi(x, y)$$

$$\alpha(x, y) = ax + by \quad \beta(x, y) = a'x + b'y.$$

Assume

$$\alpha(x, y)_{\alpha, \beta}^{n, n} = p_n x + q_n y \quad \beta(x, y)_{\alpha, \beta}^{n, n} = p'_n x + q'_n y,$$

which gives

$$p_{n+1} = ap_n + a'q_n$$

$$q_{n+1} = bp_n + b'q_n,$$

and similar equations for p'_n and q'_n . By elimination we get the same equation of differences for the determination of all four, namely,

$$T_{n+2} - (a + b')T_{n+1} + (ab' - a'b)T_n = 0,$$

or

$$T_n = Cr^n + C'r'^n,$$

where r and r' are the roots of

$$R^2 - (a + b')R + ab' - a'b = 0.$$

To determine p_n and q_n , for instance, we must suppose

$$p_0 = 1 \quad q_0 = 0 \quad p_1 = a \quad q_1 = b,$$

and the n , n th functions in question are the following,

$$\left(\frac{r^n - r'^n}{r - r'} a - \frac{r'^n - r r'^n}{r - r'}\right)x + \frac{r^n - r'^n}{r - r'} by$$

$$\frac{r^n - r'^n}{r - r'} a'x + \left(\frac{r^n - r'^n}{r - r'} b' - \frac{r'^n - r r'^n}{r - r'}\right)y.$$

Assume

$$P = \frac{ax + by - rx}{r - r'} \quad Q = \frac{ax + by - rx}{r - r'}$$

$$P' = \frac{a'x + b'y - r'y}{r - r'} \quad Q' = \frac{a'x + b'y - r'y}{r - r'}.$$

Then $Pr^n - Qr'^n$ may be assumed to be any function of $P'r^n - Q'r'^n$, rr' being $ab' - a'b$. The only way in which we can immediately draw a conclusion is that in which we suppose the first to be c times the second, which gives

$$\left(\frac{r}{r'}\right)^n = \frac{Q - cQ'}{P - cP'}$$

the particular solutions which we can obtain (being of the form of $n - n_1$, in the preceding, or in any function of it) give for the algebraical solutions all algebraical functions of

$$\frac{(Q - cQ')(P - c_1P')}{(Q - c_1Q')(P - cP')},$$

which is itself an algebraical function of x and y , which solves the equation. The corresponding transcendental solutions are functions of

$$\cos 2\pi \frac{\lambda(Q - cQ') - \lambda(P - cP')}{\lambda r - \lambda r'}.$$

(294.) We see, then, that whereas in the case of functional equations of one subject there seems to be generally, so far as our methods extend, one algebraical solution, accompanied by a class of solutions containing a particular sort of transcendental, defined as in (119.), in functions of two or more subjects there appear to be an infinite number of algebraical solutions, with a separate class involving also a transcendental defined by the algebraic solution. Or rather, if we consider the greater indeterminateness of the problem as requiring greater generality in the terms employed to express the solution, we should say that, in functions of two subjects, a general algebraical form, containing an *arbitrary function*, stands in the same relation to the solution as the single algebraical function containing an *arbitrary constant* does to the solutions of equations of functions containing one subject only. If we consider the *degree of indeterminateness* of a problem as a conception of comparison which actually must be formed, though not capable of numerical appreciation, we are justified in saying that, within the limits of common algebra, the indeterminateness of ϕx is to that of $\phi(x, y)$ in the same relation as an arbitrary constant to an arbitrary function of given forms.

(295.) If, in the general solution, it should appear that V is *per se* a function of W , the general equation $f(V, W) = 0$ gives $V = \text{constant}$. The resulting algebraical solution is reduced to a constant, but it will also appear that some of the transcendental solutions are reduced to algebraical solutions, in the same manner as in (129.) The successive n , n th functions will assume a certain periodic character at the same time.

To apply this, suppose we require that class of functions which is not changed by inverting x and y , provided that at the same time x is changed into $a'x$, and y into by . That is, required the solution of

$$\phi(by, a'x) = \phi(x, y);$$

considering this as a case of the last problem, we have $R^2 - a'b = 0$, and the successive functions involve the symbol of discontinuity $(-1)^n$. The algebraic solution

Calculus of Functions. given above is reduced to a constant, since it will be found that $P = \text{constant} \times P'$ and $Q = \text{constant} \times Q'$, and that $P r^n - Q r'^n$ is a function of $P' r^n - Q' r'^n$. To determine n , we assume then

$$P r^n - Q r'^n = \text{const.}, \text{ or } c^n \{P - Q(-1)^n\} = \text{const.},$$

where $c = \sqrt{a'b}$. The only case in which we can determine a result is where the constant vanishes, which gives

$$\begin{aligned} n &= \frac{1}{\lambda(-1)} \lambda \cdot \frac{P}{Q} = \frac{1}{\pi \sqrt{-1}} \lambda \frac{P}{Q} \\ \cos 2\pi n &= \left(\frac{P}{Q}\right) + \left(\frac{Q}{P}\right)^2 \\ &= \left(\frac{by-r'x}{by-rx}\right)^2 + \left(\frac{by-rx}{by-r'x}\right)^2 \\ &= \left(\frac{\sqrt{b}y - \sqrt{a'}x}{\sqrt{b}y + \sqrt{a'}x}\right)^2 + \left(\frac{\sqrt{b}y + \sqrt{a'}x}{\sqrt{b}y - \sqrt{a'}x}\right)^2 \end{aligned}$$

either term of which is a solution, and which gives as a general solution

$$\theta \left(\frac{\sqrt{b}y - \sqrt{a'}x}{\sqrt{b}y + \sqrt{a'}x} \right)^2,$$

where θ must not directly invert the square.

By making $b = 1$, $a' = 1$, we have thus obtained, independently of particular solutions obtained from observation, the known general form of symmetrical functions, namely, $\theta(\theta_1, \theta_2, \dots)$, θ_1, θ_2 &c. being such functions as θ just given.

(296.) We shall now show the propriety of the distinction between what we have called (for want of better names) *external* and *internal* simultaneous substitution, denoted by

$$\psi^{\alpha, \beta}(x, y) \text{ and } \psi(x, y)^{\alpha, \beta}.$$

When we propose the equation

$$\psi(\alpha, \beta) = \psi(x, y),$$

the general method requires some species of successive substitution, in which x and y shall be zero-functions. This is not the case as to the external substitution, in which the defined relation of ψ to both x and y gives, as we have seen, one zero-function only. But in the internal substitution we treat both x and y as independent, and the substitution has no dependence on the form of ψ , so that we are able to define x and y as the zero-functions of α and β . But if we propose the equation

$$\psi \alpha(x, y) = \psi \alpha^0(x, y),$$

in which α^0 is itself the zero-function of α , relatively to the notions implied in internal simultaneous substitution, we may proceed with that substitution in the same manner as in (50.) For if we now determine n from the equation

$$\alpha^{\alpha, \beta}(x, y) = F(\alpha^0, n) = \mu \cos 2\pi n, n = A \alpha^0(x, y),$$

we have evidently a solution of the equation

$$\phi \alpha = \phi \alpha^0 - 1;$$

but not of any form in which x and y enter through two functions. The equation

$$\phi(\alpha, \beta) = \phi(\alpha^0, \beta^0)$$

is reducible to the case already solved in (117.), by writing it in the form

$$\phi(\bar{\alpha} \alpha^0, \bar{\beta} \beta^0) = \phi(\alpha^0, \beta^0), \text{ or } \phi(\bar{\alpha} x, \bar{\beta} y) = \phi(x, y). \quad \text{Calculus of Functions.}$$

(297.) Let us now propose $\phi(\alpha, \alpha) = \phi(x, y)$ in the case where α is a function with a constant modulus, or an indefinite zero-function. If we substitute $\alpha(x, y)$ instead of both x and y , the equation of definition $\alpha(z, z) = \text{constant}$ gives $\phi(c, c) = \phi(\alpha, \alpha)$, or ϕ itself must be another function of the same kind. But this gives $\phi(x, y) = \text{constant}$, and the equation therefore has no continuous solution except this. That is, we cannot find a function which shall remain unchanged when, for instance, $x - y$ is substituted both for x and y . And, indeed, it would be the property of such a function to be reduced to a constant by $y = x$; it could, therefore, be only $\theta(y - x)$, which does not remain the same when y and x are both substituted for x .

When $\phi(\alpha, \alpha) = \phi(x, y)$, without the condition of a constant modulus, we obtain a solution by finding

$$n = A(x, y) \text{ from } \alpha(x, y)^{\alpha, \alpha} = F(n, x, y) = \mu \cos 2\pi n,$$

or, which is in this case the same thing, from

$$\alpha^{\alpha, \alpha}(x, y) = F(n, x, y) = \mu \cos 2\pi n;$$

the solution of the equation is then $\theta \cos 2\pi A(x, y)$; and here we have a case in which, generally speaking, there is no algebraical solution, but only a transcendental solution. This bears a great analogy with the case of subjects of a single function, which will appear still more when we show that, if the modulus of α be a periodic function, algebraical solutions may be obtained. Firstly, $\bar{\phi} z$ being the modulus of $\phi(x, y)$, it is clear that $\phi(\alpha, \alpha)$ is the same thing as $\bar{\phi} \alpha$, that is, as $\bar{\phi} \alpha \alpha^0$. We can then satisfy

$$\phi(\alpha \alpha) = \phi(x, y),$$

by means of $\bar{\phi} \alpha \alpha^0(x, y) = \bar{\phi} \phi^0(x, y)$, which, because one zero-function cannot be a function of x and y through another, gives $\phi^0 = \alpha^0$, or the solution has the same zero-function as the given function. We have then $\bar{\phi} \alpha z = \bar{\phi} z$, which admits algebraical solutions, as before described, only when $\bar{\alpha} z$ is a periodic function. For example, suppose the equation to be

$$\phi\left(\frac{1}{x+y}, \frac{1}{x+y}\right) = \phi(x, y),$$

where

$$\alpha(x, y) = \frac{1}{x+y}, \quad \bar{\alpha} z = \frac{1}{2z}, \quad \bar{\alpha}^2 z = z.$$

Hence

$$\bar{\phi} z = \xi \frac{1}{2z} = \theta\left(z + \frac{1}{2z}\right), \quad \alpha^0 = \frac{x+y}{2}$$

$$\bar{\phi} \phi^0 z = \bar{\phi} \alpha^0 z = \theta\left(\frac{x+y}{2} + \frac{1}{x+y}\right),$$

which will be found to be a solution.

Similarly,

$$\phi(\sqrt{1-xy}, \sqrt{1-xy}) = \phi(x, y)$$

gives

$$\phi(x, y) = \theta(\sqrt{xy} + \sqrt{1-xy}, \sqrt{xy} \sqrt{1-xy}).$$

(298.) The inverse of $\theta(x, y)$ or $\theta(x, y)^{\alpha, \beta}$ is the function in which α and β must be simultaneously substituted for x and y , in order that $\theta(x, y)$ may result. Let us then suppose

Calculus of
Functions.

from which let

$$\alpha(x, y) = x' \quad \beta(x, y) = y',$$

$$x = \alpha_1(x', y') \quad y = \beta_1(x', y').$$

We have then identically

$$\alpha\{\alpha_1(x', y'), \beta_1(x', y')\} = x' \quad \beta\{\alpha_1(x', y'), \beta_1(x', y')\} = y'$$

or

 $\theta\{\alpha_1(x, y), \beta_1(x, y)\}$, is the function required; that is,

$$\theta\{\alpha_1(x, y), \beta_1(x, y)\} = \theta(x, y)^{-1, -1}_{\alpha, \beta}.$$

The inverses which we mean to imply by this notation are also those which satisfy the converted forms

$$\alpha_1\{\alpha(x, y), \beta(x, y)\} = x \quad \beta_1\{\alpha(x, y), \beta(x, y)\} = y;$$

 these two equations answering to $\phi^{-1}\phi x = x$, and the first two to $\phi\phi^{-1}x = x$. Those which satisfy the first pair only may be considered as giving A and B, incon-vertible inverses.

 By observing different functions, we may ascertain, as in (34.), that every form of α_1 and β_1 may be made a convertible inverse in some particular method of considering the two fundamental equations, and also that if this particular method be retained in form, though the performance of direct upon inverse operations destroys all real difference of results, yet the final results of the operations analogous to $\phi\phi^{-1}x$ may be thus preserved. From which it appears, by reasoning similar to that in (34.), that $\alpha(A^n, B^n)$ and $\beta(A^n, B^n)$ will always represent one or other of the forms of α and β respectively: and also that, if the number of forms of α and β be finite, A and B must be periodic functions, which give for some one value of n

$$A(x, y)^{\overline{n, n}}_{A, B} = x \quad B(x, y)^{\overline{n, n}}_{A, B} = y.$$

As an instance, let us take

$$\alpha = x^2 + y = x'$$

$$\beta = 2x + y = y'.$$

 By choosing as our direct process of inversion the subtraction of the second equation from the first, and then proceeding arithmetically, we find (r being a square root of unity)

$$x = 1 + r\sqrt{1 + (x' - y')} = \alpha_1(x', y')$$

$$y = y' - 2 - 2r\sqrt{1 + x' - y'} = \beta_1(x', y').$$

This gives

$$\alpha_1(\alpha, \beta) = A(x, y) = 1 + r(x - 1)$$

$$\beta_1(\alpha, \beta) = B(x, y) = 2x + y - 2 - 2r(x - 1)$$

$$A(x, y)^{\overline{2, 2}}_{A, B} = 1 + r^2(x - 1) = x$$

$$B(x, y)^{\overline{2, 2}}_{A, B} = 2x + y - 2 - 2r^2(x - 1) = y.$$

Both A and B are then periodic, and of the second degree, and we have

$$\alpha(A, B) = r^2(x - 1)^2 + 2x + y - 1 = \alpha$$

$$\beta(A, B) = 2x + y = \beta.$$

 (299.) If $\theta(x, y)$ be considered as $\theta(x, y)^{\overline{0, 0}}_{\alpha, \beta}$ then by considering $\phi\theta(\phi^{-1}x, \phi^{-1}y)$ as $[\phi\theta(\phi^{-1}x, \phi^{-1}y)]^{\overline{0, 0}}_{\phi\alpha, \phi\beta}$ we have the equation

$$[\phi\theta(\phi^{-1}x, \phi^{-1}y)]^{\overline{n, n}}_{\phi\alpha, \phi\beta} = \phi : \theta(\phi^{-1}x, \phi^{-1}y)^{\overline{n, n}}_{\alpha, \beta};$$

 where, by the colon following the functional symbol, we understand that the change of θ into $\phi\theta$, x into $\phi^{-1}x$, and y into $\phi^{-1}y$ is the final operation, after all that is

 otherwise denoted has been performed. If then we have a pair of periodic functions, as in the last article, every pair of corresponding derivatives is periodic in the same manner. But it is obvious that we cannot, continuously at least, make $\theta_1(x, y)^{\overline{0, 0}}_{\alpha_1, \beta_1}$ a derivative of $\theta(x, y)^{\overline{0, 0}}_{\alpha, \beta}$, the six functional symbols meaning any whatever.

 (300.) If we now review articles (197.) and (235.), we shall see that the particular assumption of a change of x into αx , though essential to that case, is not in its specific character necessary to the solution. Neither the process nor the result depends upon any thing except there being some change which changes B_0 into B_1 , c_0 into c_1 , &c. simultaneously, and which is of such a kind that $B_n = B_0$, $c_n = c_0$, &c., followed by recurrence. And in the place where αx seems to enter specifically, namely, in $\xi\alpha x$, we see that this symbol merely defines a function, which does not change when the changes supposed in the problem are made. Consequently we have then solved the equation

$$P_0 = B_0 + c_0 P_1;$$

 where P_0 , B_0 , c_0 , are any functions whatsoever, which are by any hypothesis simultaneously changed into P_1 , B_1 , and c_1 , provided that successive operations reproduce P_0 , B_0 , and c_0 . By (235.) we can solve the general case.

(301.) One important application of this is the solution of the equation

$$\phi(x, y) = \beta(x, y) + \gamma(x, y)\phi(\mu, \nu);$$

where

$$\mu(x, y)^{\overline{n, n}}_{\mu, \nu} = x \quad \nu(x, y)^{\overline{n, n}}_{\mu, \nu} = y,$$

making

$$\phi(x, y) = P_0 \quad \beta(x, y) = B_0 \quad \gamma(x, y) = c_0$$

$$\phi(x, y)^{\overline{1, 1}}_{\mu, \nu} = P_1 \quad \beta(x, y)^{\overline{1, 1}}_{\mu, \nu} = B_1, \text{ \&c. ;}$$

 writing instead of $\xi(\alpha x)$ any function of

$$V, V', V'' \dots W, W', W'' \dots$$

 where $V, V', V'' \dots$ are symmetrical functions of

$$\mu(x, y) \quad \mu(x, y)^{\overline{1, 1}}_{\mu, \nu} \dots \mu(x, y)^{\overline{n-1, n-1}}_{\mu, \nu}$$

 and $W, W', W'' \dots$ are symmetrical functions of

$$\nu(x, y) \quad \nu(x, y)^{\overline{1, 1}}_{\mu, \nu} \dots \nu(x, y)^{\overline{n-1, n-1}}_{\mu, \nu},$$

we have in (197.) and (235.) solutions of the equations.

 (302.) From this it seems to us that the necessary restrictions upon the generality of Abel's Theorem (see (90.), note) may be obtained. Proceeding in the manner of that most excellent analyst, let us suppose an irreducible* algebraic equation $f(x) = 0$ of the n th degree, between two of whose roots, a_1 and a , the equation $a_1 = \theta a$ can be shown to exist. Then $f x = 0$ and $f\theta x = 0$ have a common root, and, consequently, have all their roots in common, for if not, a reducing factor might be obtained by the usual process, which is against hypothesis. Therefore θa is a root of $f\theta x = 0$, which, therefore, has a root in common with $f\theta^2 x = 0$, and so on. Consequently, $a, \theta a, \theta^2 a \dots$ are all roots of $f x = 0$. But as this has only n roots, the preceding series continued *ad infinitum* can only have a finite number of values less than n , consequently, as in (34.), it can be shown that we must have $\theta^m x = x$, where $m = n$. Suppose the first case, and let b be a root of $f x = 0$, which is not included in the preceding periodic

 * Namely, in which $f x$ cannot be divided by any rational algebraic function of a lower degree.

Calculus of
Functions.

series, then, by the same reasoning, $b, \theta b, \dots, \theta^{m-1}b$ must also be roots. If $2m$ be not so great as n , (it cannot be greater,) then $c\theta c, \dots, \theta^{m-1}c$ must be roots, and so on, that is, m must be a submultiple of n .

Now, from the process in (197.), it is evident that we can assign m of these roots, when we know one of them, and also the law of the function θ , provided we can ascertain the existence of a functional equation between θx and x which admits of solution. It is not necessary that we should absolutely know θx , for by the preceding method it is sufficient that we know B_0 and B_1 , from which θ can be inferred.

(303.) Further consideration points out to us that it is due as well to the memory of talents which we cannot sufficiently admire, as to the weight they will have, and ought to have, with our readers, to point out more explicitly than we have yet done, our objection to the theorem in question. The final process of M. Abel is as follows. To take a case, let n in (302.) be a prime number, so that m cannot be a submultiple of n , and must be equal to n itself. Now if such a relation as $a_1 = \theta a$ exist among two of the roots, it follows, as before, that the roots must be $a, \theta a, \theta^2 a, \dots, \theta^{n-1} a$. Consequently we can express any *disjunctively** periodical function of $a, \theta a$, &c. as a function of the coefficients of the equation. Consider, therefore, the function $\psi x = (x + \theta x \cdot r + \theta^2 x \cdot r^2 + \dots + \theta^{n-1} x \cdot r^{n-1})^n$, where $r^n = 1$, which gives $\psi \theta x = \psi x$, and, consequently,

$$\psi x = \frac{1}{n} (\psi x + \psi \theta x + \psi \theta^2 x + \dots + \psi \theta^{n-1} x),$$

or ψx is a symmetrical function of $x, \theta x, \theta^2 x, \dots$ and, therefore, can be expressed in terms of the coefficients of the equation. Let this be done, and let the result be U^n ; then calling $U, U_1, \dots$ the values of $(\psi x)^{\frac{1}{n}}$ corresponding to the roots $r, r_1, \dots$ of $r^n - 1$, it is easily proved that

$$x = \frac{1}{n} (U + U_1 + \dots + U_{n-1}).$$

Now the words in italics contain too much generalization: the function ψx is only periodically symmetrical, not disjunctively so: it remains the same if, when x becomes θx , θx also becomes $\theta^2 x$, and so on up to $\theta^{n-1} x$, which becomes $\theta^n x$ or x . But it is not allowable to interchange x and θx without the other interchanges: nor is

$$(a + b r + c r^2 + \dots + q r^{n-1})^n$$

such a function as can be expressed in terms of $a + b + \dots, a b + a c + \dots$ &c., without the previous solution of the equation of the n th degree. For if so, let the preceding be (remember that r^n recurs)

$$A_0 + A_1 r + \dots + A_{n-1} r^{n-1};$$

and writing $\cos t + \sqrt{-1} \sin t$ for r , we find that

* We cannot thus express any *periodically* symmetrical function in the same manner. For instance, of the functions

$$a^2 b^2 + b^2 c^2 + c^2 a^2 \text{ and } a b^2 + b c^2 + c a^2,$$

we can express the first in terms of

$$a + b + c, ab + bc + ca, abc,$$

but not the second unless we first find a, b , and c ; that is, solve the equation in question.

We must make M. Abel our excuse for having fallen into the same error in article (130.) in the use of the word symmetrical. It is true that we had previously defined the word to apply both to the disjunctive and periodical kinds; but, to the best of our recollection, we had forgotten our own distinction.

Calculus of
Functions.

$$A_0 + A_1 \cos t + A_2 \cos 2t + \dots + A_{n-1} \cos (n-1)t$$

$$\text{and } A_0 + A_1 \sin t + A_2 \sin 2t + \dots + A_{n-1} \sin (n-1)t$$

must be disjunctively symmetrical functions of $a, b, \dots$. That A_0, A_1 , &c. are not severally of this kind, except in the case of $n = 2$, may be made evident by expansion: for instance, when $n = 3$, $A_1 = 3(a^2 b + c^2 a + b^2 c)$, which is only periodically symmetrical. Neither then can the preceding expressions remain the same under partial interchanges; for in that case, by making a sufficient number of partial interchanges, we might determine $\cos t, \cos 2t$, &c. in terms of $A_0, A_1, \dots$ and a disjunctively symmetrical function; that is, we might find rational expressions for them, which cannot be. The complete development of $(a + b r + c r^2)^3$, where $r^3 = 1$ is (D_1 and D_2 standing for two disjunctively symmetrical functions)

$$D_1 = \frac{D^2}{2} + (a^2 b - a b^2 + c^2 a - c a^2 + b^2 c - b c^2) \frac{3\sqrt{-3}}{2},$$

in which the coefficient of $\sqrt{-3}$ will not admit a single interchange, and allow the whole to be still considered as symmetrical, unless we, at the same time, use a different sign for $\sqrt{-3}$, that is, suppose ourselves to have taken a different value of r at the commencement of the process.

(304.) We shall now consider some forms of simultaneously periodic functions. From (280.) it is evident that any two functions being periodic and of the n th order,* any two derivatives have the same property. From (293.) it is obvious that $a x + b y$ and $a' x + b' y$ are n thly periodic, when r and r' the roots of

$$R^2 - (a + b') R + a b - a' b = 0,$$

are also different roots of $R^n = 1$. If a, a' , &c. are to be real, then $a + b' = 2 \cos N$ $a b' - a' b = 1$ where N denotes $\frac{2m\pi}{n}$, m being a whole number. This gives

$$a' = -\frac{1}{b} (a^2 - 2a \cos N + 1) \quad b' = 2 \cos N - a;$$

but if we suppose $r = r' = 1$, then we have

$$(a x + b y)^{n-1} = (n a - n + 1) x + n b y$$

$$(a' x + b' y)^{n-1} = n a' x + (n b' - n + 1) y,$$

$$a' = -\frac{(a-1)^2}{b} \quad b' = 2 - a,$$

so that there can be no periodic function in this case except when $b = 0$ $a' = 0$ $a = 1$ $b' = 1$.

We have then a class of simultaneously periodic functions in

$$\phi(a \phi^{-1} x + b \phi^{-1} y) \quad \phi(a' \phi^{-1} x + b' \phi^{-1} y)$$

$$a' = -\frac{1}{b} (a^2 - 2a \cos N + 1) \quad b' = 2 - a,$$

for it is evident that similar derivatives of such functions have the same property.

(305.) We shall now investigate the conditions of simultaneous convertibility, or the solution of the equation

$$\phi(\psi, \psi) = \psi(\phi, \phi).$$

* The reader must recollect that what we now call the n th order answers to the $(n+1)$ th in subjects of a single function. Thus $a(x, \beta) = x$ shows periodicity of the first order; but if β were $= y$, and $a(x, y)$ were called μx , then $a(x, y)$ or $a(x, y)^{\frac{1}{n}}$ would be $\mu^2 x$.

Calculus of Functions.

1. If ϕ and ψ be both zero-functions, then this equation is reduced to $\psi = \phi$, which is the only solution.

2. If one only, ϕ , be a zero-function, the equation is reduced to $\psi = \psi(\phi, \phi)$, which has been already considered.

3. If both be functions with constant moduli it is sufficient that the two moduli should be the same.

4. If one only, ϕ , have a constant modulus, the equation is reduced to $\psi(\phi, \phi) = \text{constant}$, or the other is of the same kind.

5. If both have variable moduli, and therefore variable zero-functions, we see that $\phi(\psi, \psi)$ being $\bar{\phi}\psi$ or $\bar{\phi}\bar{\psi}\psi^0$, and $\psi(\phi, \phi)$ being $\bar{\psi}\phi$ or $\bar{\psi}\bar{\phi}\psi^0$, we must have

$$\bar{\phi}\bar{\psi}\psi^0 = \bar{\psi}\bar{\phi}\phi^0,$$

which shows that ψ^0 and ϕ^0 must be the same; for if not, one zero-function would be a function of another;

and that the moduli $\bar{\phi}$ and $\bar{\psi}$ must be convertible.

For instance, having the convertible functions $2x^2 - 1$ and $4x^3 - 3x$, and the zero-function $\sqrt{xy}$, we find the simultaneously convertible functions

$$2xy - 1 \text{ and } \sqrt{xy}(4xy - 3).$$

In a similar manner, the equation

$$\phi(x, y)_{\phi, \psi}^{\bar{n}, \bar{n}} = \psi(x, y)_{\phi, \psi}^{\bar{m}, \bar{m}}$$

may be reduced to the solution of the equation $\bar{\phi}\bar{\psi}^nz = \bar{\psi}\bar{\phi}^nz$, which we cannot solve generally. One simple solution may be obtained by finding convertible functions $\bar{\phi}$ and $\bar{\psi}$, which satisfy $\bar{\psi}^{n-1}x = x\bar{\phi}^{n-1}x = x$. Such are rx and $r'z$, where $r^{n-1} = 1$ and $r'^{n-1} = 1$; from which it follows that $\bar{\phi}x = \theta r\theta^{-1}z$ and $\theta r'\theta^{-1}z$ will be convertible periodic functions of the kind required; and $\theta r\theta^{-1}\alpha^0(x, y)$ and $\theta r'\theta^{-1}\alpha^0(x, y)$ will be solutions of the equation.

(306.) The solution of $\phi(\alpha, \beta) = \phi(x, y) - 1$, given in (291.), leads, as in the case of functions of one subject, to those of

$$\phi(\alpha, \beta) = \phi(x, y) \pm c.$$

$$\phi(\alpha, \beta) = c\phi(x, y) \text{ and } \phi(\alpha, \beta) = (\phi(x, y))^c;$$

and if we denote by $\xi(\alpha, \beta)$ a function of x and y , which is not changed by the simultaneous substitution of α for x , and β for y , we find that the general solution of $\phi(\alpha, \beta) = A \times \phi(x, y)$ is $\varpi(x, y) \times \xi(\alpha, \beta)$, where $\varpi(x, y)$ is a particular solution, and also that the solution of

$$\phi(x, y) = A(x, y) + B(x, y)\phi(\alpha, \beta)$$

is

$$\phi(x, y) = \varpi(x, y) + \varpi_1(x, y)\xi(\alpha, \beta),$$

where ϖ is a particular solution of the equation itself, and ϖ_1 of the equation deprived of the term A . The perfect generality of this solution may be established in the same manner as in (176.); but we may not assume that we can in any case arrive at the most general form of $\xi(\alpha, \beta)$.

(307.) Mr. Babbage* has given a method, amongst others, of obtaining $\xi(\alpha, \beta)$ as follows. To solve $\phi(\alpha, \beta) = \phi(x, y)$, assume $\phi_1(f, f_1)$ for ϕ , which gives

$$\phi_1(f, f_1) = \phi_1\{f(\alpha, \beta), f_1(\alpha, \beta)\};$$

if f and f_1 be particular solutions, any form of ϕ_1 is

* Essay, &c., part ii. prob. 4.

sufficient; but an entirely different class of solutions will (or, as far as we can see, may) arise if we suppose ϕ_1 to be a symmetrical function of x and y , and suppose

$$f_1(\alpha, \beta) = f(x, y) \quad f(\alpha, \beta) = f_1(x, y),$$

which give, by internal substitution of α and β ,

$$f_1\{\alpha(x, y)_{\alpha, \beta}^{1,1}, \beta(x, y)_{\alpha, \beta}^{1,1}\} = f_1(x, y)$$

$$f\{\alpha(x, y)_{\alpha, \beta}^{1,1}, \beta(x, y)_{\alpha, \beta}^{1,1}\} = f(x, y).$$

It is sufficient for these, that f and f_1 be particular solution of the given equation, for $f(\alpha, \beta) = f$ gives

$$f\{\alpha(x, y)_{\alpha, \beta}^{n,n}, \beta(x, y)_{\alpha, \beta}^{n,n}\} = f(x, y);$$

but the same condition is not necessary; for instance, the forms of f and f_1 are any whatsoever, which satisfy the more general equation just given. For example, they are altogether immaterial if

$$\alpha(x, y)_{\alpha, \beta}^{1,1} = x, \quad \beta(x, y)_{\alpha, \beta}^{1,1} = y,$$

which are satisfied if

$$\alpha(x, y) = \theta\left(a + b\theta^{-1}x + \frac{b^2 - 1}{b}\theta^{-1}y\right)$$

$$\beta(x, y) = \theta\left(\frac{ab}{1 - b} - b\theta^{-1}x - b\theta^{-1}y\right).$$

In all such cases $\phi(x, y) = \phi(\alpha, \beta)$ is identically satisfied by $\phi = \chi(\alpha, \beta)$. This proves the impossibility of deriving any two functions from any two given forms, which, by analogy, might at first be supposed possible.

The following conditions also make $\phi = \chi(\alpha, \beta)$ satisfy the equation $\phi(\alpha, \beta) = \phi(x, y)$ identically:

$$\alpha(x, y) = \theta\{(\theta^{-1}x)^{\sqrt{1-n^2}}(\theta^{-1}y)^n\}$$

$$\beta(x, y) = \theta\{(\theta^{-1}x)^n(\theta^{-1}y)^{-\sqrt{1-n^2}}\}.$$

(308.) We have already (117.) given the general solution of

$$\phi(x, y) = \phi(\alpha x, \beta y),$$

but we can divide the solutions which we may find into two classes, *disjunctive* solutions, in which either substitution satisfies the equation, and *simultaneous* solutions, in which both substitutions are necessary. The first solutions satisfy both of

$$\phi(x, y) = \phi(\alpha x, y) = \phi(x, \beta y);$$

and, therefore,

$$\phi(x, y) = \phi(\alpha x, \beta y);$$

and form the solution given by Mr. Babbage;* the second class may be found by the method given in (117.)

For instance, suppose we have to solve

$$\phi(x, y) = \phi\left(-x, \frac{1}{y}\right),$$

the class of disjunctive solutions is contained in

$$\phi(x, y) = \phi\left(x^2, y + \frac{1}{y}\right);$$

but by using the method in (117.) we may obtain such solutions as $x\left(y - \frac{1}{y}\right), \lambda y(e^x - e^{-x})$; or, generally,

$$(\alpha^2 x = x \quad \beta^2 y = y), \quad \psi(\phi(x, y)\phi(\alpha x, \beta y)),$$

where ψ is a symmetrical function of the two forms of ϕ . This solution is, generally, simultaneous.

(309.) The equation $\phi(\alpha x, \alpha y) = \phi(x, y)$ admits of

* Essay, &c., part ii. prob. 1.

Calculus of Functions.

solution as follows. Since the moduli must be the same, we have $\bar{\phi} \alpha z = \bar{\phi} z$, or $\bar{\phi} = \xi \alpha$; and, since the zero-functions must be the same, we have

$$\alpha^{-1} \bar{\phi}^0(\alpha x, \alpha y) = \bar{\phi}^0(x, y) \text{ or } \bar{\phi}^0(\alpha x, \alpha y) = \alpha \bar{\phi}^0(x, y).$$

But of the latter we can give no general solution, though we can make a general solution of the given equation serve to show in what the difficulty lies. Let $\phi(x, y) = \psi(Ax, Ay)$, where Ax is the exponential inverse of αx ; then

$$\phi(\alpha x, \alpha y) = \psi(Ax - 1, Ay - 1) = \psi(Ax, Ay),$$

or

$$\psi(x - 1, y - 1) = \psi(x, y),$$

whence

$$\psi(x, y) = \theta(y - x),$$

or

$$\phi(x, y) = \theta(Ay - Ax),$$

and therefore has a constant modulus.

If we suppose $\phi(\alpha, \alpha) = \alpha \phi(x, y)$, we find $\bar{\phi} \alpha = \alpha \bar{\phi}$, and $\alpha^{-1} \bar{\phi}^0(\alpha, \alpha)$ (the zero-function of $\phi(\alpha, \alpha)$) = $\bar{\phi}^0$, the zero-function of $\alpha \phi$. This gives $\bar{\phi}^0(\alpha, \alpha) = \alpha \bar{\phi}^0$, or the zero-function of ϕ must also be a solution of the equation. Consequently, we must find the general solution of $\phi(\alpha, \alpha) = \alpha \phi(x, y)$, in order to find that of $\phi(\alpha, \alpha) = \phi(x, y)$. By the method in (117.) the solution of the first is found by eliminating m, n , and p , between four equations of the form

$$F(\alpha^m x, \alpha^n y, \alpha^p z) = 0,$$

and then determining z in terms of x and y .

It may be useful to observe that the method already employed, which produces a result with a constant modulus, is equivalent to allowing three of the equations to be

$$\alpha^m x = c, \quad \alpha^n y = c', \quad \alpha^p z = c'',$$

and proposing as the fourth a relation between m, n , and p , of the form $p = \theta(m, n)$, where θ is then to be found.

Without affecting the generality of the process in (117.), we may suppose any relations to exist between m, n ; and p which are consistent with contemporaneous increase and decrease of these quantities by unity. Assume $n = m + c, p = m + c'$. Also, since $\alpha^m x$ is $A^{-1}(Ax - m)$, &c., and any functional relations between these may be chosen, we may write $Ax - m$ in place of $\alpha^m x$. Assume

$$Az - p + q(Ax - m) + q'(Ay - n) = 0$$

$$1 + q + q' = 0;$$

this allows us to eliminate m, n , and p , and gives

$$z = A^{-1}\{C - qAx + (1 + q)Ay\} = \phi(x, y),$$

which satisfies $\phi(\alpha, \alpha) = \alpha \phi$. The same, therefore, does its zero-function, which will be found from the equation

$$A^{-1}\{C - qAt + (1 + q)At\} \\ = A^{-1}\{C - qAx + (1 + q)Ay\}$$

to be

$$\bar{\phi}^0(x, y) = A^{-1}\{(1 + q)Ay - qAx\}.$$

Here, then, is a zero-function, which satisfies

$$\bar{\phi}^0(\alpha, \alpha) = \alpha \bar{\phi}^0(x, y),$$

whence

$$\phi(\alpha, \alpha) = \phi(x, y)$$

is satisfied by

$$\phi(x, y) = \theta \cos 2\pi \{(1 + q)Ay - qAx\}.$$

(310.) Before proceeding further, it will be convenient to recapitulate some of the forms of functions which have, or have not, constant moduli.

1. No symmetrical functions whatsoever have constant moduli.

2. No algebraical integral functions, except those of the form $(x - y)\phi(x, y) + c$.

3. All ratios of homogeneous functions of the same degree; for instance, such as

$$\frac{ax + by}{a'x + b'y} \quad \frac{x^2 + 2xy}{xy + y^2}, \text{ \&c.}$$

4. Ratios of algebraical functions, which have the sum of the coefficients of each order of terms the same, or given multiples of the same, such as

$$\frac{a + bx + cy + ex^2 + gxy - hy^2}{a' + b'x + c'y + e'x^2 + h'y^2},$$

where

$$a = m a' \quad b + c = m(b' + c')$$

$$e + g - h = m(e' + h').$$

5. All functions whatsoever of the above, and all functions of the form

$$(x - y) \chi(x, y) + c,$$

where $\theta(0) = 0$, and $\chi(x, x)$ is finite.

(311.) Let $\bar{\phi}^n(x, y) = \alpha(x, y)$, or $\bar{\phi}^n \phi(x, y) = \alpha \alpha^0(x, y)$. Hence the zero-functions must be the same, or one zero-function would be a function of another.

Hence the equation is reduced to $\bar{\phi}^n z = \alpha z$, a functional equation of one subject.

The same reasoning applies to every equation of the form

$$F(\alpha, \beta, \gamma, \dots, \phi, \bar{\phi}^2, \bar{\phi}^3, \dots, \bar{\phi}^n) = 0,$$

if the zero-functions of α, β, γ , &c. be the same.

(312.) In all cases where we have more than one known function not entering under the unknown functional sign, we may proceed as follows. Let the equation be

$$\bar{\phi}^2(x, y) = \alpha(x, y) + \beta(x, y)\phi(x, y).$$

As this must hold when $x = y$, let that supposition be made, which gives

$$\bar{\phi}^2 x = \bar{\alpha} x + \bar{\beta} x \bar{\phi} x,$$

which equation the modulus of ϕ must satisfy. Suppose ϖx to be a particular solution of the last equation, which is now known, and let V be the unknown zero-function which must be introduced, so that $\varpi(V)$ may solve the original equation. Then we have

$$\varpi^2(V) = \alpha(x, y) + \beta(x, y)\varpi(V),$$

in which every thing is known except V . Let V be thence determined. One or more of its values are zero-functions, because, by the supposition on which ϖ was obtained, $x = y$ and $V = x$ are simultaneously true. Consequently, for every different form which can be given to ϖ , a different zero-function is requisite; for it is obvious, that if we could obtain ϖ with an arbitrary function we could not solve the equation which gives V in finite terms. If in the second equation we substitute $\bar{\phi}^0(x, y)$ for x , we have

$$\bar{\phi}^2(x, y) = \bar{\alpha} \bar{\phi}^0(x, y) + \bar{\beta} \bar{\phi}^0(x, y) \phi(x, y),$$

which, with the first equation, determines both $\phi(x, y)$ and $\bar{\phi}^2(x, y)$, and therefore gives an equation which

Calculus of Functions.

Calculus of Functions. $\phi^0(x, y)$ must satisfy. The determination of the equation will involve an indeterminate fraction, which will render it not easy to use. But it thus appears that the first equation, at most, takes a certain class of the solutions of every equation which can be formed by retaining the moduli of α and β , and substituting the same zero-functions.

(313.) For brevity, write α^0 for $\bar{\alpha}^0(x, y)$, &c., and take the two equations

$$\bar{\phi}^2 = \bar{\alpha}\alpha^0 + \bar{\beta}\beta^0 \cdot \phi$$

$$\phi^2 = \bar{\alpha}\phi^0 + \bar{\beta}\phi^0 \cdot \phi,$$

omitting subjects, and putting a full stop between factors. We then have

$$\bar{\phi}^2 = \frac{\bar{\alpha}\alpha^0 \cdot \bar{\beta}\phi^0 - \bar{\alpha}\phi^0 \cdot \bar{\beta}\beta^0}{\bar{\beta}\phi^0 - \bar{\beta}\beta^0} \dots (\bar{\phi}^2)$$

$$\phi = -\frac{\bar{\alpha}\phi^0 - \bar{\alpha}\alpha^0}{\bar{\beta}\phi^0 - \bar{\beta}\beta^0} \dots (\phi).$$

Let α^0 and β^0 be as in the first two of the following equations; and let ϕ^0 be assumed as in the third.

$$\alpha^0 = \frac{x+y}{2} + A(x-y) \cdot A(x, y)$$

$$\beta^0 = \frac{x+y}{2} + B(x-y) \cdot B(x, y)$$

$$\phi^0 = \frac{x+y}{2} + F(x-y) F(x, y).$$

The determination of $\bar{\phi}^2$ and $\bar{\phi}$ from $(\bar{\phi}^2)$ and (ϕ) gives indeterminate fractions; but if we substitute from the equations just written, differentiate with respect to x , and afterwards make $x = y$ in the usual manner, we have the following values of $\bar{\phi}^2$ and $\bar{\phi}$.

$$\bar{\phi}^2 = \bar{\alpha}x - \bar{\beta}x \frac{\bar{\alpha}'x}{\bar{\beta}'x} \cdot \frac{F'_0 \bar{F} - A'_0 \bar{A}}{F'_0 \bar{F} - B'_0 \bar{B}}$$

$$\bar{\phi} = -\frac{\bar{\alpha}'x}{\bar{\beta}'x} \frac{F'_0 \bar{F} - A'_0 \bar{A}}{F'_0 \bar{F} - B'_0 \bar{B}};$$

where F'_0 means the differential coefficient of $F(x-y)$ with respect to x , for the particular value $x = y$, and $\bar{F}x$ is the modulus of $F(x, y)$; and the same for the rest. And F'_0 , &c. may be severally zero, if their vanishing ratios be substituted. Taking, therefore, $\bar{\phi}$ a solution of the modular equation, which makes the above-mentioned equations consistent, we must determine $\bar{F}$ from the second, giving any value we please to F'_0 . The two equations with which we set out then agree, and we have for every form of $\bar{\phi}$ an arbitrary function in the zero-function ϕ^0 ; but we thus only determine the modulus $\bar{F}$, and must afterwards find its zero-function. When $\bar{A} = \bar{B}$, and $A'_0 = B'_0$, we must suppose $\bar{F} = \bar{A}$ and $F'_0 = A'_0$, unless it so happen that $-\frac{\bar{\alpha}'x}{\bar{\beta}'x}$ is a solution of the modular equation, in which case $\bar{F}$ and F'_0 may be any whatsoever.

(314.) The solution just mentioned, and its consequence, arises from the following identical equation:

VOL. II.

$\int \phi x \cdot \phi'' \phi x \cdot \phi' x \cdot dx = \phi' \phi x \cdot \phi x - \phi^2 x$, which reduces every direct question of integration to the solution of the functional equation

$$\phi x \cdot \phi' x \cdot \phi'' \phi x = \alpha x.$$

(315.) All the conditions which depend on moduli being now satisfied, we must assume such a value of $F(x-y)$ as will reduce F'_0 to the value we suppose it to have in determining $\bar{F}$, and must then substitute

$$\frac{x+y}{2} + F \cdot \bar{F} \bar{F}^0$$

in the original equation. All is here known, except $\bar{F}$, which must be determined from the equation, and the result will enable us to give a solution; being, in effect, like the latter part of the process first given. A trial of this process will make it appear that we have no method of obtaining a solution to any given zero-function.

(316.) The equation

$$\bar{\phi}^2(x, y) = -x^2 y + 2xy \phi(x, y)$$

gives the modular equation

$$\bar{\phi}^2 x = -x^4 + 2x^2 \bar{\phi} x,$$

which is satisfied by

$$\bar{\phi} x = cx^2, \quad c^3 = 2c - 1,$$

and by

$$\bar{\phi} x = x^2 - 1.$$

The first gives the solution

$$\phi(x, y) = \frac{c^2 xy + c \sqrt{c^2 x^2 y^2 - (2c - 1)x^3 y}}{2c - 1};$$

the second gives

$$\phi(x, y) = xy - \sqrt{x^2 y^2 - x^3 y + 1}.$$

(317.) Among the cases of equations which are reducible to those we have considered, we may instance

$$\frac{\phi(\alpha, \beta) \frac{d\alpha}{dx} + \frac{d\beta}{dx}}{\phi(\alpha, \beta) \frac{d\alpha}{dy} + \frac{d\beta}{dy}} = \phi(x, y),$$

α and β being any functions of x and y . This is the most general equation with which we are acquainted that does not contain differential coefficients of unknown functions. Assume any function whatsoever, γz , and solve the equation

$$\psi(\alpha, \beta) = \gamma \psi(x, y), \text{ then } \phi(x, y) = \frac{\frac{d\psi}{dx}}{\frac{d\psi}{dy}}$$

is a solution of the given equation, as is sufficiently evident.

(318.) We shall now proceed to consider the formation of successions of partial substitutions, denoted by

$$\psi_{1,2}^{m,n}(x, y) \quad \psi_{2,1}^{m,n}(x, y), \text{ \&c.}$$

If we consider (the order of the letters being always that of operation, unless the contrary be denoted by underwritten numerals)

$$\phi^{2,1}(x, y) = \phi(\phi(x, y), y),$$

it appears that the modulus of $\phi^{2,1}(x, y)$ is $\phi(\bar{\phi}x, x)$. Similarly, the modulus of $\phi^{1,2}(x, y)$ is

$$\phi\{\phi(\bar{\phi}x, x), x\};$$

or if $M\phi^{n,1}$ be the modulus of $\phi^{n,1}$, we have

3 D

Calculus of
Functions.

we have also

$$M \phi^{n+1,1} = \phi (M \phi^{n,1}, x);$$

$$M \phi^{n+1,1} = \phi^{n,1}(\bar{\phi} x, x),$$

in which latter function n and 1 refer to $\bar{\phi} x$ and x .

If ϕx be itself a zero-function, it is evident that the modulus is and remains equal to x , or all successive functions of zero-functions are zero-functions.

(319.) The application of the preceding methods to equations involving partial substitutions would be of little use, and all that has been done on this subject amounts to the partial solution of a few equations of the most simple character.

But the preceding methods may be made useful in giving some idea of the difficulties which stand in the way of obtaining solutions.

Let us take the following equation, considered by Mr. Babbage:*

$$\phi^{2,1}(x, y) = \phi^{1,2}(x, y) \text{ or } \phi\{\phi(x, y), y\} = \phi\{x, \phi(x, y)\};$$

writing $\bar{\phi} \phi^0$ for ϕ , we find this is satisfied when $y = x$, by

$$\phi^0(\bar{\phi} x, x) = \phi^0(x, \bar{\phi} x),$$

which, being true for all values of x , must be true if we substitute ϕ^0 for x , which gives

$$\phi^0(\phi, \phi^0) = \phi^0(\phi^0, \phi);$$

or we must have a function, which is a symmetrical function of itself, and of a function of itself.

The several conditions here deduced can be easily satisfied, but we have not met with any method by which a particular solution can be found. Any zero-function, for instance, satisfies the two last-mentioned equations. And, if any solution can be found, any derivative of it is also a solution. But we have not succeeded in determining any solution.

The interchange of x and y in the preceding does not alter the problem; from which it should seem that the function in question must be symmetrical.† But, in this case, the question is reduced to assigning $\phi(\phi(x, y), y)$, so as to be equal to $\phi(\phi(y, x), x)$, or making $\phi(\phi, y)$ symmetrical. Now it is easy to see that this cannot be where ϕ is already symmetrical; for it gives $\phi(\phi, y) = \phi(\phi, x)$, which cannot be generally true, but assigns an equation of condition between y and x . This reasoning is not, as Mr. Babbage remarks, of a very sure character, but it furnishes a strong presumption that all the solutions of the given equation are discontinuous.

(320.) We shall now conclude with some forms in which particular solutions have been obtained.‡

Let

$$x \phi^{1,2}(x, y) = y \phi^{2,1}(x, y).$$

Assume

$$\phi = \theta \alpha(\theta^{-1}x, \theta^{-1}y),$$

which gives

$$\theta x \cdot \theta \alpha^{1,2}(x, y) = \theta y \theta \alpha^{2,1}(x, y),$$

which is satisfied if

$$\alpha^{1,2}(x, y) = y \quad \alpha^{2,1}(x, y) = x;$$

that is, we have to find a function which is periodic of the second order, both with respect to x alone and to y alone. Take any periodic function of the second order, which contains one or more arbitrary constants, being a function of x , and assume the arbitrary constants such

functions of y as will make the periodic function in question symmetrical. For instance, $\frac{a-x}{1-bx}$ being a periodic function of the second order, for a write $a-y$, and for b write by , and we have the solutions

$$\frac{a-x-y}{1-bxy}, \text{ and } \theta\left(\frac{a-\theta^{-1}x-\theta^{-1}y}{1-b\theta^{-1}x\theta^{-1}y}\right),$$

but (70.) we cannot thus obtain any form which contains more than $\theta(a-\theta^{-1}x-\theta^{-1}y)$.

The same solutions will also satisfy

$$\phi^{1,2}(x, y) \times \phi^{2,1}(x, y) = xy,$$

and generally (F being given)

$$F(x, \phi^{1,2}(x, y)) = F(\phi^{2,1}(x, y), y)$$

(321.) Let

$$F(\phi^2(x, y), x) = F(a, \phi^{2,1}(x, y)).$$

Assume $\phi = \theta \alpha(\theta^{-1}x, \theta^{-1}y)$, which gives

$$F(\theta \alpha^2(x, y), \theta x) = F(a, \theta \alpha^{2,1}(x, y)),$$

and is satisfied by $\theta \alpha^2 = a$, $\alpha^{2,1}(x, y) = x$.

Take any periodic function with an arbitrary constant, and substitute for the arbitrary constant such a function of y as will give the resulting function a constant modulus c : then assume $c = \theta^{-1}a$. For instance, take

$$\frac{m-x}{1-nx}, \text{ write } m+y \text{ for } m, \text{ and } \frac{n}{y} \text{ for } n, \text{ which gives}$$

$$\frac{(m+y-x)y}{y-nx}, \text{ whose modulus is } \frac{m}{1-n} = \theta^{-1}a.$$

Hence $\theta\left(\frac{(1-n)\theta^{-1}a + \theta^{-1}y - \theta^{-1}x}{\theta^{-1}y - n\theta^{-1}x} \theta^{-1}y\right)$ is a solution.

(322.) Let $F(x, y, \psi, \psi_1, \psi_2, \dots) = 0$; ψ, ψ_1, ψ_2 , &c. being all some successive functions of ψ or $\psi(x, y)$. The assumption $\psi(x, y) = \theta \alpha(\theta^{-1}x, \theta^{-1}y)$ reduces the preceding to

$$F(\theta x, \theta y, \theta \alpha, \theta \alpha_1, \theta \alpha_2, \dots) = 0,$$

α_1, α_2 , &c. being the corresponding successive functions of α .

Assume any given form for α , and the equation is reduced to one containing one unknown function θ . For each form of α the equation may or may not possess a solution. And, even if $\alpha(x, y)$ be a particular solution of the equation, it does not follow that θ contains an arbitrary function.

(323.) The difficulty of the preceding species of solution is this, that, when we have obtained the final equation, we have no direct means of referring it to the solution of an equation with functions of one subject, except upon two considerations, neither of which will apply. Firstly, the equation may be simplified if α, α_1 , &c. have the same zero-functions, which will not be the case. Secondly, if we take a particular form of α , and solve the equation with respect to either x or y , considering the other as an arbitrary constant, we shall obtain solutions containing the supposed arbitrary constant in a manner which requires that we should actually consider it as such, so that we cannot obtain a final result which remains true when the temporary arbitrary constant undergoes the given changes. To take a case, let us again propose the equation

$$\phi^{2,1}(x, y) = \phi^{1,2}(x, y),$$

and let us suppose $\phi(x, y) = \theta(\theta^{-1}x + \theta^{-1}y)$. This gives

$$\theta(\theta^{-1}x + 2\theta^{-1}y) = \theta(2\theta^{-1}x + \theta^{-1}y),$$

* Essay, &c. part ii. &c. † Id. Ib.
‡ Id. Ib.

Calculus of or
Functions.
or

$$\theta(x+2y) = \theta(2x+y),$$

$$\theta x = \mu \cos 2\pi \frac{\lambda(x-3y)}{\lambda 2},$$

considering y as an arbitrary constant. If we distinguish y when it comes from the supposition $\phi(x, y) = \theta(\theta^{-1}x + \theta^{-1}y)$ from y as it appears in the preceding solution, so as to suppose y to be lineally (266.) changed with respect to the manner in which it thus enters in forming $\phi^{1,2}(x, y)$, we can obtain a solution: but such is not the case supposed.

(324.) Since we have separated the considerations connected with $\phi(x, y)$ into 1. those which relate to its zero-function, 2. those which relate to its modulus; and since it appears that problems may be possible so far as one of the two is concerned, and not possible with respect to the other, we shall propose some questions in which one only of the preceding two is concerned. Suppose, for instance, it is required to find those functions in which $\phi^{m,1}(x, y)$ has a constant modulus. If we take the case of $m=2$, we have

$$\bar{\phi}^0(\bar{\phi}x, x) = \bar{\phi}^{-1}c,$$

c being the given constant. If then we assume any zero-function, we may determine $\bar{\phi}x$ in terms of x and $\bar{\phi}^{-1}c$.

Thus in the case of $\bar{\phi}^0(x, y) = \frac{1}{2}(x+y)$, we find

$$\bar{\phi}x = 2\bar{\phi}^{-1}c - x \quad \phi(x, y) = 2c - \frac{x+y}{2},$$

which satisfies the conditions. And our being thus able to determine one modulus only to a given zero-function, shows that problems, in which both are left indeterminate, will probably be of very limited solutions.

(325.) Let the modulus of $\phi^{m,n}$ be denoted by $\bar{\phi}^{m,n}$, and let the equation be

$$\bar{\phi}^{2,1}(x, y) = \bar{\phi}^{1,2}(x, y).$$

The zero-function is now arbitrary, and as in (319.) we find

$$\bar{\phi}^0(\bar{\phi}x, x) = \bar{\phi}^0(x, \bar{\phi}x),$$

and our solution is this, that the function may either be a zero-function itself, or its modulus may be any invertible solution of $f(x, y) = f(y, x)$: for the preceding equation gives $\bar{f}\bar{\phi}^0(x, y) = \bar{f}\bar{\phi}^0(y, x)$. Thus by solving the equation

$x^2 + y = y^2 + x$, we have $y = 1 - x$, and the zero-function in question is derived from

$$x^2 + z = x^2 + y, \text{ giving } z = -\frac{1}{2} + \sqrt{x^2 + y + \frac{1}{4}},$$

so that the function required is

$$\phi(x, y) = \frac{3}{2} - \sqrt{x^2 + y + \frac{1}{4}},$$

which will be found to satisfy the condition; namely, that the modulus of $\phi^{2,1}(x, y)$, or of

$$\frac{3}{2} - \sqrt{\frac{11}{4} + x^2 + 2y - 3\sqrt{x^2 + y + \frac{1}{4}}},$$

is the same as that of $\phi^{1,2}(x, y)$, or of

$$\frac{3}{2} - \sqrt{x^2 + \frac{7}{4} - \sqrt{x^2 + y + \frac{1}{4}}}.$$

So long as a condition between moduli only is given, leaving the zero-functions indeterminate, the question is frequently reduced to one of ordinary algebra: but when the modulus is given, and the zero-function is to be found, the question is reduced to one of the species solved in (309.)

(326.) Required the function as to which the result is the same, whether we differentiate with respect to x or change y into αy : that is, which gives

$$\phi(x, \alpha y) = \frac{d}{dx} \phi(x, y).$$

Assume

$$\phi(x, y) = \mu y e^{\nu y \cdot x},$$

which gives

$$\nu \alpha y = \nu y \quad \mu \alpha y = \mu y \cdot \nu y \dots (\mu, \nu),$$

by the first of which it appears that the second may be solved as if νy were constant. This gives

$$\nu y = \xi \alpha y \quad \mu y = (\xi \alpha y)^{-A y},$$

where $A y$ is the exponential inverse of αy . Hence we have

$$\phi(x, y) = \sum \{ (\xi \alpha y)^{-A y} e^{\xi \alpha y \cdot x} \}.$$

When the function αy is periodic, (μ, ν) cannot be both solved continuously, unless $(\alpha^n x = x)$,

$$\nu x \cdot \nu \alpha x \dots \nu \alpha^{n-1} x = 1, \text{ or } (\nu x)^n = 1,$$

which gives

$$\xi \alpha x = r^p (r^n = 1)$$

$$\phi(x, y) = \sum \{ (y + r^p \alpha y + \dots + r^{(n-1)p} \alpha^{n-1} y) e^{r^{-p} x} \},$$

or, more generally,

$$\phi(x, y) = \sum \{ \xi \alpha y (y + r^p \alpha y + \dots + r^{(n-1)p} \alpha^{n-1} y) e^{r^{-p} x} \}.$$

(327.) Mr. Babbage has reduced the preceding to a differential equation, when $\alpha^n y = y$, as follows: the first of these equations gives all the rest.

$$\phi(x, \alpha y) = \frac{d}{dx} \phi(x, y) \quad \phi(x, \alpha^2 y) = \frac{d}{dx} \phi(x, \alpha y) \\ \dots \dots \dots \phi(x, y) = \frac{d}{dx} \phi(x, \alpha^{n-1} y)$$

whence

$$\phi(x, y) = \frac{d^n}{dx^n} \phi(x, y),$$

from which the same solution may be obtained.

By similar methods, the equation

$$\frac{d}{dx} \phi(x, \beta y) = \frac{d}{dy} \phi(\alpha x, y)$$

may be reduced to a partial differential equation, when αx and βy are periodic of the same order. The general solution of this case is of considerable difficulty, if indeed it be practicable.

(328.) A functional equation may admit of reduction to one of a more simple form, if it can be seen that a solution is derived, by supposing one function of the unknown forms to undergo no change, when another function of the same is substituted. Such is the equation belonging* to the class of curves which have the ordinate and the normal drawn from the same point in the axis of abscissæ, equal to each other. It is

$$(\phi x)^2 + \left(\phi x \frac{d\phi x}{dx} \right)^2 = \left(\phi \left(x + \phi x \frac{d\phi x}{dx} \right) \right)^2;$$

* Euler, cited by Babbage, *Essay*, &c., part ii.
3 D 2

Calculus of
Functions.

by differentiating which, it will be seen that $\phi x \frac{d\phi x}{dx}$

does not change when $x + \phi x \frac{d\phi x}{dx}$ is substituted for x . This leads to the equation

$$y^2 = xy \frac{dy}{dx} + \phi \left(y \frac{dy}{dx} \right).$$

The connection between differential and functional equations might be advantageously examined by means of such equations, but we have already gone beyond the limits which we had prescribed to ourselves in setting out.

Remark alluded to in (48.) The reason why we may suspect that any general form, which contains all the *whole* functions, may also contain, at least, some fractional ones, lies simply in our perfect ignorance of the general form of $\phi^n x$. We see in (70.) that one purely arbitrary method of representing a whole function, gives results entirely agreeing with those which are derived from a more legitimate deduction of form; and this, even in cases where no fractional functions, or functional inverses, can be obtained from the same representation of $\phi^n x$. In making the extension on which this remark is written, we have still, in our representation of $\phi^n x$, preserved the condition $\phi(\phi^n x) = \phi^{n+1} x$, which is the definition of $\phi^n x$. If, then, we find no functional inverses among the forms so obtained, that we cannot do so can only be a result of observation of some particular cases; and the smallness of the number of forms to which we can apply observation, together with the results of (153.) and (154.), and the infinite number of

forms which we are able to give to $\phi^n x$, when $\phi x = x$, must (even taking into account any presumption to be derived from the observations in (158.), oblige us to pause before we assume the most simple method of representing $\phi^n x$ in terms of n , as the proper general expression, even in the case where n is a whole number.

Addition to Article (26.) The invention of what we may call primary or fundamental notation has been but little indebted to analogy, evidently owing to the small extent of ideas in which comparison can be made useful. But at the same time analogy should be attended to, were it for no other reason than that, by making the invention of notation an art, the exertion of individual caprice ceases to be allowable. Nothing is more easy than the invention of notation, and nothing of worse example and consequence than the confusion of mathematical expressions by unknown symbols. If new notation be advisable, either permanently or temporarily, it should carry with it some mark of distinction from that which is already in use, unless it be a demonstrable extension of the latter. For this reason, had the separation of functional symbols in (22.) occurred often, we should have proposed to write $[\phi + \psi] x$ instead of $(\phi + \psi) x$.

The mathematical parts of a work may be divided into two species: firstly, the verbal explanations which accompany the passage from step to step; secondly, the actual expressions themselves. In the first, there are circumstances of convenience or inconvenience which are not of the same, or even of an opposite character, when considered with respect to the second. We think the frequent use of $>$, $=$, and $<$ in the first as rather injurious to what is there required, namely, the consideration of the different clauses of an assertion with respect

to their mutual relation in reasoning, though of course highly convenient when the whole of an expression is to be considered rather as to what is obtained than as to how it has been obtained. But the power of the printer's types is also to be considered in the first part, though not in the second, and the following suggestion is made on a circumstance which is of considerable consequence in the printing of mathematical works

The occurrence of fractions, such as $\frac{a}{b}$, $\frac{a+b}{c+d}$, in the

verbal part of mathematical works is a source of considerable loss of room, and creates an inelegant and even confused appearance in the printed page. It is very desirable, in every point of view, except the strictly mathematical one, that some method of representation should be adopted which does not require a larger space than is usual between two successive lines. At the same time, it is by no means of very great importance that the verbal part should entirely coincide with the mathematical part in notation, so long as the latter remains to preserve the usual conventions. The symbol $\div$ has been disused for a sufficient reason, namely, the number of times which the pen must be taken off to form it. This has been, and we imagine always will be, the cause either of abandonment or abbreviation. The question is, whether a new and easy notation could not be substituted; and it is desirable that it should be derived from analogy, such as (accidentally, we believe) does exist in $>$, $=$, and $<$. If we look at $\times$ and $+$, and observe that the first is made by turning the second through half a right angle, denoting multiplication, which is primarily an extension of addition in like manner as division is an extension of subtraction, we may thus invent the symbol $/$ or $\setminus$ to denote division, which is also the symbol of subtraction turned through half a right angle. If a/b were used to denote a divided by b , and $(a+b)/(c+d)$ to denote $a+b$ divided by $c+d$, all necessity for increased spacing would be avoided; but this alteration should not be introduced into completely mathematical expressions, though it would be convenient in particular cases.

A complicated exponent might be avoided by the use of the symbol λ^{-1} . The student would soon learn to consider $\lambda^{-1} \{ (a+bx)/(c+ex) \times \lambda a \}$ as meaning the same thing as

$$\frac{a+bx}{a^c+ex}.$$

If, in the course of investigation, some plan of this kind be not adopted, the expense of printing will place a limit to analysis. A work entirely devoted to the consideration of such expressions as the preceding might easily be doubled in size and price by the frequent occurrence of them in the text, or else rendered confused and unintelligible by successive abbreviations. Considering, however, that λ is an inverse symbol in the sense of (124.), and λ^{-1} a direct one, it would, perhaps, rather be advisable that some method of denoting an exponent should be adopted which does not raise the exponent above the symbol of the root. Either of the following might be proposed, the defect of them all being that they are not derived from analogy:

$a \wedge \{ (a+bx)/(c+ex) \}$ $a : \{ (a+bx)/(c+ex) \}$, or the like. But we do not advocate the introduction of these into purely symbolical expressions, any more than in the former case.

Addition to Article (185.) In common differential

Calculus of
Functions.

Calculus of Functions. equations, the arbitrary constant may have a discontinuous value. Take any arcs which join each other from different individual curves of the species $y = \psi(x, c)$, and the broken curve so formed is in every respect a solution of the differential equation, even at the points of union, where it may be supposed to have a discontinuous tangent. As the phrase "solution of a differential equation" may contain matter of opinion, and even of convention, we state that we mean by the solution of $F\left(x, y, \frac{dy}{dx}\right) = 0$ simply this: any line whatsoever, such, that if any values a and $a + h$ be taken for x , giving b and $b + k$ for y , the expression $F\left(a, b, \frac{k}{h}\right)$ may be made as small as we please by taking a sufficiently small value of h . This appears to us nothing more than the rejection of intermediate definitions, and the statement of the conditions in terms of fundamental notions; and,

if we are right in saying this, then it certainly follows that the constant may be discontinuous. By diminishing the intervals of continuity without limit, we get for a limit the curve whose equation is the *particular solution* of the equation in question; but which might be called the *limit of the discontinuous solutions*.

The same might be extended to the case of partial differential equations and their surfaces, by making the arbitrary functions discontinuous. If the representation of a discontinuous solution be made, the intervals of discontinuity determined, and then diminished without limit, the value of the discontinuous constant is then of the form $f(a)$, where a is the value of x at the last epoch of discontinuity. The diminution of the interval of continuity will then be found to require a process for the determination of the constant, which amounts to differentiating the original equation with respect to that constant, and eliminating in the usual way.

ERRATA

Article	Line
1,	7, from the end, <i>for enables read enable.</i>
5,	5, from the end, <i>for q is of 1 read p is of 1.</i>
25,	4, supply ψ , which has dropped out.
46,	in expression beginning $u_n = \phi^n x$ <i>for cⁿ read cⁿ</i>
126,	3, <i>for and then eliminating read so as to eliminate.</i>
129,	6, from the end, <i>for which, since read and.</i>
214,	3, from the end, <i>for y read $\frac{dy}{dx}$.</i>
244,	in expression for e^x <i>for $(= 1)^e = read = (1)^e$.</i>

CONTENTS.

** The Numbers refer to the Articles.

- | | |
|--|---|
| <p>Contents. (1.) Preliminary statement.
 (2.) Definition of <i>function</i>.
 (3.) Functions of the same form.
 (4.) Similar functions.
 (5.) Comparison of functional and arithmetical symbols.
 (6.) Comparison of functional and algebraical symbols.
 (7.) Meaning of, and question in the <i>Calculus of Functions</i>.
 (8.) Recapitulation of algebraical forms, and reason for it.
 (9.) Notion analogous to that of magnitude not existing as to functional symbols. Possible consequence.
 (10.) Possibility of <i>continuous change of form</i>.
 (11.) Definition of inverse functions; comparison with those of arithmetic. List of correlative terms.
 (12.) Numbers considered as functions of zero, and their inverses.
 (13.) Numbers considered as functions of 1, 2, &c. and inverses.
 (14.) Functional symbols, and applications.
 (15.) Notation of successive functions.
 (16.) Distinction of <i>algebraical</i> and <i>functional</i> inverses. Reasons for the notation employed.
 (17.) Extension and improvement of (12.) and (13.). Analogy of (13.) with Napier's method of considering logarithms.
 (18.) Distinction between <i>differential</i> and <i>functional</i> symbols.
 (19, 20.) Separation of <i>differential</i> symbols of operation and quantity.
 (21.) Calculus of ratios.
 (22.) Separation of <i>functional</i> symbols of operation and quantity; meaning of $(\phi + \psi)x$ where $\phi x = a(x + m) - m$ $\psi x = b(x + m) - m$ $(\phi + \psi)x = \phi x + \psi x + m$.
 (23.) Other cases of the same kind.
 (24.) Remarks on the possible extent of analysis.
 (25.) General view of the Calculus of Functions.
 (26.) Remarks on the invention of notation, (see also addition at the end).
 (27.) Remarks on the zero of operation. Functional distinctions between similar algebraical forms.
 (28.) Notation of functions of two variables.
 (29.) Formation of a functional equation to a given solution. Possible extent of its solutions.
 (30.) Formation of a functional equation with two given solutions.
 (31.) Instances of algebraically inverse functions.
 (32.) Formation of all the inverses of a given function.
 (33.) Distinction of <i>convertible</i> and <i>inconvertible</i> inverses. Every inverse is convertible with one or other of the forms of the direct function.
 (34.) Notation of inconvertible inverses.
 If ψ^{-1} be an inconvertible inverse of ψ, and if $\psi^{-1}\psi = \phi$, then $\psi\phi^n$ contains all the forms of ψ, and no others.
 (35.) <i>Periodic</i> functions. The preceding an <i>a priori</i> proof of their possibility.
 (36.) Instances of the preceding theorem.
 (37.) Limit of $\phi^n x$, a solution of $\phi x = x$.
 (38.) Case in which $\phi^n x$ has successive limits.
 (39.) Direct use of the preceding theorem in algebra.
 (40.) Indirect use of the same by Newton and Lagrange.
 (41.) Illustration in commercial and other problems.
 (42.) Geometrical illustration.
 (43.) Illustration from a tabulated function.
 (44.) $\phi^n x$ considered as a function of n. On the fractional values of n.
 (45.) Definition of <i>cognate</i> forms. Classification of functions with respect to their successive functions.
 (46.) Laplace's method applied to determination of successive functions.
 Instances: $\frac{2x}{1-x^2}$ and $2x^2 - 1$.
 (47.) Fractional and negative values of n in the above.</p> | <p>(48.) Reasons for supposing that the most general form of $\phi^n x$ is not obtained in the above (see remarks at the end).
 (49.) Definition of <i>derivative</i> functions. Their importance.
 (A) Derivatives of derivatives are derivatives.
 (B) The derivative of a compound function is compounded of the derivatives of its components.
 (C) The corresponding derivatives of inverse or convertible functions are themselves inverse or convertible.
 (D) The successive functions of a derivative are the derivatives of the successive functions.
 (E) The derivatives of periodic functions are periodic.
 (F) The derivatives of cognate functions are cognate.
 (50.) To find the derivative which one given function is of another.
 (51.) Instance: $a' + b'x$ made a derivative of $a + bx$.
 (52.) Illustration of functional distinctions; failure of the above when $b' = 1$.
 (53.) Separate process for $a + bx$ and $a' + x$.
 (54.) Determination of some periodic functions.
 (55.) Known and fundamental forms whose successive functions are cognates.
 (56.) Conditions of convertibility of $a + bx$, and $p + qx$.
 (57.) Conditions of convertibility of $\frac{a + bx}{a' + b'x}$, and $\frac{p + qx}{p' + q'x}$.
 (58.) Conditions of convertibility of $a x^b$ and $p x$.
 (59.) <i>Theorem</i>. If ϕx contain constants all values of which give convertible results, then $\phi^n x$ only differs from ϕx in the value of those constants.
 (60.) Determinations of $\phi^n x$ where ϕx is $a + bx$ or $a x^b$.
 (61.) $\phi^n x$ where $\phi x = \frac{a + bx}{a' + b'x}$. First method.
 (62.) Ditto ditto ditto. Condition of periodicity.
 (63.) Ditto ditto ditto. Second method.
 (64.) Ditto ditto ditto. Third method.
 (65.) The preceding expressions give algebraical and functional inverses.
 (66.) Extension to the case of periodic functions, and remarks.
 (67.) Equation to determine functional inverses of periodic functions.
 (68.) Case in which successive functions of a periodic function are functional inverses.
 (69.) Remark on the condition of impossibility in the Calculus of Functions.
 (70.) Expression of any periodic function of the <i>second</i> order as a derivative of any other of the same order.
 (71.) Attempt to express a non-periodic function as a derivative of a periodic function. Introduction of the sign of discontinuity.
 (72.) Remarks upon an anomaly mentioned by Mr. Babbage.
 (73.) A discontinuous function of a discontinuous function may be a continuous function.
 (74.) Distinction between discontinuity of <i>value</i> and discontinuity of <i>law</i>.
 (75.) Expression of a periodic function of <i>any</i> order as a derivative of any other of the same order.
 (76.) On convertible functions. Successive functions of all kinds convertible. Limitation of the converse of (49.)
 (77.) General method of finding functions convertible with a given function.
 (78.) Derivatives with respect to a convertible function are the same as the function itself.
 (79.) Compounds of convertible periodic functions are periodic.
 (80.) Definition of symmetrical functions.
 (81.) Symmetrical functions of x and y are <i>unambiguous</i> functions of $x + y$ and xy.
 (82.) If a function be symmetrical with respect to two pairs out of three quantities, it is symmetrical with respect to the third.
 (83.) Representation of a symmetrical function of three variables.</p> |
|--|---|

- Contents.** (84.) A function of n variables is symmetrical with respect to every pair, if it be symmetrical with respect to $n-1$ pairs, containing every letter, once at least, among them.
- (85.) If $\phi x = x$, a symmetrical function of $x, \phi x$, &c., is not changed when x becomes ϕx .
- (86.) Remarks on the converse of the preceding.
- (87.) Form of functions which are periodical in form when ϕx ($\phi x = x$) is substituted for x .
- (88.) Another species of symmetry.
- (89.) More general solutions of $\psi \phi x = \psi x$, where $\phi^n x = x$.
- (90.) Remarks on a theorem of M. Abel. For further remarks, see (301.)
- (91.) Disjunctive and periodical symmetry.
- (92.) Deduction of $\phi ax = \phi x$ from $\phi^n x = \phi x$.
- (93.) Notation of symmetrical functions. Meaning given to ξax .
- (94.) Classification of functional questions:—
1. Where all the forms are known, or *common algebra*.
 2. Where some relations are to be satisfied independently of others.
 3. Where relations between known forms are to be satisfied.
- (95.) Comparison of the second and third cases. Solution of a question of the second class, where the subjects do not enter symmetrically.
- (96.) Cases of the second class reducible to $Fx = Fy$, and symmetrical cases not so reducible.
- (97.) Solution of $\phi x + \phi y = \phi(x+y)$, and theorems in connection with it.
- (98.) Equation containing one unknown form; reduction by means of a particular solution to a certain number of subordinate equations.
- (99.) Limitation of the arbitrary functions in the last solution.
- (100.) Laplace's method by equations of differences. For the consideration of Sir John Herschel's extension, see (252.)
- (101.) Recapitulation.
- (102.) Solution of $\phi ax = \phi x$:—
1. If $\phi^n x = \mu (\cos 2\pi n)$ and $\phi^n y = \nu (\cos 2\pi n)$ give $n = Ax$ and $n = By$ then $\phi x = B^{-1}Ax$ solves $\phi ax = \phi x$.
 2. If $\phi^n x = c$ give $n = Ax$, then $Aax = Ax - 1$ and $\xi ax = \nu (\cos 2\pi Ax)$.
- (103.) Solution of $\psi ax = \psi x$.
Solution of $\psi ax = \psi x + c$ and $\psi ax = (\psi x)^c$.
- (104.) Ax called the *exponential inverse* of ax . Deduction of exponential inverses of the derivatives of ax .
- (105.) On the equation $Aax = Ax + m$.
- (106.) Application to $a + bx$ and $a + x$.
- (107.) Application to $(ax)_b$ and ax .
- (108.) Application to the two forms of $\frac{a+bx}{a'+b'x}$.
- (109.) Application to the derivatives of ax .
- (110.) Connection with preceding results.
- (111.) Application to a case of $a + bx + cx^2$.
- (112.) Recapitulation of derivative forms for which $\phi ax = \phi x$ can be solved.
- (113.) Determination of $\phi^n x$ from Ax .
- (114.) Convertibility of successive functions.
- (115.) Application to finding functions convertible with a given function.
- (116.) Extensive class of algebraical functions which are convertible.
- (117.) Determination of a function which is changed in a given manner, by simultaneously changing x_1 into $a_1 x_1$, x_2 into $a_2 x_2$, &c. Want of generality of the solution.
- (118.) Example and remarks.
- (119.) Extension of the form of ξax .
- (120.) Reason for supposing the simultaneous equations $\phi ax = \phi x$, $\phi \beta x = \phi x$ to be generally insoluble with existing forms.
- (121.) Want of a process for the above analogous with one of algebra.
- (122.) Case in which the above admits of general solution.
- (123.) Meaning of the word *transcendental*; remarks on the progress of transcendentals.
- (124.) Direct and inverse functions.
- (125, 126.) Different methods of forming functional equations.
- (127, 128.) Remarks on their transcendentals.
- (129.) Exponential inverse of periodic functions of the second order, and reduction of $\xi \phi x$ thus obtained to the form in (86.)
- (130.) Extension to periodic functions of higher orders.
- (131, 132.) Methods of changing functional equations into others, by substituting functions of x for x , and also by destroying a known term by means of a particular solution.

- (133.) There cannot be two particular solutions of $\phi ax = \beta x + \phi x \gamma x$ independently of ξax . See also (176.)
- (134.) Except where ax is periodic there is one general *algebraic* solution of the preceding.
- (135.) Method of destroying the known term of a general functional equation of the first degree.
- (136.) Method of obtaining a particular solution, where the known functions are successive functions of a periodic function.
- (137.) Solution of $\phi ax = \beta x + \phi x \gamma x$, where $\phi^n x = x$.
- (138, 139.) Cases in which an arbitrary function enters the solution.
- (140.) Solution of $\phi ax = c \phi x + c'$ and instances.
- (141.) Solution of $\phi ax = \phi x + c'$.
- (142.) Solution of $\phi(x^b) = p x^m \phi x$.
- (143.) Solution of $\phi^n x = \frac{d^m \phi x}{dx^m}$.
- (144.) Definition of *compound* functions and *converse* functions.
- (145.) Given the results of a compound operation of the second order and of its converse, required the operations themselves; solution of $\phi \psi x = \cos a \cos^{-1} b$ $\psi \phi x = \cos b \cos^{-1} a$.
- (146.) Solution of $\phi \psi x = \theta \alpha \theta^{-1} x$ $\psi \phi x = \theta \beta \theta^{-1} x$.
- (147.) Solution of $\phi \psi x = ax$, $\psi \phi x = x^b$.
- (148.) Deduction of the solution of $\phi_{-1} \phi x = ax$.
- (149.) The same method applies to finding convertible functions.
- (150.) Determination of a function which is inverted with respect to one variable by the interchange of two others.
- (151.) Converse functions are simple derivatives of each other.
- (152.) $\phi^n x = ax$, distinction between it and $\phi^n x = x$.
- (153.) Extensions of particular solutions of $\phi^n x$ by use of general forms of ax . Example, $\phi x = x^a = ax$.
- (154.) Another example, $\phi x = \tan 2 \tan^{-1} x$.
- (155.) Method by which arbitrary functions might be supposed to be introduced, and remarks.
- (156.) General method of solving $\phi^n x = a^x x$ shown to lead to the method of (113.)
- (157.) Indefinite expression for $a^{\frac{1}{n}} x$ by means of the exponential inverse of x .
- (158.) Conditions of verification of the preceding.
- (159.) $\phi^n x = ax$, where ax is periodic.
- (160.) Method for $\phi^n x = a^n x$.
- (161.) Deduction of successive functions as derivatives of each other and of their functional inverse. Connection between the deriving functions in the two cases.
- (162.) Mr. Babbage's method of finding $\phi x = \xi ax = \xi \beta x$.
- (163.) Difficulties in the way of its application. Its failure when convertible inverses only are employed.
- (164.) Application to the case of $\phi x = \xi m x = \xi n x$. Ambiguity of the solution deduced. Reason why there can be no direct and unambiguous solution.
- (165.) Application to $\phi x = \xi x^m = \xi n x$.
- (166.) General method of the last two instances.
- (167, 168.) Solution of the problem in the case of periodic convertible functions.
- (169.) General insolubility in other cases.
- (170.) Solution of $(\psi a)^2 \psi \beta x a = \gamma x$.
- (171.) Method of obtaining periodic functions.
- (172.) Solution of $a_1 x \phi_1 x + a_2 x \phi_2 x + \dots = \gamma x$ with rational forms of $\phi_1 x, \dots$, when $a_1 x \div \gamma x, \dots$ are rational.
- (173.) Solution of $\psi a(x, \psi x) = x$.
- (174.) Solutions of $F(x_1 \psi x, \psi^2 ax, \dots) = 0$.
 $F(x_1 \psi x, \psi^2 x, \dots) = 0$.
and remarks upon the character of the solution.
- (175.) Direct reduction of a functional equation to an equation of differences by means of the exponential inverse.
- (176.) Process of solution of $\phi x + \beta x \phi ax = \gamma x$. Only one arbitrary function can enter the general solution.
- (177.) There may be in appearance more than one algebraical form of particular solution.
- (178.) There may be an infinite number of discontinuous solutions. Method of forming them.
- (179.) The arbitrary constant of the equation may be discontinuous.
- (180.) Remarks on the arbitrary constant.
- (181.) Conditions of the discontinuous constant.
- (182.) Application to some examples.
- (183.) *Single values* of the discontinuous constant.
- (184.) Remarks on developement into series.
- (185.) Developements represented by functional equations. A discontinuous constant may enter an equation of finite differences. See remark at the end on differential equations.

Contents.

- Contents. (186.) Solution of an algebraical difficulty by means of the discontinuous constant. See also (199.)
- (187.) Reference to the usual manner of explaining the same.
- (188.) Another instance of the same envelopment of a continued product.
- (189.) Remarks on what precedes.
- (190.) Abbreviated method of reduction to an equation of differences.
- (191.) Division of functions.
- (192.) Solution of $\phi x = \beta x + \gamma x \phi \alpha x$ where αx is periodic.
- (193—197.) Case in which the preceding fails, and way of obtaining a solution.
- (198.) Second case of (191.)
- (199.) Reason why a discontinuous constant and not a variable function enters in (186.)
- (200.) Application to equations of differences.
- (201.) Soluble cases of the general equation of the first order.
- (202.) The solution may be a definite integral.
- (203—208.) Further cases of the same kind.
- (209.) Cases in which a constant is required to be a specific value of the solution.
- (210.) Comparison of Laplace's method with that in (175.)
- (211.) Definition of the *differential inverse*. Comparison of an equation containing the differential coefficient of an unknown form with the cases already solved.
- (212.) Extensions of form of $\xi \alpha x$, and remarks on the evasion of mathematical difficulties.
- (213.) Particular case connected with the preceding remarks.
- (214.) On the equation $\psi \frac{dy}{dx} = \frac{d}{dx} \psi y$
- (215.) On the equation $\psi \alpha x = \frac{d\psi x}{dx}$
- (216.) Mr. Babbage's reduction of $\psi \alpha x = \frac{d^m \psi x}{dx^m}$ to a differential equation when αx is periodic.
- (217.) Alternative to the equation in (215.) having an infinite number of solutions.
- (218.) Partial differential equation to $z = \psi \phi^n x$.
- (219.) Relation between $\alpha^m x$ and differential coefficients when αx is periodic, and consequences.
- (220.) *Complementary* property of periodic functions.
- (221.) General method of reducing a functionally periodic differential equation to a common differential equation.
- (222.) Case in which the preceding may fail, and additional method.
- (223.) Example of the preceding method, and remark on differential equations.
- (224.) Instance of finding the functional equation to which a given series is a solution.
- (225, 226.) Fourier's theorem.
- (227.) Method in which the integrations must be made in the preceding.
- (228.) Demonstration of Fourier's theorem by M. Deflers.
- (229.) Demonstration of Fourier's theorem by M. Poisson.
- (230—233.) Examples of periodic equations, and proofs that the appearance of more than one arbitrary function is not a reality.
- (234.) Incongruous periodic equations.
- (235.) Extension of the method in (197.) to the complete equation.
- (236.) On the equation $\sum a_n \phi m_n x = \sum k_n x^n$.
- (237.) Solution of a difficulty with regard to $\phi x \cdot \phi \alpha x \dots = \beta x$.
- (238, 239.) Further remarks on the derivation of periodic functions.
- (240.) On $\sum \psi \alpha^n x = 0$, $\alpha^n x = x$.
- (241.) On $\sum \beta_n x \psi^n x = 0$.
- (242.) Example of the preceding.
- (243.) Equation containing the continued product of $(\psi \alpha^n x)^m$.
- (244.) Remarks on the case where direct functions have more values than one, and application to some proposed theorems on the logarithms of unity.
- 245.) Historical remarks on algebra, as the predecessor of the Calculus of Functions.
- (246—249.) Early history of the Calculus of Functions. Method of Monge for finding functional equations for the arbitrary functions of the Integral Calculus.
- (250.) Monge's method of extending finite differential into functional equations.
- (251.) Laplace's method. Lagrange's theorem.
- (252.) Extension of Laplace's method.
- (253.) Distinction between dependent and independent coexistence.
- (254.) Dependently coexistent differential equations require an arbitrary function.
- (255—258.) Restrictions on the method of (252.)
- (259.) On the possible arbitrary constants in the solution of a functional equation.
- (260.) Proof that a differential equation of the first degree cannot have two arbitrary constants.
- (261.) Failure of the preceding in the Calculus of Differences.
- (262.) Functions of two or more subjects. Successive substitution.
- (263.) Linear substitution.
- (264.) General and simultaneous substitution.
- (265.) Instance of a problem which might be proposed.
- (266.) Zero-functions of general substitution.
- (267.) Convertibility of simultaneous substitutions
- (268—274.) Zero-functions of simultaneous substitution.
- (275.) On functions with constant or indefinite zero-functions.
- (276.) Definition of the *modulus* of a function, and instances.
- (277.) On functions with moduli of indefinite forms.
- (278.) Sir J. Herschel's method of solving questions bearing analogy to the preceding.
- (279.) *Simultaneous derivatives*, definition and properties.
- (280.) All derivatives of zero-functions are zero-functions.
- (281.) On simultaneously periodic functions.
- (282.) On simultaneous substitution.
- (283.) The same when the zero-functions are indefinite.
- (284.) Property of zero-functions.
- (285, 286.) Method of making one function a derivative of another.
- (287.) On simultaneous inverses.
- (288.) Extension of (34.)
- (289.) On *internal* substitution.
- (290.) Transcendentals in the solution of an equation.
- (291.) Simplification of notation.
- (292.) Solutions of $\phi(\alpha, \beta) = \phi$.
- (293—296.) Examples. *A priori* deduction of the form of symmetrical functions.
- (297.) Remarks on the distinction of internal and external substitution. New case solved by the latter
- (298.) Solution of $\phi(\alpha, \alpha) = \phi(x, y)$.
- (299.) On the inverses of internal substitution.
- (300, 301.) Extension of (197.)
- (302.) Limitation of Abel's theorem.
- (303.) Developement of the objection in (90.)
- (304.) Deduction of simultaneously periodic functions.
- (305.) On simultaneous convertibility.
- (306.) Solution of several equations.
- (307.) Mr. Babbage's method of obtaining $\xi(\alpha, \beta)$.
- (308.) Distinction between disjunctive and simultaneous solutions of $\phi(\alpha, \beta) = \phi(x, y)$.
- (309.) On $\phi(\alpha x, \alpha y) = \phi(x, y)$.
- (310.) Recapitulation of functions with constant moduli.
- (312—316.) Solution of equations.
- (317.) Solution of equations, including known differential coefficients.
- (318—323.) Examination of equations involving non-simultaneous successive substitutions.
- (324, 325.) Cases in which a relation between moduli only is given for solution.
- (326—328.) Cases involving differential coefficients of unknown functions.
- Remark alluded to in (48.) On functional inverses.
- Addition to (26.) On the invention of notation.
- Addition to (185.) On the discontinuous solutions of differential equations of one variable.

THEORY OF PROBABILITIES.

Theory of
Pro-
babilities.

(1.) IF it had been our only view to write a Mathematical Treatise on the Theory of Probabilities, in the common sense of the word, we might at once have proceeded to lay down the leading principles of the science. But we have a prior duty to perform, arising out of the various opinions which are held, even by mathematicians, upon the nature of this subject and the extent to which it is to be considered as a guide to truth. No part of mathematics or mathematical physics involves considerations so strange or so difficult to handle correctly, and there is no subject upon which opinions have been more freely hazarded by the ignorant, or rational dissent more unambiguously expressed by the learned. One, an amusing and popular writer, but all whose works put together do not, from the nature of his subjects, necessarily imply an acquaintance with so much as the four rules of arithmetic, calls it, from his own stores of knowledge, an "elaborate delusion;" a second, a mathematico-physical writer of the first character, while he appears to value the results* of the theory, seems not convinced by their evidence. If we appear severe in calling attention to the maxim of Horace,

Theory of
Pro-
babilities.

Intererit multum, Davus ne loquatur, an heros,

we consider that we are justified, with respect to the former writer, by his presumptuous opinion upon an unstudied science; and with respect to our subject, by the necessity of making it clear that we do not intend these preliminary observations as an answer to every objector, but to those whose knowledge places them above that position, *inter Davum et heros*, at which the formation of an independent opinion begins to deserve respect.

At the outset, we concede this point,—that no *primary* considerations connected with the subject of *probability* can be or ought to be received if they depend upon the results of a complicated mathematical analysis. The latter, if it be any thing else but an abbreviator of operations, is indeed an "elaborate delusion," and has often been made such: for instance, by an earlier school of writers on astronomy, who contended that the form of a planet's orbit should be *deduced* from geometry only. But the concession will not amount to much; for it would equally be made to a person who was desirous of knowing the number of cannon balls in a square pyramid of 50 rows high. All the mathematics in existence will teach no more about the number in the pile than can be found by counting; and if the latter should be preferred to a rule by a man who felt no assurance of the truth of algebra, a mathematician who stood by, while he admitted the propriety of not trusting an unknown science, would congratulate himself that he could certainly know how (by taking one-sixth of the product of 50, 51, and 101) to produce the result in a minute which would cost the other an hour. But if he who counted called algebra a delusion, he would not the less deceive himself because it is possible to do without algebra.

(2.) If, upon very simple and obvious principles, we can show that a certain problem about ten subjects requires the multiplication of all whole numbers up to ten, and so on, it will be equally clear as to principle that the same problem in the case of ten thousand subjects will require the multiplication of all the whole numbers up to ten thousand; and it will at once be decided that the first is within the compass of a reasonable industry, and that the second is not. But this decision will not be correct; for the higher branches of mathematical analysis show a method of finding, not exactly but approximately, the continued product of all numbers up to any given limit, with very little trouble. That is, suppose the product were really

2376948 (with 10,000 intervening digits) 16432,

it would be easy to find, as a mathematical approximation,

2376948 (with 10,000 intervening ciphers) 00000

the second of which, though it differ enormously from the first, does not differ from it by a millionth part of itself; or, if money were the thing in question, by the tenth part of a farthing upon a hundred pounds. This will be sufficient for practical applications, and every reader may now see in what way the higher branches of mathematics are concerned in our present subject. They are the abbreviators of long and tedious operations, and it would be perfectly possible, with sufficient time and sufficient industry, to do without them in every case.

The preceding consideration naturally divides the subject into two parts. In the first, we have to consider first principles and easy applications: in the second we have to apply to *the same first principles* the heavy artillery of mathematical analysis, in the case of problems which involve large numbers. We shall therefore proceed at once to the consideration of the first principles.

(3.) The word *probable*, as commonly applied to a coming event, indicates many different degrees of that feeling with which the mind looks at the prospects of the future, and which depends upon the habits derived from looking at the past. Mark the impression which the two following assertions produce upon the mind—"The sun will not rise to-morrow," and—"The sun will rise to-morrow." The first is so excessively improbable as to be practically absolutely deniable; and the second approaches so near to certainty as to admit of positive affirmation. Take any third assertion, and compare it with the preceding two as to the effect it produces;

* "I will not undertake to say that I think the method of *minimum squares* is unexceptionable in all its applications, or that I attach much more than a relative value to the estimation of *probable errors*. But I think there is no doubt that these methods have contributed much to the accuracy of modern astronomy, &c."—*Rep. Brit. Assoc.* vol. i. p. 165.

Theory of
Pro-
babilities.

if it be such as we should incline to class with the first, rather than the second, we at once call it improbable. If we should rather class it with the second, we call it probable. If we are in doubt which to do, we say it is neither probable nor improbable. These words do not apply to any thing in the event itself, but to a state of mind produced by its contemplation; which must be remarked, because it is at once the answer to many objections, particularly those of a religious character. The theory of probabilities has been sometimes asserted to refer to chance the settled plan of God's providence, so that it cannot reasonably apply except on the supposition that events happen by chance. This entirely proceeds on the notion that we are looking at the event itself that is to come, and considering its probability as a kind of quality which it possesses, in the same sense as matter is said to possess extension, weight, &c. Now even if we granted the well-known theory of Berkeley, and referred all these qualities to the mind, even then we say that we must refer the notion of *probability* to the mind of the observer in a closer degree, as being one which belongs to subjects on which the knowledge and habits of individuals make differences. Take Berkeley's theory for granted, and carry two men to a room in which are two boxes, one small and ribbed with steel, the other large and roughly put together. Let these men have come from the two most opposite points of the earth in manners and customs, yet they will immediately, when asked, point out which is the larger of the two boxes: if they are both sane, disagreement will be impossible. Now produce a piece of gold, and ask which of the two boxes is filled with that substance. One has seen gold, and knows its value, and also that it is rarely collected in large quantities, or placed in insecure receptacles. He would say that most likely the smaller box, if either, is full of gold. Or he may think that the question and circumstances are so extraordinary, that the former would not have been put unless this case had been a departure from ordinary rules, and may therefore pronounce for the larger box. In either case it is clear that the probability or improbability is the consequence of a state of his own mind, or of an impression existing in himself, in a sense which cannot be, in any view of the case, applied to the extension of the two boxes. If the other man knew nothing of gold, he would not be able to bring his mind to either of the preceding conclusions, in preference to the other. What we mean, then, by an event being probable or improbable, is this; that with regard to that event the mind of the spectator is in a state of disposition either to doubt or believe its happening; which evidently depends in no way upon the event itself, but upon the whole train of previous ideas and associations which the mind of the spectator possesses upon such circumstances as he thinks similar. Therefore it is wrong to speak of any thing being probable or improbable in itself. The same thing may be *really* probable to one person and improbable to another. And thus men may be justified in drawing different conclusions upon the same subject.

(4.) With regard to the question above mentioned, that the theory of probabilities makes events happen by *chance*, let us consider for a moment what chance is. In common phrase, it means that we see no reason why events should have happened one way rather than another. And, if this be chance, events *do* happen by chance, for they certainly do happen when we can see no reason why they should not have been otherwise. But if by chance be meant that events happen when there is no reason why they should have happened, it is soon shown that *the rejection of the notion of chance is absolutely necessary to every theory of probabilities*. A meets B in Cheapside to-day at ten o'clock; he says, I met B *by chance* to-day, by which he means that he can show no reason why B should not have been any where else. He meets B to-morrow at the same place and hour; he immediately begins to suspect that the meeting was not *the work of chance*, but that B must have some *reason* for being in Cheapside at that hour, which renders it *probable* that he will meet him again to-morrow. The very notion of probability only exists as attached to some cause why the thing should happen; it increases with our degree of belief in a cause, and finally becomes certainty when we know there is a cause.

Suppose a native of a foreign country in which there are no forms of trial, he himself not understanding English, were to spend a week in observing the process of one of our criminal courts. He would certainly be able to find out that upon the pronouncing of the words "guilty" or "not guilty" would depend whether the prisoner would be led away in custody, or allowed to go free. Perfectly convinced as he would be of the existence of a cause for all the ceremonial he saw, what guide would he have to determine the manner in which causes act, so as to infer the result in any particular case? It is to him a problem of *chance*, as opposed to the proper object of inference. But it is not a problem of *chance* as opposed to the result of order, or of cause and effect, for he sees enough to assure him that there are reasons, but not enough to determine what they are. He is exactly in the situation in which we stand with regard to many phenomena in the moral order of things.

The word *chance* has very often been confounded with *probability*, which is the only excuse for the preceding objection. To take an ambiguous and metaphysical word, without inquiring in what sense it is used, namely, *as opposed to certainty*, and to argue thereupon as if it entered *as opposed to Providence*, is a logic every way worthy of the opponents of the sciences: we commend their fairness in not using one to the destruction of another. If we use the word chance, it will be in a common signification, that of a very low degree of probability. "It is a *mere chance* that I go to-morrow, but I shall *probably* go the day after."

(5.) The notion of *probable* is one which admits of the relation of more or less. It is *more* probable that rain will follow a fall of the barometer than a rise, *much* more probable. So far every one admits; but to the question *how much* more probable, no immediate answer can be given. But this question must be answered before mathematical reasoning can apply. Magnitude, the object of possible mathematics, is that to which the terms more or less can be applied. But *actual* mathematics cannot be useful unless we can, having applied the term *one* to a certain quantity, proceed to ascertain whether another quantity ought to be called 2, 3, 4, &c., or any intermediate fraction. If we had the same sort of knowledge of mind that we have of matter, we might then know how to weigh the force of one impression against another, just as we might in such a case make a numerical comparison of the proportions in which two men possess moral qualities of any given kind. But this is above our reach, and if we attempt to submit any state of mind to numerical analysis, it must be by means of some external manifestation of the feeling; and let it be remarked that there must be at some point or other of the passage

Theory of
Pro-
babilities.

Theory of
Pro-
babilities.Theory of
Pro-
babilities.

a convention as to the method of measurement to be adopted. It may be concealed, but still it exists; it may be considered by one as dubious, by another as nearly certain, but still it cannot be denied to be one of that class of truths which, if admitted at all, must be admitted upon the species of conviction in our own mind, which comes from its own observations, according to one sect of metaphysicians, or from its own fundamental notions, according to another, without the possibility of receiving absolute demonstration.* For it is easier to decide upon the notion itself, in the sense in which the words strike us, than to find out whether he who would demonstrate it means the same thing by the words he uses as we do ourselves; to say nothing of any other difficulty.

The notion we mean is this; we assert and require it to be granted that the feeling of probability or improbability is of the same kind, whatever may be the event in question; that the probability we attach to one event, say a fact in history, *bears a ratio* to that which we attach to any other of another kind, say the gaining of a prize in a lottery. If we were to state to any person that an advantage depended upon his guessing correctly which was the greater, the distance from London to York, or the weight of St. Paul's, he would at once answer that the question was inconceivable, for that greater or less, which apply to one length against another, or to one weight against another, do not apply to a length against a weight. By going to that universal referee, abstract number, he could tell whether the *number* of tons in St. Paul's exceeds or falls short of the number of feet in the distance from London to York, and might make this the subject of a guess with a certainty of being either right or wrong. But if his ideas on the subject be clear, he would answer that he could neither be said to be right or wrong in declaring that the length in question was greater or less than the weight. And with regard to immeasurable subjects, he would answer in the same way; for instance, if he were asked which was the greater, the musical power of Handel, or the mathematical power of Lagrange. Grant that the former was ten times the average power of musicians, and the latter nine times that of mathematicians, and the relation of these abstract numbers suggests an answer; but put one concrete quality by the side of the other for comparison, and the answer will appear impossible.† But with regard to probability, or the state of mind which produces it, if we were empowered to put the following question, we conceive that there would be but one answer. "There are two events, one past and one to come, on neither of which are you in possession of total and mathematical certainty. The first is the execution of Charles I.; the second is the drawing of a white ball from an urn which contains one white and ninety-nine black balls. Choose one of these, and let your interest in any way depend on your deciding rightly the one you select: would you rather the safety of your life should depend upon your saying correctly whether Charles I. was or was not executed, or upon your drawing the white ball, and not one of the black ones?" Every one who could make his decision under these circumstances with satisfaction to his own mind, does *bonâ fide* admit what he meant he should admit, though the words in which he would state the principle might be different from ours. He judges the impressions produced on his own mind by different events, with reference to probability, proper objects of comparison, which he would make numerical comparison, if he knew how. And he would not call his answer a guess, but the result of a rational process. We might put a case of doubt, as for instance the executions of Charles I. and Lord Strafford, on which the decision would be, that no man would sleep less soundly if his life depended on answering rightly as to the truth of either. But if a sufficient authority were to assure him that one of the two executions was a total fiction, the difficulty he would be under to say which it was would be a proof of his thinking the probabilities of the two proper objects of comparison, and of a defect of means only to decide, not of power to use them if he had them.‡

(6.) But still this question remains as much unanswered as before,—How are we to measure and compare the impressions on our own minds made by different events? To this we answer as follows. In the first place, there are two distinct considerations, of which the preceding is only one. They are

1. *How are we to measure and compare the impressions made on our minds by different prospects?*
2. *What is the rational way of using such measures or comparisons when we have got them?*

It is evident that we are liable to mistake in answering both questions; we may use a right measure wrongly or rightly, and we may obtain a wrong measure, and use it either wrongly or rightly. The first is a psychological question, the second a mathematical one. Any given answer to the first may admit of dispute; there is no fear of mathematics failing us in the second. But among the objectors we find a class who have confounded the two

* We are of no sect of metaphysicians, at least not decidedly enough to make their opinions the basis of a science. If the language used in any part of this Treatise appear to favour one opinion or another, we take this opportunity of saying that we merely endeavour to represent phenomena, as they appear to us, in phrases generally used, and to which of course any theory may be applied.

† Observe that this is distinct from the question, which should we admire most? just as the question which cost most, the road to York, or the structure of St. Paul's? is answerable, and distinct from the one first put.

‡ The mathematician may say, that the certainty of any one event is a unit of its kind, and that the probabilities are abstract fractions of it; whence he may argue that we are only engaged upon the same sort of question as that of comparing a number of miles with a number of tons. If we concede this, which we have no objection to do, our distinction must be thus stated: that we are constantly in the habit of doing the first, whether we will or no, but not the second.

A distinguished writer says, "Numerical measures are applicable only to things which depend on time or space. By such elements, in such a mode, how are we to estimate happiness and virtue, thought and will? To speak of a formula with regard to such things would be to assume that their laws must needs take the shape of those laws of the material world which our intellect most fully comprehends. A more absurd and unphilosophical assumption we can hardly imagine. We conceive, therefore, that the laws by which God governs his moral creatures, &c." The italics are of our own making: and unless for "are we" we should read "is God," here is as direct an argument from privation of human power to privation of divine power as the forms of language will allow. And from the preceding sentence to our quotation (where the author is speaking of the possibility of the divine intelligence embracing the whole of the mechanics of the universe in one formula) it is obvious that our proposed alteration, or something like it, must have been meant. But this renders the whole a train of reasoning we cannot admit. Our answer to it would be contained in maintaining the following proposition: whenever the terms *greater* and *less* can be applied, there *twice*, *thrice*, &c. can be conceived, though not perhaps measured by us.

Theory of
Pro-
babilities.

questions, and have treated those who answered the second with disdain, upon arguments which prove that they thought it was the first which was attempted to be answered. This has happened particularly with regard to the speculations of Condorcet and Laplace upon the *mathematical* consequences of testimony. These objectors rank with those who dispute Kepler's laws because they cannot conceive attraction. But is it not evident that the second question, properly answered, closes up one source of mistake, whether the first be answered or no? How many, for example, of controversial writers on all subjects have comprehended that an inference from ten equally probable premises is not itself as probable as each of the premises. Because they did not know which was two and which was three, they have declined to admit that two and three make five.

Theory of
Pro-
babilities.

(7.) We shall now enter upon the first of the two questions. Seeing that the impression which a prospect produces upon our minds is the result of a large combination of circumstances, and that it is almost impossible for us even roughly to estimate the share which each takes in the united effect, our plan must be to make circumstances for ourselves, in which all collateral considerations shall be avoided in every case in which we have any reason to suppose they exert influence. Our object being, not to investigate circumstances themselves, but the effect they produce upon our disposition to doubt or expect, we shall invent the former and examine the latter. It must of course depend upon our degree of knowledge; and, in examining the effects of partial knowledge, we must state specifically the case we wish to try. Let us suppose it to be the following. We are in a room which contains an urn. Our knowledge of its contents is limited to this much: that if it contain any thing it is a ball or balls, either black or white, the conditions being interpreted without reference to any thing in the form of expression which may make any one conceivable case more prominent than another. It is clear that a single drawing must either produce nothing, or a white ball, or a black ball. To which supposition do our minds incline? To neither; for there is no circumstance which should turn the scale. If we think for a moment that the possible contingency of ninety-nine black and one white would make a black ball very likely, we instantly remember that the equally possible contingency of ninety-nine white and one black would make a white ball as likely. Caprice, as we should call it, might decide, but this could be only interpreted as the silent whisper of some habit or association of ideas, strong enough to turn a point so nicely balanced, but too weak to be audible by itself; just as the thousandth part of a grain will turn the scales of a chemist, itself being hardly seen and not possible to be felt. But we have supposed ourselves to come into the room without this thousandth part of a grain: we are not asking whether we ever *could* come without it, but what should we do *if* we could. We should not guess in favour of either.

We have required as an axiom the right to consider the impressions produced by different events as comparable in magnitude, and still more, of course, those produced by the different contingencies which the same prospect presents. *Mathematical certainty* (a thing perhaps impossible in the strictest sense) is the terminus or limit towards which our impressions approach as our knowledge becomes greater and greater, and is never attained as long as any doubt whatsoever remains. *Practical certainty* is that high degree of probability on which the mind acts at once, without thinking the counter-probabilities sufficiently large to be taken into account; and it depends upon the character of the individual. A question must be raised as to whether the impression of probability is a fraction of the impression called certainty which varies directly as the numerical amount of circumstances. The mathematician will understand us; but we may put it in this way.

1. To take one extreme case: suppose the life of an individual depending upon his guessing the result of one drawing of the urn already mentioned; suppose a second power overruled the first, and allowed him to name two cases, granting his life if the result fell within either; suppose a third power, making one more interference, and allowing him to name three cases; which would render certain the safety contingent in the first two cases: for he would of course name as his three allowed cases, *none*, *white*, and *black*, one of which it must be. Now the impression of obligation and of benefit received being proportional, would he feel more obliged to the second or the third? We suppose it would be answered without hesitation to the third; that is, these apparently equal steps to certainty would seem to add very unequal benefits, or very unequal senses of the probability of escape. Those who have considered this subject (and appeal to experience is necessary) have never doubted this phenomenon of the human mind; and under the name of *moral probability* have distinguished that impression which has direct reference to the particular effect of the event upon the individual. Thus they have considered that two men who play for equal stakes with very unequal fortunes are not, in fact, (though staking the same sums of money,) playing equally, and ought not to regard the result with impressions of the same force.

2. Now let every thing remain the same as before, except that the risk incurred is nothing at all, or of too small consequence to upset the mind by hope or fear. The question now is, would the second or the third power be held to have conferred the greater benefit? We are afraid that in this case the third would be charged with having *spoiled the fun*. But supposing it a matter of common life, and all stimulus of gambling out of the question, it seems to be the general opinion of mankind that the second and third powers would have acted equally: this is proved by the uniform consent of those who have written on the subject, and the almost uniform reception of their opinions. To some sort of consent of mankind every theory on this subject would be absurd if it did not appeal. What would be a theory of *probabilities* (that is, of the manner in which certain impressions of the human mind are to be compared) which did not appeal to the voice of the majority? The question at issue is, Does the preceding statement fairly represent that voice? Or it might even be put in this way: Would opinions on one side and the other be such as to make the preceding notions a fair average statement? For even in that case it would be the best probable rule of action.

(8.) We now lay down the following definitions.

1. *Moral probability* is the impression existing with regard to the happening of an event depending upon the constitution of the individual, his knowledge of the circumstances, and the effect the event will produce.

2. *Mathematical probability* is the moral probability in that case, and in that case only, in which the mind is

Theory of
Pro-
babilities.Theory of
Pro-
babilities.

disposed to consider equal successive changes of favourable circumstances into unfavourable, or *vice versa*, as of equal importance; not regarding certainty as possessing any peculiar value, more than would be derived from its being the sum of certain probabilities, and therefore of course larger than either.

3. In future, *probability* means *mathematical probability*, unless the contrary be specified.

4. Absolute certainty *against* an event is denoted by the degree of probability 0 for it; and *vice versa*. Absolute certainty for an event is denoted by the degree of probability 1; and all intermediate degrees of probability are denoted by fractions, as follows.

5. The probability *in favour* of an event is such a proportion of 1 as the number of *favourable* cases is of the whole: all the cases being considered as equally probable. Thus the probability of drawing a white ball where 3 out of 10 are white is $\frac{3}{10}$.

6. The probability *against* an event is such a proportion of 1 as the number of *unfavourable* cases is of the whole. Thus the probability against drawing a white ball in the preceding case is $\frac{7}{10}$.

7. *Chance* is used synonymously with probability, but mostly in reference to problems of few cases, in which the expectation, that results will nearly agree with a general average, is not great.

8. When the number of favourable exceeds that of unfavourable cases, the *odds* (inequality of probabilities) are said to be in favour of the event, and *vice versa*. In the preceding instance the *odds* are against drawing a white ball, as 7 to 3, or $2\frac{1}{3}$ to 1.

9. The sum of all the different probabilities which any case presents must be 1: and if any different subordinate cases be collected, and the happening of either one or other of them be the condition under which an event is to be considered as having happened, the probability of that event is the sum of the probabilities of the several events of which it is disjunctively composed. Thus if there be 3 red, 2 white, and 9 black balls, the event "one of the two red or white" has a probability $\frac{3}{14} + \frac{2}{14}$ or $\frac{5}{14}$. For the event may evidently happen in five ways out of 14.

(9.) We shall now proceed to some applications.

Lemma 1. The number of ways in which n counters can be taken out of p counters, supposing each to be put back again before the next drawing, is p^n . Let this be true for any one value of n , say 5; then there are p^5 ways of drawing 5, but every one of these ways has p different ways of being converted into a drawing of six: therefore $p^5 \times p$ or p^6 is the number of ways of drawing six. That is, if this theorem be true for any one value of n , it is true for $n + 1$: but it is true for one drawing, (there being p or p^1 ways of drawing,) therefore it is true for all.

Corollary. This case remains exactly the same, if instead of p distinct counters $C_1 C_2 \dots C_p$, we suppose there to be n_1 like C_1 , n_2 like C_2 , &c., there being p different *sorts* in all, and two of a sort being considered as the same event.

Thus the number of different events arising out of four throws with a halfpenny is 2^4 or 16, as follows (H stands for head, and T for tail):

H H H H	T H H H	T H T H	T T H T
H H H T	H H T T	T T H H	T H T T
H H T H	H T H T	H T T H	H T T T
H T H H	T H H T	T T T H	T T T T

The following questions will enable the reader to see whether he understands the definitions.

Which is the most probable event, considering the same number of H and T as the same event, whatever be the order? *Answer.* H twice, and T twice, which call $(H_2 T_2)$, of which the probability is $\frac{6}{16}$, while that of $(H_1 T_3)$ and of $(H_3 T_1)$ is only $\frac{4}{16}$. That of (H_4) is $\frac{1}{16}$, and also of (T_4) .

What is the probability of H once at least? *Ans.* $\frac{15}{16}$.

What are the odds in favour of H once at least? *Ans.* 15 to 1.

What are the odds against alternate throws? *Ans.* 7 to 1.

Let it be observed that p^n is the coefficient of x^n in the developement of $\frac{1}{1 - px}$.

(10.) *Lemma 2.* If the whole number n be made up of $n_1 n_2 \dots n_p$, then the number of ways in which n drawings of p counters give n_1 of C_1 , n_2 of C_2 , &c., without reference to order, is

$$\frac{1 \cdot 2 \cdot 3 \dots n-1 \cdot n}{1 \cdot 2 \cdot 3 \dots n_1 \cdot 1 \cdot 2 \cdot 3 \dots n_2 \dots 1 \cdot 2 \cdot 3 \dots n_p} \text{ or } \frac{[n]}{[n_1][n_2] \dots [n_p]}$$

This follows from the common theory of permutations. The new definition is

$$[n] \text{ means } 1 \cdot 2 \cdot 3 \cdot 4 \dots n-1 \cdot n \quad (\Gamma(n+1) \text{ of Legendre}).$$

The preceding formula is the coefficient of $a_1^{n_1} a_2^{n_2} \dots a_p^{n_p}$ in the development of
 $(a_1 + a_2 + \dots + a_p)^n$.

Theory of
Pro-
babilities.

Theory of
Pro-
babilities.

Lemma 3. The number of combinations of n quantities which can be taken out of p quantities is (as is well known)

$$\frac{p \cdot p-1 \dots p-n+1}{1 \cdot 2 \dots n} \text{ or } \frac{[p]}{[n][p-n]},$$

and the number of permutations is $p \cdot p-1 \dots p-n+1$ or $\frac{[p]}{[p-n]}$.

(11.) There is in algebraical multiplication a resemblance to the sorting of all the events which can happen by combination of simple events, which is one of the most important principles in the application of mathematics to this subject. If the first of three urns give out either a or b , the second either c or d , the third either e or f , it is plain that the results of a drawing give one of the following:

$$a c e, a c f, a d e, a d f, b c e, b c f, b d e, b d f.$$

Now consider $a, b, &c.$ as algebraical quantities, and the sum of the preceding terms (juxtaposition denoting multiplication) is no other than the product

$$(a + b)(c + d)(e + f).$$

Write these two under one another.

$$(a + b)(c + d) = ac + ad + bc + bd$$

$$(a \text{ or } b)(c \text{ or } d) \text{ is either } ac \text{ or } ad \text{ or } bc \text{ or } bd.$$

The first is a well-known algebraical truth; the second is a truth also, if by $(a \text{ or } b)$ we mean an event which may be one or the other, and by $(a \text{ or } b)(c \text{ or } d)$ the consecutive fulfilment of two such contingencies. It is palpable that $(a \text{ or } b)$ followed by $(c \text{ or } d)$ must be either ac or ad , or bc or bd . The analogy which exists between the two preceding processes directs us to the following theorem. Let $+$ be read *or*; let $(\cdot)$, $(\cdot)$, $(\cdot) \dots$ denote a succession of events, each having the contingencies described in the brackets, so that $a + 2b + 3c$ means that we draw in a lottery* having the following tickets,

$$a \quad b \quad b \quad c \quad c \quad c,$$

and is read; a , or b in either of two ways, or c in either of three ways. This being premised, the results of the process, which under algebraical definitions would be multiplication, are true.

$$\text{Thus from } (a + 2b + 3c)^2 = a^2 + 4ab + 4b^2 + 6ac + 12bc + 9c^2$$

we gather that by a drawing from each of two urns which contain

$$a \quad b \quad b \quad c \quad c \quad c$$

we have one chance in favour of aa , four for ab , four for bb , six for ac , twelve for bc , and nine for cc ; as would be evident by developing

$$(a + b + b + c + c + c)(a + b + b + c + c + c)$$

by multiplication. The whole number of events is the sum of all the coefficients, or

$$1 + 4 + 4 + 6 + 12 + 9, \text{ or } 36;$$

this is what the preceding expression becomes when a, b , and c are severally $= 1$.

(12.) We shall now proceed to lay down some primary theorems in the manner in which Laplace has given them, but not in the order.

Principle I. Is the definition of probability already given, in the case where all events are equally possible. Let f and u be the number of favourable and unfavourable cases, and the probabilities are

$$\text{for it } \frac{f}{f+u}, \text{ against it } \frac{u}{f+u}.$$

(13.) **Principle II.** The probability of the happening of two, three, or more events, is the product of the probabilities of their happening separately. This depends upon the following; that if there be f and f' cases in favour of two events, the number of ways in which they may happen together is ff' . For any one of the f ways in which the first may happen may be combined with any one of the f' ways in which the second may happen. The whole number of possible cases is, by the same rule, $(f+u)(f'+u')$, which gives for the probability of happening

$$\frac{ff'}{(f+u)(f'+u')} \text{ or } \frac{f}{f+u} \times \frac{f'}{f'+u'}$$

the product of the simple probabilities. It is the test of correctness in many problems, that the development of the denominator furnishes numerators for the probabilities of all the possible cases. In the preceding instance,

* This is merely saying we take a hazard of any sort. All affairs of life are more or less lotteries. One merchant, for instance, has such skill that his success in business may be likened to drawing for the chance of a white ball where one of ten only is black. The ignorance of another reverses the case. The continual occurrence of illustrations by lotteries, and such contingencies, leads many to suppose that this subject is only applicable to games of chance. To any one who depends so much upon words we may present the following application of them, which is perfectly consistent with our definitions. "Certainty is the chance of drawing a white ball from a lottery of ten balls, all white."

Theory of
Pro-
babilities.

or the whole number of compound events consists of

$$(f + u)(f' + u') = ff' + fu' + f'u + uu',$$

ff' ways in which both may happen.
 fu' ways in which the second *only* may fail.
 $f'u$ ways in which the first *only* may fail.
 uu' ways in which *both* may fail.

Theory of
Pro-
babilities.

For instance, suppose the following syllogism: A is B; B is C; therefore A is C. Suppose the first assertion is considered as having ten to one in its favour, and the second three to one. Now the probability of the conclusion is entirely that of both premises being true, from which it necessarily follows. The probabilities for the single premises are $\frac{10}{11}$ and $\frac{3}{4}$, the product of which $\frac{15}{22}$ is that of the conclusion, or 15 to 7, little more

than two to one. This, it must be observed, is independent of any force the conclusion may derive from other sources, a case we shall presently consider. In this way it is obvious that a lengthened chain of arguments, unless each be a certainty, produces an inference, the probability of which weakens at every step. But at the same time it is difficult, if not impossible, to produce a chain of arguments of which the reasoner can rest the result on those arguments only. The result will of itself, without reference to its supports, produce an impression of its own, which strengthens the premises if it be favourable, and weakens them if unfavourable. Thus the propositions of geometry as they severally appear, have an evidence of their own, were it not for which, we feel quite confident that in the mind of a beginner unused to reasoning they would, after the first few propositions, rest on authority only.

(14.) *Principle IV.* If the different ways in which an event may happen are not equally probable, the probability is the sum of the probabilities of the different cases; *but it must be remembered that each method of happening must be considered as contingent upon that method being the one which actually arrives.* For instance, suppose two closed urns as follows.

(3 wh. 4 bl.) P (4 wh. 3 bl.) Q.

A white ball may be drawn in two ways, from P or from Q. Suppose the drawer has no preference in his mind for either. The common mistake would be to say that the probability of the white ball from the first is $\frac{3}{7}$, from the second $\frac{4}{7}$, and therefore from both $\frac{3}{7} + \frac{4}{7}$ or 1, that is, certainty. This result is absurd; but the reason of the mistake is that his going to one urn or another is itself a contingency. As to the urn P we should ask, 1. What is his chance of selecting that urn? This is $\frac{1}{2}$. 2. Having selected that urn, what is his chance of drawing a white ball from it? This is $\frac{3}{7}$. Consequently his chance of going to that urn, and selecting a white ball is $\frac{1}{2} \times \frac{3}{7}$, and for the other $\frac{1}{2} \times \frac{4}{7}$. The sum of these two is $\frac{1}{2}$, or the two together give an even chance. This is easily verified; for any given ball in one urn being as likely of its kind to come up as any other, we do not alter the problem by pouring the contents of one urn into the other, and thus we have

(7 wh. 7 bl.).

If the drawer have any bias on his mind which renders it two to one that he will choose P in preference to Q, a by-stander must calculate thus: the chance of P is $\frac{2}{3}$, that of a white ball from it $\frac{3}{7}$; therefore $\frac{2}{3} \times \frac{3}{7}$ or $\frac{6}{21}$ is the probability thus derived, and $\frac{1}{3} \times \frac{4}{7}$ or $\frac{4}{21}$ that of a white ball from Q, and $\frac{10}{21}$ is the total probability. We may suppose no such bias to exist, and place the cause of a similar alteration of probabilities in the circumstances themselves, by having three urns,

(3 wh. 4 bl.) P (3 wh. 4 bl.) P (4 wh. 3 bl.) Q,

the total probability for a white ball (no bias existing) is

$$\frac{1}{3} \frac{3}{7} + \frac{1}{3} \frac{3}{7} + \frac{1}{3} \frac{4}{7}, \text{ or } \frac{10}{21} \text{ as before.}$$

We may thus illustrate the dependence of the probability upon the knowledge of the spectator. We have supposed that he knows the contents of the urns, but not which is the one and which is the other. If he know nothing of the contents, the chance is even for a white ball from each, and the total probability is $\frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{1}{2}$ or $\frac{1}{2}$, an even chance, as of course it should be.

Again, the problem undergoes a change during the process to a by-stander who knows which urn has the four white balls. The probability to him varies with the motions of the drawer's hand, according as he sees reason to

Theory of
Pro-
babilities.

suppose he is more likely to take one or the other. At the beginning both parties have the probability $\frac{10}{21}$; but the moment the drawer has inserted his hand into P, the probability in the mind of the spectator is $\frac{3}{7}$ or $\frac{9}{21}$.

Theory of
Pro-
babilities.

(15.) Let us now consider the following case. It is an even chance that A is B, and the same that B is C; and therefore 1 to 3 from these grounds only, that A is C. But other considerations of themselves give an even chance that A is C. What is the resulting degree of evidence that A is C? Let us take all the possible cases.

From the premises { 1. An even chance that A is B } { 3. An even chance that A is C
 { 2. An even chance that B is C } { from other evidence.

Let (1) denote the truth of 1, and [1] its falsehood. The following are possible cases:

- | | | | |
|----------------|----------------|----------------|-----------------|
| 1. (1) (2) (3) | 2. (1) (2) [3] | 3. [1] (2) (3) | 4. [1] (2) [3] |
| 5. [1] [2] (3) | 6. [1] [2] [3] | 7. (1) [2] (3) | 8. (1) [2] [3]. |

Here are eight cases, equally probable, each consisting of a result compounded of three results, and therefore having a probability of $\frac{1}{8}$. Of these, cases 1, 2, 3, 5, and 7 make the conclusion "A is C" necessary, either by the establishment of the argument, or of the independent evidence. That is, five out of eight equally possible combinations are in favour of A being C, or the odds for it are 5 to 3.

Let us now treat the preceding question as having two contingencies, the compound argument (1 to 3 for), and the independent evidence (an even chance). We have, therefore, four possible cases.

	Probability of "A is C."
Argument and evidence both true	$\frac{1}{4} \times \frac{1}{2}$ or $\frac{1}{8}$
Argument false and evidence true	$\frac{3}{4} \times \frac{1}{2}$ or $\frac{3}{8}$
Argument true and evidence false	$\frac{1}{4} \times \frac{1}{2}$ or $\frac{1}{8}$
Argument and evidence both false	0.

The sum of these is $\frac{5}{8}$ as before.

(16.) The above, generalized, is as follows. Let a and $(1 - a)$ be the probabilities for and against the argument, and let e and $1 - e$ be the probabilities from any other source. Then the chance that both are wrong is $(1 - a)(1 - e)$, and of the contradictory, namely, that the result "A is C" follows from one or other is

$$1 - (1 - a)(1 - e) \text{ or } a + e - ae.$$

But this proceeds entirely upon the supposition that an inference can be drawn, and that the arguments which furnish that inference remain with their probabilities unaffected by the circumstances which, on other grounds, give probability or improbability to the result. For in like manner as the certain falsehood of the result proves the falsehood of one or both premises, so the improbability of a result will throw doubt of itself upon the premises, and lower their *à priori* probability. How much of such effect they do produce we cannot ascertain; but it so happens that one indisputable case will show that we can deduce a phenomenon which will, as soon as stated, be recognised and admitted to exist.

The preceding result evidently fails to represent the character of usual impressions by probable reasoning. Let there be $\frac{3}{4}$ of probability for "A is C" from argument, but suppose other grounds of evidence give only $\frac{1}{1000}$ of probability for it, or 999 to 1 against it. Can it be believed that the latter adds probability to the former, or that the total probability is

$$\frac{3}{4} + \frac{1}{1000} \left(1 - \frac{3}{4}\right), \text{ which is greater than } \frac{3}{4}?$$

This result is admissible, provided the methods by which $\frac{3}{4}$ is established have a probability which is not shaken by the smallness of the probability for the result independently of them. But this can hardly be the case, and the preceding theorem therefore omits a necessary consideration which applies with more or less force whenever e is less than a .

But there is one case in which the preceding objection does not apply. When the independent probability e is one half, the evidence for the premises cannot be affected by it. When a result of argument cannot be considered as either likely or unlikely, we do not go back to the premises with any consideration drawn from the

Theory of
Pro-
babilities.Theory of
Pro-
babilities.

nature of the result. Let $e = \frac{1}{2}$, which gives $a + e - ae = \frac{1}{2} + \frac{1}{2}a$. This is, we think, the proper representation of the force of an argument, of which a is the compound probability of the premises, and of which the result is to be considered as neither likely nor unlikely in itself, nor having any force but what it derives from the premises. The following theorem will be readily admitted on its own evidence.

If any assertion appear neither likely nor unlikely in itself, then any logical argument in favour of it, however weak the premises, makes it in some degree more likely than not.

In the manner in which writers on logic apply the calculus of probabilities, this result never is a consequence of their suppositions. For what we have called a is their resulting probability of the argument. Suppose, for instance, a writer on logic presumed that the argument from analogy gave $\frac{3}{10}$ to the probability that there is vegetation in the planets, which must be considered as a thing neither likely nor unlikely in regard of evidence derived from any other source. He would assert $\frac{3}{10}$ to be the probability of this result, that is, *less after an argument in favour of it than it was before*. We substitute $\frac{1}{2} + \frac{1}{2} \cdot \frac{3}{10}$ or $\frac{13}{20}$.

(17.) The chance of throwing at least one head in two throws is $\frac{3}{4}$. Let us suppose that the thrower is to leave off as soon as he throws once. There are then three cases. 1. H the first time, in which case he leaves off. 2. T the first time, and H the second. 3. T the first time, and T the second. These are the only three events contemplated; two are favourable, and one unfavourable: the probability of H then appears* to be $\frac{2}{3}$. But it has been forgotten that the three cases are not *equally probable*. For H the first time we have $\frac{1}{2}$, for $T_1 H_2$ we have $\frac{1}{2} \times \frac{1}{2}$ or $\frac{1}{4}$, and for $T_1 T_2$ we have also $\frac{1}{4}$. Consequently the probability of H once in two throws is $\frac{1}{2} + \frac{1}{4}$ or $\frac{3}{4}$, as before.

But at the same time, with regard to the objection of D'Alembert cited in the note, we must observe that if any individual really feel himself certain, in spite of authority and principle, as here laid down, that the preceding cases are equally probable, he is fully justified in adopting $\frac{2}{3}$ instead of $\frac{3}{4}$, till he see reason to the contrary, which it is hundreds to one he would find, if he continued playing for a stake throughout a whole morning, that is, accepting bets of two to one that H would not come up once in two throws, instead of requiring three to one. He must begin to reason mathematically from that point at which he is convinced he has the means of applying mathematical principles; and remark that the accordance of apparently different methods does not confirm the theory, but the correctness with which the theory has been applied. The individual just supposed, has applied correct mathematics to a manner in which he feels obliged to view the subject, in which we think him wrong, but the error is in the first of the two considerations in (6.), and not in the second.

(18.) All that we have hitherto done is to lay down *direct* principles, in which we reason from known causes to probable events. We now come to *inverse* principles, in which we reason from known events to probable causes.

Principle IV. Knowing the probability of a compound event, and that of one of its components, we find the probability of the other by dividing the first by the second. This is a mathematical result of the last too obvious to require further proof.

(19.) *Principle V.* When an event has happened, and may have happened in two or three different ways, that way which is most likely to bring about the event is most likely to have been the cause. This is so well known a principle that it is continually in our mouths. "A man has died; is it A, B, or C?"—"Most likely it is A, because he was in weakest health, and therefore most likely to die." This is so common a phraseology that we at first imagine we are using tautology. But on considering a moment we see that it is not so. Death has happened, either to A, B, or C; an event is given, and we ask in which out of given ways it arose. We immediately, not change the phrase, but reverse the supposition, and make the event a contingency. If death be to happen, to which is it most likely to happen?—To A, it is presumed. Then, we say and know that the death which has happened is most likely that of A.

This principle is not in reality different from the one first stated (Principle I.) If the causes of an event be all equally likely, we can give no preference to one over another, and consequently, if they be n in number, each has a probability $\frac{1}{n}$. But to say that one cause is more probable than another is merely to say it is a greater collection of *partial causes* than are contained in the other. Now what are we considering? An event has happened, say a white ball has been drawn, and it must be from one of the two,

* So said D'Alembert, "Dès que croix sera arrivé au premier coup, on n'en jouera pas un second; et ainsi les deux premiers combinaisons croix croix, croix pile, se réduisent à croix seul."—*Opuscules*, vol. ii. p. 16.

Theory of
Pro-
babilities.

7 wh. 3 bl., or 10 wh. 2 bl.

Theory of
Pro-
babilities.

A white ball has been drawn; let us, without altering the contingencies of either, change the number of balls in both, so that the number of balls shall be the same in both. Multiplying both numbers in each urn by the same number does not alter the probabilities resulting from either, and therefore does not change the character of either as a cause. We thus have

84 wh. 36 bl., or 100 wh. 20 bl.

There are now 120 balls in both; and the only question is, was the white ball actually drawn one of 84, or one of 100? Each ball has the same chance of being drawn, whichever it be in, namely $\frac{1}{120}$. Because the ball is not black, we may strike out the black balls altogether, and if we suppose the white balls supplied from a third urn which contained 184, the chance of the ball which was drawn having been in the first urn is $\frac{84}{184}$, and in the second $\frac{100}{184}$. These are then the chances of the observed event having arisen from the two causes respectively. But

$$\frac{84}{84 + 100} = \frac{\frac{84}{120}}{\frac{84}{120} + \frac{100}{120}} = \frac{\frac{7}{10}}{\frac{7}{10} + \frac{10}{12}}, \quad \frac{100}{184} = \frac{\frac{10}{12}}{\frac{7}{10} + \frac{10}{12}}$$

But $\frac{7}{10}$ and $\frac{10}{12}$ are the probabilities of a white ball appearing, when the causes are chosen, and the appearance of a white ball is the contingency. And we may thus put the principle in a mathematical form as follows. If an observed event may arise from either of the causes A, B, C, &c., and the probabilities of its happening on the assumption of the several causes be $a, b, c, \&c.$; then the probability of the cause A having been the one which really did act, is

$$\frac{a}{a + b + c + \&c.}; \text{ that of B is } \frac{b}{a + b + c + \&c.} \&c.$$

Suppose a ball to be drawn from an urn of which all we know is that it contains three balls. It is white, and is restored after drawing. The second and third drawings give a white ball also; what is the probability that all in the urn are white, &c.?

Possible causes.	Assuming this cause, the prob. of the observed event is
1 white, 2 black	$\frac{1}{27}$
2 white, 1 black	$\frac{8}{27}$
3 white, 0 black	1 or $\frac{27}{27}$
Then for 1 white, 2 black, the prob. is	$\frac{1}{1 + 8 + 27}$ or $\frac{1}{36}$
2 white, 1 black.....	$\frac{8}{1 + 8 + 27}$ or $\frac{8}{36}$
3 white, 0 black.....	$\frac{27}{1 + 8 + 27}$ or $\frac{27}{36}$

(20.) *Principle VI.* Having observed an event, and given the degree of probability arising out of it to each possible cause, what is the probability of another event happening in a given manner? This problem requires the application of Principle III. Calculate the probability of the prescribed event happening from each cause; that is, multiply the probability of each cause by the probability of the indicated event following it, and add the results together.

Thus the probability that the fourth drawing will give a white ball, in the last example, is found as follows.

Prob. of the circumstances being as below.	Prob. which these circum- stances give the event.	Prob. of the event really happen- ing from these circumstances.
1 white, 2 black $\frac{1}{36}$	$\frac{1}{3}$	$\frac{1}{3 \cdot 36}$ or $\frac{1}{108}$
2 white, 1 black $\frac{8}{36}$	$\frac{2}{3}$	$\frac{8 \cdot 2}{3 \cdot 36}$ or $\frac{16}{108}$
3 white, 0 black $\frac{27}{36}$	1 or $\frac{3}{3}$	$\frac{3 \cdot 27}{3 \cdot 36}$ or $\frac{81}{108}$
Total probability required,		Sum $\frac{98}{108}$

Theory of
Pro-
babilities.

If a' , b' , &c. be the probabilities which the several causes would give the prescribed event, and a , b , &c. those which they would give to the observed event, the probability of the prescribed event is, by this principle and the last,

Theory of
Pro-
babilities.

$$\frac{a a' + b b' + \&c.}{a + b + \&c.}$$

(21.) With regard to the preceding principles, it is clear that we may determine by them the most probable cause; out of which immediately arise two distinct branches of the subject.

1. When we propose to abide by the *absolute* event of a chance in such manner that any deviation from a prescribed method of happening shall be a total loss, we are *gambling*. Thus in throwing up a hundred half-pence, the most probable of all events is that there shall come down fifty heads and fifty tails; that is, this single combination is more probable than any other which can be named. But it is not therefore probable in itself; for if we consider all the other cases put together, their united probabilities far exceed the probability of that which, compared with other single events, is the most probable. Thus though a horse may be named which is likely to beat *any one* of the other horses in a race, it is not therefore likely that he will beat them all. Suppose a horse had been tried against nine other horses, and had been found to beat each of them five times out of six. If then we call x the probability that any one of the nine will win, it is clear that $5x$ is the probability for the favourite. But the sum of all the probabilities against the latter is $9x$; and though $5x : x$ is in favour of the first named, against any one competitor, yet it is $5x : 9x$ against him on the whole field. The neglect of this consideration in common life leads to no good results. It is the argument on which speculators, trusting to their skill in single instances, imagine it to be as likely that they shall succeed in all their undertakings, as in any one. The only way to act prudently is by decreasing the risk as the number of speculations increases. If this be done, the greater the number of speculations, properly understood, the safer is the result.

(22.) 2. This brings us to the second species of risks, in which, instead of putting all upon one event, the risk is divided among a great number. And this is the principle of prudence whether the risk of each individual event be small or great; indeed when the number of events is very great, it is indifferent (as to safety) whether the risk of each be small or great. Suppose that in the preceding case the speculator is to pay a hundred pounds if any event come up except 50 heads and 50 tails. He is gambling, even though he be to receive a sum in case of that event, large in proportion to its improbability. But if he divide his risk in this way: if he engage to pay a pound for every head, and receive one for every tail, the gambling character of the transaction is materially diminished. The odds are now against his being really a staker of ten pounds; and if he were allowed to make as many throws as he pleases, and were to receive, say a shilling certain for every throw, independently of his hazard, he might go on until he had gained any sum.

The mathematical part of the first species of risks is comparatively easy: that of the second very difficult. Hence books on this subject are generally on gambling, and in all, questions of the first sort precede those of the second. And from this arises in great measure the character which this science bears among those who are unacquainted with the higher branches of mathematics.

(23.) We shall first address ourselves to some superstitions which we have observed that ignorance on this subject allows to exist even among educated men in this day.

1. We have heard it gravely asserted by sensible men, that there is no doubt whatever of *runs of luck*, in a tone which implied that they think it is something extraordinary.* We, on the other hand, should be as much surprised if there were not runs of luck, as they are at what really takes place, which we think to be the probable order of events. On inquiring what is meant by a run of luck, we find it is the occurrence, skill being equal between two players, of a succession of events on one side. Thus the smallest run possible is that A should win two games following of B, and hardly merits the name: three games following is a run, and four very decidedly so. Let us then call three games or upwards following each other with the same result a *run of luck*. Let A and B sit down to play four games with equal skill. There are sixteen possible cases, of which six, namely, (A denotes that A wins, &c.) A A A A, B B B B, A A A B, B A A A, B B B A, A B B B, contain a run of luck. It is therefore only 10 to 6, or $1\frac{2}{3}$ to 1, that there is no such thing in any four games. If they play five games, there is an even chance for a run of three; and if they play six games, there are 38 out of 64, or 38 to 26 in favour of a run of three; that is, out of six games it is *more likely than not* that there should be a run of three. We put down the total number of possible events which six games may give; that is, we write down 32 cases, and the other 32 may be found by changing B into A and A into B.

A A A A A A	A A A B B A	A A B A A B	A B B B A A
A A A A A B	A A B B A A	A B A A B A	A A B A B B
A A A A B A	A B B A A A	B A A B A A	A B A B B A
A A A B A A	B B A A A A	A B A A A B	A B A A B B
A A B A A A	A A A B A B	B A A A B A	A B A B A B
A B A A A A	A A B A B A	B A A A A B	A B B A A B
B A A A A A	A B A B A A	A A A B B B	A A B B A B
A A A A B B	B A B A A A	A A B B B A	A B B A B A

Here 19 out of 32 exhibit runs of three or more, and when the whole process is doubled, there will be 38 out of 64, as before stated.

* We are sure that many people have a lurking belief in some mysterious sympathies or antipathies existing between the cards or dice and themselves.

Theory of
Pro-
babilities.

In this way, by taking a sufficient number of games, a run of four becomes probable, or of any number however great. It would be possible to play a number of games so great that it should be a hundred to one in favour of the series containing a run of a hundred games.

Theory of
Pro-
babilities.

When the game is partly skill and partly chance, the probabilities of a run are increased. The party who wins the first game gets confidence, and ease of mind, while the reverse takes place with the other. The first is therefore rendered more skilful than the second, or which is the same thing, placed in circumstances which favour the undisturbed application of his skill.

This belief in runs of luck is exactly the feeling of destiny which has predominated in the minds of many historical characters, and particular of the Emperor Napoleon. Granting somewhat to temperament, the greater part of the feeling appears to us to have arisen from an idea that the common order of events is regular succession of good and bad fortune, and that any thing else is to be attributed to some particular cause, to which the first cannot be traced. Let those who are inclined to believe that *runs of luck* are out of the common order of events watch the succession of carts and carriages in a crowded street. It will be found that a crossing will sometimes be clear for a minute or two; then a rush, more or less, of vehicles will approach, and the events of successive minutes will exhibit every variety of succession, from the average number up to five or six times as much, or down to nothing. A continual average would render the great thoroughfares in London impassable.

(24.) 2. We have frequently heard it asserted that a change of circumstances, such as taking another chair, or the like, *changes the luck*. If the preceding cases be observed, and if we suppose this alteration of circumstances to take place at the end of the first three games, it will be found that in 22 out of 32 cases the luck is not the same both before and after, and that in 16 out of 32 the first and last triads present decided differences. It is then an even chance at least that a change of position will happen immediately before a triad which will differ in its character from that immediately preceding; and it is always 11 to 5 at least against the approaching triad resembling the preceding one, in the number of games won and lost. But if A having *lost* three games in succession should change his chair to get rid of his ill luck, (which we have seen done,) he then may easily suspect some virtue peculiar to his new position, for it is 7 to 1 that there must be some alteration, and any alteration is in his favour. Those, therefore, who adopt this method of courting fortune, should try it on all sorts of luck; for by using it only when she is very much against them, they put the majority of possible cases on their sides as well as the magic of the new seat.

(25.) 3. A confusion very often exists in the mind between the joint probability of two events which are to happen, and the same where one *has* happened. Thus say, it is beforehand unlikely that a man should receive two letters by the post; suppose that it is an even chance he receives one, and that he has ten correspondents. Consequently, supposing the probabilities of their severally writing on any one day to be the same, the chance of

either writing on a given day is $\frac{1}{20}$. Now he knows that his correspondent A must write on a particular day: he therefore thinks it unlikely that either of the others will write on that day, for he calls it unlikely that he should receive two letters on one day. He here compares his position with that of ordinary days, in which he is certain

of no letter whatever: in which case, since there are $\frac{1}{2} 10 \cdot 9$, or 45 pairs of correspondents, either of whom may write, and since the probability of any one pair both writing is $\frac{1}{20} \times \frac{1}{20}$, the probability of his receiving two

letters *only* is $\frac{45}{400}$, and so on. But when one letter is certain from A, the probability of his receiving two letters only is exactly the probability of his receiving the second letter, which, from all the remaining nine correspondents, is $\frac{9}{20}$. Similarly, a man may say, "It is very improbable that I should be robbed twice in one day; I

have been robbed once, and therefore most likely shall not be robbed again." Here is the same fallacy; it is very unlikely that two robberies *should* take place in the same day; but when the first *has* taken place, the probability of *two* becomes the same as the probability of the remaining one.

(26.) We now proceed to the principles on which pecuniary risks are proportioned to the probabilities of the events on which they depend. First, let us suppose that the sum in question is of trifling consequence, so that it is indifferent to the speculator what the result may be. Suppose that the chances are 2 to 1 against his receiving a pound. It is clear that his chance has a value to him; he may not gain, but he is in a different situation from a person who has no risk upon the same event. We may consider the subject in two different ways.

1. As a matter of gambling. Let us suppose the pound must be given either to A (the speculator in question) or to B or C, and that all three have the same chance. It is then 2 to 1 against A, as proposed. If they all three agree to relinquish the hazard, it is clear that the pound must belong equally to all three, or if divided, must be equally divided; for no reason can be given why A should receive any sum, which is not equally applicable in favour of B or C. Hence each would receive one-third of a pound, and if the stake were to be made up by equal contributions, each must give one-third of a pound, and take his chance of receiving all or nothing. If C sell his expectation to B, then B has two chances where A has one and consequently must, if they draw stakes, receive two-thirds of the pound. Hence

Principle VII. The advantage of a contingent benefit, or the expectation which the speculator should have, must be such a proportion of the benefit, as his probability of getting it is of unity.

(27.) 2. As a matter of commercial speculation. In this case a great many such contingencies are contemplated, and the merchant is to calculate in what manner he shall probably make the gains cover the losses,

Theory of
Pro-
babilities.

and realize a profit on the money invested at the same time. Let us suppose there are a ways of gaining, and b ways of losing, and that there is a number n of speculations in question, depending on n events of the same sort. In the development of

Theory of
Pro-
babilities.

$$(a+b)^n \text{ or in } a^n + n a^{n-1} b + n \frac{n-1}{2} a^{n-2} b^2 + \dots$$

we see (11.) in the coefficients the number of ways in which all, all but one, &c., may succeed, and if we divide both sides by the whole possible number of cases $(a+b)^n$, and let

$$\frac{a}{a+b} = \alpha, \quad \frac{b}{a+b} = \beta,$$

we have

$$1 = \alpha^n + n \alpha^{n-1} \beta + n \frac{n-1}{2} \alpha^{n-2} \beta^2 + \dots,$$

the terms of which express the probabilities that all, all but one, &c., will succeed. The ratios of these successive terms to those which precede are

$$\frac{n \beta}{\alpha}, \quad \frac{n-1}{2} \frac{\beta}{\alpha}, \quad \frac{n-2}{3} \frac{\beta}{\alpha} \text{ &c.}$$

$$\frac{(p+1)\text{th term}}{p\text{th term}} = \frac{n-p+1}{p} \frac{\beta}{\alpha}; \text{ let this be called } P_p.$$

Let the p th term be the greatest; that is, let P_p be less than unity, and P_{p-1} greater than unity; or $(\beta + \alpha = 1)$

$$\begin{aligned} (n-p+1) \beta &< p \alpha & (n-p-1+1) \beta &> p-1 \alpha \\ (n+1) \beta &< p & (n+1) \beta &> p-1. \end{aligned}$$

But the p th term expresses the probability of losing $p-1$ and gaining $n-p+1$ of the speculations: consequently, in n separate events, against each of which the probability is β , the most probable of all losses is the whole number next below $(n+1) \beta$, which is more likely than any other given loss, though not more likely than any out of the other losses. The preceding relations also give

$$\frac{p-1}{n-p+1} > \frac{n \beta - \alpha}{(n+1) \alpha} < \frac{(n+1) \beta}{n \alpha - \beta},$$

when n is a great number, the two fractions just found are very near to $\frac{\beta}{\alpha}$: whence this theorem.

Of n successive, or simultaneously independent events, the combination which of all others is most likely to happen is that in which the number of losses is to the number of gains most nearly as the probability of losing any one to the probability of gaining it.

We shall afterwards show that though this most likely individual case is not therefore to be expected, yet that the greater the number of speculations in question, the smaller the *per-centage* of deviation which is to be reasonably looked for. This, then, is the primary axiom of commercial prudence, as far as what may be called accidental fluctuations are concerned. If 1000 speculations* be entered into, against each of which it is nine to one, then the undertaker must look to lose 900 and gain 100; he must therefore not adventure, unless upon terms which will make 100 pay the risk of the whole thousand. If each successful speculation would bring £10, he must only invest £1 in each of the 1000. But he must even invest less than this, to compensate the probable departure from the most probable case. There is then a contingency in what are called his *profits*, the remaining part being made up of what he would derive from simply investing his capital in the funds, and of that portion of profit which he expects for his own labour. Considering that the fluctuations will be on one side and on the other of the most probable case, he may be tolerably certain that his extra *fund for fluctuation* may be made a source of profit; but it must always exist, especially at the commencement.

(28.) The preceding views, though not *numerically* acted upon, for want of positive data, are notoriously truths of the commercial world. It is well known that large businesses can exist with lower profits than smaller ones; for the fluctuation fund must be a larger *per-centage* of the whole in the latter case than in the former. The increase of large farms, large manufactories, &c. confirms this abundantly. A large business and a small one, at the same rate of profits, are much like a line-of-battle ship and a boat in a rough sea. Where the first only rocks, the second is upset.

It must not therefore be inferred that a small business is comparatively dangerous. A risk to *some one* there must be, but there are various methods of taking this off the customer. The first is the system of credit, by which all ordinary fluctuations are equalized, and the loss of one month or year is carried to the chances of the next. Where these chances are prudently calculated there is little risk, but where the most probable case has been mistaken, a series of transactions must end by showing a permanent departure from the anticipated series of results. The second is the investment of a fund larger than is necessary for the purposes of business. Thus in the case of an incipient insurance-office, the capital which always appears as a part of the scheme is a reserve for the possible fluctuations of the first years. The largeness of the sum usually staked upon the success of such undertakings is, we suppose, the proof that those who invest it feel perfectly secure, for it is generally much more than can be needed.

* We do not mean the large transactions which *par excellence* have that name, but we consider every transaction, however small, as one case.

Theory of
Pro-
babilities.

The principle above stated being the same as that which applies in a single gambling transaction, it has been customary to consider the two together; but though the results be the same, the difference of the method of deriving them is very great. On a single throw it matters little whether the just expectation be staked or not. But the merchant is certain of ruin sooner or later, if he neglect this principle on the average of his transactions. The first adopts it because there is no reason why he should stake more or less; the second because there are positive reasons why it should be followed, to the exclusion of all other methods. So obvious is the second result to most persons that it might be made the foundation of the authority for the first. When a halfpenny is thrown up many times in succession, it is never disputed that *in the long run*, as the phrase is, there will be as many heads as tails; or if not exactly, very nearly. Consequently, the chances of a single throw are presumed to be equal for both events.

Theory of
Pro-
babilities.

(29.) A systematic gambler, who has many transactions of the same kind, is a merchant whose speculations differ from those commonly called mercantile, only in the individual risks being somewhat greater. But even this is not always the case. To take the instance of a horse-race; suppose one person can find somebody to take any bet he offers, and that four horses, A, B, C, and D, are to start. He offers four bets against each, (he must take care not to bet in favour of any,) namely, a to a' against A, b to b' against B, c to c' against C, and d to d' against D. The following then are his contingencies:

If A win, he gains $b' + c' + d' - a$.

If B win, he gains $a' + c' + d' - b$.

If C win, he gains $a' + b' + d' - c$.

If D win, he gains $a' + b' + d' - d$.

All he has to do then is to make the sum of every three consequents greater than the fourth antecedent. Suppose, for instance, he bets £10 against £10 on every horse; he must then win £20 whichever way the matter may go. Or he may bet 10 to 5 against A, 10 to 6 against B, 10 to 7 against C, and 10 to 8 against D. Then, if A win, he gains £11; if B, £10; if C, £9; if D, £8: that is, £3 worth of fair gambling, and £8 worth of—not always perhaps sheer roguery, but very often so, and generally something very like it. It is evidently a *fool-chase*, not a horse-race, which he is engaged in.

(30.) PROBLEM. To determine the proportions in which the acceptors of the four preceding bets are unfairly dealt with. Suppose the real probabilities of winning are α, β, γ , and δ ; in which case $\alpha + \beta + \gamma + \delta = 1$. The odds for A are α to $1 - \alpha$, or a to $\frac{a\alpha}{1 - \alpha}$; but $\frac{a\alpha}{1 - \alpha} - a' = \frac{(a + a')\alpha - a'}{1 - \alpha}$; if $(a + a')\alpha < a'$, he takes an advantage.

In the case where the betting is fair, the total sum risked upon A's winning is $\frac{b\beta}{1 - \beta} + \frac{c\gamma}{1 - \gamma} + \frac{d\delta}{1 - \delta} - a$.

(31.) The odium which attaches to the character of a gambler, that is, a person who lives upon mutual transactions which necessarily involve loss to one party or the other, without producing any addition to the commercial prosperity of society, does not arise, or ought not to arise, from his being considered as a spendthrift, or as a man who willingly puts to hazard the means of fulfilling his engagements. If he gamble much and skilfully, not staking too large a proportion of his capital on any one hazard, he is in truth one of the safest merchants going, and cannot be considered as running so great a risk of ruin as is incurred in many honourable and necessary affairs. If he played only with gamblers like himself in knowledge and means, whatever might be thought of the utility of his occupation, nothing could be said against its honesty. But it is evident that a society of such persons must finally come to ruin; for their transactions having no tendency to make any absolute profit, or to increase the value of any thing they handle, the deductions necessary for life must in time destroy their common fund altogether. And herein consists the villainy of their occupation, that any permanent increase of their fund must come from unequal play, that is, from offering or accepting hazards which are unduly in their favour, of course without the inequality being known to and recognised by their opponents. The principle which is carried the length of certain gain in the horse-racing trick we have mentioned must exist more or less throughout their engagements. Now it is clear that no honest man can enter into bargains with A, B, and C separately which would be fraudulent if he had offered all three to A alone, merely because neither of the three knows what has been done by the two others. This matter would be here irrelevant, but for the common notion that this theory has a tendency to promote gambling. On the contrary, it will appear sufficiently from the lower as well as the higher parts of this subject, that there is a proposition which admits of something as like mathematical demonstration as can be possible in any case of morals—namely, that when a man plays for more than he can prudently spend in pleasure with a man or a firm which lives by gambling, there is a fool on one side and a rogue on the other.

(32.) We must refer the reader to the article ANNUITIES for the principal applications of the preceding to the commercial cases of annuities, insurances, and life contingencies. Much prolixity might be saved in such works, if it were usual to lay down the general problems as preliminary lemmas, which it is preferred to repeat as often as there are particular cases. These problems we shall now give as simple applications of the preceding principles. Let the existence of any set of circumstances be called a *status*, which may be either actual, or prospectively certain, or prospectively contingent. A benefit may accrue to an individual yearly during the continuance of an actual *status* which must have an end, or it may begin to accrue yearly upon the commencement of a prospective *status* which must come, or of a *contingent status* which may come into existence. And the questions are the following.

1. What must A pay at once to entitle him to receive one pound at the expiration of every 365 days after the completion of the bargain, so long as a given status, existing at the time of the bargain shall continue to exist? This is the problem of annuities.

Theory of
Pro-
babilities.

2. What must A pay down, and continue to pay at intervals of 365 days, so long as a given status continues, to entitle him to receive a pound at the time when he would otherwise have repeated his payment, at the settling day next following the termination of the status? This contains the most common cases of life insurances.

3. What must A pay, and continue to pay as before, so long as two given status both continue, to entitle him to receive a pound, as before, when one of the two status (not either, but one which is named) has terminated?

(33.) To simplify the question, let the first of January be the only day of payment by either party to the bargain, and let it be the birthday of all the parties whose lives are concerned in the status in question. Let P_n be the probability that the status in question in the first problem is in existence at the end of n years from the day of completing the bargain. Let r be the interest of one pound for one year, interest being acquired yearly. Then, in problem 1, the chance which A has of receiving the n th pound is P_n , and its value is therefore $\mathcal{L}P_n$. Consequently, he must now give for that single chance such a sum as, improved at compound interest for n years, will give $\mathcal{L}P_n$, namely, $\mathcal{L}P_n(1+r)^{-n}$, or $\mathcal{L}P_n v^n$, if $v = (1+r)^{-1}$, the present value of one pound to be received at the end of a year. And hence, what he must give for all his prospects is $\sum P_n v^n$, from $n=1$ to such a value that $P_n v^n$ is not worth calculating, or the value of his annuity is $P_1 v + P_2 v^2 + P_3 v^3 + \dots$.

This problem generally admits of no simplification: each term must be calculated separately, or the following method must be adopted. Let us consider the state in which the buyer of the annuity will be after the first year, if the status in question continue, and he should then wish to make another bargain for another annuity. Let $Q_1, Q_2, \&c.$ be the probabilities of the status then continuing one, two, &c. years. Now it is evident that $P_1 \times Q_1 = P_2$, for the event of which P_2 is the chance must arise from the consecutive happening of the events whose chances are P_1 and Q_1 . Similarly, $P_1 \times Q_2 = P_3$ and $P_1 \times Q_n = P_{n+1}$, by which relations it will appear, that if

$$(P) = P_1 v + P_2 v^2 + P_3 v^3 + \&c.$$

$$(Q) = Q_1 v + Q_2 v^2 + Q_3 v^3 + \&c.$$

$$(P) = P_1 v \{1 + (Q)\}.$$

we must have

So that by beginning under such circumstances that the probability of the status continuing more than one year is not worth considering, we may pass to the case of the year preceding, of the year preceding that, and so on, down to the case actually in question. It is thus as easy to form a whole table as to find the present value in the earliest case; and this is usually done, and tables of annuities are thus calculated, to which all the corresponding questions of the second and third problems are referred by formulæ, as we shall proceed to show.

(34.) Let $(P), (P)_1, (P)_2, \&c.$ be the present values of an annuity, namely, (P) at present, $(P)_1$ as it will be a year hence, if the status be in existence, and the bargain be then made, $(P)_2$ the same for two years hence, and so on. We now ask what is the present value of an annuity which is to begin payment n years hence,* if the status be then in existence, and to continue during the remainder of such existence? It is plain that the value of such annuity the instant before the first payment is $1 + (P)_n$; the chance of the status on which payment depends being then existing is P_n , and $P_n \{1 + (P)_n\}$ is what should be paid if money made no interest during the period of deferment; with reference to the interest, the present value of the preceding is $P_n v^n \{1 + (P)_n\}$, an expression of easy calculation by means of tables.

(35.) We shall now consider the second problem, which is that of the present value of an insurance. The bargain made by A is made up of a series of separate bargains; he is to receive at the end of the n th year one pound, if the status determine in the n th year. Now that it will determine in the n th year, and in no other, is the fraction by which the probability of its failing in n years exceeds that of failing in $n-1$ years, or $(1 - P_n) - (1 - P_{n-1})$. For the probability of the status ending in n years is made up of the probabilities of its failing in the several years of that period. Upon this probability depends the acquisition of $\mathcal{L}1$ at the expiration of n years, and the present value of that prospect is therefore $\mathcal{L}(P_{n-1} - P_n) v^n$. Hence, since $P_0 = 1$, for at the moment of the bargain the status is in existence, the whole present value of a pound to be received at the close of what year soever the status may end in, is

$$(1 - P_1) v + (P_1 - P_2) v^2 + (P_2 - P_3) v^3 + \&c.,$$

or

$$v + v(P) - (P), \text{ or } v - (1 - v)(P);$$

so that the tables of (P) being given, the present values of insurances can be immediately constructed by a simple formula.

(36.) Now to take the next case, namely, what must A pay down, and continue to pay at intervals of a year during the status, to receive $\mathcal{L}1$ at the end of the interval in which it determines? If he paid $\mathcal{L}1$, it is clear that the present value of all his payments (premiums they would be called) is $1 + (P)$; consequently, if he pay $\mathcal{L}x$, the present value of his premiums is $\{1 + (P)\}x$. But this must be equated to the present value of the benefit he is to receive, as determined above, which gives for the premium required

$$\frac{v - (1 - v)(P)}{1 + (P)}, \text{ or } v - \frac{(P)}{1 + (P)}.$$

But if the premiums are to be paid during the continuance of another status, the probabilities of the continued existence of which at the end of the first, second, &c. years, are $P'_1, P'_2, \&c.$, and (P') the present value of the annuity, then the proper premium will obviously be

* It is important to remember that, in treating of annuities, the formulæ are so constructed that they are said to be created a year before payment is made: for instance, that if we buy an annuity January 1, 1836, to date from that day, the first payment is made January 1, 1837. Thus an annuity which begins payment January 1, 1840, is an annuity to commence on that day augmented by one year's purchase.

Theory of
Pro-
babilities.

Theory of
Pro-
babilities.

$$\frac{v - (1 - v)(P)}{1 + (P')}$$

Theory of
Pro-
babilities.

(37.) As an instance of such problems, let us suppose the following question. A father who has a life income allows an annuity to a son and his wife, and intends to allow the same to the survivor, if either of them die before him: he desires to secure that annuity to them jointly, or to the survivor; that is, he wishes to place them in the same position as they would be in if he were certain of surviving both. What should be paid to an office which would engage to fulfil his intentions?

It is plain that the status during which payment of premiums should be made continues as long as, 1. all three are alive; 2. the father and son only are alive; 3. the father and daughter-in-law only are alive; for the object has ceased if the father survive both, and the benefit is to begin if the father die. Let a_n be the probability that the father will survive n years; and p_n and q_n the several probabilities of the son and daughter surviving n years. Then the probability of all three being alive in n years is $a_n p_n q_n$; that of the father and son only surviving is $a_n p_n (1 - q_n)$; and that of the father and daughter only is $a_n (1 - p_n) q_n$. The sum of these, or $a_n p_n + a_n q_n - a_n p_n q_n$, is the probability of premium being payable at the end of n years, and consequently if £1 be paid down, and £1 of premium contemplated, the present value of all premiums is

$$\frac{1 + \sum (a_n p_n + a_n q_n - a_n p_n q_n) v^n}{1 + \sum a_n p_n v^n + \sum a_n q_n v^n - \sum a_n p_n q_n v^n};$$

or

the three latter terms of which can be found from tables, for the second is the value of an annuity to be paid so long as father and son are both alive, the third the same so long as father and daughter are both alive, and the fourth so long as all three are alive. These are technically called annuities on the *joint* lives, and where two lives are concerned, are laid down in tables, which afford a sufficient method of approximating to the case where three are concerned. We will suppose these annuities found, and called (AP), (AQ), and (APQ). That is, if x be the premium necessary, the consideration given for the benefit must be estimated at $x \{1 + (AP) + (AQ) - (APQ)\}$. Now the cases in which the office will be required to pay, at the end of the n th year, are when either of the following status exist: 1. the father dead, and both the others alive; 2. the father dead, and the son only alive; 3. the father dead, and the daughter-in-law only alive; the several probabilities of which are $(1 - a_n) p_n q_n$, $(1 - a_n) p_n (1 - q_n)$, $(1 - a_n) (1 - p_n) q_n$. The probability of one or other of these, which is the status in question, is the sum of the preceding, or $p_n + q_n - a_n p_n - a_n q_n - p_n q_n + a_n p_n q_n$, which, by similar reasoning to the preceding, gives for the present value of such a contingent annuity of £1, $(P) + (Q) - (AP) - (AQ) - (PQ) + (APQ)$; whence, if the office engage to pay £ c annually, the proper premium is

$$x = \frac{(P) + (Q) - (AP) - (AQ) - (PQ) + (APQ)}{1 + (AP) + (AQ) - (APQ)} c.$$

(38.) We shall now come to what is called the *moral expectation*, as distinguished from the mathematical expectation. And here we must take the commercial case first. We have seen that a small and a large business cannot risk the same sums in speculation, unless the former make a higher rate of profit; we shall afterwards see reason to conclude it must be a much higher rate. If we suppose that a business which can always command £10,000 considers it prudent to risk £1000 in a certain speculation, then another which can only command £100 should only risk £10 in the same. To take the extreme case, suppose one man can command pounds, where another can command only pence. What should the second do to preserve his relation unaltered to the first? It is clear that he must, 1. engage in exactly the same transactions; 2. risk one penny where the second risks a pound. In this case both, if either, must be bankrupt together, and both pay the same per-centage of their debts: or if both gain, it is the same per-centage of their invested capital. The only difference between them, mathematically speaking, is in the name of the coin they deal in. Change the word *pound* into *penny*, and the books of the first become those of the second. And the rule of conduct which prudence points out in multifarious transactions becomes, from want of sufficient reason to depart from it on one side more than on the other, the proper guide in single transactions, or in pure gambling. Hence the principle in the next article.

We omit here a circumstance which always is, and must be, in favour of the larger business. The expense of management is one of the risks of the business; now that outlay in a small business cannot be as much less than the same in the large one as the different capitals invested require, according to the preceding rule. It will always take, to manage £10,000, more than half as much as to manage £20,000. We also omit the consideration that a small business not being able to command so large a sum, has not the same range of choice for temporary investments, and must take the best out of a more limited number than the larger one can command. In this country, the increase of proprietary bodies, the funds, &c., so far as they influence the point at all, tend to equalize this disproportion; and the system of credit still more so: but the comparative disadvantage must always exist more or less. On these considerations it is evident that partnerships of small capitals are good to the holders, *cæteris paribus*.

(39.) *Principle VIII.* Two players of different fortunes stake equally when they stake the same proportion of their fortunes, other things being the same: and, all things being the same except the fortunes of two speculators, the sums they ought to risk must be the same proportions of that which they can honestly lose.

This principle will always furnish an answer to those who are pressed by friends of more means than themselves to engage equally with them in any speculation, however probable. If A, who has twice as much as B, take ten shares in a project, and recommend B to do the same, he asks his friend for twice as much confidence in his own judgment as he has himself.

(40.) There is another consideration which enters into the *moral expectation* as it is, and distinguishes it from that which it ought to be; namely, the temperament of the individual, of which we can give no other account

Theory of
Pro-
babilities.Theory of
Pro-
babilities.

than that different persons will look forward in the same circumstances with different degrees of hope. One man will consider himself better off than before when he has bartered one pound certain for an even chance of two: a second will contemplate loss more strongly than gain, and will consider himself damnified by the exchange. The following is a celebrated problem, by which any reader may test himself in this respect.

A penny is thrown up time after time, until a head appears, when the play ends. If it appear the first time, I gain £1; if not till the second, I gain £4; if not till the third £8; and if not till the n th, £ 2^n : what is the value of my expectation?

The probability of there being no head till the n th throw is $\left(\frac{1}{2}\right)^n$; which, multiplied by the benefit £ 2^n gives

£1: consequently, the mathematical expectation for each throw is £1. But there is a mere possibility of head being deferred till any throw whatsoever; whence if the play is to go on *ad infinitum*, the mathematical expectation is $1+1+1+\dots$ *ad inf.*, or infinite. But, nevertheless, no one would give £20 for his chance at this game; unless it were so small a sum that the mere caprice of amusement would be sufficient inducement.

It is plain that different men would stake a different number of throws, according to their different ideas of caution. But the chance of more than a million of millions of millions of pounds at the 60th throw would tempt no one to give a pound for the chance of 59 tails in succession followed by a head. This problem has always led those who have considered it to the conclusion that the actual expectation is not the mathematical expectation; and very justly as far as a single case is concerned. But if any one were to offer us as many sets of sixty throws as we pleased, without long delay or trouble to ourselves, with the addition of only the millionth part of a farthing for each set, over and above what its result would give us, we should propose to have 10^{100} sets of throws, at the cost of £60 each set, and should feel as certain of winning largely as it is possible to be on any subject. And if it be absurd to talk of so many throws, it is not more so than to talk of 10^{18} pounds. A case may appear absurd* when the *presqu'infini* of one of its conditions is prominently insisted upon, to the exclusion of the *presqu'infini* of another. But, nevertheless, with regard to the single case, the following principle must be admitted.

(41.) *Principle IX.* The mathematical expectation is not a sufficient approximation to the actual phenomenon of the mind, when benefits depend upon very small probabilities; even when the fortune of the player forms no part of the consideration. A great number of mathematical laws might be proposed as hypothetical representatives of the *law of expectation*; but there are no direct means of submitting them to experiment.

(42.) In (38.) it must be observed that we have only compared the proportions in which the risks of different persons are to be adapted to their fortunes. But in considering the value of the different parts of a person's fortune to himself, it is evident that the first shilling risked is of less value than the second, and so on. If we suppose F the amount which an individual should not risk at all, and F' the rest of his fortune, $F+F'$ in all, and if we divide F' into n parts, each equal to dx , and so great in number that dx is small, the values of the successive equal portions, estimated by the proportions they bear to the amount left, will be

$$\frac{m dx}{F+F'-dx}, \frac{m dx}{F+F'-2dx}, \dots, \frac{m dx}{F+F'-n dx} \text{ or } \frac{dx}{F},$$

where m is a constant to be determined.

If we diminish dx and increase n , without limit, we have by known principles the value of the first portion R of F' in the formula

$$m \int_0^R \frac{dx}{F+F'-x}, \text{ or } m \log \frac{F+F'}{F+F'-R} = M_R.$$

If we consider the value of all that it is right to risk as unity, we have, making $R=F'$,

$$m \log \frac{F+F'}{F} = 1, \quad M_R = \frac{\log(F+F') - \log(F+F'-R)}{\log(F+F') - \log(F)};$$

which is, on the preceding suppositions, the moral value of R pounds to a person who has $F+F'$, of which F' is the largest portion he ought to risk. The suppositions entirely depend upon that of the value of a small sum being inversely as the rest of the fortune; if this be admitted, all the rest is demonstration. From the formula, it is evident that any species of logarithms may be chosen.

Thus if a man may risk half his fortune, or if $F=F'$, and if $R=\frac{1}{2}F$, the moral value of the first half is

$$\frac{\log 2 - \log 1\frac{1}{2}}{\log 2}, \text{ or about } .42.$$

With these principles we shall now proceed to some cases of simple mathematical probabilities.

(43.) *PROBLEM.* If two common dice be thrown, what is the probability that the sum of the two numbers thrown is n , which of course is not less than 2, or more than 12?

We have said that mathematics is in this subject only an abbreviator of operations. To show this we shall first solve the problem by inspection, and then apply a method. The possible throws are as follows; all but the doublets (*) may occur in two different ways.

* It is recorded somewhere or other that the pendulum of a clock once revolted, upon the ground of the enormous number of times it had to vibrate in one year; but was appeased by one of the weights, a more solid thinker, who reminded it that it would have just as enormous a number of seconds to vibrate in. Report also says, that not only did the pendulum return to its duty, but that it never went too fast or too slow again; a commendable example for our readers and ourselves.

Theory of
Pro-
babilities.

*1+1=2	*2+2=4	3+4=7	*5+5=10
1+2=3	2+3=5	3+5=8	5+6=11
1+3=4	2+4=6	3+6=9	*6+6=12
1+4=5	2+5=7	*4+4=8	
1+5=6	2+6=8	4+5=9	
1+6=7	*3+3=6	4+6=10	

Theory of
Pro-
babilities.

The probability in favour of 2 is $\frac{1}{36}$ 5 is $\frac{4}{36}$ 8 is $\frac{5}{36}$ 11 is $\frac{2}{36}$
 3 is $\frac{2}{36}$ 6 is $\frac{5}{36}$ 9 is $\frac{4}{36}$ 12 is $\frac{1}{36}$
 4 is $\frac{3}{36}$ 7 is $\frac{6}{36}$ 10 is $\frac{3}{36}$

If we now consider the expression

$$x + x^2 + x^3 + x^4 + x^5 + x^6, \text{ or } x \frac{1-x^6}{1-x},$$

and multiply it by itself, the rule of multiplication (*addition* of exponents) combined with reasoning similar to that in (11.) shows us that so many times as x^n appears in the result, so many ways can n be formed by a couple taken out of 1, 2, 3, 4, 5, 6, with repetition. But

$$\left(x \frac{1-x^6}{1-x}\right)^2 = x^2 (1-2x^6+x^{12}) (1+2x+3x^2+\dots) \\ = x^2 + 2x^3 + 3x^4 + \&c.$$

by actual multiplication. The whole number of cases is 6^2 , or 36, of which 3 (the coefficient of x^3 in the preceding) favour the production of 2; whence the result in this case. This is not properly an abbreviated process in this simple instance; but it leads to the two following theorems, which will afterwards be of the highest utility.

If there be a ways of drawing a number α , b of drawing a number β , &c., all equally probable, then the probability of drawing a set of n counters, the sum of the numbers on which shall be p , is the coefficient of x^p in the development of $(ax^\alpha + bx^\beta + \dots)^n$.

The same theorem applies when any of the set α , β , &c. are negative, provided counters so marked are to be subtracted.

We shall now choose a problem in which this method will be an abbreviation and a security.

What is the chance of throwing 10 on three dice of 10 faces marked from 1 to 10? *Answer.* The number of ways is the coefficient of x^{10} in the development of

$$(x + x^2 + x^3 + \dots + x^{10})^3 \text{ or } x^3 (1-x^{10})^3 (1-x)^{-3},$$

or

$$x^3 (1-3x^{10}+3x^{20}-x^{30}) (1+3x+6x^2+10x^3+\dots).$$

This is the coefficient of x^7 in the latter term, or 36.

But the whole number of possible cases is 10^3 , or 1000, whence $\frac{9}{250}$, or nearly 1 to 27, is the probability in favour of the event.

The preceding method may be thus simplified. It is clear that the number of ways in which p can be the sum of n drawings out of 1, 2, 3, ... is exactly the same as the number of ways in which $p-n$ can be the sum of n drawings out of 0, 1, 2, 3, ..., and also that the number of ways in which a given sum can be drawn from m counters does not depend upon the number of these counters, unless some be wanting which may enter into that sum.

For example, if we would know in how many ways 30 can be drawn in four trials out (1, 2, ..., 27) and (1, 2, ..., 20), we have two different problems: there are ways of making thirty from four drawings of the first which do not exist in the second. But if we ask for the *number of ways* of drawing thirty in four trials from (1, 2, ..., 40) and (1, 2, ..., 100), the two problems are the same, as is sufficiently evident. But it is clear that the *probabilities* of drawing thirty are less in the second than in the first. In talking of a die of ten faces we mean that there are to be supposed ten ways of obtaining a number, all equally probable. A prism of ten sides, with ends too small to allow of its falling with either end upwards, would answer the purpose.

(44.) We shall now apply the resources of mathematics to the general problem, and shall proceed, in the manner of Laplace, to point out the application to some phenomena of the material universe.

Let there be $n+1$ counters, marked 0, θ , 2θ , 3θ , ..., $n\theta$; one is drawn and put back again; and so on. What is the probability that i drawings will give $s\theta$, s being a whole number? This is evidently the same problem as finding the probability of drawing s in i drawings from 0, 1, 2, ..., n .

* We have followed Mr. Lubbock (*Probability, Lib. Useful Knowl.*) in substituting the principles just laid down for the very unwieldy method of Laplace, which nevertheless is an excellent deduction from something like inspection of cases.

While on this subject we may state, that though we should in any case have thrown aside most of the problems on gambling which usually fill works on our subject, in order to make room for those principles which "have contributed much to the accuracy of modern astronomy;" yet we do this with the more satisfaction, that there exists in our language such a collection of these problems, and in so accessible a form, as that given by Mr. Lubbock. They have the additional advantage of having been, to all appearance, selected more with a view to show the manner in which mathematics may be applied, than on account of any value attached to the chances of a card or a die, considered as a result.

Theory of
Pro-
babilities.

The result required is the coefficient of x^s in the developement of $(x^0 + x^1 + x^2 + \dots + x^n)^i$ or $(1 - x^{n+1})^i$ or $(1 - x)^{-i}$, or the product of the two series

Theory of
Pro-
babilities.

$$1 - ix^{n+1} + \frac{i(i-1)}{2} x^{2n+2} - \frac{[i, i-2]}{[3]} x^{3n+3} + \&c.$$

$$1 + ix + \frac{i(i+1)}{2} x^2 + \frac{[i, i+2]}{[3]} x^3 + \frac{[i, i+3]}{[4]} x^4 + \&c.;$$

where, as before, $[i, i+z]$ means $i(i+1) \dots (i+z)$, $[z]$ means $1 \cdot 2 \cdot 3 \dots z$,

we wish to extract the term Sx^s from the product of the preceding, and we must therefore ask in what ways s can be made by taking one out of each of the sets

$$\begin{array}{cccccc} 0 & n+1 & 2n+2 & 3n+3 & \&c. \\ 0 & 1 & 2 & 3 & \&c. \end{array}$$

or to find all the solutions of $V(n+1) + W = s$, where V and W are whole numbers (0 being included.) This gives

Exponent from first series.	Exponent from second series.	Corresponding coefficient of x^s .
$V=0$	$W=s$	$1 \times \frac{[i, i+s-1]}{[s]}$
$V=1$	$W=s-n-1$	$-i \times \frac{[i, i+s-n-2]}{[s-n-1]}$
$V=2$	$W=s-2n-2$	$i \frac{(i-1)}{2} \frac{[i, i+s-2n-3]}{[s-2n-2]}$

and so on, up to

$V = \text{whole number next below } \frac{s}{n+1}, \text{ say } v;$

$$W = s - vn - v; \text{ giving } \pm \frac{[i, i-v+1]}{[v]} \cdot \frac{[i, i+s-nv-v-1]}{[s-nv-v]}$$

The sum of these $v+1$ terms is the coefficient required, or the number of ways of drawing s in i trials from $0, 1, \dots, n$; and this divided by $(n+1)^i$, the whole number of possible drawings, gives for the probability of drawing s , $v+1$ terms of

$$\frac{1}{(n+1)^i} \left\{ \frac{[i, i+s-1]}{[s]} - i \frac{[i, i+s-n-2]}{[s-n-1]} + \frac{i(i-1)}{2} \frac{[i, i+s-2n-3]}{[s-2n-2]} + \dots \right\}$$

which, by the application of the formula

$$\frac{[i, z+n]}{[z]} = \frac{[z+n]}{[i-1] \cdot [z]} = \frac{[z+1, z+n]}{[i-1]},$$

(which gives a common denominator) becomes $v+1$ terms of

$$\frac{1}{[i-1](n+1)^i} \left\{ [s+1, s+i-1] - i[s-n, s+i-n-2] + i \frac{i-1}{2} [s-2n-1, s+i-2n-3] - \&c. \right\}$$

As an instance, suppose we ask in how many ways 25 can be drawn from three drawings of $0, 1, \dots, 10$. Here $i=3, n=10, s=25$, and $v=2$; therefore the term in brackets ($v+1=3$) is

$$[26, 27] - [15, 16] \times 3 + [4, 5] \times \frac{3 \cdot 2}{2}, \text{ or } 42;$$

which divided by $[i-1]$ or 1.2 gives 21 for the number of ways in question, which may be easily verified by observing that $10+10+5$ may be drawn in *three* ways; $10+9+6$ in *six* ways; $10+8+7$ in *six* ways; $9+9+7$ in *three* ways; $9+8+8$ in *three* ways; and $3+6+6+3+3=21$. But $(n+1)^3=1331$; therefore the probability required is $\frac{21}{1331}$.

(45.) Now let us suppose the number n to be increased without limit, and θ , the *unit of the counters*, to be diminished without limit, in such manner that $n\theta$, the highest drawing, shall always be N ; and let s at the same time increase without limit, so that $s\theta$ shall always be S . The common method of the integral calculus shows that we must thus obtain the infinitely small probability* of obtaining *exactly* S for the sum of i drawings, where each drawing may give *any thing* between 0 and N , and all intermediate quantities have equal probabilities. It is evident that each product within the $\{ \}$ has $i-1$ terms, which, severally, as s increases without limit, approximate to s , in the first product, to $s-n$ in the second, to $s-2n$ in the third, and to $s-vn$ in the last; v being

* If we should suppose a point to be thrown at a straight line, all points of which it has the same chance of hitting, it is plain that the probability of hitting any given point in a finite number of trials is infinitely small. If we were here advocating the theory of infinitely small quantities, (which we only use as an abbreviation of language,) we should say that we have a conception upon which the mind can found a notion of *infinitely small* as distinguished from *nothing*. No one can say there is *no probability whatsoever* of hitting a given point; while it is clear there is only one case in favour of it against an infinite number of unfavourable cases.

Theory of
Pro-
babilities.

the whole number next below $\frac{s}{n+1}$, or $\frac{s\theta}{n\theta+\theta}$, or $\frac{S}{N+\theta}$, or $\frac{S}{N}$, since θ diminishes without limit. Hence the preceding probability continually approaches to $v+1$ terms of

Theory of
Pro-
babilities.

$$\frac{1}{[i-1](n+1)} \left\{ \left(\frac{s}{n+1} \right)^{i-1} - i \left(\frac{s-n}{n+1} \right)^{i-1} + i \frac{i-1}{2} \left(\frac{s-2n}{n+1} \right)^{i-1} - \&c. \right\},$$

which itself may have the first factor written in the form $\frac{\theta}{[i-1](N+\theta)}$, and (multiplying the numerators and denominators of the fractions in the $\{ \}$ by θ) may be shown to approach to a ratio of equality with $v+1$ terms of

$$\frac{\theta}{[i-1]N} \left\{ \left(\frac{S}{N} \right)^{i-1} - i \left(\frac{S}{N} - 1 \right)^{i-1} + i \frac{i-1}{2} \left(\frac{S}{N} - 2 \right)^{i-1} - \&c. \right\}.$$

Now θ is the differential of S , for the possible drawing next higher than S or $s\theta$ is $(s+1)\theta$ or $S+\theta$, $\therefore \theta = dS$; and we have thus the infinitely small probability in question. To determine the probability of the i drawings giving a total between 0 and a certain specific value of S , we must add the probabilities of each of the several cases, 0, dS , $2dS$, &c., up to the specific value; that is, we must integrate the preceding expression with respect to S from $S=0$ up to S = the specific value; which gives for the said probability (i resulting in the denominator from each integration, changing $[i-1]$ into $[i]$) $v+1$ terms of

$$\frac{1}{[i]} \left\{ \left(\frac{S}{N} \right)^i - i \left(\frac{S}{N} - 1 \right)^i + i \frac{i-1}{2} \left(\frac{S}{N} - 2 \right)^i - \&c. \right\}.$$

If $\frac{S}{N}$ be a whole number, it is plain from the original investigation that v is that whole number. If we then propose to find the probability of three drawings giving a total between 0 and $3N$, we must take $S=3N$, $v=3$, $i=3$, which gives

$$\frac{1}{2 \cdot 3} \{ 3^3 - 3 \cdot 2^3 + 3 \cdot 1^3 - 1 \cdot 0^3 \}, \text{ or } 1;$$

that is, *certainty*, which is evidently true. Let the student deduce this mathematical theorem independently;

$$[i] = i^i - i(i-1)^i + i \frac{i-1}{2} (i-2)^i - \&c.,$$

which is here the means of indicating the certainty of i throws, each between 0 and N , giving not more than iN

(46.) There are ten planets, besides the earth, the orbit of which is the ecliptic. Each of these ten planets has an orbit inclined in a certain degree to the ecliptic. The inclinations of these ten orbits are so small that their sum is only $91^\circ 41' 87''$ (French metrical system) less than one alone might be (100°), if the orbits of the planets were the work of chance.

Suppose it possible that these ten inclinations were not dependent on any cause, but that they were formed in a manner which would render any one inclination from 0° to 100° as likely as another. What then is the probability that we should have ten inclinations, the total of which is under 92° . In this case $S=92^\circ$, $N=100^\circ$, $v=0$, and the first term of the formula gives

$$\frac{1}{[10]} \left(\frac{92}{100} \right)^{10}, \text{ or } .00000012;$$

and if there be a reason for the inclinations being as described, the probability of the event is 1. Consequently it is 1 : .00000012, or about ten million to one that there was a necessary cause in the formation of the solar system for the inclinations being what they are. The same reasoning, applied to all the observed comets, does not give any ground for suspecting that their inclinations are similarly circumstanced.

(47.) If we suppose the ratio of the probabilities of drawing exactly P and P' (lying between 0 and N) in single trials to be p and p' , then the probability of drawing between P and $P+dP$ in a single trial must be proportional to $p dP$. If we consider p itself as a function of P , and dP as infinitely small, then any variation of probability between P and $P+dP$ will itself be infinitely small, and may be disregarded in connection with p . This principle is precisely the same as that by which, P representing the accelerating force on a particle, $P dm$ represents the moving force on the element dm .

If, then, we wish to know the probability of a drawing lying between a and b , we must integrate $p dP$ from $P=a$ to $P=b$, which we can do when p is a known function of P . For instance, if the probability of drawing exactly P be aP^2 for any value of P between 0 and N , we must have $\int_0^N aP^2 dP = 1$, if there be certainty of drawing between 0 and N , or $aN^3 = 3$: so that it is a rational supposition to lay down the case, that a drawing may be any where between 0 and N with a probability of $\frac{3P^2}{N^3}$ for the drawing being between P and $P+dP$. And the probability of drawing between a and $a+k$ is therefore

$$\frac{3}{N^3} \int_a^{a+k} P^2 dP, \text{ or } \frac{(a+k)^3 - a^3}{N^3}.$$

In just the same way, if there be two drawings, and the probabilities of their severally giving results between P and $P+dP$ and Q and $Q+dQ$ be $p dP$ and $q dQ$, the probability of both events taking place is $p q dP dQ$, from which may be found by integration the probability of both drawings lying between given pairs of limits.

Theory of
Pro-
babilities.

There is no need to insist on the various ways in which the integrations may be required to be made, as they are all sufficiently considered in the applications of the integral calculus to geometry.

Theory of
Pro-
babilities.

(48.) We now come to the question of ascertaining how to judge of the probability of future events from those which are past, on the supposition that the circumstances which produced the latter continue in force. This must be the practical problem in all cases; for we cannot command urns with a given proportion of black and white balls, nor can we open those which circumstances present, but must be content to draw what we can, and form our conclusions as to those which are to come from our experience of those which have appeared. This, as explained in (19.), must be done by referring the event to every possible cause, deducing the probability of each cause from the event, and then deducing the probability of the coming event from each of the possible causes.

PROBLEM. There is an urn containing counters marked W, X, Y, and Z. Of $m+n+p+q$ single drawings, (restoring each drawing again,) m were marked W, n were marked X, p were marked Y, and q were marked Z. What is the probability that in the next $m'+n'+p'+q'$ drawings m' shall be marked W, n' shall be marked X, &c., supposing that we have no reason whatever to imagine the contents to be in any one ratio rather than another, except what we gather from the observed event? *we are, however, supposed to know that either W, X, Y, or Z must come out.*

Let $w, x, y,$ and z be the probabilities of drawing W, X, Y, and Z at one throw. Then

$$w+x+y+z=1; \quad (1)$$

and the probability of the observed event is $w^m x^n y^p z^q$, for their coming out in any assigned order. But the whole number of possible orders is (10.) $[m+n+p+q]$ divided by $[m]. [n]. [p]. [q]$, which call A. Then $A w^m x^n y^p z^q$ is the probability of the observed event. Now let us suppose that $w, x, y,$ and z may be severally taken out of the sets $\omega, 2\omega, \dots$ &c. *ad inf.* $\xi, 2\xi, \dots$ &c., $\eta, 2\eta, \dots$ &c., $\zeta, 2\zeta, \dots$ &c., in any manner consistent with equation (1). This gives for the probability of w, x, y, z , being the real proportions, the fraction

$$\frac{A w^m x^n y^p z^q}{\sum A w^m x^n y^p z^q}, \text{ or } \frac{w^m x^n y^p z^q}{\sum w^m x^n y^p z^q}, \text{ which call } \frac{P}{\sum P},$$

in the denominator of which (19.) every possible state of the numerator is a term. If we multiply numerator and denominator by $\omega \eta \xi$, and then diminish the three factors without limit, which, as in the last problem, is equivalent to representing the last product by $dw dx dy$, we find the preceding become $\frac{P dw dx dy}{\int P dw dx dy}$; which, since the inte-

gration is to be made for every possible value of $w, x, y,$ and z or $1-x-y-w$, gives the following: suppose w and x constant, and integrate with respect to y from $y=0$ to $y=1-w-x$; then suppose w constant, and integrate with respect to x from $x=0$ to $x=1-w$; thirdly, integrate with respect to w from $w=0$ to $w=1$.

LEMMA.

$$\int_0^a v^m (a-v)^n dv = [m] \cdot [n] \div [m+n+1] \times a^{m+n+1}.$$

For

$$\int v^m (a-v)^n dv = \frac{v^{m+1} (a-v)^n}{m+1} + \frac{n}{m+1} \int v^{m+1} (a-v)^{n-1} dv.$$

Call the required integral between the given limits $V_{m,n}$; then

$$V_{m,n} = \frac{n}{m+1} V_{m+1,n-1} = \frac{n(n-1)}{(m+1)(m+2)} V_{m+2,n-2}$$

$$\dots = \frac{[n]}{[m+1, m+n]} \{ V_{m+n,0}, \text{ or } \int_0^a v^{m+n} dv \};$$

whence

$$\int_0^a v^m (a-v)^n dv = \frac{[m][n]}{[m+n+1]} a^{m+n+1};$$

because

$$\frac{[n]}{[m+1, m+n]} \frac{1}{m+n+1} = \frac{[m][n]}{[m+n+1]}$$

The steps of the application of the lemma are as follows:

Integrate	from	to	which gives
$y^p (1-w-x-y)^q dy$	$y=0$	$y=1-w-x$	$\frac{[p][q]}{[p+q+1]} (1-w-x)^{p+q+1}$
$x^n (1-w-x)^{p+q+1} dx$	$x=0$	$x=1-w$	$\frac{[n][p+q+1]}{[n+p+q+2]} (1-w)^{n+p+q+2}$
$w^m (1-w)^{n+p+q+2} dw$	$w=0$	$w=1$	$\frac{[m][n+p+q+2]}{[m+n+p+q+3]}$

whence the denominator required is $\frac{[m][n][p][q]}{[m+n+p+q+3]} = \frac{1}{C}$; or the probability of w, x, y, z being the proportions

of the counters, is $C w^m x^n y^p z^q dw dx dy dz$. If this were certain, the probability of the proposed prospect in a given order would be $w^{m'} x^{n'} y^{p'} z^{q'}$, therefore the probability that the prospect will be realized from this cause only is (20.) $A' \times C w^{m+m'} x^{n+n'} y^{p+p'} z^{q+q'} (1-w-x-y)^{q+q'} dw dx dy$, where A' is (what A was to the observed event) the number of orders in which the prospect may be realized, namely, $[m'+n'+p'+q'] \div [m'][n'][p'][q']$; and therefore, to find the whole probability, taking every possible cause into account, this must be integrated in the same manner as the preceding, which gives (restoring A' and C)

Theory of
Pro-
babilities.

or

$$\frac{[m'+n'+p'+q']}{[m'] [n'] [p'] [q']} \frac{[m+n+p+q+3]}{[m] [n] [p] [q]} \frac{[m+m'] [n+n'] [p+p'] [q+q']}{[m+n+p+q+m'+n'+p'+q'+3]},$$

$$\frac{[m'+n'+p'+q']}{[m'] [n'] [p'] [q']} \frac{[m+1, m+m'] [n+1, n+n'] [p+1, p+p'] [q+1, q+q']}{[m+n+p+q+4, m+n+p+q+m'+n'+p'+q'+3]}.$$

Theory of
Pro-
babilities.

It is evident that 3 will be replaced by $i-1$, when there are i sorts.

Suppose, for instance, the first six drawings have given four white and two black balls, (assuming only white and black to be possible,) what is the probability that the next two balls shall be both white, or both black, or white and black? These three cases suppose

$$\begin{array}{lll} i=2 & i-1=1 \\ \text{I. } m=4 & n=2; & \text{both white} \dots m'=2 \quad n'=0 \\ \text{II. } m=4 & n=2; & \text{white and black} \quad m'=1 \quad n'=1 \\ \text{III. } m=4 & n=2; & \text{both black} \dots m'=0 \quad n'=2 \end{array}$$

$$\begin{array}{lll} \text{I. } \frac{2 \cdot 1}{2 \cdot 1} \frac{[7]}{[4] [2]} \cdot \frac{[6] [2]}{[9]} = \frac{30}{72} \\ \text{II. } \frac{2 \cdot 1}{1 \cdot 1} \frac{[7]}{[4] [2]} \frac{[5] [3]}{[9]} = \frac{30}{72} \\ \text{III. } \frac{2 \cdot 1}{2 \cdot 1} \frac{[7]}{[4] [2]} \frac{[4] [4]}{[9]} = \frac{12}{72}; \end{array}$$

these are the probabilities of the three cases; their sum is unity, as the hypothesis requires.

The general formula is as follows; let there be i sorts, and let the observed event be m_1 of the first, m_2 of the second, &c., and the prospect of which the probability is required be m'_1 of the first, m'_2 of the second, &c. Let Σm and $\Sigma m'$ denote the number of drawings made and to be made: then the probability which the observed event gives to the proposed is

$$\frac{[\Sigma m']}{[m'_1] \dots [m'_i]} \cdot \frac{[m_1+1, m_1+m'_1] \dots [m_i+1, m_i+m'_i]}{[\Sigma m+i, \Sigma m+\Sigma m'+i-1]}.$$

Thus if the prospect be that of drawing one ball of the r th species, the probability is $\frac{m_r+1}{\Sigma m+i}$.

(49.) There remains, however, an important case not yet considered; suppose that having obtained i sorts in Σm drawings, and i sorts only, we do not take it for granted that these are all the possible cases, but allow ourselves to imagine there may be sorts not yet come out. This requires that in the integration we should not suppose z to be $1-w-x-y$, but treat it as an independent variable, subject only to the condition that its limits of variation must be 0 and $1-w-x-y$. That is, integrate between these limits before substitution, or begin with $z^{i+1} \div (q+1)$ instead of z^i , introducing $dw dx dy dz$ as a factor of the numerator and denominator, instead of $dw dx dy$. This process changes the result in nothing but substituting $i+1$ for i in the last denominator, and gives

$\frac{m_r+1}{\Sigma m+i+1}$ for the probability of the next drawing being of the r th sort; adding together therefore the probabilities of the first drawing in prospect being of either of the sorts already obtained, we find *not* certainty, as before, but $(\Sigma m+i) \div (\Sigma m+i+1)$, leaving one chance against $\Sigma m+i$ that the new sort may be different from any preceding. Therefore we have this theorem:

Judging of future events from the past, and in no other manner whatsoever, if Σm observed events show i varieties, it is $\Sigma m+i$ to 1 that the next event shall be a repetition of some one preceding; and, *ceteris paribus*, the greater the number of varieties the more probable is it that no others will happen. But the last part of this theorem may be mistaken. Do we mean to say, for instance, that if a continued succession of drawings gave 1, 2, 3 . . . up to 1000, it would be more probable that the next should be a repetition of some preceding number (and not 1001) than it would have been when we had only drawn 1, 2? In asking this question, the reader is taking the word event in a wrong sense. He is turning his ideas from the mere *drawing of a number* to a *relation between two successive numbers*, and constituting the latter the event in question, while he wonders at a conclusion drawn from the former. Let the *succession* not have been observed, but only the numbers, and then $\Sigma m=1000$, $i=1000$, and it is* 2000 to 1 upon this observed event that the 1001st drawing will be a repetition. But let him have observed the succession; that is, let the observed event be the repetition of the one event, *ordinal succession*; and 1. The very number of events observed is different, being 999 and not 1000. 2. Since $i=1$ and $\Sigma m=999$, it is 1000:1 in favour of *one more* succession; and the same conclusion would have been obtained if the continual difference of each drawing from the preceding had been the event observed. We are here met by the same difficulty which we stated in (15.), namely, the necessity of considering the combined probability arising from an event having different probabilities when considered in different relations, or as the result of different trains of inference. We have now got beyond the point within which we are prepared to try the results of our theory by our own impressions, or those of mankind in general. Cases so complicated as the choice out of an infinite number of causes, to be made according to the several probabilities of their yielding an observed event, are beyond the reach of unassisted judgment. Whether the mathematical theory of probabilities be a good and trustworthy guide may be a question: but it is not to be opposed on the strength of any independent impressions which exist

* We must again warn the reader that probabilities are in his mind, not in the urn from which he draws.

Theory of Probabilities. upon suppositions so complicated as those we have entered into. Let us take the theory of probabilities itself upon probability; if there be an even chance that its principles are trustworthy, we have then a full even chance that the consequences are the same; for though remote, they are connected with the first principles by absolute demonstration. Theory of Probabilities.

(50.) We shall now take some problems involving complicated possibilities, not so much for the sake of numerical solutions, as of showing various methods by which the difficulty is reduced to a purely mathematical form.

PROBLEM. Two players, A and B, of equal skill, want x and y points of winning a game, and whichever makes up his points first, wins. What are the probabilities in favour of each?

Let $\phi(x, y)$ be A's probability; if he win the next game (of which the chance is $\frac{1}{2}$) his chance will then be $\phi(x-1, y)$, and therefore of his winning the first point, and winning finally, the chance is $\frac{1}{2}\phi(x-1, y)$; similarly, of his losing the first point, and of winning the game, his chance $\frac{1}{2}\phi(x, y-1)$. But the sum of the two latter probabilities makes up his present chance; therefore

$$\phi(x, y) = \frac{1}{2}\phi(x-1, y) + \frac{1}{2}\phi(x, y-1) \dots (1).$$

Now imagine that $\phi(x, y)$ is the coefficient of $t^x v^y$ in a developement which contains all whole values of x and y , such as $\phi(0, 0) + \phi(0, 1)v + \phi(1, 0)t + \dots + \phi(x, y)t^x v^y + \dots$

Let this function be $\psi(t, v)$, or $\sum \phi(x, y)t^x v^y$; then it is plain that $t\psi(t, v)$ and $v\psi(t, v)$ will have for their developements

$$\begin{aligned} \sum \phi(x-1, y)t^x v^y & (\sum \text{not containing terms in which } x=0) = t\psi(t, v), \\ \sum \phi(x, y-1)t^x v^y & (\sum \text{not containing terms in which } y=0) = v\psi(t, v); \end{aligned}$$

whence (if $x > 0, y > 0$, both restrictions being made)

$$\begin{aligned} \psi(t, v) - \frac{1}{2}t\psi(t, v) - \frac{1}{2}v\psi(t, v) &= \sum \left(\phi(x, y) - \frac{1}{2}\phi(x, y-1) - \frac{1}{2}\phi(x-1, y) \right) t^x v^y \quad (x > 0, y > 0) \\ &+ \left\{ \begin{array}{l} \text{sum of terms of } \sum \phi(x, y)t^x v^y \text{ omitted by omitting severally } x=0, y=0, \\ \text{or } \phi(0, 0) + \phi(0, 1)v + \phi(0, 2)v^2 + \dots \\ \quad + \phi(1, 0)t + \phi(2, 0)t^2 + \dots \end{array} \right\} \\ &- \left\{ \begin{array}{l} \frac{1}{2} \text{ sum of terms of } \sum \phi(x-1, y)t^x v^y \text{ omitted by omitting } y=0, \\ \text{or } \phi(0, 0)t + \phi(1, 0)t^2 + \dots \end{array} \right\} \\ &- \left\{ \begin{array}{l} \frac{1}{2} \text{ sum of terms of } \sum \phi(x, y-1)t^x v^y \text{ omitted by omitting } x=0, \\ \text{or } \phi(0, 0)v + \phi(0, 1)v^2 + \dots \end{array} \right\} \end{aligned}$$

But $\phi(0, 0) = 0$, there is no case in which both can win; $\phi(0, y) = 1$, A has actually won when he has no points left to gain; $\phi(x, 0) = 0$, A cannot win when B wins; and $\phi(x, y) - \frac{1}{2}\phi(x-1, y) - \frac{1}{2}\phi(x, y-1) = 0$ in all cases. The preceding therefore gives

$$\left(1 - \frac{1}{2}t - \frac{1}{2}v\right)\psi(t, v) = v + v^2 + \dots - \frac{1}{2}(v^2 + v^3 + \dots) = \frac{v\left(1 - \frac{1}{2}v\right)}{1-v},$$

or

$$\psi(t, v) = \frac{v\left(1 - \frac{1}{2}v\right)}{(1-v)\left(1 - \frac{1}{2}t - \frac{1}{2}v\right)},$$

in the developement which the coefficient of $t^x v^y$ is the probability required.

The developement of this in powers of t is

$$\frac{v}{1-v} \left\{ 1 + \frac{1}{2} \frac{t}{1 - \frac{1}{2}v} + \dots + \frac{1}{2^x} \frac{t^x}{\left(1 - \frac{1}{2}v\right)^x} + \dots \right\},$$

and the coefficient of v^y in $\frac{v}{1-v} \left(1 - \frac{1}{2}v\right)^{-x}$,

or

$$(v + v^2 + v^3 + \dots) \times \left(1 + x \frac{v}{2} + x \frac{x+1}{2} \frac{v^2}{4} + \dots\right)$$

is the sum of all the values of the coefficient of

$$v^\theta \times \frac{[x, x+\lambda-1]}{[\lambda]} \frac{v^\lambda}{2^\lambda},$$

where $\theta + \lambda = y$, from $\lambda = 0, \theta = y$, to $\lambda = y-1, \theta = 1$. This multiplied by 2^{-x} , is

Theory of
Pro-
babilities.

$$\frac{1}{2^x} \left\{ 1 + x \cdot \frac{1}{2} + x \frac{x+1}{2} \frac{1}{2^2} + \dots + \frac{[x, x+y-2]}{[y-1]} \frac{1}{2^{y-1}} \right\}.$$

Theory of
Pro-
babilities.

Suppose, for instance, $x=2, y=3$, and the game must be decided in five points at most. The orders favourable to A winning are AA, BAA, ABA, ABBA, BBAA, BABA, the several probabilities of which (the chances being even) are $\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{16}, \frac{1}{16}, \frac{1}{16}$, the sum of which is $\frac{11}{16}$. This will be also the result of the preceding expression; for when $x=2, y=3$, it becomes

$$\frac{1}{4} \left(1 + \frac{2}{2} + \frac{2 \cdot 3}{2} \frac{1}{4} \right) = \frac{22}{32} = \frac{11}{16}.$$

We have introduced the preceding problem that the student may be prepared* to admit the use of Laplace's *Theory of Generating Functions*, which we now proceed to describe. The preceding is a case in which its principles are applied. It may be stated in the following words. To find P_x a required function of x , where x is a whole number, endeavour to find the series of powers of t , which shall have $P_0, P_1, P_2, \&c.$ for coefficients of $t^0, t^1, t^2, \&c.$; that is, find ϕt the finite envelopment of $P_0 + P_1 t + P_2 t^2 + \&c.$ It will then follow that every method of developing ϕt will also be a method of expressing P_x . And ϕt is then called the *generating function* of P_x , (which we shall abbreviate into *generatrix*), and P_x we shall call the *generata* of ϕt . Similarly, when $\psi(t, v) = \sum P_{x,y} t^x v^y$, ψ is the *generatrix* of $P_{x,y}$, &c. The following theorems may now be easily proved.

1. When the generatrices are identical, but in different forms only, the same is true of the generatæ, and *vice versa*.

2. The generatrix of a sum or difference is the sum or difference of the generatrices. Thus if ϕt generate P_x , and χt generate K_x , then $\phi t \pm \chi t$ generates $P_x \pm K_x$; and if $\phi t = \psi t \pm \chi t$, and $\phi t, \psi t$, and χt severally generate P_x, Q_x , and R_x , then $P_x = Q_x \pm R_x$.

3. In the preceding case $\phi t \times \chi t$ generates $K_x P_0 + K_{x-1} P_1 + \dots + K_1 P_{x-1} + K_0 P_x$.

4. If ϕt generate P_x , then $t^n \phi t$ generates P_{x-n} , beginning from $x=n$ inclusive upwards.

5. Let ϕt generate P_x , then $t^{-n} \phi t$ generates 1. the series $\frac{P_0}{t^n} + \frac{P_1}{t^{n-1}} + \dots + \frac{P_{n-1}}{t}$; 2. P_{x+n} for all values of x .

Observe, that the phrase " ϕt generates P_x " means that, x being a whole number, $P_x t^x$ is the only term of the development which contains t^x ; that is, $P_0 + P_1 t + P_2 t^2 + \&c.$ is the whole of the development, so far as whole powers are concerned. But this has no relation to any negative or fractional powers: consequently, we may not say conversely that if ϕt generate P_x , therefore the generatrix of P_x is ϕt ; for ϕt is only one of the generatrices of P_x ; others are contained in $\phi t + \psi t$, if ψt do not admit of expansion, wholly or partially, in whole powers of t . When we pass from a generatrix to the generata, the process is direct; but the passage from the latter to the former involves the addition of an arbitrary function, depending upon the particular circumstances of the case. A student who is used to the differential calculus in its various forms will easily seize this consideration. We shall now show an instance or two of the application of this theory.

Let ϕt generate P_x , then $\frac{1}{t} \phi t$ generates P_{x+1} , and $\left(\frac{1}{t} - 1\right) \phi t$ generates $P_{x+1} - P_x$, or ΔP_x .

Similarly $\left(\frac{1}{t} - 1\right)^2 \phi t$ generates $\Delta P_{x+1} - \Delta P_x$, or $\Delta^2 P_x$, and $\left(\frac{1}{t} - 1\right)^n \phi t$ generates $\Delta^n P_x$. But

$$\left(\frac{1}{t} - 1\right)^n \phi t = \frac{1}{t^n} \phi t - n \frac{1}{t^{n-1}} \phi t + n \frac{n-1}{2} \frac{1}{t^{n-2}} \phi t - \&c.;$$

and therefore by theorem (2) preceding,

$$\Delta^n x = u_{x+n} - n u_{x+n-1} + n \frac{n-1}{2} u_{x+n-2} - \&c.,$$

a well known theorem. Similarly $\frac{1}{t^n} = \left\{ 1 + \left(\frac{1}{t} - 1\right) \right\}^n$ expanded gives $u_{x+n} = u_x + n \Delta x + \&c.$; and to use the words of Laplace, which contain a theorem more easily deducible from first principles than any other of the same extent, "*Toutes les manières de développer la puissance $\frac{1}{t^r}$, donnent autant de manières différentes d'interpoler les suites.*"

For example, suppose it required to express P_{x+i} in terms of $P_x, \Delta P_x, \Delta^2 P_x, \&c.$ We can do this the moment we have expanded $\frac{1}{t^i}$ in powers of $\left(\frac{1}{t} - 1\right) t^r$. Let this be α ; then $\frac{1}{t} = 1 + \alpha \left(\frac{1}{t}\right)^r$: use Lagrange's theorem to develop $\left(\frac{1}{t}\right)^i$; pass from the generatrices to the generatæ, and we have

$$P_{x+i} = P_x + i \Delta P_x + \frac{i(i+2r-1)}{2} \Delta^2 P_x + \dots + \frac{i(i+n-1) \dots (i+n-1)}{2 \cdot 3 \cdot 4 \dots n} \Delta^n P_x + \&c.$$

* Is not the neglect of this theory a consequence of the introduction of it into books without any results of application except such as have been already obtained without it?

Theory of
Probabilities.

(51.) We have found that $\left(\frac{1}{t}-1\right)^n \phi t$ generates $\Delta^n P_x$, when n is a positive whole number. let us examine the consequences of supposing $n = -1$. In that case

Theory of
Probabilities.

$$\left(\frac{1}{t}-1\right)^{-1} \phi t = \{t+t^2+t^3+\dots\} \{P_0+P_1 t+P_2 t^2+\&c.\} \\ = P_0 t + (P_0+P_1) t^2 + (P_0+P_1+P_2) t^3 + \&c.$$

But $\Sigma P_x = P_0 + P_1 + \dots + P_{x-1}$; therefore $\left(\frac{1}{t}-1\right)^{-1} \phi t$ generates ΣP_x . If we extend the theorem with respect to $\Delta^n P_x$ to the case of negative whole numbers, we see that it is necessary that $\Delta^{-1} P_x$ should stand for ΣP_x .

Let $T = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$, $a_0, a_1, \&c.$ being constants. Then if ϕt generate P_x , it will be found that $T \phi t$ generates $a_0 P_x + a_1 P_{x-1} + \dots + a_n P_{x-n}$, from $x = n$ upwards; or universally, if we suppose P_x to stand for zero whenever x is negative. In fact, the series $T \phi t$ is $a_0 P_0 + (a_0 P_1 + a_1 P_0) t + \dots + \Sigma (a_0 P_x + \dots + a_n P_{x-n}) t^x$, where Σ includes all values of x from n inclusive upwards. If, then, we suppose P_x to be such a function that $a_0 P_x + a_1 P_{x-1} + \dots + a_n P_{x-n} = 0 \dots (1)$, for all values of x from n upwards, and if, moreover, we assume $a_0 P_0 = A_0$, $a_0 P_1 + a_1 P_0 = A_1$, $\&c.$ up to $a_0 P_{n-1} + \dots = A_{n-1}$, we have $T \phi t = A_0 + A_1 t + A_2 t^2 + \dots + A_{n-1} t^{n-1}$, and therefore this theorem: that the equation of differences (1) has for the value of P_x the genera of

$$\frac{A_0 + A_1 t + \dots + A_{n-1} t^{n-1}}{T},$$

where $A_0, A_1, \dots, A_{n-1}$ are the n arbitrary constants which it is known that integration must introduce.

In a similar way it may be proved of the function $P_{x,y}$, which satisfies the equation (from $x = n, y = n$ upwards) $a P_{x,y} + b P_{x-1,y} + c P_{x,y-1} + \dots$ up to $\{k P_{x-n,y} + \dots + s P_{x,y-n}\} = 0$, that it is the coefficient of $t^x v^y$ in the expansion of

$$\frac{A + B t + C v + \dots + (K t^{n-1} + \dots + S v^{n-1})}{a + b t + c v + \dots + (k t^n + \dots + s v^n)},$$

where $A, B, C, \&c.$ are indeterminate, or depend on the indeterminate constants.

(52.) PROBLEM. There are $n+1$ players, $C_1, C_2, \dots, C_{n+1}$, who toss a shilling together in this manner. C_1 first tosses with C_2 , the loser puts a shilling into a common stock, and the winner tosses with C_3 , the process being repeated. If C_1 beat all the rest, the game is finished; if not, the process goes on, each one coming in again in turn, till one has beat all the rest, one after another. What is the probability that this game will be finished at or before the x th throw? The player who wins at the x th throw must come into play for the last time against a winner, and at the $(x-n+1)$ th throw, which he must win, together with the $n-1$ following throws. This preceding winner has won either 1, 2, 3, $\dots$ or $n-1$ throws, not more, for he would then have won the game. Let $P_1, P_2, \dots, P_{n-1}$ be the probabilities that immediately previous to the $(x-n+1)$ th throw, the player in possession shall count 1, 2, 3, $\dots$, or $n-1$ throws won. Let m be a number not $> n-1$; then, if a player have won m throws, and win the remaining $n-m$, he wins the game; and if the former event have been completed in $x-n$ throws, he wins it, if at all, when $x-n+n-m$ (or $x-m$) throws have been made in all. The probability, therefore, of the game finishing in $x-m$ throws (being that of two events of which the probabilities are P_m and $\frac{1}{2^{n-m}}$) is $P_m \times \frac{1}{2^{n-m}}$; or if z_p be the probability that the game finishes in p throws, we have

$$P_m \times \frac{1}{2^{n-m}} = z_{x-m}, \text{ or } P_m = 2^{n-m} z_{x-m}.$$

The probability that a new player shall gain the $(x-n+1)$ th throw is $\frac{1}{2}$; and his chance of winning is evidently made up of the probabilities that he will win by entering against a player who counts 1, 2, 3, $\dots$, or $n-1$ throws. Hence that he will, 1. enter at the $(x-n+1)$ th throw against a player who has won m throws preceding; 2. win the $(x-n+1)$ th; 3. win the $n-1$ following, the probability is

$$P_m \times \frac{1}{2} \times \frac{1}{2^{n-1}}, \text{ or } 2^{n-m} z_{x-m} \times \frac{1}{2^n}, \text{ or } \frac{1}{2^m} z_{x-m};$$

the sum of these probabilities, for all values of m from 1 to $n-1$, is the total probability of his winning, or z_x ; therefore

$$z_x = \frac{1}{2} z_{x-1} + \frac{1}{2^2} z_{x-2} + \frac{1}{2^3} z_{x-3} + \dots + \frac{1}{2^{n-1}} z_{x-n+1} \dots (2)$$

Of this the solution z_x has for its generatrix (51.)

$$\frac{\psi t}{1 - 2^{-1} t - 2^{-2} t^2 - \dots - 2^{-n+1} t^{n-1}}, \text{ or } \frac{\psi t \left\{ 1 - \frac{1}{2} t \right\}}{1 - t + \left(\frac{t}{2} \right)^n},$$

where ψt is to be determined. If we develop the remaining factor (considering ψt as one) its first term is 1, consequently the lowest term of the product is the lowest term in ψt . Now, at first sight, it would appear as if the equation of differences gave $z_x = 0$: for it is obvious that the game cannot be won in $n-1$ throws or less;

Theory of
Pro-
babilities.

that is, $z_1=0, z_2=0 \dots z_{n-1}=0$, therefore (apparently) $z_n=0$ (z_n being from the equation (2) the sum of zeros.) But* the consideration on which the equation was formed excludes all supposition of the game being won at the n th throw; for the player who wins was supposed to enter against a player who had won one throw at least. Therefore the process counts from the $(n+1)$ th game. The generated series has no term corresponding to z_1 , and the $n-1$ constants depend on $z_2, z_3, \dots, z_n$: of which, by the conditions, all are zero except the last, which is

$\frac{1}{2^{n-1}}$, because the game is won if the player who wins the first game (be it either of the two) also win the $n-1$

succeeding games. Therefore, by making $\psi t = \frac{t^n}{2^{n-1}}$, as many terms of the complete series $z_1 t^1 + z_2 t^2 + \dots$ as there are arbitrary constants are made correct, and the developement will give the remainder. Now the probability of winning the game at or before the x th throw is made up of the probabilities of winning it at the n th, $n+1$ th, &c., up to the x th; that is, of $z_n + z_{n+1} + \dots + z_x$. But, as we have seen, if the series $z_n t^n + \dots$ be multiplied by $t + t^2 + \dots$ the coefficient of t^{x+1} will be $z_n + \dots + z_x$: multiply, therefore, by $1 + t + \dots$, or by $\frac{1}{1-t}$, and the coefficient of t^x will be the probability of winning at or before the x th throw. We have, therefore, to find the coefficient of t^x in

$$\frac{t^n}{2^{n-1}} \frac{1 - \frac{1}{2}t}{(1-t)\left(1-t+\frac{t^n}{2^n}\right)}$$

or that of t^x in

$$\frac{t^n}{2^n} \frac{2-t}{(1-t)(1-t+2^{-n}t^n)},$$

which is

$$\frac{1}{2^n} \frac{(2-t)t^n}{(1-t)^2} \left\{ 1 - \frac{2^{-n}t^n}{1-t} + \frac{2^{-2n}t^{2n}}{(1-t)^2} - \dots \right\}.$$

The r th term of the preceding developement (without its sign) is

$$\frac{1}{2^n} \frac{(2-t)t^n}{(1-t)^{r+1}} 2^{-rn+n}, \text{ or } \frac{1}{2^{rn}} \frac{(2-t)t^n}{(1-t)^{r+1}},$$

or

$$\frac{1}{2^{rn-1}} \frac{t^n}{(1-t)^{r+1}} - \frac{1}{2^{rn}} \frac{t^{n+1}}{(1-t)^{r+1}};$$

and the coefficient of t^x in this is

$$\frac{1}{2^{rn-1}} \frac{[r+1, r+x-rn]}{[x-rn]} - \frac{1}{2^{rn}} \frac{[r+1, r+x-rn-1]}{[x-rn-1]} \\ = \frac{1}{2^{rn}} \frac{[r+1, r+x-rn-1]}{[x-rn]} (x-rn+2r) = \frac{1}{2^{rn}} \frac{[x-rn+1, r+x-rn-1]}{[r]} (x-rn+2r),$$

and every term has a coefficient of t^x , for which x is greater than or equal to rn ; or as long as the whole number in $\frac{x}{n}$ is greater than or equal to r . Consequently the probability that the game in question will finish at or before

the x th throw is as many terms of the following series as there are units in the whole number next below $\frac{x}{n}$. (Explain the apparent anomaly in the first term, as found from the formula, by referring to the formula immediately preceding.)

$$\frac{x-n+2}{2^n} - \frac{(x-2n+1)}{1 \cdot 2 \cdot 2^{2n}} (x-2n+4) + \frac{(x-3n+1)(x-3n+2)}{1 \cdot 2 \cdot 3 \cdot 2^{3n}} (x-3n+6) - \&c.$$

If we suppose x to increase without limit, there is a set of terms (the highest powers of x in each) the ratio of which to the sum of the whole continually approximates to unity; namely,

$$\frac{x}{2^n} - \frac{x^2}{1 \cdot 2 \cdot 2^{2n}} + \frac{x^3}{1 \cdot 2 \cdot 3 \cdot 2^{3n}} - \&c., \text{ or } 1 - e^{-\frac{x}{2^n}}.$$

But this latter continually approximates to unity as x is increased without limit; consequently it is certain this game must finish if we are at liberty to continue it for any number of throws. This is a result which appears rather strange; but the extraordinary part is the phrase "it is certain." It would be better to say "it ought to be considered certain," because what has just been said proves the following:—If all the number of possible events in x throws were placed together, and those in which the game would have stopped be compared with those in which the game would not have stopped; then, by making x sufficiently great, the favourable cases may be made to exceed the unfavourable, as p to 1, where p is as great as we please. Play then a number of

* Laplace (p. 240) has omitted all allusion to this circumstance; and the omission is highly characteristic of his method of writing. No one was more sure of giving the result of an analytical process correctly, and no one ever took so little care to point out the various small considerations on which correctness depends. His *Théorie des Probabilités* is by very much the most difficult mathematical work we have ever met with, and principally from this circumstance: the *Mécanique Céleste* has its full share of the same sort of difficulty; but the analysis is less intricate.

Theory of
Pro-
babilities.

Theory of Pro-
babilities. throws so great that the ways in which there is a stoppage are 10,000,000,000,000 as great in number as the Theory of Pro-
babilities. ways in which there is none; and there is more certainty of the game being won somewhere in that number of
throws than of any event of common life which is in the smallest degree *problematical*, in any sense in which the
word is ever used. There are few events, of the happening of which to himself a man ever thinks, which
it is 10^{10} to 1 that he shall be able to avoid.

The preceding is highly illustrative of what we have said (23.) on *runs of luck*. In fact, it requires that some one player should have a run of n throws in his favour before this game is won. And it appears that, however great the number of players, the number of throws allowed may be made so much greater, as to make the chance of *some one* winning as near as we please to certainty. Those who are desirous of seeing the chance of any given player, and the value of it, may consult Laplace's *Théorie des Probabilités*, page 242, &c. We shall now take the wide question of the probability which skill gives to a run of luck, as the phrase is.

(53.) PROBLEM. Let q be the probability of a single event: required the probability of its having happened i times following at or before the x th throw. (The principal difference between this problem and the last, which if $q = \frac{1}{2}$ appear to be the same, lies in this, that every throw in the last puts some one or other one step towards the game; whereas here there is no step towards the game unless one given event happens.)

Let us first ask, what is the probability of such a run being just completed at the x th throw for the first time? The $(x-i)$ th throw must be unfavourable, or else the run of i , which is supposed not to have occurred before the x th, would in fact have occurred at the $(x-1)$ th. Prior to the following succession, (Y the event does happen, N it does not, the number of the throw being underneath,) .

N Y Y Y
..... $x-i$ $x-i+1$ $x-i+2$ x

the throws may be any whatever, provided there be no run of i . Again, the N immediately preceding the $(x-i)$ th throw may be either the $(x-i-1)$ th . . . up to the $(x-2i)$ th, but not further back, for reason as above. Now let P_m represent the probability of this compound event, that there shall have been no run of i successes preceding the $(x-i-m)$ th throw, and that the $(x-i-m)$ th shall be the first unfavourable throw next preceding the $(x-i)$ th, be that which it may. Consequently the $x-i-m+1$. . . $x-i-m+m-1$ are all Ys, $m-1$ in number, and the run will be accomplished in $i-m+1$ more throws, or at the $(x-m)$ th, if all these be Ys, of which the probability is q^{i-m+1} ; that is, the two events whose probabilities are P_m and q^{i-m+1} must happen consecutively to complete the *first* run at the $(x-m)$ th throw. Let z_p be the probability of the first run of i being completed at the p th throw; then will $P_m q^{i-m+1} = z_{x-m}$. Again, the probability of the succession above written by itself, and without reference to what precedes, is $(1-q) q^i$; but to make it the event in question, it must be preceded by one or other of the events whose probabilities are $P_1, P_2, P_3, \dots, P_i$; that one or other of these shall happen, the probability is $P_1 + P_2 + \dots + P_i$, and therefore that one or other of these shall happen, followed by the succession above written, the probability is $(P_1 + P_2 + \dots + P_i) (1-q) q^i$, which is also z_x . But

$$P_1 = z_{x-1} q^{-i} \quad P_2 = z_{x-2} q^{-i+1} \quad \dots \quad P_i = z_{x-i} q^{-1};$$

hence

$$z_x = (1-q) \cdot \{z_{x-1} + q z_{x-2} + q^2 z_{x-3} + \dots + q^{i-1} z_{x-i}\}.$$

After the i th throw, therefore, the probability required, z_x , is the coefficient of t^x in

$$\frac{\psi t}{1 - (1-q)(t + q t^2 + \dots + q^{i-1} t^i)}, \quad \text{or} \quad \frac{\psi t (1-q t)}{1 - q t - (1-q)(t - q^i t^{i+1})},$$

where ψt must be so taken that no term under t^i may appear, (the event cannot happen in less than i throws,) and that the coefficient of t^i may be q^i ; that is, ψt must be $q^i t^i$. If we then divide by $1-t$, the coefficient of t^x becomes $z_i + z_{i+1} + \dots + z_x$, which is the probability of having a run of i in x throws or less; that is, somewhere in x throws. Consequently, we must determine the coefficient of t^x in the development of

$$\frac{1}{1-t} \frac{q^i t^i (1-q t)}{1 - t + (1-q) q^i t^{i+1}},$$

$$\text{or} \quad \frac{1-q t}{(1-t)^2} q^i t^i - \frac{1-q t}{(1-t)^3} (1-q) q^{2i} t^{2i+1} + \frac{(1-q t)}{(1-t)^4} (1-q)^2 q^{3i} t^{3i+2} - \&c.,$$

the r th term (without its sign) being

$$\frac{(1-q t)^{r-1} t^{r-1}}{(1-t)^{r+1}} \times (1-q)^{r-1} q^{ri};$$

and the coefficient of t^x in the first factor is ($x =$ or $> ri + r - 1$),

$$\frac{[r+1, x-ri+1]}{[x-ri-r+1]} - q \frac{[r+1, x-ri]}{[x-ri-r]},$$

$$\text{or} \quad \frac{[r+1, x-ri]}{[x-ri-r+1]} \{x-ri+1 - q(x-ri-r+1)\}, \quad \text{or} \quad \frac{[x-ri-r+2, x-ri]}{[r]} \{(1-q)(x-ri-r+1) + r\}.$$

Multiply this by $(1-q)^{r-1} q^{ri}$, make $r=1, 2, \&c.$, and so on as long as x is greater than $ri + r - 1$, or up to $x=ri+r-1$; make the terms alternately positive and negative, and we have (settling the first term by itself, as before, and making $q'=1-q$.) for the probability required

$$q^i \{q'(x-i)+1\} - q^{2i} q' \frac{x-2i}{1 \cdot 2} \{q'(x-2i-1)+2\} \\ + q^{3i} q'^2 \frac{(x-3i-1)(x-3i)}{1 \cdot 2 \cdot 3} \{q'(x-3i-2)+3\} - \&c.$$

Let us suppose the number of throws to be $6=x$; the number in the run $2=i$; and the chances even, $q=q'=\frac{1}{2}$. We have then $6>3i-1$, or i must not exceed 2. Therefore the probability of having a run of 2 heads in six throws is

$$\frac{1}{4} \left(\frac{1}{2} \cdot 4 + 1 \right) - \frac{1}{16} \cdot \frac{1}{2} \cdot \frac{6-4}{2} \left(\frac{1}{2} \cdot 1 + 2 \right), \text{ or } \frac{3}{4} - \frac{5}{64}, \text{ or } \frac{43}{64}.$$

This may be easily verified, for the cases in which runs of two heads occur, are 1. Where the heads exceed the tails, or $1+6+\frac{6 \cdot 5}{2}$, or 22 cases. 2. Where there are 3 heads and 3 tails, $\frac{6 \cdot 5 \cdot 4}{2 \cdot 3}$, or 20 cases, excepting only HTHHT, THTHT, HTHTH, HTHTH, or 4 cases, remainder 16 cases. 3. Where there are 2 heads and 4 tails, in the cases HHTTTT, THHTT, &c., 5 cases; or $22+16+5$, or 43 cases out of 2^6 , that is, 43 out of 64.

Now comes this question: if two players, whose chances are as q and q' ($q+q'=1$), play on indefinitely, what is the highest run which the first is ever to expect in his favour? If we take out of the preceding the highest powers of x , and thus form the series to which the preceding perpetually approximates as x increases without limit, we find

$$q^i q' x - q^{2i} q'^2 \frac{x^2}{1 \cdot 2} + q^{3i} q'^3 \frac{x^3}{1 \cdot 2 \cdot 3} - \&c., \text{ or } 1 - \varepsilon^{-q^i q'^x}.$$

Consequently we have this singular result, that however much the chances may be in favour of one player, on a single game, so many games can be played that the odds shall be as great as we please to name, in favour of the one who has the smaller chance winning as many games following as we like to name, somewhere in the series.

We may also establish this result by returning to the generatrix of z_x ; or

$$\frac{q^i t^i (1-qt)}{1-t+(1-q)q^i t^{i+1}} = z_i t^i + z_{i+1} t^{i+1} + \dots$$

If we now suppose $t=1$, we have $1=z_i+z_{i+1}+\dots$ *ad inf.*, which verifies the preceding. But if we ask, in a very great number of throws, what are the runs for which the two have equal chances, we see, reversing q and q' , to obtain the chance of a run of i' , (which substitute for i), in favour of the other player, that we must have

$$1 - \varepsilon^{-q^{i'} q'^x} = 1 - \varepsilon^{-q^i q'^x}, \text{ or } q^i q' = q^{i'} q',$$

giving

$$\frac{i-1}{i'-1} = \frac{\log q'}{\log q}.$$

Thus if one player be nine times as skilful as another, $q'=\frac{9}{10}$, $q=\frac{1}{10}$, we have

$$\frac{i-1}{i'-1} = \frac{.954-1}{0-1} = .046, \text{ so that if } i=2, i'=22.+$$

Or a player nine times as skilful as another has a better chance (slightly) of winning 22 games running in a very large number than the other has of winning two games running.

(54.) PROBLEM. There is a lottery containing n numbers, $1, 2, 3, \dots, n$. Of these r are drawn at once, and then replaced. What is the probability that all the n numbers shall have been drawn in i drawings?

Let $z_{n,q}$ be the number of sets of i drawings, in which q and all under it can be drawn. It is evident that the number of sets of i which contain up to q fall short of the number which contain up to $q-1$ by a parcel of possible drawings, each of which contains up to $q-1$ without the appearance of q ; the latter being precisely the number of sets of i drawings of a lottery from which ticket q has been removed, and the problem altered from demanding $1 \dots q$ into $1 \dots q-1$. That is, the number of sets by which $z_{n,q}$ falls short of $z_{n,q-1}$ is $z_{n-1,q-1}$, or

$$z_{n,q} = z_{n,q-1} - z_{n-1,q-1} = \Delta z_{n-1,q-1},$$

where Δ refers to n . But by the same reasoning $z_{n-1,q-1} = \Delta z_{n-2,q-2}$, and so on, whence $z_{n,q} = \Delta^2 z_{n-2,q-2} = \dots = \Delta^i z_{n-q,0}$. The whole number of ways of drawing r out of $n-q$ is $[n-q, n-q-r+1] \div [r]$, and as i trials may combine these in any number of ways, we have the i th power of the preceding for the whole number of cases. And this itself is $z_{n-q,0}$, for the preceding must be interpreted as the number of sets of i drawings, in which all up to 0 appear, or the number of sets of i drawings without restriction. (This ought not to be perfectly satisfactory at first to a beginner; we leave it to him to furnish a method which contains no extension.) Consequently

$$z_{n,q} = \Delta^i \left(\frac{[n-q, n-q-r+1]}{[r]} \right)^i.$$

But the total number of sets of i drawings of r out of n is $\{[n, n-r+1] \div [r]\}^i$; whence the probability of all the q numbers coming out, as required in the enunciation, is

Theory of
Pro-
babilities.

$$\frac{\Delta^i([n-q, n-q-r+1])^i}{([n, n-r+1])^i}.$$

Theory of
Pro-
babilities.

To find the probability of all coming out, we must make $q=n$ in this result, which is, when developed,

$$1 - q \left(\frac{n-1}{n-r+1} \right)^i + q \frac{q-1}{2} \left(\frac{n-1}{n-r+1} \right)^i \left(\frac{n-2}{n-r} \right)^i - \dots \mp q \left(\frac{n-1}{n-r+1} \right)^i \dots \left(\frac{n-q+1}{n-q-r+3} \right)^i \pm \left(\frac{n-1}{n-r+1} \right)^i \dots \left(\frac{n-q}{n-q-r+2} \right)^i.$$

This formula is not easily applied, except in cases where no denominator becomes negative. But the preceding expression may be applied. For instance, required the probability of drawing all (q) out of four (n) in two (i) drawings of two (r) each. In the product whose difference is taken there are two (r) terms, and if we therefore begin forming products sufficient for the fourth difference, we have

$$\begin{array}{rcl} 4^2 \cdot 3^2 & = & 144 \\ 3^2 \cdot 2^2 & = & 36 \\ 2^2 \cdot 1^2 & = & 4 \\ 1^2 \cdot 0^2 & = & 0 \\ 0^2 \cdot (-1)^2 & = & 0 \end{array} \quad \begin{array}{rcl} 108 & 76 & 48 \\ 32 & 28 & 24 \\ 4 & 4 & 4 \\ 0 & 0 & 0 \end{array} \quad \frac{24}{144} = \frac{1}{6} \text{ prob. required.}$$

This may be verified as follows. The drawings of ABCD which favour the event are AB . CD, CD . AB, AC . BD, BD . AC, AD . BC, BC . AD, six in all; the total number of drawings is the square of the combinations of two out of four, or 36, and $\frac{6}{36}$ is $\frac{1}{6}$, as before.

(55.) All the formulæ to which we have hitherto been led are perplexing to calculate for moderate numbers, and absolutely out of the question for high numbers. We now come to the part of our subject which may be considered as the most important of the whole, being the most difficult of the indispensable considerations; namely, the approximation to functions of very high numbers. But we have first to consider the species of expressions which have been hitherto obtained. And here we cannot have avoided observing that in every case variable products have been obtained, in which one of the variables is the *number of factors*, which varies in a manner depending upon the number of events in question.

Now an event of any kind is not like a continuous magnitude, capable of division into parts which resemble each other and the whole, in every thing except size. All the parts of a line are complete in the possession of every property which belongs to the whole line; yet if three drawings of a lottery gave severally an inch, an inch, and half an inch, we could not, in common use of language, say that only two events and a half had happened, but *three* events, the third of which had a numerical accident of only half the magnitude of the first two. If we look at the word *individuality*, the same in etymology as *indivisibility*, we may trace to the notion above given the foundation of our way of making a general word which shall express a collection of circumstances or properties, as making a whole distinct from other wholes of the same kind. And because we say the first event, or the second event, &c., we are not therefore to talk of fractions of events; for it will be readily seen that there are here *pronouns of succession*, rather than numeral nouns. *One* as distinguished from *another*, is not the same as *one* as distinguished from *one* and *one* put together. The first is only *this*, as distinguished from *that*; or the *first* as distinguished from the *second*. Now *first*, *second*, *third*, &c. are really pronouns,* whatever grammarians may say to the contrary; that is, they are referable to the same class as *this*, *that*, and *the other*; and they do not admit ideas of partition.

We cannot, therefore, speak of $2\frac{1}{2}$ events, except by an extension, which, though perceptible enough in itself, will not be any guide to proper deductions. The product $1 \cdot 2 \cdot 3 \dots x$, when x is a whole number, is readily understood; but what is its meaning when x is $3\frac{1}{2}$? Is it $1 \cdot 2 \cdot 3 \cdot 3\frac{1}{2}$? We shall show that by laying down this definition we do not get a continuous function of x . Let α be a small fraction; and suppose $[3-\alpha]$ means $1 \cdot 2 \cdot (3-\alpha)$, while $[3+\alpha]$ means $1 \cdot 2 \cdot 3 (3+\alpha)$. Therefore

$$\frac{[3+\alpha]}{[3-\alpha]} = \frac{3(3+\alpha)}{3-\alpha},$$

and when $\alpha=0$, $1=3$, which shows that $[3+\alpha]$, &c., as thus defined, cannot have the properties of a continuous algebraical function. We now ask this question: What is that function of x , which, whenever x is a whole number, reduces itself to $1 \cdot 2 \cdot 3 \dots x$, and which varies continuously from $1 \cdot 2 \cdot 3 \dots n$ to $1 \cdot 2 \cdot 3 \dots n+1$, when x varies continuously from n to $n+1$?

Take an axis of coordinates, and measure off $1, 2, 3 \dots$ for values of x , upon which construct $1, 1 \cdot 2, 1 \cdot 2 \cdot 3$, &c. for the values of the ordinates of y perpendicular to x . We may thus lay down an infinite number of points in the curve, and through the first two we can draw the line $y=a_0+a_1x$ by properly assuming a_0 and a_1 . Through the first three we can draw $y=a_0+a_1x+a_2x^2$, and so on; whence through all we can draw $y=a_0+a_1x+\dots$ *ad. inf.*; or if this form should give anomalous results, we can draw $y=a_0\phi_0x+a_1\phi_1x+a_2\phi_2x+\dots$ *ad. inf.*, where ϕ_0x, ϕ_1x , &c. are functions of x to be determined. Having got this, we can now

* Our definition of the distinction between an adjective and a pronoun would be as follows: the former is a word denoting an accident absolutely considered, and which therefore is not definite until the quantity, &c. &c. is defined; and the word is used in a manner which would render a word for the *grade* of quality desirable, if it could be given. Thus when we say a green ball, meaning to describe an individual ball, the meaning is not complete until we know what degree of greenness we are speaking of. But when we speak of a lottery of green and blue balls, it matters nothing how green or how blue the balls may be, for any shades of the two contain the distinction required. The greenness and blueness being here made the distinction between nouns, are, we assert, *pronominally* used, as much as *this* or *that*.

Theory of
Pro-
babilities.

assign the value of y , corresponding to fractional values of x , *not the fractional values of* $1.2.3 \dots x$, but the fractional values of the function whose whole values are contained in $1.2.3 \dots x$.

Theory of
Pro-
babilities.

(56.) The method adopted by Laplace is that of expressing such functions as $[s]$ by means of definite integrals, in which the variable of integration is not s , but an independent quantity whose limits are given. Nothing better tends to illustrate the calculus of variations than this method, which repeats its process and result with different meanings and hypotheses. We shall give one case at length as an exposition of the method. Required a continuous function of x , which shall, whenever x is a whole number, be equivalent to $1.2.3 \dots s$. If we call this y , we have $y_{s+1} = (s+1)y_s$, which we assume to be a universal property of the values of x . If there be any definite integral of the form $\int_a^b \phi(x, s) dx$, which is $\psi(s, a, b)$, then by the well-known properties of constants in integration, we have $\int_a^b \phi(x, s+1) dx = \psi(s+1, a, b)$. If we assume $y_s = \int_a^b x^s \cdot \phi \cdot dx$, where ϕ is to be a function of x only, we find

$$\begin{aligned} y_{s+1} - (s+1)y_s &= \int_a^b (x^{s+1} - (s+1)x^s) \phi dx \\ &= \int_a^b x^{s+1} \phi dx - x^{s+1} \phi + \int_a^b x^{s+1} \frac{d\phi}{dx} dx, \text{ from } x=a \text{ to } x=b \\ &= a^{s+1} \phi_a - b^{s+1} \phi_b + \int_a^b \left(\phi + \frac{d\phi}{dx} \right) x^{s+1} dx. \end{aligned}$$

Now if we propose for ϕ such a function that the last term shall be integrable, we may make the whole expression $=0$, and shall have done what is required, subject to this condition, that one of the two, a or b , must be a function of the other and of s . Our object is to have a and b fixed and independent of s , which can be done by assuming $\phi + \frac{d\phi}{dx} = 0$, or $\phi = C \varepsilon^{-x}$, which leaves $C a^{s+1} \varepsilon^{-a} - C b^{s+1} \varepsilon^{-b}$; and this can be made equal to 0 consistently with y_s being finite,* by $a^{s+1} = 0$, and $\varepsilon^{-b} = 0$, or $a = 0$, $b = \infty$. This gives the following:

$$\begin{aligned} \text{If} \quad y_{s+1} &= (s+1)y_s \quad y_s = C \int_0^\infty x^s \varepsilon^{-x} dx \quad y_0 = C \int_0^\infty \varepsilon^{-x} dx = C, \\ \text{or} \quad y_s &= y_0 \int_0^\infty x^s \varepsilon^{-x} dx. \end{aligned}$$

But when s is a whole number, it is obvious that $y_s = y_0 \cdot 2 \cdot 3 \cdot 4 \dots s$, whence $\int_0^\infty \varepsilon^{-x} x^s dx = [s]$, when s is a whole number, and is a continuous function, such that $y_{s+1} = (s+1)y_s$ in all cases where s is positive.

In the preceding process, we naturally ask why the form $\int x^s \phi dx$ was preferred to any other? Observe, that the question, generalized, is as follows: given

$$\psi s + \psi_1 s \cdot y_s + \psi_2 s \cdot y'_s + \dots \quad (A),$$

required an expression of this function as an integral, in such a manner that the integral may have certain definite limits independent of the value of s . Now in the calculus of variations, which we here mention not to recall the object of that science, but only its process, we have the solution of the following problem. Given y and x , of which the former is an indeterminate function of the latter, and f being a given functional form required the separation of

$$\int f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots\right) dx \dots (B)$$

into a function of the form

$$P_0 + P_1 \frac{d\omega}{dx} + P_2 \frac{d^2\omega}{dx^2} + \dots + \int \chi\left(x, y, \frac{dy}{dx}, \dots\right) \times \omega dx \quad (C),$$

where ω is an indeterminate function of x , and $P_0, P_1, \dots$ and χ are determined. This being done, the equation $\chi = 0$ makes y such a function of x as leaves the disappearance of the preceding between given limits depending only on satisfying the equation $\Pi_2 - \Pi_1 = 0$, where Π_2 and Π_1 are the values of the terms of (C) which precede $\int$, for the superior and inferior limits. In the calculus of variations those limits are given, or at least subject to one or more relations. In the present subject they are part of the *quæsitæ* of the problem. And thus the preceding expression is made to disappear, independently of the general form of ω , though values of ω at the limits may be specified.

To reduce the expression (A) to the form (B), we must observe that s is not to be a function of any continuous variable mentioned, but if y_s , &c. be expressed by definite integral functions of x , must be an arbitrary constant. And the most obvious method is to take such a function of x and s that all the terms in (A) may be easily represented by differential coefficients of the functions, combined with functions of x ; so that s may not appear except in the functions just mentioned. Let $\gamma(x, s)$ be one of those functions, giving readily ψs in terms of γ , its differential coefficients with respect to x , and x itself. If, then, we assume

$$y_s = \int \gamma \times \phi \times dx, \quad y'_s = \int \gamma \times \phi' \times dx, \text{ \&c.,}$$

where ϕ, ϕ' , &c. are functions of x only, it is plain that we may, under the preceding supposition with respect to $\psi s, \psi_1 s$, &c., express (A), which now becomes

$$\psi s + \int \psi_1 s \cdot \gamma \cdot \phi \cdot dx + \int \psi_2 s \cdot \gamma \cdot \phi' \cdot dx + \text{\&c.,}$$

in the form $\int f\left(x, \phi, \phi' \dots \gamma, \frac{d\gamma}{dx}, \dots\right) dx$, and from thence, choosing γ itself for the function ω , we may reduce

* If we attain the condition by making $a=b$, we have $y_s = 0$, which is one solution, seeing that $0 = (s+1) \times 0$.

Theory of
Pro-
babilities.

the preceding to the form (C), determine the forms of ϕ , ϕ' , &c. under which the integral will vanish independently of its limits, and the limits of integration under which the preceding part will satisfy given conditions. In general, χ , in the expression (C), will be a function of γ , ϕ , ϕ' , &c., and their differential coefficients, which will lead to the preliminary solution of a differential equation.

Theory of
Pro-
babilities.

(57.) The only cases of ψs , $\psi_1 s$, &c. which it will be necessary to consider are rational and integral functions of s . Now a rational and integral function of s may be expressed in either of the following forms, consistently with easy reference to usual theorems:

$$\begin{aligned}\psi s &= A_0 + A_1 s + A_2 s(s-k) + A_3 s(s-k)(s-2k) + \&c. \\ \psi s &= B_0 + B_1 s + B_2 s^2 + B_3 s^3 + \&c. \\ A_n &= \frac{1}{1.2.3\dots n} \times \left(\frac{\Delta^n \psi s}{k^n} \right) \quad B_n = \frac{1}{1.2.3\dots n} \times \left(\frac{d^n \psi s}{ds^n} \right),\end{aligned}$$

where $\Delta \psi s = \psi(s+k) - \psi s$, and the brackets indicate the value for $s=0$. In the case of common differences, where $k=1$, the assumption of $\gamma(x, s) = x^s$ gives

$$\begin{aligned}s(s-1)(s-2)\dots(s-n+1) &= \frac{d^n \gamma}{dx^n} \times \frac{x^n}{\gamma} \\ \psi s &= \frac{1}{\gamma} \left(A_0 \gamma + A_1 x \frac{d\gamma}{dx} + A_2 x^2 \frac{d^2 \gamma}{dx^2} + \dots \right).\end{aligned}$$

In the second form given to ψs the assumption of $\gamma(x, s) = \varepsilon^{-sx}$ (where ε^{sx} would do equally well, but it is desirable to have a function which vanishes when x is infinite and *positive*, for convenience merely) gives

$$\begin{aligned}s^n &= (-1)^n \frac{1}{\gamma} \frac{d^n \gamma}{dx^n} \\ \psi s &= \frac{1}{\gamma} \left(B_0 \gamma - B_1 \frac{d\gamma}{dx} + B_2 \frac{d^2 \gamma}{dx^2} - \&c. \right)\end{aligned}$$

If in the first case we assume $y_s = \int x^s \cdot \phi \cdot dx$, we have, by the convertibility of independent differentiations and integrations,

$$\begin{aligned}\frac{dy_s}{ds} &= \int x^s \cdot \phi \cdot \log x \, dx \dots \frac{d^n y_s}{ds^n} = \int x^s \cdot \phi \cdot (\log x)^n \, dx \\ \Delta y_s &= y_{s+1} - y_s = \int x^{s+1} \phi \cdot dx \dots \Delta^n y_s = \int x^{s+n} \phi \cdot dx,\end{aligned}$$

the limits being those employed in y_s .

When we assume $y_s = \int \varepsilon^{-sx} \phi \, dx$, we have

$$\begin{aligned}\frac{dy_s}{ds} &= - \int \varepsilon^{-sx} x \phi \, dx \dots \frac{d^n y_s}{ds^n} = (-1)^n \int \varepsilon^{-sx} x^n \phi \, dx \\ \Delta y_s &= \int \varepsilon^{-sx} (\varepsilon^{-x} - 1) \phi \, dx \dots \Delta^n y_s = \int \varepsilon^{-sx} (\varepsilon^{-x} - 1)^n \phi \, dx.\end{aligned}$$

(58.) We shall apply this method to the expression of s^{-i} and $\Delta^n \cdot s^{-i}$. The reader must remember that the reason of this apparently complicated method of expressing simple formulæ is, that there is a method to be presently explained of expressing the definite integrals so obtained, in series which are very convergent when the numbers n and i are large, which is usually the case in the theory of probabilities.

$$y_s = s^{-i} \text{ gives } s \frac{dy_s}{ds} + i y_s = 0. \text{ Let } y_s = \int \varepsilon^{-sx} \phi \cdot dx, \varepsilon^{-sx} = \gamma$$

$$\int \left\{ \frac{1}{\gamma} \frac{d\gamma}{dx} x \gamma \phi \, dx + i \gamma \phi \, dx \right\} = 0$$

$$\int \left(\frac{d\gamma}{dx} x \phi + i \gamma \phi \right) dx = \gamma \cdot x \phi + \int \left(i \phi - \frac{d(x\phi)}{dx} \right) \gamma \, dx = 0.$$

Let

$$i \phi - \frac{d(x\phi)}{dx} = 0, \text{ or } \phi = A x^{i-1}, \text{ which gives } \gamma \cdot x \phi = \varepsilon^{-sx} \cdot A x^i.$$

The limits of the integration being $x=0$, $x=\infty$, the term $\gamma x \phi$ vanishes at both limits, and the equation is satisfied; we have then

$$s^{-i} = A \int_0^\infty \varepsilon^{-sx} x^{i-1} \, dx, \text{ and } 1 = A \int_0^\infty \varepsilon^{-sx} x^{i-1} \, dx,$$

the latter being what the former becomes when $s=1$; and we have, therefore,

$$s^{-i} = \frac{\int_0^\infty \varepsilon^{-sx} x^{i-1} \, dx}{\int_0^\infty \varepsilon^{-sx} x^{i-1} \, dx}, \quad \Delta^n \cdot s^{-i} = \frac{\int_0^\infty \varepsilon^{-sx} (\varepsilon^{-x} - 1)^n x^{i-1} \, dx}{\int_0^\infty \varepsilon^{-sx} x^{i-1} \, dx},$$

the denominator being constant with respect to s .

(59.) Having thus shown that such functions as the preceding can be reduced to definite integrals, we shall for the present leave this part of the subject, and proceed to inquire in what way these integrals may be expanded into convergent series. Since the object is to expand functions containing high powers or many factors, we have first to ask *what functions of high powers are necessarily small*, in order that we may endeavour to obtain series arranged in powers of such small quantities.

Theory of
Pro-
babilities.

The terms great and small have, relatively to each other, a determinate conventional meaning, though, absolutely speaking, they have no fixed meaning. If s be called great, $\frac{1}{s}$ is to be called small, and *vice versa*.

Theory of
Pro-
babilities.

This is definition. But further, the terms great and small are never applied except in cases where the terms "as great as we please" or "as small as we please" may supply their places, by carrying the initial supposition, out of which greatness and smallness arose, "as far as we please." For instance, "when x is small, $\sin x$ and $x - \frac{1}{6}x^3$ are nearly equal," is a common theorem. But this theorem implies more; it is really equivalent to the

following: "By taking x sufficiently small, we may make $\sin x$ and $x - \frac{1}{6}x^3$ as nearly equal as we please; or $\frac{\sin x}{x - \frac{1}{6}x^3}$ as near to 1 as we please."

And in all that follows the student is carefully to watch that he is never led into admitting* that p is small when q is nearly $=r$, except upon proof that p may be made as small as he pleases, by making q as near to r as may be requisite. With this accompaniment, we may freely use the indeterminate terms small and great, without the implied circumlocutions.

Let n be great, then u^n is neither small nor great, necessarily; being the first when u is less than unity, and the second when u is greater than unity. But its logarithm, $n \log u$, and the differential coefficient $\frac{n}{u} \frac{du}{dx}$, are great, if u be independent of n . (Remember, this means that $\frac{1}{u} \frac{du}{dx}$ being a fixed quantity, n times as much may be made as great as we please by making n sufficiently great.) If the preceding be then called $\frac{1}{v}$, we have a small quantity in v ; and if $y = u^s u^{s'} \dots \times \phi$, where $s, s', \&c.$, any or all, are great, we have a small quantity in

$$\frac{y}{\frac{dy}{dx}}, \text{ or } \frac{1}{\frac{s}{u} \frac{du}{dx} + \frac{s'}{u'} \frac{du'}{dx} + \dots + \frac{1}{\phi} \frac{d\phi}{dx}} = v.$$

(60.) PROBLEM. To expand $\int y dx$ in terms of v . We have then

$$\begin{aligned} \int y dx &= \int v dy = vy - \int y \frac{dv}{dx} dx = vy - \int v \frac{dv}{dx} dy \\ &= vy - vy \frac{dv}{dx} + \int y \frac{d}{dx} \frac{v dv}{dx} dx = vy - vy \frac{dv}{dx} + vy \frac{d}{dx} \frac{v dv}{dx} - \int y \frac{d}{dx} v \frac{d}{dx} v \frac{dv}{dx} dx; \end{aligned}$$

and, proceeding in this way, we find

$$\int y dx = vy \left(1 - \frac{dv}{dx} + \frac{d}{dx} v \frac{dv}{dx} - \frac{d}{dx} v \frac{d}{dx} v \frac{dv}{dx} + \&c. \right),$$

which is to be taken in the usual manner between the limits, when they are given. It will be observed that the differential coefficients of $\frac{dv}{dx}$ are also small quantities; and, in general, we know that if a function be necessarily small, on account of a great constant which appears in the denominator, the differential coefficients will also be small. So that if we say that v is of the first order of smallness, $v \frac{dv}{dx}$ will be of the second; $v \frac{d}{dx} v \frac{dv}{dx}$ of the third, and so on: that is, the preceding series will be convergent. But we must except the case in which one of the limits is near that value of x which makes y a maximum or a minimum, for then $\frac{dy}{dx}$ may be small, and v not small. For instance, if $y = x^n \epsilon^{-x}$, we have $v = \frac{x}{n-x}$, which is, generally, small when n is great, except only when x is nearly equal to n . For practical use, x must be small as compared with n ; and the series is altogether inapplicable when $n = x$, in which case $\frac{dy}{dx} = 0$. As it is this latter case which is most useful, a special method has been applied by Laplace.

(61.) Previous to entering on this method it may be useful to show in a more common way the principle on which a series becomes convergent when the *exponents* of a previous equation are great numbers. Suppose we have

$$(A - Bx^{n+1} + Cx^{n+2} - \dots)',$$

in which all powers of x are wanting up to the n th inclusive. If we assume the preceding $= A' \epsilon^{-x^{n+1}}$, and take the s th root of both sides, expanding the exponential, we have

* It is quite impossible a beginner can clearly understand what follows unless he attend to this point.

Theory of
Pro-
babilities.

$$A - Bx^{n+1} + Cx^{n+2} - \dots = A \left(1 - \frac{t^{n+1}}{s} + \frac{t^{2n+2}}{2s^2} - \&c. \right),$$

Theory of
Pro-
babilities.

of which the common reversion of series shows that x can be expanded in the form $Pt + Qt^2 + \dots$, and that P , Q , &c. decrease rapidly when s is great compared with n , or when $\frac{1}{s^{n+1}}$ is considerable. And to determine P we have

$$BP^{n+1} = \frac{A}{s} \quad P = \sqrt[n+1]{\frac{A}{Bs}} \quad x = Pt \text{ nearly.}$$

(62.) Let $x=a$ make $\frac{1}{y} \frac{dy}{dx} = 0$; whence it follows that $(x-a)^\mu$ (μ being positive) is a factor of $\frac{1}{y} \frac{dy}{dx}$, and $(x-a)^{\mu+1}$ is a factor of the integral of the preceding, or of $\log y + C$, provided C be so taken that the preceding shall vanish when $x=a$. Both factors are the highest of their kind. Let Y be the value of y corresponding to $x=a$, then $(x-a)^{\mu+1}$ is a factor of $\log y - \log Y$. If, then, the latter* be called $-t^{\mu+1}$, we find that $x-a$ is a factor of t . Let

$$t = \frac{x-a}{v}, \text{ or } x = a + vt \dots (1),$$

where v does not become infinite when $x=a$; for if it did, we should know that $(x-a)^\nu$ is a factor of $\frac{1}{v}$, and $(x-a)^\nu$ is not then the highest power of $x-a$ which is a factor of t . The preceding also gives

$$v = \overline{x-a} (\log Y - \log y)^{-\frac{1}{\mu+1}}, \text{ a function of } x;$$

so that from (1), by Lagrange's theorem, we have

$$x = a + (v)t + \left(\frac{d \cdot v^2}{dx} \right) \frac{t^2}{2} + \left(\frac{d^2 \cdot v^3}{dx^2} \right) \frac{t^3}{2 \cdot 3} + \&c.,$$

where the brackets are placed to denote that x is made $=a$ after the differentiations. The bracketed terms are then functions of a only. If we differentiate the last equation, and multiply by

$$y = Y \varepsilon^{-t^{\mu+1}}, \text{ making } G_n \text{ stand for } \int t^n \varepsilon^{-t^{\mu+1}} dt,$$

we have

$$\int y dx = Y \left((v) G_0 + \left(\frac{d \cdot v^2}{dx} \right) G_1 + \left(\frac{d^2 \cdot v^3}{dx^2} \right) \frac{G_2}{2} + \&c. \right).$$

The question is, therefore, reduced to the determination of G_0 , G_1 , &c., or in general of G_n ; belonging to a succeeding part of the subject, the actual investigation of definite integrals. But, as these integrals are of a degree of importance in this science almost corresponding to that of successive powers in algebra, we shall at once proceed to show how they may be found. We must observe, however, that we shall have little, if any thing, to do with any case except where $\mu=1$; that is, with the form

$$G_n = \int t^n \varepsilon^{-t^2} dt.$$

Integrate the preceding one step by parts, which gives

$$G_n = -\frac{1}{2} t^{n-1} \varepsilon^{-t^2} + \frac{n-1}{2} G_{n-2}.$$

Write $n-2$ for n , which gives G_{n-2} in terms of G_{n-4} , and so on; the ultimate forms being G_1 when n is odd, G_0 when n is even; of which G_1 is algebraically integrable, for

$$G_1 = \int \varepsilon^{-t^2} t dt = -\frac{1}{2} \varepsilon^{-t^2};$$

and G_0 or $\int \varepsilon^{-t^2} dt$ remains to be found. We have then

$$\begin{aligned} G_{2n+1} &= -\frac{\varepsilon^{-t^2}}{2} \left\{ t^{2n} + n t^{2n-2} + n \cdot \overline{n-1} t^{2n-4} + \dots + [n, 2] t^2 + [n, 1] \right\} \\ G_{2n} &= -\frac{\varepsilon^{-t^2}}{2} \left\{ t^{2n-1} + \frac{2n-1}{2} t^{2n-3} + \frac{2n-1}{2} \frac{2n-3}{2} t^{2n-5} + \dots + \frac{2n-1}{2} \frac{2n-3}{2} \dots \frac{3}{2} t \right\} \\ &\quad + \frac{2n-1}{2} \cdot \frac{2n-3}{2} \dots \frac{3}{2} \frac{1}{2} \int \varepsilon^{-t^2} dt. \end{aligned}$$

To determine G_0 from $t=0$ to $t=\infty$ let

$$U = \int_0^\infty \int_0^\infty \varepsilon^{-(1+x^2)} dx ds = \int_0^\infty \frac{dx}{1+x^2} = \frac{\pi}{2};$$

the integrations being made, first for s , then for x , and the given limits used. Now perform the integrations in the inverse order denoted by

$$U = \int_0^\infty (\varepsilon^{-s} ds \int_0^\infty \varepsilon^{-sx^2} dx).$$

But

$$\int_0^\infty \varepsilon^{-sx^2} dx = \frac{1}{\sqrt{s}} \int_0^\infty \varepsilon^{-(\sqrt{s} \cdot x)^2} d(\sqrt{s} \cdot x) = \frac{1}{\sqrt{s}} \int_0^\infty \varepsilon^{-t^2} dt$$

* The negative sign is an anticipation. It will hereafter be found that Y is generally greater than y , so that an explicitly negative equivalent form of $\log y - \log Y$ will be convenient.

Theory of Probabilities. for a definite integral does not depend on the letter used for the variable, but on its limits. We have then

Theory of Probabilities.

$$U = \int_0^\infty G_0 \varepsilon^{-t} s^{-\frac{1}{2}} ds = G_0 \int_0^\infty \varepsilon^{-v^2} 2dv, \text{ if } v^2 = s \\ = G_0 \times 2G_0 = 2G_0^2; \text{ whence } 2G_0^2 = \frac{\pi}{2} \quad G_0 = \frac{\sqrt{\pi}}{2}$$

or

$$\int_0^\infty \varepsilon^{-t^2} dt = \frac{1}{2} \sqrt{\pi} \quad \int_{-\infty}^{+\infty} \varepsilon^{-t^2} dt = \sqrt{\pi};$$

the last being a consequence of a well-known property of integrals of the form $\int \phi(t^2) dt$. It now remains to show how to approximate to the value of this same integral from $t=0$ to $t=a$, where a is any number or fraction. By simply expanding ε^{-t^2} in powers of t^2 , and integrating, the first of the following series is found: the second is found by integrating by parts, as in

$$G_0 = \varepsilon^{-t^2} \cdot t + \frac{2}{3} \int \varepsilon^{-t^2} d \cdot t^3, \text{ \&c. ;}$$

and the third by the same method, applied as follows:

$$G_0 = -\frac{\varepsilon^{-t^2}}{2t} - \frac{1}{2} \int \frac{\varepsilon^{-t^2}}{t^2} dt, \text{ \&c} \\ \int_0^a \varepsilon^{-t^2} dt = a - \frac{a^3}{3} + \frac{a^5}{2 \cdot 5} - \frac{a^7}{2 \cdot 3 \cdot 7} + \frac{a^9}{2 \cdot 3 \cdot 4 \cdot 9} - \text{\&c.} \\ \int_0^a \varepsilon^{-t^2} dt = a \varepsilon^{-a^2} \left(1 + \frac{2a^2}{3} + \frac{4a^4}{3 \cdot 5} + \frac{8a^6}{3 \cdot 5 \cdot 7} + \text{\&c.} \right) \\ \int_a^\infty \varepsilon^{-t^2} dt = \frac{\varepsilon^{-a^2}}{2a} \left(1 - \frac{1}{2a^2} + \frac{1 \cdot 3}{4a^4} - \frac{1 \cdot 3 \cdot 5}{8a^6} + \text{\&c.} \right)$$

The first two are always convergent, but the convergence is distant and slow if a be greater than 2. The third is never convergent, and is not from $t=0$ to $t=a$, but from $t=a$ to $t=\infty$. But we know that

$$\int_0^\infty \phi dt = \int_0^a \phi dt + \int_a^\infty \phi dt,$$

whence

$$\int_0^a \varepsilon^{-t^2} dt = \frac{1}{2} \sqrt{\pi} - \int_a^\infty \varepsilon^{-t^2} dt.$$

(63.) The third series may be converted into a continued fraction, which shall become more convergent the more divergent the series; as follows. Let

$$u = \frac{1}{x} - \frac{q}{x^3} + \frac{3q^2}{x^5} - \frac{3 \cdot 5q^3}{x^7} + \text{\&c.}$$

which gives

$$q \frac{du}{dx} = ux - 1.$$

Let u be developed in powers of $1-x$, as follows:

$$u = y_0 + y_1(1-x) + y_2(1-x)^2 + \text{\&c.}$$

By comparing $q \frac{du}{dx}$, and $ux - 1$, or $u - (1-x)u - 1$, we find

$$y_0 - 1 = -qy_1, \quad y_1 - y_0 = -2qy_2, \quad y_2 - y_1 = -3qy_3,$$

and, generally,

$$y_{r+1} - y_r = -(r+2)qy_{r+2}.$$

Hence

$$y_0 = \frac{1}{1 + q \frac{y_1}{y_0}} \quad \frac{y_1}{y_0} = \frac{1}{1 + 2q \frac{y_2}{y_1}} \text{ \&c., or } y_0 = \frac{1}{1 + q \frac{1}{1 + 2q \frac{1}{1 + 3q \frac{1}{1 + \text{\&c.}}}}}$$

But y_0 is the value of u when $1-x=0$, or when $x=1$; the series for which from the first series is

$$1 - q + 3q^2 - 3 \cdot 5q^3 + \text{\&c.}$$

Let $q = \frac{1}{2a^2}$, and we have from the third series for G_0

$$\int_a^\infty \varepsilon^{-t^2} dt = \frac{\varepsilon^{-a^2}}{2a} \cdot \frac{1}{1 + q \frac{1}{1 + 2q \frac{1}{1 + 3q \frac{1}{1 + \text{\&c.}}}}}$$

The successive approximations to the continued fraction being

$$1, \quad \frac{1}{1+q}, \quad \frac{1+2q}{1+3q}, \quad \frac{1+5q}{1+6q+3q^2}, \quad \frac{1+9q+8q^2}{1+10q+15q^2}, \quad \text{\&c. ;}$$

Theory of
Pro-
babilities.

which, if q be less than $\frac{1}{4}$, or a greater than $\sqrt{2}$, converge rapidly. For values, see the Table at the end of this Work.

Theory of
Pro-
babilities.

(64.) If we proceed in the same manner as in (62.), with

$$\int_0^\infty \int_0^\infty \varepsilon^{-s(1+x^n)} ds dx, \text{ or } \int_0^\infty \frac{dx}{1+x^n}, \text{ or } \frac{\pi}{n \sin \frac{\pi}{n}},$$

we find the following result, by a method analogous to that just cited:

$$n \int_0^\infty \varepsilon^{-t^n} dt \times \int_0^\infty t^{n-2} \varepsilon^{-t^n} dt = \frac{\pi}{n \sin \frac{\pi}{n}}$$

For n write $\frac{n}{r-1}$, and for t write t^{r-1} , which gives

$$n^2 \int_0^\infty t^{r-2} \varepsilon^{-t^n} \times \int_0^\infty t^{n-r} \varepsilon^{-t^n} dt = \frac{\pi}{\sin \left(\frac{r-1}{n} \cdot \pi \right)}.$$

Adopting the notation of Legendre, namely $\Gamma(a+1)$ for $\int_0^\infty \varepsilon^{-x} x^a dx$, and making $n=1$ in the preceding, and $r-1=v$, this becomes

$$\Gamma(v) \times \Gamma(1-v) = \frac{\pi}{\sin \pi v}.$$

(65.) Having found G_0 we may see from (62.) that

$$\int_0^\infty t^{2n+1} \varepsilon^{-t^2} dt = \frac{1}{2} n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1 = G_{2n+1}$$

$$\int_0^\infty t^{2n} \varepsilon^{-t^2} dt = \frac{2n-1}{2} \cdot \frac{2n-3}{2} \dots \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \sqrt{\pi} = G_{2n}$$

$$\int_{-\infty}^{+\infty} t^{2n+1} \varepsilon^{-t^2} dt = 0 \quad \int_{-\infty}^{+\infty} t^{2n} \varepsilon^{-t^2} dt = \frac{2n-1}{2} \dots \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi};$$

whence, in (62.), $\int y dx$, taken between such values of x as make $t=-\infty$ and $t=+\infty$, (the most common case,) we have

$$\int y dx = Y \sqrt{\pi} \left((v) + \frac{1}{2} \cdot \frac{1}{1 \cdot 2} \left(\frac{d^3 \cdot v^3}{dx^2} \right) + \frac{1 \cdot 3}{2^2} \cdot \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \left(\frac{d^4 \cdot v^4}{dx^4} \right) + \&c. \right)$$

$$v = \frac{(x-a)}{\sqrt{\log Y - \log y}} \quad t = \sqrt{\log Y - \log y}.$$

On looking at $y = Y \varepsilon^{-t^2}$, we see that $t=-\infty$ gives $y=0$; $t=0$ gives $y=Y$ a maximum; $t=+\infty$ gives $y=0$; so that, in fact, the preceding is the expression for the integral throughout the interval, at the beginning and end of which $y=0$, and Y is the intermediate maximum or minimum value of y . And $t=0$ gives $x=a$. If $x=a+\theta$, and if $Y, Y'=0, Y'', \&c.$ be the values of $y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \&c.$ corresponding to $x=a$, we have

$y = Y + \frac{1}{2} Y'' \theta^2 + \&c.$ by Taylor's theorem. Consequently,

$$\log Y - \log y = -\log \left(1 + \frac{1}{2} \frac{Y''}{Y} \theta^2 + \dots \right) = -\frac{1}{2} \frac{Y''}{Y} \theta^2 + \dots$$

$$\sqrt{\log Y - \log y} = \theta \sqrt{-\frac{1}{2} \frac{Y''}{Y} + \dots} \quad v = \sqrt{\frac{2Y}{-Y''} + A_1 \theta + A_2 \theta^2 + \dots}$$

Consequently (v) , made by making $x=a$ or $\theta=0$, is $\sqrt{\frac{2Y}{-Y''}}$, giving (very nearly)

$$\int y dx = \pm \sqrt{2\pi} \frac{Y^{\frac{3}{2}}}{\sqrt{-Y''}} \dots \dots (A).$$

If Y be positive, then it is an algebraical maximum, and Y'' is negative. If Y be negative, it is an algebraical minimum, and Y'' is positive. In either case the preceding expression is possible. In the first case the positive sign must be taken; in the second case the negative.

(66.) The preceding process may be thus more synthetically performed. Let Y be the maximum or minimum value of y corresponding to $x=a$, and let $y = Y \varepsilon^{-t^2}$, there being two values of x for which $y=0$, one on each side of Y , and if t be determined by the preceding equation, answering to $t=-\infty$ and $t=+\infty$. Assume $x=a+\theta$, giving

Theory of
Pro-
babilities.Theory of
Pro-
babilities.

$$y = Y + Y'' \frac{\theta^2}{2} + Y''' \frac{\theta^3}{2 \cdot 3} + \dots$$

or

$$-t^2 = \log \left(1 + \frac{Y''}{Y} \frac{\theta^2}{2} + \frac{Y'''}{Y} \frac{\theta^3}{2 \cdot 3} + \dots \right) = -A_2 \theta^2 - A_3 \theta^3 - \&c.$$

from which, by reversion, let

$$\theta = B_1 t + B_2 t^2 + \dots \quad d\theta = dx = B_1 dt + 2B_2 t dt + \dots$$

$$\int y dx = Y (B_1 G_0 + 2B_2 G_1 + 3B_3 G_2 + \&c.);$$

the integrals being taken from $t = -\infty$ to $t = +\infty$. This gives

$$\int y dx = Y \sqrt{\pi} \left(B_1 + \frac{3}{2} B_2 + \frac{15}{4} B_3 + \&c. \right)$$

It is easily ascertained that $B_1 Y \sqrt{\pi}$ is the term (A), as given in the last article.

(67.) If in the equation

$$1 + \frac{Y''}{Y} \frac{\theta^2}{2} + \frac{Y'''}{Y} \frac{\theta^3}{2 \cdot 3} + \dots = e^{-t^2}$$

we substitute $B_1 t + B_2 t^2 + B_3 t^3 + \dots$ for θ , and expand the first and second sides, we find by equation of coefficients

$$B_1 = \sqrt{-\frac{2Y}{Y''}} \quad B_2 = \frac{1}{3} \frac{Y''' Y}{Y''^2} \quad B_3 = \frac{9Y'''^2 Y + 5Y''' Y''^2 - 3Y^{iv} Y'' Y^2}{18Y''^4} \sqrt{-\frac{Y}{2Y''}}$$

(68.)* The preceding principles are applied to the determination of $\int y dx dx'$ in the following manner. Let this integral be required between limits which severally give $y=0$, and let $x=a$, $x'=a'$ be the values which give the maximum intermediate value of y . Let Y be this maximum, and assume

$$y = Y e^{-t^2 - t'^2},$$

 t and t' being new variables. Let $x=a+\theta$, $x'=a'+\theta'$, which gives, remembering that Y' and Y'' , or $\frac{dy}{dx}$ and $\frac{dy}{dx'}$, (when $x=a$, $x'=a'$), are severally $=0$,

$$Y + A \theta^2 + 2B \theta \theta' + C \theta'^2 = Y e^{-t^2 - t'^2}$$

where A , B , and C are functions of θ and θ' ; the whole of Taylor's series, from terms of the second degree upwards, being capable of such an arrangement in an infinite number of different ways. For instance, the term $P \theta^6 \theta'^3$ might be written either as $P \theta^4 \theta'^3 \cdot \theta^2$, or $P \theta^3 \theta'^4 \cdot \theta \theta'$, or $P \theta^6 \theta'^3 \cdot \theta'^3$, and might be brought into either of the three last terms. Divide by Y , take the logarithms of both sides, and change the signs, which gives an equation of the form

$$M \theta^2 + 2N \theta \theta' + P \theta'^2 = t^2 + t'^2,$$

which may be satisfied by an infinite number of relations between t , θ , θ' , and t' , θ , θ' . Resolve the first side into the sum of two squares, which equate to t^2 and t'^2 , giving

$$t = M^{-\frac{1}{2}} (M\theta + N\theta') \quad t' = M^{-\frac{1}{2}} (MP - N^2)^{\frac{1}{2}} \theta' \dots (1),$$

and by differentiation

$$dt = L d\theta + I d\theta' \quad dt' = L' d\theta + I' d\theta'.$$

These latter equations follow necessarily from the preceding, independently of all suppositions as to the method of differentiating, it not even being requisite that this should be the same in both. It only remains to see what peculiar forms of them are required by the method of differentiating implied in $\int y dx dx'$, which is given, and its equal $\int T dt dt'$, which is to be found. In the first $dx = d\theta$, $dx' = d\theta'$. Let $\int y d\theta d\theta' = V$, and suppose θ' expressed in terms of θ and t from the first of (1), which requires that we should determine $d\theta'$ from

$$d\theta' = \frac{dt}{I} - \frac{L}{I} d\theta,$$

or from this as it will be when its value has been substituted for θ' in L and I .But $d\theta'$ in $\int y d\theta \cdot d\theta'$ was obtained on the supposition that $d\theta = 0$, whence $d\theta' = \frac{dt}{I}$ or $\int y d\theta d\theta' = \int \frac{y d\theta dt}{I}$, which contains θ and t without θ' or t' . Now substitute for θ its value in terms of t and t' derived from both equations in (1), remembering that dt' is to be considered as obtained simultaneously with $dt=0$, that is, from

$$0 = L d\theta + I d\theta', \quad dt' = L' d\theta + I' d\theta',$$

or

$$d\theta = \frac{I dt'}{L'I - LI'}, \text{ giving } \int y d\theta d\theta' = Y \int \frac{e^{-t^2 - t'^2} dt dt'}{L'I - LI'}.$$

The denominator of the integral is a function of t and t' derived from the inversion of (1); and in the same manner as in (65.), where it is shown that the first term of the series gives a sufficient approximation, we may use instead of M , N , and P , their first terms, which are constants, namely,

$$M = -\frac{Y''}{2Y} \quad 2N = -\frac{Y'}{Y} \quad P = -\frac{Y}{2Y'}$$

* Any reader who is not very well versed in analysis is recommended to omit this Article.

Theory of
Pro-
babilities.

$Y'', Y',$ and $Y_{//}$ being the values of $\frac{d^2y}{dx^2}$, $\frac{d^2y}{dx dx'}$, and $\frac{d^2y}{dx'^2}$, when $x=a$, $y=a'$. Hence it is easily found that

Theory of
Pro-
babilities.

$$L = \sqrt{M} \quad I = \frac{N}{\sqrt{M}} \quad L' = 0 \quad I' = \sqrt{\frac{PM - N^2}{M}}$$

$$\int y dx dx' = \pm \frac{Y}{\sqrt{PM - N^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varepsilon^{-t^2 - t'^2} dt dt' = \pm \frac{2\pi Y^2}{\sqrt{Y'' Y_{//} - (Y')^2}}$$

on the sign of which remarks similar to those in (65.) apply.*

(69.) The following results will be easily obtained :

$$\int \varepsilon^{-t^2} t dt = -\frac{1}{2} \varepsilon^{-t^2}$$

$$\int \varepsilon^{-t^2} t^2 dt = -\frac{1}{2} t \varepsilon^{-t^2} + \frac{1}{2} \int \varepsilon^{-t^2} dt$$

$$\int \varepsilon^{-t^2} t^3 dt = -\frac{1}{2} \varepsilon^{-t^2} (t^2 + 1)$$

&c. &c. &c.

Hence, when the series in (62.) is to be taken from $t=T$ to $t=T'$, we have (if we make

$$\left(\frac{d^{n-1} \cdot v^n}{dx^{n-1}} \right) = Y_n,$$

and let $a-\theta$ and $a+\theta'$ be the values of x corresponding to $t=T$ and $t=T'$ the following equation :

$$\int_{a-\theta}^{a+\theta'} y dx = Y \left(Y_1 + \frac{1}{2} \frac{Y_3}{1 \cdot 2} + \frac{1 \cdot 3}{2^2} \frac{Y_5}{1 \cdot 2 \cdot 3 \cdot 4} + \&c. \right) \int_T^{T'} \varepsilon^{-t^2} dt$$

$$+ \frac{Y}{2} \varepsilon^{-T^2} \left(Y_2 + T \frac{Y_3}{2} + (T^2 + 1) \frac{Y_4}{2 \cdot 3} + \&c. \right)$$

$$- \frac{Y}{2} \varepsilon^{-T'^2} \left(Y_2 + T' \frac{Y_3}{2} + (T'^2 + 1) \frac{Y_4}{2 \cdot 3} + \&c. \right)$$

The first line of the preceding is sufficiently convergent when y contains large exponents. the second and third are convergent if T and T' be not too great. To examine this further, observe that if $\frac{1}{\alpha}$ be a large quantity of the same order as the exponents in question, in which case α will be a small quantity, then v , containing $\log y$ under a square root in the denominator, will be of the order $\sqrt{\alpha}$, so that $Y_1, Y_2, Y_3, \&c.$ will be of the order $\sqrt{\alpha}, \alpha, \sqrt{\alpha^3}, \&c.$ Consequently T and T' should not be of an order much superior to $\frac{1}{\sqrt{\alpha}}$, or else the series will converge very slowly.

(70.) We shall now proceed to some examples of the methods here laid down. We have shown (56.) that

$$1 \cdot 2 \cdot 3 \dots s = \int_0^{\infty} \varepsilon^{-x} x^s dx \quad y = \varepsilon^{-x} x^s.$$

Now $\varepsilon^{-x} x^s$ is a maximum when $x=s$, and is 0 when $x=0$, or when $x=\infty$. For as x^s increases by increase of x , ε^{-x} increases in continually greater ratio, as appears sufficiently evident, if we consider the effect of increasing an exponent, compared with that of increasing the root of a power. The student should apply the common methods of the differential calculus. Let then $x=s+\theta$, $Y=s^s \varepsilon^{-s}$, and assume

$$\varepsilon^{-(s+\theta)^2} = \varepsilon^{-s^2} \varepsilon^{-s\theta} \varepsilon^{-\theta^2} \dots (1).$$

To follow our method we must differentiate $\varepsilon^{-x} x^s$ continually, and make $x=s$ in the several results, in order to obtain $Y', Y'', \&c.$ The shortest way to do this is to differentiate $y \varepsilon^x = x^s$, which gives, calling $y^{(n)}$ the n th differential coefficient of y ,

$$y' + y = s x^{s-1} \varepsilon^{-x}, \quad y'' + 2y' + y = s(s-1) x^{s-2} \varepsilon^{-x}, \quad \&c.$$

$$y^{(n)} + n y^{(n-1)} + \dots + n y' + y = [s, s-n+1] x^{s-n} \varepsilon^{-x},$$

from which we get $Y' = 0, \quad Y'' = -\frac{1}{s} Y, \quad Y''' = \frac{2}{s^2} Y, \quad Y^{(4)} = \left(\frac{3}{s^2} - \frac{6}{s^3} \right) Y, \quad \&c.,$

or from (67.)

$$B_1 = \sqrt{2s}, \quad B_2 = \frac{2}{3}, \quad B_3 = \frac{1}{9} \cdot \frac{1}{\sqrt{2s}}, \quad \&c.$$

$$\int_0^{\infty} x^s \varepsilon^{-x} dx = s^s \varepsilon^{-s} \sqrt{\pi} \left\{ \sqrt{2s} + \frac{3}{2} \cdot \frac{1}{9} \cdot \frac{1}{\sqrt{2s}} + \&c. \right\}$$

* We have only indicated the principal steps of this result, as its complete developement would require more space than its importance entitles us to allow it. The reason of the difficulty is the incomplete manner in which the comparison of differentiations under different circumstances is treated in all the elementary works on the Differential Calculus.

Theory of
Pro-
babilities.Theory of
Pro-
babilities.

$$1 \cdot 2 \cdot 3 \dots s = s^{s+\frac{1}{2}} \varepsilon^{-s} \sqrt{2\pi} \left\{ 1 + \frac{1}{12s} + \&c. \right\}.$$

When s is a large number this series converges rapidly. If s be so great that the second and following terms may be neglected, we have what we should have obtained from the approximate value in (65.), namely,

$$1 \cdot 2 \cdot 3 \dots s = s^{s+\frac{1}{2}} \varepsilon^{-s} \sqrt{2\pi} = \sqrt{2\pi s} \left(\frac{s}{\varepsilon} \right)^s \dots (P).$$

But this result may be, in this particular case, more simply obtained from (1) by dividing both sides by $\varepsilon^{-s} s^s$, and taking the logarithm of both sides, which gives

$$-\theta + s \log \left(1 + \frac{\theta}{s} \right) = -t^2,$$

or

$$t^2 = \frac{\theta^2}{2s} - \frac{\theta^3}{3s^2} + \frac{\theta^4}{4s^3} - \&c.,$$

by the reversion of which we obtain

$$\theta = \sqrt{2s} \cdot t + \frac{2}{3} t^3 + \frac{1}{9\sqrt{2s}} t^5 + \&c.;$$

from hence we may find $d\theta$ or dx , and hence $\int y dx$, as before.

(71.) This series is also a simple result of the well-known theorem which expresses Σy , or $y_0 + y_1 + \dots + y_{s-1}$ in terms of $\int y ds$ and differential coefficients of y , namely, if $t(\varepsilon^t - 1)^{-1}$ be expanded as follows,

$$\frac{t}{\varepsilon^t - 1} = 1 - \frac{t}{2} + \beta_1 \frac{t^2}{2} - \beta_2 \frac{t^3}{2 \cdot 3 \cdot 4} + \&c.,$$

which developement must necessarily* be the true form: then it may be shown that

$$\Sigma y = \int y ds - \frac{1}{2} y + \frac{\beta_1}{2} \frac{dy}{ds} - \frac{\beta_2}{2 \cdot 3 \cdot 4} \frac{d^2 y}{ds^2} + \&c.$$

In this theorem it is most important to remember that an alteration in the meaning of Σ , as to the terms with which it shall begin, is equivalent to nothing but an alteration of the arbitrary constant in the integral $\int y ds$.

This theorem may be shown thus: if Σy can be expanded in a series of the form

$$\Sigma y = A_0 \int y ds + A_1 y + A_2 \frac{dy}{ds} + \dots (A),$$

where $A_0, A_1, A_2, \&c.$ are the same for all functions, we may determine them by substituting any function of y , of which we know the developement. Let ε^{ks} be the function chosen; then

$$\Sigma \varepsilon^{ks} = 1 + \varepsilon^k + \varepsilon^{2k} + \dots + \varepsilon^{(s-1)k}, \text{ or } \frac{\varepsilon^{ks} - 1}{\varepsilon^k - 1},$$

which gives

$$\frac{\varepsilon^{ks} - 1}{\varepsilon^k - 1} = A_0 \left(\frac{\varepsilon^{ks}}{k} + \text{constant} \right) + A_1 \varepsilon^{ks} + A_2 k \varepsilon^{ks} + \&c.$$

As this constant of integration is independent of s , let s be infinite and negative, or $\varepsilon^{ks} = 0$, which gives

$$-\frac{1}{\varepsilon^k - 1} = A_0 \times \text{constant}.$$

Subtract this from the former, and divide by ε^{ks} , which gives

$$\frac{1}{\varepsilon^k - 1} = \frac{A_0}{k} + A_1 + A_2 k + A_3 k^2 + \dots;$$

which, compared with the preceding developement, gives

$$A_0 = 1 \quad A_1 = -\frac{1}{2} \quad A_2 = \frac{\beta_1}{2} \quad A_3 = 0, \&c.$$

It only remains then to show that Σy can be expressed by such a series as that assumed. We know that, in all cases,

$$\Delta y = \frac{dy}{ds} + \frac{1}{2} \frac{d^2 y}{ds^2} + \frac{1}{2 \cdot 3} \frac{d^3 y}{ds^3} + \&c.$$

For y , write $\int y ds$, and then sum both sides, or perform the operation Σ on both, which gives

$$\int y ds = \Sigma y + \frac{1}{2} \Sigma \frac{dy}{ds} + \frac{1}{2 \cdot 3} \Sigma \frac{d^2 y}{ds^2} + \&c.$$

$$\Sigma y = \int y ds - \frac{1}{2} \Sigma \frac{dy}{ds} - \&c.$$

* This function gives $\phi t - \phi(-t) = -t$; and if $a_0 + a_1 t + \dots$ be made to satisfy this condition, we find that $a_1 = -\frac{1}{2}$, and that $a_2, a_3, \&c.$ disappear.

Theory of
Pro-
babilities.Substitute $\frac{dy_s}{ds}$ for y_s , successively, which gives

$$\begin{aligned}\sum \frac{dy_s}{ds} &= y_s - \frac{1}{2} \sum \frac{d^2 y_s}{ds^2} - \&c. \\ \sum \frac{d^2 y_s}{ds^2} &= \frac{dy_s}{ds} - \frac{1}{2} \sum \frac{d^3 y_s}{ds^3} - \&c.,\end{aligned}$$

Theory of
Pro-
babilities.

by substituting each of which in the preceding the first is reduced to the form in (A).

The numbers $\beta_1, \beta_2, \&c.$ are called the numbers of Bernoulli, and would be a very important element of our subject, if it were not that the very great convergence of the series renders many terms unnecessary.

The most simple method of obtaining them is from the expression given by Sir J. Herschel, (*Examples Calc Diff.* p. 81), namely,

$$\beta_{2x-1} = (-1)^x \left\{ \frac{\Delta \cdot 0^{2x}}{2} - \frac{\Delta^2 \cdot 0^{2x}}{3} + \frac{\Delta^3 \cdot 0^{2x}}{4} - \dots \pm \frac{\Delta^{2x} \cdot 0^{2x}}{2x+1} \right\};$$

then

$$\beta_1 = \frac{1}{6} \quad \beta_2 = \frac{1}{30} \quad \beta_3 = \frac{1}{42}.$$

To calculate the last, observe that $x=3$, $(-1)^x = -1$, and the successive differences of the first term of the series $0^6, 1^6, 2^6, \dots$ are 1, 62, 540, 1560, 1800, 720, whence

$$\beta_3 = - \left\{ \frac{1}{2} - \frac{62}{3} + \frac{540}{4} - \frac{1560}{5} + \frac{1800}{6} - \frac{720}{7} \right\} = \frac{1}{42}.$$

If we apply the theorem to the case of $y_s = \log p_s$, we find

$$\begin{aligned}\log p_0 + \log p_1 + \dots + \log p_s &= \sum \log p_s + \log p_s \\ &= \int \log p_s ds + \frac{1}{2} \log p_s + \frac{\beta_1}{2} \frac{d \cdot \log p_s}{ds} - \&c. \\ &= - \int \frac{s}{p_s} \frac{dp_s}{ds} ds + \left(s + \frac{1}{2} \right) \log p_s + \frac{\beta_1}{2} \frac{d \cdot \log p_s}{ds} - \&c.,\end{aligned}$$

and the inverse logarithmic process gives

$$p_0 p_1 \dots p_s = \varepsilon^{-V} p_s^{s+\frac{1}{2}} \left\{ 1 + \frac{1}{12 p_s} \frac{d \cdot p_s}{ds} + \&c. \right\};$$

putting V for $\int \frac{s}{p_s} \frac{dp_s}{ds} ds$, and $\frac{1}{6}$ for β_1 . The theorem in (70.) (making $p_s = s$) may be confirmed by this in every point except the factor $\sqrt{2\pi}$, which comes from the arbitrary constant to be added to V , namely, $\log \sqrt{2\pi}$ in this case. In fact, the difficulty of determining this constant renders the preceding theorem practically useless, though, as we have seen, the necessity does not occur in the method of Laplace.

(72.) To form an idea of the degree of approximation which the equation (P) makes to exactness, write $s+1$ for s , and divide the result by (P), which gives

$$s+1 = (s+1) \left(1 + \frac{1}{s} \right)^{s+\frac{1}{2}} \varepsilon^{-1}, \text{ or } \varepsilon = \left(1 + \frac{1}{s} \right)^s \sqrt{1 + \frac{1}{s}}.$$

This equation evidently becomes true when s is infinite, from a well-known property of $(1+s^{-1})^s$, namely, that its limit is ε when s increases without limit. Taking the common logarithms of both sides of the last, we have

$$.4342945 = \left(s + \frac{1}{2} \right) (c \cdot \log s + 1 - c \cdot \log s).$$

We shall not be very far from the truth, considering how little approximation we ought then to expect, if we make $s=1$, which gives .45154 for the second side of the equation. If $s=9$, we get .4348 on the second side. The approximation is therefore very rapid: the result being a little too great.

(73.) The method of (70.) applied to $y_s = 1.3.5 \dots 2s-1$ will give

$$1.3.5 \dots 2s-1 = \frac{2^{s+1}}{\sqrt{\pi}} \int_0^\infty x^{2s} \varepsilon^{-x^2} dx = 2^{s+\frac{1}{2}} \left(\frac{s}{\varepsilon} \right)^s \text{ nearly.}$$

This we leave to the reader, and also the deduction of Wallis's theorem, namely, that

$$\frac{\pi}{4} = \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \dots \text{ad. inf.}$$

(74.) We shall now consider the case alluded to in (22.), where a large number of events being to happen, each of which must happen in one of two ways, with known probabilities in favour of either, it is required to know the probability in favour of a given degree of departure from the most likely collection of events. Suppose, for example, the chance of drawing a white ball to be $\frac{2}{3}$, and that of a black ball $\frac{1}{3}$, we are going to solve generally a question of this kind:—in 3000 drawings which are relatively more likely to give 2000 white balls

Theory of
Pro-
babilities.

than any other individual number, what is the probability that the number of white balls will lie between 2000 + 100 and 2000 - 100? Let these two possible events be denoted by P and Q, and let p and q be their probabilities; whence $p + q = 1$. Let the number of drawings be $x + x'$ where $x : x' :: p : q$, so that the most likely case is that P shall happen x times and Q shall happen x' times. Let $x + x' = n$, so that $np = x$, $nq = x'$; and let P_{-l} be the probability that P shall happen exactly $x - l$ times out of n , and Q, $x' + l$ times. Then P_{-l} is that term of the developement of $(p + q)^{x+x'}$, which contains $p^{x-l} q^{x'+l}$.

Theory of
Pro-
babilities.

$$P_{-l} = \frac{1.2.3 \dots n}{1.2.3 \dots (x-l) \times 1.2.3 \dots (x'+l)} p^{x-l} q^{x'+l}$$

$$= \frac{n^{n+\frac{1}{2}} \varepsilon^{-n} \sqrt{2\pi}}{(x-l)^{x-l+\frac{1}{2}} \varepsilon^{-x+l} \sqrt{2\pi} \cdot (x'+l)^{x'+l+\frac{1}{2}} \varepsilon^{-x'-l} \sqrt{2\pi}} \times \left(\frac{x}{n} \right)^{x-l} \left(\frac{x'}{n} \right)^{x'+l} \left. \vphantom{\frac{n^{n+\frac{1}{2}} \varepsilon^{-n} \sqrt{2\pi}}{(x-l)^{x-l+\frac{1}{2}} \varepsilon^{-x+l} \sqrt{2\pi} \cdot (x'+l)^{x'+l+\frac{1}{2}} \varepsilon^{-x'-l} \sqrt{2\pi}}} \right\} \text{very nearly,}$$

$$= \sqrt{\frac{n}{2\pi}} (x-l)^{-x+l-\frac{1}{2}} (x'+l)^{-x'-l-\frac{1}{2}} x^{x-l} x'^{x'+l};$$

which is incorrect by terms of the order $\frac{1}{x}, \frac{1}{x'}$, at most. Now the logarithm of the second factor is

$$\left(-x + l - \frac{1}{2} \right) \log(x-l), \text{ or } \left(-x + l - \frac{1}{2} \right) \left(\log x - \frac{l}{x} - \frac{l^2}{2x^2} \right) \text{ nearly;}$$

or $\left(-x + l - \frac{1}{2} \right) \log x + l - \frac{l^2}{2x} + \frac{l}{2x},$ rejecting $\frac{1}{x^2}$, &c.;

or the factor is $x^{-x+l-\frac{1}{2}} \varepsilon^{\frac{l^2}{2x}} \left(1 + \frac{l}{2x} \dots \right).$

To make the third factor, change x into x' , and l into $-l$, which gives, rejecting $\frac{1}{xx'}$, the following for the approximate value of P_{-l} ,

$$\sqrt{\frac{n}{2\pi}} x^{-x+l-\frac{1}{2}+x-l} x'^{-x'-l-\frac{1}{2}+x'+l} \varepsilon^{\frac{l^2}{2x} - \frac{l^2}{2x'}},$$

or

$$\sqrt{\frac{n}{2\pi xx'}} \varepsilon^{-\frac{nl^2}{2xx'}}.$$

And the same for P_l , the probability of P happening $x + l$ times exactly, since the preceding is a function of l^2 . Hence the probability of the number of white balls being either $x - l, x - l + 1, \dots, x + l - 1, x + l$, is the following

$$P_{-l} + P_{-l+1} + \dots + P_{l-1} + P_l = 2 \sqrt{\frac{n}{2\pi xx'}} \Sigma \varepsilon^{-\frac{nl^2}{2xx'}} - P_0 + 2P_l,$$

where Σ includes every value of l from 0 to $l-1$. Without the subtraction, the term answering to $l=0$ would have been repeated twice on the second side.

If l be under $\sqrt{x}$ and $\sqrt{x'}$, or thereabouts, the exponent of ε will be of the order $\frac{1}{x}$ at most, and its differential coefficients decrease rapidly; we have, therefore,

$$\Sigma \varepsilon^{-\frac{nl^2}{2xx'}} = \int \varepsilon^{-\frac{nl^2}{2xx'}} dl - \frac{1}{2} \varepsilon^{-\frac{nl^2}{2xx'}} + \text{constant};$$

and, since Σ and $\int$ are to begin together at $l=0$, we have constant $= \frac{1}{2}$. We have, therefore,

$$2 \sqrt{\frac{n}{2\pi xx'}} \Sigma \varepsilon^{-\frac{nl^2}{2xx'}} - P_0 + 2P_l = 2 \sqrt{\frac{n}{2\pi xx'}} \int_0^l \varepsilon^{-\frac{nl^2}{2xx'}} dl + \sqrt{\frac{n}{2\pi xx'}} \varepsilon^{-\frac{nl^2}{2xx'}}.$$

But the first side represents the probability of the number of Ps being between $x-l$ and $x+l$. Suppose

$$l \sqrt{\frac{n}{2xx'}} = t \dots (1),$$

then the second side of the preceding becomes

$$\frac{2}{\sqrt{\pi}} \int_0^t \varepsilon^{-t^2} dt + \frac{t}{\sqrt{\pi}} \varepsilon^{-t^2} \dots$$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_t^\infty \varepsilon^{-t^2} dt + \frac{t}{\sqrt{\pi}} \varepsilon^{-t^2} \dots (A).$$

We shall give an instance (being that whose result is stated, Laplace, p. 281) as follows. Supposing 18 boys are born to 17 girls, in which case out of 14,000 births the most likely individual case is, that 7200 should be boys, and 6800 girls, what is the probability that the number of boys shall fall between 7200 ± 163 ?

Theory of
Pro-
babilities.

$$\begin{aligned}x &= 7200 \\ x' &= 6800 \\ n &= 14000 \\ l &= 163\end{aligned}$$

Theory of
Pro-
babilities.

$$\begin{aligned}\log x &= 3.85733 \\ \log x' &= 3.83251 \\ \log 2 &= 0.30103 \\ \hline &7.99087 \\ \log n &4.14613 \\ \hline &2) 4.15526 \\ \log t \div l &= 2.07763 \\ \log l &2.21219 \\ \hline \log t &0.28982 \quad t=1.949 \\ &2 \\ \hline &0.57964 \\ \log .43429 &1.63778 \\ \hline &.21742 \\ \\ .43459 t^2 &= 1.6498 \\ - .43459 t^2 &= 2.35020 \quad = \log . \varepsilon^{-t^2} \\ \log t \div l &= 2.07763 \\ \hline \text{Comp. log } \frac{\sqrt{\pi}}{2} &0.05246 \\ \text{Comp. log } 2 &1.69897 \\ \hline .00015 &4.17926\end{aligned}$$

Table II. ($t=1.949$)

$$\begin{aligned}&1.95 \quad 3.71252 \\ \text{Diff. } .001 &- .00002 \\ \hline &1.949 \quad 3.71250 \\ \log \frac{1}{2} \sqrt{\pi} &1.94754 \\ \hline &.00582 \quad 3.76496 \\ \\ 1 &1.00000 \\ t \div (l \sqrt{\pi} \varepsilon^t) &.00015 \\ \hline &1.00015 \\ \\ \frac{2}{\sqrt{\pi}} \int \varepsilon^t dt &.00582 \\ \hline &.99433\end{aligned}$$

It is, therefore, .99433 to .00567, or 175 to 1, that the number of male births shall be within the limits specified. Such a problem as the preceding will explain how it is that offices dealing in calculable contingencies are not precarious speculations. The truth is that, properly managed, and with a real knowledge of their contingencies, they are, so soon as they command steady supplies of business, by many times more safe than any other species of mercantile adventure whatsoever. If we suppose, for example, an office which engages to give a certain sum to families on the birth of each son, on receiving a suitable consideration. Putting other risks aside, and attending to this one only, it appears that, if they can command a continual supply of business to such an amount as will give them the chances of 14,000 children, and if they are correct in calculating the relative births in the whole country at 18 boys to 17 girls, they will, by calculating their premiums on the supposition that 163 out of 7200 cases will happen contrary to expectation, make it extremely unlikely that they should sustain any loss by what would be called the accidental fluctuations of the proportion.

(75.) Let l be a certain fraction of n , namely, en , or 100 e per cent. of n . We have then

$$t = en \sqrt{\frac{n}{2xx'}} = e \sqrt{n} \times \sqrt{\frac{n^2}{2xx'}} = \frac{e \sqrt{n}}{\sqrt{2pq}};$$

and the probability of an excess or defect not exceeding 100 e per cent. in n drawings is

$$\Pi_e = 1 - \frac{2}{\sqrt{\pi}} \int_t^\infty \varepsilon^{-t^2} dt + \frac{\varepsilon^{-t^2}}{\sqrt{2\pi pqn}}.$$

When the chances are even, we have $t = e \sqrt{2n}$,

$$\Pi_e = 1 - \frac{2}{\sqrt{\pi}} \int_t^\infty \varepsilon^{-t^2} dt + \frac{\varepsilon^{-t^2}}{\sqrt{\frac{1}{2}\pi n}}.$$

In both these formulæ t varies as $e \sqrt{n}$, and the third term is very small when compared with the second; whence in drawing rough conclusions we shall consider the two first terms only. The greater t is, the smaller is the second term, and when t is as great as 3 the tables show that the second term is very small. But $t=3$ gives $e=3 \div \sqrt{2n}$; and if we were to suppose $n=5000$, we should have $e=.03$, or 3 per cent. It is, therefore, next to impossible, as the phrase is, that out of 5000 even risks of the same kind the total loss or gain should amount to 3 per cent. of the whole sum staked. But if $x=200$, then $e=.15$, or 15 per cent. The probability of this loss or gain from fluctuation is then excessively small. If we make $e=.10$, and $n=128$, we have $t=1.6$, and $\Pi_e=.024$ nearly, or about 40 to 1 against accidental fluctuation causing loss or gain of 10 per cent.

It must be borne in mind that the absolute safety of life contingency offices depends upon two circumstances. 1. The correctness of their tables, or rather their being on the right side of correctness. 2. The quantity of the variations from the assumed standard of events. Now considering that such offices always set out with tables

Theory of
Pro-
babilities.

confessedly favourable to themselves, (as they ought to do,) and add at least 10 per cent. for fluctuation and commercial profit, it is clear from the preceding that any idea of their being insecure from the course of events, independently of improper management, should be out of the question altogether. The preceding instances are fair representatives of the mass of their transactions, as follows. In an insurance-office, for instance, some die before they have paid enough, while some pay their policies perhaps twice over before they die. Supposing the tables to be correct, and the premiums calculated accordingly, it is, as far as can be known, an even chance whether any given individual shall fall among their good or their bad bargains. And, though the preceding is not precisely the problem of determining the question of probable excess or defect to a given amount, it is one of the same kind, and will serve to show perfectly well the order of probability which exists for the gain or loss in question.

(76.) It is evident from the preceding that, t remaining the same, e varies inversely as the root of n (in all cases). Consequently, if for a given number of events it be prudent to count upon a fluctuation of 10 per cent. from the most probable case, it is not *twice* as many events that will justify calculating upon 5 per cent. of fluctuation, but *four* times as many.

(77.) In what precedes we have the probability that the number of times P will happen out of n throws will lie between l times more and l times less than the number which is individually most probable. We shall now consider the inverse question, namely, supposing the event to have happened x times out of $x+x'$, or n , what

presumption* is there that the probability of P happening lies between $\frac{x}{n}+k$ and $\frac{x}{n}-k$? Let p be the probability in question, then, making

$$[x'] \times [x] \div [x+x'] = X,$$

it is evident that the probability of the observed event is $X p^x (1-p)^{x'}$, whence the presumption of the probability

lying between p and $p+dp$ is $\left(\frac{x}{n} = \omega\right)$,

$$\frac{X p^x (1-p)^{x'} dp}{X \int_0^1 p^x (1-p)^{x'} dp}, \text{ and } \frac{\int_{\omega-k}^{\omega+k} p^x (1-p)^{x'} dp}{\int_0^1 p^x (1-p)^{x'} dp}$$

is the presumption required. Let us then suppose $y = p^x (1-p)^{x'}$, considered as a function of p , and by (66.) this presumption is very nearly

$$\frac{B_1 \int \epsilon^{-t^2} dt + 2B_2 \int t \epsilon^{-t^2} dt}{B_1 \int_{-\infty}^{+\infty} \epsilon^{-t^2} dt + 2B_2 \int_{-\infty}^{+\infty} t \epsilon^{-t^2} dt};$$

which, since the second term of the denominator vanishes, and the first is $\sqrt{\pi} B_1$, is the same as

$$\frac{1}{\sqrt{\pi}} \int \epsilon^{-t^2} dt + \frac{2B_2}{B_1 \sqrt{\pi}} \int t \epsilon^{-t^2} dt \dots (1);$$

the indefinite integrals in both cases being taken between those values of t , which correspond to $p = \omega - k$ and $p = \omega + k$. We must now find B_1 and B_2 . The maximum value of y is ω , consequently $p = \omega$ and $t = 0$ are simultaneous; and by differentiation it will be ascertained that $y, y', y'',$ and y''' in the case of $p = \omega$, or $Y, Y', Y'',$ and Y''' , may be thus obtained. Differentiate the equation

$$\log y = x \log p + x' \log (1-p)$$

three times successively; giving $\left(\omega = \frac{x}{n}, 1-\omega = \frac{x'}{n}\right)$,

$$\frac{y'}{y} = \frac{x}{p} - \frac{x'}{1-p}, \quad \frac{Y'}{Y} = 0$$

$$\frac{y y'' - y'^2}{y^2} = -\frac{x}{p^2} - \frac{x'}{(1-p)^2}, \quad \frac{Y''}{Y} = -\frac{n^2}{x x'}$$

$$\frac{y^2 (y y''' - y' y'') - (y y'' - y'^2) 2y y'}{y^4} = \frac{2x}{p^3} - \frac{2x'}{(1-p)^3}, \quad \frac{Y'''}{Y} = -\frac{2n^2 (x-x')}{x^2 x'^2}$$

$$B_1 = \sqrt{-\frac{2Y}{Y''}} = \sqrt{\frac{2x x'}{n^2}}, \quad B_2 = \frac{1}{3} \frac{Y''' Y}{Y'^2} = -\frac{2}{3} \frac{x-x'}{n^2}, \quad Y = \left(\frac{x}{n}\right)^x \left(\frac{x'}{n}\right)^{x'}$$

Hence if $p = \omega + \theta$, we have $\theta = B_1 t + B_2 t^2 + \dots$; and for small values of θ we have $\theta = B_1 t$. Whence, supposing k to be small, we have to integrate from

$$t = -k \sqrt{\frac{n^2}{2x x'}} \text{ to } t = +k \sqrt{\frac{n^2}{2x x'}}$$

* To avoid using such a phrase as the probability of a probability, writers on this subject usually employ a synonyme. Thus Laplace talks of the probability of the *facility* of P happening, and English writers speak of the *possibility* of P having its probability. We have used the phrase in the text as being more English; we cannot reconcile our ears to *degrees of possibility*.

Theory of
Pro-
babilities.

Theory of
Pro-
babilities.

which makes the second integral in (1) vanish, and reduces the first to

$$\frac{1}{\sqrt{\pi}} \int_{-t}^{+t} \varepsilon^{-t^2} dt, \text{ or } \frac{2}{\sqrt{\pi}} \int_0^k \sqrt{\frac{n^3}{2xx'}} \varepsilon^{-t^2} dt.$$

Theory of
Pro-
babilities.

If we suppose the variation of probability k to be such as would lead us to expect $x+l$ throws where there are now x , out of n , then would $x+l=n(\omega+k)$, or $l=nk$, so that the superior limit of integration would be

$l\sqrt{\frac{n}{2xx'}}$, the same as before, and the expression obtained is the same as the first term of that which precedes (A) in (74.)

The preceding expression may be thus stated, to accord with our tables: if P happen x times out of $x+x'$, then the presumption that the probability of P happening in any one instance lies between $\frac{x}{n}+k$ and $\frac{x}{n}-k$ (k being small) is nearly

$$1 - \frac{2}{\sqrt{\pi}} \int_t^\infty \varepsilon^{-t^2} dt, \quad t = k\sqrt{\frac{n^3}{2xx'}}.$$

The preceding process fails from the commencement when $x'=0$, and y becomes p^x , both by giving an expression of a different form, and also by presuming a probability $1+k$ greater than 1. But in this case the presumption that p lies between 1 and $1-k$ may be found from the expression

$$\int_{1-k}^1 p^x dp \div \int_0^1 p^x dp, \text{ or } 1 - (1-k)^{x+1}.$$

If 100 balls out of 101 be white, and the remaining one black, the presumption that the probability of drawing a white ball lies between $\frac{100}{101} \pm .01$ is .68. If there be 100 balls drawn, all white, the last formula makes the presumption that the same lies between 1 and $1-.01$ to be .63.

(78.) We have introduced the preceding articles in order that the student who does not wish to go very far into the subject may stop here with some view of the manner in which the higher branches of analysis become the abbreviators of operation which we have described them to be in (1.) We now resume the preliminary mathematics of the subject, preparatory to more extensive applications.

The following results of integration are readily obtained. The limits of integration are written after them, the inferior limit below.

$$\begin{aligned} \int_a^b \varepsilon^{-rt^2} dt &= \frac{1}{\sqrt{r}} \int_{a\sqrt{r}}^{b\sqrt{r}} \varepsilon^{-v^2} dv & \int_{-\infty}^{+\infty} t^{2n} \varepsilon^{-rt^2} dt &= \frac{2n-1}{2r} \cdot \frac{2n-3}{2r} \cdots \frac{3}{2r} \frac{1}{2r} \frac{\sqrt{\pi}}{\sqrt{r}} \\ \int_a^b \varepsilon^{-rt^2+st+c} dt &= \frac{1}{\sqrt{r}} \varepsilon^{\frac{s^2}{4r}+c} \int_{v=\sqrt{r}\left(a-\frac{s}{2r}\right)}^{v=\sqrt{r}\left(b-\frac{s}{2r}\right)} \varepsilon^{-v^2} dv & \int_{-\infty}^{+\infty} t^{2n} \varepsilon^{-rt^2+st} dt &= \varepsilon^{\frac{s^2}{4r}} \int_{-\infty}^{+\infty} \left(t+\frac{s}{2r}\right)^{2n} \varepsilon^{-rt^2} dt. \end{aligned}$$

(79.) Previously to proceeding further, the character of definite integrals will require some discussion, particularly as to those in which impossible quantities enter. By $\int y dx$, taken from $x=a$ to $x=b$, is generally understood $\phi b - \phi a$ where ϕx is that function of x which, differentiated, gives y . But, 1. This definition is not the point of view under which integration is considered in its applications either to geometry or physics. 2. It is not a proper assumption to speak of $\int y dx$ as a function obeying the laws of algebra, in those instances in which we have not the most remote idea what such function may be, and when, in all probability, it is a transcendental of a class which algebra has no present means of expressing, or of knowing whether it has not peculiar properties. May it not, for example, have some properties which set ordinary theorems of algebra at defiance? (See the Calculus of Functions, art. 72.) We require a definition peculiar to a definite integral, and we find it in the view which actually exists of such a function in geometry and mechanics. The definite integral of $\phi x dx$, taken from $x=a$ to $x=b$, means the following: divide the interval $b-a$ into n equal parts, each $=\theta$; then form

$$\theta \{ \phi a + \phi(a+\theta) + \phi(a+2\theta) + \dots + \phi(a+n\theta, \text{ or } b) \}.$$

The limit of this, made by diminishing θ or increasing n without limit, is the definite integral in question. We do not here inquire into the general form of the function for all values of a , but we require only the expression for n terms of a given series, or, where that cannot be found, its limit as n increases without limit, subject to $n\theta=b-a$. But it must be observed that this definition only applies in cases where ϕx does not become infinite in the interval between $x=a$ and $x=b$. In this case the general term $\theta\phi(a+m\theta)$ does not always necessarily diminish without limit, and other considerations are necessary to be introduced, which have been considered by M. Poisson.* But such cases will not need to be considered in the integrals which this subject introduces. It may be shown† that this definition of a definite integral agrees with the common one in all cases in which an inverse

* Journal de l'Ecole Polytechnique, cahier 18, vol. xi.

† Also Differential Calculus, (Library of Useful Knowledge.)

Theory of
Pro-
babilities.

differential coefficient can be found, to which Taylor's theorem may be applied. What we are now concerned with is the passage from possible to impossible functions, and most particularly in the case of exponential functions, which we may now consider, in common algebra, as freed from anomalous results.

Theory of
Pro-
babilities.

(80.) But we must first remark (as we suppose others have done, though we have not met with it) that we have no right to assume a definite integral to be properly expressed in analysis by the symbol ∞ , because the numerical result is infinite: such assumption being precisely the same as that $1+2+4+8+\dots$ is infinite, because its arithmetical summation is so. We know the latter to be expressed by -1 , and the reasons are explained in works on algebra. We cannot doubt that the successive acquisition of 1, 2, 4, 8, $\dots$ counters would lead to the acquisition of any number in time; or that the area of the curve whose ordinate is ε^x might be made to contain as many square feet as we please by taking the abscissa x sufficiently great. What we say is, that those who would contend for $\int_0^\infty \varepsilon^x dx = \infty$ must be required to show cause why $1+2+4+\dots$ should not be regarded as also infinite; while those who abide by the result $1+(1+p)+(1+p)^2+\dots = -\frac{1}{p}$ may be made to know that $\int_0^\infty \varepsilon^{px} dx = -\frac{1}{p}$ on the same grounds. For it is obvious that

$$\theta (\varepsilon^{p\theta} + \varepsilon^{2p\theta} + \varepsilon^{3p\theta} + \dots \text{ad inf}) = \frac{\theta \varepsilon^{p\theta}}{1 - \varepsilon^{p\theta}} = -\frac{1}{p}, \text{ when } \theta = 0,$$

by the common rules of the differential calculus. When p is negative, this result is also arithmetically true, in like manner as $1 + \frac{1}{2} + \frac{1}{4} + \dots = 2$. And if it be necessary to ask of every algebraical *infinite*, whether it is numerically infinite, or has become negative by passing through infinity, the same question must be put as to a result of the integral calculus. And if any should contend for the infinity of a definite integral, by referring its value to that obtained from the ordinary definition, we should ask them how, in any particular problem, they prove the continuity of the arbitrary constant which forms a part of the general expression, in the precise case where the variable is infinite? (See Calculus of Functions, art. 181, &c.)

(81.) We shall now show the consistency of the results derived from passing from possible to impossible quantities. Let us take the cases of $\int_0^\infty \cos x dx$ and $\int_0^\infty \sin x dx$, as derived from the definition. By taking

$$\theta (\cos \theta + \cos 2\theta + \cos 3\theta + \dots), \text{ or } -\frac{\theta}{2}$$

$$\theta (\sin \theta + \sin 2\theta + \sin 3\theta + \dots), \text{ or } \frac{1}{2} \frac{\theta \sin \theta}{1 - \cos \theta},$$

which severally approach to 0 and 1 when θ is diminished without limit, we have

$$\int_0^\infty \cos x \cdot dx = 0, \quad \int_0^\infty \sin x \cdot dx = 1.$$

By using the following equation,

$$\int_0^\infty = \int_0^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^\pi + \int_\pi^{\frac{3\pi}{2}} + \&c.,$$

we obtain for the first integral $1-1-1+1+1-\&c.$, or $1-2\left(\frac{1}{2}\right)=0$. For the second, we have $+1+1-1-1+\&c.$, or $2\left(\frac{1}{2}\right)$, or 1. These results will be better understood, by taking the arithmetical cases of which they are limits, that is, $\int_0^\infty \varepsilon^{-px} \cos x dx$ and $\int_0^\infty \varepsilon^{-px} \sin x dx$, p being positive. By integrating these, whether by common process or by finding

$$\sum_{m=0}^{\infty} \varepsilon^{-mp\theta} \cos m\theta \cdot \theta, \text{ and } \sum_{m=0}^{\infty} \varepsilon^{-mp\theta} \sin m\theta \cdot \theta,$$

it will be found that, for every positive value of p , however small, the first ($\theta=0$) is arithmetically $\frac{p}{1+p^2}$, and the second $\frac{1}{1+p^2}$; the limits of which (p diminishing without limit) are 0 and 1. And the same results have been deduced by M. Poisson and others, in various other ways. We shall now apply these integrals to the case of

$$\int_0^\infty \varepsilon^{\pm x \sqrt{-1}} = \int_0^\infty \cos x dx \pm \sqrt{-1} \int_0^\infty \sin x dx,$$

the second side of which, treated in the manner just shown, gives $\pm \sqrt{-1}$, or $\mp \frac{1}{\sqrt{-1}}$, a result agreeing with that

obtained from the right-hand side in the usual manner, by means of the equation $\int_0^\infty \varepsilon^{px} dx = -\frac{1}{p}$, before established in all cases.

The integral $y = \int_0^\infty \varepsilon^{-a^2 x^2} \cos rx dx$ may be thus obtained without impossible quantities. Differentiate with respect to r , which gives

Theory of
Pro-
babilities.

or

$$\frac{dy}{dr} = -\int_0^\infty \varepsilon^{-a^2 x^2} \sin rx \cdot x dx = -\frac{r}{2a^2} \int_0^\infty \varepsilon^{-a^2 x^2} \cos rx dx$$

$$\frac{dy}{dr} + \frac{r}{2a^2} y = 0, \quad y = B \varepsilon^{-\frac{r^2}{4a^2}},$$

Theory of
Pro-
babilities.

where B must be the value of y when $r=0$; that is, $\int_0^\infty \varepsilon^{-a^2 x^2} dx$, or (78.)

$$\int_0^\infty \varepsilon^{-a^2 x^2} \cos rx dx = \frac{\sqrt{\pi}}{2a} \varepsilon^{-\frac{r^2}{4a^2}}.$$

Now substitute the exponential value for $\cos rx$, find $\int_0^\infty \varepsilon^{-a^2 x^2 \pm rx \sqrt{-1}} dx$, as in (78.), and the same result will be obtained.

(82.) We shall now proceed to find approximate values for coefficients of given powers of a , in the expression

$$(a^{-n} + a^{-n+1} + \dots + a^{-1} + 1 + a + \dots + a^{n-1} + a^n)^s \dots (1),$$

where n and s are high numbers. If we call P_l the coefficient of a^l in this formula, we see firstly that a^{-l} has the same coefficient, so that the formula may be written

$$P_0 + P_1(a + a^{-1}) + P_2(a^2 + a^{-2}) + \dots (2);$$

or, if we make $a + a^{-1} = 2 \cos \varpi$, the formula becomes

$$P_0 + 2P_1 \cos \varpi + 2P_2 \cos 2\varpi + \dots (3).$$

But, if we take the following equation, (v and w being whole numbers,) multiply both sides by $d\varpi$, and integrate from $\varpi=0$ to $\varpi=\pi$,

$$\cos v \varpi \cos w \varpi = \frac{1}{2} \cos (v-w) \varpi + \frac{1}{2} \cos (v+w) \varpi,$$

we find that, according as v differs from or is equal to w ,

$$\int_0^\pi \cos v \varpi \cos w \varpi d\varpi = 0, \text{ or } \frac{1}{2} \pi.$$

This holds also when $w=0$. Hence, if we multiply (3) by $\cos l\varpi$, and integrate, we find every term of the result evanescent, except only that which arises from $2P_l \cos l\varpi$, which becomes πP_l . But when $l=0$, owing to the first term not being multiplied by 2, the result is $\frac{1}{2} \pi P_0$: and the multiplication is by $\cos 0\varpi$, or 1. But if in (1), the equivalent of (3), we make the obvious summation by substitution of

$$1 + \frac{a^{-1} - a^{-n-1}}{1 - a^{-1}} + \frac{a - a^{n+1}}{1 - a}, \text{ or } \frac{a^n + a^{-n} - (a^{n+1} + a^{-(n+1)})}{2 - (a + a^{-1})},$$

or

$$\frac{2 \cos n\varpi - 2 \cos (n+1)\varpi}{2 - 2 \cos \varpi}, \text{ or } \frac{\sin \frac{1}{2} (2n+1) \varpi}{\sin \frac{1}{2} \varpi},$$

and

$$\pi P_l = \int_0^\pi \left(\frac{\sin \frac{1}{2} (2n+1) \varpi}{\sin \frac{1}{2} \varpi} \right)^s \cos l\varpi d\varpi \dots (l)$$

for all whole values of l except $l=0$, in which case

$$\frac{1}{2} \pi P_0 = \int_0^\pi \left(\frac{\sin \frac{1}{2} (2n+1) \varpi}{\sin \frac{1}{2} \varpi} \right)^s d\varpi \dots (0).$$

Let A' be the power of the ratio of the sines in question. Then it is shown immediately that this function is never infinite, for if $\theta = w\pi \pm \phi$ (ϕ being small) $m\theta = mw\pi \pm m\phi$, and

$$\frac{\sin m\theta}{\sin \theta} = \pm, \text{ or } \mp \frac{\sin m\phi}{\sin \phi} = \pm m, \text{ when } \phi=0.$$

Now m being an odd number, as θ proceeds from 0 to $\frac{1}{2} \pi$, (its extreme value since θ is here $\frac{1}{2} \pi$), $m\theta$ proceeds through an odd number of right angles, in each of which the ratio may have an algebraical maximum or minimum, determined by the equation

$$\tan m\theta = m \tan \theta$$

for

$$\frac{d}{d\theta} \cdot \frac{\sin m\theta}{\sin \theta} = \frac{\cos m\theta \cdot \cos \theta}{\sin^2 \theta} (m \tan \theta - \tan m\theta),$$

which can only vanish when the last factor vanishes, and does not become infinite from $\theta=0$ to $\theta=\frac{1}{2} \pi$. There is a change from $+$ to $-$ in this differential coefficient from before to after $\theta=0$, indicating a maximum, and in the first right angle $m \tan \theta$ is always less than $\tan m\theta$. The whole continues negative till θ has so increased that $m\theta$ is out of the second right angle, but while $m\theta$ is in the third right angle (the tangent having become

Theory of
Pro-
babilities.

positive again) there is an angle for which $\tan m\theta = m \tan \theta$, and a minimum is indicated. Similarly, when $m\theta$ is in the fifth, seventh, &c. right angles, there are maxima and minima alternately. It remains to find superior and inferior limits for the values of θ which determine them.

Theory of
Pro-
babilities.

The first, a maximum, is when $\theta = 0$; for the second, $m\theta$ being in the third right angle, lies between π and $\frac{3}{2}\pi$, or

$$\text{For the second } \dots \theta > \frac{\pi}{m} < \frac{3}{2} \frac{\pi}{m}$$

$$\text{For the third } \dots \theta > \frac{2\pi}{m} < \frac{5}{2} \frac{\pi}{m}$$

$$\dots \dots \dots$$

$$\text{For the } m\text{th } \dots \theta > \frac{(m-1)\pi}{m} < \text{, or not } > \frac{\pi}{2}.$$

To find closer limits, observe that $m \tan \theta$ being always $> m\theta$ when $\theta < \frac{1}{2}\pi$, we must have in the cases under discussion $\tan m\theta > m\theta$; if, then, $m\theta$ be greater than $v\pi$, $\tan m\theta$ is also $> v\pi$, and $m\theta$ therefore greater than $v\pi + \frac{\pi}{4}$, for all tangents of angles between $v\pi$ and $v\pi + \frac{1}{4}\pi$ are less than unity, and therefore less than $v\pi$. That is, at the second value above mentioned, $m\theta$ is greater than $\frac{5}{4}\pi$; at the third, $m\theta$ is greater than $\frac{9}{4}\pi$, and so on.

Now, as to the value of the function itself, we have in the case of a maximum or minimum (for numerical values, neglecting signs)

$$\frac{\sin m\theta}{\sin \theta} = m \frac{\cos m\theta}{\cos \theta} = \frac{m}{\cos \theta \sqrt{1 + m^2 \tan^2 \theta}} = \frac{m}{\sqrt{\cos^2 \theta + m^2 \sin^2 \theta}} < \frac{1}{\sin \theta}, \text{ or } \frac{m}{m\theta} \cdot \frac{\theta}{\sin \theta}, \text{ which } < \frac{m}{m\theta} \cdot \frac{\pi}{2};$$

for it may readily be shown that $\theta \div \sin \theta$ is an increasing function of θ from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, and therefore $\frac{1}{2}\pi$ is greater than any other value within those limits. Substituting for $m\theta$ any inferior limit, *a fortiori*, the last obtained disparity is increased; that is,

$$\text{At the second maximum* } \frac{\sin m\theta}{\sin \theta} < \frac{2}{5} m$$

$$\dots \text{ third } \dots < \frac{2}{9} m$$

$$\dots \text{ fourth } \dots < \frac{2}{13} m.$$

The maxima and minima of A' , or $(\sin m\theta \div \sin \theta)^s$, depend upon the same value of θ as those of A ; and the first value, when $\theta = 0$ is m^s ; the remaining values are severally less than $\left(\frac{2}{5}m\right)^s$, $\left(\frac{2}{9}m\right)^s$, &c.

Of these numerical maxima, it is evident that their values diminish with extreme rapidity when s is even a moderate whole number. If s be 10, the second is little more than the ten-thousandth part of the first. If s be 100, the second is a fraction .0000... of the first, where first significant figure is in the fortieth decimal place. Consequently, since $\sin m\theta \div \sin \theta$ is a maximum when $\theta = 0$, and becomes 0 when

$$\theta = \frac{\pi}{m}, \text{ or } \frac{2\pi}{m}, \text{ or } \frac{3\pi}{m}, \dots, \text{ or } \frac{1}{2}(m-1)\pi,$$

we have the equation $\left(\frac{\varpi}{2} = \theta\right)$,

$$\int_0^\pi A^s d\varpi = \int_0^{\frac{2\pi}{m}} A^s d\varpi + \int_{\frac{2\pi}{m}}^{\frac{4\pi}{m}} A^s d\varpi + \dots + \int_{\frac{(m-1)\pi}{m}}^\pi A^s d\varpi.$$

The first integration is made from a maximum to a vanishing value of A ; the second, &c. (with the exception of the last) from one vanishing value of A to the next: if, then, we apply the method in (64.), and call $Y_1, Y_2, \dots$ the maxima of A^s , we shall have

$$\int_0^\pi A^s d\varpi = Y_1 T_1 + Y_2 T_2 + \dots$$

$T_1, T_2, \dots$ being certain convergent series. But, owing to the very small value of $Y_2, Y_3, \dots$, as compared with Y_1 , the first of them only need be retained, or it is only necessary to integrate $A^s d\varpi$ from $\varpi = 0$ to the first value of ϖ , which gives $\sin(2n+1)\frac{\varpi}{2}$, or $\sin m\theta = 0$; that is, to $\varpi = \frac{2\pi}{2n+1}$, or $\theta = \frac{\pi}{m}$.

* We call them all *maxima*, because it is numerical value only which we are considering. Taking the alternate *numerical maxima* with their negative signs, they are algebraic minima.

Theory of
Pro-
babilities.

Let us then assume

$$y = \left(\frac{\sin m\theta}{\sin \theta} \right)^s \quad Y = m^s \log y = s \log \sin m\theta - s \log \sin \theta \quad \frac{y'}{y} = s \left(\frac{m}{\tan m\theta} - \frac{1}{\tan \theta} \right)$$

$$\frac{y'' y - y'^2}{y^3} = s \left(-\frac{m^2 (1 + \tan^2 m\theta)}{\tan^2 m\theta} + \frac{1 + \tan^2 \theta}{\tan^2 \theta} \right) = -s(m^2 - 1) + s \frac{\tan^2 m\theta - m^2 \tan^2 \theta}{\tan^2 m\theta \cdot \tan^2 \theta};$$

Theory of
Pro-
babilities.

of which the second term vanishes at every maximum except the first, when $\theta=0$, in which case, substituting $\theta + \frac{1}{3}\theta^3 + \dots$ for $\tan \theta$, &c., this same second term becomes $\frac{2}{3}s(m^2-1)$; so that at the first maximum we have $\frac{y''}{y} = -\frac{1}{3}(m^2-1)s$, and $-(m^2-1)s$ at every other.

Now since θ and $-\theta$ give the same values to A , and therefore to A' , we may integrate from $\theta=0$ to $\theta=\frac{\pi}{m}$, by taking the half of the approximate integral from $\theta=-\frac{\pi}{m}$ to $\theta=+\frac{\pi}{m}$, which is the integral taken between the two vanishing points which contain the maximum m' . This approximation (65.) is

$$\int_{-\frac{\pi}{m}}^{+\frac{\pi}{m}} A' d\theta = \sqrt{\pi} Y \sqrt{-\frac{2Y}{Y''}} = m' \sqrt{\frac{6\pi}{(m^2-1)s}};$$

therefore

$$\int_0^{\frac{\pi}{m}} A' d\theta = \frac{1}{2} m' \sqrt{\frac{6\pi}{(m^2-1)s}} = \int_0^{\frac{\pi}{m}} A' d\omega;$$

so that the term independent of a in (1) is very nearly expressed by dividing the preceding result by $\frac{1}{2}\pi$, and putting $2n+1$ for m , which gives

$$P_0 = (2n+1)' \sqrt{\frac{3}{2n(n+1)s\pi}},$$

a very near approximation, when s is a large number.

The passage to any other term is easily made by means of the integral in (81.) In the process, of which the preceding is an application, we have (66.) (since the maximum is when $\theta=0$, whence θ in this case is $0+\theta$ there; that is, θ is the same in both cases)

$$\theta = B_1 t = \sqrt{-\frac{2Y}{Y''}} \cdot t \text{ very nearly.}$$

And it may be easily shown that $A' \cos 2l\theta$ is a function with alternate positive and negative maxima, decreasing with great rapidity. But, without entering into this question, we have, from what immediately precedes,

$$\cos 2l\theta = \cos \left(2l \sqrt{\frac{6}{(m^2-1)s}} \cdot t \right) \text{ very nearly,}$$

$$A' = m' \varepsilon^{-t^2} \text{ (the fundamental hypothesis,)}$$

$$d\theta = \sqrt{\frac{6}{(m^2-1)s}} \cdot dt \text{ very nearly.}$$

Whence, on account of the smallness of A' after $\theta=\frac{\pi}{m}$, and the still greater smallness of $A' \cos 2l\theta$, we need only integrate from $\theta=0$ to $\theta=\frac{\pi}{m}$ instead of to $\theta=\pi$, which answers to integrating from $t=0$ to $t=\infty$, and gives

$$\int_0^{\frac{\pi}{m}} A' \cos 2l\theta \cdot d\theta = \sqrt{\frac{6}{(m^2-1)s}} m' \int_0^{\infty} \varepsilon^{-t^2} \cos \left(2l \sqrt{\frac{6}{(m^2-1)s}} \cdot t \right) \cdot dt$$

$$(81.) = \sqrt{\frac{6}{(m^2-1)s}} \frac{m' \sqrt{\pi}}{2} \varepsilon^{-\frac{6l^2}{(m^2-1)s}};$$

which (if we multiply by 2, then substitute ω for 2θ , and $2n+1$ for m , and use this integral, for reasons above stated, instead of integrating from $\omega=0$ to $\omega=\pi$) will, when divided by π , give P nearly. Or we have

$$P_l = \sqrt{\frac{3}{2n(n+1)s\pi}} (2n+1)' \varepsilon^{-\frac{3l^2}{2n(n+1)s}}.$$

(83.) We shall now proceed as in (74.) to express

$$S = P_{-l} + P_{-l+1} + \dots + P_{-1} + P_0 + P_1 + \dots + P_{l-1} + P_l;$$

as before, we have, supposing $P_0 + P_1 + \dots + P_{l-1} = \Sigma P_l$, to find $2\Sigma P_l - P_0 + 2P_l$ or

$$2 \left\{ \int_0^{\pi} P_l d\omega - \frac{1}{2} P + \frac{1}{12} \frac{dP}{d\omega} - \&c. \right\} - P_0 + 2P_l + \text{constant}; \quad (71.)$$

Theory of
Pro-
babilities.

and, if we suppose l^2 small with respect to n^2s , we may reject the differential coefficient, and we have, therefore, for the sum required,

$$2 \int_0^l P_l dl + P_l - P_0 + \text{constant};$$

which, if $l=0$ is to reduce the series to P_0 , (which is the case supposed,) requires that the constant $= P_0$, or we have

$$S = 2 \int P_l dl + P_l, \text{ or } \frac{(2n+1)^s \sqrt{6}}{\sqrt{n(n+1)s\pi}} \left\{ \int_0^l \varepsilon^{-\frac{3l^2}{2n(n+1)s}} dl + \frac{1}{2} \varepsilon^{-\frac{3l^2}{2n(n+1)s}} \right\}.$$

(84.) We now come to the application of the Theory of Probabilities to the estimation of the errors which may be expected in the results of observation.

The numerical result derived from any observation whatsoever must be expected to contain an error, depending for its magnitude upon the distinctness of the phenomenon itself, the general and individual qualities of the instrument employed, the habits and temperament of the observer, and similar circumstances. Each observation must be considered as a result most probably greater or less than the truth, and in the former case is said to be effected with a positive, in the latter with a negative, error.

In defining what we here mean by an *error*, it must be understood that we include only those variations from truth of which no account can be given, except that they are found to exist. Under any other conditions they cease to be errors, and become phenomena, the laws and causes of which are objects of inquiry. Thus, for instance, upon a principle analogous to that in (49.), if any circumstance connected with the observation excite a remark, that remark is itself a new observation, which must be taken into account. Suppose, for example, the numerical results of a set of observations continually increase, no one being less than its predecessor. The probabilities become insuperable, either that the phenomenon or the instrument is in a state of regulated change; something which was imagined to be constant is not constant; and the laws of a phenomenon present themselves for investigation, not the probable amount of a result. Nothing is to be called an error which is more likely to be of one sign than the other. Suppose an observer should find it to be the case, in comparing his observations, say of right ascensions of stars, with the results of a catalogue, that the transits he observes make the stars almost always pass the meridian too soon according to the catalogue, and seldom too late. He will then feel strongly assured, either that he is in the habit of noting phenomena too soon, or that the observer who formed the catalogue had a contrary habit, or that one or other of the instruments was not well fixed in the meridian, or that there is some permanent mistake in the method of reducing the results of the first catalogue to his own time, arising, it may be, from a yet undetected change in right ascensions, or from error of principle or computation. But if he only find a moderate excess of positive departures over negative ones, on comparing his results with those of the catalogue, it will be proper for him to try, by the method in (74.), what would have been the probability of such an excess upon the supposition that positive and negative departures were equally likely. If he find his observed event very improbable, then (77.) the supposition of the differences being really errors, in our sense of the word, is also very unlikely; and the contrary.

(85.) An instrument for observation consists of two parts; the observer himself, and the inert matter which he has put together for his purpose. The combination is subject to errors at each observation: but upon which part of the apparatus they are to be laid is a question which cannot be altogether settled. We suppose that the results of different observations do not differ much from each other; so that any one will be near to the truth, and the extreme differences will give some idea of the probable limits of error. But they cannot be made to furnish any presumption as to what is the law of probability of the error, until the most probable result of the observations has been obtained, as it should be, supposing the tendencies of the instrument to be entirely unknown. We may afterwards apply knowledge gained from a first set of observations to finding the (with that knowledge) most probable result of a second set: and here new circumstances may create new impressions and new probabilities. But, as there must be one set at least antecedent to any formation of opinion upon the instrument, it is to this set that we must first turn our attention.

(86.) By the *law of facility* of the errors is meant $\phi x dx$, which expresses the probability that, before an observation is made, its coming error will lie between x and $x+dx$: so that $\int \phi x dx$ between any limits will express the chance of the error lying between these limits. The cases which arise in practice are always those in which considerable error is extremely improbable, that is, ϕx is a function which diminishes rapidly as x increases. The most convenient mathematical representation of the law of facility is one which supposes any error, however great, positive or negative, to be *possible*, but errors of a certain magnitude to be so *improbable*, not that they may be rejected, but that they may be taken into account; the reason being, that the integrals which the subject will present are by that supposition made more manageable. If, then, a be an amount of error which we are almost certain of avoiding, it must have such a function ϕx for its law of facility, that $\int_{-a}^{+a} \phi x dx$ is very nearly the same as $\int_{-\infty}^{+\infty} \phi x dx$.

(87.) Considering an error as a disadvantage in proportion to its magnitude, we may, on the same principle as that in which we ascertain the value of a pecuniary risk, determine the total risk arising from all possible errors. If any sum might be gained between a and b , and if $\phi x dx$ were the chance of its lying between x and $x+dx$, the value of the contingent advantage would be $\int x \phi x dx$ from $x=a$ to $x=b$. If there might be gain or loss to any amount, the sum to be given or received, according as it is positive or negative, would be $\int_{-a}^{+a} x \phi x dx$. Similarly, where $\phi x dx$ expresses the law of facility of errors, the amount of error on which it would be most safe to reckon has the same expression. And this is called the mean risk of error: or, generally, the mean risk arising from all the errors between a and b is $\int x \phi x dx$ from $x=a$ to $x=b$. But, if we should ask for the probable mean n th power* of the error, it must be found from the expression $\int x^n \phi x dx$.

* Observe, that the most probable square, &c. of the error, and the square, &c. of the most probable error, are distinct things.

Theory of
Pro-
babilities.

Theory of
Pro-
babilities.

(88.) Now no previous impression which we have of the law of facility is better worth attention than this, that we expect the most probable n th power of the error to be a finite quantity, whatever n may be. No one who has ever observed, or seen the results of observation, will admit any other axiom. If the errors of all the transit observations that ever were made, a tenth of a second counting as unity, had their n th powers *arithmetically* added together, and if the sum were divided by the number of observations, which would be an approximation to the order of magnitude of the probable n th power, it would appear as unlikely to an observer that the result should be as great as 10,000ⁿ as that he himself should mistake by ten minutes the mere noting of a single phenomenon. We shall take the impression as good for this much at least; namely, to make it worth while to inquire what are the laws of facility for which it is true that $\int_0^\infty x^n \phi x dx$ is finite for all values of n , on the supposition that ϕx is positive and less than unity for all values of x . This requires that $x^n \phi x$ should diminish without limit as x increases without limit; for if $x^n \phi x$ were always greater than a , from and after $x=p$, then $\int_p^\infty x^n \phi x dx$ would be greater than $\int_p^\infty a dx$, and would be either infinite (which is against the supposition) or negative, which cannot be, seeing that the most probable of all the positive errors must be positive. We have, then, to investigate a class of functions for which $x^n \phi x = 0$ when $x = \infty$, for all values of n . By the common rules of the Differential Calculus, we must have in such a case

$$0 = x^n \phi x = \frac{n \cdot n-1 \cdot n-2 \dots 3 \cdot 2 \cdot 1}{\frac{d^n}{dx^n} \frac{1}{\phi x}} \text{ for all values of } n, x \text{ being infinite.}$$

Consequently $\frac{1}{\phi x}$ and all its differential coefficients must be infinite when x is infinite. The most general of known

and ordinary forms which satisfies such a condition is $\varepsilon^{\psi x} \cdot \chi x$, where ψx and x become infinite together, and which may be represented in the form $\varepsilon^{ax+bx^2+\dots} \{A+Bx+\dots\}$, which may be reduced to $\varepsilon^{ax} \{A'+B'x+\dots\}$. This is not the most general form which might be given; but it will be sufficient for our purpose. This being the reciprocal of ϕx , we have for ϕx the form $\varepsilon^{-ax} \{A''+B''x+\dots\}$, and if positive and negative errors of the same magnitude are to be equally likely, we must give ϕx the form $\varepsilon^{-ax^2} (A+Bx^2+\dots)$. In all these cases a must be positive.

If we assume the form just obtained for ϕx , it is evident that $\int_0^\infty x^n \phi x dx$ is a series of terms of the form $K \int_0^\infty x^p \varepsilon^{-ax^2} dx$, where p is always odd or always even. And from the general formulæ,

$$\begin{aligned} \int_0^\infty x^{2n+1} \varepsilon^{-ax^2} dx &= \frac{1}{2} \frac{1 \cdot 2 \cdot 3 \dots n}{a^{n+1}} & \int_0^\infty x^{2n} \varepsilon^{-ax^2} dx &= \frac{\sqrt{\pi}}{2} \frac{1 \cdot 3 \cdot 5 \dots 2n-1}{2^n a^{n+\frac{1}{2}}} \\ \int_0^\infty x^{2n+1} \phi x dx &= \frac{A}{2} \frac{1 \cdot 2 \cdot 3 \dots n}{a^{n+1}} + \frac{B}{2} \frac{1 \cdot 2 \cdot 3 \dots n+1}{a^{n+2}} + \dots & \int_0^\infty x^{2n} \phi x dx &= \frac{\sqrt{\pi}}{2} \left\{ \frac{A}{2^n} \frac{1 \cdot 3 \dots 2n-1}{a^{n+\frac{1}{2}}} + \frac{B}{2^{n+1}} \frac{1 \cdot 3 \dots 2n+1}{a^{n+\frac{3}{2}}} + \dots \right\} \end{aligned}$$

These series may be made convergent in an infinite number of different ways, by giving a sufficiently rapid diminution to $A, B, C, \dots$: so that we have an extensive class of possible laws of facility which will satisfy all our preconceived notions of the conditions necessary to be fulfilled. (See also remark at the end.)

(89.) When we have no reason to prefer any one observation to any other, the universal practice is to take the average or arithmetical mean of the results, and to consider that average as the most probable value of the numerical phenomenon in question. Where positive and negative errors are equally likely, we may show, as in (74.), that the most probable sum of errors (before the observations are made) is 0; or L being the true result, and A_n that of the n th and last observation, both unknown, that $\Sigma (A_p - L) = 0$ is the most probable equation, or

$$L = \frac{1}{n} \Sigma A_p. \text{ We therefore conclude, that if it were requisite to construct a method of treating the observations}$$

before they were made, the best method would be that of the average. But when the observations *have been made*, and the errors, though unknown, are no longer contingencies, the position of the observer is changed. It is no longer necessary to reason as upon an urn containing known balls, from which drawings are to be made, since they have been made, and the question is, what, judging by the drawings, are the probable contents of the urn. Let ϕe represent the comparative probability of an error e being made at a single observation. If, then, x had been the true result of the observations $A_1, A_2, \dots, A_n$, the antecedent probability of the observed event, namely, the succession of errors $A_1 - x, A_2 - x, \&c.$ would have been $\phi(A_1 - x) \phi(A_2 - x) \dots \phi(A_n - x)$, which call

y . And the probability resulting from the observed event to the value of x will be (19.) $\frac{y dx}{\int y dx}$, taken between the

extreme limits of the possible value of x ; for every different value of x may be considered as a possible state of things under which the observed event might have happened. If, as in (86.), we suppose $-\alpha$ and $+\alpha$ to be the limits of the value of x , and make $H = 1 \div \int_{-\alpha}^{+\alpha} y dx$, we have $H y dx$ for the probability that x is true. Now the absolute truth is not the object of investigation. Let L be the value which, whatever may be the reason, we take as the most probable result. Consequently $L - x$ is the error, and the risk of this error is exactly that of x being true, or $(L - x) H y dx$ is the disadvantage.* The sum of all the possible disadvantages is $\int_{-\alpha}^{+\alpha} (L - x) H y dx$, which can be made equal to nothing by taking for L the value $\frac{\int_{-\alpha}^{+\alpha} x y dx}{\int_{-\alpha}^{+\alpha} y dx}$; that is, given the law of facility of error, the most probable value, or at least the value which gives the least total risk of error, is determined. Now if we represent the law of facility ϕe by $\varepsilon - \psi(e^2)$, where ψ is not ambiguous in form,

* Meaning the opposite of advantage, as used when we say that the advantage of an even chance of £2 is £1.

Theory of
Pro-
babilities.

which may be made to include every case in which positive and negative errors are equally likely; and if, more-
over, we assume $x = L + z$, we have

Theory of
Pro-
babilities.

$$y = \varepsilon - \psi(A_1 - x)^2 - \psi(A_2 - x)^2 - \dots = \varepsilon - M - 2Nz - Pz^2 - \dots$$

$$M = \sum \psi(L - A)^2 \quad N = \sum \{(L - A) \psi'(L - A)^2\} \quad P = \sum \{\psi'(L - A)^2 + 2(L - A)^2 \psi''(L - A)^2\}, \text{ \&c.}$$

$$L = \frac{\int_{-\infty}^{+\infty} (L + z) \varepsilon - M - 2Nz - \dots dz}{\int_{-\infty}^{+\infty} \varepsilon - M - 2Nz - \dots dz}, \text{ or } \int_{-\infty}^{+\infty} z \varepsilon - M - 2Nz - Pz^2 - \dots dz = 0;$$

which is the necessary condition of a law of facility, in which positive and negative errors are equally likely, being coexistent with any result of *observations actually made*, and such that the sums of all the risks of error is nothing. The preceding condition is evidently satisfied *per se*, if there be no odd powers of z in the exponent: but the following consideration will allow us to disregard them all except Nz . Since the observations are near together, and since the supposed law of facility is such, that the probability of the result not lying among them is excessively small, y becomes excessively small when z is not very small; that is, the part of the integral $\int_{-\infty}^{+\infty} zy dz$, which arises from all the sensible values of z , is inconsiderable compared with that which arises from the small values of z . Hence the terms following Pz^2 may be neglected; for when z is small they are of inconsiderable effect, when z increases the whole becomes inconsiderable. But we have

$$\int_{-\infty}^{+\infty} z \varepsilon - M - 2Nz - Pz^2 dz = \varepsilon^{-M + \frac{N^2}{P}} \int_{-\infty}^{+\infty} \varepsilon - \left(\frac{N}{\sqrt{P}} + \sqrt{P}z \right)^2 z dz = -\frac{N\sqrt{\pi}}{P^{\frac{3}{2}}} \varepsilon^{-M + \frac{N^2}{P}};$$

whence $N=0$ is the necessary condition of the law of facility; or $\sum (L - A) \psi'(L - A)^2 = 0$. This is a relation between the value of L and the form of ψ , which cannot be explicitly deduced, but it leads to the following remark, namely, that the average of observations is not *necessarily* the most probable result, when the observations themselves are given, though it is so for unknown or prospective observations. If, then, we assume the average as the most probable result, we do either assume a law of facility, or the necessity of choosing one law out of a particular class, defined by the preceding equation. We now proceed to inquire what this law must be. Let us

suppose the preceding equation to be satisfied in all cases by $L = \frac{1}{s} \sum (A_1 + \dots + A_s)$, s being the number of observations. Let i of the observations give A_1 , and the remaining $s-i$ give A_2 , then we have

$$L = \frac{i A_1 + (s-i) A_2}{s}, \quad L - A_1 = (s-i) \frac{A_2 - A_1}{s}, \quad L - A_2 = i \frac{A_1 - A_2}{s}; \text{ make } A_1 - A_2 = \Delta;$$

$$i(s-i) \frac{\Delta}{s} \psi' \left\{ (s-i)^2 \frac{\Delta^2}{s^2} \right\} - (s-i) i \frac{\Delta}{s} \psi' \left\{ i^2 \frac{\Delta^2}{s^2} \right\} = 0, \text{ or } \psi' \left\{ (s-i)^2 \frac{\Delta^2}{s^2} \right\} = \psi' \left\{ i^2 \frac{\Delta^2}{s^2} \right\};$$

which being true for all values of s and i , gives $\psi'x = \text{constant}$, or $\psi x = cx + c'$. And this may be shown to be true for s *different* observations, since N is then $\sum c(L - A)$, or $c(nL - \sum A)$, which is $=0$. Consequently, the chance of an error lying between e and $e+de$ is $\varepsilon - ce^2 - c'de$, or $c''\varepsilon - ce^2 de$; and since this integrated from $-\infty$ to $+\infty$ must give 1, we have $c''\sqrt{\pi} \div c = 1$, whence the law of facility of errors, which gives to the average of observed results the sum of all the risks of positive error = the same for negative errors, may be thus expressed:

The probability of the error of a single observation lying between p and q is $= \sqrt{\frac{c}{\pi}} \int_p^q \varepsilon^{-cx^2} dx$.

(90.) The supposition of the real law of facility (which depends on the instrument, the observer, &c.) being a consequence of a common notion as to the best method of treating results, is altogether out of the question as an assumption, upon its own merits; but we must observe, 1. That the indeterminateness of the constant c will allow of an adjustment of this law, with a considerable degree of nearness, to any law which can be proposed. 2. That, as we shall hereafter see, *the assumption of this law will give in fact a mathematical approximation to any other law, and the more nearly the greater the number of observations in question.*

(91.) Returning to the suppositions in (89.) we have now (for s observations)

$$y = H \left(\frac{c}{\pi} \right)^{\frac{s}{2}} \varepsilon^{-c \sum (x - A)^2}, \text{ and is of the form } H' \varepsilon^{-csu^2}, \text{ where } x = \frac{\sum A}{s} + u.$$

This, for any value of u , is the presumption which the collective observations give to the supposition that u is the error of the average. Integrated from $-\infty$ to $+\infty$ it gives 1, whence $H' = \sqrt{cs \div \pi}$, and

The probability of an average having an error between p and q is $\sqrt{\frac{cs}{\pi}} \int_p^q \varepsilon^{-csu^2} du$.

(92.) The infinitely small probabilities of two sets of s and s' observations giving the exact truth are as $\sqrt{s}$ to $\sqrt{s'}$; and since we have

$$\sqrt{\frac{cs}{\pi}} \int_{-\infty}^{+\infty} \varepsilon^{-csu^2} du = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} \varepsilon^{-t^2} dt, \quad \sqrt{cs}u = t, \quad \sqrt{cs}du = dt,$$

the errors which different sets of observations have the same chance of not exceeding are inversely as the values of

Theory of
Pro-
babilities.

$\sqrt{cs}$: for different sets of the same observer, inversely as the square roots of the numbers of observations; and for the same numbers of observations by different observers, inversely as the square roots of the values of c . Hence the probability of s observations, averaged, giving an error between $\pm e$ is the same as that of a single observation giving one between $\pm \sqrt{s} \cdot e$. Suppose, for instance, the extreme of error on which it is worth while to reckon in the noting of the passage of a star over a single wire of a transit instrument to be two-tenths of a second, or 0.2 , then if $\sqrt{5}e = .2$, we have $e = .09$, or nine-hundredths of a second is as improbable in the mean of five wires, as two-tenths of a second on one wire. The present Astronomer Royal,* speaking of his observations at Cambridge, remarks that he has "been accustomed to consider the error of a transit over seven wires as not greater than 0.06 ." Now $\sqrt{7} \times .06 = .158$ nearly, which is the consequent estimate of the error at a single wire.

Theory of
Pro-
babilities.

(93.) The multiplication of indifferent observations does not give results so much exceeding in value those of a few good ones as is sometimes supposed. Let there be 10,000 observations each as likely to have ± 1 for the limits of error as 100 others to have each an error lying within ± 1 . In the first, $\sqrt{10,000} \cdot e = 1$, or $e = .01$; in the second, $\sqrt{100}e = .1$, or $e = .01$; that is, the first set does not exceed the second in merit.

(94.) In Table I. it is not $\int_{-a}^{+a} \varepsilon^{-t^2} dt$ which is given, but $\int_a^\infty \varepsilon^{-t^2} dt$, the two being connected by the equation

$$\int_{-a}^{+a} \varepsilon^{-t^2} dt = \sqrt{\pi} - 2 \int_a^\infty \varepsilon^{-t^2} dt, \text{ or } P_e = 1 - \frac{2}{\sqrt{\pi}} I(\sqrt{c}e),$$

where P_e denotes the probability of the error lying within $\pm e$ in a single observation, and $I(\sqrt{c}e)$ means the number opposite to $\sqrt{c}e$ in Table I. If P_e can be ascertained, c can be found by inverse process from the Table.

And it will facilitate all operations with these Tables to remember that $\frac{1}{2}\sqrt{\pi}$ is itself $I(0)$, being $\int_0^\infty \varepsilon^{-t^2} dt$.

For instance, let us assume it as 4 to 1 that the error on a single wire of a transit does not exceed $\pm .158 (=e)$,

or $P_e = \frac{4}{5}$. We have then

$1 - P_e = .2$	$\log \bar{1}.30103$	$.9062$	$\log \bar{1}.95722$
$\frac{1}{2}\sqrt{\pi}$, or $I(0)$	$I(0) \log \bar{1}.94754$	$e = .158$	$\log \bar{1}.19866$
$\sqrt{c}e = .9062$	$I(\sqrt{c}e) \bar{1}.24857$		$\bar{0}.75856$
			$\underline{2}$
		$c = 32.89$	$\bar{1}.51712$

(95.) If in (91.) we suppose all the different observations to have been made under different laws of facility, whose constants are $c_1, c_2, \&c.$, the same process will give

$$\frac{\sum cA}{\sum c} \text{ for the average; } \sqrt{\frac{\sum c}{\pi}} \int_{-e}^{+e} \varepsilon^{-\sum c \cdot u^2} du \text{ for the chance of an error between } \pm e,$$

which substitutes

$$\frac{c_1 A_1 + c_2 A_2 + \dots + c_s A_s}{c_1 + c_2 + \dots + c_s} \text{ instead of } \frac{A_1 + A_2 + \dots + A_s}{1 + 1 + \dots + 1}.$$

This observers call giving different *weights* to the observations, and say that the weight of A_1 is to that of A_2 in the proportion of c_1 to c_2 . They are also in the habit of assigning weights by judging of the relative goodness of observations from the impressions on their minds at the time of observing. The soundness of this practice entirely depends upon the truth of the proposition which it tacitly assumes, namely, that there is a high probability that the observation which the observer *thinks* the better of two *is* the better. We cannot either confirm or refute this notion; but the practice derived from it should be used with great caution.

We may, however, call the constant c the *specific weight* of the observations to which it applies, and $\sum cA \div \sum c$ the *weighted mean*; and thus we may express the following theorem: the weight of a weighted mean is the sum of the weights of its component observations.

(96.) If we consider both the positive and negative errors, the most probable individual result of a coming observation is that the error should be nothing, and the most probable risk of error in an average is also nothing, since $\int_{-\infty}^{+\infty} x \cdot \phi x \cdot dx = 0$. But if we consider positive errors alone, we find the risk of error to be $\int_0^\infty x \phi x dx$, which,

when $\phi x = \sqrt{\frac{c}{\pi}} \varepsilon^{-cx^2}$, becomes $\frac{1}{2\sqrt{c\pi}}$. This is what Laplace calls *l'erreur moyenne à craindre en plus*, and

the corresponding error *en moins* is of the same magnitude with a different sign. We shall call it the *risk* of the observation, the sign of the error not being considered. It is what a man should in equity pay for being relieved of the liability of losing any possible sum, under such circumstances that the chances of the loss lying between x and $x+dx$ is $\sqrt{\frac{c}{\pi}} \varepsilon^{-cx^2} dx$. It appears, then, that the risk of an observation is inversely as the square root of the specific weight.

(97.) The average $\frac{1}{s} \sum A$ is evidently the value of x which makes $\sum (x - A)^2$ a minimum. If x be assumed as

* *Memoirs of the Royal Astronomical Society*, vol. vi. p. 97.

Theory of
Pro-
babilities.

the probable truth, then $x-A$ is the error of A , or the *method of averaging* is a particular case of the *method of least squares*, meaning the method by which such a result is obtained as will make the sum of the squares of the errors the least possible. And the weighted mean is the value of x , which makes $\sum c(x-A)^2$ a minimum. If we consider every determination as the solution of an equation, the case we have discussed is simply that of $x=a$, or $x-a=0$, where a is determined by observation. But let us suppose the equation in question to be $px+h=b$, where p , h , and b are to be determined by observation. We are not now at liberty to form the value of x , namely $(b-h)\div p$, and to proceed as before; for the chance of x thus obtained is a complicated function of the separate possibilities of error of p , h , and b . Two terms of an equation which is subject to error are not to be combined without considering the risk of the result, and the proper method by which it ought to be deduced from the risks of the components.

Theory of
Pro-
babilities.

(98.) Let $f(x_1, x_2, x_3, \dots)$ be a function of $x_1, x_2, \&c.$, which are to be severally observed. We shall confine ourselves to the case of three variables, and inquire what is the risk of the expression. From any number of observations obtain approximate values a_1, a_2, a_3 , differing from the truth by quantities of the same order as the risks of the several species of observations. Then the correction to be applied to $f(a_1, a_2, a_3)$ is of the form $m\Delta x_1 + n\Delta x_2 + p\Delta x_3 + \dots$ if $x_1 = a_1 + \Delta x_1, \&c.$, and we need only consider terms of the first order. Let θ_1, θ_2 , and θ_3 be the errors of Δx_1 or $x_1, \&c.$, then $m\theta_1 + n\theta_2 + p\theta_3$ is the error of f very nearly: and the probability of this lying between r and $r+dr$, or of $\frac{m}{p}\theta_1 + \frac{n}{p}\theta_2 + \theta_3$ lying between $\frac{r}{p}$ and $\frac{r}{p} + d \cdot \frac{r}{p}$ is the sum of every possible value of

$$\frac{\sqrt{c_1 c_2 c_3}}{\pi \sqrt{\pi}} e^{-c_1 \theta_1^2 - c_2 \theta_2^2 - c_3 \theta_3^2} d\theta_1 d\theta_2, \text{ which consists with } \frac{m}{p}\theta_1 + \frac{n}{p}\theta_2 + \theta_3 = \frac{r}{p} \dots \dots (1)$$

Write m instead of $\frac{m}{p}$, &c. for the present, substitute $\theta_3 = r - m\theta_1 - n\theta_2$; we want then

$$dr \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(A\theta_1^2 + 2B\theta_1\theta_2 + C\theta_2^2) + 2D\theta_1 + 2E\theta_2 + F} d\theta_1 d\theta_2 \quad \left| \begin{array}{lll} A = c_1 + c_3 m^2 & B = c_3 m n & C = c_2 + c_3 n^2 \\ D = c_3 r m & E = c_3 r n & F = -c_3 r^2 \end{array} \right.$$

$$\text{Assume} \quad v = \theta_1 + \frac{B}{A} \theta_2, \quad M = \frac{AC - B^2}{A}, \quad N = \frac{AE - BD}{A};$$

which reduces the exponent to $-M\theta_2^2 - 2N\theta_2 - Av^2 + 2Dv + F$, in which the variables are separated. Let $dv = d\theta_1$, or let v and θ_1 vary together in the first integration, and integrate, which gives

$$dr \int_{-\infty}^{+\infty} e^{-M\theta_2^2 - 2N\theta_2} d\theta_2 \int_{-\infty}^{+\infty} e^{-Av^2 + 2Dv + F} dv = \frac{\pi dr}{\sqrt{AM}} e^{F + \frac{D^2}{A} + \frac{N^2}{M}} = \frac{\pi dr}{\sqrt{AC - B^2}} e^{F + \frac{CD^2 + AE^2 - 2BDE}{AC - B^2}}.$$

1. Substitute for $A, \&c.$ their values; and 2. restore $\frac{r}{p}$ for r , $\frac{dr}{p}$ for dr , &c.; which gives for (1) above

$$1. \frac{\sqrt{c_1 c_2 c_3} dr}{\sqrt{\pi} (c_1 c_2 + c_2 c_3 m^2 + c_3 c_1 n^2)^{\frac{1}{2}}} e^{-\frac{c_1 c_2 c_3 r^2}{c_1 c_2 + c_2 c_3 m^2 + c_3 c_1 n^2}}; \quad 2. \sqrt{\frac{V}{\pi}} e^{-Vr^2} dr, \quad V^2 = \frac{c_1 c_2 c_3}{c_1 c_2 p^2 + c_2 c_3 m^2 + c_3 c_1 n^2};$$

and the risk required is

$$\frac{1}{2\sqrt{V\pi}} = \frac{1}{2\sqrt{\pi}} \sqrt{\frac{m^2}{c_1} + \frac{n^2}{c_2} + \frac{p^2}{c_3}},$$

or the squared risk of a sum is the sum of the squared risks of its components. For the risk of Δx_1 being $1 \div 2\sqrt{c_1\pi}$, that of $m\Delta x_1$, m not being subject to error, is $m \div 2\sqrt{c_1\pi}$, and so on. This theorem applies to any number of variables, for the preceding shows that, by considering a sum of two terms as one observed result, instead of its components as two observed results, we must give to the result the risk expressed by the square root of the sum of the squares of the risks of the components; and the converse. Whence the general theorem readily follows. And we see also that the risk of any function f is that of $\frac{df}{dx_1} \cdot x_1 + \frac{df}{dx_2} \cdot x_2 + \dots$, where the partial differential coefficients are considered as given, and $x_1, x_2, \&c.$ subject to error. Hence, if we denote by $\rho f(x_1, x_2, \dots)$ the risk of $f(x_1, x_2, \dots)$, we have

$$\begin{aligned} \rho(x_1 + x_2 + \dots) &= \sqrt{(\rho x_1)^2 + (\rho x_2)^2 + \dots} & \rho \sum ax &= \sqrt{\sum a^2 (\rho x)^2} & \rho(xy) &= \sqrt{x^2 (\rho y)^2 + y^2 (\rho x)^2} \\ \rho\left(\frac{x}{y}\right) &= \sqrt{\frac{1}{y^2} + \frac{x^2}{y^4}} \cdot \rho x, \text{ if } \rho x = \rho y & \rho \frac{x_1 + \dots + x_s}{y_1 + \dots + y_s} &= \sqrt{\frac{s}{(\sum y)^2} + \frac{s(\sum x)^2}{(\sum y)^4}} \cdot \rho x. \end{aligned}$$

(99.) But if the above method be incautiously used, we become liable to the mistake of supposing the same variable determined by different observations in different parts of the same expression. For instance, the risk of xy being as just ascertained, that of $x+xy$ is $\sqrt{(\rho x)^2 + x^2 (\rho y)^2 + y^2 (\rho x)^2}$; but treating $x+xy$ as one expression, which has the partial differential coefficients $1+y$ and x , we find its risk to be $\sqrt{(1+y)^2 (\rho x)^2 + x^2 (\rho y)^2}$. Both are correct on their different suppositions: in the first we observe xy , and then x , and take the sum, the observations by which x is determined in xy and in x not being the same; but in the second we observe x and y , and from these values we compute $x+xy$. The risks are different in the two cases, and, as might be supposed, that of the first process is the less of the two; there being a chance of compensation in the first, which does not

Theory of Probabilities. exist in the second. Similarly, the risk of $2x$ is $2\rho x$, if x be observed, and $2x$ computed; but the risk of $x+x$, Theory of Probabilities, where both are observed, is only $\sqrt{2} \cdot \rho x$; and that of $x-x$ is the same.

(100.) We have hitherto considered the mean risk of error, to which a probable approximation might be made by taking the average amount of a large number of errors. The name of the *probable* error has sometimes been given to this result, but incorrectly. If by probable error be meant the most probable of all errors, the answer should be nothing, or the correct truth is individually more probable than any given error. At the same time, if we take the common meaning of the word probable (that is, more likely than not), we see that there is a certain amount of error positive or negative, which is probable, all lower amounts being improbable. For c being the

specific weight, if we determine e , so that $\sqrt{c/\pi} \int_{-e}^{+e} e^{-cx^2} dx = \frac{1}{2}$, we have the following theorem: it is exactly an even chance that the error lies between $+e$ and $-e$; more than an even chance that it lies between $e+e'$ and $-e-e'$; and less than an even chance that it lies between $e-e'$ and $-e+e'$. In calling e , thus obtained, the probable error, we mean then that any less superior limit of error is improbable, or any greater inferior limit. It is improbable that the error should be less than $e-e'$, or that it should be greater than $e+e'$.

The solution of the equation $\sqrt{\frac{1}{\pi}} \int_{-e}^{+e} e^{-x^2} dx = \frac{1}{2}$ gives $e = .4769360$;

consequently, (92.), we have $\frac{.4769360}{\sqrt{c}} = \text{probable error to specific weight } c$; and the probable error to the mean risk ρ is $1.690694 \times \rho$.

(101.) The distinction between the mean risk of error and the probable error is exactly analogous to the one for want of which the *mean duration* of life has been called the *expectation* of life in English Tables. The mean duration of life is found practically by summing the number of years enjoyed by a large number of persons of the same age: theoretically, it is $\int x \phi x dx$, where, of a very large number of persons attaining the age $a+x$ (a being the common age of commencement) ϕx represents the proportion which those who die between the ages of $a+x$ and $a+x+dx$ bear to those who began at the age a : the limits being from $x=0$ to $x=$ the greatest possible term of survivorship. This is the average duration of life; but it is not therefore the term of existence which each of the individuals may *expect*; that is to say, it is not the term through which each is as likely to live as not. For the proportion surviving at x years from the epoch of commencement is evidently $1 - \int_0^x \phi x \cdot dx$, and the value of x which makes this $= \frac{1}{2}$ is the probable life, and is what should have been called the *expectation* of life.

In De Moivre's celebrated hypothesis the two coincide; in the Northampton Table, which very much resembles it at the ages of adult youth and manhood, the two are nearly the same: in the Carlisle Table there is sensible difference, as will appear by the following comparison:

Age.	Northampton.		Carlisle.	
	Mean duration.	Probable life.	Mean duration.	Probable life.
20	33.4	33.3	41.5	44.8
30	28.3	28.1	34.3	36.6
40	23.1	22.7	27.6	28.9
50	18.0	17.5	21.1	21.6

(102.) We have dwelt thus much upon one law of errors, because, as already observed, it is not only the law which is necessarily connected with the *assumption* of the average of observations made and finished, but also the approximate result of any other supposition which preserves the essential conditions (88.). This we now proceed to show.

That we may pass from finite to infinite, as in (45.), let all the possible errors be multiples of θ ; and to preserve general terms, which may render the process more perceptibly applicable to other questions, let counters be thrown into an urn, marked $-n\theta, -(n-1)\theta, \dots, -\theta, 0, \theta, \dots, (n-1)\theta, n\theta$, the number of counters of each

mark $x\theta$, being proportional to $\psi\left(\frac{x}{n}\right)$ the comparative probability of that error; the greatest positive and negative errors being of course $\pm n\theta$.

Then to express that one of these errors must appear, we have $\sum \psi\left(\frac{x}{n}\right) = 1$, the sign $\sum$ extending from $x=-n$ to $x=+n$. And if s drawings be made, and the result of each drawing restored before proceeding to the next, the probability of the sum of the s drawings, attending to their signs, being exactly $l\theta$, is the coefficient of a^l in the development of

$$\left\{ \psi\left(\frac{-n}{n}\right)a^{-n} + \psi\left(\frac{-n-1}{n}\right)a^{-(n-1)} + \dots + \psi\left(\frac{0}{n}\right) + \psi\left(\frac{1}{n}\right)a + \dots + \psi\left(\frac{n}{n}\right)a^n \right\}^s$$

or, which is the same thing, the term independent of l in the preceding development multiplied by $a^l + a^{-l}$. If then, as in (82.), we make $a + a^{-1} = 2\cos\omega$, and if positive and negative errors of the same magnitude be equally likely, or if $\psi(-z) = \psi(+z)$, we find by integration for the coefficient in question

$$\frac{1}{\pi} \int_0^\pi \cos l\omega d\omega \left\{ \psi\left(\frac{0}{n}\right) + \psi\left(\frac{1}{n}\right) \cdot 2\cos\omega + \dots + \psi\left(\frac{n}{n}\right) \cdot 2\cos n\omega \right\}^s$$

Theory of
Pro-
babilities.

Expand $\cos \varpi$, &c. in the term in $\{ \}$; the term containing ϖ^z in that expression is

$$\pm \frac{2\varpi^z}{2.3\dots z} \left(\psi\left(\frac{1}{n}\right) \cdot 1^z + \psi\left(\frac{2}{n}\right) 2^z + \dots + \psi\left(\frac{n}{n}\right) \cdot n^z \right).$$

Theory of
Pro-
babilities.

(103.) When n increases without limit, and θ decreases without limit, let E be the greatest error (a given quantity); so that $n\theta = E$: and let the sum of the drawings, $l\theta$, be always $= L$. And as the probabilities of

individual errors diminish in such case without limit, let $\psi\left(\frac{x}{n}\right)$ be written $\theta\phi\left(\frac{X}{E}\right)$, where $X = x\theta$: the ultimate

law being that the probability of a single drawing lying between p and q is $\int \phi\left(\frac{X}{E}\right) dX$ between these limits;

for θ being $(x+1)\theta - x\theta$, is the difference between consecutive values of X , or is dX . Again, as l increases, let ϖ diminish without limit, but in such manner that $l\varpi = Lv$, which is a fixed quantity throughout the preliminary transformations, but which, when we come to integrate with respect to ϖ , gives $Ldv = l d\varpi$, or $l\theta \cdot dv = l d\varpi$, or $d\varpi = \theta dv$. We have also $\varpi = v\theta$; consequently, as ϖ , &c. diminish, the limits of v answering to those of ϖ (0 and π) become 0 and ∞ . Make these substitutions in the last expression of (102.), which becomes

$$\pm \frac{2v^z \theta^z}{2.3\dots z} \left\{ \theta\phi\left(\frac{\theta}{E}\right) \cdot 1^z + \dots + \theta\phi\left(\frac{E}{E}\right) \cdot n^z \right\}, \text{ or } \pm \frac{2v^z}{2.3\dots z} \left\{ \theta\phi\left(\frac{\theta}{E}\right) \cdot \theta^z + \dots + \theta\phi\left(\frac{E}{E}\right) \cdot (n\theta)^z \right\},$$

the limit of which is
$$\pm \frac{2v^z}{2.3\dots z} \int_0^E dX \cdot \phi\left(\frac{X}{E}\right) X^z, \text{ or } \pm \frac{2v^z E^{z+1}}{2.3\dots z} \int_0^1 \phi x \cdot x^z \cdot dx$$

On this we remark; 1. That to render the last expression in (102.) perfectly general, the last but one should have $\psi\left(\frac{0}{n}\right)$ altered into $2\psi\left(\frac{0}{n}\right)$, but the term so added decreases without limit. 2. That if the greatest possible error be infinite, as, with the remark in (86.), it may be, the preceding appears to fail; but it may readily be transformed thus. Let $\phi\left(\frac{X}{E}\right)$ be written $\chi(X)$; the integral in $\{ \}$ of the transformation becomes $\int_0^E \chi(X) \cdot X^z dX$, or, by the supposition, $\int_0^\infty \chi(X) \cdot X^z dX$. Whence the term in question becomes $2v^z \div 2.3\dots z \times \int_0^\infty \chi(X) X^z dX$; or, in the processes following, if we would pass from the supposition of limited errors to that of unlimited errors, we must write unity for E , except where it is the limit of an integral, in which case we must write ∞ for E .

Let $\int_0^\infty \phi x \cdot x^z \cdot dx$, or $\int_0^\infty \chi(X) X^z dX$, as the case may be, be called $k^{(z)}$, except only where $z=0$, in which case let $k=2\int_0^\infty \phi x \cdot dx$. (We preserve the notation of Laplace.) That is, let $k=2\int_0^\infty \phi x \cdot dx$, $k'=\int_0^\infty \phi x \cdot x dx$, &c. Substitute in the last but one of (102.) Lv for $l\varpi$, θdv for $d\varpi$, &c., and the probability of the sum of s drawings giving L is

$$\frac{\theta}{\pi} \int_0^\infty \cos Lv \left\{ Ek - \frac{2E^3 v^2}{2} k'' + \frac{2E^5 v^4}{2.3.4} k^{(4)} - \dots \right\}^s dv.$$

But

$$Ek = 2E \int_0^\infty \phi x \cdot dx = 2 \int_0^\infty \phi\left(\frac{x}{E}\right) dx = \int_{-E}^{+E} \phi\left(\frac{x}{E}\right) dx = 1,$$

since the error must lie between $-E$ and $+E$ (inclusive.) Assume $\varepsilon = t^2$ for the s th power just obtained, remembering that $Ek=1$, and if s be considerable ($s=10$ is sufficient), v can be expanded in a very convergent series of powers of t . Take the logarithms of both sides, and the first term of the expansion of that on the first side, which will give

$$s E^3 v^2 k'' = t^2 \text{ nearly, or the probability required is } \frac{\theta}{\pi} \int_0^\infty \cos\left(\frac{Lt}{\sqrt{sk''E^3}}\right) \cdot \varepsilon^{-t^2} \frac{dt}{\sqrt{sk''E^3}}.$$

Let

$$\beta = \frac{1}{4sk''E^3}, \text{ and this becomes } \theta \sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{+\infty} \varepsilon^{-\beta L^2} dL,$$

which is the probability that the sum of s drawings will be L .

In this expression θ is dL , and thus we have $\sqrt{\frac{\beta}{\pi}} \int_{-\infty}^{+\infty} \varepsilon^{-\beta L^2} dL$ for the probability of the same lying between $\pm e$.

For L write Ls , and remember that the probability of the average drawing being L is the same as that of the sum being Ls , and make $c = \beta s^2$, we have then

$$\sqrt{\frac{c}{\pi}} \int_{-\infty}^{+\infty} \varepsilon^{-cL^2} dL, \text{ where } c = \frac{s}{4k''E^3}.$$

for the probability that the average of s drawings will lie between $\pm e$: that is to say, whatever the law of the errors may be, the results of the law already established as necessary when the average is assumed to be the most probable result, become more and more nearly true as the number of observations is increased, and are practically true for moderate numbers of observations. That is, we may proceed as if the specific weight of a single observation were $(4k''E^3)^{-1}$, the mean risk of error $\sqrt{k''E^3 \div \pi}$, and the probable error 1.69 of the last.

(104.) This method fails to show that the average of *two* observations is the most probable result; but if we go

Theory of Probabilities. back to (89.), we find that, on any supposition as to ψ , the substitution of an average of two results for L gives $N=0$; but not so for an average of three. So that if we diminish the accuracy of the preceding approximation by giving smaller and smaller values to s , the approximation does not become more and more defective; but when s has diminished to a certain extent, begins again to approach the true result of the same principles, as they would appear if the approximations were made perfectly rigorous, and finally coincides with them when $s=2$. Theory of Probabilities.

(105.) If in (103.) we write t^2 for cL^2 , in which case the limits of t are $\pm\sqrt{c}.e$, we have, making $T=\sqrt{c}.e$,

$$\frac{1}{\sqrt{\pi}} \int_{-T}^{+T} \varepsilon^{-t^2} dt, \text{ or } \frac{2}{\sqrt{\pi}} \int_0^T \varepsilon^{-t^2} dt$$

for the probability that the true result lies within the average of s observations $\pm \frac{2T\sqrt{k'E^s}}{\sqrt{s}}$.

(106.) The preceding is the result of Laplace, with a difference of notation, as follows, arising from the necessity of simplifying the peculiar species of infinitesimals which he has employed. To compare any result with the form given in the *Théorie des Probabilités*, for k' , k'' , &c. in this work write $k' \div k$, $k'' \div k$, &c., and for E write 1.

(107.) We now proceed to consider errors without reference to sign, that is, counting them as positive in both cases; and we suppose it required to determine the sum of a certain function of the errors, αe , or to find the probability that $\alpha e_1 + \alpha e_2 + \dots + \alpha e_s$ will be contained within given limits. It is evident that the probability of this sum being exactly $(l+\mu s)\theta$ (the errors being $n\theta$, $(n-1)\theta$, $\dots$, 0 , $\dots$, $n-1\theta$, $n\theta$, since the negative errors are reckoned as positive) is the coefficient of $\alpha^{(l+\mu s)\theta}$ in

$$\left(\psi\left(\frac{0}{n}\right) \alpha^0 + 2\psi\left(\frac{1}{n}\right) \alpha^{\alpha\theta} + \dots + 2\psi\left(\frac{x}{n}\right) \alpha^{\alpha(x\theta)} + \dots + 2\psi\left(\frac{n}{n}\right) \alpha^{\alpha(n\theta)} \right)^s \dots (1).$$

If α be $\varepsilon^{\omega\sqrt{-1}}$, the series above written has for the term containing ω^s (or would have, if for $\psi\left(\frac{0}{n}\right)$ we wrote $2\psi\left(\frac{0}{n}\right)$, which makes no difference when we take the limit),

$$\pm \frac{2(\omega\sqrt{-1})^s}{2.3\dots s} \left(\psi\left(\frac{0}{n}\right) (\alpha 0)^s + \psi\left(\frac{1}{n}\right) (\alpha\theta)^s + \dots + \psi\left(\frac{n}{n}\right) (\alpha n\theta)^s \right) \dots (2),$$

in which let $\psi\left(\frac{x}{n}\right)$ be convertible into $\theta\phi\left(\frac{X}{E}\right)$ as before, so that when n increases without limit, the limit of the last factor is $\int_0^E \phi\left(\frac{X}{E}\right) (\alpha X)^s dX$. Write this Σ_s until the limits are taken. Consequently, the preceding s th power is

$$\left(2\Sigma_0 + 2\Sigma_1 \omega\sqrt{-1} - \frac{2\Sigma_2}{1.2} \omega^2 - \frac{2\Sigma_3}{1.2.3} \omega^3 \sqrt{-1} + \dots \right)^s \dots (3).$$

But in the original expression we can detect the coefficient required by multiplying by $\varepsilon^{-(l+\mu s)\theta\omega\sqrt{-1}} d\theta\omega$, and integrating from $\theta\omega = -\pi$ to $\theta\omega = +\pi$; for $\int_{-\pi}^{+\pi} \varepsilon^{x\theta\omega\sqrt{-1}} d\theta\omega$ is 0 between these limits, except only when $x=0$, in which case it is 2π . We have then for the probability in question

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \varepsilon^{-(l+\mu s)\theta\omega\sqrt{-1}} d\theta\omega \left(\psi\left(\frac{0}{n}\right) \varepsilon^{\alpha 0 \cdot \omega\sqrt{-1}} + \dots \right)^s, \text{ or } \frac{1}{2\pi} \int_{-\pi}^{+\pi} \varepsilon^{-(l+\mu s)\theta\omega\sqrt{-1}} d\theta\omega (2\Sigma_0 + 2\Sigma_1 \omega\sqrt{-1} + \dots)^s$$

Now previously to the operation of integrating with respect to ω , comes that of diminishing θ without limit. Let θ be always $=L$, and $\mu\theta = M$; we may then represent θ by dL , and $d(\theta\omega)$ by $dL d\omega$, the rejected term being of an inferior order. But the limits of ω , which are $\pm \frac{\pi}{\theta}$, now become $\pm \infty$. Represent the limit of $2\Sigma_0$,

$$2 \int_0^E \phi\left(\frac{X}{E}\right) dX, \text{ or } 2E \int_0^1 \phi x dx \text{ by } Ek \text{ as before } (=1); \text{ and let } \alpha X = E^{\omega} \alpha_1 \left(\frac{X}{E}\right),$$

$$\text{and let } \int_0^E \phi\left(\frac{X}{E}\right) (\alpha X)^s dX, \text{ or } E^{s+1} \int_0^1 \phi x (\alpha_1 x)^s dx \text{ be } E^{s+1} K^{(s)}.$$

The probability required is $\frac{dL}{2\pi} \int_{-\infty}^{+\infty} \varepsilon^{-(L+Ms)\omega\sqrt{-1}} d\omega (1 + 2K'E^{s+1}\omega\sqrt{-1} - K''E^{2s+1}\omega^2 - \dots)^s$. If, as before, we assume ε^{-t^2} for the s th power in question, we have

$$-t^2 = s \log(1 + 2K'E^{s+1}\omega\sqrt{-1} - K''E^{2s+1}\omega^2 - \dots) = s(2K'E^{s+1}\omega\sqrt{-1} - K''E^{2s+1}\omega^2 + 2K'^2E^{2s+2}\omega^3),$$

as far as ω^2 inclusive. Substitute this for $-t^2$, and make $-Ms + 2sK'E^{s+1} = 0$, or assume $M = 2K'E^{s+1}$, which gives for the probability required

$$\frac{dL}{2\pi} \int_{-\infty}^{+\infty} \varepsilon^{-L\omega\sqrt{-1} - s(K''E^{2s+1} - 2K'^2E^{2s+2})\omega^2} d\omega = \frac{dL}{2\sqrt{\pi s(K'E^{2s+1} - 2K'^2E^{2s+2})}} \varepsilon^{-\frac{L^2}{4s(K'E^{2s+1} - 2K'^2E^{2s+2})}},$$

which gives the probability that $\Sigma \alpha e$ shall lie between $2K'E^{s+1}s + L$ and $2K'E^{s+1}s + L + dL$. This probability is

Theory of Probabilities. greatest when $L=0$; or errors being all positive, the most probable individual value of $\Sigma \alpha e$ is $2K'E^{s+1}s$. And Theory of Probabilities. we also have the following theorem: the most probable value of $\Sigma \alpha e$ for s terms is $2K'E^{s+1}s$; and the probability of a departure from this value lying between $\pm L$ is that of a single error lying between the same limits, in an observation having the specific weight $\{4s(K'E^{2s+1} - 2K'^2E^{2s+2})\}^{-1}$. And E is the greatest value of e which is

possible, K' and K'' being respectively $\int_0^1 \phi x \alpha_1 x dx$ and $\int_0^1 \phi x (\alpha_1 x)^2 dx$, where $\phi\left(\frac{X}{E}\right) dX$ is the probability of $e=X$ and $\alpha_1 x = E^{-s} \alpha(Ex)$. But in the case where E is infinite (103.), put unity for E , and make the limit of the integrals 0 and ∞ .

The probable value admits of a confirmatory explanation, as follows: in $2K'E^{s+1}$, or

$$2E^{s+1} \int_0^1 \phi x \cdot \alpha_1 x \cdot dx, \text{ or } 2 \int_0^E \phi\left(\frac{x}{E}\right) E^s \alpha_1\left(\frac{x}{E}\right) d \cdot X, \text{ or } 2 \int_0^E \phi\left(\frac{X}{E}\right) \alpha X dX,$$

we see the sum of all the values of the function, each multiplied by its probability: and s times as much is the probable sum of s such results.

The preceding may also be written thus: the probability of

$$\Sigma \alpha e \text{ lying between } 2K'E^{s+1}s \pm 2L\sqrt{s(K'E^{2s+1} - 2K'^2E^{2s+2})} \text{ is } \frac{1}{\sqrt{\pi}} \int_{-L}^{+L} e^{-t^2} dt.$$

(108.) Let $\alpha e = e$. Then $\alpha X = E \frac{X}{E}$, or $\alpha_1 X = X$, and $\omega = 1$; whence $K' = k'$, and $K'' = k''$. The probability of the sum of s errors lying between

$$2k'E^2s \pm 2L\sqrt{s(k'E^3 - 2k'^2E^4)} \text{ is } \frac{1}{\sqrt{\pi}} \int_{-L}^{+L} e^{-t^2} dt.$$

(109.) Let $\alpha e = e^2$. Then $\alpha X = E^2 \frac{X^2}{E^2}$, or $\alpha_1 X = X^2$ and $\omega = 2$; whence $K' = k''$, and $K'' = k'$. The probability of the sum of the squares of s errors lying between

$$2k''E^3s \pm 2L\sqrt{s(k''E^3 - 2k'^2E^4)} \text{ is } \frac{1}{\sqrt{\pi}} \int_{-L}^{+L} e^{-t^2} dt.$$

(110.) The last two theorems give the means of determining k' and k'' experimentally with sufficient accuracy. Add a large number of errors, and also their squares, and equate the results to $2k'E^2s$ and $2k''E^3s$. We thus obtain approximately $k'E^2$ and $k''E^3$, the necessary elements for the application of the results in (105.) and (108.), which are the cases of most practical utility.

(111.) If in the preceding problem we suppose all the functions of the errors to be different, or propose to find the probability of $\alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_s e_s$ being contained within given limits, the only alteration required in the preceding is this, that the generating function is not an s th power, or the product of s identical factors, but the product of s factors which are similar functions of α_1, α_2 , &c. Consequently the logarithm will not be s &c., but Σ &c., or for sK' we must write $K'_1 + K'_2 + \dots + K'_s$, and the same for the rest. The probable mean is derived from $-Ms + 2\Sigma(K'E^{2s+1})$, where K', E , and ω may be different in all the different functions. The transformation may then be very easily made.

(112.) We shall now take the problem, supposing the negative errors are to count as negative, and firstly when all the functions are the same. In this case, in expression (1) of (107.), instead of $2a^{s(x^2)}$, or $a^{s(x^2)} + a^{s(x^2)}$ doubled because errors of both kinds enter with positive effect, we must write $a^{s(x^2)} + a^{-s(x^2)}$, the consequence of which

in (2.) is that instead of $2\Sigma_0$ &c. we have $\Sigma_0 + \bar{\Sigma}_0$, &c. where $\bar{\Sigma}_0$, &c. have the limits $\int_{-E}^0 \phi\left(\frac{X}{E}\right) \alpha(-X) dX$,

$\int_{-E}^0 \phi\left(\frac{X}{E}\right) \{\alpha(-X)\}^2 dX$ &c., which, if positive and negative errors be equally likely, or $\phi\left(-\frac{X}{E}\right) = \phi\left(\frac{X}{E}\right)$ gives

$\int_{-E}^0 \phi\left(\frac{X}{E}\right) (\alpha X)^2 dX$ for the limit of $\bar{\Sigma}_2$, or $\int_{-E}^{+E} \phi\left(\frac{X}{E}\right) (\alpha X)^2 dX$ must be substituted for $2\Sigma_2$ in (3). If, then,

we wish to retain the final notation of (107.), we must assume

$$\text{for } z=0 \quad E \int_{-1}^{+1} \phi x dx = Ek = 1, \quad \text{for } z=1, 2, \&c. \quad \int_{-1}^{+1} \phi x (\alpha_1 x)^z dx = 2K^{(z)};$$

and the rest of the process with the result is then as before. The probable sum is $s \int_{-E}^{+E} \phi\left(\frac{X}{E}\right) \alpha X dX$, which can be confirmed as in (107.)

Next, let all the functions be different. We have then, as in (111.), to change $sK^{(s)}$ or $K^{(s)} + K^{(s)} + \dots$ into $\Sigma K^{(s)}$ or $K_1^{(s)} + K_2^{(s)} + \dots$, similarly derived from α_1, α_2 , &c.

(113.) If the limits of positive and negative error be different, and also their facilities, suppose E_1, E_2 , &c. to be the limits of positive error, and E_1, E_2 , &c. to be those of negative error. Let νn , and n' be whole numbers always proportional to $E + E$, E and E ; then (107.) expression (1) becomes the s th power of

$$\psi\left(\frac{0}{\nu}\right) a^{n^0} + \psi\left(\frac{1}{\nu}\right) a^{n^1} + \dots + \psi\left(\frac{n}{\nu}\right) a^{n^{(n^0)}} + \psi\left(\frac{0}{\nu}\right) a^{n^0} + \psi\left(\frac{-1}{\nu}\right) a^{n^{(-n^0)}} + \dots + \psi\left(\frac{-n}{\nu}\right) a^{n^{(-n^0)}},$$

which divides (2), as it there stands, by 2, and substitutes for the last factor of (2) a sum whose limit is

Theory of Pro-
babilities. $\int_{-E}^{+E} \phi\left(\frac{X}{E+E}\right) (\alpha X)^2 dX$. When $z=0$, this is $=1$, in other cases let it be $2k^{(s)}$, and the final result is that the probability of $\alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_s e_s$, or $\Sigma \alpha e$, lying between

Theory of Pro-
babilities.

$$2\Sigma k' \pm 2L \sqrt{\Sigma k'' - 2\Sigma k'^2} \text{ is } \frac{1}{\sqrt{\pi}} \int_{-L}^{+L} \varepsilon^{-t^2} dt.$$

Let $\alpha_1 e_1 = m_1 e_1$, $\alpha_2 e_2 = m_2 e_2$, &c., the limits of negative error being the same throughout, and also those of positive error. If, then, $\int_{-E}^{+E} \phi\left(\frac{X}{E+E}\right) \cdot X^2 dX$ be called $k_{(s)}$, for distinction between this and the $k^{(s)}$ of (103.), we have $\Sigma k' = \frac{1}{2} \Sigma m \times k$, $\Sigma k'^2 = \frac{1}{4} \Sigma m^2 \times k^2$, and $\Sigma k'' = \frac{1}{2} \Sigma m^2 \times k''$, or the probability of the preceding sum lying between

$$\Sigma m \cdot k \pm L \sqrt{2(k'' - k'^2) \Sigma m^2} \text{ is } \frac{1}{\sqrt{\pi}} \int_{-L}^{+L} \varepsilon^{-t^2} dt. \text{ (Laplace, p. 332.)}$$

(114.) It very frequently happens *sensibly* (always, we presume, more or less) that the means of sets of observations made by two different observers will differ sensibly, and the same way for every set: that is, that the average of A's observations will always be a little greater, or always a little less, than those of B, even for the same species of phenomena, observed with the same instrument, under the same circumstances. Undoubtedly this arises from a difference of temperament in the two observers, which induces one to suppose a phenomenon actually arrived a little before it appears so to the other, or from a difference of constitution of the eyes of the two, or something which depends on the observer only. It is called the *personal equation*. We may take account of it thus: suppose that the numerical phenomenon is A, and that if the observers could be sure of avoiding casual errors, or those which are equally likely to be positive and negative, the phenomenon would be to them A and a , so that positive and negative departures from these are equally likely. Let E and $-E$, and e and $-e$, be the limits of their errors, and S and s the number of observations in a set of each observer. The probability of error in $\Sigma A - \Sigma a$, or $\Sigma A + \Sigma(-a)$, is easily found from (113.) Let Φ and ϕ be the functional symbols of the laws of error: then

$$\begin{aligned} & S \int_{-E}^{+E} \Phi\left(\frac{z}{E+E}\right) dz - s \int_{-e}^{+e} \phi\left(\frac{z}{e+e}\right) dz \text{ being } 2SK' - 2sk', \\ \text{and} & S \int_{-E}^{+E} \Phi\left(\frac{z}{E+E}\right) z^2 dz + s \int_{-e}^{+e} \phi\left(\frac{z}{e+e}\right) z^2 dz \text{ being } 2SK'' + 2sk'', \end{aligned}$$

the probability of the error of $\Sigma A - \Sigma a$ lying between

$$2SK' - 2sk' \pm 2L \sqrt{(SK'' + sk'') - 2(SK'^2 + sk'^2)} \text{ is } \frac{1}{\sqrt{\pi}} \int_{-L}^{+L} \varepsilon^{-t^2} dt.$$

This result is expressed in a form which will allow of our immediately passing to the supposition of infinite limits of possible error. Let T be the real phenomenon, and T+P and T+p the *personal* phenomena; that is, the phenomena which the constitution of the observers would lead them to assert, using the same instrument, &c. Supposing positive and negative departures from these to be equally likely in single observations, the laws of facility may be expressed as follows:

$$\Phi\left(\frac{z}{E+E}\right) = \sqrt{\frac{C}{\pi}} \varepsilon^{-C(P-z)^2} \quad \phi\left(\frac{z}{e+e}\right) = \sqrt{\frac{c}{\pi}} \varepsilon^{-c(p-z)^2},$$

which gives

$$2K' = P \quad 2k' = p \quad 2K'' = P^2 + \frac{1}{2C} \quad 2k'' = p^2 + \frac{1}{2c};$$

and the probability of S observations of the first observer exceeding s of the second by a quantity lying between

$$SP - sp \pm L \sqrt{\frac{S}{C} + \frac{s}{c}} \text{ is } \frac{1}{\sqrt{\pi}} \int_{-L}^{+L} \varepsilon^{-t^2} dt.$$

When the number of observations made by both observers is the same, the difference of the sums enables us to approximate to nothing but $P - p$, the advance of one upon the other. But by taking different sets of observations, a number of equations may be obtained of the form $SP - sp = \text{obs. diff.}$, from which, by the method of least squares presently to be obtained, the values of P and p may be obtained, and their probabilities.

(115.) The considerations which have been applied to the errors of observation may be applied to the observations themselves, which will amount to the use of a process similar to that of changing the origin of coordinates in geometry. Instead of supposing positive and negative departures from zero equally likely, in which the theory of errors consists, suppose positive and negative departures from A equally likely, up to $A \pm E$.

Let $\psi\left(\frac{x-A}{E}\right) dx$ represent the probability that the observation shall lie between x and $x+dx$, or the error between $x-A$ and $x+dx-A$; then the equation

Theory of
Pro-
babilities.

$$\int_p^q \psi(x-A) dx = \int_{p-A}^{q-A} \psi x dx$$

Theory of
Pro-
babilities.

immediately shows that the theory of errors may be reduced to that of *discordant observations*. Thus the case of (105.) in the theory of errors is the same as that in (108.), when we use the observations instead of the errors: since all the observations are supposed to have one sign. Thus, from the theory of errors and (105.) we learn that the probability of the error of an average of s observations lying between

$$\pm \frac{2T\sqrt{\frac{1}{2} \text{ mean square of an error}}}{\sqrt{s}} \text{ is } \frac{2}{\sqrt{\pi}} \int_0^T e^{-t^2} dt:$$

while from the theorem on discordant observations answering to that on errors in (108.), we find that the probability of the truth lying between

$$\text{Mean of obs. } \pm \frac{2T\sqrt{\frac{1}{2} \{ \text{mean sq. of an obs.} - (\text{mean of obs.})^2 \}}}{\sqrt{s}} \text{ is } \frac{2}{\sqrt{\pi}} \int_0^T e^{-t^2} dt.$$

Let V be the mean of the observations, and $e_1, e_2, \&c.$ the errors; then

$$\frac{\sum (V+e)^2}{s} - \left\{ \frac{\sum (V+e)}{s} \right\}^2 = \frac{\sum e^2}{s} - \left(\frac{\sum e}{s} \right)^2, \text{ and } \sum e = 0.$$

Whence

$$\text{mean sq. of obs.} - (\text{mean of obs.})^2 = \text{mean sq. of error},$$

which shows the coincidence of the two methods.

(116.) If in (115.) we take that value of T which gives an even chance for the probability in question, or, (100.) $T = .4769360$, we have the following theorems, which may easily be applied without any knowledge, except that of common arithmetic, by any one who desires to know the relative merit of any sets of observations.

To determine the probable error of the mean of any sets of observations, take either of the two following methods, whichever is most convenient.

Find the mean of all the observations.

Take the excess or defect of each observation from the mean of the whole, and add together the squares of all these *reputed errors*.

Divide the square root of this sum by the number of observations, and multiply the quotient by .67449, whose logarithm is .8289751 - 1.

Find the mean of all the observations, and square it.

Square all the observed numbers, and find the mean of these squares.

From the mean square subtract the square of the mean; divide the difference by the number of observations, and multiply the square root of the quotient by .67449, whose logarithm is .8289751 - 1.

If the result of the preceding be e , it is an even chance that the error of the mean is less than e ; or as follows:

1	e	4	$1.91 e$	7	$2.27 e$	10	$2.51 e$	100	$3.80 e$
2	$1.44 e$	5	$2.05 e$	8	$2.36 e$	20	$2.93 e$	200	$4.20 e$
3	$1.71 e$	6	$2.17 e$	9	$2.44 e$	30	$3.18 e$	300	$4.90 e$

This Table signifies that it is 1 to 1 against the error of the mean exceeding e as found above; 2 to 1 against its exceeding 1.44 times as much, &c., &c., and 300 to 1 against its being 4.90 times as much.

EXAMPLE. Observed readings of the microscopes A, B, &c. of the mural circle during two observations (direct and by reflexion) of a star, the microscopes being supposed correctly placed.

	A	B	C	D	E	F
Reflected	$61''.9$	$60''.3$	$64''.3$	$58''.3$	$64''.3$	$55''.0$
Direct	31.8	23.8	29.1	24.0	27.3	24.3

which of these two observations must be considered the best?

The mean of the first readings is 60.7 , and the errors are $1.2, -4, 3.6, -2.4, 3.6$, and -5.7 , the sum of the squares of which is 65.77 . The mean of the second readings is 26.7 , and the errors are $5.1, -2.9, 2.4, -2.7, .6$, and -2.4 , the sum of the squares of which is 53.59 . The square roots of 65.77 and 53.59 are 8.11 and 7.32 , the sixth parts of which are 1.35 and 1.22 , and seven-tenths, (which is near enough for our purpose) of these results give .95 and .85. So far, then, as this one comparison can justify the formation of any opinion, the direct observation is a little better than the reflected one, and it is more than an even chance against either being wrong by so much as one second.

(117.) The Tables at the end are intended to facilitate the determination of all probabilities depending on $\int e^{-t^2} dt$. When we began to write this Article, we were not in possession of any other Table of the integral except that given by Kramp, in his *Analyse des Réfractions*, Strasburg, 1799. We afterwards found the article on the method of least squares in the Berlin *Astronomisches Jahrbuch* for 1834, appended to which are Tables in a more convenient form for this specific subject. But considering the great and increasing importance of the integral $\int e^{-t^2} dt$, and the difficulty of obtaining Kramp's work in this country, we have considered it advisable to insert the Tables from both works.

Theory of
Pro-
babilities.

Tables I. and II. (which run together) contain the values of the integral from t to infinity, together with the logarithms of the values, and the differences of both, and are taken from Kramp. They have already been used (74.), and need no further explanation. They were calculated (Kramp, p. 135.) from the following approximate formula, ($\Delta t = r = .01$), obtained from Taylor's theorem,

Theory of
Pro-
babilities.

$$\Delta \int_t^\infty e^{-t^2} dt = -r e^{-t^2} \left(1 - rt + \frac{2t^2 - 1}{3} r^2 - \frac{2t^3 - 3t}{6} r^3 \right).$$

(118.) Table III. contains the values of the integral from *nothing* to the values of t which appear as arguments, multiplied by $2 \div \sqrt{\pi}$, thus giving the values of the function at which we have so frequently arrived. It is from the *Jahrbuch* cited above, but we do not know whether it depends on the Table of Kramp, or was calculated independently.

(119.) Table IV. (from the same work) contains the integral of Table III., the limits being, not t the argument, but $.4769360 \times t$, so that it gives by inspection the probability of the error being within $t \times e$, in which e is the probable error, or error which there is an even chance of not exceeding. The following is an example of its use.

If the odds for success in a single trial be as 7 to 3, required the probability that, out of 1000 trials, the successes shall lie between $700 \pm a$. We ascertain first the value of a , for which there is an even chance. This (74.) must be found by equating $a \sqrt{1000 \div 2 \times 700 \times 300}$ to $.476936$, which gives $a = 10$ (in round numbers), or it is about an even chance that out of 1000 trials the successes lie between 690 and 710. Looking in Table IV., opposite to 2, we find .82: that is, it is 82 to 18, or more than 4 to 1, that the successes lie between $700 \pm 2 \times 10$, or between 680 and 740.

(120.) Table V. is abridged from C. F. Degen, *Tabularum ad faciliorem et breviorum probabilitatis computationem utilium Enneas*, Copenhagen, 1824. It contains the logarithms of the values of $1.2.3 \dots x$, or $[x]$, as we have called it, from $x=1$ to $x=1200$; and its utility is sufficiently apparent.

(121.) The method of correction known by the name of that of *least squares* is comparatively of recent date. It was proposed by Legendre* in 1806, not as having any foundation in the Theory of Probabilities, but as a very convenient method of reducing equations which are more in number than the unknown quantities to a set which, while they give only one result, will make that result partake of the peculiarities of all the equations in question.

If, for instance, a , b , and c be variable elements which are connected together by the equation $ax + by = c$, where x and y are constant, but unknown, it is plain that two correct simultaneous observations of a , b , and c will accurately determine x and y . But suppose twenty sets of observations to be made, affected with errors, it is equally clear that the use of different pairs of observations will give different values of x and y ; so that 190 different results might be obtained, the average of which, it might be supposed, would give values very near the truth. But if the results of observation be $a_1, b_1, c_1; a_2, b_2, c_2$, &c.; and if α_n, β_n , and γ_n be the errors of the n th observations, it is plain that $a_n x + b_n y - c_n$, instead of being 0, as it should be, will be $\alpha_n x + \beta_n y - \gamma_n$, if x and y have their true values. Let this be called e_n ; we have then sets of equations of the form $a_n x + b_n y - c_n = e_n$, where a_n, b_n, c_n are observed, e_n is unknown but small, and as likely to be positive as negative; and the question is, to determine the most probable values of x and y . The most probable value of the sum of the errors is 0, and $\sum e_n = 0$ gives $x \sum a_n + y \sum b_n - \sum c_n = 0$, an equation of a high degree of probability as compared with any of the components from which it is formed, but still only one, whereas two are wanted to determine x and y . If we take the equations

$$x + \frac{b_n}{a_n} y - \frac{c_n}{a_n} = \frac{e_n}{a_n}, \quad \frac{a_n}{b_n} x + y - \frac{c_n}{b_n} = \frac{e_n}{b_n}, \quad \text{and assume } \sum \frac{e_n}{a_n} = 0, \quad \sum \frac{e_n}{b_n} = 0,$$

we have (s being the whole number of observations)

$$sx + \sum \frac{b}{a} \cdot y - \sum \frac{c}{a} = 0, \quad \sum \frac{a}{b} \cdot x + sy - \sum \frac{c}{b} = 0,$$

which are two equations of higher probability than any of the originals, and from which values of x and y may be obtained. But the hypothesis on which we here proceed is pure assumption; and we are not to be content with two equations simply more probable than any of those derived from single observations; but must endeavour to find those which are more probable than any others whatsoever.

(122.) Legendre observed, that by giving x and y those values which make $\sum e^2$ the least possible, 1. as many equations are obtained as there are unknown quantities, and no more; 2. the method reduces itself in several remarkably simple cases to that which the unassisted judgment most approves. For 1. The rules of the differential calculus give $\sum (ax + by - c)^2$ a *minimum*, when $\sum a^2 \cdot x + \sum ab \cdot y - \sum ac = 0$ and $\sum abx + \sum b^2 \cdot y - \sum bc = 0$; and 2. when $x=a$ is the equation, and observation of a is to determine x , then $\sum (x-a)^2$ a *minimum* gives $x =$ the average of a_1, a_2 , &c. And Legendre also observed, that an old proposition relative to the position of the centre of gravity of a number of points in space is a theorem of this method. If equal weights be applied at each point,

* *Nouvelles méthodes pour la détermination des orbites des comètes*, Paris, 1806. La méthode qui me paroît la plus simple et la plus générale, consiste à rendre minimum la somme des quarrés des erreurs. On obtient ainsi autant d'équations qu'il y a des coefficients inconnus, &c. Preface.

Par ce moyen, il s'établit entre les erreurs une sorte d'équilibre qui empêchant les extrêmes de prévaloir, est très propre à faire connaître l'état du système le plus proche de la vérité. Appendix.

It must be stated, however, that Gauss, in the *Theoria Motus Corporum*, &c. Hamburg, 1809, mentions his having used the same method from the year 1795. *Ceterum principium nostrum, quo jam inde ab Anno 1795, usi sumus, nuper etiam a clar. Legendre in opere Nouvelles Méthodes, &c., prolutum est, &c.*—p. 221.

Theory of
Pro-
babilities.

the centre of gravity is a point so placed, that the sum of the squares of its distances from all the points is a minimum; that is, a_n, b_n , and c_n being the n th observation of the coordinates, the choice of the centre of gravity as the most probable mean of all the observed points is the assumption of those values of x, y and z , which make $\Sigma(x-a)^2 + \Sigma(y-b)^2 + \Sigma(z-c)^2$ a minimum. Circumstances of this kind brought the method into use as one which had as fair claims as any other, and could not be far wrong, at least if any whatever could be right.

Theory of
Pro-
babilities.

(123.) The method of least squares was shown by Laplace to give the most probable, as well as the most convenient result. As a demonstration has not, to the best of our knowledge, found its way into any elementary work, we shall confine ourselves, in the first instance, to the most simple case, $ax=b$, where a and b are to be determined by observation, and s sets of observations have been made. We have as well to establish the method itself as to meet the objections which have been taken to it. These, for the most part, seem to resolve themselves into a supposition, that the formation of an average is a method of which the soundness is almost self-evident and universally admitted, while the method of least squares has a hard and mathematical name, and is a deduction from dubious principles. With regard to the soundness of the average, we have shown (89.) that, in an absolute sense, it is derived from a pure notion of the mind, independent of the individual qualities of the observations; and that it is by no means a necessary truth, being only admissible as such when one particular law of the facility of errors exists. Consequently, when we only know the observations themselves, and have nothing to guide us as to the law of facility, if we assume the average, we assume with it the law of facility in question. We leave out of view the subsequent demonstration, that whatever the law of facility may be, the most probable result of a considerable number of observations cannot differ sensibly from the average, because that consideration, rendering the probable truth of the average itself a matter of deduction as complicated as the method of least squares, cannot be a ground of objection against the latter as compared with the former.

(124.) There are two obvious ways in which the average equation of $a_1x=b_1, a_2x=b_2$, &c. may be formed: the first by averaging a and b separately for the probable values, and then solving the equation; the second by forming the solutions first, and then averaging. We leave to the objectors the decision between these two methods, as to which is the one which unassisted judgment would point out.

The first gives $x = \frac{\Sigma b}{\Sigma a}$; the second gives $x = \frac{1}{s} \Sigma \left(\frac{b}{a} \right)$. Both are particular cases of the following, in which we shall consider them. Let the equations have been previously multiplied by $m_1 m_2 \dots m_s$, so that the solution of the average equation $\Sigma ma.x = \Sigma mb$ is $x = \frac{\Sigma mb}{\Sigma ma}$, the risk of which is (supposing a and b to have equal risks)

$$\sqrt{\frac{\Sigma m^2}{(\Sigma ma)^2} + \frac{\Sigma m^2 (\Sigma mb)^2}{(\Sigma ma)^4}} \rho a, \text{ or } \sqrt{1 + \left(\frac{\Sigma mb}{\Sigma ma} \right)^2} \cdot \frac{\sqrt{\Sigma m^2}}{\Sigma ma} \cdot \rho a.$$

The second average, of which the result was given above, is a particular case of this, derived from $m_1 = \frac{1}{a_1}$, &c.; the first supposes $m_1 = m_2$, &c. = 1. It remains to determine m_1, m_2 , &c. in such a way that the preceding risk may be the least possible.

Now $\Sigma mb \div \Sigma ma$ must lie between the extreme values of $b \div a$, and therefore the factor in which it appears only differs from the truth by a quantity of the order of the squares of the errors, and may be considered as constant. The remaining factor is therefore to be made a minimum. The differentiation of $\Sigma m^2 \div (\Sigma ma)^2$ with respect to m_1 gives, when the result is equated to zero, $2m_1 \cdot (\Sigma ma) - 2a_1 \Sigma m^2 = 0$, whence

$$\frac{m_1}{a_1} = \frac{m_2}{a_2} = \frac{m_3}{a_3} = \dots = \frac{\Sigma m^2}{\Sigma ma}$$

expresses all the requisite conditions, and m_1, m_2 , &c. are proportional to a_1, a_2 , &c. This gives for the final result $\Sigma ab \div \Sigma a^2$, which is the value of x necessary to make $\Sigma (ax-b)^2$ a minimum. The result is the same as arises from assuming $m_1 = a_1, m_2 = a_2$, &c., and the mean risk of error is therefore (x being the obtained result)

$$\frac{\text{risk of a single observation}}{\sqrt{1+x^2} \sqrt{\Sigma a^2}},$$

on which we have to observe, that when x is very small, $\rho a \div \Sigma a^2$ expresses the absolute value of the risk; but that when x is considerable, the same fraction expresses the proportion of the error to the whole result: whence it is always best, if possible, to observe the coefficients of equations which give small results. This is made still more desirable by observing that, *cæteris paribus*, the greater the value of a , the smaller the risk of error.

(125.) The preceding is entirely based upon the supposition, that the risk of a single error being between e and $e+de$ is $\sqrt{c \div \pi} \varepsilon - e^2 de$, to which, as we have seen, every admissible law approximates (and we may add, with a closeness unknown in most other approximations) for a moderately great number of observations. The result of Laplace (p. 320, changing the notation) is $E \sqrt{k'' \div k \pi} \div \sqrt{\Sigma a^2}$, in which (103.) $E=1$, as the limits of possible errors are infinite, and Ek is also $=1$, whence $k=1$. But, in Laplace, the expressions k'' and k have reference to $ax-b$, considered as a single result; whence if c be the specific weight of that result, we have $k'' = \int_0^\infty \phi e \cdot e^2 de$, which, if ϕe be $\sqrt{c \div \pi} \varepsilon - ce^2$, is $1 \div 4c$. Hence $\sqrt{k'' \div \pi}$ is $1 \div 2 \sqrt{c \pi}$, the mean risk of $ax-b$. But, the mean risks of a and b being the same, that of $ax-b$ is (98.) $\sqrt{(\rho a)^2 x^2 + (\rho b)^2}$, or $\sqrt{1+x^2} \times$ mean risk of a or b ; whence the accordance of the two results* is manifest.

* This proof of the method of least squares appears very much more simple than that of Laplace, which is owing to our not

Theory of
Pro-
babilities.

(126.) The preceding also specifically depends upon the supposition, that the weights of all the observations are equal: and we have sometimes seen the method of least squares, in the most usual meaning of the words, applied to cases in which the different a s and b s were themselves means of different numbers of observations, and therefore not equally good. Let us now suppose every a and every b to be an observation with a distinct specific weight. Hence the risk of the average $\Sigma mb \div \Sigma ma$ is

Theory of
Pro-
babilities.

$$\sqrt{\frac{\Sigma(m\rho b)^2}{(\Sigma ma)^2} + \frac{\Sigma(m\rho a)^2 (\Sigma mb)^2}{(\Sigma ma)^4}}, \text{ or } \sqrt{\frac{\Sigma(m\rho b)^2 + x^2 \Sigma(m\rho a)^2}{(\Sigma ma)^2}},$$

x being $\Sigma mb \div \Sigma ma$, the presumed value, and $\rho a_1, \rho b_1$, &c. the mean risks of a_1, b_1 , &c. Differentiate with respect to m_1 , and we have

$$\frac{m_1}{a_1} \{(\rho b_1)^2 + x^2 (\rho a_1)^2\} = \frac{\Sigma(m\rho b)^2 + x^2 \Sigma(m\rho a)^2}{\Sigma(ma)}, \text{ and so on;}$$

this is satisfied by $m_1 = a_1 \div \{(\rho b_1)^2 + x^2 (\rho a_1)^2\}$, &c. &c.: that is, the probable mean is the value of x which makes $\Sigma \frac{(ax-b)^2}{(\rho b)^2 + x^2 (\rho a)^2}$ a minimum, where x_1 is a near approximation; and the mean risk of error of this result is the reciprocal of the square root of $\Sigma \left(\frac{a^2}{(\rho b)^2 + x_1^2 (\rho a)^2} \right)$, which, as should be, agrees with the result in (124.), when $\rho a_1 = \rho a_2 = \rho b_1 = \&c.$

(127.) Since the mean risks are inversely as the square roots of the specific weights, let α_1, α_2 , &c. be the specific weights of a_1, a_2 , &c., and β_1, β_2 , &c. those of b_1, b_2 , &c. Then the most probable result of $a_1 x = b_1$, &c. is that value which makes the first of the following expressions a minimum; and its mean risk is the second:

$$\Sigma \frac{\alpha\beta}{\alpha + \beta x_1^2} (ax-b)^2: \quad \sqrt{1 \text{ divided by } \Sigma \frac{\alpha\beta a^2}{\alpha + \beta x_1^2}}.$$

The expression $ax-b$, considered independently, has the risk $\sqrt{x^2 (\rho a)^2 + (\rho b)^2}$, whence a case of the following remarkable theorem, which is true of all functions whatsoever.

Let $U_1=0, U_2=0$, &c. be equations which give nearly the same value of x , and which should give exactly the same, if certain data of observation could be made absolutely correct. Let c_1, c_2 , &c. be the specific weights of U_1, U_2 , &c., determined from (mean risk) $\times \sqrt{\text{specific weight}} = 1 \div 2\sqrt{\pi}$. Then the most probable value of x is that which makes $\Sigma c U^2$ a minimum. The mean risk of this result is the reciprocal square root of $\Sigma 4\pi c \left(\frac{dU}{dx} \right)^2$, or its

specific weight is $\Sigma c \left(\frac{dU}{dx} \right)^2$. The generalization is easily made by the process in (98.): the correction of x (which call ξ) being reducible to the solution of an equation of the form $m\xi + n = 0$, where m and n are to be observed.

(128.) In the preceding process it is to be remarked, that it is not necessary for a and b to be derived from observations of the same phenomena; it is sufficient that they give nearly the same values of x . The point of difficulty is the determination of the relative weights. So long as we really appeal to independent observations of the same a and b , made under the same circumstances, the most simple case of the formula may be applied. But if, as frequently happens, the coefficients a and b are under different circumstances, it must be remembered that the making of ΣU^2 a minimum, or tacitly supposing an equal weight throughout the observations, does not give a result which is necessarily better than an ordinary average, except when the weights are nearly equal.

This point can be best tested by approximating to $k''E^3$, or $1 \div 4c$, remembering that $2k''E^3$ s, or in this case $2k''$ s, since the possible limits are infinite, that is, $s \div 2c$, will be nearly equal to the sum of the squares of a large number of errors. Or the specific weight of a single observation may be assumed at half the reciprocal of the average square of an error. Nor will this result be sensibly affected if we take reputed errors derived from the mean, as in (116.). Thus, if the average square of 20 errors give .01, the specific weight is about 50. This rough test may be generally applied with great ease; and it would be highly advisable in series of observations to examine the sums of the squares of the errors for each day's work: from which we are persuaded a much better notion might be formed of the relative value of the different sets, than from the unassisted estimation of the observer. It is very well known, that nothing is more common than for an observer entirely to mistake the relative merits of his individual observations; we know of nothing to show that any one is a better judge by estimation of a collective set, than of the individual cases composing it. The application of the preceding test would either afford to every person an irresistible confirmation of his own judgment, or would gradually lead him to better habits of estimation.

(129.) Applying the results of the last article to (124.), and making $ax-b=e$, when x has its real value, and a and b are observed (in which case all e is error), we find $\rho e \div \sqrt{\Sigma a^2}$ for the mean risk of the most probable result.

virtually repeating the process in (98.), as is done by our illustrious guide, not once or twice, but always. The fact is, that the greater part of the *Théorie des Probabilités* is a reprint of papers in the *Memoirs of the Academy*, which appear to contain the contents of the first papers on which he set down his processes. These, with preliminary chapters, descriptive, not of what follows, but of the general methods which he drew from the following parts, make up the whole work.

Theory of Pro-
babilities. But the specific weight of e is $s \div 2\Sigma e^2$ very nearly, where s is the number of equations. Substitute for pe its value $1 \div 2\sqrt{\text{sp. w.} \times \pi}$, which gives $(x = \Sigma ab \div \Sigma a^2)$, Theory of Pro-
babilities.

$$\frac{pe}{\sqrt{\Sigma a^2}} = \sqrt{\frac{\Sigma(ax-b)^2}{2s\pi \Sigma a^2}} = \sqrt{\frac{\Sigma(a \cdot \Sigma ab - b \Sigma a^2)^2}{2s\pi (\Sigma a^2)^3}} = \frac{\sqrt{\Sigma a^2 \Sigma b^2 - (\Sigma ab)^2}}{\Sigma a^2 \sqrt{2s\pi}};$$

since $\Sigma(a^2(\Sigma ab)^2) = \Sigma a^2 \times (\Sigma ab)^2$, &c. We have thus an approximate value of the mean risk in terms of the data of observation only.

(130.) Now let there be any number, l , of unknown quantities, $x, x', x'',$ &c. to be determined from a set of s equations ($s > l$) of the form

$$a_1x + a'_1x' + a''_1x'' + \dots = h_1, \text{ or } u_1 = 0 \\ a_2x + a'_2x' + a''_2x'' + \dots = h_2, \text{ or } u_2 = 0, \text{ \&c.,}$$

where $a, a', \dots, h$ are data of observation. If we were to multiply these equations by $m_1, m_2,$ &c., $n_1, n_2,$ &c., and thus form l equations $\Sigma m a_1 x + \dots = \Sigma m h_1$, $\Sigma n a_1 x + \dots = \Sigma n h_1$, and then deduce the mean risk of $x, x', \dots$ derived from these equations, we should find that risk to be a minimum when $m_1 = a_1, m_2 = a_2,$ &c., $n_1 = a'_1, n_2 = a'_2,$ &c., or when $x, x', \dots$ have the values which make Σu^2 a minimum. We might thus extend the process of (124.) and (126.), which we should recommend the student to do in the case of $l=2$. Instead of this, suppose it granted that the method of least squares is true up to $l-1$ quantities. It is obvious that the most probable values of $x', x'', \dots$ must also be their most probable values derived from any set of equations which are mathematically equivalent to the preceding.

LEMMA. If $u, u', u'', \dots$ be severally functions of $x, y, z, \dots$, and if the value of x derived from $u=0$, be substituted throughout $\phi(u, u', u'', \dots)$, called also $\alpha(x, y, z, \dots)$, thereby making it become $\beta(y, z, \dots)$: the values of $y, z, \dots$, which make β a minimum, are also those which, with the proper value of x , make α a minimum. Let $d\phi = Adu + A'du' + \dots$, $du = m dx + n dy + p dz + \dots$, $du' = m' dx + n' dy + p' dz + \dots$; then the values of $y, z, \dots$, which make β a minimum, are those determined from $u=0, du=0, A'du' + A''du'' + \dots = 0$, for all ratios of $dy, dz, \dots$. From this we find

$$A' \left(n' - m' \frac{n}{m} \right) + A'' \left(n'' - m'' \frac{n}{m} \right) + \dots = 0 \quad A' \left(p' - m' \frac{p}{m} \right) + \dots = 0, \text{ \&c. \&c.,}$$

$$\text{or} \quad An + A'n' + \dots - \frac{n}{m} (Am + A'm' + \dots) = 0, \text{ or } \Sigma An = \frac{n}{m} \Sigma Am, \quad \Sigma Ap = \frac{p}{m} \Sigma Am, \text{ \&c.}$$

If, then, to the system of equations which makes β a minimum, we add $\Sigma Am = 0$, we have $\Sigma Am = 0, \Sigma An = 0, \Sigma Ap = 0$, &c., which are the conditions under which α is a minimum.

This lemma contains almost obviously the means of showing the truth of the method of least squares as to l equations from its truth as to $l-1$ equations. Assuming $u_1 = 0$, eliminate x out of $u_2 = 0$, &c. We thus get $l-1$ equations, containing $x', x'', \dots$, of which we presume the most probable values are those which make $u_2^2 + u_3^2 + \dots$ a minimum, after x is eliminated. But these values are those which make Σu^2 a minimum with regard to $x', x'', \dots$. Assuming, therefore, these values, we have now only to ask what value of x is the most probable result of $u_1 = 0$, &c., after these most probable values of $x',$ &c. have been substituted. The answer is, that value (124.), which makes Σu^2 a minimum with respect to x alone; the condition of which, combined with the other conditions, shows that the most probable set of values of the whole l quantities is derived from the same method as that of the $l-1$.

(131.) We now pass to the case of the last article, with this addition, that $a_1, a_2,$ &c., $a'_1, a'_2,$ &c., $h_1, h_2,$ &c. all have different risks. Looking at the last process, we see that it enables us to assert our proposition with respect to l quantities by means of $l-1$ quantities, to every extent to which it can be asserted of the latter. In (124.), it is shown, that in the case of one unknown quantity, the condition $d \cdot \Sigma u^2 = 0$ gives the true result, not only in the case where $a_1, a_2,$ &c., $b_1, b_2,$ &c. have all the same risks, but also when they have different risks, provided only that $x^2(\rho a)^2 + (\rho b)^2$ be the same in all the equations. That is, our result is true, if $u_1, u_2,$ &c. all have the same risk, in whatever manner the component risks may differ. Again, k being a determined constant, the risk of kz is $k\rho z$. Divide, then, the equations $u_1 = 0, u_2 = 0,$ &c. by such quantities as will make their risks equal; that is, k being arbitrary, divide by $k\rho u_1, k\rho u_2,$ &c. If we were to divide by $\rho u_1, \rho u_2,$ &c. we should reduce the risks to unity, which may appear not to be a small risk: but the risks are now all equal to $1 \div k$, which may be made as small as we please. Consequently, the theorem in the last article applies to $u_1 \div k\rho u_1$, in place of u_1 , &c., and the most probable result is derived from making $\Sigma (u^2 \div k^2 (\rho u)^2)$, or $\Sigma (u^2 \div (\rho u)^2)$ a minimum. This answers to the generalization in (126.), which might have been established in the same way.

(132.) It remains to inquire, what is the mean risk of the result thus obtained? First, let the risks of all the expressions $u_1, u_2,$ &c. be the same; let $x, x',$ &c. be the probable values of the unknown quantities. Then we must have the following equations, deduced from the equations which the method of least squares gives.

$$\Sigma a^2 x + \Sigma a a' x' + \dots = \Sigma a h, \quad \Sigma a a' x + \Sigma a'^2 x' + \dots = \Sigma a' h, \text{ \&c. \&c.}$$

Now, in virtue of the risks of $a_1, a_2,$ &c., $a'_1, a'_2,$ &c., let e_1 be the error of u_1 , &c., so that $u_1 = e_1, u_2 = e_2,$ &c. would give the true results, taking $a_1, a_2, \dots, h_1, h_2, \dots$, as they have been observed. Let $\xi, \xi',$ &c. be the corrections of $x, x',$ &c., in consequence of these total errors: then we have

$$\Sigma a^2 \xi + \Sigma a a' \xi' + \dots = \Sigma a e, \quad \Sigma a a' \xi + \Sigma a'^2 \xi' + \dots = \Sigma a' e, \text{ \&c. \&c.}$$

From which it follows, that changing $h_1, h_2,$ &c. into $e_1, e_2,$ &c., the values of $x, x',$ &c. become those of $\xi, \xi',$ &c. And they assume the form

Theory of
Pro-
babilities.Theory of
Pro-
babilities.

where $M, A, A', \&c.$ are easily (in principle) obtained from the solution of the linear equations. Now, in calculating the risks of $\xi, \xi', \&c.$, it is unnecessary to consider those of $a, a', \&c.$, both because the risks of these quantities are already considered in $e_1, e_2, \&c.$; and because, were it not so, $e_1, e_2, \&c.$ being already of the order of the errors, the errors of $a_1, a_2, \&c.$ would only enter in terms of the second order. Consequently the risks of $e_1, e_2, \&c.$ being the same (let each be ρe), we have for the several mean risks

$$\rho\xi = \rho x = \frac{\rho e}{M} \sqrt{(Aa_1 + A'a'_1 + \dots)^2 + (Aa_2 + A'a'_2 + \dots)^2 + \dots},$$

$$\rho\xi' = \rho x' = \frac{\rho e}{M} \sqrt{(Ba_1 + B'a'_1 + \dots)^2 + (Ba_2 + B'a'_2 + \dots)^2 + \dots}$$

where the expressions under the radical signs are $A^2 \Sigma a^2 + A'^2 \Sigma a'^2 + \dots + 2AA' \Sigma aa' + \dots$, and so on.

(133.) To apply this to the case of two quantities, observe that $a_1 x + a'_1 x' = h_1$, &c. give $\Sigma a^2 \cdot x + \Sigma aa' \cdot x' = \Sigma ah$ and $\Sigma aa' \cdot x + \Sigma a'^2 \cdot x' = \Sigma a'h$, from which we deduce

$$x = \frac{\Sigma a'^2 \cdot \Sigma ah - \Sigma aa' \cdot \Sigma a'h}{\Sigma a^2 \cdot \Sigma a'^2 - (\Sigma aa')^2} \quad x' = \frac{\Sigma a^2 \cdot \Sigma a'h - \Sigma aa' \cdot \Sigma ah}{\Sigma a^2 \cdot \Sigma a'^2 - (\Sigma aa')^2}$$

$$M = \Sigma a^2 \cdot \Sigma a'^2 - (\Sigma aa')^2, \quad A = \Sigma a'^2, \quad A' = -\Sigma aa', \quad B = -\Sigma aa', \quad B' = \Sigma a^2,$$

$$\rho x = \frac{\rho e \sqrt{\Sigma a'^2}}{\sqrt{\Sigma a^2 \cdot \Sigma a'^2 - (\Sigma aa')^2}} \quad \rho x' = \frac{\rho e \sqrt{\Sigma a^2}}{\sqrt{\Sigma a^2 \cdot \Sigma a'^2 - (\Sigma aa')^2}}$$

For ρe write its probable value $\sqrt{\Sigma e^2 \div 2s\pi}$, derived as in (128.), and we have the expressions of Laplace.

(134.) There are several equations of connection between $A, A', \&c.$, $B, B', \&c.$, which it would be worth while to investigate only if (which never happens) more than three or four unknown quantities are considered together. For instance, the quantities under the radical sign in the results of (132.) all contain M as a factor. But the following is the only property which will be of use here, as by it we pass from the risk in the case of $l-1$ to that in the case of l quantities. Suppose we eliminate one of the variables (t , whose coefficients in $u_1, u_2, \&c.$ are $\theta, \theta', \&c.$) from the s equations in (130.) We have then a set of $s-1$ equations of the following form:

$$\left\{ \Sigma a^2 - \frac{(\Sigma a\theta)^2}{\Sigma \theta^2} \right\} x + \left\{ \Sigma aa' - \frac{\Sigma a\theta \cdot \Sigma a'\theta}{\Sigma \theta^2} \right\} x' + \dots = \Sigma ah - \frac{\Sigma a\theta \cdot \Sigma h\theta}{\Sigma \theta^2};$$

and so on; using the equation in which t has the coefficient $\Sigma \theta^2$, as the eliminator. And since the value of x , and its risk, so far as the same depends upon forms derived from the solution, will be the same from whichever set it is derived, we may ascertain the successive risks of two, three, &c. unknown quantities, by changing Σz^2 into

$\Sigma z^2 - \frac{(\Sigma z\theta)^2}{\Sigma \theta^2}$, and $\Sigma zz'$ into $\Sigma zz' - \frac{\Sigma z\theta \cdot \Sigma z'\theta}{\Sigma \theta^2}$; where Σ after the change contains one more term than before, as

does Σe^2 in the coefficient. Thus, having $\sqrt{1 \div \Sigma a^2}$ for the coefficient of ρe in the case of one unknown quantity, we immediately obtain

$$\sqrt{1 \div \left(\Sigma a^2 - \frac{(\Sigma aa')^2}{\Sigma a'^2} \right)}, \text{ or } \sqrt{\Sigma a'^2} \div \sqrt{\Sigma a^2 \cdot \Sigma a'^2 - (\Sigma aa')^2}$$

for two unknown quantities, and

$$\left(\Sigma a'^2 - \frac{(\Sigma a'a'')^2}{\Sigma a''^2} \right) \div \sqrt{\left\{ \Sigma a^2 - \frac{(\Sigma aa')^2}{\Sigma a'^2} \right\} \left\{ \Sigma a'^2 - \frac{(\Sigma a'a'')^2}{\Sigma a''^2} \right\} - \left\{ \Sigma aa' - \frac{\Sigma aa'' \Sigma a'a''}{\Sigma a''^2} \right\}^2},$$

or $\sqrt{\Sigma a'^2 \cdot \Sigma a''^2 - (\Sigma a'a'')^2} \div \sqrt{\Sigma a^2 \cdot \Sigma a'^2 \Sigma a''^2 + 2\Sigma aa' \Sigma a'a'' \Sigma a''a - \Sigma a^2 \cdot (\Sigma a'a'')^2 - \Sigma a'^2 (\Sigma a''a) - \Sigma a''^2 (\Sigma aa')^2}$,

for three unknown quantities, and so on. To extend the same process to equations of different weights, divide each coefficient by the square of the risk of its equation.

(135.) Upon reviewing the whole process of the method of least squares, it seems to us, 1. That this method is at least as worthy of confidence as the notion that the average of observations of a single phenomenon gives the most probable result. 2. That its weak point in practice is the difficulty of ascertaining with great precision the weights to be given to the several equations. With regard to the latter point, it may be shown that errors in the estimation of mean risks must be considerable before they can much affect the mean risk of the result.

Let $\lambda_1, \lambda_2, \&c.$ be the presumed multipliers of $a_1 x - b_1, a_2 x - b_2, \&c.$ as in (131.), which are to reduce the equations in (124.) to equations of equal risk; derived from the general form $\lambda = 1 \div \sqrt{x^2 (\rho a)^2 + (\rho b)^2}$. We may then consider $\lambda a x - \lambda b = 0$ as giving a new set of equations in which the coefficients λa and λb are affected both by the error of observation and that of estimation. If we could determine $\rho \lambda$, the risk of the preceding expression might then be ascertained from

$$\lambda^2 \{ (\rho a)^2 x_1^2 + (\rho b)^2 \} + (\rho \lambda)^2 (ax - b)^2, \text{ or } 1 + (\rho \lambda)^2 (ax_1 - b)^2.$$

Consequently (126.), we must make $\Sigma \left\{ \frac{\lambda^2}{1 + (ax_1 - b)^2 (\rho \lambda)^2} (ax - b)^2 \right\}$ a minimum.

Now, since it is not the absolute errors of $\lambda_1, \lambda_2, \&c.$ which affect our result, but only the errors of their proportions, and since

Theory of
Pro-
babilities.

$$(ar_1 - b)^2 (\rho\lambda)^2 = \frac{(ax_1 - b)^2}{(\rho a)^2 x_1^2 + (\rho b)^2} \cdot \left(\frac{\rho\lambda}{\lambda}\right)^2, \text{ which is of the order } \frac{1}{x_1^2} \left(\frac{\rho\lambda}{\lambda}\right)^2,$$

Theory of
Pro-
babilities.

we see that the method is abundantly secure if $\rho\lambda \div \lambda$ be of the same order as ρa , &c., and the more so the greater the value of the unknown quantity.

(136.) The methods which precede imply that we are obliged to make use of all the equations, so that if one or more should give results very different from the mean as compared with the rest, it is not therefore to be either rejected or modified. But it would not be difficult, having determined the most probable value, or even any kind of mean, to return with that value to the equations, and to select a set from which, if a new probable value were deduced, its risk with reference to its own equations, or even to all the equations, should be less than the most probable value of all. This Laplace has done in a method on which he lays some stress, having explained it in three places.* He calls it the *method of situation*, and still continuing to style that of least squares the *most advantageous*, we accordingly find conditions under which it is stated in so many words, that the method of situation is *better* than the *most advantageous* method. But it must be remembered that Laplace did not speedily come to the consideration of a method of least squares *with equations of different weights*; which appears for the first time in the third supplement. His method, properly considered, is only one of the infinite number of ways in which presumptions drawn from the most probable result might be allowed to affect the presumed weights of the equations: and we cannot discover any reason why his particular method should be preferred. At the same time we cannot refuse to allow that the establishment of a most probable result places us in the circumstances alluded to in (15.), where we have to return upon the premises with new probabilities forced upon us by the conclusion. The probability of an equation is that of its result being exactly true: consequently, X being the most probable value of x , and x_n that derived from the n th equation, the reputed probability of that equation is as $\varepsilon^{-c(X-x_n)^2}$, where c is the specific weight of the result, derived from the mean risks. When we have more than one unknown quantity, we may either ascertain its probability with respect to x , where x_n is the value of x deduced after substitution of the probable values of x' , x'' , &c.: or we may ascertain the joint probability of any l equations (l being the number of unknown quantities), which will be as the product of $\varepsilon^{-c(X-x)^2}$, $\varepsilon^{-c'(X'-x')^2}$, &c. The principle explained in the article CALCULUS OF FUNCTIONS, § 37, furnishes a method by which the method of successive substitution may be appended to that of least squares. Assuming the weights as nearly as they can be found, ascertain the most probable result, from which find the weights of the equations. If these agree with the assumed weights, the process is finished: if not, repeat the process with the new weights (or if we wish to give more importance to the first weights, with the mean proportionals between them and the new weights), and so on, until a result is obtained for which the assumed and deduced weights of the equations are sufficiently near to equality. This process would involve considerable labour, but the time must arrive when the method of least squares will require to be modified in this or some equivalent manner.

(137.) The method of least squares coincides with the ordinary solution when the number of equations is the same as that of unknown quantities; but when the unknown quantities are in excess, it gives an indeterminate problem, since, though as many equations as unknown quantities are produced, they are reducible to the original equations, and are not independent.

(138.) The method of least squares holds true when positive and negative errors are not equally likely: the only difference being in the specific weight of the result, and in $\sqrt{c \div \pi} \int \varepsilon^{-c^2} de$, from $+e$ to $-e$, no longer denoting the probability of an error between these limits, but between $M \pm e$, where M is the mean error obtained on considerations similar to those in (107., &c.) But it is hardly worth while to enter into detail on this point, and we therefore pass to other applications of the theory.

(139.) Dividing all discrepancies into errors of observation, and unexplained phenomena, the former of which are as likely to be positive as negative, we may immediately ascertain what, as respects two average results, is the relative probability of their difference being referable to one or the other class. Let there be two phenomena A and B , which, being at first supposed under the same circumstances, are found to differ in s observations, the sum of all the A 's exceeding that of the B 's by k . If we suppose $A - B$ to be a quantity as likely to be positive as negative, it follows from (103.), that the probability of $s(A - B)$ being as great as h , or greater, is

$$\int_h^\infty \varepsilon^{-\beta x^2} dx \times \sqrt{\beta \div \pi}, \text{ where } \beta = 1 \div 4sk''E^2,$$

$$\text{or } \frac{1}{2} \left(1 - \frac{2}{\sqrt{\pi}} \int_0^t \varepsilon^{-t^2} dt \right), \text{ from } t=0 \text{ to } t=h \div \sqrt{4sk''E^2}.$$

This is the probability of the result arising from ordinary error; if it be very small, then the complement to unity, which is very near unity, is the probability that there is some other cause of the difference. Supposing k'' unknown, the case most unfavourable to an external cause (excluding the supposition that the error and its probability increase together) is that all errors from $-E$ to $+E$ are equally likely. This gives

$$\phi\left(\frac{X}{E}\right) = \text{const.} = \frac{1}{2E} \quad k''E^2 = \int_0^E \frac{1}{2E} \cdot x^2 dx = \frac{E^2}{6},$$

or the most unfavourable value of the limit of integration is $h \div \sqrt{\frac{2}{3}sE^2}$. To take the instance given by Laplace: 400 days were chosen, throughout which the barometer did not vary 4 millimetres: so that, in this case,

* *Théorie*, &c. p. 343, 2ième. *Supplément*, &c. p. 42. *Méc. Cél.* liv. iii. No. 39, 40.

Theory of
Pro-
babilities.

we suppose $E=4$. The sum of the heights at nine in the morning exceeded those at four in the afternoon by 400 millimetres, giving an average of one millimetre. That this should have happened from mere error of observation, has only the probability $\frac{1}{2} \left(1 - \frac{2}{\sqrt{\pi}} \int \varepsilon^{-t^2} dt \right)$, from $t=0$ to $t=\sqrt{37\frac{1}{2}}$; a result of such exceeding

Theory of
Pro-
babilities.

smallness that the supposition of a diurnal variation of the barometer becomes insuperably probable.

(140.) Common sense dictates to us the notion that, of two causes which are not equally probable, the one most likely beforehand to have caused an event is most likely to have been the cause. From this notion (shown in (19.) to accord with the fundamental principles of the theory), we have the following theorem, already used: that if we consider each possible degree of probability as belonging to a state of things under which the event observed might have happened (which is all we here imply by a *cause*), and if the assumption of $x_1, x_2, \dots$ for the probabilities of the components of an event give $y=f(x_1, x_2, \dots)$ for the probability of the event happening under that status, the relative probability of such status having really existed is $f(x_1, x_2, \dots) \div \int f(x_1, x_2, \dots) dx_1 dx_2 \dots$. Or rather, $f \times dx_1 dx_2 \dots$ divided by $\int f \times dx_1 dx_2 \dots$ is the probability of the causal status being such that the chances of the components lie between x_1 and x_1+dx_1, x_2 and $x_2+dx_2, \&c.$ If $x_1, \&c.$ be independent of each other, the integral in the denominator must extend from $x_1=0$ to $x_1=1$, from $x_2=0$ to $x_2=1$, &c.; but if some of the variables depend on the rest, only those which are independent must be used as integrators. The most probable cause is that status which makes f a maximum: and it must be our object to determine (at least for one and two variables) what is the probability that $a_1, a_2, \&c.$ being the values at the maximum, the causal status is one in which the component probabilities lie between $a_1-\theta_1$ and $a_1+\theta'_1, a_2-\theta_2$ and $a_2+\theta'_2, \&c.$

(141.) But first, it is worth while to observe, that the argument in (89.) is reinforced by proving that the result is that* which makes the observed event most probable, as well as that which gives the estimated disadvantage equal to nothing. The maximum value of $\phi(A_1-x) \cdot \phi(A_2-x) \dots \phi(A_n-x)$ is ascertained from the following equations: $\chi x = \phi'x \div \phi x$ and $\sum \chi(A-x) = 0$. Assuming that the average satisfies this, we may by the same method, as in (89.), ascertain that $\chi x = 2cx$ or $\phi x = \varepsilon^{cx^2+c'}$, in which the condition of a maximum (as distinguished from a minimum) requires that c should be negative.

(142.) Let θ and θ' be so near and so small that $-T$ and T' (in (69.), altering the sign of the inferior limit), the values of t answering to $x=a-\theta$ and $x=a+\theta'$, differ by no more than a quantity of the order $1 \div \sqrt{\alpha}$: where α , as in the article cited, is such that the α th root of y does not contain considerable exponents. The equation there given then becomes

$$\int_{a-\theta}^{a+\theta'} y dx = \frac{1}{\sqrt{\pi}} \int_{-T}^{+T'} y dx \times \int_{-T}^{+T'} \varepsilon^{-t^2} dt + \text{terms of the order } \frac{1}{\alpha}, \&c.$$

The values of y corresponding to the limits of t are both zero: and it will generally happen that the values of x which make $y=0$ are 0 and 1: since practical cases are mostly those in which the observed event could not have happened with absolute certainty either for or against the event whose probability is x . It is sufficient if the

extreme values of y be very small. Let $\sqrt{\frac{1}{2}(T^2+T'^2)} = \tau$, and observe that using $-\tau$ and $+\tau$ for $-T$ or

$-\sqrt{\frac{1}{2}(T^2+T'^2)} + \frac{1}{2}(T^2-T'^2)$, and $+T'$ or $+\sqrt{\frac{1}{2}(T^2+T'^2)} - \frac{1}{2}(T^2-T'^2)$ is equivalent to a rejection of terms which are only of the order $1 \div \alpha$. Reject all such terms, and we have

$$\int_{a-\theta}^{a+\theta'} y dx \div \int_0^1 y dx = \frac{1}{\sqrt{\pi}} \int_{-\tau}^{+\tau} \varepsilon^{-t^2} dt.$$

But assuming $y=Y \varepsilon^{-t^2}$, as in (65.), and observing that $x=a$ gives $t=0, y=Y$, we have

$$t^2 = -\frac{1}{2} \frac{Y''}{Y} \theta^2 + A\theta^3 + \dots, \text{ whence } \tau = \theta \sqrt{-\frac{1}{2} \frac{Y''}{Y}} \text{ very nearly;}$$

or the probability of x lying between $a \pm L \sqrt{-\frac{2Y}{Y''}}$ is $\frac{1}{\sqrt{\pi}} \int_{-L}^{+L} \varepsilon^{-t^2} dt$.

(143.) When the observed event is compounded of two independent events, and y is the probability† that the observed event would have *à priori*, if x and x' were those of the components, then the probability which the observed event gives to x being contained between p and q , and x' between p' and q' , is (48.), (68.),

$$\frac{\int y dx dx' \text{ (from } x=p, x'=p' \text{ to } x=q, x'=q')}{\int y dx dx' \text{ (from } x=0, x'=0 \text{ to } x=1, x'=1)} \text{ or } \frac{\int \varepsilon^{-t^2-t'^2} dt dt' \text{ (limits specified below)}}{\int \varepsilon^{-t^2-t'^2} dt dt' \text{ (from } t=-\infty, t'=-\infty \text{ to } t=+\infty, t'=+\infty)}$$

(neglecting a factor which should multiply both the latter integrals) where $t=M^{-1}(M\theta+N\theta')$, $t'=M^{-1}(MP-N^2)^{-1}\theta'$, (supposing $x=a+\theta, x'=a'+\theta'$) are equations which connect simultaneous values of x and x' with those of t and t' . To this it is a sufficient approximation that $2MY, 2NY$, and $2PY$ should be $-Y'', -Y'$, and $-Y''$, being the values of the second differential coefficients of y , when $x=a, x'=a'$, or when y is a maximum ($=Y$). This gives us

* This is the foundation adopted by Gauss, in his *Theoria Motus*, &c.

† It is supposed, as generally happens, that y is 0, or very small when $x=0, x'=0$, or when $x=1, x'=1$.

Theory of
Pro-
babilities.

$$\theta' = \left(\frac{-2Y''Y}{Y''Y'' - Y'^2} \right)^{\frac{1}{2}} t', \quad \theta = \left(-\frac{2Y}{Y''} \right)^{\frac{1}{2}} t - \frac{(2Y)^{\frac{1}{2}} Y' t'}{\{-Y''(Y''Y'' - Y'^2)\}^{\frac{1}{2}}};$$

Theory of
Pro-
babilities.

whence the probability that

$$\left. \begin{array}{l} x \text{ will lie between } a \text{ and } a + \frac{t}{\sqrt{M}} - \frac{Nt'}{\sqrt{M(MP - N^2)}} \\ x' \text{ will lie between } a' \text{ and } a' + \frac{\sqrt{M} \cdot t'}{\sqrt{MP - N^2}} \end{array} \right\} \text{ is } \frac{1}{\pi} \int_0^t \int_0^{t'} e^{-t^2 - t'^2} dt dt'.$$

(144.) We shall now apply the preceding analysis to some instances connected with statistical inquiries. And first let there be two events possible, P and Q, and p being greater than q , let P have happened p times, and Q, q times. We can from (77.) ascertain the presumption, that the probability of P lies between $p \div (p+q) \pm k$, where k is not considerable. But we ask what is the presumption that the chance of P in a case yet to come is greater than that of Q. Let x be chance of P happening in any one instance. The probability of the observed event is $[p+q] \div [p] \cdot [q] \times x^p (1-x)^q$, and as in (142.) the presumption that the chance of P lies between 0 and $\frac{1}{2}$ is $\int_0^{\frac{1}{2}} x^p (1-x)^q dx \div \int_0^1 x^p (1-x)^q dx$, where the first integral is taken from $x=0$ to $x=\frac{1}{2}$. The method of (61.) will not apply to this case, in most instances, and that of (60.) must be employed. When $y = x^p (1-x)^q$, we find $v = x(1-x) \div \{p - (p+q)x\}$, which gives

$$\int_0^{\frac{1}{2}} y dx = \frac{1}{2^{p+q+1}(p-q)} \left\{ 1 - \frac{p+q}{(p-q)^2} + \dots \right\}; \text{ but } \int_0^1 y dx = \sqrt{2\pi} \frac{p^{p+\frac{1}{2}} q^{q+\frac{1}{2}}}{(p+q)^{p+q+\frac{1}{2}}},$$

from (48.) and (70.) We have, therefore, for the presumption that the event which has happened most often is *not* the more probable of the two,

$$\frac{(p+q)^{p+q+\frac{1}{2}}}{2^{p+q+1}} \cdot \frac{1}{p^{p+\frac{1}{2}} q^{q+\frac{1}{2}}} \cdot \frac{1}{\sqrt{\pi}(p-q)} \left\{ 1 - \frac{p+q}{(p-q)^2} \right\} \text{ very nearly.}$$

The most easy method of computing this is by the formula

$$\begin{aligned} \log \left\{ \left(\frac{p+q}{2} \right)^{p+q} \div p^p q^q \right\} &= -p \log \left(1 + \frac{p-q}{p+q} \right) - q \log \left(1 - \frac{p-q}{p+q} \right) \\ &= -(p+q) \left\{ \frac{1}{1.2} \left(\frac{p-q}{p+q} \right)^2 + \frac{1}{3.4} \left(\frac{p-q}{p+q} \right)^4 + \frac{1}{5.6} \left(\frac{p-q}{p+q} \right)^6 + \dots \right\}, \end{aligned}$$

formed by expansion and reduction of the terms in pairs, the first and second, third and fourth, &c. In the case where p and q differ so much that $p+q$ is small compared with $(p-q)^2$, the resulting presumption is very nearly

$$\frac{(p+q) e^{-c^2}}{4c \sqrt{pq\pi}}, \text{ where } c = \frac{p-q}{\sqrt{2}(p+q)} : \text{ which is a very small quantity.}$$

The remainder of unity is the presumption that the event which happens most often is the more probable. But when p and q are so near each other that the preceding does not apply, we find, as in (142.), that $\int_0^{\frac{1}{2}} y dx \div \int_0^1 y dx$ is $\int_0^{\frac{1}{2}} e^{-t^2} dt \div \sqrt{\pi}$, where the limits of the integral require examination. Taking $y = x^p (1-x)^q$, where p is the greater, we find that in $y = Y e^{-t^2}$, Y is derived from $x = p \div (p+q)$, or x is more than $\frac{1}{2}$. If, then, we integrate from $x=0$ to $x=\frac{1}{2}$, we do not get so far as the maximum value of y ; whence we must assume $t = +\infty$ as the limit corresponding to $x=0$, and $t = -\infty$ to $x=1$. Since $Y = p^p q^q \div (p+q)^{p+q}$, we have $\left\{ \frac{1}{2} (p+q) \right\}^{p+q} \div p^p q^q$ for the value of e^{-t^2} , corresponding to $x = \frac{1}{2}$. Integrate (to preserve a positive sign,

reverse* the limits) from $t = (p-q) \div \sqrt{2}(p+q)$ to $t = +\infty$, using the value of t derived from the first term of the preceding series. We have then the presumption, that the most frequent event is not the most probable.

(145.) Let there be two sets of $p+q$ and $p'+q'$ events, and let p and p' be the number of times which P happens, and q and q' the number of times which Q happens. Also let $p' \div q'$ be greater than $p \div q$. Required the presumption, that the probability of P under the first circumstances is greater than under the second.

Return to (77.), making $c = \sqrt{(p+q)^3 \div 2pq}$ and $c' = \sqrt{(p'+q')^3 \div 2p'q'}$, we find that the several probabilities of P lying between $a+\theta$ and $a+\theta+d\theta$ in the first, and between $a'+\theta'$ and $a'+\theta'+d\theta'$ in the second, are $\sqrt{c} \div \pi e^{-c^2 \theta} d\theta$ and $\sqrt{c'} \div \pi e^{-c'^2 \theta'} d\theta'$, a and a' being $p \div (p+q)$ and $p' \div (p'+q')$. If, then, we multiply these together, make $a'+\theta' = a+\theta+t$, and make θ' and t vary together from $t=0$ to $t=\infty$, we have, for a given value of θ , the presumption that $a+\theta$ has the corresponding value, and $a'+\theta'$ a greater. Then, integrating with respect to θ , from $\theta=-\infty$ to $\theta=+\infty$, we have the total presumption that $a'+\theta'$ is greater than $a+\theta$. That is, we must find (78.)

* The ambiguity of sign in the value of $\int y dx$ in (65.) has reference to this circumstance.

Theory of
Pro-
babilities.Theory of
Pro-
babilities.

$$\frac{cc'}{\pi} \int_{-\infty}^{+\infty} \int_0^{\infty} \epsilon^{-c^2 t^2 - c'^2 (\theta + t - a' + a)^2} d\theta dt, \text{ or } \sqrt{\frac{k}{\pi}} \int_0^{\infty} \epsilon^{-k^2 (t - a' + a)^2} dt,$$

(where $k^2 = c^2 c'^2 \div (c^2 + c'^2)$); this is $\frac{1}{\sqrt{\pi}} \int_{-L}^{\infty} \epsilon^{-t^2} dt$, where $L = k(a' - a)$, which is also

$$\frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^L \epsilon^{-t^2} dt.$$

(146.) The preceding questions may be easily applied to the determination of the presumption, that the probability of a child to be born shall be a male rather than a female, by means of the recorded births in any country. But this is a question of pure curiosity compared with the following. The method of forming tables of mortality is (where it can be done) by observing what proportions of a number of individuals remain after one, two, &c. years. Supposing a table to be formed which should thus represent the career of 1000 individuals, it is obvious, 1. That the value of that table goes on diminishing as the number of individuals composing it diminishes. 2. That it must be a question how far that number itself is a sufficient guarantee for the presentation of something like an average of the whole of the class in question. We now ask, supposing that of p status of the same kind, q or λp are left at the end of a given time, what presumption is thereby afforded that at another trial, of p' status, some number between $q' \pm z$, or $\lambda p' \pm z$ a given number, shall be left at the end of a second such time. If x had been the *a priori* probability that any one of the first status should have lasted out the time, the consequent probability of the observed event would have been (No. of combinations of q out of p) $\times x^q (1-x)^{p-q}$, and the probability of the proposed event (on the supposition that $q' + z$ exactly are left) arising out of the possibility of that particular presumption being correct, will be

$$(\text{No. of combinations of } q' + z \text{ out of } p') \times \frac{x^q (1-x)^{p-q}}{\int_0^1 x^q (1-x)^{p-q} dx} \times x^{q'+z} (1-x)^{p'-q'-z};$$

so that the required result is

$$\frac{[p]}{[q' + z][p' - q' - z]} \frac{\int_0^1 x^{q+q'+z} (1-x)^{p+p'-q-q'-z} dx}{\int_0^1 x^q (1-x)^{p-q} dx}.$$

Write for q and q' their values λp and $\lambda p'$, and use the formulæ

$$[a] = \sqrt{2\pi} a^{a+1} \epsilon^{-a}, \quad \int_0^1 x^p (1-x)^q dx = p^{p+1} q^{q+1} (p+q)^{-p-q-3}$$

which gives, after reduction, the following result:

$$\frac{1}{\sqrt{2\pi}} \sqrt{\frac{p}{p'(p+p')(\lambda-\lambda^2)}} \left(1 + \frac{z}{\lambda(p+p')}\right)^{\lambda(p+p')+1+z} \left(1 + \frac{z}{\lambda p'}\right)^{-\lambda p' - 1 - z} \times \{\text{The same functions of } 1-\lambda \text{ and } -z\}.$$

But $(1 + Az)^{Bz+C} = \epsilon^{(Bz+C)(Az - \frac{1}{2}A^2z^2)} = \epsilon^{(AB - \frac{1}{2}A^2C)z^2} (1 + ACz)$ nearly;

and in the four factors preceding, making $c = p \div 2p'(p+p')(\lambda-\lambda^2)$,

$$\Sigma \left(AB - \frac{1}{2} A^2 C \right) = -c - \frac{1}{4} \left(\frac{1}{(p+p')^2} - \frac{1}{p'^2} \right) \left(\frac{1}{\lambda^2} + \frac{1}{(1-\lambda)^2} \right), \quad \Sigma AC = \frac{1}{2} \frac{p^2}{p'(p+p')} \cdot \frac{2\lambda-1}{\lambda-\lambda^2};$$

and rejecting the term which follows $-c$, as being of an order inferior to c , the probability in question becomes $\sqrt{c \div \pi} \epsilon^{-cz^2}$. We reject the term containing ΣAC , both on account of its smallness, and because it presently disappears in integration. Therefore $\sqrt{c \div \pi} \epsilon^{-cz^2} dz$ is the probability that the number left will lie between $\lambda p' + z$ and $\lambda p' + z + dz$, and $\sqrt{c \div \pi} \int_{-L}^{+L} \epsilon^{-cz^2} dz$, that it will lie between $\lambda p' \pm L$.

If we suppose the original number p from which λ is determined to be very considerable with respect to p' , then $c = 1 \div 2p'(\lambda - \lambda^2)$, and we may easily draw the following conclusions: 1. That a table of mortality formed from finite data may be applied to any contingencies, and that the theory of the probable errors is the same as that of the errors of observations. 2. That the specific weight of a table is the greater, the less the number of persons to whose lives it is to be applied. 3. That presuming an application of the table to a number of persons bearing a given proportion to the number in the table, the specific weight is inversely as the number of deaths in the table $\times (\lambda - \lambda^2)$. If λ be near to unity, as is usually the case, let $\xi = 1 - \lambda$, and the specific weight is then as inversely as (number of deaths) $\times \xi$, nearly.

This result seems contrary to common sense: but examination will show that it coincides with former results. For a given amount of business, the contingency being the number who shall remain alive at the end of a given term, the greater the number of previous observations made, the greater is the confidence which may be placed in the results. But if an office dealing in such contingencies, and adding but a small security fund to the results of tables, were to double its engagements on the strength of having a double quantity of observations, without increasing the per-centage of the security fund, it would not be so safe as before. The observations must be increased in a greater proportion than it is proposed to increase the engagements. But 1. Such contingencies are not common. 2. The per-centage which is added to the results of tables is at present amply sufficient. Nevertheless, an office whose gains or losses depend entirely on the number of survivors being within or without given limits at the end of a given term, might be ruined by over confidence in its tables, derived from the extensiveness of the observations.

Theory of
Pro-
babilities.

(147.) Now suppose it to have been observed that there were p status at the commencement of an observation, of which λp lasted through a first interval, μp through a second, and νp through a third. The probability that of p' of such status, $\lambda p' + z$ shall last through the first interval, has been determined. This event having happened,

Theory of
Pro-
babilities.

the probability that $\frac{\mu}{\lambda}(\lambda p' + z) + u$ shall last through the second interval has the specific weight $\lambda p \div 2p' (p + p')$ ($\mu\lambda - \mu^2$), if z be neglected as compared with p' or $\lambda p'$. Again, this second event having happened, the probability that $\frac{\nu}{\mu}(\mu p' + \frac{\mu}{\lambda}z + u) + v$ shall last to the end of the third interval has the specific weight $\mu p \div 2p' (p + p')$ ($\nu\mu - \nu^2$). Hence the probability of the three events happening (within dz , du , and dv), calling these specific weights c , c' , c'' , is

$$\sqrt{\frac{c c' c''}{\pi^3}} \varepsilon^{-cz^2 - c'u^2 - c''v^2} dz du dv, \text{ or } \sqrt{\frac{c c' c''}{\pi^3}} \varepsilon^{-cz^2 - c'(z' - \frac{\mu}{\lambda}z)^2 - c''(z'' - \frac{\nu}{\mu}z')^2} dz dz' dz'',$$

making $\lambda p' + z$, $\mu p' + z'$, and $\nu p' + z''$ the limits.

(148.) The problem in (146.) may be applied to any event whatever, of which the single probabilities (supposed equal) can be approximately found. For instance, n marriages are contracted, in which all the men are of the same age, and also all the women. If the law of mortality be that λ and λ' be the surviving fractions of the numbers of men and women at the end of a years, the probability of any given marriage still continuing at the end of that term is $\lambda\lambda'$, and $n\lambda\lambda'$ is the most probable number to be then in existence. The probability that the number lies between $n\lambda\lambda' \pm L$ is $\sqrt{c \div \pi} \int \varepsilon^{-ct^2} dt$ from $t = -L$ to $t = +L$, where c is the reciprocal of $2n(\lambda\lambda' - \lambda^2\lambda'^2)$. The result of Laplace, corrected,* is equivalent to this.

(149.) We shall now consider the single or joint duration of a number of lives. If, at a given age, the probability of a life lasting x years be ϕx , we have to alter the result of (107.) as follows. In the article cited, there were errors of either sign, both to be interpreted as positive. Consequently, the real law of facility there employed was $2\phi x dx$, which is the $\phi x dx$ of this article. Hence, for $2k'$ and $2k''$ we must write k' , k'' , and (the limits of possible drawing (86.) being infinite) we have $1 \div \sqrt{\pi} \int \varepsilon^{-t^2} dt$, from $t = -L$ to $t = +L$, for the probability of the sum of s lives lying between $k's \pm L \sqrt{2s(k'' - k'^2)}$. But $k's$ is the sum presumed from the law of the tables, whence the probability of the departure from tables being $L \sqrt{2s(k'' - k'^2)}$ is $1 \div \sqrt{\pi} \int \varepsilon^{-L^2} dL$. The specific weight is, therefore, the reciprocal of $2s(k'' - k'^2)$, and the mean risk of error is the square root of $s(k'' - k'^2) \div 2\pi$. Dividing this by s , to find the mean risk on a single life, and remembering the value of k'' (the $2k''$ of (106.)), we have the following result: s being the number alive at the commencement, and $s_1, s_2, \&c.$ those remaining at the end of successive years,

$$\text{Mean risk of duration of life} = .39894 \sqrt{\frac{\text{mean square of duration} - (\text{mean duration})^2}{s}}.$$

See (115.), of which this is a case, with the mean risk deduced instead of the probable error. It appears that correctness of observation is of more importance than large numbers. *Cæteris paribus*, one table is twice as good as another, when there are four times as many events recorded in it.

(150.) The same result applies to other cases, and by way of recapitulation we may now state the following theorem.

Let there be s status, each of which has the chance λ of outlasting a given term: it being necessary that λ should diminish without limit as the term increases without limit. Then the probability that at the end of the term the number of status remaining should lie between $s\lambda \pm L \sqrt{2s(\lambda - \lambda^2)}$ is $(1 \div \sqrt{\pi}) \int \varepsilon^{-t^2} dt$, from $t = -L$ to $t = +L$. The mean duration of the status is $\int_0^\infty \lambda x dx$, λ being a function of x , the time elapsed from the common commencement; and the probability of any one status lasting for a time lying between

$$k' \pm L \sqrt{\frac{2(k'' - k'^2)}{s}} \text{ is } \frac{1}{\sqrt{\pi}} \int_{-L}^{+L} \varepsilon^{-t^2} dt \quad \{k' = \int_0^\infty \lambda x dx, \quad k'' = \int_0^\infty \lambda x^2 dx\},$$

where k' and k'' are approximately found by taking the actual mean, and the actual mean square, of a large number of observed durations. Again, the mean risk of error in estimating a single duration is $.39894 \sqrt{(k'' - k'^2) \div s}$, and the probable error is $.67449 \sqrt{(k'' - k'^2) \div s}$.

(151.) It is to be regretted that the consequences of this general theory have never been put in so popular a form as to be accessible to the majority of those who are engaged in the construction of tables† of contingencies for practical use. In most contingency offices, all their different kinds of dealings are considered as equally likely to depart from the tables one way or the other, in providing the necessary overplus for security. This prevents

* The solution of Laplace (p. 415—418) is highly characteristic of this most powerful, but unsymmetrical, architect. In page 418, the solution of a definite problem actually appears with an indeterminate quantity (p'), altogether independent of the conditions of the problem. But a previous supposition (subsequently neglected) requires that this quantity should be very great, which reduces the result to the one which we have given.

† Not being well able to decide upon the relative importance of small details, calculators on this subject have hitherto judiciously presented their results such as they ought to be if the tables were mathematically exact, and to the nearest farthing. But more extended views on the subject of probabilities, and on the nature of observations in general, would have caused the time which has been wasted in carrying out annuities to many more decimals than the data are good for, to be employed in apportioning the risks of fluctuation, by estimation of the mean risks of the tables.

Theory of
Pro-
babilities.Theory of
Pro-
babilities.

an equitable partition of the risks of fluctuation. Suppose, for example, to simplify the question, an office which undertakes to grant annuities single and joint, upon lives aged twenty. The observations on which the office tables are founded are the history of a large number s of such lives, containing $\frac{1}{2}s(s-1)$ pairs. The mean duration, single and joint, and the mean square of duration, are ascertained sufficiently for the purpose without any

great difficulty. Calling these k'_1 and k'_2 , and k''_1 and k''_2 , we have $\sqrt{(k''_1 - k'^2_1) \div s} : \sqrt{(k''_2 - k'^2_2) \div \frac{1}{2}s(s-1)}$

for the proportion of the mean fluctuations against the office in the case of single and joint lives. Commercial prudence points out that the fluctuations should be treated as if they must necessarily be against the office. This proportion is the proportion of the per-centages to be added to the results of the tables, for security. It is desirable, for the sake of equitable distribution, that something like this proportion should be preserved; as it certainly will be when insurers of all kinds come to be aware that, as the matter stands, those concerned in simple risks pay a part of a security fund which should fall somewhat more heavily on the complicated risks.

(152.) We have hitherto considered nothing but the probabilities arising out of proposed circumstances: we now come to the appreciation of the probable gain or loss, on the supposition that every possible case which can happen is attended with a given amount of one or the other. Returning to the principle enunciated in (26.), we lay it down that the *mathematical* advantage arising from the chance p of a gain v is expressed by pv ; and also that it is only when the parties opposed (for such there must be, since the gain of one must be the loss of another) are similarly situated in regard to the value of an absolute gain or loss, that the mathematical advantage is a fair estimate of the thing required.

(153.) We may also consider questions of absolute gain or loss as questions of simple chances, in which certainty is no longer represented by 1, but by v : it being known that when v is negative, the contingent gain becomes a contingent loss, and *vice versa*. We may also see that a negative answer to a question of probabilities denotes that the solution gives a chance of an event diametrically opposed to the one presumed in the mathematical reduction of the problem.

(154.) If we multiply every possible loss or gain by its chance, and add the results with their proper signs, it follows that the sum is the value of the prospect, or what we may call the *mean advantage*. In the case of a succession of single events, each of which must be either A or B, the chances for these being q and $1-q$, let us suppose that A and B bring the gains v and μ every time they occur; consequently, in s trials, if A occur i times and B occur $s-i$ times, the resulting gain is $iv + (s-i)\mu$, or $s\mu + i(v-\mu)$. Multiply this by the probability of the event, or by $\{[s] \div [i] \cdot [s-i]\} q^i (1-q)^{s-i}$, which taken for every value of i from $i=0$ to $i=s$, and each case multiplied by its resulting benefit, gives the same result as the following expression, if t be made $=1$ after differentiation,

$$s\mu (qt+1-q)' + (v-\mu) \frac{d}{dt} (qt+1-q)'; \text{ which is } s\mu + sq(v-\mu),$$

or $q \cdot sv + (1-q)s\mu$ is the mean advantage required.

In the case of mathematical equality, μ and v must be of opposite names, and in the proportion of q to $1-q$. Again, from (74.), the probability of A occurring a number of times, which lies between $sq \pm l$ (extremes inclusive) is

$$\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt + \frac{t e^{-t^2}}{l\sqrt{\pi}}, \text{ where } t = \frac{l}{\sqrt{2sq(1-q)}}.$$

But this is the probability of the actual benefit derived lying between $(sq \pm l)v + (s-sq \mp l)\mu$, or $M \pm l(v-\mu)$, where M is the mean advantage.

(155.) The most usual case is where v and μ are of opposite names; and we may suppose μ to be a loss, and therefore negative. We are now led to the appreciation of the advantage arising from the *division of risks*. Let us suppose that, instead of making s ventures, each of which is either a gain of v , or a loss of $-\mu$, a merchant makes ks ventures, each of which is either a gain of $v \div k$, or a loss of $-\mu \div k$. The mean advantage remains unaltered, but the probability of a given departure from it is altered in his favour, as follows. Let l' be the value of l , which gives the same value of t as before: then $l' = l\sqrt{k}$ and $l(v-\mu)$, and $l\sqrt{k}(v-\mu) \div k$, or $l(v-\mu) \div \sqrt{k}$ are the limits of departure for which there are the same chances. Consequently whatever may be the risk of ruinous fluctuation in embarking a certain quantity of capital in each of a certain number of similar speculations, it is halved by employing one-fourth as much in each of four times as many speculations, and so on.

(156.) Mercantile men are perfectly aware of the general truth of this principle, but they very often imagine that they *halve* the risk by *doubling* the number of ventures, which is not true. And the rest of the world, in judging of the various commercial schemes which arise, regards them as a symptom of a propensity to gamble. But this is not the case: a monied man who takes the run of the market, and makes large numbers of different investments, has only to consider the general features of the time, and to ascertain whether the prevailing disposition is on the whole regulated by prudence and calculation. If he see this attested by the decided success of a large majority of undertakings, it is, though not unimportant, yet of minor importance, whether he judge wisely or not in one or two instances. He can, with no unusual degree either of prudence or knowledge, make his success highly probable.

(157.) We shall now suppose that any number of different events may happen at every trial, of which the probabilities are a_1, b_1, c_1 , &c. at the first, a_2, b_2, c_2 , &c. at the second, and so on; so that we have generally $a + b + c + \dots = 1$. Again, suppose that at the first trial the benefits contingent on the different ways of

Theory of Probabilities. happening are λ_1, μ_1, ν_1 , &c., at the second λ_2, μ_2, ν_2 , &c. If, then, we suppose s trials, and multiply together the Theory of Probabilities. s expressions of the form $ax^\lambda + bx^\mu + cx^\nu + \dots$, we shall have a series of terms of the form Ax^K , where K is one of the possible results as to benefit, and A the probability of obtaining it. Let M be the mean advantage of s trials; whence, as in (102.), we find the chance of the advantage of s trials being $M+l$ to be the term independent of x in the development of

$$x^{-(M+l)} (a_1 x^{\lambda_1} + b_1 x^{\mu_1} + \dots) (a_2 x^{\lambda_2} + b_2 x^{\mu_2} + \dots) (\dots) (a_s x^{\lambda_s} + \dots),$$

or, assuming $x = \varepsilon^{\sqrt{-1}}$, we must integrate the preceding from $\omega = -\pi$ to $\omega = +\pi$, and divide by 2π . But since $a+b+\dots=1$, the several factors preceding are all of the form $1 + (a\lambda + b\mu + \dots) \omega \sqrt{-1} - \frac{1}{2} (a\lambda^2 + b\mu^2 + \dots) \omega^2 - \dots$, and their logarithms are of the form $(a\lambda + b\mu + \dots) \omega \sqrt{-1} + \frac{1}{2} \{ (a\lambda + b\mu + \dots)^2 - (a\lambda^2 + b\mu^2 + \dots) \} \omega^2 + \dots$. Consequently, passing from logarithms to their exponentials, and remembering that M must be $\Sigma (a\lambda + b\mu + \dots)$, where Σa means $a_1 + a_2 + \dots$; and making

$$2sk^2 = \Sigma \{ (a\lambda^2 + b\mu^2 + \dots) - (a\lambda + b\mu + \dots)^2 \},$$

we have for the probability in question

$$\frac{1}{2\pi} \int_{-\pi}^{+\pi} \varepsilon^{-l\omega\sqrt{-1} - sk^2\omega^2 - \dots} d\omega,$$

the integration of which, when s is a considerable number, will not be materially affected by our making the limits infinite; thus in $\int \varepsilon^{-sx^2} dx$, from $x = -h$ to $x = +h$ depends upon $\int \varepsilon^{-x^2} dx$, from $x = -h\sqrt{s}$ to $x = +h\sqrt{s}$. And we may neglect the terms which contain ω^3 , &c. on principles explained in (89.) Consequently (78.), the probability of the total resulting gain being $M+l$ is

$$\frac{1}{2\sqrt{\pi sk^2}} \varepsilon^{-\frac{l^2}{4sk^2}}: \text{ which gives } \frac{1}{\sqrt{4\pi sk^2}} \int \varepsilon^{-\frac{t^2}{4sk^2}} dt, \text{ from } t = -l \text{ to } t = +l,$$

for the probability of the resulting gain lying between $M \pm l$. But this last step is on the supposition that the number of partial contingencies composing each event is so large, and the gain of each so small, that the definite integration will be nearly equivalent to the summation which suggests it: or else that the number s is so large that any gain lying between $M \pm l$ may be obtained either exactly or nearly, in many different ways. The result is that the probability of the resulting gain lying between

$$\Sigma (a\lambda + b\mu + \dots) \pm 2k\sqrt{s}.l \text{ is } \frac{1}{\sqrt{\pi}} \int_{-l}^{+l} \varepsilon^{-t^2} dt$$

$$k^2 = \frac{1}{2s} \Sigma \{ (a\lambda^2 + b\mu^2 + \dots) - (a\lambda + b\mu + \dots)^2 \}.$$

In (154.) we see a particular case of this, with the first rejected term, owing to the process in (74.) having been carried further than is necessary for sufficient approximation. This term, as in the example of (74.), is always a small part of the whole, when s is considerable.

(158.) Where there are only two contingencies in each trial, we find that the probability of the result lying between

$$\Sigma \{ a(\lambda - \mu) + \mu \} \pm l \sqrt{2a(1-a)(\lambda - \mu)^2 s} \text{ is } \frac{1}{\sqrt{\pi}} \int_{-l}^{+l} \varepsilon^{-t^2} dt,$$

and the substitution of the factor s for the symbol Σ converts the general case into that in which all the trials are repetitions of the same.

(159.) Suppose s lives of the same age to be insured for £Q each, of which the law of mortality is as follows: of K lives of the given age, K_1 remain at the end of a year and K_n at the end of n years: now P being the premium paid by each, what is the probability that the present value of the ultimate gain or loss of the insurance office will lie between given limits?

It is impossible here to consider the life of each individual, and its total result, as one event with respect to the office, for it is made up of all the chances connected with the different years. Considering an individual A , and the period of time between n and $n+1$ years elapsed from the moment of completing the insurances, we have the three following cases: 1. A may overlive the period, in which case the office gains P at the beginning of the year: or, looking at every loss and gain with reference to its value at the time of making the bargain, it gains Pv^n , where v is the present value of £1, to be paid a year hence. The chance of this event is $K_{n+1} \div K$, which call p_{n+1} . 2. A may die during the year; in which case the office gains P at the beginning, and loses Q at the end, of the period; that is, loses $Qv^{n+1} - Pv^n$. The chance of the event is $(K_n - K_{n+1}) \div K$, or $p_n - p_{n+1}$. 3. A may have died before the period begins; in which case the office neither gains nor loses, or this case may be omitted. Hence the portion of the mean result contributed by every individual is the sum of all his terms of the form

$$p_{n+1} Pv^n - (p_n - p_{n+1}) (Qv^{n+1} - Pv^n), \text{ or } Pp_n v^n - (p_n - p_{n+1}) Qv^{n+1}.$$

This is exclusive of $P+E$ certainly paid down at the beginning, where E is a fee demanded at entrance. And P

Theory of Probabilities. itself is not the gross premium actually paid, but the remainder after a certain proportion is deducted for expenses of management. The mean result, therefore, from each individual is ($p_0=1$), Theory of Probabilities.

$$P + E + Pp_1v + Pp_2v^2 + \dots - (1-p_1)Qv - (p_1-p_2)Qv^2 \dots$$

The tabular value of an annuity of £1 on the life of each party is $p_1v + p_2v^2 + \dots$. Call this A : then the preceding becomes $E + P(1+A) - Qv(1+A) + QA$. The office premium is generally calculated as in (36.), with the addition of a per-centage. Let f be the fraction of unity in this per-centage, and we have $P = \{Qv(1+A) - QA\} (1+f) \div (1+A)$; where in f may be included the diminution allowed for expenses of management. This reduces the preceding expression to $E + fQ(v + vA - A)$, and s times this latter sum is the mean result.

The component of k^2 , in (157.), which arises from the period between the completion of n and $n+1$ years, is

$$p_{n+1}P^2v^{2n} + (p_n - p_{n+1})(Qv^{n+1} - Pv^n)^2 - \{Pp_nv^n - (p_n - p_{n+1})Qv^{n+1}\}^2 \\ = v^{2n} \{ (Qv - P)^2 p_n - Qv(Qv - 2P)p_{n+1} + 2Qv(Qv - P)p_n p_{n+1} - (Qv - P)^2 p_n^2 - Q^2 v^2 p_{n+1}^2 \}.$$

Summing this expression with respect to an individual insurer, an easy method presents itself of providing what we will call *fluctuation tables*, in addition to the common tables. A new rate of interest must be taken, such that v^2 , and not v , may be the present value of one pound receivable a year hence (for 4 per cent. 8 per cent. will do.) With this rate must be calculated tables of annuities on single lives, on two joint lives of equal ages, and on two joint lives in which the older is a year in advance of the younger. Let $\bar{A}$, $\bar{\bar{A}}$, $\bar{AA}$, and $\bar{\bar{AA}}$ represent the values of these annuities for £1 per annum, increased by one year's purchase, (see Note to (34.)), where the single bar indicates the age of a party being that of the problem, and the double bar a year older. Hence we have, S denoting summation with respect to successive years of the same individual (from $n=0$ upwards),

$$Sp_nv^{2n} = \bar{A}, \quad Sp_{n+1}v^{2n} = p_1\bar{\bar{A}}, \quad Sp_np_{n+1}v^{2n} = p_1\bar{AA}, \quad Sp_n^2v^{2n} = \bar{AA}, \quad Sp_{n+1}^2v^{2n} = p_1^2\bar{\bar{AA}}.$$

Let the following expression be denoted by K^2 ,

$$(Qv - P)^2 \bar{A} + 2Qv(Qv - P)p_1\bar{\bar{AA}} - Qv(Qv - 2P)p_1\bar{\bar{A}} - (Qv - P)^2 \bar{AA} - (Qv)^2 p_1^2 \bar{\bar{AA}};$$

then sK^2 is the factor Σ (&c.) in the value of k^2 in (157.), and therefore $k^2 = \frac{1}{2}K^2$. Hence the probability that the ultimate gain of the office from s individuals shall lie between

$$sE + sfQ(v + vA - A) \pm Kl\sqrt{2s} \text{ is } \frac{1}{\sqrt{\pi}} \int_{-l}^{+l} e^{-t^2} dt,$$

and the same that the average profit from each individual lies between $E + fQ(v + vA - A) \pm Kl\sqrt{2/s}$.

(160.) Let us now see how the case would be altered if each individual paid down the whole present value of his insurance, instead of yearly premiums. This case may be deduced from the former by supposing $P=0$, and E to be the present value paid down + its former value.

Beginning the problem again on the supposition that the present value demanded is $Q(v + vA - A)(1+f)$, we find the mean advantage of the office the same as before; while the value of K^2 , which call K_1^2 , is now deduced from K^2 by making $P=0$, which gives

$$K^2 - K_1^2 = P^2(\bar{A} - \bar{AA}) + 2PQvp_1(\bar{\bar{A}} - \bar{AA}) - 2PQv(\bar{A} - \bar{AA}).$$

Now the difference of $\bar{A} - \bar{AA}$ and $\bar{\bar{A}} - \bar{AA}$ (necessarily positive quantities) will be trifling; and the sign of $P - 2Qv(1-p_1)$ will decide that of the preceding. And l and s remaining the same, the limits of fluctuation for which there is a given probability increase with K . When the preceding is *positive*, premiums are most favourable to fluctuation: when *negative*, total payments. And *cæteris paribus*, the larger p_1 is, that is, the better the class of lives, the more will the former method favour fluctuation, as compared with the latter. But these conclusions must not be applied to very old lives, without examination of $\bar{A}$, &c. This question is immaterial at present, owing to the rarity of the occurrence of an insurance being completely paid for at the moment of completing the bargain.

The same considerations may be applied to the question of a number of annuities purchased, or any other of the kind, and generally the facilitation of the question of probable fluctuations will require the calculation of ordinary life-tables at different rates of interest, connected with the one on which the primitive tables are constructed. At present, however, owing to the tables in use being always indicative of a much lower (or in the case of reversions, higher) rate of mortality than is expected, and the rarity of knowledge upon the higher parts of this theory, it is useless to expect any adoption of results founded upon the calculation of fluctuations.

(161.) From the preceding it appears that the *mathematical* advantage is greatly in favour of the insurance offices, and most certainly it ought to be so. It is by giving the office mathematical certainty of a large profit that the individual insurers buy the moral security which they have of receiving the stipulated benefits. But while we take this opportunity of declaring that the directors of any contingency office would be most justly open to censure if they attempted to hit the line of mathematical security exactly, or very nearly: we must also say, that there is no need to help the proprietors of a nominal capital to any part of the large profits which are accumulated to deaden the effects of fluctuation. Let the insurers share this among themselves; that is, let them be mutual insurers, fearless of fluctuation, because they have established a sufficient fund to meet it; and withdrawing the surplus of this fund, as it accumulates, for their own use and benefit.

(162.) Instead of the probabilities being absolutely given, let p events have been first observed, of which

Theory of
Pro-
babilities.

θp have happened, so as each to give a gain ν , and the rest so as each to give a gain μ , these being the only possible cases. Then, by (146.), the probability that of p' cases yet to come the number which gives ν will lie between

Theory of
Pro-
babilities.

$$\theta p' \pm L \sqrt{\frac{2p'(p+p')(\theta-\theta^2)}{p}} \text{ is } \frac{1}{\sqrt{\pi}} \int_{-L}^{+L} e^{-t^2} dt,$$

which is also the chance of the total result, as to benefit, lying between

$$\{\theta\nu + (1-\theta)\mu\} \pm L(\nu-\mu) \sqrt{\frac{2p'(p+p')(\theta-\theta^2)}{p}}.$$

On comparing this with the similar case in (154.), and supposing θ absolutely given as the probability of the gain ν in each instance, we see that the limits of fluctuation for which there is the probability just found, are $\pm L(\nu-\mu)\sqrt{2p'\theta(1-\theta)}$, or less than the preceding in the ratio of $\sqrt{p}$ to $\sqrt{p+p'}$. That is, the assumption of a probability from observation of p past events, made an hypothesis for calculating the chances of p' events yet to come, must be considered as likely to produce larger fluctuations than happen in cases where the same probability is absolutely known, in the proportion of $\sqrt{p+p'}$ to $\sqrt{p}$.

(163.) We now come to the question of gain or loss, relatively and not absolutely considered, as in (42.). The value of money must be considered absolutely in all transactions between man and man; that is, the terms of every bargain must express definite sums. But in the previous consideration, which every one must give, as to the policy of a proposed bargain, it is notoriously false that any practical view of the subject considers money absolutely. The supposition, therefore, of the mere mathematical gain or loss must be abandoned, whatever may be thought as to the propriety of one or another substitute. The theorem in (42.) gives results decidedly nearer the truth than any results of the view just abandoned: but a little consideration will show that it must be modified before it will yield the best representation which we can give of relative value. That a man with five hundred pounds will think *less* of five shillings than another of the same temperament who has only one hundred pounds, is readily admitted: but that the first will look upon five shillings just as the second does upon one, may be doubted. But we must observe, that no person looks upon an outlay of money with reference to the whole of his own property, but considers the purpose for which the expense is to be incurred, and the proportion which it bears to the greatest possible sum he can devote to that or similar objects. An amateur of painting, of very large fortune, will give a thousand pounds for one work of a great master; while another person, of the same fortune, intending to lay out one thousand pounds in pictures, will not buy the single picture, but will prefer twenty at a cheaper price. In the two cases, the value of a thousand pounds is different; in the first it is only a small part of the fund out of which it is to come; in the second it is the whole. Suppose it to be admitted that the relative values of small and equal additions to the similar funds of two persons are inversely as the amounts of the funds, which is not only the most natural supposition, but one which seems not more liable to objection as giving too much, than as giving too little. In that case, the fund being x , the value of dx is of the form $kdx \div x$, and the value of the whole fund is $k \log x + h$, where h is a constant to be determined. But it must be observed, that the whole process fails when x becomes very small, and it is besides easily shown that in such case the natural and obvious character of the supposition entirely disappears.

Imagine a small fund in the possession of an individual, out of which an expense of serious magnitude is proposed. If this fund be the sole means of the individual, the value of its several parts will be large. But if the purpose in question be the support of life, we must place in the fund a money representation of all that the individual can do to maintain himself when his money is gone. If that small fund were really the whole support of the individual, so that he must undoubtedly perish after its extinction, there would be no objection to the formula $k \log x + h$, taken from $x=0$, considered as representing the whole value of the fund. But no such case arises, so that in fact when we speak of $kdx \div x$ as representing the relative value of dx , we are considering dx as an accession to a fund already existing, and of such magnitude as not to present an extreme case. Under these circumstances, the above valuation of dx is of a highly probable character, and $k \log x + h$ is the necessary mathematical consequence of it, as the representative of the relative value of x . Laplace calls x the *fortune physique*, and $k \log x + h$ the *fortune morale*, for which we shall substitute the terms absolute and relative values.

(164.) The preceding method of estimation will require us to reconsider the method of estimating probable losses and gains. When we speak only of absolute values, the formula $(a+b)-b=a$, and the fundamental axiom from which it springs, cause many simplifications of process. For instance, suppose a man who has a , has also the chance p of gaining b . We estimate his absolute advantage at $a+pb$. But we may also proceed as follows. An event to come will leave him either with a or $a+b$, the chances of which are $1-p$ and p . Hence his expectation is $(1-p)a + p(a+b)$, which is $a+pb$. But suppose that the values of a and $a+b$ are represented by ϕa and $\phi(a+b)$; then, except only where $\phi a = \text{const.} \times a$, the value of a sum is no longer the sum of the values. We have then a lottery in which a proportion $1-p$ of the tickets is marked ϕa , and the remaining portion p is marked $\phi(a+b)$. If a very large number of persons draw, the average drawing will have a very high probability of being excessively near to $(1-p)\phi a + p\phi(a+b)$. In this form we must state the preceding sentence, for otherwise it would appear as if we were adding values to values to produce the value of the sum. We must then solve the equation $(1-p)\phi a + p\phi(a+b) = \phi x$, and x is the absolute sum corresponding to the present state of the party's expectation. Thus suppose the formula which expresses the value of x to be $k \log x + h$, and that a person who has now z must gain either $\alpha, \beta, \gamma, \dots$, of which the chances are $p, q, r, \dots$. The solution of

$$pk \log(z+\alpha) + ph + qk \log(z+\beta) + qh + \dots = k \log x + h$$

Theory of
Pro-
babilities.

Theory of
Pro-
babilities

gives $x = (z + \alpha)^p (z + \beta)^q (z + \gamma)^r \dots$ since $p + q + r + \dots = 1$,
and $(z + \alpha)^p (z + \beta)^q (z + \gamma)^r \dots - z$ is the benefit which the chances give.

(165.) If $\alpha, \beta, \gamma \dots$ be very small, the preceding is very nearly equal to $p\alpha + q\beta + r\gamma + \dots$, the expression for the mathematical expectation. That is to say, the absolute and relative methods may be considered as coincident for very small additions to the absolute sum proposed.

(166.) All risks, if only mathematically balanced, are disadvantageous. Suppose A, whose fund is a , stakes the sum μ , with the chance p of winning. He ought therefore to win $(1-p)\mu \div p$, and the value of his expectation is therefore

$$\left(a + \frac{1-p}{p}\mu\right)^p (a - \mu)^{1-p} - a, \text{ or } a \left\{ \left(1 + \frac{1-p}{p} \frac{\mu}{a}\right)^p \left(1 - \frac{\mu}{a}\right)^{1-p} - 1 \right\}.$$

The product in the second factor has the logarithm

$$p \log \left(1 + \frac{1-p}{p} \frac{\mu}{a}\right) + (1-p) \log \left(1 - \frac{\mu}{a}\right), \text{ or } (1-p) \int_0^{\mu} \left\{ \frac{p}{ap + (1-p)\mu} - \frac{p}{ap - p\mu} \right\} d\mu,$$

in which the coefficient of $d\mu$ is evidently negative for all positive values of μ . The logarithm is therefore negative, or the function less than unity: whence the expectation itself is negative.

To compensate the disadvantage, suppose t to be the sum against which μ is stated. Then

$$(a+t)^p (a-\mu)^{1-p} = a, \text{ or } t = \left\{ a^{\frac{1}{p}} \div (a-\mu)^{\frac{1-p}{p}} \right\} - a.$$

In the case where $p = \frac{1}{2}$ this is $\mu a \div (a - \mu)$. Thus a person should not stake half his fortune upon an even chance, unless the contingent gain is to be double of the stake.

(167.) In using the formula $k \log x + h$, to signify the moral value of a sum x , it is obvious that, though the initial supposition in (163.) may seem reasonable, the result deduced from it is of a strange and unusual appearance. But at the same time it may be shown that the results of common notions upon subjects which are unconnected with any principle of estimation of relative values, present an accordance with the consequences of the preceding method, which, even on the supposition of a mere accidental coincidence, shows that the formula itself, which exhibits such an agreement with conclusions drawn from other sources, must not be slightly thought of on account of its unusual form. Let us take the case of an insurance upon an event which must happen at some time or other, and may happen at any time, such as a common life assurance. In looking at the grounds of such a contract, we see that it is not gain which is contemplated, for the insured party is as likely to have over paid the value of his insurance as to have fallen short of it. Indeed, it is obvious that, if the insuring parties did not receive as much of surplus from those who live beyond the calculated term as they have to pay in the shape of loss to those who do not come up to the term, they could not fulfil their contracts. It is not, therefore, gain, but freedom from the consequences of uncertainty, which is the benefit contemplated by those who insure. It would have precisely the same effect if the uncertainty were removed by a power which could insure, not money in case of death, but actual life. We have before mentioned the tendency which exists to consider the average as the certainty which is to be assumed, when the effects of fluctuation or discordance are to be removed by assumption: and, in the present case, we are sure of being joined by a very large majority of those who would express any opinion on the subject, when we say that the practical effect of an insurance office must be, on the whole, to secure to each of its customers *his average life*: so that the moral effect of it would not be altered if each were made to live as long as, one with another, each does live. Without either adopting or rejecting this notion, let us remember that just as in (89.) the adoption of the method of averaging was found to be necessarily connected with the assumption of one particular law of facility of error, so in the present instance may the apparently fundamental proposition just stated be equally connected with some particular method of estimating the value of the contingent benefits of insurance. Now supposing a sum x to be paid at the end of n years from the present time, and v^n to be the present value of £1 due at that time, let $\phi(xv^n)$ be that function of the present value of x by which its value is to be estimated, as a possible payment of a contingent insurance. Upon the assumptions in (35.), the present value of an insurance of £1 is

$$(1 - P_1) \phi v + (P_1 - P_2) \phi v^2 + (P_2 - P_3) \phi v^3 + \dots \dots \dots (A).$$

Again, supposing s status to exist at first, the number left at the end of 1, 2, 3, &c. years (or other terms) is $sP_1, sP_2, sP_3, \&c.$, whence, provided that the duration of the different status through fractions of years may be neglected,* shows us that $s(1 - P_1)$ enjoy a complete year, $s(P_1 - P_2)$ enjoy two complete years, and so on: whence the average duration of each status is

$$(1 - P_1) + 2(P_1 - P_2) + 3(P_2 - P_3) + \dots \dots \dots = 1 + \Sigma P.$$

Insuring this duration, the present value of £1 at the end of the term, estimated according to the same formula, is $\phi(v^{1+\Sigma P})$. On equating this to the expression (A), the resulting functional equation is found to be satisfied by $\phi v = k \log v + h$. For in that case (A) becomes

$$(1 - P_1 + 2P_1 - 2P_2 + \dots) k \log v + (1 - P_1 + P_1 - P_2 + \dots) h,$$

or $(1 + \Sigma P) k \log v + h$. And this is the only solution which is equally true of any number of terms. On looking over the preceding, however, we remark that we have made the terms $v, v^2, v^3, \&c.$ enter the formula independently of any property which the insured possesses beyond his interest in the insurance. This is equivalent to supposing

* The common correction for this supposition amounts to adding half a year to the average duration.

Theory of
Pro-
babilities.

that his remaining interests, if any, are inconsiderable when compared with the insurance itself. Hence we see that, in the case of a person who depends entirely upon an insurance, the assumption that he is placed by the insurance office in precisely the position in which he would be if he were certain of living his average term, is not correct, unless accompanied by the supposition that the present moral value of every interest x is of the form $k \log x + h$, x being the mathematical present value.

Theory of
Pro-
babilities.

(168.) The principle of a common insurance is, as already observed, a moral security given to the insured, to whom his own particular case is every thing, in consideration of a mathematical advantage given to the insurers, who run no risk, seeing that they have the average of a large number of similar transactions. The amount which an insurer can afford to pay for the security, over and above its mathematical value, may be determined in a simple case, as follows. Let a man's property in hand, or otherwise certain, be denoted by 1, and let θ be the expected return of an existing venture, the chance of which turning out favourably is p . On the supposition that he does not insure, the relative value of his present expectation is $kp \log (1+\theta) + h$; but on the supposition that he insures, paying $(1-p)\theta + \alpha$, or the mathematical risk increased by an office profit α , he is then made certain of $1+p\theta - \alpha$, the relative value of which is $k \log (1+p\theta - \alpha) + h$. The equation of these two gives

$$\alpha = 1 + p\theta - (1+\theta)^p,$$

which is always a positive quantity, when θ is positive and p less than unity. For

$$p > p(1+\theta)^{p-1}, \text{ and } \int_0^\theta p d\theta > \int_0^\theta p(1+\theta)^{p-1} d\theta, \text{ or } p\theta > (1+\theta)^p - 1.$$

This value of α is the highest which can be prudently paid upon the hypothesis as to value which we are now considering. Let us suppose $\theta = \frac{1}{20}$, $p = \frac{49}{50}$, and we have $\alpha = .000025$, $(1-p)\theta$ being .001. That is, two and a half per cent on the mathematical value of the insurance is the greatest profit which should be allowed to the office.

In the case in which the risk p is not known, but it is equally likely to be of any values between, say p'' and p' , then the value of θ , resulting from the probability of success, is $\int_{p'}^{p''} \frac{dp}{p'' - p'} p\theta$, or $\frac{1}{2}(p' + p'')\theta$, so that the average risk may be taken.

(169.) Having thus made it appear that there is a method of estimating relative values more in accordance with the first notions on the subject than the assumption of absolute values, and which, in several cases, represents common opinion in its results, it becomes of interest to inquire how far this new hypothesis may be dispensed with in considering a multitude of transactions together. The truth is, as we shall proceed to show, that the division of risks not only diminishes the chances of fluctuation, as in (155.), but also tends to bring the relative value of a prospect near to its mathematical value: so that, though isolated transactions may frequently be looked upon with a degree of caution arising from the preceding considerations, yet there is no necessity to recommend the adoption of any new rule in calculations which involve multifarious dealings of a nearly similar kind.

Let us suppose the fraction θ in the last article to be invested in r similar risks, each risk having the probability p of turning out favourably. The chance of z being successful, and the rest otherwise, is therefore

$$\frac{[r, r-z+1]}{[z]} p^z (1-p)^{r-z}, \text{ and the result is } 1 + \frac{z}{r} \theta,$$

since $(z \div r)\theta$ is added by this event to 1 already possessed. Multiplying the relative value of each result by its probability, we find the following expression for the relative value of the prospect, the summation extending from $z=r$ to $z=0$, both inclusive,

$$\sum \left\{ \frac{[r, r-z+1]}{[z]} p^z (1-p)^{r-z} \left(k \log \left\{ 1 + \frac{z}{r} \theta \right\} + h \right) \right\};$$

which, as the logarithm independent of z , or made from $z=0$, vanishes, and the coefficient of h is $(p+1-p)^r$, becomes

$$kp \sum \int_0^\theta \frac{[r, r-z+1]}{[z]} \cdot \frac{p^{z-1}(1-p)^{r-z} z d\theta}{r+z\theta} + h \dots \dots (1);$$

but when the whole fraction θ is in one risk, the preceding integral becomes

$$kp \int_0^\theta \frac{d\theta}{1+\theta}, \text{ or } kp \sum \int_0^\theta \frac{[r-1, r-z+1]}{[z-1]} \frac{p^{z-1}(1-p)^{r-z} d\theta}{1+\theta},$$

since $(p+1-p)^{r-1} = 1$. Subtracting this from (1.), we have

$$kp \sum \int_0^\theta \frac{[r-1, r-z+1]}{[z-1]} \frac{p^{z-1}(1-p)^{r-z} (r-z) \theta d\theta}{(1+\theta)(r+z\theta)},$$

a quantity which is necessarily positive, since all its terms are positive. And this is the excess of the relative value of the divided risks over that of the single risk.

The expression (1) may be transposed into

$$kp \sum \int_0^\theta \frac{[r-1, r-z+1]}{[z-1]} \frac{p^{z-1}(1-p)^{r-z} d\theta}{1 + \frac{\theta}{r} + \frac{z-1}{r} \theta} + h,$$

Theory of
Pro-
babilities.which, as appears by the equation $\int_0^\infty e^{-cx} dx = \frac{1}{c}$, is

$$kp \int_0^\infty \int_0^\infty e^{-(1+\frac{\theta}{r})x} \{p e^{-\frac{\theta}{r}x} + 1 - p\}^{r-1} dx d\theta + h.$$

Let us first consider this expression with respect to x , that is, as $\int y dx$. There is no maximum or minimum of y between $x=0$ and $x=\infty$, for we have

$$\frac{dy}{dx} = -e^{-(1+\frac{\theta}{r})x} \left\{ p e^{-\frac{\theta}{r}x} + 1 - p \right\}^{r-2} \left\{ p(1+\theta) e^{-\frac{\theta}{r}x} + (1-p) \left(1 + \frac{\theta}{r} \right) \right\},$$

which is always negative between the limits specified. Proceeding as in (60.) we find

$$v = - \frac{p e^{-\frac{\theta}{r}x} + 1 - p}{p(1+\theta) e^{-\frac{\theta}{r}x} + (1-p) \left(1 + \frac{\theta}{r} \right)} \quad \frac{dv}{dx} = - \frac{p(1-p)(r-1)\theta^2 e^{-\frac{\theta}{r}x}}{r^2 \left\{ p(1+\theta) e^{-\frac{\theta}{r}x} + (1-p) \left(1 + \frac{\theta}{r} \right) \right\}^2}$$

Now y vanishes when $x=\infty$, and is $=1$ when $x=0$. In the latter case

$$v = - \frac{1}{1+p\theta + (1-p)\frac{\theta}{r}} \quad \frac{dv}{dx} = - \frac{p(1-p)(r-1)\theta^2}{r^2 \left\{ 1+p\theta + (1-p)\frac{\theta}{r} \right\}^2}$$

$$\int_0^\infty y dx = \frac{1}{1+p\theta + (1-p)\frac{\theta}{r}} \left\{ 1 + \frac{p(1-p)(r-1)\theta^2}{r^2 \left\{ 1+p\theta + (1-p)\frac{\theta}{r} \right\}^2} + \&c. \right\},$$

which is now to be integrated with respect to θ . But the expression, rejecting the higher powers of $\frac{1}{r}$, becomes

$$\frac{1}{1+p\theta} - \frac{1-p}{r} \frac{\theta}{(1+p\theta)^2} : \text{ and } kp \int_0^\infty \int_0^\infty y dx d\theta = k \log(1+p\theta) - kp \frac{(1-p)}{r} \int_0^\infty \frac{\theta d\theta}{(1+p\theta)^3}.$$

Adding h to both sides, it now appears that, when r is considerable, the absolute fortune corresponding to this result is nearly $1+p\theta$, that is, the sum already possessed together with the mathematical advantage of the prospect. At the same time the relative prospect is always a little less than the absolute one.

(170.) We shall now resume the problem in (40.), in which, as there shown, the mathematical advantage of the player is infinite. This will be a test of the practical value of the hypothesis now under consideration; since it is clear that no theory which professes to produce results in accordance with common notions, in cases where they are worth considering, ought to indicate that any one could prudently stake a large proportion of his fortune at such a game. Let a be the fortune at the outset, and x what the possessor should give for the chance of n throws; that is, his sole chance of gain is on the first time head is thrown, and if this happen at the v th throw, and v be less than or equal to n , he is to receive 2^v pounds sterling.

The relative value of his fortune before the stake is $k \log a + h$, after it $k \log(a-x) + h$, which there is the chance 2^{-n} that it will remain. At the same time there is the chance 2^{-v} that the absolute fortune will become $a-x+2^v$, and this for every value of v from $v=1$ to $v=n$. Hence the relative value of the prospect is

$$k \sum 2^{-v} \log(a-x+2^v) + h \sum 2^{-v} + 2^{-n} k \log(a-x) + 2^{-n} h,$$

or

$$k \sum 2^{-v} \log(a-x+2^v) + k 2^{-n} \log(a-x) + h,$$

or

$$\log \left\{ (a-x) \left(1 + \frac{2}{a-x} \right)^{\frac{1}{2}} \left(1 + \frac{4}{a-x} \right)^{\frac{1}{4}} \dots \left(1 + \frac{2^n}{a-x} \right)^{\frac{1}{2^n}} \right\} + h,$$

which, equated to $k \log a + h$, gives

$$a = (a-x) \left(1 + \frac{2}{a-x} \right)^{\frac{1}{2}} \left(1 + \frac{4}{a-x} \right)^{\frac{1}{4}} \dots \left(1 + \frac{2^n}{a-x} \right)^{\frac{1}{2^n}}.$$

These factors approximate without limit to unity: for the limit of

$$\frac{1}{2^n} \log \left(1 + \frac{2^n}{a-x} \right), \text{ or } \frac{n \log 2}{2^n} + \frac{1}{2^n} \log \left(\frac{1}{2^n} + \frac{1}{a-x} \right) \text{ is } = 0.$$

If these logarithms be summed, from $n=1$, until $2^n \div (a-x)$ is not small (say >10), the sum of the remainder of the logarithms *ad infinitum* from n inclusive, will be nearly

$$\sum \left(\frac{1}{2^n} \log \left(\frac{1}{a-x} \right) + \frac{a-x}{2^{2n}} + \frac{n \log 2}{2^n} \right), \text{ or } \frac{1}{2^{n-1}} \log \left(\frac{1}{a-x} \right) + \frac{a-x}{2^{2n-1}} + \frac{(n+1) \log 2}{2^{n-1}},$$

in which the tabular logarithms may be used, if the second term be multiplied by $\cdot 4342945$. We thus have, in the case where the mathematical advantage is infinite, the fortune which a person should have before the stake, in terms of his fortune after the stake, supposed given; that is, the proportion of his whole fortune which, on preceding principles, he may risk at this game. For $a-x=100$ it gives $a=108$, and for $a-x=200$ it gives $a=209$; that is, out of £108, not more than £8 should be risked, nor more than £9 out of £209.

(171.) All the problems which have hitherto been proposed are those in which an answer either can be given, or may be readily conceived to be given, to both of the questions proposed in (6.). We now proceed to a

Theory of
Pro-
babilities.

Theory of
Pro-
babilities.

class of investigations in which we are only concerned with the method of using the measures of probability, without being able to point out any practical mode of obtaining them. We refer to questions connected with evidence and matters of moral probability, in the usual sense of the words. But we shall begin by pointing out a mode in which the theory may be applied to a sort of intermediate question, which at first sight will appear utterly insoluble, but will nevertheless be shown to be perfectly within the theory (and, were it worth while, to a great extent within the practice) of the methods previously laid down.

Theory of
Pro-
babilities.

(172.) PROBLEM. A floor is ruled with equidistant parallel lines, and a thin rod of less length than the distance between two parallels is thrown down at hazard, and which therefore may or may not fall on one of the parallels, but cannot cut two of them: required the probability of its falling on one of the parallels.

Let a be the distance between the parallels, and $2r$ the length of the rod ($2r < a$). The problem will not be altered if we suppose that the centre of the rod must fall on one given perpendicular to the parallels, and also between two given parallels (inclusive): for in whatever proportion the number of favourable cases is increased by removing this restriction, the whole number of possible cases is increased in the same proportion. On the principle applied in (44.), let the distance a be divided into n equal parts, and a whole revolution into m equal parts, reckoning from the direction of one of the parallels. Let us suppose, for the present, that the centre of the rod must fall on one of the subdivisions of a , and that the angle its direction makes with that of the parallels must be a multiple of $2\pi \div m$. Then upon the supposition that its centre falls at a distance y from one parallel, the number of positions in which it can cut that parallel corresponds to the number of multiples of $2\pi \div m$, contained in the angle 4ϕ , where $r \cos \phi = y$ (it being possible that either side of the rod may cut the parallel), and is in the case of the limit proportional to 4ϕ ; that is, when m is increased without limit. Taking this result for all the subdivisions of a , we have, when n is increased without limit, the total number of favourable cases proportional

to $4 \int \phi dy$, or $4\phi y - 4 \int y d\phi$; that is, $4\phi y - 4r \sin \phi$, from $y=0$ and $\phi = \frac{\pi}{2}$ to $y=r$ and $\phi=0$; that is, $4r$. From $y=r$ to $y=a-r$ there are no favourable cases; while from $y=a-r$ to $y=a$ there are chances of the rod meeting the other parallel, which are, on the same suppositions, also represented by $4r$. Now the whole number of cases in each revolution are as 2π , and for all values of y , as $2\pi a$; whence $8r \div 2\pi a$, or $4r \div \pi a$ is the probability of the rod intersecting one of the parallels.

Hence, two out of the three, r , a , and π , being given, the third can be approximately found by observation of a large number of hazards. Thus an approximation to the value of the ratio between the circumference and diameter of a circle may be made by throwing a stick on a floor a sufficient number of times, and registering the number of times which the stick cuts one of the divisions between the boards. Supposing n throws to be made, and that $r \div a = \theta$, the probability that the number of throws in which this is the case lies between

$$\frac{4\theta}{\pi} n \pm \frac{2t}{\pi} \sqrt{2\theta(\pi - 4\theta)n} \text{ is (74.) } = \frac{1}{\sqrt{\pi}} \int_{-t}^{+t} e^{-t^2} dt.$$

To make the probable variation the least fraction of the whole, *cæteris paribus*, $(\pi - 4\theta) \div \theta$ must have the least value* consistent with $\theta < \frac{1}{2}$, or $= \frac{1}{2}$, which will admit of the application of the process. Hence we must take $\theta = \frac{1}{2}$, or the rod must be equal in length to the distance between the parallels. In this case m out of n being the observed number of times which the rod strikes one of the parallels, the probability that m lies between

$$\frac{2n}{\pi} \pm \frac{2\sqrt{(\pi - 2)n}}{\pi} \cdot t \text{ is } \frac{1}{\sqrt{\pi}} \int_{-t}^{+t} e^{-t^2} dt.$$

In this way it is sufficiently obvious that, were such a process necessary, the value of π might be determined with any degree of accuracy short of that with which lines can be actually measured. And many problems might be suggested in which the determination of given constants by comparison of the results of large numbers of hazards, strange as it may appear at first sight, is free from any difficulty of principle.

(173.) We now consider a class of questions in which we can only form a notion of the method of using data, without absolute knowledge of what the data may be in any particular case. We refer to the probabilities of evidence, and of judicial or other decisions, considered as depending upon the separate probabilities of each witness, or of each judge. It would much facilitate a proper use of the theory, if its particular province in these matters were better defined than is usual in mathematical writings.

All application of the present subject requires firstly, that we should not be in possession of any absolute cause from which we can deduce the quantity or character of an effect; and secondly, that there should be (4.) some relation between the successive phenomena in question, which makes us certain that, happen what may, there is some particular reason for the event being what it is and no other. The observation of physical phenomena shows us that one circumstance, which call A, is either always accompanied by another, B, in which case the connection between the two is the object of mathematico-physical research; or else it shows that A is more or less frequently followed by B. In the latter case, the phenomena having reference to quantity, we do not stop here and consider the connection between A and B as more or less probable: for our means of further inquiry may be such as to enable us to seek an absolute reason why B should follow in some cases, and not in others; that is, to detect all the relations of cause and effect which exist among the various particular phenomena which

* The result of Laplace (p. 360) makes the fluctuation for which there is a given probability a *maximum* instead of a *minimum*.

Theory of
Pro-
babilities.

were examined. But the boundary between absolute knowledge and absolute ignorance is not a mathematical line: there is an intermediate state in which our partial knowledge of relations, and means of observation, oblige us to consider results as having more or less probability, which we do not therefore imagine to be themselves fortuitous or unfixed. It is precisely the same with most of the questions on which neither the certainty of physical measurement nor of mathematical deduction can be obtained: the only difference lies in the very different character of the means of discovery which it is necessary to employ.

Theory of
Pro-
babilities.

Accurate investigation, or absolute physical or mathematical demonstration, being supposed impossible, it might be imagined that the next method of inquiry, in order of merit, would be the absolute collection and comparison of large numbers of facts. Such a process would lead direct to the application of the theory of probabilities: and the little use which has been made of the latter arises from the neglect to procure the former. All attempts of this kind have been of comparatively recent date. It seems to have been one of the most difficult conceptions which could be arrived at, that by possibility there might exist general laws among masses of phenomena, however uncertain might be the results of any particular case. Take two men aged forty, and it is very possible that the life of one of them should be double that of the other; but it is extremely improbable, and is never found to be the fact, that the sum of the lives of one hundred men, each aged forty, differs materially from the same sum for another hundred lives of the same class. Statistical observations are making it more clear from day to day, that, taking a whole community, there is hardly any one circumstance which admits of measurement, in which the average of one year differs greatly from that of another, or in which a gradual progression in either direction exists without a sufficient and assignable cause.

(174.) This power of deriving knowledge from numerical observation of facts, which exists incontestably with regard to natural and social phenomena, is denied by many in regard to questions the elements of which are functions of the moral and intellectual state of mankind. Independently of the theoretical, we see also a practical denial of such power, in the indisposition to collect any specific information. Perhaps no question of its kind has been more debated, than the English policy of requiring absolute unanimity among the individuals composing a jury. The *tendency* arguments, the *à priori* deductions of reason for or against, from the nature of man, and the "state of society," the discussion of the "principle" of starvation, &c., &c., &c. are very amusing drawing-room topics, and will, we doubt not, with many, supersede the necessity of counting cases. But for ourselves we should feel much better able to form an opinion if we could procure some statistical knowledge of this kind. All the world knows that when a jury retires for two or more hours, the necessity for an unanimous verdict is the proximate cause, in nine cases out of ten at least. Now presuming, which we doubt not is the fact, that in most cases where new trials are granted there is some reason which people in general would be inclined to admit for disturbing the previous verdict, a test presents itself which must be acknowledged to be very likely to afford a valuable result. Take five hundred trials, in which the jury have delivered their verdict at once, and five hundred more in which they have deliberated for two hours. Does the per-centage of new trials granted from the conduct of the jury in the second set exceed that in the first set? No doubt it does would be the answer; for the second five hundred would include all the difficult cases. But, not allowing any thing to these antecedent presumptions, we should like to be informed *as to the fact*. In the first set there is real unanimity; in the second there is the forced unanimity of the law. If it should turn out (and no one knows the contrary, for it has never been tried) that the per-centage of error in the second set does not materially exceed that in the first, there arises an argument for the continuance of the present system by very much stronger than any which has been produced. But better still, such information might arise during the process of examination, as would show in what direction further inquiries should be made.

(175.) But though the preceding process have one definite practical end, the consequences fairly deducible from it may be carried further. Suppose it to be established from the first set of cases, that unbiassed unanimity, where it occurs, is liable to a certain chance of error. The questions next arise, what chance of error is thence to be attributed to each individual of the twelve? Would the chances of error be materially altered if the number of the jury were reduced to six? Such questions as these are mathematical deductions, and can be clearly shown to be such. Let us suppose, for instance, that we put the following case. Knowing the absolute impossibility of judging with certainty from such evidence as it is generally practicable to procure, we must rest with the conviction that a certain proportion of judicial decisions (let us confine ourselves to criminal cases, where the verdict can be only one out of two) will be wrong. Let us admit that society is secure if only one decision out of a hundred be

wrong; that is, if the probability of a correct decision be $\frac{99}{100}$. Let us further admit, which seems reasonable,

that, owing to the manner in which unanimity is obtained in our courts, the verdict of a jury of twelve men is not entitled to more confidence than would be a majority of eight to four, all unbiassed by each other, and each worthy of the average degree of confidence. It then becomes a purely mathematical question what that average degree

of confidence should be, in order that the aforesaid majority may invest its decision with a probability of $\frac{99}{100}$.

(176.) Let x be the probability that any one individual will form a right decision. Then the observed event being, that eight are of one opinion and four of another, two cases arise which may be considered as causes: 1. That the majority is right and the minority wrong, which gives $x^8(1-x)^4$ for the probability of the observed event. 2. That the minority is right, which gives $(1-x)^8x^4$ for the same probability. Hence the probability that the decision is correct is

$$\frac{x^8(1-x)^4}{x^8(1-x)^4 + (1-x)^8x^4}, \text{ which in (175.) is supposed } = \frac{99}{100}; \text{ giving } x = \frac{1000}{1317}.$$

Theory of
Pro-
babilities.

This problem will serve to illustrate the particular province of our theory in regard to such a question. No mathematician, as such, can say whether or no the confidence due to the verdict of an English jury is or is not equivalent to an unbiassed majority of eight over four: nor can he undertake, as such, to say, whether one wrong verdict out of a hundred is within or without the limits of necessary security. But he *can* undertake to say that, upon the preceding suppositions, the preceding amount of security cannot be attained, unless the individual jurymen, supposed equally worthy of confidence, be such as can be depended upon for correctly deciding about ten cases out of thirteen, each one by himself.

Theory of
Pro-
babilities.

This individual probability of correctness being admitted, the reduction of the jury to six, and the requirement of unanimity, considered as equivalent in value to the verdict of four against two, would give for the probability of the verdict

$$\frac{x^4(1-x)^2}{x^4(1-x)^2 + x^2(1-x)^4}, \text{ which when } x = \frac{1000}{1317} \text{ is } = \frac{10}{11} \text{ nearly:}$$

or such a jury would decide one case out of eleven wrongly.

(177.) The same distinctions may be drawn with regard to the value of evidence, and they tend to separate the worth of an hypothesis, which may be small, from that of the conclusions drawn from it, which may be absolutely demonstrable. For instance, there are very few cases in which different witnesses who actually assert the same fact are perfectly independent of each other, so that neither could have known of the existence of the other's evidence. In such cases we give immediate credence to the account. But it more usually happens that there is mixed up with the consideration of the question, 1. a possibility of collusion between the witnesses; 2. the circumstance of the evidence of all not being the same, but exhibiting only more or less of agreement. These two characteristics of evidence are in some degree contradictory of each other; perfect agreement strengthens the probability of collusion; * while to draw the line, and determine where disagreement ceases to be presumption of honesty, and to become that of perjury, is the most delicate part of a decision. Taking all such circumstances into account, it becomes impossible even to frame a mathematical hypothesis, the results of which it would be worth while to examine. Nevertheless there are circumstances connected with the mathematical theory of independent evidence which it may be useful to examine.

In this, as in several other preceding investigations, it is not so much our wish to deduce and *impose* results, as to inquire whether these results really coincide with the methods of judging which our reason, unassisted by exact comparison, has already made us adopt. The use of the process is, that both our theory and our preconceptions thus either assist or destroy each other: in the former case we feel able to trust this science for *further directions*; in the latter, a useful new inquiry is opened. For when we consider the very imposing character of the first principles of the science of probabilities, and the mathematical necessity which connects those simple first principles with their results, we feel convinced that, even on the supposition that the main conclusions of the present treatise are altogether fallacious, there must arise a necessity for investigating the reason why a *methodical* treatment of certain notions should lead to results inconsistent with the *vague* application of them on which we are accustomed to rely. For it must not be imagined that opposition to the principles laid down in this treatise is always conducted on other principles: on the contrary, it frequently happens that it is only a result of themselves obtained without calculation, which is arrayed against arithmetical deduction.

(178.) A witness, whose *veracity* (meaning here the probability that he is willing to speak truth) is p , and whose judgment (meaning the probability that he is correct in what he supposes) is r , asserts a circumstance to have happened. Say that there is a lottery with n balls, and he asserts number *one* to have been drawn, it being known that some one or other has been drawn. Required the probability that number *one* actually was drawn. There are two cases in which number one was drawn: 1. The witness tells† the truth, and is correct in his idea of it. 2. The witness intends to deceive, and to state a number which was not drawn, but mistakes the number drawn, and selects the real one, thinking it not to be such. The probability of the first supposition is pr : that of the second depends on three circumstances, namely, that he intends to deceive, is wrong in his judgment, and chooses number *one* out of $n-1$ balls which might equally have been chosen. Of this triple event the probability is $(1-p)(1-r) \div (n-1)$, and therefore the probability that the event stated really happened is

$pr + \frac{(1-p)(1-r)}{n-1}$. If the possible number of events be very great, then pr is the probability in question; but if there be only two, then $1 + 2pr - p - r$ is the probability.

(179.) The preceding question is solved in a different manner by Laplace, as follows. Here is an observed event, namely, the *announcement* of the drawing of number one, and the actual truth of the statement is one among the various causes (or precedent circumstances) of the observed event. The hypotheses which can be made as to the preceding circumstances are as follows: 1. The witness is both honest and correct. 2. He is dishonest, but not mistaken. 3. He is honest, but mistaken. 4. He is both dishonest and mistaken. In either case he may announce number one: and the probability of this event being determined from each hypothesis, the sum of such probabilities as necessarily involve the truth of the fact, divided by the sum of all the probabilities, will

* Cases may arise in which disagreement happens by collusion; but the whole history of judicial evidence tends to show that cross-examination must render not only the disagreement, but the agreement, of collusion, almost certain to be detected. At the same time, such detection may become difficult, if the circumstance on which the deception turns be extremely simple. There was, it is said, at one time an Old Bailey method of proving an *alibi*, which often succeeded. The parties called for the prisoners swore honestly to an *alibi* which really existed some hours or days before the committal of the offence, taking care to state no untruth whatsoever, except only the *time* of the circumstances to which they deposed.

† Observe that this phrase has throughout reference to his intentions: he means to tell the truth.

Theory of Probabilities. give the probability that the precedent circumstances are one or other of the sets which involve the truth of the fact; that is, that the announcement is true. Theory of Probabilities.

Now in the principle of (19.), it is easily shown that, if the causes be not *a priori* equally probable, the probability which each cause gives to the observed event must be multiplied by the probability of the cause itself. If, for instance, a white ball be drawn from one or other of two lotteries, which have the fractions a and b of white balls out of the whole of each, the probability of the first lottery having been the one selected is $a \div (a+b)$. But if there be m of the first lotteries, and n of the second, then though the probability of any one of the first lotteries having been chosen is $a \div (ma+nb)$, yet that of one or other of the first having been chosen is

$$\frac{ma}{ma+nb}, \text{ or } \frac{\{m \div (m+n)\} a}{\{m \div (m+n)\} a + \{n \div (m+n)\} b}, \text{ as was asserted.}$$

Reasoning in this way, we give the results of the four several cases mentioned above, as exercises in the subject:

$$(1.) \frac{pr}{n}, \quad (2.) \frac{n-1}{n} \cdot \frac{1}{n-1} (1-p)r \quad (3.) \frac{n-1}{n} \frac{1}{n-1} p(1-r) \quad (4.) \frac{n-2}{n(n-1)} (1-p)(1-r) + \frac{1}{n(n-1)} (1-p)(1-r).$$

In the fourth case the first term has reference to the case where number one is not drawn; and the second to that in which number one is drawn. And the sum of the probabilities corresponding to the truth of the announcement, $pr \div n + (1-p)(1-r) \div n(n-1)$, divided by the sum of all (which is $1 \div n$), gives the result of the preceding article.

(180.) Let α and β be the probabilities of a white and black ball ($\alpha + \beta = 1$): and a witness has announced a white ball; his veracity and judgment being as before. On the first hypothesis of the preceding article (of which the chance is pr) the white ball must have appeared, and the relative probability of this hypothesis is αpr . On the second hypothesis, the same is $\beta(1-p)r$; on the third $\beta(1-r)p$; and on the fourth $\alpha(1-r)(1-p)$; for this case can only happen when the announcement is true (there being but two possible cases). Hence $\{q = pr + (1-p)(1-r)\}$.

The probability that an event whose chance is α , asserted by a witness whose veracity is p and judgment r , really did happen is $\frac{\alpha q}{\alpha q + (1-\alpha)(1-q)}$.

This is in accordance, as to result, with the universal impression that a very extraordinary event, asserted by a single witness, is not worthy of belief unless the presumptions in favour of the honesty of the witness be proportionally strong. It must be observed, that q is the probability that the witness states what happened (whether by intention or by mistaken method of attempting to deceive), without reference to the probability of the event. And the preceding probability becomes an even chance when $q + \alpha = 1$: that is, we may by easily admissible conventions make the following sentence a mathematical truth. An extraordinary fact, *asserted by one witness*, becomes as likely as not, if the character of the witness for honesty and judgment together be such that his being incorrect would be as extraordinary as the fact itself.

It is here worth while to notice the preceding result as a specimen of the manner in which a mathematical expression points out the most convenient meaning of a term. Thus in the Article LIGHT, the inverse of the focal length of a lens is called its power; not because there are any *a priori* notions which point out the propriety of such a definition, but because it appears that, subsequently to the deduction of a complicated proposition, such a definition will lead to the *desirable* theorem, "the power of a complex lens is the sum of the powers of the simple lenses;" a notion which any one would be likely to form, but which could not be formed correctly unless the definition of power were as above stated.

It is not of unfrequent occurrence that such a choice is made of the meanings to be assigned to terms as will render a common and vague notion strictly true. Thus it is universally admitted that the extraordinary nature of an asserted event must be overcome by evidence of such strength that its failure would be at least as extraordinary as the fact. This proposition becomes a mathematical truth under the following definition. The *strength* of the evidence of a single witness is measured by $pr + (1-p)(1-r)$, where p measures his veracity, and r the correctness of his judgment. We shall resume this point after having considered the evidence of two or more witnesses who agree in asserting the same thing.

(181.) Let a number of ν witnesses, whose veracities are p_1, p_2 , &c., unite in asserting that number one was drawn out of a lottery containing n different numbers. Let it also be supposed, for simplicity, that they are not liable to mistake: whence it follows either that all tell the truth, or that all deceive; the probabilities of which hypotheses are $p_1 p_2 \dots$ and $(1-p_1)(1-p_2) \dots$. In the first case, number one was drawn, of which the probability is $1 \div n$: in the second all the witnesses have separately chosen the same false number, of which, supposing them to have no bias in favour of one rather than another, the probability is $1 \div n(n-1)^{\nu-1}$, where ν is the number of witnesses; for $(n-1) \div n$ is the probability that number one is not drawn, and $1 \div (n-1)^{\nu}$ is the chance of ν individuals fixing on the same number out of $n-1$. Hence, as in the preceding articles, the probability which the evidence gives to the fact asserted is

$$\frac{(n-1)^{\nu-1} p_1 p_2 p_3 \dots}{(n-1)^{\nu-1} p_1 p_2 p_3 \dots + (1-p_1)(1-p_2)(1-p_3) \dots}$$

From this we see* the necessity of distinguishing between the case in which there are only two events to choose out of, and that in which there are three or more. If a hundred witnesses of very bad character (say of a veracity of one-tenth) unite in asserting that A happened where only A or B could happen, the multiplication of

* By which we mean that we see such a consequence to result from the mathematical formula. We are not proving these well-known things, but simply showing that the results of the formula coincide with those which are universally admitted.

Theory of
Pro-
babilities.

evidence weakens the probability of the fact, which is only $1 \div (1 + 9^{100})$. Here the only question is, Have these witnesses all spoken truth, or all spoken false? and it is clear that though the union be an extraordinary fact, yet it is much less extraordinary on the second supposition than on the first. But if there had been ten possible cases, A, B, C . . . the question is, Have these witnesses all spoken truth, or have they all, not only spoken false, but independently fixed upon A as the false assertion out of nine equally probable ones? The probability of the assertion now becomes $9^{99} \div (9^{99} + 9^{100})$, or one-tenth. But it is evident that, however low the veracity of the witnesses may be, the number of events out of which they have to choose may be so great that (collusion being impossible) the truth of an unanimous assertion becomes insuperably probable. The result is, that in cases where a very large number of different events might have been chosen, the unanimity of many witnesses (or even of two witnesses) entirely destroys all question about their veracity, and throws the whole difficulty of deciding upon this question, Was there, or was there not, collusion? Whether they be good or bad witnesses individually can only affect the rational probability of their assertion in the manner in which it affects our disposition to suspect combination. And here are two distinct elements to be employed: firstly, what, all circumstances considered, is the probability that there *could have been* collusion? secondly, what, from the known character of the witnesses, is the probability that there *was* collusion?

(182.) We have said that the mere unanimity of witnesses upon an assertion chosen out of a very large number, throws their veracity out of the question, until we come to ask whether there was or was not collusion. There is a celebrated argument with regard to miracles (that of Hume), of which it may be said that it was a fallacy answered by fallacies, so far as the mathematical bearing of it was concerned. Many who reject the mathematical theory of probabilities nevertheless read Hume's argument and Paley's answer with serious attention, never dreaming that they are considering a mathematical formula expressed in words, and that $q + \alpha = 1$ (see (180.)) contains Hume's assertion, tacitly admitted by Paley.

Hume, (*Essay on Miracles.*)

When *any one* tells me that he saw a dead man restored to life, I immediately consider with myself whether it be more probable that this person should either deceive or be deceived, or that the fact which he relates should really have happened. I weigh the one miracle against the other, and according to the superiority which I discover, I pronounce my decision, and always reject the greater miracle. If the falsehood of his testimony would be more miraculous than the event which he relates; then, and not till then, can he pretend to command my belief or opinion.

Paley, (*Evidences of Christianity.*)

Mr. Hume states the case of miracles to be a contest of opposite improbabilities, that is to say, a question whether it be more improbable that the miracle should be true, or the testimony false: and this I think a fair account of the controversy.

Now from the preceding articles it appears, 1. That not only is Hume's principle rational *in the case of a single witness*, but that he himself would have been (had he understood his own assertion) of a morbid degree of faith, willing to believe a miracle the moment more than an even chance was made out in its favour. 2. That in the case of *two or more witnesses*, his "contest of opposite improbabilities" fails entirely to represent the state of the case, and should not have been admitted as "a fair account of the controversy."

(183.) Generally speaking, however, the favourable presumptions which evidence presents do not arise from the selection by several witnesses of the same circumstance out of many, but from their agreement in several circumstances, each of which presents only a pair of alternatives. And it may be fairly assumed that, before an assertion is made, that is, before the nature of the fact asserted affects the character of the witness, it is more likely than not that the witness will speak the truth. Let us assume that all degrees of veracity between $\frac{1}{2}$ and 1 have equal presumptions. It is observed, that n witnesses have all agreed in asserting m distinct facts, of each of which the antecedent probability is α , (let $\alpha + \beta = 1$). Required the probability that all these facts shall be true.

The veracity of a witness being x , the probability of n witnesses agreeing in the assertion of any one of the facts is the sum of the chances in the two following cases: 1. The fact happens, and all the witnesses speak truth. 2. The fact did not happen, and all the witnesses speak falsely. This probability is, therefore, $\alpha x^n + \beta (1 - x)^n = P$. Consequently the probability of m such occurrences is P^m , and the presumption which the observed event gives to any value of x is $P^m dx$, divided by $\int P^m dx$, taken between the widest limits of the question, that is, from *one half* to *one*.

Hence the presumption for any value of x between one half and one is $P^m dx \div \int_{\frac{1}{2}}^1 P^m dx$. Again, a value of x being assumed, the resulting probability of the several facts being all true is $(\alpha x^n \div P)^m$; whence it follows that the probability of x having a given value, and of the facts being true, is

$$\frac{P^m dx}{\int_{\frac{1}{2}}^1 P^m dx} \times \frac{\alpha^n x^{mn}}{P^m}, \text{ or } \frac{\alpha^n \int_{\frac{1}{2}}^1 x^{mn} dx}{\int_{\frac{1}{2}}^1 P^m dx} \text{ is the total probability.}$$

The same process may be applied to the case in which out of n witnesses ($= p + q$) p unite in affirming and q in denying m different facts.

(184.) We shall notice one more case, taking the same sort of definite mathematical assumption which has already preceded. *Tradition*, in the strictest sense, is the delivery of a statement by a first witness to a second, by the second to a third, and so on, no one witness having any authority except that which he derives from the

Theory of
Pro-
babilities.

Theory of
Pro-
babilities.

Theory of
Pro-
babilities.

one immediately preceding. In the more common signification of the term, it implies the successive delivery of a fact from generation to generation, each race having not only the authority of that immediately preceding, but also the confirmation of written or other monuments: to such an extent, that the only pure *tradition* is that of the writings, customs, or other modes of testimony.

Suppose that any number of successive witnesses pass the statement that, out of a lottery of n balls, number *one* was drawn; that is, the observed event is, that the r th witness affirms that he was told this by his predecessor. Let the veracity of the r th witness be p_r , and let y_r be the probability of the fact so soon as stated by this witness. Then, on the delivery of the statement to a new witness (the $(r+1)$ th), the probability of the fact (y_{r+1}) is the sum of the two following probabilities. 1. That the r th witness is correct, and the $(r+1)$ th reports him truly, (or $p_{r+1} y_r$). 2. That the r th witness is incorrect, and that the $(r+1)$ th witness reports his account falsely, choosing number one out of $n-1$, not named by the predecessor, (or $(1-p_{r+1})(1-y_r) \div (n-1)$). Hence we have

$$y_{r+1} = p_{r+1} y_r + \frac{(1-p_{r+1})(1-y_r)}{n-1} = \frac{(np_{r+1}-1)y_r + 1 - p_{r+1}}{n-1},$$

or

$$y_{r+1} - \frac{1}{n} = \frac{np_{r+1}-1}{n-1} \left(y_r - \frac{1}{n} \right),$$

whence

$$y_r = \frac{1}{n} + \frac{n-1}{n} \cdot \frac{(np_1-1)(np_2-1)\dots(np_r-1)}{(n-1)^r},$$

where the constant of integration $(n-1) \div n$ is so taken that when $r=1$ $y_1=p_1$. From this it appears, that so long as the veracity of the witnesses is greater than the probability of the fact, their evidence makes the latter more probable than it is in itself. But a single witness less credible than the fact itself makes the whole chain of evidence weaken the intrinsic probability, and so does the occurrence of any *odd* number of such witnesses; while an *even* number of the same has the contrary effect. The mathematical treatment of this problem serves to show how little the common notion of tradition coincides with the strict definition of it. That no received fact of history is really accompanied by a distinct perception and admission of its having passed through many successive witnesses, will be readily admitted: and we may add, looking at the manner in which the general results of the theory of probabilities coincide with maxims to which universal assent is given, that disagreement between any mathematical result and general opinion is of itself a strong presumption that the first principles on which the former is obtained are not those on which the latter is formed. Thus we see that, in the result of the preceding investigation, a succession of witnesses of no credibility whatsoever (or who always speak false), if *even* in number, increases the intrinsic probability of the fact delivered by the last.

If the number n be extremely great, the result of the preceding is very nearly the product $p_1 p_2 p_3 \dots p_r$.

On the preceding investigation, the same observations may be made which have suggested themselves with regard to the evidence of a single witness. When there is only one chain of tradition, the evidence for the fact asserted depends materially on the character of the witnesses. But when there are two or more such chains, which end by agreeing in one fact, out of a great many which might have been chosen, the character of the witnesses becomes immaterial, except so far as it affects the chances of previous collusion. If it be impossible that there can have been concerted agreement between any of the witnesses in the two chains, then their final agreement is insuperable evidence of the truth of the fact which they assert: including under the case of concerted agreement, that in which the separation of the two chains is not entire throughout their whole continuance, or in which the same witness or witnesses are part of both chains.

(185.) We feel at liberty to say, that though a result of the theory of probabilities, upon a moral question, is not to be lightly or easily adopted, when it differs from usual notions, yet on the other hand it is not therefore to be immediately rejected. When both the ordinary notion and the mathematical result are derived from the same hypothesis, the latter must be the more correct: and in those numerous cases in which the difficulty lies in reducing the original circumstances to a mathematical form, there is nothing to show that we are less liable to error in deducing a common sense result from principles too indefinite for calculation, than we should be in attempting to define more closely, and to apply numerical reasoning.

(186.) *Remark on Article (88.)*—The theory of series, in the case of terms which are alternately positive and negative, receives an important addition from the theorem given by Lagrange on the limits of Maclaurin's theorem; namely,

$$f(x) = f(0) + f'(0) \cdot x + f''(0) \frac{x^2}{2} + \dots + f^{(n)}(0) \frac{x^n}{2 \cdot 3 \dots n} + f^{(n+1)}(\theta x) \frac{x^{n+1}}{2 \cdot 3 \dots (n+1)},$$

where θ is less than unity. It is easily proved that, in the case of *decreasing* terms *ad infinitum*, alternately positive and negative, the sum of any number of terms is in error by less than the first term rejected; and this whether the whole number of terms be finite or infinite. Whence it follows, that if such a series as $A - Bx + Cx^2 - \dots$ begin by diminishing terms, the same proposition is true as long as the terms diminish, even though they should afterwards increase without limit. Supposing we stop at the term $f^{(n)}(0) \times x^n \div 2 \cdot 3 \dots n$, all the terms up to that point having been as above described, the theorem stated must be true for all values of x which make $f^{(n)}(0)$ greater than $f^{(n+1)}(\theta x) \times x \div (n+1)$, provided $f^{(n+1)}(\theta x)$ be always of a contrary sign to $f^{(n)}$ from 0 to x . Thus, in the third series of (62.) for $\int \varepsilon^{-t^2} dt$, we see a series which though always divergent at last, yet begins by convergency whenever $2a^2$ is greater than 1. Thus when $a=2$, we may infer that

$$\int_0^\infty \varepsilon^{-t^2} dt \text{ does not differ from } \frac{\varepsilon - a^2}{2a} \left(1 - \frac{1}{2a^2} \right) \text{ by so much as } \frac{1 \cdot 3 \varepsilon - a^2}{8a^5}.$$

ERRATA.

Article 5, line 9 from the end, *for* what he meant *read* what we meant.
14, line 1, *for* IV. *read* III.
24, line 8 from the end, *for* particular *read* particularly.
60, line 5, *for* u *read* v .
79, last line of the page, *for* of a definition *read* of an integral.

INDEX TO THEORY OF PROBABILITIES.

Index.

(1—42.) *Principles of the subject: sketch of methods, and solution of simple questions.*

- (1.), (2.) Object of mathematics in the theory of probabilities.
- (3.) Meaning of the word probability.
- (4.) Distinction between *chance* and *probability*.
- (5.) Homogeneity of the notion of probability arising out of different kinds of events.
- (6.) Distinction between the psychological and mathematical part of the subject.
- (7.) Method of arriving at the measure of probability.
- (8.) Definitions connected with the measure of probability.
- (9.) Simple application for practice.
- (10.) Theorems on combinations connected with the subject.
- (11.) Principle on which algebraical developement distributes the whole number of the possible events into the several numbers of different cases.
- (12.) *Principle I.* Definition of probability.
- (13.) *Principle II.* Probability of compound events determined.
- (14.) *Principle III.* Method of (12.) extended to the case where the simple probabilities are not the same.
- (15.) Probability of a conclusion deduced from that of the premises.
- (16.) Action of the probability deduced from one argument upon the premises of another. Objection to the method followed by writers on logic.
- (17.) Consideration of D'Alembert's objection.
- (18.) *Principle IV.* Converse operation to that in (13.).
- (19.) *Principle V.* Probability of an antecedent determined from the observed consequent.
- (20.) *Principle VI.* Deduction of the probability of an event to come from an observed event.
- (21.) Definition of gambling. (22.) How avoided by division of risks.
- (23.) Discussion of *runs of luck*.
- (24.) On the notion that the change of an indifferent circumstance changes luck.
- (25.) Mistake arising from confounding an event past with one to come.
- (26.) *Principle VII.* On the value of a contingent gain.
- (27.) Application of the preceding to cases of many risks.
- (28.) Remarks on the application of the preceding in commerce.
- (29.), (30.), (31.) A gambling trick. On gambling in general.
- (32.) Statement of different cases of life contingencies.
- (33.) Problem of life annuities.
- (34.) Problem of deferred life annuities.
- (35.) Present value of a reversion or an insurance.
- (36.) Premium to be paid for the last.
- (37.) Instance of the more complicated problems of life contingencies.
- (38.) Moral expectation as distinguished from mathematical.
- (39.) *Principle VIII.* Relative stakes necessary for fair play.
- (40.) Paradox connected with the mathematical expectation.
- (41.) *Principle IX.* Insufficiency of the mathematical expectation.
- (42.) Plausible substitute for the latter.
- (43.—54.) *Complicated problems solved without the aid of the integral calculus.*
- (43.) Simple problem solved independently, and also by the method in (11.)
- (44.) Probability of the amount of the sum of the drawings in a lottery.
- (45.) Extension of the last to the case of an infinite number of possible drawings.
- (46.) Application to a point connected with the theory of the solar system.
- (47.) Extension of (14.) by the integral calculus.
- (48.) On the probable contents of a lottery, as inferred from drawings, the possible cases being given.
- (49.) The same where the number of possible cases is unlimited. Occurrence of the difficulty mentioned in (15.)
- (50.) Chances of two players, each of whom wants a given number of points. General principles of the theory of generating functions, and of the interpolation of series.

Index.

- (51.) Method of solving an equation of differences by generating functions.
- (52.) Chance of one player beating n others in succession in less than x trials.
- (53.) Problem of the chance of *runs of luck*, the skill of the players being different.
- (54.) Probability of all the numbers in a lottery coming out in n trials.
- (55.—83.) *Method of applying the calculus of definite integrals to the valuation of functions of high numbers.*
- (55.) Remarks on functions which contain a variable number of factors.
- (56.) Sketch of the method of expressing such functions by a definite integral.
- (57.) Application to the cases of rational functions. Expression of their differential coefficients and differences.
- (58.) Case of s^{-1} and $\Delta^n.s^{-1}$.
- (59.) Investigation of a function of y which is necessarily small, when n contains many factors.
- (60.) Expansion of $\int y dx$ in powers of the functions of (59.)
- (61.) Way in which great exponents make convergent series.
- (62.) Expansion of $\int y dx$ in the case where one of the limits makes y a maximum. Investigation of the integrals on which the result depends. Methods of determining $\int \epsilon^{-t} dt$ from $t=0$.
- (63.) Conversion of the last integral into a continued fraction.
- (64.) Extensions of the process of definite integration in (62.)
- (65.) Determination of $\int y dx$, where y contains many factors, between two values of x , which make $y=0$.
- (66.) Another method. (67.) Determination of the indeterminate coefficients.
- (68.) Application to integrals of the form $\int y dx dx'$.
- (69.) Developement of the result of (62.) for any limits, and investigation of the convergency of the result.
- (70.) Application to the case of $1.2.3\dots s$.
- (71.) Deduction of the preceding from the common developement of Σy .
- (72.) Proof of the nearness of the approximation.
- (73.) Application to $1.3.5\dots(2s-1)$.
- (74.) Probability of a given amount of fluctuation on a large number of similar events whose probabilities are given. Example.
- (75.) Change of the form of the preceding result. Remarks on contingency offices.
- (76.) The probable limits of fluctuation vary as the square root of the number of events.
- (77.) Inverse problem to that in (74.)
- (78.) Various integrals deduced from (62.)
- (80.), (81.) Remarks on definite integrals.
- (82.), (83.) Coefficient and sum of the coefficients between given limits, of the s th power of Σa^n , determined.
- (84.)—(138.) *Application of the theory to the estimation of the probable result of discordant observations.*
- (84.) Conditions under which discrepancy is called error.
- (85.) The observer and the instrument the united source of error.
- (86.) Definition of the *law of facility*.
- (87.) Estimation of the *disadvantage* of an error.
- (88.) *A priori* conditions which the law of facility must fulfil.
- (89.) The average not *theoretically* the most probable result for all laws of facility. Determination of the law under which it is so.
- (90.) Reason why the average is *practically* a sufficient approximation.
- ** From (91.) to (98.) the law of facility deduced in (89.) is adopted.
- (91.) Probability that the error of an average lies between given limits, determined.
- (92.) Comparison of the errors of the averages of different sets.

Index.

- (93.) Bad observations in large numbers not preferable to few and good ones.
- (94.) Example of the determination of the constant of the law of facility.
- (95.) Problem of (91.), the observations having different laws of facility. Definition of the term *specific weight*.
- (96.) Definition and expression of the *risk* or *mean risk* (*l'erreur moyenne*).
- (97.) The preceding results are included in the method of least squares.
- (98.) Risk of a function deduced from the risks of its variables.
- (99.) Caution in the use of the preceding process.
- (100.) Definition of *probable error*. Distinction between it and *mean risk*. The former always about 1.69 of the latter.
- (101.) Analogous distinction in tables of mortality: *mean life* and *probable life*.
- (102.), (103.) Problem of (91.) in the case of *any* law of facility. Justification of (90.)
- (104.) The mean of *two* observations *always* the most probable result: remark on the approximation of (103.).
- (105.) Alteration of form of the result of (103.)
- (106.) Comparison of preceding notation with that of Laplace.
- (107.) Determination of the probability of a given amount in the sum of any functions of errors, all errors being considered as positive.
- (108.) The preceding in the case of the sums of the errors.
- (109.) The same in the case of the sums of the squares of the errors.
- (110.) Approximate determination of the necessary elements.
- (111.) Problem of (107.) in the case where all the functions are different.
- (112.) The same where the negative errors are to be counted as negative.
- (113.) The same where the limits of positive and negative error are different.
- (114.) Consideration of the question of *personal* equation.
- (115.) Consideration of discordant observations, compared with that of their errors. Coincidence of results shown.
- (116.) Rule for the determination of the probable error in an independent form, with example.
- (117.), (118.), (119.), (120.) Description of the tables at the end of the work.
- (121.) Introductory account of the method of least squares.
- (122.) Its coincidence with several common notions.
- (123.) Comparison of the method of least squares with that of averaging.
- (124.) Demonstration in the case of one element to be determined.
- (125.) Coincidence of the result with that of Laplace.
- (126.) Extension to the case where the weights of the observations are not equal.
- (127.) Extension to the case where the unknown element is not determined from an equation of the first degree.
- (128.) Practical method of determining the specific weight of observations.
- (129.) Determination of the mean risk of the result of (124.)
- (130.) Method of least squares, the number of elements to be determined being more than one.
- (131.) The same in the case where the equations are not equally probable.
- (132.) Method of determining the mean risk of the result.
- (133.) Application to the case of two quantities.
- (134.) Artifice for passing from two to three or more elements.
- (135.) Errors in the estimation of the risks of the equation must be considerable before they can materially affect the result of the method of least squares.
- (136.) Remarks on Laplace's *method of situation*.
- (137.) Cases in which the method of least squares fails.
- (138.) The method holds good when positive and negative errors are not equally likely.

(139.—151.) *Miscellaneous applications of the theory of probable fluctuations.*

- (139.) On the question whether observed discrepancies are consequences of a general law, or accidental fluctuations.
- (140.) Method of proceeding in the determination of the most probable cause.
- (141.) Gauss's method of establishing the result in (89.).
- (142.) Application of (69.) to the question of (139.) in the case of one single cause.
- (143.) Application of (68.) to the case of two causes.
- (144.) Application of (60.) to the determination of the presumption that the event which has most often happened is the most probable.
- (145.) Determination of the presumption that increased frequency of an event has a particular cause.
- (146.) Presumption that a table of mortality which represents the past shall represent the future within given limits, on a single set of instances.
- (147.) The same in several instances.
- (148.) Application of the same principle to other problems.
- (149.) Mean risk of fluctuation in the duration of lives
- (150.) Recapitulation of the preceding Articles.
- (151.) Method of estimating the apportionment of *security funds*.
- (152.—162.) *On the mathematical advantage or disadvantage arising from any given prospect.*

- (152.) Statement of principle.
- (153.) Meaning of a negative answer.
- (154.) Determination of the *mean advantage*, and probability of a given amount of fluctuation, in a simple succession of chances.
- (155.), (156.) Principle of the division of risks.
- (157.), (158.) Extension of (154.).
- (159.) Method of constructing tables for the estimation of the probable fluctuations in the gain of an insurance office.
- (160.) Comparison of premiums and single payments.
- (161.) Advantage of *mutual* insurance.
- (162.) Case of (154.), where the probabilities are inferred from given previous observations.

(163.—170.) *On the moral or relative advantage or disadvantage of any given prospect.*

- (163.) Further consideration of (42.).
- (164.) Determination of the moral benefit of a prospect.
- (165.) Case in which this coincides nearly with the mathematical benefit.
- (166.) Risks, mathematically balanced, may be disadvantageous.
- (167.) Coincidence of a result of the method under consideration with a common notion.
- (168.) On the quantity of mathematical advantage which can be given to an insurance office in return for the moral security.
- (169.) The division of risks raises the moral advantage nearly to the mathematical one.
- (170.) Further consideration of the paradox in (40.).

(171.—186.) *Miscellaneous, principally on the probabilities of testimony.*

- (171.), (172.) Application of the theory to a peculiar class of problems.
- (173.), (174.), (175.) Remarks on the application of the calculus of probabilities to moral questions.
- (176.) On the decisions of a jury.
- (177.), (178.), (179.), (180.) On the testimony of a single witness.
- (181.) On the testimony of two or more witnesses.
- (182.) On Hume's argument with respect to miracles.
- (183.) Extension of formulæ by the integral calculus.
- (184.) On tradition.
- (185.) General remarks.
- (186.) Remark on (88.)

Index.

Theory of
Pro-
babilities.Theory of
Pro-
babilities.TABLES I. and II.—Values of $\int_0^t e^{-t^2} dt = G$, and of their logarithms, for intervals of t each $= .01$, from $t=0$ to $t=3$. (From Kramp's *Treatise on Refraction*, corrected.)

t	G	Δ	Δ^2	Δ^3	Δ^4	$\log G$	Δ	Δ^2	t
0.00	0.88622692	999968	201	199		9.9475448	49280	558	0.00
0.01	0.87622724	999767	400	199		9.9426168	49838	557	0.01
0.02	0.86622957	999367	599	200		9.9376330	50395	562	0.02
0.03	0.85623590	998768	799	199		9.9325935	50957	563	0.03
0.04	0.84624822	997969	998	197		9.9274978	51520	567	0.04
0.05	0.83626853	996971	1195	199		9.9223458	52087	567	0.05
0.06	0.82629882	995776	1394	196		9.9171371	52654	573	0.06
0.07	0.81634106	994382	1590	195		9.9118717	53227	572	0.07
0.08	0.80639724	992792	1785	194		9.9065490	53799	577	0.08
0.09	0.79646932	991007	1979	195		9.9011691	54376	580	0.09
0.10	0.78655925	989028	2174	192		9.8957315	54956	580	0.10
0.11	0.77666897	986854	2366	190		9.8902359	55536	583	0.11
0.12	0.76680043	984488	2556	189		9.8846823	56119	587	0.12
0.13	0.75695555	981932	2745	188		9.8790704	56706	588	0.13
0.14	0.74713623	979187	2933	186		9.8733998	57294	591	0.14
0.15	0.73734436	976254	3119	184		9.8676704	57885	594	0.15
0.16	0.72758182	973135	3303	183		9.8618819	58479	595	0.16
0.17	0.71785047	969832	3486	180		9.8560340	59074	598	0.17
0.18	0.70815215	966346	3666	175		9.8501266	59672	600	0.18
0.19	0.69848869	962680	3841	178		9.8441594	60272	604	0.19
0.20	0.68886189	958839	4019	173		9.8381322	60876	602	0.20
0.21	0.67927350	954820	4192	171		9.8320446	61478	610	0.21
0.22	0.66972530	950628	4363	168		9.8258968	62088	607	0.22
0.23	0.66021902	946265	4531	166		0.8196880	62695	613	0.23
0.24	0.65075637	941734	4697	163		9.8134185	63308	613	0.24
0.25	0.64133903	937037	4860	160		9.8070877	63921	617	0.25
0.26	0.63196866	932177	5020	157		9.8006956	64538	617	0.26
0.27	0.62264689	927157	5177	155		9.7942418	65155	620	0.27
0.28	0.61337532	921980	5332	151		9.7877263	65775	623	0.28
0.29	0.60415552	916648	5483	149		9.7811488	66398	624	0.29
0.30	0.59498904	911165	5632	145		9.7745090	67022	627	0.30
0.31	0.58587739	905533	5777	142		9.7678068	67649	629	0.31
0.32	0.57682206	899756	5919	138		9.7610419	68278	629	0.32
0.33	0.56782450	893837	6057	136		9.7542141	68907	635	0.33
0.34	0.55888613	887780	6193	131		9.7473234	69542	632	0.34
0.35	0.55000833	881587	6324	129		9.7403692	70174	638	0.35
0.36	0.54119246	875263	6453	125		9.7333518	70812	640	0.36
0.37	0.53243983	868810	6578	121		9.7262706	71452	639	0.37
0.38	0.52375173	862232	6699	118		9.7191254	72091	642	0.38
0.39	0.51512941	855533	6817	114		9.7119163	72733	646	0.39
0.40	0.50657408	848716	6931	110		9.7046430	73379	645	0.40
0.41	0.49808692	841785	7041	107		9.6973051	74024	650	0.41
0.42	0.48966907	834744	7148	103		9.6899027	74674	648	0.42
0.43	0.48132163	827596	7251	98		9.6824353	75322	653	0.43
0.44	0.47304567	820345	7349	97		9.6749031	75975	654	0.44
0.45	0.46484222	812996	7446	90		9.6673056	76629	655	0.45
0.46	0.45671226	805550	7536	88		9.6596427	77284	659	0.46
0.47	0.44865676	798014	7624	85		9.6519143	77943	663	0.47

Theory of
Pro-
babilities.

TABLES I. and II. continued.

Theory of
Pro-
babilities.

t	G	Δ	Δ^2	Δ^3	Δ^4	log G.	Δ	Δ^2	t
0.48	0.44067662	790390	7709	78		9.6441200	78606	656	0.48
0.49	0.43277272	782681	7787	77		9.6362598	79262	662	0.49
0.50	0.42494591	774894	7864	71		9.6283336	79924	666	0.50
0.51	0.41719697	767030	7935	69		9.6203412	80590	666	0.51
0.52	0.40952667	759095	8004	64		9.6122822	81256	668	0.52
0.53	0.40193572	751091	8068	61		9.6041566	81924	669	0.53
0.54	0.39442481	743023	8129	55		9.5959642	82593	672	0.54
0.55	0.38699458	734894	8184	54		9.5877049	83265	673	0.55
0.56	0.37964564	726710	8238	48		9.5793784	83938	673	0.56
0.57	0.37237854	718472	8286	45		9.5709846	84611	678	0.57
0.58	0.36519382	710186	8331	40		9.5625235	85289	678	0.58
0.59	0.35809196	701855	8371	38		9.5539946	85967	678	0.59
0.60	0.35107341	693484	8409	34		9.5453979	86645	683	0.60
0.61	0.34413857	685075	8443	28		9.5367334	87328	681	0.61
0.62	0.33728782	676632	8471	27		9.5280006	88009	684	0.62
0.63	0.33052150	668161	8498	22		9.5191997	88693	687	0.63
0.64	0.32383989	659663	8520	19		9.5103304	89380	687	0.64
0.65	0.31724326	651143	8539	14		9.5013924	90067	688	0.65
0.66	0.31073183	642604	8553	12		9.4923857	90755	690	0.66
0.67	0.30430579	634051	8565	6		9.4833102	91445	693	0.67
0.68	0.29796528	625486	8571	6		9.4741657	92138	692	0.68
0.69	0.29171042	616915	8577	0		9.4649519	92830	696	0.69
0.70	0.28554127	608338	8577	2		9.4556689	93526	695	0.70
0.71	0.27945789	599761	8575	7		9.4463163	94221	697	0.71
0.72	0.27346028	591186	8568	8		9.4368942	94918	700	0.72
0.73	0.26754842	582618	8560	14		9.4274024	95618	700	0.73
0.74	0.26172224	574058	8546	15		9.4178406	96318	700	0.74
0.75	0.25598166	565512	8531	20		9.4082088	97018	706	0.75
0.76	0.25032654	556981	8511	21		9.3985070	97724	703	0.76
0.77	0.24475673	548470	8490	25		9.3887346	98427	706	0.77
0.78	0.23927203	539980	8465	29		9.3788919	99133	707	0.78
0.79	0.23387223	531515	8436	31		9.3689786	99839	710	0.79
0.80	0.22855708	523079	8405	33		9.3589947	100549	709	0.80
0.81	0.22332629	514674	8372	37		9.3489398	101258	711	0.81
0.82	0.21817955	506302	8335	39		9.3388140	101969	712	0.82
0.83	0.21311653	497967	8296	42		9.3286171	102681	714	0.83
0.84	0.20813686	489671	8254	45		9.3183490	103395	714	0.84
0.85	0.20324015	481417	8209	46		9.3080095	104109	718	0.85
0.86	0.19842598	473208	8163	50		9.2975986	104827	716	0.86
0.87	0.19369390	465045	8113	52		9.2871159	105543	719	0.87
0.88	0.18904345	456932	8061	54		9.2765616	106262	719	0.88
0.89	0.18447413	448871	8007	56		9.2659354	106981	722	0.89
0.90	0.17998542	440864	7951	58		9.2552373	107703	721	0.90
0.91	0.17557678	432913	7893	61		9.2444670	108424	724	0.91
0.92	0.17124765	425020	7832	62		9.2336246	109148	724	0.92
0.93	0.16699745	417188	7770	65		9.2227098	109872	727	0.93
0.94	0.16282557	409418	7705	66		9.2117226	110599	724	0.94
0.95	0.15873139	401713	7639	67		9.2006627	111323	731	0.95
0.96	0.15471426	394074	7572	71		9.1895304	112054	726	0.96
0.97	0.15077352	386502	7501	70		9.1783250	112780	733	0.97
0.98	0.14690850	379001	7431	74		9.1670470	113513	729	0.98

Theory of
Pro-
babilities.

TABLES I. and II. continued.

Theory of
Pro-
babilities.

t	G	Δ	Δ^2	Δ^3	Δ^4	log G.	Δ	Δ^2	t
0.99	0.14311849	371570	7357	74		9.1556957	114242	734	0.99
1.00	0.13940279	364213	7283	75		9.1442715	114976	732	1.00
1.01	0.13576066	356930	7208	77		9.1327739	115708	736	1.01
1.02	0.13219136	349722	7131	80		9.1212031	116444	733	1.02
1.03	0.12869414	342591	7051	78		9.1095587	117177	739	1.03
1.04	0.12526823	335540	6973	81		9.0978410	117916	736	1.04
1.05	0.12191283	328567	6892	81		9.0860494	118652	741	1.05
1.06	0.11862716	321675	6811	83		9.0741842	119393	737	1.06
1.07	0.11541041	314864	6728	83		9.0622449	120130	742	1.07
1.08	0.11226177	308136	6645	85		9.0502319	120872	741	1.08
1.09	0.10918041	301491	6560	85		9.0381447	121613	743	1.09
1.10	0.10616550	294931	6475	86		9.0259834	122356	744	1.10
1.11	0.10321619	288456	6389	85		9.0137478	123100	743	1.11
1.12	0.10033163	282067	6304	88		9.0014378	123843	747	1.12
1.13	0.09751096	275763	6216	87		8.9890535	124590	746	1.13
1.14	0.09475333	269547	6129	89		8.9765945	125336	748	1.14
1.15	0.09205786	263418	6040	87		8.9640609	126084	748	1.15
1.16	0.08942368	257378	5953	89		8.9514525	126832	750	1.16
1.17	0.08684990	251425	5864	89		8.9387693	127582	749	1.17
1.18	0.08433565	245561	5775	89		8.9260111	128331	751	1.18
1.19	0.08188004	239786	5686	88		8.9131780	129082	753	1.19
1.20	0.07948218	234100	5598	91		8.9002698	129835	753	1.20
1.21	0.07714118	228502	5507	88		8.8872863	130588	753	1.21
1.22	0.07485616	222995	5419	90		8.8742275	131341	755	1.22
1.23	0.07262621	217576	5329	88		8.8610934	132096	755	1.23
1.24	0.07045045	212247	5241	90		8.8478838	132851	758	1.24
1.25	0.068327982	2070061	51513	888		8.8345987	133609	756	1.25
1.26	0.066257921	2018548	50625	886		8.8212378	134365	758	1.26
1.27	0.064239373	1967923	49739	884		8.8078013	135123	759	1.27
1.28	0.062271450	1918184	48855	880		8.7942890	135882	759	1.28
1.29	0.060353266	1869329	47975	876		8.7807008	136641	762	1.29
1.30	0.058483937	1821354	47099	872		8.7670367	137403	761	1.30
1.31	0.056662583	1774255	46227	868		8.7532964	138164	761	1.31
1.32	0.054888328	1728028	45359	863		8.7394800	138925	764	1.32
1.33	0.053160300	1682669	44496	859		8.7255875	139689	764	1.33
1.34	0.051477631	1638173	43637	850		8.7116186	140453	764	1.34
1.35	0.049839458	1594536	42787	847		8.6975733	141217	765	1.35
1.36	0.048244922	1551749	41940	840		8.6834516	141982	766	1.36
1.37	0.046693173	1509809	41100	833		8.6692534	142748	768	1.37
1.38	0.045183364	1468709	40267	826		8.6549786	143516	766	1.38
1.39	0.043714655	1428442	39441	819		8.6406270	144282	769	1.39
1.40	0.042286213	1389001	38622	812		8.6261988	145051	769	1.40
1.41	0.040897212	1350379	37810	802		8.6116937	145820	770	1.41
1.42	0.039546833	1312569	37008	797		8.5971117	146590	770	1.42
1.43	0.038234264	1275561	36211	786		8.5824527	147360	771	1.43
1.44	0.036958703	1239350	35425	779		8.5677167	148131	773	1.44
1.45	0.035719353	1203925	34646	771		8.5529036	148904	771	1.45
1.46	0.034515428	1169279	33875	760		8.5380132	149675	774	1.46
1.47	0.033346149	1135404	33115	751		8.5230457	150449	774	1.47
1.48	0.032210745	1102289	32364	743		8.5080008	151223	775	1.48
1.49	0.031108456	1069925	31621	734		8.4928785	151998	775	1.49

Theory of
Pro-
babilities.

TABLES I. and II. continued.

Theory of
Pro-
babilities.

t	G	Δ	Δ^2	Δ^3	Δ^4	log G.	Δ	Δ^2	t
1.50	0.030038531	1038304	30837	723		8.4776787	152773	776	1.50
1.51	0.0290002273	10074170	301638	7133		8.4624014	153549	777	1.51
1.52	0.0279928103	9772532	294505	7036		8.4470465	154326	777	1.52
1.53	0.0270155571	9478027	287469	6937		8.4316139	155103	777	1.53
1.54	0.0260677544	9190558	280532	6834		8.4161036	155880	781	1.54
1.55	0.0251486986	8910026	273698	6740		8.4005156	156661	779	1.55
1.56	0.0242576960	8636328	266958	6624		8.3848495	157440	778	1.56
1.57	0.0233940632	8369370	260334	6527		8.3691055	158218	781	1.57
1.58	0.0225571262	8109036	253807	6422		8.3532837	158999	781	1.58
1.59	0.0217462226	7855229	247385	6318		8.3373838	159780	782	1.59
1.60	0.0209606997	7607844	241067	6213		8.3214058	160562	783	1.60
1.61	0.0201999153	7366777	234854	6104		8.3053496	161345	783	1.61
1.62	0.0194632376	7131923	228750	6003	108	8.2892151	162128	784	1.62
1.63	0.0187500453	6903173	222747	5895	107	8.2730023	162912	783	1.63
1.64	0.0180597280	6680426	216852	5788	104	8.2567111	163695	784	1.64
1.65	0.0173916854	6463574	211064	5684	110	8.2403416	164479	787	1.65
1.66	0.0167453280	6252510	205380	5574	102	8.2238937	165266	785	1.66
1.67	0.0161200770	6047130	199806	5472	100	8.2073671	166051	787	1.67
1.68	0.0155153640	5847324	194334	5372	122	8.1907620	166838	787	1.68
1.69	0.0149306316	5652990	188962	5250	89	8.1740782	167625	788	1.69
1.70	0.0143653326	5464028	183712	5161	110	8.1573157	168413	787	1.70
1.71	0.0138189298	5280316	178551	5051	103	8.1404744	169200	790	1.71
1.72	0.0132908982	5101765	173500	4948	100	8.1235544	169990	789	1.72
1.73	0.0127807217	4928265	168552	4848	107	8.1065554	170779	792	1.73
1.74	0.0122878952	4759713	163704	4741	99	8.0894775	171571	785	1.74
1.75	0.0118119239	4596009	158963	4642	102	8.0723204	172356	795	1.75
1.76	0.0113523230	4437046	154321	4540	98	8.0550848	173151	790	1.76
1.77	0.0109086184	4282725	149781	4442	102	8.0377697	173941	792	1.77
1.78	0.0104803459	4132944	145339	4340	95	8.0203756	174733	792	1.78
1.79	0.0100670515	3987605	140999	4245	99	8.0029023	175525	795	1.79
1.80	0.0096682910	3846606	136754	4146	96	7.9853498	176320	792	1.80
1.81	0.0092836304	3709852	132608	4050	96	7.9677178	177112	794	1.81
1.82	0.0089126452	3577244	128558	3954	91	7.9500066	177906	794	1.82
1.83	0.0085549208	3448686	124604	3863	96	7.9322160	178700	797	1.83
1.84	0.0082100522	3324082	120741	3767	88	7.9143460	179497	793	1.84
1.85	0.0078776440	3203341	116974	3679	95	7.8963963	180290	798	1.85
1.86	0.0075573099	3086367	113295	3584	86	7.8783673	181088	795	1.86
1.87	0.0072486732	2973072	109711	3498	90	7.8602585	181883	797	1.87
1.88	0.0069513660	2863361	106213	3408	84	7.8420702	182680	799	1.88
1.89	0.0066650299	2757148	102805	3324	90	7.8238022	183479	796	1.89
1.90	0.0063893151	2654343	99481	3234	79	7.8054543	184275	799	1.90
1.91	0.0061238808	2554862	96247	3155	85	7.7870268	185074	800	1.91
1.92	0.0058683946	2458615	93092	3070	83	7.7685194	185874	797	1.92
1.93	0.0056225331	2365523	90022	2987	80	7.7499320	186671	801	1.93
1.94	0.0053859808	2275501	87035	2907	76	7.7312649	187472	801	1.94
1.95	0.0051584307	2188466	84128	2831	79	7.7125177	188273	800	1.95
1.96	0.0049395841	2104338	81297	2752	84	7.6936904	189073	801	1.96
1.97	0.0047291503	2023041	78545	2668	53	7.6747831	189874	801	1.97
1.98	0.0045268462	1944496	75877	2615	94	7.6557957	190675	801	1.98
1.99	0.0043323966	1868619	73262	2521	65	7.6367282	191476	804	1.99
2.00	0.00414553469	17953570	707406	24558	700	7.6175806	192280	803	2.00

Theory of
Pro-
babilities.TABLES I. and II. *continued.*Theory of
Pro-
babilities.

t	G	Δ	Δ^2	Δ^3	Δ^4	log G.	Δ	Δ^2	t
2.01	0.00396599899	17246164	682848	23858	697	7.5983526	193083	802	2.01
2.02	0.00379353735	16563316	658990	23161	674	7.5790443	193885	804	2.02
2.03	0.00362790419	15904326	635829	22487	658	7.5596558	194689	804	2.03
2.04	0.00346886093	15268497	613342	21829	651	7.5401869	195493	805	2.04
2.05	0.00331617596	14655155	591513	21178	646	7.5206376	196298	804	2.05
2.06	0.00316962441	14063642	570335	20532	603	7.5010078	197102	806	2.06
2.07	0.00302898799	13493307	549803	19929	622	7.4812976	197908	806	2.07
2.08	0.00289405492	12943504	529874	19307	591	7.4615068	198714	805	2.08
2.09	0.00276461988	12413630	510567	18716	574	7.4416354	199519	807	2.09
2.10	0.00264048358	11903063	491851	18142	582	7.4216835	200326	807	2.10
2.11	0.00252145295	11411212	473709	17560	543	7.4016509	201133	808	2.11
2.12	0.00240734083	10937503	456149	17017	537	7.3815376	201941	806	2.12
2.13	0.00229796580	10481354	439132	16480	540	7.3613435	202747	809	2.13
2.14	0.00219315226	10042222	422652	15940	510	7.3410688	203556	808	2.14
2.15	0.00209273004	9619570	406712	15430	496	7.3207132	204364	809	2.15
2.16	0.00199653434	9212858	391282	14934	497	7.3002768	205173	809	2.16
2.17	0.00190440576	8821576	376348	14437	478	7.2797595	205982	809	2.17
2.18	0.00181619000	8445228	361911	13959	457	7.2591613	206791	810	2.18
2.19	0.00173173772	8083317	347952	13502	459	7.2384822	207601	812	2.19
2.20	0.00165090455	7735365	334450	13043	439	7.2177221	208413	809	2.20
2.21	0.00157355090	7400915	321407	12604	432	7.1968808	209222	812	2.21
2.22	0.00149954175	7079508	308803	12172	411	7.1759586	210034	809	2.22
2.23	0.00142874667	6770705	296631	11761	408	7.1549552	210843	814	2.23
2.24	0.00136103962	6474074	284870	11353	399	7.1338709	211657	812	2.24
2.25	0.00129629888	6189204	273517	10954	380	7.1127052	212469	811	2.25
2.26	0.00123440684	5915687	262563	10574	375	7.0914583	213280	814	2.26
2.27	0.00117524997	5653124	251989	10199	359	7.0701303	214094	812	2.27
2.28	0.00111871873	5401135	241790	9840	354	7.0487209	214906	814	2.28
2.29	0.00106470738	5159345	231950	9486	340	7.0272303	215720	814	2.29
2.30	0.00101311393	4927395	222464	9146	331	7.0056583	216534	813	2.30
2.31	0.00096383998	4704931	213318	8815	324	6.9840049	217347	815	2.31
2.32	0.00091679067	4491613	204503	8491	309	6.9622702	218162	815	2.32
2.33	0.00087187454	4287110	196012	8182	303	6.9404540	218977	814	2.33
2.34	0.00082900344	4091098	187830	7879	292	6.9185563	219791	816	2.34
2.35	0.00078809246	3903268	179951	7587	287	6.8965772	220607	815	2.35
2.36	0.00074905978	3723317	172364	7300	271	6.8745165	221422	816	2.36
2.37	0.00071182661	3550953	165064	7029	268	6.8523743	222238	819	2.37
2.38	0.00067631708	3385889	158035	6761	260	6.8301505	223057	813	2.38
2.39	0.00064245819	3227854	151274	6501	245	6.8078448	223870	819	2.39
2.40	0.00061017965	3076580	144773	6256	244	6.7854578	224689	816	2.40
2.41	0.00057941385	2931807	138517	6012	234	6.7629889	225505	818	2.41
2.42	0.00055009578	2793290	132505	5778	226	6.7404384	226323	819	2.42
2.43	0.00052216288	2660785	126727	5552	216	6.7178061	227142	817	2.43
2.44	0.00049555503	2534058	121175	5336	212	6.6950919	227959	818	2.44
2.45	0.00047021445	2412883	115839	5124	204	6.6722960	228777	819	2.45
2.46	0.00044608562	2297044	110715	4920	196	6.6494183	229596	820	2.46
2.47	0.00042311518	2186329	105795	4724	191	6.6264587	230416	818	2.47
2.48	0.00040125189	2080534	101071	4533	183	6.6034171	231234	821	2.48
2.49	0.00038044655	1979463	96538	4350	177	6.5802937	232055	819	2.49
2.50	0.00036065192	1882925	92188	4173	171	6.5570882	232874	818	2.50
2.51	0.00034182267	1790737	88015	4002	164	6.5338008	233692	823	2.51

Theory of
Pro-
babilities.TABLES I. and II. *continued.*Theory of
Pro-
babilities.

t	G	Δ	Δ^2	Δ^3	Δ^4	log G.	Δ	Δ^2	t
2.52	0.00032391530	1702722	84013	3838	160	6.5104316	234515	821	2.52
2.53	0.00030688808	1618709	80175	3678	152	6.4869801	235336	819	2.53
2.54	0.00029070099	1538534	76497	3526	148	6.4634465	236155	823	2.54
2.55	0.00027531565	1462037	72971	3378	142	6.4398310	236978	821	2.55
2.56	0.00026069528	1389066	69593	3236	137	6.4161332	237799	822	2.56
2.57	0.00024680462	1319473	66357	3099	131	6.3923533	238621	821	2.57
2.58	0.00023360989	1253116	63258	2968	128	6.3684912	239442	823	2.58
2.59	0.00022107873	1189858	60290	2840	121	6.3445470	240265	823	2.59
2.60	0.00020918015	1129568	57450	2719	119	6.3205205	241088	823	2.60
2.61	0.00019788447	1072118	54731	2600	112	6.2964117	241911	822	2.61
2.62	0.00018716329	1017387	52131	2488	108	6.2722206	242733	823	2.62
2.63	0.00017698942	965256	49643	2380	105	6.2479473	243556	825	2.63
2.64	0.00016733686	915613	47263	2275	101	6.2235917	244381	823	2.64
2.65	0.00015818073	868350	44988	2174	95	6.1991536	245204	826	2.65
2.66	0.00014949723	823362	42814	2079	94	6.1746332	246030	822	2.66
2.67	0.00014126361	780548	40735	1985	88	6.1500302	246852	825	2.67
2.68	0.00013345813	739813	38750	1897	85	6.1253450	247677	824	2.68
2.69	0.00012606000	701063	36853	1812	83	6.1005773	248501	827	2.69
2.70	0.00011904937	664210	35041	1729	78	6.0757272	249328	825	2.70
2.71	0.00011240727	629169	33312	1651	76	6.0507944	250153	825	2.71
2.72	0.00010611558	595857	31661	1575	71	6.0257791	250978	826	2.72
2.73	0.00010015701	564196	30086	1504	70	6.0006813	251804	824	2.73
2.74	0.00009451505	534110	28582	1434	67	5.9755009	252628	829	2.74
2.75	0.00008917395	505528	27148	1367	64	5.9502381	253457	825	2.75
2.76	0.00008411867	478380	25781	1303	59	5.9248924	254282	828	2.76
2.77	0.00007933487	452599	24478	1244	60	5.8994642	255110	827	2.77
2.78	0.00007480888	428121	23234	1184	56	5.8739532	255937	826	2.78
2.79	0.00007052767	404887	22050	1128	53	5.8483595	256763	828	2.79
2.80	0.00006647880	382837	20922	1075	51	5.8226832	257591	828	2.80
2.81	0.00006265043	361915	19847	1024	49	5.7969241	258419	827	2.81
2.82	0.00005903128	342068	18823	975	48	5.7710822	259246	829	2.82
2.83	0.00005561060	323245	17848	927	43	5.7451576	260075	827	2.83
2.84	0.00005237815	305397	16921	884	45	5.7191501	260902	828	2.84
2.85	0.00004932418	288476	16037	839	39	5.6930599	261730	831	2.85
2.86	0.00004643942	272439	15198	800	40	5.6668869	262561	827	2.86
2.87	0.00004371503	257241	14398	760	38	5.6406308	263388	831	2.87
2.88	0.00004114262	242843	13638	722	34	5.6142920	264219	827	2.88
2.89	0.00003871419	229205	12916	688	36	5.5878701	265046	831	2.89
2.90	0.00003642214	216289	12228	652	31	5.5613655	265877	829	2.90
2.91	0.00003425925	204061	11576	621	32	5.5347778	266706	829	2.91
2.92	0.00003221864	192485	10955	589	29	5.5081072	267535	832	2.92
2.93	0.00003029379	181530	10366	560	29	5.4813537	268367	830	2.93
2.94	0.00002847849	171164	9806	531	27	5.4545170	269197	831	2.94
2.95	0.00002676685	161358	9275	504	25	5.4275973	270028	831	2.95
2.96	0.00002515327	152083	8771	479	24	5.4005945	270859	830	2.96
2.97	0.00002363244	143312	8292	455		5.3735086	271689	831	2.97
2.98	0.00002219932	135020	7837			5.3463397	272520	832	2.98
2.99	0.00002084912	127183				5.3190877	273352		2.99
3.00	0.00001957729					5.2917525			3.00

Theory of
Pro-
babilities.TABLE III.—Values of $\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt = H$, for intervals of t each $= .01$, from $t=0$ to $t=2$.Theory of
Pro-
babilities.

t	H	Δ	Δ^2	t	H	Δ	Δ^2	t	H	Δ	Δ^2
0.00	0.00000 00	1128 33	22	0.55	0.56332 33	829 24	9 23	1.10	0.88020 50	332 80	7 31
0.01	0.01128 33	1128 11	45	0.56	0.57161 57	820 01	9 30	1.11	0.88353 30	325 49	7 21
0.02	0.02256 44	1127 66	67	0.57	0.57981 58	810 71	9 35	1.12	0.88678 79	318 28	7 12
0.03	0.03384 10	1126 99	90	0.58	0.58792 29	801 36	9 40	1.13	0.88997 07	311 16	7 01
0.04	0.04511 09	1126 09	1 12	0.59	0.59593 65	791 96	9 45	1.14	0.89308 23	304 15	6 91
0.05	0.05637 18	1124 97	1 35	0.60	0.60385 61	782 51	9 49	1.15	0.89612 38	297 24	6 82
0.06	0.06762 15	1123 62	1 58	0.61	0.61168 12	773 02	9 53	1.16	0.89909 62	290 42	6 72
0.07	0.07885 77	1122 04	1 79	0.62	0.61941 14	763 49	9 55	1.17	0.90200 04	283 70	6 61
0.08	0.09007 81	1120 25	2 01	0.63	0.62704 63	753 94	9 59	1.18	0.90483 74	277 09	6 52
0.09	0.10128 06	1118 24	2 24	0.64	0.63458 57	744 35	9 62	1.19	0.90760 83	270 57	6 42
0.10	0.11246 30	1116 00	2 46	0.65	0.64202 92	734 73	9 63	1.20	0.91031 40	264 15	6 31
0.11	0.12362 30	1113 54	2 67	0.66	0.64937 65	725 10	9 65	1.21	0.91295 55	257 84	6 22
0.12	0.13475 84	1110 87	2 88	0.67	0.65662 75	715 45	9 66	1.22	0.91553 39	251 62	6 11
0.13	0.14586 71	1107 99	3 10	0.68	0.66378 20	705 79	9 68	1.23	0.91805 01	245 51	6 02
0.14	0.15694 70	1104 89	3 31	0.69	0.67083 99	696 11	9 67	1.24	0.92050 52	239 49	5 91
0.15	0.16799 59	1101 58	3 52	0.70	0.67780 10	686 44	9 68	1.25	0.92290 01	233 58	5 81
0.16	0.17901 17	1098 06	3 72	0.71	0.68466 54	676 76	9 68	1.26	0.92523 59	227 77	5 71
0.17	0.18999 23	1094 34	3 93	0.72	0.69143 30	667 08	9 66	1.27	0.92751 36	222 06	5 61
0.18	0.20093 57	1090 41	4 14	0.73	0.69810 38	657 42	9 66	1.28	0.92973 42	216 45	5 52
0.19	0.21183 98	1086 27	4 34	0.74	0.70467 80	647 76	9 65	1.29	0.93189 87	210 93	5 41
0.20	0.22270 25	1081 93	4 53	0.75	0.71115 56	638 11	9 62	1.30	0.93400 80	205 52	5 32
0.21	0.23352 18	1077 40	4 73	0.76	0.71753 67	628 49	9 61	1.31	0.93606 32	200 20	5 22
0.22	0.24429 58	1072 67	4 92	0.77	0.72382 16	618 88	9 57	1.32	0.93806 52	194 98	5 11
0.23	0.25502 25	1067 75	5 12	0.78	0.73001 04	609 31	9 56	1.33	0.94001 50	189 87	5 02
0.24	0.26570 00	1062 63	5 29	0.79	0.73610 35	599 75	9 52	1.34	0.94191 37	184 85	4 93
0.25	0.27632 63	1057 34	5 49	0.80	0.74210 10	590 23	9 48	1.35	0.94376 22	179 92	4 82
0.26	0.28689 97	1051 85	5 67	0.81	0.74800 33	580 75	9 45	1.36	0.94556 14	175 10	4 74
0.27	0.29741 82	1046 18	5 84	0.82	0.75381 08	571 30	9 41	1.37	0.94731 24	170 36	4 63
0.28	0.30788 00	1040 34	6 01	0.83	0.75952 38	561 89	9 36	1.38	0.94901 60	165 73	4 55
0.29	0.31828 34	1034 33	6 19	0.84	0.76514 27	552 53	9 31	1.39	0.95067 33	161 18	4 45
0.30	0.32862 67	1028 14	6 36	0.85	0.77066 80	543 22	9 26	1.40	0.95228 51	156 73	4 35
0.31	0.33890 81	1021 78	6 52	0.86	0.77610 02	533 96	9 21	1.41	0.95385 24	152 38	4 27
0.32	0.34912 59	1015 26	6 67	0.87	0.78143 98	524 75	9 16	1.42	0.95537 62	148 11	4 18
0.33	0.35927 85	1008 59	6 84	0.88	0.78668 73	515 59	9 09	1.43	0.95685 73	143 93	4 09
0.34	0.36936 44	1001 75	6 98	0.89	0.79184 32	506 50	9 04	1.44	0.95829 66	139 84	3 99
0.35	0.37938 19	994 77	7 14	0.90	0.79690 82	497 46	8 97	1.45	0.95969 50	135 85	3 91
0.36	0.38932 96	987 63	7 29	0.91	0.80188 28	488 49	8 91	1.46	0.96105 35	131 94	3 82
0.37	0.39920 59	980 34	7 42	0.92	0.80676 77	479 58	8 83	1.47	0.96237 29	128 12	3 74
0.38	0.40900 93	972 92	7 55	0.93	0.81156 35	470 75	8 77	1.48	0.96365 41	124 38	3 65
0.39	0.41873 85	965 37	7 69	0.94	0.81627 10	461 98	8 70	1.49	0.96489 79	120 73	3 57
0.40	0.42839 22	957 68	7 82	0.95	0.82089 08	453 28	8 61	1.50	0.96610 52	117 16	3 49
0.41	0.43796 90	949 86	7 95	0.96	0.82542 36	444 67	8 55	1.51	0.96727 68	113 67	3 40
0.42	0.44746 76	941 91	8 07	0.97	0.82987 03	436 12	8 46	1.52	0.96841 35	110 27	3 32
0.43	0.45688 67	933 84	8 17	0.98	0.83423 15	427 66	8 39	1.53	0.96951 62	106 95	3 25
0.44	0.46622 51	925 67	8 30	0.99	0.83850 81	419 27	8 30	1.54	0.97058 57	103 70	3 16
0.45	0.47548 18	917 37	8 40	1.00	0.84270 08	410 97	8 22	1.55	0.97162 27	100 54	3 09
0.46	0.48465 55	908 97	8 51	1.01	0.84681 05	402 75	8 13	1.56	0.97262 81	97 45	3 01
0.47	0.49374 52	900 46	8 61	1.02	0.85083 80	394 62	8 05	1.57	0.97360 26	94 44	2 94
0.48	0.50274 98	891 85	8 69	1.03	0.85478 42	386 57	7 96	1.58	0.97454 70	91 50	2 86
0.49	0.51166 83	883 16	8 78	1.04	0.85864 99	378 61	7 86	1.59	0.97546 20	88 64	2 79
0.50	0.52049 99	874 38	8 88	1.05	0.86243 60	370 75	7 77	1.60	0.97634 84	85 85	2 73
0.51	0.52924 37	865 50	8 96	1.06	0.86614 35	362 97	7 68	1.61	0.97720 69	83 12	2 64
0.52	0.53789 87	856 54	9 03	1.07	0.86977 32	355 29	7 60	1.62	0.97803 81	80 48	2 59
0.53	0.54646 41	847 51	9 10	1.08	0.87332 61	347 69	7 49	1.63	0.97884 29	77 89	2 51
0.54	0.55493 92	838 41	9 17	1.09	0.87680 30	340 20	7 40	1.64	0.97962 18	75 38	2 45

Theory of
Pro-
babilities.TABLE III. *continued.*Theory of
Pro-
babilities.

<i>t</i>	H	Δ	Δ ²	<i>t</i>	H	Δ	Δ ²	<i>t</i>	H	Δ	Δ ²
1.65	0.98037 56	72 93	2 38	1.77	0.98769 10	48 32	1 68	1.89	0.99247 93	31 11	1 16
1.66	0.98110 49	70 55	2 31	1.78	0.98817 42	46 64	1 65	1.90	0.99279 04	29 95	1 12
1.67	0.98181 04	68 24	2 26	1.79	0.98864 06	44 99	1 59	1.91	0.99308 99	28 83	1 08
1.68	0.98249 28	65 98	2 20	1.80	0.98909 05	43 40	1 54	1.92	0.99337 82	27 75	1 06
1.69	0.98315 26	63 78	2 12	1.81	0.98952 45	41 86	1 50	1.93	0.99365 57	26 69	1 01
1.70	0.98379 04	61 66	2 08	1.82	0.98994 31	40 36	1 44	1.94	0.99392 26	25 68	99
1.71	0.98440 70	59 58	2 01	1.83	0.99034 67	38 92	1 41	1.95	0.99417 94	24 69	95
1.72	0.98500 28	57 57	1 96	1.84	0.99073 59	37 51	1 36	1.96	0.99442 63	23 74	91
1.73	0.98557 85	55 61	1 90	1.85	0.99111 10	36 15	1 33	1.97	0.99466 37	22 83	89
1.74	0.98613 46	53 71	1 85	1.86	0.99147 25	34 82	1 27	1.98	0.99489 20	21 94	85
1.75	0.98667 17	51 86	1 79	1.87	0.99182 07	33 55	1 24	1.99	0.99511 14	21 09	82
1.76	0.98719 03	50 07	1 75	1.88	0.99215 62	32 31	1 20	2.00	0.99532 23		

TABLE IV.—Values of $\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt = K$ (where $\rho = .4769360$, the solution of $\frac{2}{\sqrt{\pi}} \int_0^t e^{-t^2} dt = \frac{1}{2}$), for intervals of t each = .01, from $t=0$ to $t=3.40$, and each = .10, from $t=3.40$ to $t=5$.

<i>t</i>	K	Δ	<i>t</i>	K	Δ	<i>t</i>	K	Δ	<i>t</i>	K	Δ
0.00	0.00000	538	0.35	0.18662	523	0.70	0.36317	481	1.05	0.52119	418
0.01	0.00538	538	0.36	0.19185	522	0.71	0.36798	479	1.06	0.52537	415
0.02	0.01076	538	0.37	0.19707	522	0.72	0.37277	478	1.07	0.52952	414
0.03	0.01614	538	0.38	0.20229	520	0.73	0.37755	476	1.08	0.53366	412
0.04	0.02152	538	0.39	0.20749	519	0.74	0.38231	474	1.09	0.53778	410
0.05	0.02690	538	0.40	0.21268	519	0.75	0.38705	473	1.10	0.54188	407
0.06	0.03228	538	0.41	0.21787	517	0.76	0.39178	471	1.11	0.54595	406
0.07	0.03766	537	0.42	0.22304	517	0.77	0.39649	469	1.12	0.55001	403
0.08	0.04303	537	0.43	0.22821	515	0.78	0.40118	468	1.13	0.55404	402
0.09	0.04840	538	0.44	0.23336	515	0.79	0.40586	466	1.14	0.55806	399
0.10	0.05378	536	0.45	0.23851	513	0.80	0.41052	465	1.15	0.56205	397
0.11	0.05914	537	0.46	0.24364	512	0.81	0.41517	462	1.16	0.56602	396
0.12	0.06451	536	0.47	0.24876	512	0.82	0.41979	461	1.17	0.56998	393
0.13	0.06987	536	0.48	0.25388	510	0.83	0.42440	459	1.18	0.57391	391
0.14	0.07523	536	0.49	0.25898	509	0.84	0.42899	458	1.19	0.57782	389
0.15	0.08059	535	0.50	0.26407	508	0.85	0.43357	456	1.20	0.58171	387
0.16	0.08594	535	0.51	0.26915	506	0.86	0.43813	454	1.21	0.58558	384
0.17	0.09129	534	0.52	0.27421	506	0.87	0.44267	452	1.22	0.58942	383
0.18	0.09663	534	0.53	0.27927	504	0.88	0.44719	450	1.23	0.59325	380
0.19	0.10197	534	0.54	0.28431	503	0.89	0.45169	449	1.24	0.59705	378
0.20	0.10731	533	0.55	0.28934	502	0.90	0.45618	446	1.25	0.60083	377
0.21	0.11264	532	0.56	0.29436	500	0.91	0.46064	445	1.26	0.60460	373
0.22	0.11796	532	0.57	0.29936	499	0.92	0.46509	443	1.27	0.60833	372
0.23	0.12328	532	0.58	0.30435	498	0.93	0.46952	441	1.28	0.61205	370
0.24	0.12860	531	0.59	0.30933	497	0.94	0.47393	439	1.29	0.61575	367
0.25	0.13391	530	0.60	0.31430	495	0.95	0.47832	438	1.30	0.61942	366
0.26	0.13921	530	0.61	0.31925	494	0.96	0.48270	435	1.31	0.62308	363
0.27	0.14451	529	0.62	0.32419	492	0.97	0.48705	434	1.32	0.62671	361
0.28	0.14980	528	0.63	0.32911	491	0.98	0.49139	431	1.33	0.63032	359
0.29	0.15508	527	0.64	0.33402	490	0.99	0.49570	430	1.34	0.63391	356
0.30	0.16035	527	0.65	0.33892	488	1.00	0.50000	428	1.35	0.63747	355
0.31	0.16562	526	0.66	0.34380	486	1.01	0.50428	425	1.36	0.64102	352
0.32	0.17088	526	0.67	0.34866	486	1.02	0.50853	424	1.37	0.64454	350
0.33	0.17614	524	0.68	0.35352	483	1.03	0.51277	422	1.38	0.64804	348
0.34	0.18138	524	0.69	0.35835	482	1.04	0.51699	420	1.39	0.65152	346

TABLE IV. *continued.*Theory of
Pro-
babilities.Theory of
Pro-
babilities.

<i>t</i>	K	Δ	<i>t</i>	K	Δ	<i>t</i>	K	Δ	<i>t</i>	K	Δ
1.40	0.65498	343	1.95	0.81158	225	2.50	0.90825	129	3.05	0.96033	65
1.41	0.65841	341	1.96	0.81383	224	2.51	0.90954	128	3.06	0.96098	63
1.42	0.66182	339	1.97	0.81607	221	2.52	0.91082	126	3.07	0.96161	63
1.43	0.66521	337	1.98	0.81828	220	2.53	0.91208	124	3.08	0.96224	62
1.44	0.66858	335	1.99	0.82048	218	2.54	0.91332	124	3.09	0.96286	60
1.45	0.67193	333	2.00	0.82266	215	2.55	0.91456	122	3.10	0.96346	60
1.46	0.67526	330	2.01	0.82481	214	2.56	0.91578	120	3.11	0.96406	60
1.47	0.67856	328	2.02	0.82695	212	2.57	0.91698	119	3.12	0.96466	58
1.48	0.68184	326	2.03	0.82907	210	2.58	0.91817	118	3.13	0.96524	58
1.49	0.68510	323	2.04	0.83117	207	2.59	0.91935	116	3.14	0.96582	56
1.50	0.68833	322	2.05	0.83324	206	2.60	0.92051	115	3.15	0.96638	56
1.51	0.69155	319	2.06	0.83530	204	2.61	0.92166	114	3.16	0.96694	55
1.52	0.69474	317	2.07	0.83734	202	2.62	0.92280	112	3.17	0.96749	55
1.53	0.69791	315	2.08	0.83936	201	2.63	0.92392	111	3.18	0.96804	53
1.54	0.70106	313	2.09	0.84137	198	2.64	0.92503	110	3.19	0.96857	53
1.55	0.70419	310	2.10	0.84335	196	2.65	0.92613	108	3.20	0.96910	52
1.56	0.70729	309	2.11	0.84531	195	2.66	0.92721	107	3.21	0.96962	51
1.57	0.71038	306	2.12	0.84726	193	2.67	0.92828	106	3.22	0.97013	51
1.58	0.71344	304	2.13	0.84919	190	2.68	0.92934	104	3.23	0.97064	50
1.59	0.71648	301	2.14	0.85109	189	2.69	0.93038	103	3.24	0.97114	49
1.60	0.71949	300	2.15	0.85298	188	2.70	0.93141	102	3.25	0.97163	48
1.61	0.72249	297	2.16	0.85486	185	2.71	0.93243	101	3.26	0.97211	48
1.62	0.72546	295	2.17	0.85671	183	2.72	0.93344	99	3.27	0.97259	47
1.63	0.72841	293	2.18	0.85854	182	2.73	0.93443	98	3.28	0.97306	46
1.64	0.73134	291	2.19	0.86036	180	2.74	0.93541	97	3.29	0.97352	45
1.65	0.73425	289	2.20	0.86216	178	2.75	0.93638	96	3.30	0.97397	45
1.66	0.73714	286	2.21	0.86394	176	2.76	0.93734	94	3.31	0.97442	44
1.67	0.74000	285	2.22	0.86570	175	2.77	0.93828	94	3.32	0.97486	44
1.68	0.74285	282	2.23	0.86745	172	2.78	0.93922	92	3.33	0.97530	43
1.69	0.74567	280	2.24	0.86917	171	2.79	0.94014	91	3.34	0.97573	42
1.70	0.74847	277	2.25	0.87088	170	2.80	0.94105	90	3.35	0.97615	42
1.71	0.75124	276	2.26	0.87258	167	2.81	0.94195	89	3.36	0.97657	41
1.72	0.75400	274	2.27	0.87425	166	2.82	0.94284	87	3.37	0.97698	40
1.73	0.75674	271	2.28	0.87591	164	2.83	0.94371	87	3.38	0.97738	40
1.74	0.75945	269	2.29	0.87755	163	2.84	0.94458	85	3.39	0.97778	39
1.75	0.76214	267	2.30	0.87918	160	2.85	0.94543	84	3.40	0.97817	359
1.76	0.76481	265	2.31	0.88078	159	2.86	0.94627	84	3.50	0.98176	306
1.77	0.76746	263	2.32	0.88237	158	2.87	0.94711	82	3.60	0.98482	261
1.78	0.77009	261	2.33	0.88395	155	2.88	0.94793	81	3.70	0.98743	219
1.79	0.77270	258	2.34	0.88550	155	2.89	0.94874	80	3.80	0.98962	185
1.80	0.77528	257	2.35	0.88705	152	2.90	0.94954	79	3.90	0.99147	155
1.81	0.77785	254	2.36	0.88857	151	2.91	0.95033	78	4.00	0.99302	129
1.82	0.78039	252	2.37	0.89008	149	2.92	0.95111	76	4.10	0.99431	108
1.83	0.78291	251	2.38	0.89157	147	2.93	0.95187	76	4.20	0.99539	88
1.84	0.78542	248	2.39	0.89304	146	2.94	0.95263	75	4.30	0.99627	73
1.85	0.78790	246	2.40	0.89450	145	2.95	0.95338	74	4.40	0.99700	60
1.86	0.79036	244	2.41	0.89595	143	2.96	0.95412	73	4.50	0.99760	48
1.87	0.79280	242	2.42	0.89738	141	2.97	0.95485	72	4.60	0.99808	40
1.88	0.79522	239	2.43	0.89879	140	2.98	0.95557	71	4.70	0.99848	31
1.89	0.79761	238	2.44	0.90019	138	2.99	0.95628	70	4.80	0.99879	26
1.90	0.79999	236	2.45	0.90157	136	3.00	0.95698	69	4.90	0.99905	21
1.91	0.80235	234	2.46	0.90293	135	3.01	0.95767	68	5.00	0.99926	
1.92	0.80469	231	2.47	0.90428	134	3.02	0.95835	67			
1.93	0.80700	230	2.48	0.90562	132	3.03	0.95902	66			
1.94	0.80930	228	2.49	0.90694	131	3.04	0.95968	65			

Theory of Pro-
babilities. TABLE V.—Logarithms of the products $1.2.3\dots x$ for every value up to $x=1200$. (N.B. The last decimal place is not corrected.) Theory of Pro-
babilities.

x	$\log \Gamma(x+1).$	x	$\log \Gamma(x+1).$	x	$\log \Gamma(x+1).$	x	$\log \Gamma(x+1).$	x	$\log \Gamma(x+1).$
1	0.000000	56	74.851868	111	180.246240	166	297.954420	221	423.703061
2	0.301029	57	76.607743	112	182.295458	167	300.177137	222	426.049414
3	0.778151	58	78.371171	113	184.348537	168	302.402446	223	428.397719
4	1.380211	59	80.142023	114	186.405441	169	304.630333	224	430.747967
5	2.079181	60	81.920174	115	188.466139	170	306.860781	225	433.100150
6	2.857332	61	83.705504	116	190.530597	171	309.093778	226	435.454258
7	3.702430	62	85.497896	117	192.598783	172	311.329306	227	437.810284
8	4.605520	63	87.297236	118	194.670665	173	313.567352	228	440.168219
9	5.559763	64	89.103416	119	196.746212	174	315.807901	229	442.528054
10	6.559763	65	90.916330	120	198.825393	175	318.050939	230	444.889782
11	7.601155	66	92.735874	121	200.908179	176	320.296452	231	447.253394
12	8.680336	67	94.561948	122	202.994539	177	322.544425	232	449.618882
13	9.794280	68	96.394457	123	205.084444	178	324.794845	233	451.986238
14	10.940408	69	98.233306	124	207.177865	179	327.047698	234	454.355454
15	12.116499	70	100.078405	125	209.274775	180	329.302971	235	456.726522
16	13.320619	71	101.929663	126	211.375146	181	331.560649	236	459.099434
17	14.551068	72	103.786995	127	213.478950	182	333.820721	237	461.474182
18	15.806341	73	105.650318	128	215.586160	183	336.083172	238	463.850759
19	17.085094	74	107.519550	129	217.696749	184	338.347990	239	466.229157
20	18.386124	75	109.394611	130	219.810693	185	340.615162	240	468.609368
21	19.708343	76	111.275425	131	221.927964	186	342.884674	241	470.991385
22	21.050766	77	113.161916	132	224.048538	187	345.156516	242	473.375201
23	22.412494	78	115.054010	133	226.172390	188	347.430674	243	475.760807
24	23.792705	79	116.951637	134	228.299494	189	349.707136	244	478.148197
25	25.190645	80	118.854727	135	230.429828	190	351.985889	245	480.537363
26	26.605619	81	120.763212	136	232.563367	191	354.266923	246	482.928298
27	28.036982	82	122.677026	137	234.700088	192	356.550224	247	485.320995
28	29.484140	83	124.596104	138	236.839967	193	358.835781	248	487.715447
29	30.946538	84	126.520383	139	238.982981	194	361.123583	249	490.111646
30	32.423660	85	128.449802	140	241.129109	195	363.413618	250	492.509586
31	33.915021	86	130.384301	141	243.278329	196	365.705874	251	494.909260
32	35.420171	87	132.323820	142	245.430617	197	368.000340	252	497.310660
33	36.938685	88	134.268303	143	247.585953	198	370.297005	253	499.713781
34	38.470164	89	136.217693	144	249.744315	199	372.595858	254	502.118614
35	40.014232	90	138.171935	145	251.905683	200	374.896888	255	504.525155
36	41.570535	91	140.130977	146	254.070036	201	377.200084	256	506.933395
37	43.138736	92	142.094765	147	256.237354	202	379.505436	257	509.343328
38	44.718520	93	144.063247	148	258.407615	203	381.812932	258	511.754947
39	46.309585	94	146.036375	149	260.580802	204	384.122562	259	514.168247
40	47.911645	95	148.014099	150	262.756893	205	386.434316	260	516.583220
41	49.524428	96	149.996370	151	264.935870	206	388.748183	261	518.999861
42	51.147678	97	151.983142	152	267.117713	207	391.064153	262	521.418162
43	52.781146	98	153.974368	153	269.302405	208	393.382217	263	523.838118
44	54.424599	99	155.970003	154	271.489926	209	395.702363	264	526.259722
45	56.077811	100	157.970003	155	273.680257	210	398.024582	265	528.682968
46	57.740569	101	159.974325	156	275.873382	211	400.348865	266	531.107849
47	59.412667	102	161.982925	157	278.069282	212	402.675200	267	533.534361
48	61.093908	103	163.995762	158	280.267939	213	405.003580	268	535.962496
49	62.784104	104	166.012795	159	282.469336	214	407.333994	269	538.392248
50	64.483074	105	168.033985	160	284.673456	215	409.666432	270	540.823612
51	66.190645	106	170.059290	161	286.880282	216	412.000886	271	543.256581
52	67.906648	107	172.088674	162	289.089797	217	414.337346	272	545.691150
53	69.630924	108	174.122098	163	291.301984	218	416.675802	273	548.127312
54	71.363318	109	176.159524	164	293.516828	219	419.016246	274	550.565063
55	73.103680	110	178.200917	165	295.734312	220	421.358669	275	553.004396

Theory of
Pro-
babilities.TABLE V. *continued.*Theory of
Pro-
babilities.

x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$
276	555.445305	331	691.970705	386	832.477506	441	976.394942	496	1123.295752
277	557.887785	332	694.491843	387	835.065217	442	979.040365	497	1125.992108
278	560.331829	333	697.014288	388	837.654049	443	981.686768	498	1128.689337
279	562.777434	334	699.538034	389	840.243999	444	984.334151	499	1131.387438
280	565.224592	335	702.063079	390	842.835063	445	986.982511	500	1134.086408
281	567.673298	336	704.589418	391	845.427240	446	989.631846	501	1136.786246
282	570.123547	337	707.117048	392	848.020526	447	992.282154	502	1139.486949
283	572.575333	338	709.645965	393	850.614919	448	994.933432	503	1142.188517
284	575.028652	339	712.176164	394	853.210415	449	997.585678	504	1144.890948
285	577.483497	340	714.707643	395	855.807012	450	1000.238890	505	1147.594239
286	579.939863	341	717.240398	396	858.404707	451	1002.893067	506	1150.298390
287	582.397745	342	719.774424	397	861.003498	452	1005.548205	507	1153.003398
288	584.857137	343	722.309718	398	863.603381	453	1008.204304	508	1155.709262
289	587.318035	344	724.846276	399	866.204354	454	1010.861359	509	1158.415979
290	589.780433	345	727.384095	400	868.806414	455	1013.519371	510	1161.123550
291	592.244326	346	729.923172	401	871.409558	456	1016.178336	511	1163.831970
292	594.709709	347	732.463501	402	874.013784	457	1018.838252	512	1166.541240
293	597.176576	348	735.005080	403	876.619089	458	1021.499117	513	1169.251358
294	599.644924	349	737.547906	404	879.225471	459	1024.160930	514	1171.962321
295	602.114746	350	740.091974	405	881.832926	460	1026.823688	515	1174.674128
296	604.586037	351	742.637281	406	884.441452	461	1029.487389	516	1177.386778
297	607.058794	352	745.183823	407	887.051046	462	1032.152031	517	1180.100268
298	609.533010	353	747.731598	408	889.661706	463	1034.817612	518	1182.814598
299	612.008681	354	750.280601	409	892.273429	464	1037.484130	519	1185.529765
300	614.485803	355	752.830830	410	894.886213	465	1040.151583	520	1188.245769
301	616.964369	356	755.382280	411	897.500055	466	1042.819969	521	1190.962607
302	619.444376	357	757.934948	412	900.114952	467	1045.489286	522	1193.680277
303	621.925819	358	760.488831	413	902.730902	468	1048.159531	523	1196.398779
304	624.408692	359	763.043926	414	905.347903	469	1050.830704	524	1199.118110
305	626.892992	360	765.600228	415	907.965951	470	1053.502802	525	1201.838269
306	629.378713	361	768.157735	416	910.585044	471	1056.175823	526	1204.559255
307	631.865852	362	770.716444	417	913.205180	472	1058.849765	527	1207.281066
308	634.354403	363	773.276350	418	915.826357	473	1061.524626	528	1210.003700
309	636.844361	364	775.837452	419	918.448571	474	1064.200404	529	1212.727155
310	639.335723	365	778.399745	420	921.071820	475	1066.877098	530	1215.451431
311	641.828483	366	780.963226	421	923.696102	476	1069.554705	531	1218.176526
312	644.322638	367	783.527892	422	926.321414	477	1072.233223	532	1220.902437
313	646.818182	368	786.093740	423	928.947755	478	1074.912651	533	1223.629164
314	649.315112	369	788.660766	424	931.575121	479	1077.592987	534	1226.356706
315	651.813422	370	791.228968	425	934.203510	480	1080.274228	535	1229.085060
316	654.313109	371	793.798342	426	936.832919	481	1082.956373	536	1231.814224
317	656.814169	372	796.368885	427	939.463347	482	1085.639420	537	1234.544199
318	659.316596	373	798.940593	428	942.094791	483	1088.323367	538	1237.274981
319	661.820386	374	801.513465	429	944.727248	484	1091.008213	539	1240.006570
320	664.325536	375	804.087496	430	947.360717	485	1093.693954	540	1242.738963
321	666.832041	376	806.662684	431	949.995194	486	1096.380591	541	1245.472161
322	669.339897	377	809.239025	432	952.630678	487	1099.068120	542	1248.206160
323	671.849100	378	811.816517	433	955.267165	488	1101.756539	543	1250.940960
324	674.359645	379	814.395156	434	957.904655	489	1104.445848	544	1253.676559
325	676.871528	380	816.974940	435	960.543144	490	1107.136044	545	1256.412955
326	679.384746	381	819.555865	436	963.182631	491	1109.827126	546	1259.150148
327	681.899294	382	822.137928	437	965.823112	492	1112.519091	547	1261.888135
328	684.415167	383	824.721127	438	968.464586	493	1115.211938	548	1264.626916
329	686.932363	384	827.305458	439	971.107051	494	1117.905665	549	1267.366488
330	689.450877	385	829.890919	440	973.750504	495	1120.600270	550	1270.106851

TABLE V. *continued.*Theory of
Pro-
babilities.Theory of
Pro-
babilities.

x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$
551	1272.848002	606	1424.786340	661	1578.893757	716	1734.989471	771	1892.920543
552	1275.589941	607	1427.569528	662	1581.714615	717	1737.844991	772	1895.808161
553	1278.332667	608	1430.353432	663	1584.536129	718	1740.701115	773	1898.696340
554	1281.076176	609	1433.138049	664	1587.358297	719	1743.557844	774	1901.585081
555	1283.820469	610	1435.923379	665	1590.181118	720	1746.415176	775	1904.474383
556	1286.565544	611	1438.709420	666	1593.004593	721	1749.273112	776	1907.364245
557	1289.311399	612	1441.496172	667	1595.828718	722	1752.131649	777	1910.254666
558	1292.058033	613	1444.283632	668	1598.653495	723	1754.990787	778	1913.145645
559	1294.805445	614	1447.071801	669	1601.478921	724	1757.850526	779	1916.037183
560	1297.553633	615	1449.860676	670	1604.304996	725	1760.710864	780	1918.929277
561	1300.302596	616	1452.650256	671	1607.131718	726	1763.571800	781	1921.821928
562	1303.052333	617	1455.440542	672	1609.959088	727	1766.433335	782	1924.715135
563	1305.802841	618	1458.231530	673	1612.787103	728	1769.295466	783	1927.608897
564	1308.554120	619	1461.023221	674	1615.615763	729	1772.158194	784	1930.503213
565	1311.306168	620	1463.815612	675	1618.445066	730	1775.021517	785	1933.398083
566	1314.058985	621	1466.608704	676	1621.275013	731	1777.885434	786	1936.293505
567	1316.812568	622	1469.402494	677	1624.105602	732	1780.749945	787	1939.189480
568	1319.566916	623	1472.196982	678	1626.936831	733	1783.615049	788	1942.086006
569	1322.322029	624	1474.992167	679	1629.768701	734	1786.480745	789	1944.983083
570	1325.077903	625	1477.788047	680	1632.601210	735	1789.347032	790	1947.880710
571	1327.834540	626	1480.584621	681	1635.434357	736	1792.213910	791	1950.778887
572	1330.591936	627	1483.381889	682	1638.268142	737	1795.081378	792	1953.677612
573	1333.350090	628	1486.179849	683	1641.102562	738	1797.949434	793	1956.576885
574	1336.109002	629	1488.978499	684	1643.937618	739	1800.818078	794	1959.476706
575	1338.868670	630	1491.777840	685	1646.773309	740	1803.687310	795	1962.377073
576	1341.629092	631	1494.577869	686	1649.609633	741	1806.557128	796	1965.277986
577	1344.390268	632	1497.378586	687	1652.446590	742	1809.427532	797	1968.179444
578	1347.152196	633	1500.179990	688	1655.284178	743	1812.298521	798	1971.081447
579	1349.914875	634	1502.982079	689	1658.122397	744	1815.170094	799	1973.983994
580	1352.678303	635	1505.784853	690	1660.961247	745	1818.042250	800	1976.887084
581	1355.442479	636	1508.588310	691	1663.800725	746	1820.914989	801	1979.790716
582	1358.207402	637	1511.392449	692	1666.640831	747	1823.788310	802	1982.694891
583	1360.973070	638	1514.197270	693	1669.481564	748	1826.662211	803	1985.599606
584	1363.739483	639	1517.002771	694	1672.322923	749	1829.536693	804	1988.504862
585	1366.506639	640	1519.808951	695	1675.164908	750	1832.411754	805	1991.410658
586	1369.274537	641	1522.615809	696	1678.007517	751	1835.287394	806	1994.316993
587	1372.043175	642	1525.423344	697	1680.850750	752	1838.163612	807	1997.223867
588	1374.812552	643	1528.231555	698	1683.694606	753	1841.040407	808	2000.131278
589	1377.582667	644	1531.040441	699	1686.539083	754	1843.917779	809	2003.039227
590	1380.353519	645	1533.850001	700	1689.384181	755	1846.795725	810	2005.947712
591	1383.125107	646	1536.660233	701	1692.229899	756	1849.674247	811	2008.856732
592	1385.897429	647	1539.471137	702	1695.076236	757	1852.553343	812	2011.766288
593	1388.670483	648	1542.282712	703	1697.923191	758	1855.433012	813	2014.676379
594	1391.444270	649	1545.094957	704	1700.770764	759	1858.313254	814	2017.587003
595	1394.218787	650	1547.907870	705	1703.618953	760	1861.194068	815	2020.498161
596	1396.994033	651	1550.721451	706	1706.467758	761	1864.075452	816	2023.409851
597	1399.770007	652	1553.535699	707	1709.317177	762	1866.957407	817	2026.322073
598	1402.546708	653	1556.350612	708	1712.167210	763	1869.839932	818	2029.234827
599	1405.324135	654	1559.166190	709	1715.017857	764	1872.723025	819	2032.148110
600	1408.102286	655	1561.982431	710	1717.869115	765	1875.606687	820	2035.061924
601	1410.881161	656	1564.799335	711	1720.720985	766	1878.490915	821	2037.976267
602	1413.660757	657	1567.616900	712	1723.573465	767	1881.375711	822	2040.891139
603	1416.441075	658	1570.435126	713	1726.426554	768	1884.261072	823	2043.806539
604	1419.222112	659	1573.254012	714	1729.280252	769	1887.146998	824	2046.722466
605	1422.003867	660	1576.073556	715	1732.134558	770	1890.033489	825	2049.638920

TABLE V. *continued.*

x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$	x	$\log \Gamma(x+1)$
826	2052.555900	881	2213.781955	936	2376.499325	991	2540.620323	1046	2706.067004
827	2055.473406	882	2216.727424	937	2379.471065	992	2543.616834	1047	2709.086951
828	2058.391436	883	2219.673385	938	2382.443268	993	2546.613784	1048	2712.107312
829	2061.309991	884	2222.619837	939	2385.415933	994	2549.611170	1049	2715.128087
830	2064.229069	885	2225.566780	940	2388.389061	995	2552.608993	1050	2718.149277
831	2067.148670	886	2228.514214	941	2391.362651	996	2555.607253	1051	2721.170879
832	2070.068793	887	2231.462137	942	2394.336702	997	2558.605948	1052	2724.192895
833	2072.989438	888	2234.410550	943	2397.311213	998	2561.605078	1053	2727.215323
834	2075.910604	889	2237.359452	944	2400.286185	999	2564.604644	1054	2730.238164
835	2078.832291	890	2240.308842	945	2403.261617	1000	2567.604644	1055	2733.261416
836	2081.754497	891	2243.258720	946	2406.237508	1001	2570.605078	1056	2736.285080
837	2084.677222	892	2246.209085	947	2409.213858	1002	2573.605946	1057	2739.309155
838	2087.600466	893	2249.159936	948	2412.190667	1003	2576.607246	1058	2742.333641
839	2090.524228	894	2252.111274	949	2415.167933	1004	2579.608980	1059	2745.358537
840	2093.448508	895	2255.063097	950	2418.145657	1005	2582.611146	1060	2748.383843
841	2096.373304	896	2258.015405	951	2421.123837	1006	2585.613744	1061	2751.409558
842	2099.298616	897	2260.968197	952	2424.102474	1007	2588.616774	1062	2754.435683
843	2102.224443	898	2263.921473	953	2427.081567	1008	2591.620234	1063	2757.462216
844	2105.150786	899	2266.875233	954	2430.061115	1009	2594.624125	1064	2760.489158
845	2108.077642	900	2269.829476	955	2433.041119	1010	2597.628447	1065	2763.516507
846	2111.005013	901	2272.784200	956	2436.021577	1011	2600.633198	1066	2766.544264
847	2113.932896	902	2275.739407	957	2439.002488	1012	2603.638378	1067	2769.572429
848	2116.861292	903	2278.695095	958	2441.983854	1013	2606.643988	1068	2772.601000
849	2119.790200	904	2281.651263	959	2444.965673	1014	2609.650026	1069	2775.629978
850	2122.719619	905	2284.607912	960	2447.947944	1015	2612.656492	1070	2778.659362
851	2125.649548	906	2287.565040	961	2450.930667	1016	2615.663386	1071	2781.689151
852	2128.579988	907	2290.522647	962	2453.913842	1017	2618.670707	1072	2784.719346
853	2131.510937	908	2293.480733	963	2456.897469	1018	2621.678454	1073	2787.749946
854	2134.442395	909	2296.439297	964	2459.881546	1019	2624.686628	1074	2790.780950
855	2137.374361	910	2299.398338	965	2462.866073	1020	2627.695229	1075	2793.812358
856	2140.306835	911	2302.357857	966	2465.851050	1021	2630.704254	1076	2796.844171
857	2143.239815	912	2305.317852	967	2468.836477	1022	2633.713705	1077	2799.876386
858	2146.173303	913	2308.278322	968	2471.822352	1023	2636.723581	1078	2802.909005
859	2149.107296	914	2311.239269	969	2474.808676	1024	2639.733881	1079	2805.942027
860	2152.041794	915	2314.200690	970	2477.795447	1025	2642.744605	1080	2808.975450
861	2154.976798	916	2317.162585	971	2480.782667	1026	2645.755752	1081	2812.009276
862	2157.912305	917	2320.124954	972	2483.770333	1027	2648.767323	1082	2815.043503
863	2160.848316	918	2323.087797	973	2486.758446	1028	2651.779316	1083	2818.078132
864	2163.784829	919	2326.051113	974	2489.747005	1029	2654.791731	1084	2821.113161
865	2166.721845	920	2329.014900	975	2492.736009	1030	2657.804568	1085	2824.148591
866	2169.659363	921	2331.979160	976	2495.725459	1031	2660.817827	1086	2827.184421
867	2172.597382	922	2334.943891	977	2498.715354	1032	2663.831507	1087	2830.220650
868	2175.535902	923	2337.909093	978	2501.705693	1033	2666.845607	1088	2833.257279
869	2178.474922	924	2340.874765	979	2504.696475	1034	2669.860127	1089	2836.294307
870	2181.414441	925	2343.840906	980	2507.687701	1035	2672.875068	1090	2839.331733
871	2184.354459	926	2346.807517	981	2510.679370	1036	2675.890428	1091	2842.369558
872	2187.294976	927	2349.774597	982	2513.671482	1037	2678.906206	1092	2845.407781
873	2190.235990	928	2352.742145	983	2516.664035	1038	2681.922404	1093	2848.446401
874	2193.177501	929	2355.710161	984	2519.657030	1039	2684.939019	1094	2851.485418
875	2196.119510	930	2358.678644	985	2522.650467	1040	2687.956053	1095	2854.524832
876	2199.062014	931	2361.647593	986	2525.644344	1041	2690.973503	1096	2857.564643
877	2202.005013	932	2364.617009	987	2528.638661	1042	2693.991371	1097	2860.604850
878	2204.948508	933	2367.586891	988	2531.633418	1043	2697.009655	1098	2863.645452
879	2207.892497	934	2370.557238	989	2534.628614	1044	2700.028356	1099	2866.686450
880	2210.836979	935	2373.528050	990	2537.624249	1045	2703.047472	1100	2869.727842

TABLE V. *continued.*

x	$\log \Gamma(x+1).$	x	$\log \Gamma(x+1).$	x	$\log \Gamma(x+1).$	x	$\log \Gamma(x+1).$	x	$\log \Gamma(x+1).$
1101	2872.769630	1121	2933.687702	1141	2994.760680	1161	3055.985850	1181	3117.360591
1102	2875.811811	1122	2936.737696	1142	2997.818347	1162	3059.051056	1182	3120.433208
1103	2878.854387	1123	2939.788075	1143	3000.876393	1163	3062.116636	1183	3123.506193
1104	2881.897356	1124	2942.838841	1144	3003.934819	1164	3065.182589	1184	3126.579545
1105	2884.940718	1125	2945.889993	1145	3006.993624	1165	3068.248915	1185	3129.653263
1106	2887.984473	1126	2948.941532	1146	3010.052809	1166	3071.315614	1186	3132.727348
1107	2891.028621	1127	2951.993456	1147	3013.112372	1167	3074.382684	1187	3135.801799
1108	2894.073161	1128	2955.045765	1148	3016.172314	1168	3077.450127	1188	3138.876615
1109	2897.118092	1129	2958.098459	1149	3019.232634	1169	3080.517942	1189	3141.951797
1110	2900.163415	1130	2961.151537	1150	3022.293332	1170	3083.586128	1190	3145.027344
1111	2903.209129	1131	2964.205000	1151	3025.354407	1171	3086.654685	1191	3148.103256
1112	2906.255234	1132	2967.258846	1152	3028.415860	1172	3089.723612	1192	3151.179532
1113	2909.301729	1133	2970.313076	1153	3031.477689	1173	3092.792910	1193	3154.256172
1114	2912.348614	1134	2973.367689	1154	3034.539895	1174	3095.862578	1194	3157.333177
1115	2915.395889	1135	2976.422685	1155	3037.602477	1175	3098.932616	1195	3160.410545
1116	2918.443553	1136	2979.478063	1156	3040.665435	1176	3102.003024	1196	3163.488276
1117	2921.491607	1137	2982.533824	1157	3043.728768	1177	3105.073800	1197	3166.566370
1118	2924.540048	1138	2985.589966	1158	3046.792477	1178	3108.144945	1198	3169.644827
1119	2927.588878	1139	2988.646490	1159	3049.856560	1179	3111.216459	1199	3172.723646
1120	2930.638096	1140	2991.703395	1160	3052.921018	1180	3114.208341	1200	3175.802827

DEFINITE INTEGRALS.

Definite Integrals. *Definite Integrals.* Fx being a function of x , whose differential coefficient is fx , and C an arbitrary constant, $Fx+C$ is the **GENERAL INTEGRAL** of fx . If the arbitrary constant C be determined on the hypothesis that this general integral shall vanish when $x=a$, and the *definite* value b be assigned to x , this integral will assume the particular value $Fb-Fa$, and it will have become a **DEFINITE INTEGRAL**: said to be taken between the limits a and b , and written

$\int_a^b fx dx$, the superior limit being placed above and the inferior beneath the sign of integration.

A meaning will be attached to this process of integration, from which its fundamental properties may be deduced, if we revert to that conception of the Integral Calculus in which it originated, and to which many of the terms and symbols of its operations are to be traced, but which has now become a theorem no longer embracing our conception of an integral, but deducible from it.

THEOREM I.—The definite integral $\int_a^b fx dx$ is the limit of the sums of the values severally assumed by the product $fx \cdot \Delta x$, as x is made to vary by successive equal increments of Δx , from a to b , and as each such equal increment is continually and infinitely diminished, and their number therefore continually and infinitely increased.

To prove this, let us suppose that fx does not become infinite for any value of x between a and b , and let any two such values be x and $x+\Delta x$: therefore by Taylor's theorem $F(x+\Delta x) = Fx + \Delta x fx + (\Delta x)^{1+\lambda} M$, where the exponent $1+\lambda$ is given to the third term of the expansion instead of the exponent 2, that the case may be included in which the second differential coefficient of $Fx \frac{d^2fx}{dx^2}$ is infinite, and in which the exponent of Δx in that term is, therefore, a fraction less than 2.

Let the difference between a and b be divided into n equal parts, and let each be represented by Δx , so that

$$\frac{b-a}{n} = \Delta x.$$

Giving to x , then, the successive values $a, a+\Delta x, a+2\Delta x, \dots, a+(n-1)\Delta x$, and adding

$$F(a+n\Delta x) = Fa + \Delta x \sum_1^n f\{a+(n-1)\Delta x\} + (\Delta x)^{1+\lambda} \sum M_n,$$

$$\therefore Fb - Fa = \Delta x \sum_1^n f\{a+(n-1)\Delta x\} + (\Delta x)^{1+\lambda} \sum M_n.$$

Now none of the values of M are infinite, since for none of these values is fx infinite. If, therefore, M be the greatest of these values, then is $\sum M_n$ less than nM , and therefore

$$Fb - Fa - \Delta x \sum_1^n f\{a+(n-1)\Delta x\} < (b-a)M(\Delta x)^\lambda.$$

The difference of the definite integral $Fb - Fa$, and the sum $\sum_1^n \Delta x f\{a+(n-1)\Delta x\}$ is always, therefore, less than $(b-a)M(\Delta x)^\lambda$. Now M is finite, and $(b-a)$ is given, and as n is increased, Δx is diminished continually, and therefore $(\Delta x)^\lambda$ is diminished continually, λ being positive.

Thus by increasing n indefinitely, the difference of the definite integral and the sum may be diminished indefinitely, and therefore in the limit the definite integral is equal to the sum, (*i. e.*)

$$Fb - Fa = \text{limit } \sum_1^n \Delta x \cdot f\{a+(n-1)\Delta x\},$$

or, interpreting this formula, $Fb - Fa$ is the sum of the values of $\Delta x \cdot fx$, when x is made to pass by infinitesimal increments, each represented by Δx , from a to b .*

* This demonstration fails in the case in which fx becomes infinite for any of the supposed values of x ; but M. Poisson has remarked that, analytically, this difficulty will be overcome, provided the transition of x from a to b be made through a series of imaginary values of that variable, in which case none of the corresponding values of fx will be infinite, and the definite integral may receive its ordinary interpretation. Thus, in respect to the integral $\int_{-1}^{+1} \frac{1}{x} dx$, in which $\frac{1}{x}$ becomes infinite for the value 0, through which x passes in its transition from -1 to $+1$, and upon which the ordinary interpretation cannot, therefore, be put in its present form, if we assume $x = -\{\cos z + \sin \sqrt{-1} z\}$; the extreme values ± 1 of x will correspond to the values 0 and $(2n+1)\pi$ of z , and between these the function of $z = \frac{-1}{\cos z + \sin z \sqrt{-1}}$ will not pass through infinity; the definite integral will have, therefore, this substitution being made, its ordinary meaning. Substituting

$$\begin{aligned} \int_{-1}^{+1} \frac{dx}{x} &= \int_0^{(2n+1)\pi} \frac{d\{\cos z + \sin \sqrt{-1} z\}}{\cos z + \sin \sqrt{-1} z} \\ &= \int_0^{(2n+1)\pi} \frac{-\sqrt{-1} dz}{\cos z + \sin \sqrt{-1} z} \\ &= -(2n+1)\pi \sqrt{-1}. \end{aligned}$$

Definite Integrals.

Where the general integral of a function may be determined, the discussion of any definite integral of that function is unnecessary. There are, however, *no* methods of integration which include the whole order of GENERAL INTEGRALS; they extend only to functions of the form of rational fractions, or reducible to that form by a slight transformation. Definite Integrals.

There are, moreover, no *general* methods of DEFINITE INTEGRATION in respect to given limits known. There is, indeed, reason to believe that a large proportion of the functions which are included under the general designation of definite integrals cannot be represented analytically;* and so few, in comparison of what remain, are the definite integrals, including results in physics, which have yet been determined, that our ignorance in this matter may be considered to interpose a barrier on the very threshold of science.

Out of the infinite variety of algebraical formulæ which fx is put to represent, there may, nevertheless, be formed orders or classes of functions, having in a greater or less degree common properties in respect to the process of integration, and reducible in some cases to common methods.

Among these there is an extensive class which it will be convenient to distinguish as periodical functions.

THEOREM II.—A definite integral being considered as the SUM of the products of the infinitesimal quantity beneath the sign of integration by all the successive values of the finite quantity fx , when x is made to pass, by successive equal increments, from one limit to another; it is manifest that, if the finite part be periodical, so that as x is increased by the same quantity a , the same values of fx will return, then will the definite integral, when taken between any two consecutive values of x , differing by a , be the same as when taken between any other two. For each of these definite integrals is a *sum*, and the quantities composing each sum are the same.

Moreover, the integral taken between values of x , differing by n intervals of a , will be n times the value of that integral when taken between limits which differ only by one interval of a . Thus, if fx be a periodical function of x ,

$$\int_0^{na} fx dx = n \int_0^a fx dx \quad \int_b^{b+na} fx dx = n \int_b^{b+a} fx dx.$$

If the value of fx be periodical with respect to the increment a , with this reservation, that for each successive increment it changes its sign, then supposing a series of definite integrals of this function to be taken, whose limits have, in each, an interval of a , it is manifest that these will, by reason of the variation of the sign of fx , have in succession different signs but the same numerical value. Taken then in respect to limits which differ by an even number of intervals of a ,

$$\int_b^{b+na} fx dx = 0,$$

equals nothing; and taken between limits which differ by an odd number of intervals of a ,

$$\int_b^{b+na} fx dx = \pm \int_b^{b+a} fx dx.$$

These properties of periodical integrals appear to point them out as the first to be investigated. And to this class belong the integrals called Elliptic Functions.

The FIRST FIVE sections of the following paper will contain a discussion of the theory of elliptic functions.

In the SIXTH will be discussed the relations of a class of integrals of extensive application, called Eulerian integrals.

The SEVENTH will contain the discussion of formulæ for the expression of the sums of certain series, and the functions of which certain series are the expansions, by definite integrals.

The EIGHTH (conversely), of certain methods by which the value of a function may, between certain limits of the variable on which it depends, be expressed by the sum of a series of periodical definite integrals taken also between those limits. This section will contain the demonstration of the celebrated theorems of Poisson and Fourier.

The NINTH section will be devoted to certain applications of a method of definite integration by differentiation and integration in respect to an arbitrary constant under the sign of integration.

In the TENTH section will be discussed what is known of the theory of ULTRA ELLIPTIC FUNCTIONS.

ELLIPTIC FUNCTIONS.

The definite integrals to which the name of elliptic functions has been given are comprised under the three following forms:

$$\int_0^{\phi_1} \frac{d\phi}{\sqrt{1-k^2 \sin^2 \phi}}, \quad \int_0^{\phi_1} \sqrt{1-k^2 \sin^2 \phi} \cdot d\phi, \quad \int_0^{\phi_1} \frac{d\phi}{(1+n \sin^2 \phi) \sqrt{1-k^2 \sin^2 \phi}};$$

and according as they are included under the FIRST, SECOND, or THIRD of these forms, they are said to be of the FIRST, SECOND, or THIRD ORDERS.

By a similar substitution the definite integral $\int_{-1}^{+1} \frac{dx}{x^m}$ may be determined to be $\frac{(-1)^m}{m-1} \{ \cos(m-1)(2u+1)\pi - 1 \}$, and the ordinary interpretation may be put upon it.

* Laplace states, in the *Mécanique Céleste*, that he has proved the definite integrals, on which depends the theory of the attraction of elliptic spheroids, to be impossible. He has not published this demonstration, and it has not been found among his papers.

Definite
Integrals.

In each order the superior limit ϕ_1 of the integral is called the **AMPLITUDE** of the function, and the arbitrary constant k , supposed always to be less than unity, is called the **MODULUS** of the function.

 Definite
Integrals.

Into functions of the third order there enters a second arbitrary constant n , and this is called the **PARAMETER** of those functions.*

Where the modulus of an elliptic function of either of the three orders is zero or unity, or nearly equal to zero or unity, the value of the integral may be approximated to by the ordinary methods with any required degree of accuracy.

The value of a function having any other modulus is determined either by establishing a relation between it and another having a less modulus, by which its value is made to depend upon that of the other, that upon a third whose modulus is yet less, and so on until its value is eventually made to depend upon that of a function whose modulus is nearly evanescent; or the value of the function is determined by causing this transition from modulus to modulus to be made through an ascending instead of a descending scale, until it is at length made to depend upon the value of a function whose modulus is so near to unity that it may be approximated to by the ordinary methods.

The elliptic function of the **FIRST ORDER** is represented by $Fk\phi_1$; that of the **SECOND ORDER** by $E_k\phi_1$; and that of the **THIRD ORDER** by $\Pi nk\phi_1$.

When the amplitude of a function of either order is a right angle, the function is called a **COMPLETE** function, and written thus,

$$F_k, \quad E_k, \quad \Pi_{nk},$$

The quantity $\sqrt{1 - k^2 \sin^2 \phi}$ is not unfrequently represented by the symbol Δ , or by $\Delta k\phi$.

When an elliptic function F is considered as a function of its amplitude, that amplitude is represented thus amF . The complement of a function is the difference between it and a complete function; and from certain analogies between the properties of these amplitudes of elliptic functions and trigonometrical functions, the amplitude of the complement of a function F has come to be called the **coamplitude** of that function, and written $coam F$.

The first step taken in the theory of elliptic functions was the determination of a relation between the amplitudes of three functions of either order, such that there should exist an algebraical relation between the three functions themselves, of which these were the amplitudes. It is one of the most remarkable of the discoveries which science owes to the illustrious **EULER**. In the year 1761 he gave to the world the complete integration of an equation of two terms, each an elliptic function of the first or second order, not *separately* integrable;† of which integration the arbitrary constant, introduced a third function, related to the first two by a given equation between the amplitudes of the three.

This integration involved, in respect to functions of the second order, a property of the ellipse or the hyperbola, before discovered by **FAGANI**,‡ by which property two arcs differing from one another by an assignable algebraical quantity may be taken on either of these curves in an infinite number of different ways; and the properties of the lemniscata in reference to a function of the first order having the modulus $\frac{1}{\sqrt{2}}$, discovered by the same Italian philosopher.

In 1775 **LANDEN**, an English mathematician, published his celebrated theorem, showing that any arc of an hyperbola may be measured by two arcs of an ellipse. An important element of the theory of elliptic functions as it has since been developed, but *then* an isolated result.

The great problem of the comparison of elliptic functions having different moduli remained unsolved; and it is a remarkable fact that, although Euler had in a measure exhausted the comparison of functions of the same modulus, this of functions having different moduli he left untouched. It was completed in 1784 by **LAGRANGE**,§ and in respect to the determination of the numerical values of elliptic functions the comparison of Lagrange may be considered to leave little to be wished for. The value of a function may be determined by it in terms of a series of functions whose moduli continually increase or diminish, until at length this relation becomes one between the proposed function and a function having a modulus indefinitely near to unity or zero.

For all practical purposes this was sufficient; and all that remained in respect to functions of the first and second orders was to calculate for a series of values of k , extending by sufficiently small intervals from 0 to k , the numerical value of each function corresponding to a given amplitude, and to do this for each amplitude between 0 and 90°.

This herculean task was undertaken by **LEGENDRE**. At enormous labour, and with his own hand, he completed tables of elliptic functions, in which are to be found their numerical values for each degree of amplitude between 0 and 90°, and for every value of the modulus k determined by a degree of the angle $\sin^{-1} k$ between the same limits.

The labours of Legendre in the advancement of this branch of analytical knowledge were not, however, confined

* Any function of the second order may be represented by the arc of an ellipse; and it is from this circumstance that the name of elliptic functions has been made to distinguish the whole class of which this order of functions forms a part.

All integrals embraced under the form $\int_0^x \frac{fx dx}{\sqrt{a+bx+cx^2+fx^4}}$, where fx is any rational function of x , may be made to depend for their integration upon the integration of elliptic functions. See Legendre, *Fonctions Elliptiques*, vol i. ch. i.; or Hymer's *Integ. Cal.*, p. 184.

† *Novi Com. Petrop.* tom. vi. 7.

‡ *Produzione Matematiche*, tom. ii. 1750.

§ *Nouveaux Mémoires de Turin*, ans 1784 et 1785, tom. ii.

Definite
Integrals.

to the calculation of these tables. There is none of the discoveries of his predecessors in it which has not received more or less of perfection at his hands; and it was he who first supplied to the whole that connection and arrangement which have constituted it an independent science. Although to the numerical calculation of the values of elliptic integrals the method of Lagrange was sufficient, yet to the theory of elliptic functions it was insufficient. His relation between two functions having different moduli had about it all the characteristics of a particular case of some more general theorem, of which general theorem a second particular case was afterwards discovered by Legendre.

Definite
Integrals.

Here, nevertheless, the science remained. From the year 1786, when Legendre took it up, until the year 1827, when the second volume of his work (*Traité des Fonctions Elliptiques*) was published, the labours of no other philosopher of eminence were superadded. It had been in a great measure the business of his life, and he had pursued it alone. Scarcely, however, had his work passed from the hands of the printer, when it became known to the scientific world that the theory of elliptic functions was placed upon a new basis, and had received a new and extraordinary developement by the independent labours of two mathematicians, whose names were comparatively new in science. The researches of JACOBI, professor of mathematics in Königsburg, in the theory of elliptic functions were first published in the 123d Number of the *Journal of Schumacher*, and those of ABEL, professor of mathematics at Christiana, in the 3d Number of the *Journal of Crelle* for the year 1827. Three supplements to the great work of Legendre contain an account of these discoveries, the publication of the last of which but a little preceded his death.

The researches of Jacobi have for their principal object the developement of that general relation of functions of the first order having different moduli, of which the scales of Lagrange and Legendre are particular cases. This branch of the subject he has pursued with extraordinary sagacity, and he may be said almost to have perfected it.

It was to Abel that the idea first occurred of treating the elliptic integral as a function of its amplitude; and, proceeding from this new point of view, he embraced in his speculations all the principal results of Jacobi. In some of the most important of these, as in the developement of the properties of functions directly and inversely by the introduction of imaginary quantities, he undoubtedly preceded him; but in that particular theorem which is called the theorem of Jacobi, by which so much light is thrown on the investigations of preceding mathematicians, although the discovery of it appears to have been made by both nearly at the same time, yet the priority of publication is undoubtedly with Jacobi. It is not, however, by discoveries, in which he divides the honour with Jacobi, that Abel has rendered his name immortal, but by his contributions to this branch of analysis.

Having undertaken to develop the principle on which rests the fundamental proposition of Euler, establishing an algebraical relation between three functions which have the same modulus, dependent upon a certain relation of their amplitudes, he has extended it from three to an indefinite number of functions, and from elliptic functions to an infinity of other functions embraced under an indefinite number of classes, of which that of elliptic functions is but one, and of which each class has a division analogous to that of elliptic functions into three orders having common properties.

The discovery of Abel is of infinite moment, as presenting the first step of approach towards a more complete theory of the infinite class of ultra-elliptic functions, destined probably ere long to constitute one of the most important of the branches of transcendental analysis, and to include among the integrals of which it effects the solution some of those which at present arrest the researches of the philosopher in the very elements of physics.

Abel died in 1828, at Christiansand, in Norway, at the age of twenty-seven. Science has had in our times no greater loss to deplore.

In the FIRST SECTION of the following paper it will be shown, that if there be three elliptic functions having the same modulus k between the amplitudes ϕ_1, ϕ_2, ϕ_3 , of which there obtains the relation

$$\cos \phi_3 = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \Delta \phi_1 \dots \dots \dots (1.);$$

then among the functions themselves, of which these are the amplitudes, will there exist, according to the orders to which they severally belong, the following transcendental equations:

$$F\phi_1 + F\phi_2 - F\phi_3 = 0 \dots \dots \dots (2.)$$

$$E\phi_1 + E\phi_2 - E\phi_3 = k^2 \sin \phi_1 \sin \phi_2 \sin \phi_3 \dots \dots \dots (3.)$$

$$\Pi\phi_1 + \Pi\phi_2 - \Pi\phi_3 = \sqrt{\frac{n}{(n+1)(n+k^2)}} \tan^{-1} \left\{ \frac{\{n(n+1)(n+k^2)\}^{\frac{1}{2}} \sin \phi_1 \sin \phi_2 \sin \phi_3}{1+n(1-\cos \phi_1 \cos \phi_2 \cos \phi_3)} \right\} \dots \dots \dots (4.)$$

Moreover that this relation will obtain in respect to the equations (1.) and (2.), if the sign of ϕ_2 be changed.

It will be shown in the SECOND SECTION that these are but particular cases of a more extensive theorem, embracing, with the necessary modifications, every order of ultra elliptic functions, and extending to any number of functions in each order.

In the THIRD SECTION, the general relations of two functions which have different moduli will be investigated.

In the FOURTH, these relations will be applied to the determination of the numerical values of elliptic functions.

The FIFTH will contain the discussion of certain relations of elliptic functions to the parts of a spherical triangle, and to the lengths of the arcs of certain curves.

The equation of condition (1.) admits of the following transformations, all of which have their application in the theory of elliptic functions, and which are therefore given here once for all.

$$^{(1.)} \cos^2 \phi_1 + \cos^2 \phi_2 + \cos^2 \phi_3 - 2 \cos \phi_1 \cos \phi_2 \cos \phi_3 = 1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2 \sin^2 \phi_3 \dots \dots \dots (5.)*$$

* ^(1.) This equation is obtained simply by clearing equation (1.) of the radicals which enter into it, and transposing it.

^(2.) Adding $\cos^2 \phi_1 \cos^2 \phi_2$ to both sides of equation (5.), and transposing, we have

Definite
Integrals.

 Definite
Integrals.

$$^{(2)} \cos \phi_1 = \cos \phi_2 \cos \phi_3 + \sin \phi_2 \sin \phi_3 \Delta \phi_1 \dots \dots \dots (6.)$$

$$^{(3)} \cos \phi_2 = \cos \phi_1 \cos \phi_3 + \sin \phi_1 \sin \phi_3 \Delta \phi_2 \dots \dots \dots (7.)$$

$$^{(4)} \sin \phi_3 = \frac{\sin \phi_1 \cos \phi_2 \Delta \phi_2 + \sin \phi_2 \cos \phi_1 \Delta \phi_1}{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2} \dots \dots \dots (8.)$$

$$^{(5)} \cos \phi_3 = \frac{\cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \Delta \phi_1 \Delta \phi_2}{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2} \dots \dots \dots (9.)$$

$$^{(6)} \Delta \phi_1 \Delta \phi_2 - \Delta \phi_3 = k^2 \sin \phi_1 \sin \phi_2 \cos \phi_3 \dots \dots \dots (10.)$$

$$^{(7)} \Delta \phi_1 \Delta \phi_2 \Delta \phi_3 = k'^2 + k^2 \cos \phi_1 \cos \phi_2 \cos \phi_3 \dots \dots \dots (11.)$$

SECTION I.

THEOREM III.—If there exist between the amplitudes ϕ_1, ϕ_2, ϕ_3 of three functions of the first order the relation

$$\cos \phi_3 = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \Delta \phi_3,$$

then will there exist between the functions themselves the relations

$$F\phi_1 + F\phi_2 - F\phi_3 = 0.$$

Dividing the equation of condition by $\sin \phi_1 \sin \phi_2$, and differentiating with respect to ϕ_1 and ϕ_2 , these being considered functions of a third variable ϕ ,

$$\frac{d\phi_1}{d\phi} \left\{ \frac{\cos \phi_2 - \cos \phi_1 \cos \phi_3}{\sin \phi_1} \right\} + \frac{d\phi_2}{d\phi} \left\{ \frac{\cos \phi_1 - \cos \phi_2 \cos \phi_3}{\sin \phi_2} \right\} = 0.$$

Substituting in this equation the values of

$$\frac{\cos \phi_1 - \cos \phi_2 \cos \phi_3}{\sin \phi_2} \quad \text{and} \quad \frac{\cos \phi_2 - \cos \phi_1 \cos \phi_3}{\sin \phi_1}$$

from equations (6.) and (7.) we have

$$\frac{\left(\frac{d\phi_1}{d\phi} \right)}{\Delta \phi_1} + \frac{\left(\frac{d\phi_2}{d\phi} \right)}{\Delta \phi_2} = 0;$$

$\therefore$ integrating $F\phi_1 + F\phi_2 - F\phi_3 = 0$, ϕ_3 being the value of ϕ_1 when $\phi_2 = 0$.

THEOREM IV.—If there exist between the amplitudes of three functions of the FIRST ORDER the relation

$$\cos \phi_3 = \cos \phi_1 \cos \phi_2 + \sin \phi_1 \sin \phi_2 \Delta \phi_3,$$

then will there exist between the functions themselves the relation

$$\begin{aligned} \{ \cos \phi_1 - \cos \phi_2 \cos \phi_3 \}^2 &= 1 - \cos^2 \phi_2 - \cos^2 \phi_3 + \cos^2 \phi_2 \cos^2 \phi_3 - k^2 \sin^2 \phi_1 \sin^2 \phi_2 \sin^2 \phi_3 \\ &= \sin^2 \phi_2 \sin^2 \phi_3 - k^2 \sin^2 \phi_1 \sin^2 \phi_2 \sin^2 \phi_3 \\ &= \sin^2 \phi_2 \sin^2 \phi_3 \Delta^2 \phi_1 \\ \cos \phi_1 &= \cos \phi_2 \cos \phi_3 + \sin \phi_2 \sin \phi_3 \Delta \phi_1, \end{aligned}$$

using the sign + because it appears by equation (2.) that when $\phi_1 = 0$ $\phi_2 = \phi_3$.

^(3.) This equation is obtained by a similar transposition to the last, *mutatis mutandis*.

^(4.) Multiplying (6.) by $\cos \phi_1$, and (7.) by $\cos \phi_2$, and adding, we have

$$\cos^2 \phi_1 + \cos^2 \phi_2 - 2 \cos \phi_1 \cos \phi_2 \cos \phi_3 - \sin \phi_3 \{ \cos \phi_1 \sin \phi_2 \Delta \phi_2 + \cos \phi_2 \sin \phi_1 \Delta \phi_1 \} = 0.$$

Subtracting this equation from (5.),

$$\sin^2 \phi_3 = \sin \phi_3 \{ \cos \phi_1 \sin \phi_2 \Delta \phi_2 + \cos \phi_2 \sin \phi_1 \Delta \phi_1 \} + k^2 \sin^2 \phi_1 \sin^2 \phi_2 \sin^2 \phi_3;$$

$$\therefore \sin \phi_3 = \frac{\cos \phi_1 \sin \phi_2 \Delta \phi_2 + \cos \phi_2 \sin \phi_1 \Delta \phi_1}{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2}$$

^(5.) Substituting this value of $\sin \phi_3$ in (6.),

$$\cos \phi_1 = \cos \phi_2 \cos \phi_3 + \frac{\sin \phi_1 \sin \phi_2 \cos \phi_2 \Delta \phi_1 \Delta \phi_2 + \sin^2 \phi_2 \cos \phi_1 (\Delta \phi_1)^2}{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2};$$

$$\therefore \cos \phi_2 \cos \phi_3 = \frac{\cos \phi_1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2 \cos \phi_1 - \sin \phi_1 \sin \phi_2 \cos \phi_2 \Delta \phi_1 \Delta \phi_2 - \sin^2 \phi_2 \cos \phi_1 (\Delta \phi_1)^2}{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2}$$

$$= \frac{\cos \phi_1 - \sin^2 \phi_2 \cos \phi_1 - \sin \phi_1 \sin \phi_2 \cos \phi_2 \Delta \phi_1 \Delta \phi_2}{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2}$$

$$\therefore \cos \phi_3 = \frac{\cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \Delta \phi_1 \Delta \phi_2}{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2}$$

^(6.) Transposing (7.),

$$\cos \phi_3 - \cos \phi_1 \cos \phi_2 = k^2 \sin^2 \phi_1 \sin^2 \phi_2 \cos \phi_3 - \sin \phi_1 \sin \phi_2 \Delta \phi_1 \Delta \phi_2;$$

$\therefore$ by equation (4.),

$$-\sin \phi_1 \sin \phi_2 \Delta \phi_3 = k^2 \sin^2 \phi_1 \sin^2 \phi_2 \cos \phi_3 - \sin \phi_1 \sin \phi_2 \Delta \phi_1 \Delta \phi_2;$$

$$\therefore \Delta \phi_1 \Delta \phi_2 - \Delta \phi_3 = k^2 \sin \phi_1 \sin \phi_2 \cos \phi_3.$$

^(7.) Multiplying equation (10.) by $\Delta \phi_3$,

$$\Delta \phi_1 \Delta \phi_2 \Delta \phi_3 - \Delta^2 \phi_3 = k^2 \sin \phi_1 \sin \phi_2 \cos \phi_3 \Delta \phi_3;$$

also by (1.)

$$-k^2 \cos^2 \phi_3 = -k^2 \cos \phi_1 \cos \phi_2 \cos \phi_3 + k^2 \sin \phi_1 \sin \phi_2 \cos \phi_3 \Delta \phi_3;$$

$\therefore$ by subtraction

$$\Delta \phi_1 \Delta \phi_2 \Delta \phi_3 - 1 + k^2 = k^2 \cos \phi_1 \cos \phi_2 \cos \phi_3;$$

or

$$\Delta \phi_1 \Delta \phi_2 \Delta \phi_3 = k'^2 + k^2 \cos \phi_1 \cos \phi_2 \cos \phi_3, \text{ assuming } k'^2 = 1 - k^2.$$

Definite
Integrals.

$$F\phi_1 - F\phi_2 - F\phi_3 = 0.$$

The proof of this theorem is, *mutatis mutandis*, precisely the same as that of the last.*

Definite
Integrals.

* Hence therefore, generally, in respect to functions of the first order, it is shown that, if

$$\cos \phi_3 = \cos \phi_1 \cos \phi_2 \mp \sin \phi_1 \sin \phi_2 \Delta \phi_3,$$

then $F\phi_1 \pm F\phi_2 - F\phi_3 = 0$. The converse of this proposition may be proved. It may be shown that, if these obtain the relation $F\phi_1 \pm F\phi_2 - F\phi_3 = 0$ between three functions, then will there obtain the relation $\cos \phi_3 = \cos \phi_1 \cos \phi_2 \mp \sin \phi_1 \sin \phi_2 \Delta \phi_3$ between the amplitudes of those functions. The following is the integration of Euler.

Differentiating with respect to ϕ_1, ϕ_2 assumed to be functions of a third variable ϕ , we have

$$\frac{\frac{d\phi_1}{d\phi}}{\sqrt{1-k^2 \sin^2 \phi_1}} \pm \frac{\frac{d\phi_2}{d\phi}}{\sqrt{1-k^2 \sin^2 \phi_2}} = 0.$$

Now ϕ may be any whatever, ϕ_1 may then be any function whatever of ϕ . Let ϕ_1 be that function of ϕ which satisfies the equation

$$\left. \begin{aligned} \frac{\frac{d\phi_1}{d\phi}}{\sqrt{1-k^2 \sin^2 \phi_1}} &= 1 \\ \frac{\frac{d\phi_2}{d\phi}}{\sqrt{1-k^2 \sin^2 \phi_2}} &= \mp 1 \end{aligned} \right\} \dots\dots (\alpha),$$

$$\therefore \frac{d\phi_1}{d\phi} = \sqrt{1-k^2 \sin^2 \phi_1}, \quad \frac{d\phi_2}{d\phi} = \mp \sqrt{1-k^2 \sin^2 \phi_2}.$$

Squaring these equations, and differentiating them, we have

$$2 \frac{d\phi_1}{d\phi} \frac{d^2 \phi_1}{d\phi^2} = -2k^2 \sin \phi_1 \cos \phi_1 \frac{d\phi_1}{d\phi}$$

$$2 \frac{d\phi_2}{d\phi} \frac{d^2 \phi_2}{d\phi^2} = -2k^2 \sin \phi_2 \cos \phi_2 \frac{d\phi_2}{d\phi},$$

or

$$\frac{d^2 \phi_1}{d\phi^2} = -\frac{1}{2} k^2 \sin 2\phi_1, \quad \frac{d^2 \phi_2}{d\phi^2} = -\frac{1}{2} k^2 \sin 2\phi_2;$$

$$\therefore \frac{d^2 (\phi_1 \pm \phi_2)}{d\phi^2} = -\frac{1}{2} k^2 (\sin 2\phi_1 \pm \sin 2\phi_2);$$

$$\therefore \frac{d^2 (\phi_1 + \phi_2)}{d\phi^2} = -k^2 \sin (\phi_1 + \phi_2) \cos (\phi_1 - \phi_2),$$

$$\frac{d^2 (\phi_1 - \phi_2)}{d\phi^2} = -k^2 \sin (\phi_1 - \phi_2) \cos (\phi_1 + \phi_2).$$

Also

$$\frac{d(\phi_1 + \phi_2)}{d\phi} \cdot \frac{d(\phi_1 + \phi_2)}{d\phi} = \left(\frac{d\phi_1}{d\phi} \right)^2 - \left(\frac{d\phi_2}{d\phi} \right)^2$$

$$= -k^2 (\sin^2 \phi_1 - \sin^2 \phi_2)$$

$$= -k^2 \sin (\phi_1 + \phi_2) \sin (\phi_1 - \phi_2);$$

$$\therefore \left\{ \frac{d^2 (\phi_1 + \phi_2)}{d\phi^2} \right\} = \left\{ \frac{d(\phi_1 - \phi_2)}{d\phi} \right\} \cdot \frac{\cos (\phi_1 - \phi_2)}{\sin (\phi_1 - \phi_2)}$$

$$\left\{ \frac{d^2 (\phi_1 - \phi_2)}{d\phi^2} \right\} = \frac{d(\phi_1 + \phi_2)}{d\phi} \cdot \frac{\cos (\phi_1 + \phi_2)}{\sin (\phi_1 + \phi_2)}.$$

$$\left. \begin{aligned} \frac{d(\phi_1 + \phi_2)}{d\phi} &= C_1 \sin (\phi_1 - \phi_2) \\ \frac{d(\phi_1 - \phi_2)}{d\phi} &= C_2 \sin (\phi_1 + \phi_2) \end{aligned} \right\} \dots\dots (\beta).$$

$\therefore$ integrating

Reversing the lower of these equations, and multiplying, we have

$$C_2 \sin (\phi_1 + \phi_2) \frac{d(\phi_1 + \phi_2)}{d\phi} = C_1 \sin (\phi_1 - \phi_2) \frac{d(\phi_1 - \phi_2)}{d\phi};$$

$$\therefore C_2 \cos (\phi_1 + \phi_2) = C_1 \cos (\phi_1 - \phi_2) + C_3;$$

$$\therefore (C_2 - C_1) \cos \phi_1 \cos \phi_2 - (C_2 + C_1) \sin \phi_1 \sin \phi_2 = C_3 \dots\dots (\gamma).$$

It remains to determine the constants. Adding and subtracting equation (β), and observing that by equation (α)

$$\frac{d\phi_1}{d\phi} = \Delta \phi_1, \quad \frac{d\phi_2}{d\phi} = \mp \Delta \phi_2,$$

we have

$$(C_2 + C_1) \sin \phi_1 \cos \phi_2 + (C_2 - C_1) \sin \phi_2 \cos \phi_1 = 2 \Delta \phi_1$$

$$-(C_2 + C_1) \sin \phi_2 \cos \phi_1 - (C_2 - C_1) \sin \phi_1 \cos \phi_2 = \mp 2 \Delta \phi_2.$$

Definite
Integrals.

THEOREM V.—If there exist between the amplitudes of three functions of the SECOND ORDER the relation

$$\cos \phi_3 = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \Delta \phi_3,$$

then will there exist between the function the relation

$$E\phi_1 + E\phi_2 - E\phi_3 = k^2 \sin \phi_1 \sin \phi_2 \sin \phi_3.$$

Let

$$E\phi_1 + E\phi_2 - E\phi_3 = \Phi;$$

$$\therefore \frac{d\phi_1}{d\phi} \Delta \phi_1 + \frac{d\phi_2}{d\phi} \Delta \phi_2 = \frac{d\Phi}{d\phi};$$

 $\therefore$ substituting the values of $\Delta \phi_1$ and $\Delta \phi_2$ from equations (6.) and (7.),

$$\frac{d\phi_1}{d\phi} \left\{ \frac{\cos \phi_1 - \cos \phi_2 \cos \phi_3}{\sin \phi_2 \sin \phi_3} \right\} + \frac{d\phi_2}{d\phi} \left\{ \frac{\cos \phi_2 - \cos \phi_1 \cos \phi_3}{\sin \phi_1 \sin \phi_3} \right\} = \frac{d\Phi}{d\phi};$$

$$\therefore \frac{\frac{d\phi_1}{d\phi} \{ \sin \phi_1 \cos \phi_1 - \cos \phi_2 \cos \phi_3 \sin \phi_1 \} + \frac{d\phi_2}{d\phi} \{ \sin \phi_2 \cos \phi_2 - \cos \phi_1 \cos \phi_3 \sin \phi_2 \}}{\sin \phi_1 \sin \phi_2 \sin \phi_3} = \frac{d\Phi}{d\phi};$$

$$\therefore \frac{\frac{d}{d\phi} \left\{ \frac{1}{2} \sin^2 \phi_1 + \frac{1}{2} \sin^2 \phi_2 + \cos \phi_1 \cos \phi_2 \cos \phi_3 \right\}}{\sin \phi_1 \sin \phi_2 \sin \phi_3} = \frac{d\Phi}{d\phi}.$$

But, by equation (5.),

$$\sin^2 \phi_1 + \sin^2 \phi_2 + 2 \cos \phi_1 \cos \phi_2 \cos \phi_3 = 1 + \cos^2 \phi_3 + k^2 \sin^2 \phi_1 \sin^2 \phi_2 \sin^2 \phi_3;$$

$$\therefore \frac{1}{2} k^2 \frac{d}{d\phi} \{ \sin^2 \phi_1 \sin^2 \phi_2 \sin^2 \phi_3 \} = \frac{d\Phi}{d\phi}.$$

$$\therefore k^2 \frac{d \{ \sin \phi_1 \sin \phi_2 \sin \phi_3 \}}{d\phi} = \frac{d\Phi}{d\phi};$$

$$\therefore k^2 \sin \phi_1 \sin \phi_2 \sin \phi_3 = \Phi,$$

no constant being added, because Φ , that is, $E\phi_1 + E\phi_2 - E\phi_3$ vanishes when $\phi_2 = 0$; and $\therefore$, by the equation of condition, $\phi_1 = \phi_3$. Therefore, if

$$\cos \phi_3 = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \Delta \phi_3,$$

then

$$E\phi_1 + E\phi_2 - E\phi_3 = k^2 \sin \phi_1 \sin \phi_2 \sin \phi_3.*$$

THEOREM V.—If there exist between the amplitudes of three functions of the THIRD ORDER the relation

$$\cos \phi_3 = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \Delta \phi_3,$$

then will there exist between the functions the relation

$$\Pi \phi_1 + \Pi \phi_2 - \Pi \phi_3 = \frac{n}{N} \tan^{-1} \left\{ \frac{N \sin \phi_1 \sin \phi_2 \sin \phi_3}{1 + n (1 - \cos \phi_1 \cos \phi_2 \cos \phi_3)} \right\},$$

where $N = \sqrt{n(n+1)(n+k^2)}$.

Let

$$\Pi \phi_1 + \Pi \phi_2 - \Pi \phi_3 = \Phi;$$

 $\therefore$ differentiating with respect to ϕ_1 and ϕ_2 ,

$$\frac{\left(\frac{d\phi_1}{d\phi} \right)}{(1 + n \sin^2 \phi_1) \Delta \phi_1} + \frac{\left(\frac{d\phi_2}{d\phi} \right)}{(1 + n \sin^2 \phi_2) \Delta \phi_2} = \frac{d\Phi}{d\phi} \dots \dots \dots (A.)$$

Now if

$$\phi_2 = 0, F\phi_1 - F\phi_3 = 0; \therefore \phi_1 = \phi_3;$$

$$\therefore (C_2 + C_1) \sin \phi_3 = 2 \Delta \phi_3$$

$$(C_2 - C_1) \sin \phi_3 = \pm 2 \Delta 0 = \pm 2.$$

Substituting these values in equation (7),

$$\pm 2 \cos \phi_1 \cos \phi_2 - 2 \sin \phi_1 \sin \phi_2 \Delta \phi_3 = C_3 \sin \phi_3;$$

and again, making $\phi_2 = 0$,

$$\pm 2 \cos \phi_3 = C \sin \phi_3;$$

$$\therefore \cos \phi_3 = \cos \phi_1 \cos \phi_2 \mp \sin \phi_1 \sin \phi_2 \Delta \phi_3.$$

* The converse of this proposition is not true. Differentiating equation (3.) with respect to ϕ ,

$$\frac{d\phi_1}{d\phi} \Delta \phi_1 + \frac{d\phi_2}{d\phi} \Delta \phi_2 = k^2 \left\{ \sin \phi_2 \cos \phi_1 \frac{d\phi_1}{d\phi} + \sin \phi_1 \cos \phi_2 \frac{d\phi_2}{d\phi} \right\} \sin \phi_3;$$

 $\therefore$, see the preceding note,

$$(\Delta \phi_1)^2 - (\Delta \phi_2)^2 = \{ \sin \phi_2 \cos \phi_1 \Delta \phi_1 - \sin \phi_1 \cos \phi_2 \Delta \phi_2 \} \sin \phi_3;$$

$$\therefore \cos^2 \phi_1 - \cos^2 \phi_2 = \sin \phi_2 \sin \phi_3 \cos \phi_1 \Delta \phi_1 - \sin \phi_1 \sin \phi_3 \cos \phi_2 \Delta \phi_2,$$

subtracting $\cos \phi_1, \cos \phi_2, \cos \phi_3$, from both sides, and transposing

$$\cos \phi_1 \{ \cos \phi_1 - (\cos \phi_2 \cos \phi_3 + \sin \phi_2 \sin \phi_3 \Delta \phi_1) \} - \cos \phi_2 \{ \cos \phi_2 - (\cos \phi_1 \cos \phi_3 + \sin \phi_1 \sin \phi_3 \Delta \phi_2) \} = 0.$$

Now this equation is satisfied, if equation (1.) obtain, because then equations (6.) and (7.) will obtain. But the converse of this is not necessarily true. The relation expressed in this equation does not necessarily imply that of equation (1.). Equation (1.), between the amplitudes, is therefore sufficient to equation (3.) between the functions, but it is not necessary to it.

Definite
Integrals.

Definite
Integrals.

Now by equations (2.) and (3.), to which the equation of condition has been shown to be sufficient,

Definite
Integrals.

$$\frac{\left(\frac{d\phi_1}{d\phi}\right)}{\Delta\phi_1} + \frac{\left(\frac{d\phi_2}{d\phi}\right)}{\Delta\phi_2} = 0 \dots\dots\dots (B.),$$

$$\left(\frac{d\phi_1}{d\phi}\right)\Delta\phi_1 + \left(\frac{d\phi_2}{d\phi}\right)\Delta\phi_2 = k^2 \frac{d\{\sin\phi_1 \sin\phi_2 \sin\phi_3\}}{d\phi} \dots\dots\dots (C.);$$

 $\therefore$ by (A.) and (B.)

$$\frac{\left(\frac{d\phi_1}{d\phi}\right)}{\Delta\phi_1} \left\{ \frac{1}{(1+n\sin^2\phi_1)} - \frac{1}{(1+n\sin^2\phi_2)} \right\} = \frac{d\Phi}{d\phi}$$

or

$$\frac{\left(\frac{d\phi_1}{d\phi}\right)}{\Delta\phi_1} \left\{ \frac{n(\sin^2\phi_2 - \sin^2\phi_1)}{(1+n\sin^2\phi_1)(1+n\sin^2\phi_2)} \right\} = \frac{d\Phi}{d\phi};$$

but by (B.) and (C.)

$$\{(\Delta\phi_1)^2 - (\Delta\phi_2)^2\} \frac{d\phi_1}{d\phi} = k^2 \frac{d\{\sin\phi_1 \sin\phi_2 \sin\phi_3\}}{d\phi},$$

$$\therefore \{\sin^2\phi_2 - \sin^2\phi_1\} \frac{d\phi_1}{d\phi} = \frac{d\{\sin\phi_1 \sin\phi_2 \sin\phi_3\}}{d\phi};$$

$$\therefore n \frac{\frac{d}{d\phi} \{\sin\phi_1 \sin\phi_2 \sin\phi_3\}}{(1+n\sin^2\phi_1)(1+n\sin^2\phi_2)} = \frac{d\Phi}{d\phi}.$$

Let

$$\sin\phi_1 \sin\phi_2 = \frac{1}{P};$$

by equation (1.)

$$\therefore \cos\phi_1 \cos\phi_2 = \cos\phi_3 + \frac{\Delta\phi_3}{P};$$

$$\therefore (1 - \sin^2\phi_1)(1 - \sin^2\phi_2) = \left\{ \cos\phi_3 + \frac{\Delta\phi_3}{P} \right\}^2;$$

$$\therefore \sin^2\phi_1 + \sin^2\phi_2 = \sin^2\phi_3 - 2 \frac{\cos\phi_3 \Delta\phi_3}{P} + \frac{k^2 \sin^2\phi_3}{P^2};$$

$$\therefore (1 + n\sin^2\phi_1)(1 + n\sin^2\phi_2) = 1 + n\sin^2\phi_3 - 2n \frac{\cos\phi_3 \Delta\phi_3}{P} + \frac{n^2 + nk^2 \sin^2\phi_3}{P^2};$$

$$\therefore \frac{n \sin\phi_3 \left(\frac{dP^{-1}}{d\phi}\right)}{1 + n\sin^2\phi_3 - 2n \frac{\cos\phi_3 \Delta\phi_3}{P} + \frac{n^2 + nk^2 \sin^2\phi_3}{P^2}} = \frac{d\Phi}{d\phi};$$

$$\therefore \frac{-n \sin\phi_3 \left(\frac{dP}{d\phi}\right)}{P^2(1 + n\sin^2\phi_3) - 2Pn \cos\phi_3 \Delta\phi_3 + (n^2 + nk^2 \sin^2\phi_3)} = \frac{d\Phi}{d\phi}.$$

Integrating by the ordinary rules,

$$\Phi = \left\{ \frac{n}{(n+1)(n+k^2)} \right\}^{\frac{1}{2}} \cot^{-1} \left\{ \frac{P(1 + n\sin^2\phi_3) - n \cos\phi_3 \Delta\phi_3}{\{n(n+1)(n+k^2)\}^{\frac{1}{2}} \sin\phi_3} \right\},$$

or

$$\Phi = \frac{n}{N} \cot^{-1} \left\{ \frac{1 + n\sin^2\phi_3 - n \sin\phi_1 \sin\phi_2 \cos\phi_3 \Delta\phi_3}{N \sin\phi_1 \sin\phi_2 \sin\phi_3} \right\}.$$

Now

$$1 + n\sin^2\phi_3 - n \sin\phi_1 \sin\phi_2 \cos\phi_3 \Delta\phi_3 = 1 + n - n \cos\phi_3 \{\cos\phi_3 - \sin\phi_1 \sin\phi_2 \Delta\phi_3\}$$

$$\dots\dots\dots = 1 + n - n \cos\phi_1 \cos\phi_2 \cos\phi_3;$$

$$\therefore \Phi = \frac{n}{N} \tan^{-1} \left\{ \frac{N \sin\phi_1 \sin\phi_2 \sin\phi_3}{1 + n(1 - \cos\phi_1 \cos\phi_2 \cos\phi_3)} \right\};$$

 $\therefore$ &c.** In the case in which n is negative, the integral assumes a logarithmic form.

Let

$$n = -k^2 \sin^2\theta;$$

$$\therefore \frac{N}{n} = \cot\theta \Delta\theta \sqrt{-1};$$

$$\therefore \Phi = \frac{\tan\theta}{\Delta\theta} \cdot \frac{1}{\sqrt{-1}} \tan^{-1} \left\{ \sqrt{-1} \cdot \frac{\cot\theta \Delta\theta \sin\phi_1 \sin\phi_2 \sin\phi_3}{1 + n(1 - \cos\phi_1 \cos\phi_2 \cos\phi_3)} \right\};$$

$$\therefore \Phi = \frac{1}{2} \frac{\tan\theta}{\Delta\theta} \log \left\{ \frac{1 + \left\{ \frac{\cot\theta \Delta\theta \sin\phi_1 \sin\phi_2 \sin\phi_3}{1 + n(1 - \cos\phi_1 \cos\phi_2 \cos\phi_3)} \right\}}{1 - \left\{ \frac{\cot\theta \Delta\theta \sin\phi_1 \sin\phi_2 \sin\phi_3}{1 + n(1 - \cos\phi_1 \cos\phi_2 \cos\phi_3)} \right\}} \right\}.$$

Definite
Integrals.

SECTION II.—THE THEOREM OF ABEL.

 Definite
Integrals.

The preceding theorems are particular cases of a general theorem, extending to any number of functions, and to any order of functions included under the formula $\int \frac{fx \cdot dx}{(x-\alpha) \sqrt{\phi x}}$, where fx and ϕx are any function of x .

THEOREM VI.—Let $\psi x_1, \psi x_2, \dots$ represent the definite values of the integral $\int \frac{fx \cdot dx}{(x-\alpha) \sqrt{\phi x}}$ when taken between values of x , 0, and $x_1, x_2, x_3, \dots$ respectively.

Also let ϕx be the product of any two factors $\phi_1 x$ and $\phi_2 x$, respectively of n_1 and n_2 dimensions in x . Let $\theta_1 x$ and $\theta_2 x$ be polynomials ascending by positive integral powers of x , and subject to the condition

$$(\theta_1 x)^2 (\phi_1 x) - (\theta_2 x)^2 (\phi_2 x) = P_\mu (x - x_\mu) \dots \dots (D),$$

$P_\mu (x - x_\mu)$ representing the product of the μ factors $(x - x_1) (x - x_2) (x - x_3) \dots (x - x_\mu)$,

Then
$$\Sigma \pm \psi x = C \frac{1}{x} \frac{\Psi x f x}{(x-\alpha)} - \Psi \alpha f \alpha + C \dots \dots (11),$$

where
$$\Psi x = \frac{1}{\sqrt{\phi x}} \log \left\{ \frac{\theta_1 x \sqrt{\phi_1 x} + \theta_2 x \sqrt{\phi_2 x}}{\theta_1 x \sqrt{\phi_1 x} - \theta_2 x \sqrt{\phi_2 x}} \right\} \dots \dots (E),$$

and $C \frac{1}{x} \left\{ \frac{\psi x f x}{(x-\alpha)} \right\}$ is put to represent the coefficient of $\frac{1}{x}$ in the expansion of $\frac{\Psi x f x}{x-\alpha}$, according to the negative powers of x . This is the celebrated theorem of ABEL.

Let

$$\theta_1 x = a_1 + b_1 x + c_1 x^2 + \dots + l_1 x^{m_1}$$

$$\theta_2 x = a_2 + b_2 x + c_2 x^2 + \dots + l_2 x^{m_2}.$$

By substituting these values of $\theta_1 x$ and $\theta_2 x$ in equation (D), and equating like powers of x , equations will be obtained establishing certain relations between the $m_1 + m_2 + 2$ quantities $a_1 b_1 c_1 \dots a_2 b_2 c_2 \dots$, and the μ quantities $x_1 x_2 \dots$; and if the dimensions $m_1 m_2, n_1 n_2$ be properly assumed in respect to μ , the values of these quantities will be determined in terms of $x_1 x_2 \dots$, and there will remain certain equations of condition between these quantities themselves, further necessary to the relation (D), which has been supposed.

Let one or more of the quantities $x_1 x_2 \dots$ be conceived to be variable; and assume the variable element of the polynomial $(\theta_1 x)^2 (\phi_1 x) - (\theta_2 x)^2 (\phi_2 x)$, which results from this supposition, to enter into it under the form of a function of y . Moreover, let the polynomial itself be represented under these circumstances by Φxy .

If, then, x be supposed to represent one of the quantities $x_1 x_2 x_3 \dots$ $\Phi xy = 0$;

$$\therefore \frac{d\Phi xy}{dx} dx + \frac{d\Phi xy}{dy} dy = 0 \dots \dots (F).$$

Now

$$\frac{d\Phi xy}{dy} dy = 2(\phi_1 x)(\theta_1 x) \frac{d\theta_1 x}{dy} dy - 2(\phi_2 x)(\theta_2 x) \frac{d\theta_2 x}{dx} dx.$$

But since

$$(\theta_1 x)^2 (\phi_1 x) - (\theta_2 x)^2 (\phi_2 x) = 0;$$

$$\therefore \theta_1 x \sqrt{\phi_1 x} = \pm \theta_2 x \sqrt{\phi_2 x};$$

where the sign $\pm$ is to be taken according as $\theta_1 x$ and $\theta_2 x$ have for the assumed value of x the same or different signs;

$$\therefore (\theta_1 x)(\phi_1 x) = \pm \theta_2 x \sqrt{\phi_1 x \phi_2 x} = \pm \theta_2 x \sqrt{\phi x}$$

$$(\theta_2 x)(\phi_2 x) = \pm \theta_1 x \sqrt{\phi_1 x \phi_2 x} = \pm \theta_1 x \sqrt{\phi x};$$

$$\therefore \frac{d\Phi xy}{dy} dy = \pm 2\sqrt{\phi x} \left\{ \theta_2 x \frac{d\theta_1 x}{dy} - \theta_1 x \frac{d\theta_2 x}{dy} \right\} dy \dots \dots (G);$$

Hence, reducing and assuming by equation (8.),

$$\sin \Psi_1 = \frac{\sin \theta \cos \phi_3 \Delta \phi_3 - \sin \phi_3 \cos \theta \Delta \theta}{1 - k^2 \sin^2 \theta \sin^2 \phi_3}$$

$$\sin \Psi_2 = \frac{\sin \theta \cos \phi_3 \Delta \phi_3 + \sin \phi_3 \cos \theta \Delta \theta}{1 - k^2 \sin^2 \theta \sin^2 \phi_3},$$

we have

$$\Phi = \frac{1}{2} \frac{\tan \theta}{\Delta \theta} \log \left\{ \frac{1 + k^2 \sin \theta \sin \Psi_1 \sin \phi_1 \sin \phi_2}{1 + k^2 \sin \theta \sin \Psi_2 \sin \phi_1 \sin \phi_2} \right\};$$

where, it will be observed, that the subsidiary angles, Ψ_1 and Ψ_2 , are the amplitudes of functions which satisfy the conditions

$$F\theta - F\phi_3 - F\Psi_1 = 0,$$

$$F\theta + F\phi_3 - F\Psi_2 = 0.$$

Since irrational quantities of the form $\sqrt{-1} \tan^{-1} \sqrt{-1} \Psi$ may be immediately converted into logarithmic functions, it is considered unnecessary to write the relation of three functions of the third order under any other form than that given in the text.

Definite
Integrals.

∴ by (F),

$$\frac{d\Phi xy}{dx} dx = \pm 2\sqrt{\phi x} \left\{ \theta_1 x \frac{d\theta_2 x}{dy} - \theta_2 x \frac{d\theta_1 x}{dy} \right\} dy;$$

$$\therefore \frac{\pm f x \cdot dx}{(x-\alpha) \sqrt{\phi x}} = \frac{2fx}{(x-\alpha) \frac{d\Phi xy}{dx}} \cdot \left\{ \theta_1 x \frac{d\theta_2 x}{dy} - \theta_2 x \frac{d\theta_1 x}{dy} \right\} dy.$$

In this expression x represents any one of the quantities $x_1, x_2, \dots, x_p$. Let it be made in succession to represent each of them, and let the sum of the formulæ which result from this substitution be taken;

$$\therefore \sum \frac{\pm d\psi x}{dy} = \sum \frac{2fx}{(x-\alpha) \frac{d\Phi xy}{dx}} \left\{ \theta_1 x \frac{d\theta_2 x}{dy} - \theta_2 x \frac{d\theta_1 x}{dy} \right\} \dots \dots \dots (H).$$

Let

$$2fx \cdot \left\{ \theta_1 x \frac{d\theta_2 x}{dy} - \theta_2 x \frac{d\theta_1 x}{dy} \right\} = \Lambda xy,$$

where Λxy is a polynomial ascending by positive integral powers of x , having for certain of their coefficients functions of y .

We may assume

$$\Lambda xy = \Lambda \alpha y + (x-\alpha) \Lambda_1 xy \dots \dots \dots (I).$$

By (H), therefore,

$$\sum \pm \frac{d\psi x}{dy} = \Lambda \alpha y \cdot \sum \frac{1}{(x-\alpha) \frac{d\Phi xy}{dx}} + \sum \frac{\Lambda_1 xy}{\frac{d\Phi xy}{dx}} \dots \dots \dots (K).$$

Now

$$\Phi xy = (x-x_1)(x-x_2)(x-x_3) \dots (x-x_p);$$

therefore*

$$\frac{1}{\Phi xy} = \sum \frac{1}{(x-x_n) \frac{d\Phi x_n y}{dx_n}} \dots \dots \dots (L);$$

$$\therefore \frac{1}{\Phi \alpha y} = \sum \frac{1}{(\alpha-x_n) \frac{d\Phi x_n y}{dx_n}};$$

$$\therefore \sum \pm \frac{d\psi x}{dy} = -\frac{\Lambda \alpha y}{\Phi \alpha y} + \sum \frac{\Lambda_1 xy}{\frac{d\Phi xy}{dy}} \dots \dots \dots (M).$$

It remains now to determine the value of the last term of this equation. Expanding $\frac{1}{x-x_n}$, or $\frac{1}{x} \left(1 - \frac{x_n}{x}\right)^{-1}$ in each term of the second member of equation (L),

$$\begin{aligned} \frac{1}{\Phi xy} &= \frac{1}{x} \sum \frac{1}{\frac{d\Phi x_n y}{dx_n}} + \frac{1}{x^2} \sum \frac{x_n}{\frac{d\Phi x_n y}{dx_n}} + \frac{1}{x^3} \sum \frac{x_n^2}{\frac{d\Phi x_n y}{dx_n}} + \dots \dots \dots + \frac{1}{x^{m+1}} \sum \frac{x_n^m}{\frac{d\Phi x_n y}{dx_n}} + \dots \\ \therefore \frac{x^m}{\Phi xy} &= \dots \dots \dots + \frac{1}{x} \sum \frac{x_n^m}{\frac{d\Phi x_n y}{dx_n}} + \dots \end{aligned}$$

Assuming in this equation for m the different values of the exponents of x in the terms of $\Lambda_1 xy$ successively, multiplying the equations which thus result by the respective coefficients of those terms, and adding

$$\frac{\Lambda_1 xy}{\Phi xy} = \dots \dots \dots + \frac{1}{x} \sum \frac{\Lambda_1 x_n y}{\frac{d\Phi x_n y}{dx_n}} + \dots \dots \dots$$

Thus, then, the last term of equation (M) is the coefficient of $\frac{1}{x}$ in the expansion of $\frac{\Lambda_1 xy}{\Phi xy}$, or

* Let

$A_1, A_2, \dots$ being independent of x ;

$$\frac{1}{\Phi xy} = \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2} + \frac{A_n}{x-x_n} \dots \dots \dots \frac{A_p}{x-x_p},$$

$$\therefore A_n = \frac{x_n - x_n}{\frac{d\Phi x_n y}{dx_n}} = \frac{1}{\frac{d\Phi x_n y}{dx_n}};$$

$$\therefore \frac{1}{\Phi xy} = \sum \frac{1}{(x-x_n) \frac{d\Phi x_n y}{dx_n}}.$$

Definite
Integrals.Definite
Integrals.

$$\sum \frac{\Lambda_1 xy}{d\Phi xy} = C^{\frac{1}{2}} \frac{\Lambda_1 xy}{\Phi xy};$$

$$\therefore \text{by (I),} \quad \sum \frac{\Lambda_1 xy}{d\Phi xy} = C^{\frac{1}{2}} \left\{ \frac{\Lambda xy}{(x-\alpha)\Phi xy} - \frac{\Lambda \alpha y}{(x-\alpha)\Phi xy} \right\}.$$

Now the least negative exponent in the expansion of $\frac{\Lambda \alpha y}{(x-\alpha)\Phi xy}$ is $\mu+1$, the highest positive exponent in Φxy being μ . No portion of the coefficient of $\frac{1}{x}$ is therefore found in this term of the expression;

$$\therefore \sum \frac{\Lambda_1 xy}{d\Phi xy} = C^{\frac{1}{2}} \frac{\Lambda xy}{(x-\alpha)\Phi xy};$$

$$\therefore \text{by (M)} \quad \sum \pm \frac{d\psi x}{dy} = C^{\frac{1}{2}} \frac{\Lambda xy}{(x-\alpha)\Phi xy} - \frac{\Lambda \alpha y}{\Phi \alpha y}.$$

which expression, it will be observed, does not involve x . Also

$$\frac{\Lambda \alpha y}{\Phi \alpha y} = 2f\alpha \cdot \frac{\theta_1 \alpha \cdot \frac{d\theta_2 \alpha}{dy} - \theta_2 \alpha \cdot \frac{d\theta_1 \alpha}{dy}}{(\phi_1 \alpha)(\theta_1 \alpha)^2 - \phi_2 \alpha \cdot (\theta_2 \alpha)^2},$$

in which expression $f\alpha$, $\phi_1 \alpha$, $\phi_2 \alpha$ do not involve y ;

$$\therefore \int \frac{\Lambda \alpha y}{\Phi \alpha y} dy = \frac{f\alpha}{\sqrt{\phi \alpha}} \log \left\{ \frac{\theta_1 \alpha \sqrt{\phi_1 \alpha} + \theta_2 \alpha \sqrt{\phi_2 \alpha}}{\theta_1 \alpha \sqrt{\phi_1 \alpha} - \theta_2 \alpha \sqrt{\phi_2 \alpha}} \right\};$$

also

$$\int C^{\frac{1}{2}} \frac{\Lambda xy dy}{(x-\alpha)\Phi xy} = C^{\frac{1}{2}} \frac{fx}{(x-\alpha)\sqrt{\phi x}} \log \left\{ \frac{\theta_1 x \sqrt{\phi_1 x} + \theta_2 x \sqrt{\phi_2 x}}{\theta_1 x \sqrt{\phi_1 x} - \theta_2 x \sqrt{\phi_2 x}} \right\};$$

both these integrations being made in respect to y , which is independent of x .

$$\text{By equation (H)} \quad \therefore \sum \pm \psi x = C^{\frac{1}{2}} \left\{ \frac{\Psi x \cdot fx}{x-\alpha} \right\} - \Psi \alpha f\alpha + C.$$

Of this beautiful theorem a very remarkable feature is the arbitrary character of the functions $\theta_1 x$ and $\theta_2 x$, resulting in an arbitrary superior limit of the number μ of the functions included under the symbol Σ . The further discussion of it will be reserved for that portion of this paper which treats of the theory of ultra-elliptic functions, and we shall here confine ourselves to two of the most elementary applications.

EXAMPLE 1. Let

$$fx = (x-\alpha)$$

$$\phi x = (1-x^2)(1-k^2 x^2);$$

$$\therefore \psi x = \int \frac{dx}{(1-x^2)^{\frac{1}{2}}(1-k^2 x^2)^{\frac{1}{2}}};$$

or if $x = \sin \phi$,

In this case

$$\psi x = Fk\phi.$$

$$f\alpha = \alpha - \alpha = 0,$$

$\therefore$ by Abel's theorem

$$\sum \pm \psi x = C^{\frac{1}{2}} \Psi x + C \dots \dots \dots (a).$$

Now in this case it is easily seen that

$$C^{\frac{1}{2}} \Psi x = 0.*$$

$$\therefore \sum \pm Fk\phi = C \dots \dots \dots (b).$$

Let

$$\phi_1 x = 1 - x^2$$

$$\phi_2 x = 1 - k^2 x^2$$

$$\theta_1 x = a_1 + b_1 x$$

$$\theta_2 x = a_2 + b_2 x$$

$$\mu = 4;$$

$$\text{By equation (D)} \quad \therefore (a_1 + b_1 x)^2 (1-x^2) - (a_2 + b_2 x)^2 (1-k^2 x^2) = P_4(x - \sin \phi) \dots \dots \dots (c).$$

Assuming in equation (c)

$$x = 1 \quad x = -1$$

$$x = \frac{1}{k} \quad x = -\frac{1}{k}$$

* The conditions on which the evanescence of this term depends will be found fully discussed in the tenth section of this Article.

Definite Integrals. we have

Definite Integrals.

$$\begin{aligned} -(a_2 + b_2)^2 k'^2 &= P_4 (1 - \sin \phi) \\ -(a_2 - b_2)^2 k'^2 &= P_4 (1 + \sin \phi) \\ -(b_1 + a_1 k)^2 k'^2 &= P_4 (1 - k \sin \phi) \\ -(b_1 - a_1 k)^2 k'^2 &= P_4 (1 + k \sin \phi); \\ \therefore (a_2^2 - b_2^2) k'^2 &= \pm P_4 \cos \phi \\ (b_1^2 - a_1^2 k^2) k'^2 &= \pm P_4 \Delta \phi. \end{aligned}$$

Also equating the coefficients of x^4 and x^0 in (c),

$$\begin{aligned} -b_1^2 + b_2^2 k^2 &= 1 \\ a_1^2 - a_2^2 &= P_4 \sin \phi; \\ \therefore \pm k^2 P_4 \cos \phi \pm P_4 \Delta \phi &= \{b_1^2 + a_2^2 k^2 - (a_1^2 + b_2^2) k^2\} k'^2 \\ 1 + k^2 P_4 \sin \phi &= -\{b_1^2 + a_2^2 k^2 - (a_1^2 + b_2^2) k^2\}; \\ \therefore \mp P_4 \Delta \phi \mp k^2 P_4 \cos \phi &= k'^2 + k'^2 k^2 P_4 \sin \phi. \end{aligned}$$

Now in the first term of this equation the sign $+$ must be taken, since the whole of the first member has the same sign with that term, and since moreover the second member of the equation is essentially positive;

$$\therefore P_4 \Delta \phi = k'^2 + k'^2 k^2 P_4 \sin \phi \pm k^2 P_4 \cos \phi \dots \dots \dots (d),$$

which is the equation of condition annexed to the relation

$$F\phi_1 + F\phi_2 + F\phi_3 + F\phi_4 = C.*$$

To determine the constant, suppose $\phi_1 = \phi_2 = \phi_3 = \frac{\pi}{2}$, then will the equation of condition (d) be satisfied if we assume $\phi_4 = -\frac{\pi}{2}$.

Since, then, the values $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}$ satisfy the equation (c);

$$\therefore F\frac{\pi}{2} + F\frac{\pi}{2} + F\frac{\pi}{2} - F\frac{\pi}{2} = C;$$

$$\therefore C = 2F\frac{\pi}{2};$$

$$\therefore F\phi_1 + F\phi_2 + F\phi_3 + F\phi_4 = 2F\frac{\pi}{2} \dots \dots \dots (e).$$

EXAMPLE 2. Let

$$\begin{aligned} fx &= (x - \alpha)(1 - k^2 x^2) \\ \phi x &= (1 - x^2)(1 - k^2 x^2); \\ \therefore \psi x &= \int \frac{(1 - k^2 x^2)^{\frac{1}{2}} dx}{(x - \alpha)^{\frac{1}{2}}} = Ek\phi, \end{aligned}$$

In this case, as before, $f\alpha = 0$;

$$\therefore \frac{\Psi x fx}{x - \alpha} = (1 - k^2 x^2) \Psi x;$$

$\therefore$ by Abel's theorem,

$$\Sigma \pm \psi x = C \frac{1}{x} \{1 - k^2 x^2\} \Psi x + C \dots \dots \dots (f).$$

Assume

$$\begin{aligned} \theta_1 x &= a_1 + b_1 x \\ \theta_2 x &= a_2 + b_2 x \\ \phi_1 x &= (1 - x^2) \\ \phi_2 x &= (1 - k^2 x^2) \\ u &= 3. \end{aligned}$$

Let

$$x = \frac{1}{x'}.$$

$$\therefore \{1 - k^2 x^2\} \Psi x = \left\{ \frac{1 - x'^2}{k^2 - x'^2} \right\}^{\frac{1}{2}} \log \left\{ \frac{1 + \left(\frac{b_2 + a_2 x'}{b_1 + a_1 x'} \right) \left(\frac{k^2 - x'^2}{1 - x'^2} \right)^{\frac{1}{2}}}{1 - \left(\frac{b_2 + a_2 x'}{b_1 + a_1 x'} \right) \left(\frac{k^2 - x'^2}{1 - x'^2} \right)^{\frac{1}{2}}} \right\};$$

* Let ϕ_3 and ϕ_4 become negative, the equation of condition (d) will then remain unchanged, and $F\phi_1 + F\phi_2 - F\phi_3 - F\phi_4 = C$. Suppose $\phi_4 = 0$; $\therefore F\phi_1 + F\phi_2 - F\phi_3 = 0$, and $P_3 \Delta \phi = k'^2 + k'^2 P_3 \cos \phi$, which agrees with equation 11, page 495. It will be observed that in equation (e) the ambiguous sign of equation (b) is taken positively: this assumption will be found discussed in a Note at the end of the Article.

Definite
Integrals.

 Definite
Integrals.

$$\begin{aligned} \therefore C^{\frac{1}{2}} \{1 - k^2 x^2\} \Psi x &= k C^{\frac{1}{2}} \log. \left\{ \frac{1 + \left(\frac{b_2 + a_2 x'}{b_1 + a_1 x'} \right) k}{1 - \left(\frac{b_2 + a_2 x'}{b_1 + a_1 x'} \right) k} \right\} \\ &= k C^{\frac{1}{2}} \log. \left\{ \frac{1 + \left(\frac{a_1 + a_2 k}{b_1 + b_2 k} \right) x'}{1 + \left(\frac{a_1 - a_2 k}{b_1 - b_2 k} \right) x'} \right\} \\ &= k \left\{ \left(\frac{a_1 + a_2 k}{b_1 + b_2 k} \right) - \left(\frac{a_1 - a_2 k}{b_1 - b_2 k} \right) \right\}; \\ \therefore C^{\frac{1}{2}} \{1 - k^2 x^2\} \Psi x &= k^2 \frac{2(a_1 b_2 - a_2 b_1)}{b_1^2 - b_2^2 k^2}. \end{aligned}$$

Now by Example 1, page (503.), $b_1^2 - b_2^2 k^2 = -1$. Also if $\phi_4 = 0$, $a_1^2 - a_2^2 = 0$;

$$\therefore a_1 b_2 - a_2 b_1 = -(a_1 b_1 - a_2 b_2);$$

and equating the coefficients of x^2 in equation (c), $2(a_1 b_1 - a_2 b_2) = -P_3 \sin \phi$;

$$\therefore C^{\frac{1}{2}} \{1 - k^2 x^2\} \Psi x = k^2 P_3 \sin \phi;$$

$$\therefore E\phi_1 + E\phi_2 - E\phi_3 = C + k^2 P_3 \sin \phi,$$

which agrees with equation 3, page 496.

SECTION III.

ON THE GENERAL RELATIONS OF ELLIPTIC FUNCTIONS WHICH HAVE DIFFERENT MODULI.

It is proposed in this section to investigate the relation of the amplitudes ϕ and Φ , and the moduli k and K of two functions; such that

$$Fk\phi = \mu FK\Phi \dots \dots (12.),$$

where μ is a constant quantity.

Let

$$\sin \phi = x \quad \sin \Phi = X$$

$$\chi_{kx} = \int \frac{dx}{(1-x^2)^{\frac{1}{2}}(1-k^2 x^2)^{\frac{1}{2}}} \quad \chi_{KX} = \int \frac{dX}{(1-X^2)^{\frac{1}{2}}(1-K^2 X^2)^{\frac{1}{2}}};$$

$$\therefore \chi_{kx} = \mu \chi_{KX} \dots \dots (N).$$

1. The relation $d\chi_{kx} = \mu d\chi_{KX}$ admits of the double substitution of $\frac{1}{kx}$ for x , and $\frac{1}{KX}$ for X .

For

$$\begin{aligned} d\chi_k \frac{1}{kx} &= \frac{-\frac{dx}{kx^2}}{\left(1 - \frac{1}{k^2 x^2}\right)^{\frac{1}{2}} \left(1 - \frac{k^2}{k^2 x^2}\right)^{\frac{1}{2}}} \\ &= \frac{-dx}{(1-x^2)^{\frac{1}{2}}(1-k^2 x^2)^{\frac{1}{2}}}. \end{aligned}$$

Similarly,

$$\begin{aligned} d\chi_K \frac{1}{KX} &= \frac{-dX}{(1-X^2)^{\frac{1}{2}}(1-K^2 X^2)^{\frac{1}{2}}}; \\ \therefore d\chi_k \frac{1}{kx} &= \mu d\chi_K \frac{1}{KX}. \end{aligned}$$

2. This condition of the double substitution of $\frac{1}{kx}$ and $\frac{1}{KX}$ for x and X extends to the integral

$$\chi_{kx} = \mu \chi_{KX}.$$

For let this integral solved with respect to x be represented by $x = \theta X$, $\therefore$ substituting, $d\chi_k \theta X = \mu d\chi_K$. And this is true for all values of X ;

$$\therefore d\chi_k \theta \frac{1}{KX} = \mu d\chi_K \frac{1}{KX};$$

$$\therefore d\chi_k \theta \frac{1}{KX} = d\chi_k \frac{1}{kx};$$

$$\therefore \frac{1}{kx} = \theta \frac{1}{KX}.$$

3. Let

$$\theta X = \mu X \left(\frac{1-X^2}{1-K^2 X^2} \right)^{\frac{1}{2}},$$

then will the equation $x = \theta X$ be subject to the condition of the double substitution, provided that

Definite
Integrals.Definite
Integrals.

for

$$\mu = \frac{K}{\sqrt{k}} \dots \dots \dots (O)$$

$$\begin{aligned} \theta \frac{1}{KX} &= \frac{\mu}{KX} \left(\frac{1 - \frac{1}{K^2 X^2}}{1 - \frac{K^2}{X^2}} \right)^{\frac{1}{2}} \\ &= \frac{\mu}{K^2 X} \left(\frac{1 - K^2 X^2}{1 - X^2} \right)^{\frac{1}{2}}. \end{aligned}$$

But by (O)

$$\begin{aligned} \frac{\mu}{K^2} &= \frac{1}{\mu k}; \\ \therefore \theta \frac{1}{KX} &= \frac{1}{\mu k X} \left(\frac{1 - K^2 X^2}{1 - X^2} \right)^{\frac{1}{2}} \\ &= \frac{1}{k \theta K X} \\ &= \frac{1}{k x}. \end{aligned}$$

4. The equation

$$x = \frac{KX}{\sqrt{k}} \left(\frac{1 - X^2}{1 - K^2 X^2} \right)^{\frac{1}{2}} \dots \dots \dots (P)$$

thus satisfying one condition necessary to the relation $d\chi kx = \mu d\chi KX$, let it be required to determine whether any and what other conditions are necessary to that relation.

Squaring equation (P), and transposing

$$\begin{aligned} K^4 X^4 - (1 + kx^2) K^2 X^2 + kx^2 &= 0 \dots \dots \dots (Q); \\ \therefore \text{differentiating} \quad \{2X^2 - (kx^2 + 1)\} K^2 X dX &+ \{1 - K^2 X^2\} kx dx = 0; \\ \therefore \frac{K^2}{k} \frac{XdX}{x(1 - K^2 X^2)} + \frac{xdx}{2X^2 - (kx^2 + 1)} &= 0; \end{aligned}$$

but by (Q),

$$2X^2 - (kx^2 + 1) = - \left\{ (1 + kx^2)^2 - \frac{4kx^2}{K^2} \right\}.$$

Substituting this value and that of $\frac{X}{x}$,

$$\therefore \frac{K}{\sqrt{k}} \frac{dX}{(1 - x^2)^{\frac{1}{2}} (1 - K^2 X^2)^{\frac{1}{2}}} - \frac{dx}{\left\{ 1 + 2 \left(k - \frac{2k}{K^2} \right) x^2 + k^2 x^4 \right\}^{\frac{1}{2}}} = 0.$$

Now

$$1 + 2 \left(k - \frac{2k}{K^2} \right) x^2 + k^2 x^4 = (1 - x^2) (1 - k^2 x^2),$$

if

$$-2 \left(k - \frac{2k}{K^2} \right) = 1 + k^2;$$

or if

$$K = \frac{2\sqrt{k}}{1 + k} \dots \dots \dots (R).$$

This condition being satisfied then, we have $\mu d\chi KX - d\chi kx = 0$. Thus then it appears on the whole that to the relation $\chi kx = \mu \chi KX$ the conditions (P) and (R) are sufficient. The equations (N), (P), and (R) may be otherwise written thus:

$$\left. \begin{aligned} Fk\phi &= KFK\Phi \\ \sin \phi &= \frac{\sin 2\Phi}{(1 + k) \Delta K\Phi} \\ K &= \frac{2\sqrt{k}}{1 + k} \end{aligned} \right\} \dots \dots \dots (13.)$$

5. The relation (P) is not the only one which will satisfy the condition of the double substitution.

If $\mathbf{P} \left\{ \frac{1 - \frac{x^2}{c_n^2}}{1 - k^2 c_n^2 x^2} \right\}^m$ represent the product of any number of factors obtained by giving different values to the index n of the constant entering into the formulæ enclosed within the brackets, and any value or values whatever whole fraction positive or negative to the exponent m .

Then will

$$X = \frac{x}{\mu} \mathbf{P} \left\{ \frac{1 - \frac{x^2}{c_n^2}}{1 - k^2 c_n^2 x^2} \right\}^m \dots \dots \dots (S)$$

Definite Integrals. satisfy the condition of the double substitution, provided there be given to the constant μ a certain value dependent upon the values of K, k, c, m .
Let this function of x be represented by θx ;

Definite Integrals.

$$\begin{aligned}\therefore \theta \frac{1}{kx} &= \frac{1}{\mu kx} \mathbf{P} \left\{ \frac{1 - \frac{1}{k^2 c_n^2 x^2}}{1 - \frac{k^2 c_n^2}{x^2}} \right\}^m \\ \theta \frac{1}{kx} &= \frac{1}{\mu kx} \mathbf{P} \left\{ \frac{1}{k^2 c_n^4} \right\}^m \left\{ \frac{1 - k^2 c_n^2 x^2}{1 - \frac{x^2}{c_n^2}} \right\}^m \\ &= \frac{1}{\mu^2 k} \mathbf{P} \left\{ \frac{1}{k^2 c_n^4} \right\}^m \frac{\mu}{x} \mathbf{P} \left\{ \frac{1 - k^2 c_n^2 x^2}{1 - \frac{x^2}{c_n^2}} \right\} \\ &= \frac{K}{\mu^2 k} \mathbf{P} \left\{ \frac{1}{k^2 c_n^4} \right\}^m \cdot \frac{1}{KX}.\end{aligned}$$

The condition of the double substitution holds therefore, provided*

$$\frac{K}{\mu^2 k} \mathbf{P} \left\{ \frac{1}{k^2 c_n^4} \right\}^m = 1 \dots \dots \dots (T).$$

To determine therefore the required relation between X and x , to this general form of which we are directed by the condition of the double substitution, all that is required is to assign to the constants c_n and to the exponent m in the product (S) such values as shall cause the value of X expressed by that product to satisfy the other conditions of the equation $d\chi kx = \mu d\chi KX$.

THE THEOREM OF JACOBI.

6. Now this will be the case, 1st, if we assume m equal to unity and $c_n = \sin \operatorname{am} \frac{4n}{p} F_1$, where $\operatorname{am} \frac{4n}{p} F_1$ represents the amplitude of a function equal to the fraction $\frac{4n}{p}$ of the complete function F_1 or $Fk \frac{\pi}{2}$; and 2dly, if we take the number of factors under the symbol $\mathbf{P}$ to be $\frac{p-1}{2}$, the values of n ascending from 1 to $\frac{p-1}{2}$.
These substitutions being made, equation (S) will become

$$X = \frac{x}{\mu} \mathbf{P} \left\{ \frac{1 - \frac{x^2}{\sin^2 \operatorname{am} \frac{4n}{p} F_1}}{1 - k^2 x^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1} \right\} \dots \dots \dots (14.),$$

from which equation the following is deducible,†

$$\mu = (-1)^{\frac{p-1}{2}} \mathbf{P} \left\{ \frac{\sin \operatorname{coam} \frac{4n}{p} F_1}{\sin \operatorname{am} \frac{4n}{p} F_1} \right\}^2 \dots \dots \dots (15.);$$

whence, by elimination in (T), $K = k^p \mathbf{P} \left\{ \sin \operatorname{coam} \frac{4n}{p} F_1 \right\}^4 \dots \dots \dots (16.).$

* It will be remembered that the symbol $\mathbf{P}$ indicates here, as before, the product of any number of the factors included between the brackets or of any powers or roots of any of them.

† It will hereafter be shown, that when $x=1$, $X=1$; $\therefore$ by equation (14.),

$$\mu = \mathbf{P} \left\{ \frac{1 - \frac{1}{\sin^2 \operatorname{am} \frac{4n}{p} F_1}}{1 - k^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1} \right\}.$$

Also by equation (2.) and (7.), assuming $\phi_3 = \frac{\pi}{2}$ and $\phi_2 = \operatorname{am} \frac{4n}{p} F_1$, $\sin^2 \operatorname{coam} \frac{4n}{p} F_1 = \frac{1 - \sin^2 \operatorname{am} \frac{4n}{p} F_1}{1 - k^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1}$;

$$\therefore \mu = (-1)^{\frac{p-1}{2}} \mathbf{P} \frac{\sin^2 \operatorname{coam} \frac{4n}{p} F_1}{\sin^2 \operatorname{am} \frac{4n}{p} F_1}.$$

Definite
Integrals.

This is the celebrated theorem of Jacobi.

LEMMA I.

$$\sin \operatorname{am} \left\{ Fk\phi \pm 2mFk \frac{\pi}{2} \right\} = (-1)^m \sin \operatorname{am} Fk\phi$$

$$\cos \operatorname{am} \left\{ Fk\phi \pm 2mFk \frac{\pi}{2} \right\} = (-1)^m \cos \operatorname{am} Fk\phi.$$

Let

$$\operatorname{am} \left\{ Fk\phi \pm 2mFk \frac{\pi}{2} \right\} = \phi_1;$$

$$\therefore Fk\phi \pm 2mFk \frac{\pi}{2} = Fk\phi_1.$$

Now

$$\frac{1}{\Delta\phi} = \frac{1}{\Delta\left(\phi + 2m \frac{\pi}{2}\right)};$$

therefore, by a known property of periodical definite integrals (see Theorem II.)

$$2mFk \frac{\pi}{2} = Fkm\pi;$$

$$\therefore Fk\phi \pm Fkm\pi = Fk\phi_1;$$

$$\sin \phi_1 = (-1)^m \sin \phi,$$

$$\cos \phi_1 = (-1)^m \cos \phi.$$

 $\therefore$ by equation (8.)

by equation (9.)

LEMMA II. Let

$$Fk\phi_1 + Fk\phi_2 = Fk\sigma_1$$

$$Fk\phi_1 - Fk\phi_2 = Fk\sigma_2;$$

 $\therefore$ by equation (8.)

$$\sin \sigma_1 + \sin \sigma_2 = \frac{2 \sin \phi_1 \cos \phi_2 \Delta\phi_2}{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2}$$

$$\sin \sigma_1 \sin \sigma_2 = \frac{\sin^2 \phi_1 - \sin^2 \phi_2}{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2} \dots \dots \dots (U);$$

$$\therefore (1 \mp \sin \sigma_1) (1 \mp \sin \sigma_2) = \frac{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2 + \sin^2 \phi_1 - \sin^2 \phi_2 \mp 2 \sin \phi_1 \cos \phi_2 \Delta\phi_2}{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2}.$$

Let ϕ'_2 be the coamplitude of $Fk\phi$; $\therefore$ by equation (8.)

$$\sin \phi_2 = \frac{\cos \phi'_2}{\Delta\phi'_2}, \quad \cos \phi_2 = k' \frac{\sin \phi'_2}{\Delta\phi'_2};$$

$$k' = (\Delta\phi_2) (\Delta\phi'_2).$$

and $\therefore$

Substituting these values, and reducing,

$$(1 \mp \sin \sigma_1) (1 \mp \sin \sigma_2) = \frac{k'^2 \{ \sin \phi'_2 \mp \sin \phi_1 \}^2}{(\Delta\phi'_2)^2 \{ 1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2 \}};$$

$$\therefore \frac{(1 \mp \sin \sigma_1) (1 \mp \sin \sigma_2)}{\cos^2 \phi_2} = \frac{\left\{ 1 \mp \frac{\sin \phi_1}{\sin \phi'_2} \right\}^2}{1 - k^2 \sin^2 \phi_1 \sin^2 \phi_2} \dots \dots \dots (V).$$

Let now

$$\phi_1 = \operatorname{am} F, \quad \phi_2 = \operatorname{am} \frac{4n}{p} F_1; \quad \therefore \phi'_2 = \operatorname{coam} \frac{4n}{p} F_1$$

$$\sigma_1 = \operatorname{am} \left(F + \frac{4n}{p} F_1 \right) \quad \sigma_2 = \operatorname{am} \left(F - \frac{4n}{p} F_1 \right);$$

 $\therefore$ by Lemma I. $\therefore$ by equation (U)

$$\frac{\sin \operatorname{am} \left(F + \frac{4n}{p} F_1 \right) \sin \operatorname{am} \left(F + \frac{4(p-n)}{p} F_1 \right)}{-\sin^2 \operatorname{am} \frac{4n}{p} F_1} = \frac{1 - \frac{\sin^2 \operatorname{am} F}{\sin^2 \operatorname{am} \frac{4n}{p} F_1}}{1 - k^2 \sin^2 \operatorname{am} F \sin^2 \operatorname{am} \frac{4n}{p} F_1} \dots \dots \dots (W);$$

by equation (V)

$$\frac{\left\{ 1 - \sin \operatorname{am} \left(F + \frac{4n}{p} F_1 \right) \right\} \left\{ 1 - \sin \operatorname{am} \left(F + \frac{4(p-n)}{p} F_1 \right) \right\}}{\cos^2 \operatorname{am} \frac{4n}{p} F_1} = \frac{\left\{ 1 - \frac{\sin \operatorname{am} F}{\sin \operatorname{coam} \frac{4n}{p} F_1} \right\}^2}{1 - k^2 \sin^2 \operatorname{am} F \sin^2 \operatorname{am} \frac{4n}{p} F_1} \dots \dots \dots (X).$$

Definite
Integrals.

Definite Integrals.

Since $x = \sin \operatorname{am} F$, $X = \sin \operatorname{am} \frac{F}{\mu}$, we have from equations (14.) and (W.)

$$\sin \operatorname{am} \frac{F}{\mu} = \frac{\sin \operatorname{am} F}{\mu} \prod_{\frac{p-1}{2}}^1 \left\{ \frac{\sin \operatorname{am} \left(F + \frac{4n}{p} F_1 \right) \sin \operatorname{am} \left(F + \frac{4(p-n)}{p} F_1 \right)}{-\sin^2 \operatorname{am} \frac{4n}{p} F_1} \right\};$$

$\therefore$ eliminating μ by equation (15.),

$$\sin \operatorname{am} \frac{F}{\mu} = \sin \operatorname{am} F \prod_{\frac{p-1}{2}}^1 \frac{\sin \operatorname{am} \left(F + \frac{4n}{p} F_1 \right) \sin \operatorname{am} \left(F + \frac{4(p-n)}{p} F_1 \right)}{\sin^2 \operatorname{coam} \frac{4n}{p} F_1};$$

$$\therefore \text{ by equation (16.) } \sin \operatorname{am} \frac{F}{\mu} = \left(\frac{k^p}{K} \right)^{\frac{1}{2}} \prod_p^1 \sin \operatorname{am} \left(F + \frac{4n}{p} F_1 \right) \dots \dots \dots (17.).$$

Now this equation will remain unchanged if for F we substitute in the second number $F + \frac{4m}{p} F_1$, where m is any number less than $p-1$. For by this substitution each factor of the product will be advanced m in the series of factors, except the m last factors, which are

$$\sin \operatorname{am} \left(F + \frac{4(p-m+1)}{p} F_1 \right), \sin \operatorname{am} \left(F + \frac{4(p-m+2)}{p} F_1 \right) \dots \dots \sin \operatorname{am} \left(F + \frac{4(p)}{p} F_1 \right);$$

and these become, by making the proposed substitution, and reducing by the first lemma,

$$\sin \operatorname{am} \left(F + \frac{4}{p} F_1 \right), \sin \operatorname{am} \left(F + \frac{8}{p} F_1 \right) \dots \dots \sin \operatorname{am} \left(F + \frac{4(m)}{p} F_1 \right).$$

That is, they become the first m factors of the series, and complete it.

Since, then, the value of the second member of this equation is the same, whether for F we take F , or $F + \frac{4m}{p} F_1$; and since this is, moreover, true for all values of F , it follows that it is true whether for F we take 0 or $\frac{4m}{p} F_1$. But when $F=0$ the second member of the equation vanishes, therefore it vanishes when $F = \frac{4m}{p} F_1$, for all values of m included between 1 and p ; and these are all different values of F , and they are the same in number (p) as the factors of the product $\prod_p^1$

Now this being the case, it follows that another form of the above relation (17.) is the following:

$$1 - \sin \operatorname{am} \frac{F}{\mu} = \prod_p^1 \frac{1 - \sin \operatorname{am} \left(F + \frac{4n}{p} F_1 \right)}{\cos \operatorname{am} \frac{4n}{p} F_1} \dots \dots \dots (18.);$$

for, by what has been said before, the second member of this equation retains the same value if for F we substitute in it $F + \frac{4m}{p} F_1$; and this is true for all values of F , therefore it is true whether we substitute for F the values 0 or $\frac{4m}{p} F_1$. But when for F , 0 is substituted, the second member becomes

$$\begin{aligned} & \prod_p^1 \frac{1 - \sin \operatorname{am} \frac{4n}{p} F_1}{\cos \operatorname{am} \frac{4n}{p} F_1} \\ &= \prod_{\frac{p-1}{2}}^1 \frac{\left\{ 1 - \sin \operatorname{am} \frac{4n}{p} F_1 \right\} \left\{ 1 - \sin \operatorname{am} \frac{4(p-n)}{p} F_1 \right\}}{\cos \operatorname{am} \frac{4n}{p} F_1 \cos \operatorname{am} \frac{4(p-n)}{p} F_1} \\ &= \prod_{\frac{p-1}{2}}^1 \frac{\left\{ 1 - \sin \operatorname{am} \frac{4n}{p} F_1 \right\} \left\{ 1 + \sin \operatorname{am} \frac{4n}{p} F_1 \right\}}{\cos \operatorname{am} \frac{4n}{p} F_1 \cos \operatorname{am} \frac{4n}{p} F_1} \\ &= 1; \end{aligned}$$

Definite Integrals.

Definite
Integrals. $\therefore$ in this case

$$1 - \sin \operatorname{am} \frac{F}{\mu} = 1, \text{ or } \sin \operatorname{am} \frac{F}{\mu} = 0.$$

Definite
Integrals.

Since, then, in this equation $\sin \operatorname{am} \frac{F}{\mu}$ vanishes when $F=0$, it vanishes for all the p values $\frac{4m}{p} F_1$ of F , and therefore the relation (17.) and (18.) are different forms of the same relation. Now

$$\mathbf{P}_p \frac{1 - \sin \operatorname{am} \left(F + \frac{4n}{p} F_1 \right)}{\cos \operatorname{am} \frac{4n}{p} F_1} = (1 - \sin \operatorname{am} F) \mathbf{P}_{\frac{p-1}{2}} \frac{\left\{ 1 - \sin \operatorname{am} \left(F + \frac{4n}{p} F_1 \right) \right\} \left\{ 1 - \sin \operatorname{am} \left(F + \frac{4(p-n)}{p} F_1 \right) \right\}}{\cos \operatorname{am} \frac{4n}{p} F_1 \cos \operatorname{am} \frac{4(p-n)}{p} F_1};$$

 $\therefore$ by equation (X), Lemma II.,

$$1 - \sin \operatorname{am} \frac{F}{\mu} = (1 - \sin \operatorname{am} F) \mathbf{P}_{\frac{p-1}{2}} \frac{\left\{ 1 - \frac{\sin \operatorname{am} F}{\sin \operatorname{coam} \frac{4n}{p} F_1} \right\}^2}{1 - k^2 \sin^2 \operatorname{am} F \sin^2 \operatorname{am} \frac{4n}{p} F_1} \dots\dots\dots (19.),$$

or

$$1 - X = (1 - x) \mathbf{P}_{\frac{p-1}{2}} \frac{\left\{ 1 - \frac{x}{\sin \operatorname{coam} \frac{4n}{p} F_1} \right\}^2}{1 - k^2 x^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1} \dots\dots\dots (20.).$$

Now by equation (14.) it appears that when X becomes $-X$, x becomes $-x$;

$$\therefore 1 + X = (1 + x) \mathbf{P}_{\frac{p-1}{2}} \frac{\left\{ 1 + \frac{x}{\sin \operatorname{coam} \frac{4n}{p} F_1} \right\}^2}{1 - k^2 x^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1};$$

$$\therefore (1 - X^2) = (1 - x^2) \mathbf{P}_{\frac{p-1}{2}} \frac{\left\{ 1 - \frac{x^2}{\sin^2 \operatorname{coam} \frac{4n}{p} F_1} \right\}^2}{1 - k^2 x^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1} \dots\dots\dots (Y).$$

Also when X becomes $\frac{1}{KX}$, x becomes $\frac{1}{kx}$;

$$\therefore 1 - \frac{1}{K^2 X^2} = \left(1 - \frac{1}{k^2 x^2} \right) \mathbf{P}_{\frac{p-1}{2}} \frac{\left\{ 1 - k^2 x^2 \sin^2 \operatorname{coam} \frac{4n}{p} F_1 \right\}^2}{1 - \frac{x^2}{\sin^2 \operatorname{am} \frac{4n}{p} F_1}} \left\{ \mathbf{P}_{\frac{p-1}{2}} \frac{1}{k^4 \sin^4 \operatorname{coam} \frac{4n}{p} F_1 \sin^4 \operatorname{am} \frac{4n}{p} F_1} \right\}^2$$

But $\mathbf{P}_{\frac{p-1}{2}} k^4 \sin^4 \operatorname{coam} \frac{4n}{p} F_1 \sin^4 \operatorname{am} \frac{4n}{p} F_1 = k^{-2} \left\{ \mathbf{P}_{\frac{p-1}{2}} \sin^4 \operatorname{coam} \frac{4n}{p} F_1 \right\}^2 \left\{ \mathbf{P}_{\frac{p-1}{2}} \frac{\sin^2 \operatorname{am} \frac{4n}{p} F_1}{\sin^2 \operatorname{coam} \frac{4n}{p} F_1} \right\}^2 = \frac{K^2}{k^2 \mu^2}$

by equations (15.), (16.);

$$\therefore \frac{1 - K^2 X^2}{X^2} = \mu^2 \frac{1 - k^2 x^2}{x^2} \mathbf{P}_{\frac{p-1}{2}} \frac{\left\{ 1 - k^2 x^2 \sin^2 \operatorname{coam} \frac{4n}{p} F_1 \right\}^2}{1 - \frac{x^2}{\sin^2 \operatorname{am} \frac{4n}{p} F_1}};$$

$$\therefore \text{by equation (15.) } 1 - K^2 X^2 = (1 - k^2 x^2) \mathbf{P}_{\frac{p-1}{2}} \frac{\left\{ 1 - k^2 x^2 \sin^2 \operatorname{coam} \frac{4n}{p} F_1 \right\}^2}{1 - k^2 x^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1} \dots\dots\dots (21.);$$

 $\therefore$ by equation (Y)

Definite Integrals.

Definite Integrals.

$$(1 - K^2 X^2)(1 - X^2) = (1 - k^2 x^2)(1 - x^2) \prod_{\frac{p-1}{2}}^1 \frac{\left\{1 - k^2 x^2 \sin^2 \operatorname{coam} \frac{4n}{p} F_1\right\}^2 \left\{1 - \frac{x^2}{\sin^2 \operatorname{coam} \frac{4n}{p} F_1}\right\}^2}{\left\{1 - k^2 x^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1\right\}^4}.$$

Let the numerator of the product in the second member $= M^2$, and the denominator $= N^2$;

$$\therefore (1 - K^2 X^2)(1 - X^2) = (1 - k^2 x^2)(1 - x^2) \frac{M^2}{N^2} \dots \dots (Z)$$

Let

$$X_1 = NX = \frac{x}{\mu} \prod_{\frac{p-1}{2}}^1 \left\{1 - \frac{x^2}{\sin^2 \operatorname{am} \frac{4n}{p} F_1}\right\};$$

$$\therefore (N^2 - K^2 X_1^2)(N^2 - X_1^2) = (1 - k^2 x^2)(1 - x^2) M^2.$$

The first member of this equation is the product of the factors

$$N + KX_1 \quad N - KX_1 \quad N + X_1 \quad N - X_1,$$

which are polynomials of p dimensions, N being of $p-1$, and X_1 of p dimensions. Now these four factors have no common divisor, since N and X have none. Also their product is divisible by each of the factors

$$1 + kx \quad 1 - kx \quad 1 + x \quad 1 - x;$$

And after such division each factor becomes a square, since their product is a square M^2 ; or, in other words, each of the four factors

$$N + KX_1 \quad N - KX_1 \quad N + X_1 \quad N - X_1$$

is a square, or the product of a simple quantity and a square.

Now

$$X = \frac{X_1}{N};$$

$$\therefore N^2 \frac{dX}{dx} = N \frac{dX_1}{dx} - X_1 \frac{dN}{dx} = X_2 \text{ suppose};$$

$$\therefore X_2 = (N - AX_1) \frac{dX_1}{dx} - X_1 \frac{d(N - AX_1)}{dx},$$

where A is any constant quantity.

If, then, $N - AX_1$ have a square factor, then is X_2 divisible by the simple factor of which it is the square; X_2 is therefore divisible by all the simple quantities whose squares are factors of

$$N + KX_1 \quad N - KX_1 \quad N + X_1 \quad N - X_1.$$

But the square factors of these quantities compose by their product the square M^2 , the factors of which these are the squares compose therefore M ; X_2 is therefore divisible by M .

But X_1 being of p dimensions, and N of $p-1$ dimensions, X_2 , which equals $N \frac{dX_1}{dx} - X_1 \frac{dN}{dx}$, is of $2p-2$ dimensions: also M is of $2p-2$ dimensions; $\therefore \frac{X_2}{M}$ is of no dimension in x , or it is a constant quantity.

Let x be exceeding small; in this case

$$X_1 = \frac{x}{\mu} \quad N = 1 \quad M = 1 \quad X_2 = \frac{dX_1}{dx} = \frac{1}{\mu}; \quad \therefore \frac{X_2}{M} = \frac{1}{\mu};$$

$\therefore$ for all values of x

$$\frac{X_2}{M} = \frac{1}{\mu};$$

$$\therefore M = \mu X_2 = \mu N^2 \frac{dX}{dx}; \quad \therefore \frac{M}{N^2} = \mu \frac{dX}{dx};$$

by equation (Z)

$$\therefore \frac{(1 - K^2 X^2)^{\frac{1}{2}} (1 - X^2)^{\frac{1}{2}}}{(1 - k^2 x^2)^{\frac{1}{2}} (1 - x^2)^{\frac{1}{2}}} = \mu \frac{dX}{dx}$$

$$\therefore \frac{dx}{(1 - k^2 x^2)^{\frac{1}{2}} (1 - x^2)^{\frac{1}{2}}} = \frac{\mu dx}{(1 - K^2 X^2)^{\frac{1}{2}} (1 - X^2)^{\frac{1}{2}}}.$$

The values of X , K , μ , assumed in equations (14.), (15.), (16.), satisfy, therefore, the equation $\chi kx = \mu \chi KX$, and the demonstration of the theorem of Jacobi is thus completed.

7. There are certain other forms of equation (14.) under which it will now be convenient to place it.

By equation (14.)

$$\sin \operatorname{am} \frac{F}{\mu} = \frac{\sin \operatorname{am} F}{\mu} \prod_{\frac{p-1}{2}}^1 \left\{ \frac{1 - \frac{x^2}{\sin^2 \operatorname{am} \frac{4n}{p} F_1}}{1 - k^2 x^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1} \right\};$$

Definite
Integrals.Definite
Integrals.

and by equation (Y)

$$\cos \operatorname{am} \frac{F}{\mu} = \cos \operatorname{am} F \prod_{\frac{p-1}{2}}^1 \left\{ \frac{1 - \frac{x^2}{\sin^2 \operatorname{coam} \frac{4n}{p} F_1}}{1 - k^2 x^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1} \right\}.$$

Also by Lemma II.

$$\sin \operatorname{am} \left(F + \frac{4n}{p} F_1 \right) \sin \operatorname{am} \left(F - \frac{4n}{p} F_1 \right) = - \sin^2 \operatorname{am} \frac{4n}{p} F_1 \frac{1 - \frac{x^2}{\sin^2 \operatorname{am} \frac{4n}{p} F_1}}{1 - k^2 x^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1};$$

and proceeding by a similar method to that given in Lemma II., with respect to $\cos \sigma_1$ and $\cos \sigma_2$,

$$\cos \operatorname{am} \left(F + \frac{4n}{p} F_1 \right) \cos \operatorname{am} \left(F - \frac{4n}{p} F_1 \right) = - \cos^2 \operatorname{am} \frac{4n}{p} F_1 \frac{1 - \frac{x^2}{\sin^2 \operatorname{coam} \frac{4n}{p} F_1}}{1 - k^2 x^2 \sin^2 \operatorname{am} \frac{4n}{p} F_1};$$

$$\therefore \tan \operatorname{am} \frac{F}{\mu} = \frac{\tan \operatorname{am} F}{\mu} \prod_{\frac{p-1}{2}}^1 \cot^2 \operatorname{am} \frac{4n}{p} F_1 \prod_{\frac{p-1}{2}}^1 \tan \operatorname{am} \left(F + \frac{4n}{p} F_1 \right).$$

Also by equation (21.), making $X=x=1$, since by equation (20.) these are corresponding values of X and x ,

$$K' = k' \prod_{\frac{p-1}{2}}^1 \left\{ \frac{\Delta \operatorname{coam} \frac{4n}{p} F_1}{\Delta \operatorname{am} \frac{4n}{p} F_1} \right\} = k' \prod_{\frac{p-1}{2}}^1 \frac{\left\{ \Delta \operatorname{coam} \frac{4n}{p} F_1 \right\}^{4*}}{k'^2}$$

$$K' = k' \prod_{\frac{p-1}{2}}^1 k'^2 \left\{ \frac{\sin \operatorname{coam} \frac{4n}{p} F_1}{\cos \operatorname{am} \frac{4n}{p} F_1} \right\}^4 = k'^p \prod_{\frac{p-1}{2}}^1 \left\{ \frac{\sin \operatorname{coam} \frac{4n}{p} F_1}{\sin \operatorname{am} \frac{4n}{p} F_1} \right\}^4 \prod_{\frac{p-1}{2}}^1 \left\{ \tan \operatorname{am} \frac{4n}{p} F_1 \right\}^4;$$

$$\text{by equation (15.)} \quad \therefore K' = k'^p \mu^2 \prod_{\frac{p-1}{2}}^1 \left\{ \tan \operatorname{am} \frac{4n}{p} F_1 \right\}^4; \quad \therefore \prod_{\frac{p-1}{2}}^1 \cot^2 \operatorname{am} \frac{4n}{p} F_1 = \mu \left(\frac{k'^p}{K'} \right)^{\frac{1}{2}};$$

$$\therefore \tan \operatorname{am} \frac{F}{\mu} = \left(\frac{k'^p}{K'} \right)^{\frac{1}{2}} \prod_{\frac{p-1}{2}}^1 \tan \operatorname{am} \left(F + \frac{4n}{p} F_1 \right) \dots \dots \dots (22.);$$

$$\text{also by equation (19.)} \quad \frac{1 - \sin \operatorname{am} \frac{F}{\mu}}{1 + \sin \operatorname{am} \frac{F}{\mu}} = \frac{1 - \sin \operatorname{am} F}{1 + \sin \operatorname{am} F} \prod_{\frac{p-1}{2}}^1 \left\{ \frac{\sin \operatorname{coam} \frac{4n}{p} F_1 - \sin \operatorname{am} F}{\sin \operatorname{coam} \frac{4n}{p} F_1 + \sin \operatorname{am} F} \right\}.$$

$$\therefore \tan \frac{1}{2} \left\{ \frac{\pi}{2} - \operatorname{am} \frac{F}{\mu} \right\} = \tan \frac{1}{2} \left\{ \frac{\pi}{2} - \operatorname{am} F \right\} \prod_{\frac{p-1}{2}}^1 \frac{\tan \frac{1}{2} \left\{ \operatorname{coam} \frac{4n}{p} F_1 - \operatorname{am} F \right\}}{\tan \frac{1}{2} \left\{ \operatorname{coam} \frac{4n}{p} F_1 + \operatorname{am} F \right\}} \dots \dots \dots (23.).$$

The m th, from the termination of the series of values of the form $\sin \operatorname{am} \frac{4m}{p} F_1$, obtained by giving to m the values 1 to $\frac{p-1}{2}$, is

$$\sin \operatorname{am} \frac{4 \left\{ \frac{p-1}{2} - m - 1 \right\}}{p} F_1,$$

or it is by Lemma I.

$$- \sin \operatorname{am} \frac{2(2m-1)}{p} F_1.$$

Also the m th from the beginning of the series is

$$+ \sin \operatorname{am} \frac{2(2m)}{p} F_1.$$

These are, therefore, consecutive terms included under the form

$$(-1)^n \sin \operatorname{am} \frac{2n}{p} F_1.$$

* For the relations of the amplitude and coamplitude of a function on which these transformations depend, see Lemma II.

Definite Integrals.

Wherever, therefore, in the preceding formulæ there occur factors which differ only by values of n , ascending from 1 to $\frac{p-1}{2}$, in the form $\sin \operatorname{am} \frac{4n}{p} F$, we may substitute for this form this other, $(-1)^n \sin \operatorname{am} \frac{2n}{p} F$; and the same is true of the coamplitude. Equations (14.), (20.), (15.), (16.) thus become

Definite Integrals.

$$\begin{aligned} \sin \operatorname{am} \frac{F}{\mu} &= \frac{\sin \operatorname{am} F}{\mu} \prod_{\frac{p-1}{2}}^1 \left\{ \frac{1 - \frac{\sin^2 \operatorname{am} F}{\sin^2 \operatorname{am} \frac{2n}{p} F}}{1 - k^2 \sin^2 \operatorname{am} F \sin^2 \operatorname{am} \frac{2n}{p} F} \right\} \dots\dots\dots (A'), \\ 1 - \sin \operatorname{am} \frac{F}{\mu} &= (1 - \sin \operatorname{am} F) \prod_{\frac{p-1}{2}}^1 \left\{ \frac{1 - (-1)^n \frac{\sin \operatorname{am} F}{\sin \operatorname{coam} \frac{2n}{p} F}}{1 - k^2 \sin^2 \operatorname{am} F \sin^2 \operatorname{am} \frac{2n}{p} F} \right\} \dots\dots\dots (B'), \\ \mu &= (-1)^{\frac{p-1}{2}} \prod_{\frac{p-1}{2}}^1 \left\{ \frac{\sin \operatorname{coam} \frac{2n}{p} F}{\sin \operatorname{am} \frac{2n}{p} F} \right\} \dots\dots\dots (C'), \\ K &= k^p \prod_{\frac{p-1}{2}}^1 \left\{ \sin \operatorname{coam} \frac{2n}{p} F \right\}^4 \dots\dots\dots (D') \end{aligned}$$

8. We come now to the discussion of another class of functions, included under the form

$$\sqrt{-1} Fk\phi.$$

Functions under this imaginary form having the modulus k may be transformed into others under the form of real functions, with this difference, that the modulus which was before k will now be k' .

This transformation is the object of JACOBI'S SECOND THEOREM.

$$\text{THEOREM VIII. } \sqrt{-1} Fk\phi = \int \frac{\sqrt{-1} d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int \frac{\sqrt{-1} d\phi}{\sqrt{\cos^2 \phi + k'^2 \sin^2 \phi}} = \int \frac{\sqrt{-1} \frac{d\phi}{\cos \phi}}{\sqrt{1 + k'^2 \tan^2 \phi}}.$$

Let

$$\begin{aligned} \tan \phi &= \sqrt{-1} \sin \phi'; \dots\dots\dots (E'); \\ \therefore \cos \phi &= \sec \phi'; \\ \therefore d\phi &= \sqrt{-1} \frac{d\phi'}{\cos \phi}; \\ \therefore \sqrt{-1} Fk\phi &= Fk'\phi' \dots\dots\dots (F'). \end{aligned}$$

Thus, then, the imaginary function $\sqrt{-1} Fk\phi$ will become a real function $Fk'\phi'$, provided we change k into its suppléant k' , and give to ϕ' a value determined by the relation (E'). Thus then

$$\left. \begin{aligned} \sqrt{-1} \tan \operatorname{am} Fk\phi &= \sin \operatorname{am} \sqrt{-1} Fk\phi = \sin \operatorname{am} Fk'\phi' \\ \sqrt{-1} \tan \operatorname{am} \left(F + \frac{4n}{p} F_1 \right) &= \sin \operatorname{am} \left(F' + \frac{4n}{p} F'_1 \right) \end{aligned} \right\} \dots\dots\dots (G'),$$

where F' is written for $\sqrt{-1} Fk\phi$, or $Fk'\phi'$; $\therefore$ substituting in equation (22.),

$$\sin \operatorname{am} \frac{F'}{\mu} = (-1)^{\frac{p-1}{2}} \left(\frac{k'^p}{K'} \right) \prod_{\frac{p-1}{2}}^1 \sin \operatorname{am} \left(F' + \frac{4n}{p} F'_1 \right) \dots\dots\dots (24.).$$

Also

$$\begin{aligned} \sin \operatorname{coam} F &= \frac{\cos \operatorname{am} F}{\Delta \operatorname{am} F} = \frac{1}{\Delta \operatorname{am} F'}; \\ \therefore \frac{\sin \operatorname{am} F}{\sin \operatorname{coam} F} &= -\sqrt{-1} \tan \operatorname{am} F' \cdot \Delta \operatorname{am} F' = -\frac{\sqrt{-1} \sin \operatorname{am} F'}{\sin \operatorname{coam} F'}; \end{aligned}$$

by equation (15.)

$$\therefore \mu = \prod_{\frac{p-1}{2}}^1 \left\{ \frac{\sin \operatorname{coam} \frac{4n}{p} F'_1}{\sin \operatorname{am} \frac{4n}{p} F'_1} \right\} \dots\dots\dots (25.);$$

and since

$$K' = k'^p \mu^2 \prod_{\frac{p-1}{2}}^1 \left\{ \tan \operatorname{am} \frac{4n}{p} F'_1 \right\}^4;$$

$$\therefore K' = k'^p \mu^{\frac{p-1}{2}} \prod_{\frac{p-1}{2}}^1 \left\{ \sin \operatorname{am} \frac{4n}{p} F_1 \right\}^4;$$

$$\therefore K' = k'^p \prod_{\frac{p-1}{2}}^1 \left\{ \sin \operatorname{coam} \frac{4n}{p} F_1 \right\}^4 \dots \dots \dots (26.).$$

The equations (24.), (25.), (26.) serve to determine the quantities Φ' , μ , K' , in terms of ϕ' , k' , subject to the condition $Fk'\phi' = \mu FK'\Phi'$. Comparing these equations with equations (17.), (22.), (15.), (16.), it appears "that the same formulæ which serve for the transformation of a function having the modulus k into another having the modulus K , apply equally to the transformation of a function having the modulus k' into one having the modulus K' , provided that in these last F be replaced by $\frac{F'}{\sqrt{-1}}$, and μ by $(-1)^{\frac{p-1}{2}} \mu$.

These last formulæ, which serve for the transformation of functions whose moduli are complements to the moduli of the first, are called by Jacobi complementary formulæ.

9. If in equation (B') we write, after the symbol of multiplication, the n th term from the end instead of from the beginning of the series of factors, we have

$$1 - \sin \operatorname{am} \frac{F}{\mu} = (1 - \sin \operatorname{am} F) \prod_{\frac{p-1}{2}}^1 \left\{ \frac{1 - \frac{(-1)^{\frac{p-2n+1}{2}} \sin \operatorname{am} F}{\sin \operatorname{coam} \frac{p-2n+1}{p} F_1}}{1 - k^2 \sin^2 \operatorname{am} F \sin^2 \operatorname{am} \frac{p-2n+1}{p} F_1} \right\},$$

$$\text{or } 1 - \sin \operatorname{am} \frac{F}{\mu} = (1 - \sin \operatorname{am} F) \prod_{\frac{p-1}{2}}^1 \left\{ \frac{1 - \frac{(-1)^{\frac{p-2n+1}{2}} \sin \operatorname{am} F}{\sin \operatorname{am} \frac{2n-1}{p} F_1}}{1 - k^2 \sin^2 \operatorname{am} F \sin^2 \operatorname{coam} \frac{2n-1}{p} F_1} \right\} \dots \dots \dots (H').$$

$$\text{If, therefore, } (-1)^{\frac{p-2n+1}{2}} F = \frac{2n-1}{p} F_1, \quad 1 - \sin \operatorname{am} \frac{F}{\mu} = 0,$$

and this is true for all values of n , from 1 to $\frac{p-1}{2}$. Moreover, by equation (12.), $\operatorname{am} \frac{F}{\mu}$ increases with $\operatorname{am} F$, therefore when

$$(-1)^{\frac{p-1}{2}} F = +\frac{1}{p} F_1, \quad -\frac{3}{p} F_1, \quad +\frac{5}{p} F_1, \dots \dots \dots \pm \frac{p-2}{p} F_1,$$

$$(-1)^{\frac{p-1}{2}} \operatorname{am} \frac{F}{\mu} = \frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{5\pi}{2}, \dots \dots \dots \frac{p-2}{p} F_1;$$

$$\therefore \frac{1}{p} F_1 k = \mu F_1 K \dots \dots \dots (27.),$$

$F_1 K$ and $F_1 k$ representing the complete functions $FK \frac{\pi}{2}$ and $Fk \frac{\pi}{2}$.

THEOREM IX.—By equations (14.) and (Y) it appears tha.

$$\tan \operatorname{am} \frac{F}{\mu} = \frac{\tan \operatorname{am} F}{\mu} \prod_{\frac{p-1}{2}}^1 \left\{ \frac{1 - \frac{\sin^2 \operatorname{am} F}{\sin^2 \operatorname{am} \frac{4n}{p} F_1}}{1 - \frac{\sin^2 \operatorname{am} F}{\sin^2 \operatorname{coam} \frac{4n}{p} F_1}} \right\};$$

$$\therefore \sin \operatorname{am} \frac{F'}{\mu} = \frac{\sin \operatorname{am} F'}{\mu} \prod_{\frac{p-1}{2}}^1 \left\{ \frac{1 + \frac{\tan^2 \operatorname{am} F'}{\sin^2 \operatorname{am} \frac{4n}{p} F_1}}{1 + \frac{\tan^2 \operatorname{am} F'}{\sin^2 \operatorname{coam} \frac{4n}{p} F_1}} \right\}.$$

Definite
Integrals.

Let now

$$\text{am } F' = (-1)^{\frac{p-1}{2}} \frac{\pi}{2};$$

Definite
Integrals.

$$\therefore \sin \text{am } \frac{F'}{\mu} = \frac{1}{\mu} \prod_{\frac{p-1}{2}}^1 \frac{\sin^2 \text{coam } \frac{4n}{p} F_1}{\sin^2 \text{am } \frac{4n}{p} F_1} = (-1)^{\frac{p-1}{2}} \text{ by equation (15.);}$$

 $\therefore$ when

$$\text{am } F' = (-1)^{\frac{p-1}{2}} \frac{\pi}{2}, \quad \text{am } \frac{F'}{\mu} = (-1)^{\frac{p-1}{2}} \frac{\pi}{2};$$

$$\therefore F_1 k' = \mu F_1 K' \dots \dots \dots (28.);$$

by equation (27.)

$$\therefore \frac{F_1 k}{F_1 k'} = p \frac{F_1 K}{F_1 K'} \dots \dots \dots (29.).$$

By this remarkable theorem a direct relation is established between the modulus k and the modulus K of two functions, whose amplitudes are connected by the relation (14.). For a discussion of the various applications which may be made of it in the theory of elliptic functions, the reader is referred to the third Supplement of Legendre, § vii., or to the work of Jacobi—*Fundamenta nova Theoriæ functionum ellipticarum*.

SECTION IV.

ON A METHOD OF APPROXIMATING TO THE NUMERICAL VALUES OF ELLIPTIC FUNCTIONS.

From the equation

$$K = k^p \prod_{\frac{p-1}{2}}^1 \sin^4 \text{coam } \frac{2n F_1}{p}$$

it appears that since k is less than unity, therefore K is less than k .

Thus, then, the function $Fk\phi$ may be ascertained by Jacobi's first theorem in terms of another $FK\phi$, having a less modulus. This again may be made to depend upon a third, whose modulus is yet less, and so on; and thus we may, ultimately, find the value of a function in terms of another having a modulus as much less as we like.

Again, by the second theorem, the value of a function whose modulus is k' , the complement of k , is made to depend upon one whose modulus is K' , the complement of K , and since k is greater than K , therefore k' is less than K' . By this theorem, therefore, we may make the value of one function to depend upon that of another having as much greater a modulus as we like.

And in thus making the values of functions depend upon those of others whose moduli are so much less as to be considered nearly evanescent, or so much greater as to be considered nearly equal to unity, in both which cases these values may be determined by known rules, consists the method of approximating to the numerical values of elliptic functions.

The formulæ of Jacobi supply an infinite number of relations* between k and K , by means of which this may be effected, as many indeed as there may be taken odd numbers p ; and Mr. Ivory has given analogous formulæ for the case in which p is an even number in the *Philosophical Transactions*, 1831.

Of all the infinite number of scales of moduli, however, which may thus be found, there is probably none which so well serves the purposes of calculation as that which we have first investigated (art. 9. equation 13.); as moreover this scale of moduli is that by which the existing tables of elliptic functions have been calculated, we shall enter fully on the discussion of it.

The equations (13.) admit of the following transformations:

$$\left. \begin{aligned} K &= \frac{2\sqrt{k}}{1+k} & k' &= \frac{2\sqrt{K'}}{1+K'} \\ k &= \frac{1-K'}{1+K'} & K' &= \frac{1-k}{1+k} \\ (1+K')(1+k) &= 2 \end{aligned} \right\} \dots \dots \dots (K').$$

$$\left. \begin{aligned} \sin \phi &= \frac{\sin 2\Phi}{(1+k) \Delta K \Phi} \\ \sin (2\Phi - \phi) &= k \sin \phi^\dagger \\ \tan (\phi - \Phi) &= K' \tan \phi \end{aligned} \right\} \dots \dots \dots (L').$$

* These relations have been discussed by Jacobi in his work *Fundamenta Nova*, &c. for the cases $p=3$ and $p=5$; by Guetzlaff in the twelfth volume of Crelle's *Journal*, page 173, for the case of $p=7$; and in the sixteenth number of that journal, page 97, the equations of the moduli have been placed under a variety of new forms by Professor Sohnke, and applied to the values of $p, 3, 5, 7, 11, 13, 21$.

† By equation (13.)

$$\begin{aligned} \sin^2 2\Phi &= (1+k)^2 \sin^2 \phi \left\{ 1 - \frac{4k}{(1+k)^2} \sin^2 \Phi \right\} \\ \sin^2 2\Phi &= \sin^2 \phi \{ 1 + k^2 + 2k \cos 2\Phi \}; \\ \therefore \cos 2\Phi &= \cos \phi \Delta k \phi - k \sin^2 \phi \dots \dots \dots (*); \end{aligned}$$

Definite
Integrals.Definite
Integrals.

$$\left. \begin{aligned} Fk\phi &= \frac{K}{\sqrt{k}} FK\Phi \\ Fk\phi &= \frac{2}{1+k} FK\Phi \\ Fk\phi &= (1+K') FK\Phi \end{aligned} \right\} \dots\dots\dots (M').$$

Now K^* is greater than k , and K' is less than k' ; therefore the equations

$$\left. \begin{aligned} K &= \frac{2\sqrt{k}}{1+k} \\ \sin(2\Phi - \phi) &= k \sin \phi \\ Fk\phi &= \frac{2}{1+k} FK\Phi \end{aligned} \right\} \dots\dots\dots (30.),$$

which serve to determine the value of $Fk\phi$ in terms of $FK\Phi$, make the integration of a function having a less modulus to depend on the integration of one having a greater modulus. And the equations

$$\left. \begin{aligned} k &= \frac{1-K'}{1+K} \\ \tan(\phi - \Phi) &= K' \tan \Phi \\ FK\Phi &= \frac{1}{1+K'} Fk\phi \end{aligned} \right\} \dots\dots\dots (31.),$$

which serve to determine the value of $FK\Phi$ in terms of $Fk\phi$, make the integration of a function having a greater modulus to depend upon that of a function having a less.

And by a series of equations such as these, the value of an elliptic function may be made to depend upon that of another, as near to unity or as near to zero as we like.

If k^2 be greater than $\frac{1}{2}$, it will manifestly be more convenient to use the ascending scale, and if less the descending scale.

Let $K, K_1, K_2, \dots, K_n$ be a series of moduli, and $\Phi, \Phi_1, \Phi_2, \dots, \Phi_n$ a series of amplitudes taken in an ascending scale according to the conditions of equation (30.), so that

$$K = \frac{2\sqrt{k}}{1+k}, \quad K_1 = \frac{2\sqrt{K}}{1+K}, \quad K_2 = \frac{2\sqrt{K_1}}{1+K_1}, \text{ \&c.}$$

$$\Phi = \frac{1}{2} \phi + \frac{1}{2} \sin^{-1} k \sin \phi, \quad \Phi_1 = \frac{1}{2} \Phi + \frac{1}{2} \sin^{-1} K \sin \Phi, \quad \Phi_2 = \frac{1}{2} \Phi_1 + \frac{1}{2} \sin^{-1} K_1 \sin \Phi_1, \text{ \&c.} = \text{\&c.}$$

$$\text{Then} \quad Fk\phi = \frac{2}{1+k} FK\Phi, \quad FK\Phi = \frac{2}{1+K} FK_1\Phi_1, \quad FK_1\Phi_1 = \frac{2}{1+K_1} FK_2\Phi_2, \text{ \&c.} = \text{\&c.};$$

$$\therefore Fk\phi = \frac{2^{n+1} FK_n\Phi_n}{\prod_{i=1}^n (1+K_{i-1})^\dagger} \dots\dots\dots (N').$$

If, similarly, $k, k_1, k_2, k_3, \dots$ be moduli, and $\phi, \phi_1, \phi_2, \phi_3, \dots$ amplitudes determined according to the conditions of the descending scale of equations (31.), so that

$$k = \frac{1-K'}{1+K}, \quad k_1 = \frac{1-k'}{1+k'}, \quad k_2 = \frac{1-k'_1}{1+k'_1}, \quad k_3 = \frac{1-k'_2}{1+k'_2}, \text{ \&c.} = \text{\&c.}$$

$$\phi = \Phi + \tan^{-1} K' \tan \Phi, \quad \phi_1 = \phi + \tan^{-1} k' \tan \phi, \quad \phi_2 = \phi_1 + \tan^{-1} k'_1 \tan \phi_1, \text{ \&c.} = \text{\&c.}$$

$$\therefore FK\Phi = \frac{1}{1+K'} Fk\phi, \quad Fk\phi = \frac{1}{1+k'} Fk_1\phi_1, \quad Fk_1\phi_1 = \frac{1}{1+k'_1} Fk_2\phi_2, \quad Fk_2\phi_2 = \frac{1}{1+k'_2} Fk_3\phi_3, \text{ \&c.} = \text{\&c.}$$

$$\therefore \sin^2 2\Phi = \sin^2 \phi \{1 + k^2 + 2k \cos \phi \Delta k\phi - 2k^2 \sin^2 \phi\}$$

$$\sin 2\Phi = \sin \phi \{ \Delta k\phi + k \cos \phi \} \dots\dots\dots (\xi);$$

$$\therefore \sin(2\Phi - \phi) = k \sin \phi \cos(2\Phi - \phi) = \Delta k\phi$$

$$\sin \{ \Phi + (\Phi - \phi) \} + \sin \{ \Phi - (\Phi - \phi) \} = k \sin \phi + \sin \phi$$

$$\sin \{ \Phi + (\Phi - \phi) \} - \sin \{ \Phi - (\Phi - \phi) \} = k \sin \phi - \sin \phi$$

$$2 \sin \Phi \cos(\Phi - \phi) = (1+k) \sin \phi;$$

$$2 \cos \Phi \sin(\Phi - \phi) = -(1-k) \sin \phi; \therefore \tan(\Phi - \phi) = \frac{1-k}{1+k} \tan \Phi.$$

$$* \quad 1 > \sqrt{k}; \therefore 1 + \frac{1}{\sqrt{k}} > \sqrt{k} + \frac{1}{\sqrt{k}}; \therefore 1 + \frac{2}{\sqrt{k}} + \frac{1}{k} > k + 2 + \frac{1}{k}; \therefore \frac{2}{\sqrt{k}} > 1+k; \therefore \frac{2\sqrt{k}}{1+k} > k; \therefore K > k.$$

† K_{n-1} is here supposed to represent k .

Definite
Integrals.Definite
Integrals.

$$\therefore FK\Phi = \frac{Fk_n\phi_n}{\prod_n (1+k'_{n-1})^*} \dots\dots\dots (O').$$

or since

$$\frac{1}{1+k'_{n-1}} = \frac{1+k_n}{2};$$

$$\therefore FK\Phi = \frac{Fk_n\phi_n \cdot \prod_n (1+k_n)}{2^{n+1}} \dots\dots\dots (P'),$$

Suppose that n is taken so great that in the first case K_n may equal unity, and in the other case k_n may equal zero;

$$\therefore FK_n\Phi_n = \int \frac{d\Phi_n}{\cos \Phi_n} = \log. \tan \left(\frac{\pi}{4} + \frac{1}{2} \Phi_n \right),$$

$$Fk_n\phi_n = \int d\phi_n = \phi_n,$$

$$\therefore Fk\phi = \frac{2^{n+1} \log. \tan \left(\frac{\pi}{4} + \frac{1}{2} \Phi_n \right)}{\prod_n (1+K_{n-1})} \dots\dots\dots (32.).$$

$$FK\Phi = \frac{\phi_n}{\prod_n (1+k'_{n-1})} \dots\dots\dots (33.).$$

These two equations are sufficient to determine the numerical value of any elliptic function of the first order.

12. Now the values of all elliptic functions of the second and third orders may be determined in terms of those of elliptic functions of the first order.

By equation (13.) we have†

$$d\Phi \Delta K\Phi = \frac{d\phi}{\Delta k\phi} \frac{\{k \cos \phi + \Delta k\phi\}^2}{2(1+k)},$$

$$\text{or} \quad 2(1+k) EK\Phi = \int \frac{d\phi}{\Delta k\phi} \{k \cos \phi + \Delta k\phi\}^2 = \int \frac{d\phi}{\Delta k\phi} \{2(\Delta k\phi)^2 + 2k \cos \phi \Delta k\phi - k'^2\};$$

$$\therefore (1+k) EK\Phi = Ek\phi + k \sin \phi - \frac{1}{2} k'^2 Fk\phi,$$

$$\text{or} \quad EK\Phi = \frac{Ek\phi}{1+k} - \frac{1}{2} (1-k) Fk\phi + \frac{k \sin \phi}{1+k} \dots\dots\dots (34.).$$

Thus the value of $EK\Phi$ is made to depend upon that of $Ek\phi$, and that of $Fk\phi$; K, Φ , depending upon k, ϕ , by the same relation which connects the amplitudes and moduli of functions of the first order.

By a *scale* of such amplitudes and moduli, we may, therefore, determine the value of $EK\Phi$ in terms of $Ek_n\phi_n$, where k_n approximates as nearly as we like to zero, and $Ek_n\phi_n$ to ϕ_n ; or in terms of $EK_n\Phi_n$, where K_n approximates as nearly as we like to unity, and $EK_n\Phi_n$ to $-\cos \Phi_n$. This process of reduction will be greatly facilitated if we assume

$$GK\Phi = EK\Phi - FK\Phi;$$

for subtracting from equation (34.) the equation

$$FK\Phi = \frac{1+k}{2} Fk\phi,$$

* k'_{n-1} is here supposed to represent K' .

† Differentiating equation (ε), see note, page 513,

$$2 \sin 2\Phi \frac{d\Phi}{d\phi} = \sin \phi \frac{\{k \cos \phi + \Delta k\phi\}^2}{\Delta k\phi};$$

$\therefore$ dividing by (ξ)

$$2 \frac{d\Phi}{d\phi} = \frac{k \cos \phi + \Delta k\phi}{\Delta k\phi}.$$

Also by (ε)

$$K^2 \sin^2 \Phi = \frac{K^2 (1 - \cos 2\Phi)}{2} = \frac{2k}{(1+k)^2} \{1 + k \sin^2 \phi - \cos \phi \Delta k\phi\};$$

$$\therefore \Delta K\Phi = \frac{\Delta k\phi + k \cos \phi}{1+k};$$

$$\therefore 2 \Delta K\Phi \frac{d\Phi}{d\phi} = \frac{\{k \cos \phi + \Delta k\phi\}}{(1+k) \Delta k\phi}.$$

Definite
Integrals.

we have

$$GK\Phi = \frac{1}{1+k} \{Gk\phi - kFk\phi + k \sin \phi\}.$$

Similarly,

$$Gk\phi = \frac{1}{1+k_1} \{Gk_1\phi_1 - k_1Fk_1\phi_1 + k_1 \sin \phi_1\}$$

&c. = &c.

$$Gk_{n-1}\phi_{n-1} = \frac{1}{1+k_n} \{Gk_n\phi_n - k_nFk_n\phi_n + k_n \sin \phi_n\};$$

∴ by elimination

$$GK\Phi = \sum_n^0 \left\{ \frac{k_n \{ \sin \phi_n - Fk_n \phi_n \}}{\mathbf{P}_n(1+k_n)} \right\} + \frac{Gk_n \phi_n}{\mathbf{P}_n(1+k_n)},$$

Now, by equation (P)

$$\frac{Fk_n \phi_n}{\mathbf{P}_n(1+k_n)} = \frac{2^{n+1} FK\Phi}{\mathbf{P}_n(1+k_n)^2};$$

also

$$\frac{k_{n-1}^2}{k_n} = \frac{2^2}{(1+k_n)^2};$$

$$\therefore \frac{2^{n+1} k_{n+1}}{\mathbf{P}_n(1+k_n)^2} = \frac{k_n}{2^{n+1}} \mathbf{P}_n \frac{k_{n-1}^2}{k_n} = \frac{K^2}{2^{n+1}} \mathbf{P}_n k_{n-1};$$

$$\therefore GK\Phi = \sum_n^0 \left\{ \frac{K \sin \phi_n \mathbf{P}_n k_{n-1} - K^2 \cdot FK\Phi \cdot \mathbf{P}_n k_{n-1}}{2^{n+1}} \right\} + \frac{KGk_n \phi_n}{2^{n+1} k} \mathbf{P}_n k_{n-1} \dots \dots \dots (35.).$$

LEMMA III. Let $Q = \frac{\tan \phi}{\Delta k \phi}$, and let ι be an indeterminate quantity;

$$\therefore \frac{dQ}{1+\iota Q^2} = \frac{d\phi}{\Delta k \phi} \cdot \frac{1 - k^2 \sin^4 \phi}{(1 - k^2 \sin^2 \phi) \cos^2 \phi + \iota \sin^2 \phi}.$$

Let the denominator of this last fraction equal the product of the factors

$$(1 + n \sin^2 \phi) \left(1 + \frac{k^2}{n} \sin^2 \phi \right),$$

then will

$$\iota = (1+n) \left(1 + \frac{k^2}{n} \right);$$

$$\therefore \frac{1}{\sqrt{\iota}} \tan^{-1} \sqrt{\iota} \left(\frac{\tan \phi}{\Delta k \phi} \right) = \int \frac{d\phi}{\Delta k \phi} \frac{1 - k^2 \sin^4 \phi}{(1 + n \sin^2 \phi) \left(1 + \frac{k^2}{n} \sin^2 \phi \right)} \dots \dots \dots (Q').$$

The second member of this equation may be put under the form

$$\int \frac{d\phi}{\Delta k \phi} \left\{ \frac{1}{1 + n \sin^2 \phi} + \frac{1}{1 + \frac{k^2}{n} \sin^2 \phi} - 1 \right\};$$

$$\therefore \Pi n k \phi + \Pi \frac{k^2}{n} k \phi = Fk\phi + \frac{1}{\sqrt{\iota}} \tan^{-1} \sqrt{\iota} \frac{\tan \phi}{\Delta k \phi} \dots \dots \dots (R').$$

LEMMA IV. If we assume

$$\sin \phi = \frac{(1+K) \sin \Psi}{1 + K \sin^2 \Psi},$$

then will

$$Fk\phi = (1+K) FK\Psi,$$

where

$$\frac{2\sqrt{K}}{1+K} = k,$$

for

$$d\phi = \frac{(1+K) d\Psi}{\sqrt{1-K^2 \sin^2 \Psi}} \cdot \frac{1 - K \sin^2 \Psi}{1 + K \sin^2 \Psi},$$

but

$$\frac{1 - K \sin^2 \Psi}{1 + K \sin^2 \Psi} = \sqrt{1 - k^2 \sin^2 \phi};$$

$$\therefore \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = (1+K) \frac{d\Psi}{\sqrt{1 - K^2 \sin^2 \Psi}},$$

∴ &c.

Definite
Integrals.

Definite
Integrals.

13. On the reduction of elliptic functions of the third order.

 Definite
Integrals.

Assume

$$\sin \phi = \frac{(1+K) \sin \Psi}{1+K \sin^2 \Psi};$$

$$\therefore \Pi n k \phi = \int \frac{1}{1+n \sin^2 \phi} \frac{d\phi}{\Delta k \phi} = (1+K) \int \frac{d\Psi}{\Delta K \Psi} \frac{\{1+K \sin^2 \Psi\}^2}{(1+K \sin^2 \Psi)^2 + n(1+K)^2 \sin^2 \Psi}.$$

Let the denominator of this fraction be assumed equal to the product

$$(1+M \sin^2 \Psi)(1+N \sin^2 \Psi)$$

Equating the coefficients

$$\begin{cases} M+N=2K+n(1+K)^2 \\ MN=K^2 \end{cases}$$

where M and N may be found. Of these let N be the less, then is N less than K.

$$\begin{aligned} \text{Now } \{1+K \sin^2 \Psi\}^2 &= (1+K \sin^2 \Psi) + K(1+K \sin^2 \Psi) \sin^2 \Psi \\ &\quad + \kappa(1-K^2 \sin^4 \Psi) - \kappa(1-K \sin^2 \Psi)(1+K \sin^2 \Psi), \end{aligned}$$

 where κ may be any whatever;

$$\therefore \{1+K \sin^2 \Psi\}^2 = \kappa(1-K^2 \sin^4 \Psi) + (1+K \sin^2 \Psi) \{(1-\kappa) + (1+\kappa)K \sin^2 \Psi\}$$

$$\therefore \Pi n k \phi = \kappa(1+K) \int \frac{d\Psi (1-K^2 \sin^4 \Psi)}{\Delta K \Psi (1+M \sin^2 \Psi)(1+N \sin^2 \Psi)} + (1+K) \int \frac{d\Psi \{1+K \sin^2 \Psi\} \{(1-\kappa) + (1+\kappa)K \sin^2 \Psi\}}{\Delta K \Psi \{1+M \sin^2 \Psi\} \{1+N \sin^2 \Psi\}}.$$

Assume

$$(1+\kappa)K = M(1-\kappa).$$

The second term of the above equation becomes then

$$(1+K)(1-\kappa) \int \frac{d\Psi}{\Delta K \Psi} \frac{1+K \sin^2 \Psi}{1+N \sin^2 \Psi}$$

$$= (1+K)(1-\kappa) \frac{K}{N} \int \frac{d\Psi}{\Delta K \Psi} \frac{\left(\frac{N}{K}-1\right) + (1+N \sin^2 \Psi)}{1+N \sin^2 \Psi}$$

$$= (1-\kappa) \frac{(1+K)(N-K)}{N} \Pi N K \Psi + (1-\kappa) \frac{(1+K)K}{N} F K \Psi$$

$$= -2\kappa(1+K) \Pi N K \Psi + (1+\kappa)(1+K) F K \Psi.$$

Also by Lemma III.

$$\int \frac{d\Psi}{\Delta K \Psi} \frac{1-K^2 \sin^4 \Psi}{(1+N \sin^2 \Psi)\left(1+\frac{K^2}{N} \sin^2 \Psi\right)} = \frac{1}{\sqrt{\epsilon}} \tan^{-1} \frac{\sqrt{\epsilon} \tan \Psi}{\Delta K \Psi},$$

where

$$\epsilon = (1+N)\left(1+\frac{K^2}{N}\right) = (1+N)(1+M) = 1+(M+N)+MN = (1+K)^2(1+n);$$

 $\therefore$ on the whole,

$$\Pi n k \phi = \frac{\kappa}{\sqrt{1+n}} \tan^{-1} \frac{\sqrt{\epsilon} \tan \Psi}{\Delta K \Psi} - 2\kappa(1+K) \Pi N K \Psi + (1+\kappa)(1+K) F K \Psi \dots (36.).$$

Also

$$(1+\kappa)K = (1-\kappa)M;$$

$$\therefore \kappa = \frac{M-K}{M+K} = \frac{M-\sqrt{MN}}{M+\sqrt{MN}} = \frac{M+N-2\sqrt{MN}}{M-N},$$

and

$$M-N = (1+K)\{4K+n(1+K)^2\}\sqrt{n};$$

$$\therefore \kappa = \frac{\sqrt{n}}{\left\{\frac{4K}{(1+K)^2} + n\right\}^{\frac{1}{2}}} \quad \kappa = \left\{\frac{n}{n+k^2}\right\}^{\frac{1}{2}}.$$

Thus, then, the value of a function of the third order may be found in terms of a function of the same order having a less modulus K, and a less parameter N.

14. On the theory of complete functions.

 If in equations (28.) we assume $\Phi = \frac{\pi}{2}$ and $\phi = \pi$, the second of these equations will be satisfied. Also from the third,

$$F k \pi = 2 F k \frac{\pi}{2} = \frac{2}{1+k} F K \frac{\pi}{2};$$

$$\therefore F k \frac{\pi}{2} = \frac{1}{1+k} F K \frac{\pi}{2};$$

$$\therefore F K \frac{\pi}{2} = \frac{2}{1+K'} F k \frac{\pi}{2}.$$

Definite
Integrals.

Also similar substitutions in the subsequent equations of the scale will satisfy, in each, the relation of the amplitudes.

Definite
Integrals.

$$\therefore Fk \frac{\pi}{2} = \frac{FK_n \frac{\pi}{2}}{\prod_n (1 + K_{n-1})} \dots \dots \dots (S'),$$

$$FK \frac{\pi}{2} = \frac{2^{n+1} Fk_n \frac{\pi}{2}}{\prod_n (1 + k'_{n-1})} \dots \dots \dots (T').$$

In equation (35.) let Φ be assumed $= \frac{\pi}{2}$, and $\phi = \pi$;

$$\therefore GK \frac{\pi}{2} = -\sum_n \left(\frac{K^2}{2^{n+1}} \right) FK \frac{\pi}{2} \prod_n k_{n-1};$$

since when $k_n = 0$, $Gk\phi_n = 0$;

$$\therefore EK \frac{\pi}{2} = FK \frac{\pi}{2} \left\{ 1 - \sum_n \frac{K^2}{2^{n+1}} \prod_n k_{n-1} \right\} \dots \dots \dots (U').$$

15. To determine the values of $Fk \frac{\pi}{2}$ and $Ek \frac{\pi}{2}$ in converging series when k is small.

Let

$$k = \frac{2\sqrt{K}}{1+K}$$

$$\therefore Fk\phi = F \frac{2\sqrt{K}}{1+K} \phi = \int \left\{ 1 - \frac{4K}{(1+K)^2} \sin^2 \phi \right\}^{-\frac{1}{2}} d\phi = (1+K) \int \{ 1 + 2K \cos 2\phi + K^2 \}^{-\frac{1}{2}} d\phi.$$

Let

$$\begin{aligned} \varepsilon^{2\phi} \sqrt{-1} &= x; \therefore 2 \cos 2\phi = x + x^{-1}; \\ \therefore Fk\phi &= (1+K) \int (1+Kx)^{-\frac{1}{2}} (1+Kx^{-1})^{-\frac{1}{2}} d\phi \\ &= (1+K) \int \left\{ 1 + \left(\frac{1}{2} \right)^2 K^2 + \left(\frac{1.3}{2.4} \right)^2 K^4 + \left(\frac{1.3.5}{2.4.6} \right)^2 K^6 + \dots \right. \\ &\quad \left. + A(x+x^{-1}) + B(x^3+x^{-3}) + \dots \right\} d\phi, \end{aligned}$$

where A, B, &c. are certain functions of K;

$$\therefore Fk\phi = (1+K) \left\{ 1 + \left(\frac{1}{2} \right)^2 K^2 + \left(\frac{1.3}{2.4} \right)^2 K^4 + \left(\frac{1.3.5}{2.4.6} \right)^2 K^6 + \dots \right\} \phi + (1+K) \left\{ A \sin 2\phi + \frac{1}{2} B \sin 4\phi + \dots \right\}$$

$$\therefore Fk \frac{\pi}{2} = (1+K) \left\{ 1 + \left(\frac{1}{2} \right)^2 K^2 + \left(\frac{1.3}{2.4} \right)^2 K^4 + \left(\frac{1.3.5}{2.4.6} \right)^2 K^6 + \dots \right\} \frac{\pi}{2} \dots \dots \dots (37.).$$

Similarly $Ek\phi = (1+K)^{-1} \int \{ 1 + 2K \cos 2\phi + K^2 \}^{\frac{1}{2}} d\phi = (1+K)^{-1} \int (1+Kx)^{\frac{1}{2}} (1-Kx^{-1})^{\frac{1}{2}} d\phi$

$$= (1+K)^{-1} \int \left\{ 1 + \left(\frac{1}{2} \right)^2 K^2 + \left(\frac{1.3}{2.4} \right)^2 K^4 + \left(\frac{1.3.5}{2.4.6} \right)^2 K^6 + \dots + 2A \cos 2\phi + 2B \cos 4\phi + \dots \right\} d\phi;$$

$$\therefore Ek \frac{\pi}{2} = (1+K)^{-1} \left\{ 1 + \left(\frac{1}{2} \right)^2 K^2 + \left(\frac{1.3}{2.4} \right)^2 K^4 + \left(\frac{1.3.5}{2.4.6} \right)^2 K^6 + \dots \right\} \frac{\pi}{2} \dots \dots \dots (38.).$$

Now K is always less than k ; when, therefore, k is small, we may neglect the powers of K:

$$\therefore Fk \frac{\pi}{2} = \frac{\pi(1+K)}{2} \dots \dots \dots (V');$$

$$\therefore Ek \frac{\pi}{2} = \frac{\pi(1+K)^{-1}}{2} \dots \dots \dots (W');$$

$$\therefore Fk \frac{\pi}{2} \times Ek \frac{\pi}{2} = \frac{\pi^2}{4} \dots \dots \dots (X').$$

16. To determine the value of a complete function in a convergent series when its modulus very nearly equals unity.

$$Fk\phi = \int \sec \phi \{ 1 + k^2 \tan^2 \phi \}^{-\frac{1}{2}} d\phi.$$

Expanding and applying the formulæ of reduction

$$\int \sec \phi \tan^n \phi d\phi = \frac{1}{n} \sec \phi \tan^{n-1} \phi - \frac{n-1}{n} \int \sec \phi \tan^{n-2} \phi d\phi,$$

$$\begin{aligned} Fk\phi &= \int \sec \phi d\phi - \frac{1}{2} k^2 \left\{ \frac{1}{2} \sec \phi \tan \phi - \frac{1}{2} \int \sec \phi d\phi \right\} \\ &\quad + \frac{1.3}{2.4} k^4 \left\{ \frac{1}{4} \sec \phi \tan^3 \phi - \frac{3}{8} \sec \phi \tan \phi + \frac{3}{8} \int \sec \phi d\phi \right\} + \dots \end{aligned}$$

$$\underbrace{\text{Definite Integrals.}} = \left\{ 4 + \frac{1}{4} k'^2 + \frac{1.3}{2.4} \cdot \frac{3}{8} k'^4 + \dots \right\} \log. \{ \tan \phi + \sec \phi \} - \frac{1}{4} k'^2 \sec \phi \tan \phi \left\{ 1 - \frac{1.3}{2.4} k'^2 \tan^2 \phi + \frac{1.3}{2.4} \cdot \frac{3}{2} k'^2 + \dots \right\} \underbrace{\text{Definite Integrals.}}$$

Now let $2Fk\phi = Fk \frac{\pi}{2}$; $\therefore$ by equation (1.) $\tan \phi = \frac{1}{\sqrt{k'}}$;

$$\therefore \frac{1}{2} Fk \frac{\pi}{2} = \left\{ 1 + \frac{1}{4} k'^2 + \frac{1.3}{2.4} \cdot \frac{3}{8} k'^4 + \dots \right\} \log. \left\{ \frac{1}{\sqrt{k'}} + \sqrt{1 + \frac{1}{k'}} \right\} - \frac{1}{4} k' \sqrt{1 + k'} \left\{ 1 - \frac{1.3}{2.4} k' + \frac{1.3}{2.4} \cdot \frac{3}{2} k'^2 + \dots \right\}.$$

Now as k approaches unity, k' diminishes continually, therefore when k is very nearly equal to unity

$$\frac{1}{2} Fk \frac{\pi}{2} = \log. \frac{2}{\sqrt{k'}} \quad Fk \frac{\pi}{2} = \log. \left(\frac{4}{k'} \right) \dots \dots \dots (U').$$

Again,

$$Ek\phi = \int \cos \phi \{ 1 + k'^2 \tan^2 \phi \} d\phi;$$

whence expanding and applying the formulæ of reduction

$$\int \cos \phi \tan^n \phi d\phi = \frac{1}{n-2} \sin \phi \tan^{n-2} \phi - \frac{n-1}{n-2} \int \cos \phi \tan^{n-2} \phi d\phi;$$

$$\therefore Ek\phi = \left\{ \frac{k'^2}{2} + \frac{1.1}{2.4} \cdot \frac{3}{2} k'^4 + \dots \right\} \log. \{ \tan \phi + \sec \phi \} + \sin \phi \left\{ 1 + \frac{1}{2} k'^2 - \frac{1.1}{2.4} \cdot \frac{1}{2} k'^4 \tan^2 \phi - \frac{1.1}{2.4} \cdot \frac{3}{2} k'^4 \dots \right\}.$$

Assuming as before in equation (1.) $\phi_2 = \frac{\pi}{2} \phi_1 = \phi_2 = \phi \quad \tan \phi = \frac{1}{\sqrt{k'}}$.

Also by equation (3.) $2Ek\phi - Ek \frac{\pi}{2} = (1 - k'^2) \sin^2 \phi$; $\therefore 2Ek\phi - Ek \frac{\pi}{2} = 1 - k'$;

$$\therefore Ek \frac{\pi}{2} + (1 - k') = 2 \left\{ \frac{1}{2} k'^2 + \frac{1.1}{2.4} \cdot \frac{3}{2} k'^4 + \dots \right\} \log. \left\{ \frac{1 + \sqrt{1 + k'}}{\sqrt{k'}} \right\} + \frac{2}{\sqrt{1 + k'}} \left\{ 1 - \frac{1}{2} k'^2 - \frac{1.1}{2.4} \cdot \frac{1}{2} k'^3 - \dots \right\}.$$

Let now k very nearly equal unity, or k' very nearly equal zero;

$$\therefore Ek \frac{\pi}{2} + (1 - k') = \frac{1}{2} k'^2 \log. \frac{4}{k'} + 2 \left\{ 1 - \frac{1}{2} k' - \frac{1}{8} k'^2 + \dots \right\};$$

$$\therefore Ek \frac{\pi}{2} = 1 + \frac{1}{2} k'^2 \left\{ \log. \frac{4}{k'} - \frac{1}{2} \right\} \dots \dots \dots (V').$$

17. The remarkable relation $E_1 k \cdot E_1 k' - G_1 k \cdot G_1 k' = \frac{\pi}{2}$ between functions of the first and second orders, whose functions are complements of one another, was discovered by Legendre. It may be demonstrated as follows:

$$\frac{k^2 \sin \phi \cos \phi}{\Delta k \phi} = \int \frac{k^2 \cos^2 \phi - k^2 \sin^2 \phi + k^4 \sin^4 \phi}{(\Delta k \phi)^3} d\phi = \int \frac{-k'^2 + (\Delta k \phi)^4}{(\Delta k \phi)^3} d\phi = -k'^2 \int \frac{d\phi}{(\Delta k \phi)^3} + Ek\phi.$$

$$\text{Also } \frac{dEk\phi}{dk} = - \int \frac{k \sin^2 \phi}{\Delta k \phi} d\phi = - \frac{1}{k} \frac{1 - (\Delta k \phi)^2}{\Delta k \phi} = + \frac{1}{k} (Ek\phi - Fk\phi) = + \frac{1}{k} Gk\phi; \text{ similarly } \frac{dEk'\phi}{dk'} = + \frac{1}{k'} Gk'\phi$$

$$\frac{dFk\phi}{dk} = \int \frac{k \sin^2 \phi}{(\Delta k \phi)^3} d\phi = \frac{1}{k} \int \frac{1 - (\Delta k \phi)^2}{(\Delta k \phi)^3} d\phi = \frac{1}{k} \int \frac{d\phi}{(\Delta k \phi)^3} - \frac{1}{k} Fk\phi;$$

$$\therefore \frac{dF_1 k}{dk} = \frac{1}{k k'^2} \{ E_1 k - k'^2 F_1 k \} \quad \frac{dG_1 k}{dk} = \frac{d(E_1 k - F_1 k)}{dk} = - \left(\frac{k}{k'^2} \right) E_1 k.$$

$$\text{Similarly } \frac{dG_1 k'}{dk'} = - \left(\frac{k}{k'} \right) \frac{dG_1 k'}{dk'} = + \left(\frac{k}{k'} \right) \left(\frac{k'}{k^2} \right) E_1 k' = \frac{1}{k} E_1 k'; \text{ also } \frac{dE_1 k'}{dk} = - \left(\frac{k}{k'} \right) \frac{dE_1 k'}{dk'} = - \left(\frac{k}{k'^2} \right) G_1 k'$$

Now let $E_1 k \cdot E_1 k' - G_1 k \cdot G_1 k' = H$; differentiating in respect to k , and substituting

$$\frac{dH}{dk} = \frac{1}{k} G_1 k E_1 k' - \frac{k}{k'^2} G_1 k' E_1 k - G_1 k' \frac{dG_1 k}{dk} - G_1 k \frac{dG_1 k'}{dk'} = G_1 k \left\{ \frac{1}{k} E_1 k' - \frac{dG_1 k'}{dk_1} \right\} - G_1 k' \left\{ \frac{k}{k'^2} E_1 k - \frac{dG_1 k}{dk} \right\} = 0;$$

H does not then contain k . Let then k be indefinitely small; $\therefore$ by equations (R') (S')

$$F_1 k = \frac{1}{2} \pi (1 + K), \quad E_1 k = \frac{1}{2} \pi (1 + K)^{-1} = \frac{1}{2} \pi (1 - K); \quad \therefore G_1 k = -\pi K.$$

Also $E_1 k' = 1$, and by equations (U') $F_1 k' = \log. \left(\frac{4}{k} \right)$; $\therefore G_1 k' = 1 - \log. \left(\frac{4}{k} \right)$;

$$\therefore H = \frac{1}{2} \pi (1 - K) + \pi K \left\{ 1 - \log. \left(\frac{4}{k} \right) \right\} = \frac{\pi}{2} - K\pi \left\{ \log. \left(\frac{4}{k} \right) - \frac{1}{2} \right\} = \frac{\pi}{2} - \frac{K\pi}{k^2} \{ Ek' - 1 \} \text{ by equation (V')};$$

$$\therefore E_1 k \cdot E_1 k' - G_1 k G_1 k' = \frac{\pi}{2}.$$

Definite
Integrals.Definite
Integrals.

Table of the numerical values of COMPLETE Elliptic Functions of the FIRST and SECOND orders for values of the modulus k corresponding to each degree of the angle $\sin^{-1}k$.

$\sin^{-1}k$	F_1	E_1	$\sin^{-1}k$	F_1	E_1
0°	1.57079 63267 95	1.57079 63267 95	46°	1.86914 75460 31	1.34180 60581 31
1	1.57091 59581 27	1.57067 67091 28	47	1.88480 86573 80	1.33286 99541 18
2	1.57127 49523 72	1.57031 79198 97	48	1.90108 30334 65	1.32384 21844 82
3	1.57187 36105 14	1.56972 01504 24	49	1.91799 75464 38	1.31472 95602 65
4	1.57271 24349 95	1.56888 37196 08	50	1.93558 10960 06	1.30553 90942 97
5	1.57379 21309 25	1.56780 90739 77	51	1.95386 48092 52	1.29627 80079 94
6	1.57511 36077 77	1.56649 67877 60	52	1.97288 22662 75	1.28695 37387 83
7	1.57667 79815 93	1.56494 75629 69	53	1.99266 97557 35	1.27757 39482 50
8	1.57848 65776 89	1.56316 22295 18	54	2.01326 65652 01	1.26814 65310 64
9	1.58054 09338 96	1.56114 17453 51	55	2.03471 53121 86	1.25867 96247 78
10	1.58284 28043 38	1.55888 71966 02	56	2.05706 23227 97	1.24918 16206 07
11	1.58539 41637 75	1.55639 97977 71	57	2.08035 80666 92	1.23966 11752 89
12	1.58819 72125 27	1.55368 08919 37	58	2.10465 76584 91	1.23012 72241 86
13	1.59125 43820 14	1.55073 19509 84	59	2.13002 14383 99	1.22058 89957 54
14	1.59456 83409 32	1.54755 45758 70	60	2.15651 56475 00	1.21105 60275 68
15	1.59814 20021 13	1.54415 04969 15	61	2.18421 32169 49	1.20153 81841 14
16	1.60197 85300 87	1.54052 15741 28	62	2.21319 46949 81	1.19204 56765 80
17	1.60608 13494 10	1.53666 97975 69	63	2.24354 93416 99	1.18258 90849 45
18	1.61045 41537 90	1.53259 72877 46	64	2.27537 64296 12	1.17317 93826 84
19	1.61510 09160 68	1.52830 62960 54	65	2.30878 67981 67	1.16382 79644 93
20	1.62002 58991 24	1.52379 92052 60	66	2.34390 47244 47	1.15454 66775 25
21	1.62523 36677 59	1.51907 85300 26	67	2.38087 01906 04	1.14534 78566 81
22	1.63072 91016 31	1.51414 69174 93	68	2.41984 16537 39	1.13624 43646 84
23	1.63651 74093 36	1.50900 71479 15	69	2.46099 94583 04	1.12724 96377 57
24	1.64260 41437 13	1.50366 21353 53	70	2.50455 00790 02	1.11837 77379 70
25	1.64899 52184 79	1.49811 49284 22	71	2.55073 14496 27	1.10964 34135 43
26	1.65569 69263 10	1.49236 87111 24	72	2.59981 97300 61	1.10106 21687 58
27	1.66271 59584 91	1.48642 68037 43	73	2.65213 80046 30	1.09265 03455 36
28	1.67005 94262 70	1.48029 26638 26	74	2.70806 76145 90	1.08442 52193 74
29	1.67773 48840 81	1.47396 98872 42	75	2.76806 31453 69	1.07640 51130 76
30	1.68575 03548 13	1.46746 22093 39	76	2.83267 25829 18	1.06860 95329 78
31	1.69411 43573 06	1.46077 35062 13	77	2.90256 49406 70	1.06105 93337 54
32	1.70283 52364 12	1.45390 77960 65	78	2.97856 89511 81	1.05377 69204 07
33	1.71192 46951 56	1.44686 92406 95	79	3.06172 86120 39	1.04678 64993 45
34	1.72139 08313 74	1.43966 21471 15	80	3.15338 52518 88	1.04011 43957 06
35	1.73124 51756 57	1.43229 09693 07	81	3.25530 29421 43	1.03378 94623 91
36	1.74149 92344 27	1.42476 03101 25	82	3.36986 80266 68	1.02784 36197 41
37	1.75216 52364 69	1.41707 49233 72	83	3.50042 24991 73	1.02231 25881 68
38	1.76325 61840 59	1.40923 97160 46	84	3.65185 59694 79	1.01723 69183 41
39	1.77478 59091 64	1.40125 97507 85	85	3.83174 19997 66	1.01266 35062 34
40	1.78676 91348 85	1.39314 02485 23	86	4.05275 81695 49	1.00864 79569 07
41	1.79922 15440 50	1.38488 65913 75	87	4.33865 39760 00	1.00525 85872 09
42	1.81215 98536 62	1.37650 43257 72	88	4.74271 72652 79	1.00258 40855 28
43	1.82560 18981 36	1.36799 91658 73	89	5.43490 98296 25	1.00075 15777 02
44	1.83956 67210 94	1.35937 69972 75			
45	1.85407 46773 01	1.35064 38810 48			

ON THE RELATIONS OF ELLIPTIC FUNCTIONS TO THE PARTS OF A SPHERICAL TRIANGLE AND TO THE LENGTHS OF CERTAIN ALGEBRAICAL CURVES.

Let $A B C$ be a spherical triangle, whose sides a, b, c are respectively equal to the amplitudes ϕ_1, ϕ_2, ϕ_3 of three functions $Fk\phi_1, Fk\phi_2, Fk\phi_3$, such that

$$Fk\phi_1 \pm Fk\phi_2 - Fk\phi_3 = 0.$$

Then by the principles of spherical trigonometry

$$\cos C = \frac{\cos \phi_3 - \cos \phi_1 \cos \phi_2}{\sin \phi_1 \sin \phi_2} = \mp * \Delta k \phi_3 \text{ by equation (1.)}$$

$$\cos B = \frac{\cos \phi_2 - \cos \phi_1 \cos \phi_3}{\sin \phi_1 \sin \phi_3} = \pm * \Delta k \phi_2,$$

$$\cos A = \frac{\cos \phi_1 - \cos \phi_2 \cos \phi_3}{\sin \phi_2 \sin \phi_3} = \pm * \Delta k \phi_1;$$

$$\therefore \sin A = k \sin \phi_1, \quad \sin B = k \sin \phi_2, \quad \sin C = k \sin \phi_3.$$

Thus then the constant ratio of the sines of the sides and angles of a spherical triangle, thus formed, is equal to the common modulus of the function of which the sides are the amplitudes.

If there be taken a triangle polar or supplemental to this, then will the cosines of the sides of that triangle be represented by

$$\Delta k \phi_1, \quad \Delta k \phi_2, \quad \Delta k \phi_3,$$

and their sines will have to the sines of the opposite angles the ratio $\frac{1}{k}$ instead of k .

From C let fall a perpendicular CD upon AB ; $\therefore$ by Napier's analogies,

$$\tan AD = \cos A \tan \phi_3,$$

$$\tan BD = \cos B \tan \phi_1;$$

$$\therefore \phi_3 = \tan^{-1} (\tan \phi_1 \Delta k \phi_2) \pm \tan^{-1} (\tan \phi_2 \Delta k \phi_1) \dots \dots (W).$$

ANY ELLIPTIC FUNCTION MAY BE REPRESENTED BY THE ARC OF AN ALGEBRAICAL CURVE.

LEMMA V. To represent any given integral $\int P d\phi$ by the arc of an algebraical curve.

Let CY be a perpendicular from the origin C upon the tangent to the point P of a curve whose coordinates CM, MP , are x and y . Let $CY = PMPT = \phi$, $AP = s$;

$$\therefore P = y \sin \phi + x \cos \phi;$$

and observing that

$$\frac{dy}{d\phi} = \frac{ds}{d\phi} \cos \phi, \quad \frac{dx}{d\phi} = -\frac{ds}{d\phi} \sin \phi,$$

$$\frac{dP}{d\phi} = y \cos \phi - x \sin \phi.$$

From these two equations we have

$$\left. \begin{aligned} x &= P \cos \phi - \frac{dP}{d\phi} \sin \phi \\ y &= P \sin \phi + \frac{dP}{d\phi} \cos \phi \end{aligned} \right\} \dots \dots (X');$$

$$\therefore \frac{dx}{d\phi} = -\left(P + \frac{d^2P}{d\phi^2}\right) \sin \phi; \text{ and } \therefore \frac{ds}{d\phi} = P + \frac{d^2P}{d\phi^2};$$

$$\therefore s = \int P d\phi + \frac{dP}{d\phi} + C \dots \dots (Y).$$

By which equation the integral $\int P d\phi$ is expressed in terms of the arc s of an algebraical curve, and of the algebraical function $\frac{dP}{d\phi}$.

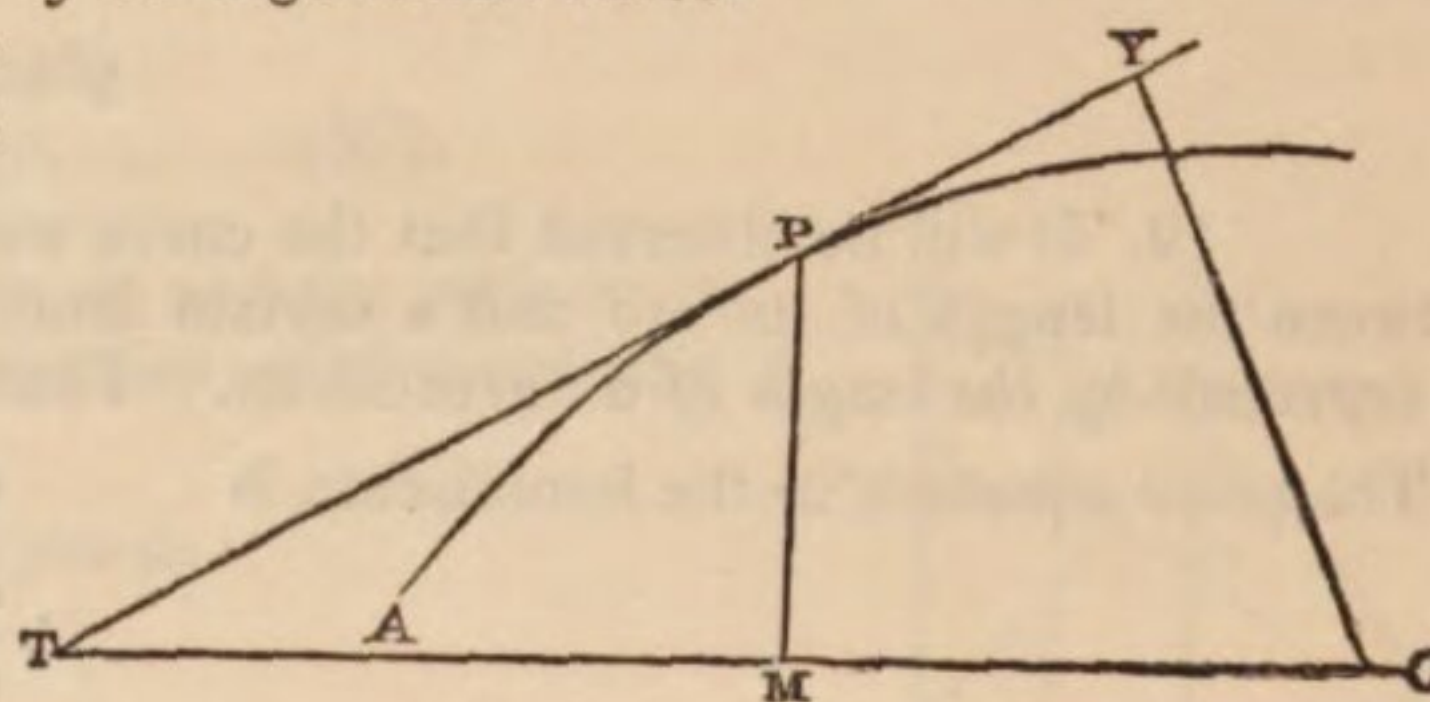
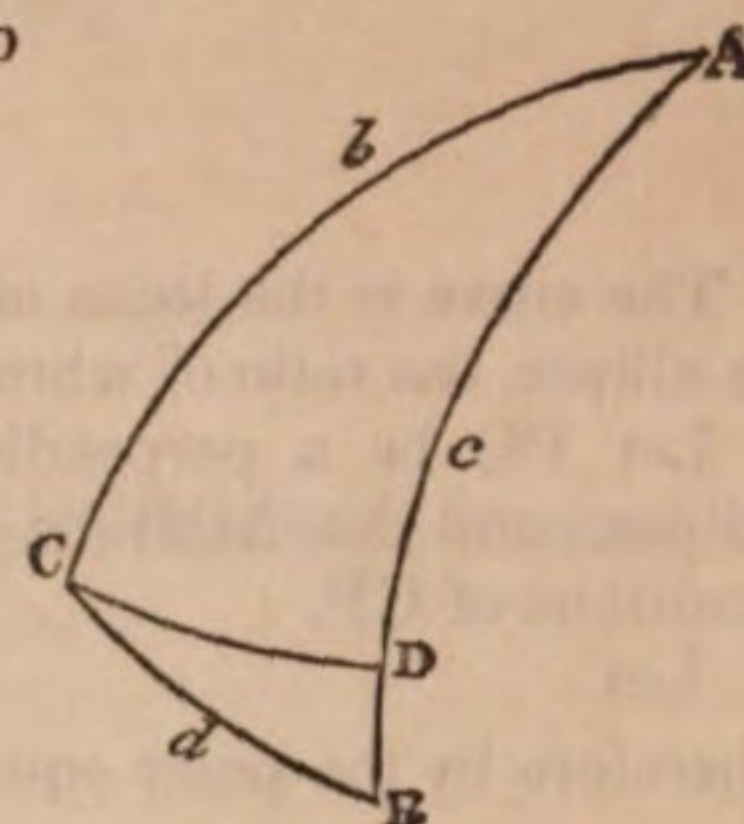
18. To represent an elliptic function of the FIRST ORDER by an algebraical curve.

Assume

$$P = \frac{1}{(1 - k^2 \sin^2 \phi)^{\frac{1}{2}}}; \quad \therefore \frac{dP}{d\phi} = \frac{k^2 \sin \phi \cos \phi}{(1 - k^2 \sin^2 \phi)^{\frac{3}{2}}};$$

by equation (Y')

$$\therefore Fk\phi = s - \frac{k^2 \sin \phi \cos \phi}{(1 - k^2 \sin^2 \phi)^{\frac{1}{2}}}$$



* The negative sign of the cosine of either angle indicates the angle to be an obtuse angle.

Definite
Integrals.

assuming the arc s to commence from the point where $\phi=0$. Equations (X') give the following values of x and y by which this curve may be traced :

Definite
Integrals.

$$x = \frac{\cos \phi}{(1 - k^2 \sin^2 \phi)^{\frac{3}{2}}} \{k'^2 + k^2 \cos 2\phi\},$$

$$y = \frac{\sin \phi}{(1 - k^2 \sin^2 \phi)^{\frac{3}{2}}} \{1 + k^2 \cos 2\phi\}.$$

The curve is the locus of the intersections of perpendiculars to the extremities of the consecutive diameters of an ellipse, the ratio of whose eccentricity to its semiaxis major is k , and whose semiaxis minor is unity.

Let PQ be a perpendicular to the extremity of any diameter CP of such an ellipse, and let AQB be the locus of the intersections of PQ in the consecutive positions of CP.

Let $PCB = \phi$, $CP = r$;
therefore by the polar equation to an ellipse from the centre

$$r = \frac{1}{(1 - k^2 \sin^2 \phi)^{\frac{1}{2}}}.$$

Now

$$\text{arc BQ} = \text{PQ} + \int r d\phi.$$

Also

$$\text{PQ} = \frac{dr}{d\phi} = \frac{k^2 \sin \phi \cos \phi}{(1 - k^2 \sin^2 \phi)^{\frac{3}{2}}};$$

$$\therefore \text{arc BQ} = \frac{k^2 \sin \phi \cos \phi}{(1 - k^2 \sin^2 \phi)^{\frac{3}{2}}} + Fk\phi,$$

or

$$Fk\phi = \text{arc BQ} - \frac{k^2 \sin \phi \cos \phi}{(1 - k^2 \sin^2 \phi)^{\frac{3}{2}}}.$$

As k is diminished the arc AQB continually approximates to the elliptic arc APB, from which indeed for small values of k it does not sensibly differ.

In the particular case in which $k^2 = \frac{1}{2}$, we obtain by elimination between equations (X') for the general equation to the curve

$$y^2 = 2 - \frac{1}{2}x^2 - \frac{3}{2}x^4.$$

19. It will be observed that the curve we have used serves to express the function $Fk\phi$ by the difference between the length of its arc and a certain function of the angle ϕ . In this particular case the function may be expressed by the length of a curve alone. That curve is the lemniscata.

The polar equation to the lemniscata is

$$r^2 = a^2 \cos 2\theta;$$

$$\therefore \frac{dr}{d\theta} = -\frac{a \sin 2\theta}{\sqrt{\cos 2\theta}};$$

$$\therefore s = \int \left\{ \left(\frac{dr}{d\theta} \right)^2 + r^2 \right\}^{\frac{1}{2}} d\theta = a \int \left\{ \frac{\sin^2 2\theta}{\cos 2\theta} + \cos 2\theta \right\}^{\frac{1}{2}} d\theta = a \int \frac{d\theta}{\sqrt{\cos 2\theta}}.$$

Let

$$\cos 2\theta = \cos^2 \phi; \quad \therefore \frac{d\theta}{d\phi} = \frac{\cos \phi \sin \phi}{\sqrt{1 - \cos^4 \phi}};$$

$$\therefore s = a \int \frac{\frac{d\theta}{d\phi} d\phi}{\cos \phi} = a \int \frac{\sin \phi d\phi}{\sqrt{1 - \cos^4 \phi}}$$

$$s = a \int \frac{d\phi}{\sqrt{1 + \cos^2 \phi}} = \frac{a}{\sqrt{2}} \int \frac{d\phi}{\sqrt{1 - \frac{1}{2} \sin^2 \phi}};$$

$$\therefore s = \frac{a}{\sqrt{2}} F \frac{1}{\sqrt{2}} \phi \dots \dots (Y').$$

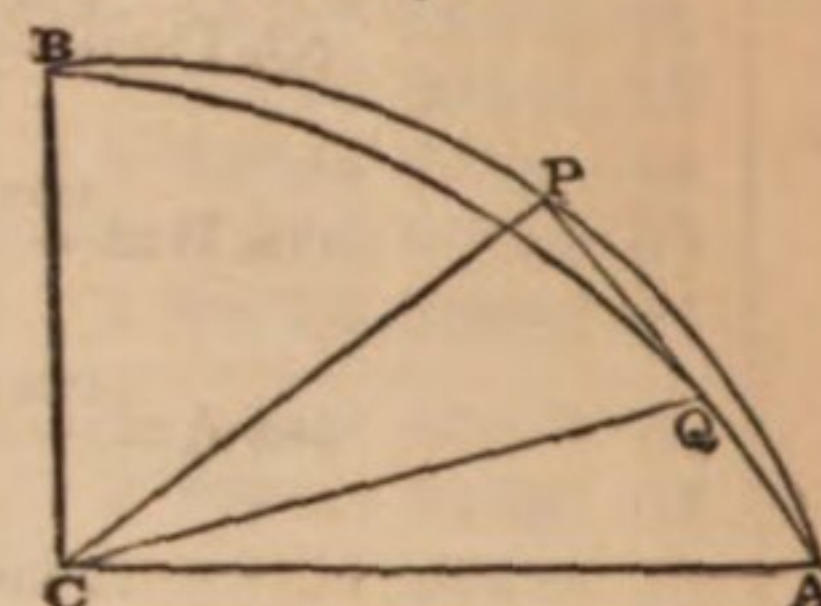
Since an elliptic function of the first order may be added to, or subtracted from, another, or multiplied any number of times, or divided into any number of equal parts, therefore the arc of a lemniscata may likewise be added or subtracted, or be multiplied as many times as we like, or divided into as many parts as we like.

Let AP_1, AP_2, AP_3 be three arcs of a lemniscata whose semiaxis is a , and at the points P_1, P_2, P_3 let the radii vectores be r_1, r_2, r_3 .

Also let

$$\phi_1 = \cos^{-1} \frac{r_1}{a}, \quad \phi_2 = \cos^{-1} \frac{r_2}{a}, \quad \phi_3 = \cos^{-1} \frac{r_3}{a};$$

and let the points P_1, P_2, P_3 be so taken that



Definite
Integrals.

 Definite
Integrals.

$$\cos \phi_3 = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \Delta \frac{1}{\sqrt{2}} \phi_3;$$

$$\therefore AP_3 = AP_1 + AP_2.$$

From this formula we may double the arc of a lemniscata, or halve it.

Suppose, for example, that it is required to halve the arc AP_3 . Let P_1 and P_2 coincide;

$$\therefore AP_1 = \frac{1}{2} AP_3 \quad \cos^2 \phi_1 = \frac{\cos \phi_3 + \Delta \frac{1}{\sqrt{2}} \phi_3}{1 + \Delta \frac{1}{\sqrt{2}} \phi_3},$$

from which equation ϕ_1 , and therefore the position of P_1 , is known in terms of ϕ_3 . If the arc to be bisected is the whole semi lemniscata, then $r_3 = 0$; $\therefore \phi_3 = \frac{\pi}{2}$;

$$\therefore \cos^2 \phi_1 = \frac{\frac{1}{\sqrt{2}}}{1 + \frac{1}{\sqrt{2}}};$$

$$\therefore \cos \phi_1 = \sqrt{\sqrt{2} - 1}, \quad r_1 = a\sqrt{\sqrt{2} - 1}.$$

19. To represent an elliptic function of the SECOND ORDER by the length of a given curve.

Assume

$$P = (1 - k^2 \sin^2 \phi)^{\frac{1}{2}};$$

$\therefore$ by equation (Y').

$$s = Ek\phi - \frac{k^2 \sin \phi \cos \phi}{(1 - k^2 \sin^2 \phi)^{\frac{1}{2}}} + C.$$

Also by equations (X') we have

$$x = \frac{\cos \phi}{P}, \quad y = \frac{k' \sin \phi}{P},$$

between which, eliminating ϕ ,

$$y^2 = k'^2 (1 - x^2).$$

The equation to an ellipse whose semiaxis major is unity, and the ratio of its eccentricity to its semiaxis major k . If we suppose s to be measured from the extremity of the axis major, the constant disappears, and we have

$$Ek\phi = s + \frac{k^2 \sin \phi \cos \phi}{(1 - k^2 \sin^2 \phi)^{\frac{1}{2}}}, \dots \dots \dots (Z').$$

Let P_1 be any point in an ellipse, whose semiaxis major is a , and the ratio of its semiaxis major to its eccentricity k , and let Q_1 be the intersection of the ordinate M_1P_1 , produced, with the circumscribing circle.

Let $Q_1CB = \phi_1$; $\therefore CM_1 = x = a \sin \phi_1$, $M_1P_1 = y = b \cos \phi_1$;

$\therefore$ calling E_1 the length of the arc BP_1 ,

$$\frac{dE_1}{d\phi_1} = (a^2 \cos^2 \phi_1 + b^2 \sin^2 \phi_1)^{\frac{1}{2}} = a(1 - k^2 \sin^2 \phi_1)^{\frac{1}{2}};$$

$$\therefore E_1 = a Ek\phi_1 \dots \dots \dots (A'');$$

the constant disappearing, since when $E_1 = 0$ $\phi_1 = 0$. If ϕ_2, ϕ_3 be similarly taken in respect to two other points P_2 and P_3 of the ellipse; then if there exist between ϕ_1, ϕ_2, ϕ_3 the relation

$$\cos \phi_3 = \cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \Delta k \phi_3$$

there will exist between the arcs BP_1, BP_2, BP_3 the relation

$$*BP_1 + BP_2 - BP_3 = ak^2 \sin \phi_1 \sin \phi_2 \sin \phi_3 \dots \dots \dots (B'');$$

$$\therefore BP_1 + P_2P_3 = ak^2 \sin \phi_1 \sin \phi_2 \sin \phi_3 \dots \dots \dots (C'');$$

or if ϕ_2 be $> \phi_3$,

$$BP_1 - P_2P_3 = ak^2 \sin \phi_1 \sin \phi_2 \sin \phi_3.$$

Let P_3 coincide with A ;

$$\therefore \phi_3 = \frac{\pi}{2}, \quad k' \tan \phi_1 \tan \phi_2 = 1, \quad BP_1 - P_2A = ak^2 \sin \phi_1 \sin \phi_2,$$

or eliminating ϕ_2

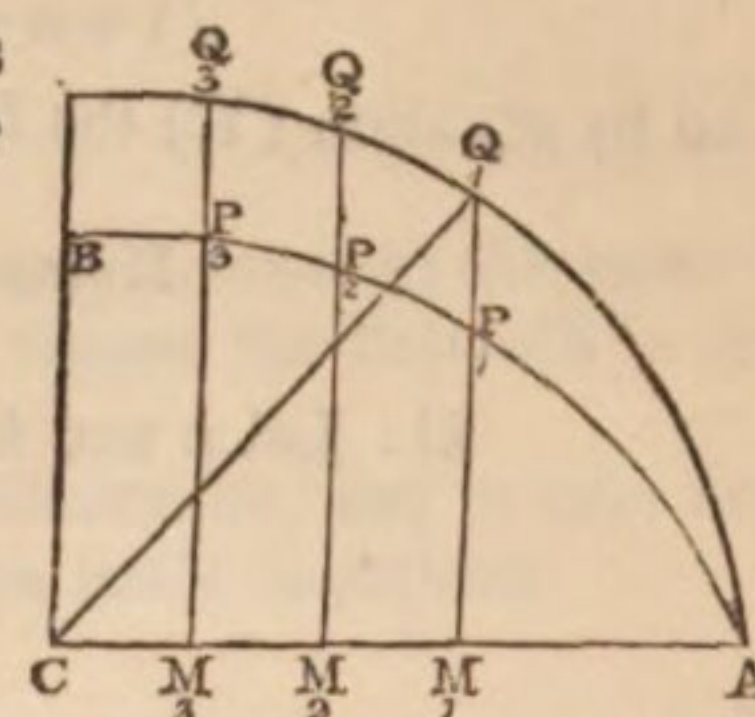
$$BP_1 - AP_2 = \frac{ak^2 \sin \phi_1 \cos \phi_1}{\Delta k \phi_1},$$

therefore the difference of the arcs BP_1 and AP_2 is equal to the perpendicular from C on a tangent to P_1 ; a

*

$$BP_1 + BP_2 - BP_3 = \frac{k^2 M_1Q_1 \times M_2Q_2 \times M_3Q_3}{a^2}.$$

Thus we can determine the arc of an ellipse which shall differ in length from the sum of two others by a given straight line, or conversely.



Definite
Integrals.

property which was discovered by Fagani, and bears his name. It may also be deduced from a comparison of equations (Z') and (A').

Definite
Integrals.

Suppose the points P_1 and P_2 to coincide;

$$\therefore BP_1 - AP_1 = ak^2 \sin^2 \phi_1,$$

and

$$\sin^2 \phi_1 = \frac{1}{1+k'};$$

$$\therefore BP_1 - AP_1 = a(1-k') = a-b.$$

If, therefore, the point P_1 be taken according to the condition $\tan \phi_1 = \frac{1}{\sqrt{k'}}$, then will the difference of the arcs

BP_1 and AP_1 equal that of the semiaxes of the ellipse. Let it be required to determine two arcs of the same quadrant of an ellipse which shall be equal to one another. Call ϕ_2, ϕ_3 the angles corresponding to the extremities of one of the arcs P_2, P_3 , and ϕ'_2, ϕ'_3 those corresponding to the extremities of the other P'_2, P'_3 . Also let there exist between the quantities $\phi_2, \phi_3, \phi'_2, \phi'_3$ the relation

$$\frac{\sin \phi_2 \cos \phi_3 \Delta \phi_3 + \sin \phi_3 \cos \phi_2 \Delta \phi_2}{1-k^2 \sin^2 \phi_2 \sin^2 \phi_3} = \frac{\sin \phi'_2 \cos \phi'_3 \Delta \phi'_3 + \sin \phi'_3 \cos \phi'_2 \Delta \phi'_2}{1-k^2 \sin^2 \phi'_2 \sin^2 \phi'_3} \dots \dots \dots (D''),$$

and let each of these equal $\sin \phi_1$; $\therefore$ by equation (8.) and equation (C'')

$$BP_1 + P_2 P_3 = ak^2 \sin \phi_1 \sin \phi_2 \sin \phi_3, \quad BP_1 + P'_2 P'_3 = ak^2 \sin \phi_1 \sin \phi'_2 \sin \phi'_3.$$

If, then,

$$\sin \phi_2 \sin \phi_3 - \sin \phi'_2 \sin \phi'_3 = 0. \dots \dots \dots (E'')$$

$$P_2 P_3 = P'_2 P'_3.$$

The relation (D'') becomes by means of (E'')

$$\sin \phi_2 \cos \phi_3 \Delta \phi_3 + \sin \phi_3 \cos \phi_2 \Delta \phi_2 = \sin \phi'_2 \cos \phi'_3 \Delta \phi'_3 + \sin \phi'_3 \cos \phi'_2 \Delta \phi'_2 \dots \dots \dots (F'').$$

There are two equations of condition, (E'') and (F''), between the four quantities $\phi_2, \phi_3, \phi'_2, \phi'_3$. Any two of these are, therefore, indeterminate: and hence it follows that *any* are being given we can determine another, and only one other, in the same quadrant which shall be of the same length.

20. To represent an elliptic function of the THIRD ORDER by the length of a given curve.

Assume

$$P = (1+n \sin^2 \phi)^{-1} (1-k^2 \sin^2 \phi)^{-\frac{1}{2}};$$

substitution in equations (X') will then give the following values for the coordinates of the curve:

$$x = \frac{\cos \phi}{(1+n \sin^2 \phi)(1-k^2 \sin^2 \phi)^{\frac{3}{2}}} \{1 + (3n-2k^2) \sin^2 \phi - 4k^2 n \sin^4 \phi\}$$

$$y = \frac{\sin \phi}{(1+n \sin^2 \phi)(1-k^2 \sin^2 \phi)^{\frac{3}{2}}} \{1 + k^2 - 2n + (3n-1)(1+k^2) \sin^2 \phi - 4k^2 n \sin^4 \phi\};$$

and by equation (Y') the function will be determined in terms of the length of the curve by the equation

$$\Pi n k \phi = s + \frac{2n \sin \phi \cos \phi}{(1+n \sin^2 \phi)^2 (1-k^2 \sin^2 \phi)^{\frac{3}{2}}} - \frac{k^2 \sin \phi \cos \phi}{(1+n \sin^2 \phi)(1-k^2 \sin^2 \phi)^{\frac{3}{2}}}.$$

21. Let k and k' be the semiaxes of an hyperbola; then will its equation be

$$y^2 = \frac{k'^2}{k^2} (x^2 - k^2).$$

Let

$$y = k'^2 \tan \phi;$$

$$\therefore x = k \{1 + k'^2 \tan^2 \phi\}^{\frac{1}{2}};$$

$\therefore$ taking Y to represent an arc of this hyperbola,

$$dY = \frac{k'^2 \sec^2 \phi d\phi}{\sqrt{1-k'^2 \sin^2 \phi}}.$$

Now

$$d\{\tan \phi \Delta k \phi\} = \Delta k \phi \sec^2 \phi d\phi - \frac{k'^2 \sin^2 \phi d\phi}{\Delta k \phi}$$

$$= \Delta k \phi d\phi + \frac{k'^2 \tan^2 \phi}{\Delta k \phi} d\phi$$

$$= \Delta k \phi d\phi + \frac{k'^2 \sec^2 \phi d\phi}{\Delta k \phi} - k'^2 \frac{d\phi}{\Delta k \phi};$$

$$\therefore \tan \phi \Delta k \phi = Ek\phi + Y - k'^2 Fk\phi$$

$$Y = \tan \phi \Delta k \phi - Ek\phi + k'^2 Fk\phi \dots \dots \dots (G'').$$

If L , a perpendicular from the centre C of the hyperbola, on the point whose coordinates are x, y , then will

$$L = \tan \phi \Delta k \phi;$$

$$\therefore Y = L - Ek\phi + k'^2 Fk\phi \dots \dots \dots (H'').$$

Let

$$Yk\phi - Lk\phi = Hk\phi$$

Definite Integrals. that

Definite Integrals.

$$Hk\phi_1 + Hk\phi_2 - Hk\phi_3 = Ek\phi_1 + Ek\phi_2 - Ek\phi_3;$$

$$\therefore Hk\phi_1 + Hk\phi_2 - Hk\phi_3 = k^2 \sin \phi_1 \sin \phi_2 \sin \phi_3 \dots (I'');$$

or the same relation obtains between the functions H as between the functions E ; or if a curve be supposed to be taken whose length shall always equal the difference between the length of an hyperbolic arc and the perpendicular let fall upon its tangent, then will the arcs of this curve have to one another the same relation as those of an ellipse. By equation (32.)

$$2(1+k)EK\Phi = 2Ek\phi - k^2 Fk\phi + 2k \sin \phi;$$

also by equation (G'')

$$Yk\phi = k^2 Fk\phi - Ek\phi + \tan \phi \Delta \phi;$$

$$\therefore Yk\phi = Ek\phi - 2(1+k)EK\Phi + \{2k \sin \phi + \tan \phi \Delta \phi\}.$$

Thus, then, the length of an hyperbolic arc may always be represented by a straight line, increased by the difference of two elliptic arcs, whose eccentricities are connected by the equation (K'), and their amplitudes by (L').

This is the theorem of Landen.

EULERIAN INTEGRALS.

SECTION VI.

The definite integrals

$$\int_0^1 \frac{x^{p-1} dx}{(1-x^q)^{\frac{n-q}{q}}} \text{ and } \int_0^1 \left\{ \log \frac{1}{x} \right\}^{\frac{p}{q}-1} dx$$

have received the name of the integrals of Euler, by whom they were first treated as functions of their exponents p and q , and shown under this form to have certain remarkable relations to one another, and an extensive application in various branches of analysis.

In respect to integrals embraced under the first form, and called those of the first class, the investigations of Euler did not extend beyond that case in which the exponents n, p, q , are whole numbers, and his comparison of different integrals was made on the supposition that n remained the same. In respect to integrals of the second class, he confined himself to the case in which p and q are whole numbers, and compared only those in which q remained the same.

As our object here is, however, to present the discussion under the form in which the more recent researches of Legendre, Poisson, and Jacobi have left it, we shall consider these integrals (after Legendre) as embraced under the following more general forms:

$$\int_0^1 (1-x)^{q-1} x^{p-1} dx \text{ represented by the symbol } [p, q]$$

$$\int_0^1 \left\{ \log \frac{1}{x} \right\}^{p-1} dx \text{ represented by the symbol } \Gamma p.$$

Both these classes of functions are considered as continuous. The latter are commonly known by the name of the functions *Gamma*, from the symbol which distinguishes them, and for a similar reason the first will in this paper be designated *Parenthetical Functions*.

It will be convenient to include both these classes of integrals under the same discussion, and to take first in order the functions *gamma*, on which the values of the *parenthetical* function may be made to depend.

Let

$$u = x \left\{ \log \frac{1}{x} \right\}^p;$$

$$\therefore \frac{du}{dx} = \left\{ \log \frac{1}{x} \right\}^p - p \left\{ \log \frac{1}{x} \right\}^{p-1};$$

and observing that between the limits 0 and 1, $u=0$, we obtain by integration

$$0 = \int_0^1 \left\{ \log \frac{1}{x} \right\}^p dx - p \int_0^1 \left\{ \log \frac{1}{x} \right\}^{p-1} dx;$$

$$\therefore \Gamma(p+1) = p \Gamma p \dots (1.).$$

Also this is true for all positive values of p , therefore if q be any positive integer,

$$\Gamma(p+q-1) = p(p+1)(p+2) \dots (p+q-2) \Gamma p \dots (2.).$$

Let

$$p=1; \therefore \Gamma p = \int_0^1 dx = 1;$$

$$\therefore \Gamma q = 1.2.3.4 \dots (q-1) \dots (3.).$$

To determine a relation between the class of parenthetical functions and the functions *gamma*.

Assume

$$u = x^p (1-x)^{q-1};$$

$$\therefore \frac{du}{dx} = (p+q-1)x^{p-1}(1-x)^{q-1} - (q-1)x^p(1-x)^{q-2}.$$

Integrating and observing that, between the limits 0 and 1, $u=0$,

Definite
Integrals.

$$\gamma = (p+q-1) \int_0^1 x^{p-1}(1-x)^{q-1} dx - (q-1) \int_0^1 x^{p-1}(1-x)^{q-1} dx;$$

Definite
Integrals.

$$\therefore [p, q] = \frac{(q-1)}{(p+q-1)} [p, q-1] \dots \dots \dots (4.).$$

And this is true for all positive values of q . Supposing it, therefore, an integer, and observing that

$$[p, 1] = \int_0^1 x^{p-1} dx = \frac{1}{p},$$

we have

$$[p, q] = \frac{(q-1)(q-2)\dots 1}{(p+q-1)(p+q-2)\dots (p+1)} \cdot \frac{1}{p} \dots \dots \dots (5.)$$

Now by equations (2.) and (3.) the product in the numerator of this fraction equals Γq , and that in the denominator $\frac{\Gamma(p+q)}{\Gamma q}$;

$$\therefore [p, q] = \frac{\Gamma p \Gamma q}{\Gamma(p+q)} \dots \dots \dots (6.).*$$

This equation has been proved only in the case in which q is an integer; containing, however, no longer the indefinite product $(1.2.3\dots q-1)$, it may be considered continuous. Now the value of the second member will not be altered if p and q be interchanged therefore generally.

$$p, q] = [q, p] \dots \dots \dots (7.).$$

The integral $[p, q]$ may be exactly determined when $p+q=1$; for in this case

$$[p, q] = [p, 1-p] = \int_0^1 x^{p-1}(1-x)^{-p} dx.$$

Assume $1-x = \frac{x}{u}$; $\therefore x = \frac{u}{1+u}$; and the values of u corresponding to $x=0$ and $x=1$ are 0 and ∞ ;

$$\therefore \text{substituting,} \quad [p, 1-p] = \int_0^\infty \frac{u^{p-1} du}{1+u}.$$

Now from the *general* value of the integral $\int_0^\infty \frac{u^{p-1} du}{1+u^m}$ it may be shown that $\int_0^\infty \frac{u^{p-1} du}{1+u} = \frac{\pi}{\sin p\pi}$. Also by

$$\text{equation (6.) } [p, 1-p] = \frac{\Gamma p \Gamma(1-p)}{\Gamma 1} = \Gamma p \Gamma(1-p);$$

$$\therefore \Gamma p \Gamma(1-p) = \frac{\pi}{\sin p\pi} \dots \dots \dots (8.).$$

$\Gamma(1-p)$ is called the complement of Γp

$$\text{For } p \text{ substitute } \left(\frac{1}{2} + p\right); \therefore \Gamma\left(\frac{1}{2} + p\right) \Gamma\left(\frac{1}{2} - p\right) = \frac{\pi}{\cos p\pi}.$$

Moreover, by equation (1.),

$$\begin{aligned} \Gamma\left\{1 + \left(\frac{1}{2} + p\right)\right\} &= \left(\frac{1}{2} + p\right) \Gamma\left(\frac{1}{2} + p\right) \\ \Gamma\left\{1 + \left(\frac{1}{2} - p\right)\right\} &= \left(\frac{1}{2} - p\right) \Gamma\left(\frac{1}{2} - p\right); \end{aligned}$$

$$\therefore \Gamma\left\{1 + \left(\frac{1}{2} + p\right)\right\} \Gamma\left\{1 + \left(\frac{1}{2} - p\right)\right\} = \left\{\left(\frac{1}{2}\right)^2 - p^2\right\} \Gamma\left(\frac{1}{2} + p\right) \Gamma\left(\frac{1}{2} - p\right).$$

$$\text{Similarly,} \quad \Gamma\left\{2 + \left(\frac{1}{2} + p\right)\right\} \Gamma\left\{2 + \left(\frac{1}{2} - p\right)\right\} = \left\{\left(\frac{3}{2}\right)^2 - p^2\right\} \Gamma\left(1 + \frac{1}{2} + p\right) \Gamma\left(1 + \frac{1}{2} - p\right);$$

&c. = &c.

$$\therefore \Gamma\left(m - \frac{1}{2} + p\right) \Gamma\left(m - \frac{1}{2} - p\right) = \left\{\left(\frac{1}{2}\right)^2 - p^2\right\} \left\{\left(\frac{3}{2}\right)^2 - p^2\right\} \dots \dots \left\{\left(m - \frac{3}{2}\right)^2 - p^2\right\} \frac{\pi}{\cos p\pi} \dots \dots (9.);$$

where m is any positive integer, and the number of factors in the product is $(m-1)$. From which equation it appears that, dividing the values of Γp into periods, of which the first contains those between $\Gamma 0$ and $\Gamma 1$, the second those between $\Gamma 1$ and $\Gamma 2$, &c., the m th those between $\Gamma m-1$ and Γm ; then those which are contained in either half of each of these periods may be determined in terms of those in the other half. If we make $p=0$, we shall obtain for the middle term of the m th period,

$$\Gamma\left(m - \frac{1}{2}\right) = 1.3.5.7\dots(2m-3) \frac{\sqrt{\pi}}{2^{m-1}}.$$

* For a direct proof of this equation, including the positive fractional values of p and q , see Note 2 at the end of this paper.

Definite Integrals. Let $m=1$

$$\therefore \Gamma \frac{1}{2} = \sqrt{\pi}.$$

Definite Integrals.

In the function $[p, q]$ suppose $p=q$; $\therefore [p, p] = \int_0^1 x^{p-1} (1-x)^{p-1} dx$. Assume $x = \frac{1}{2}(1+u)$;

$$\therefore [p, p] = 2^{1-2p} \int_{-1}^1 (1-u^2)^{p-1} du; \text{ or } [p, p] = 2^{1-2p} \int_0^1 (1-u^2)^{p-1} du;$$

or, substituting x for u^2 ,

$$[p, p] = 2^{1-2p} \int_0^1 (1-x)^{p-1} x^{-\frac{1}{2}} dx;$$

$$\therefore [p, p] = 2^{1-2p} \left[p, \frac{1}{2} \right], \dots \dots (10.).$$

But by equation (6.)

$$[p, p] = \frac{\Gamma p \Gamma p}{\Gamma 2p} \text{ and } \left[p, \frac{1}{2} \right] = \frac{\Gamma \frac{1}{2} \Gamma p}{\Gamma \left(\frac{1}{2} + p \right)};$$

$$\therefore \Gamma p \Gamma \left(\frac{1}{2} + p \right) = 2^{(1-2p)} \sqrt{\pi} \cdot \Gamma 2p \dots \dots (11.).$$

The equations numbered from (1.) to (11.) include the whole theory of the functions gamma, and the parenthetical functions $[p, q]$; all the formulæ given by Euler and Legendre may be deduced from them.

The relation of Euler's functions, $\int_0^1 \frac{x^{p-1} dx}{(1-x)^{\frac{p+q}{n}}}$ represented by $\left(\frac{p}{q} \right)$, to the functions $[p, q]$ is expressed by the following equation:

$$\left[\frac{p}{n}, \frac{q}{n} \right] = n \left(\frac{p}{q} \right) \dots \dots (12.);$$

for

$$\left[\frac{p}{n}, \frac{q}{n} \right] = \int_0^1 x^{\frac{p}{n}-1} (1-x)^{\frac{q}{n}-1} dx.$$

Assuming therefore $x=u^n$,

$$\left[\frac{p}{n}, \frac{q}{n} \right] = n \int_0^1 u^{p-1} (1-u^n)^{\frac{q}{n}-1} du = n \left(\frac{p}{q} \right); \therefore \&c.$$

Also by equation (6.)

$$\left[\frac{p}{n}, \frac{q}{n} \right] = \frac{\Gamma \frac{p}{n} \Gamma \frac{q}{n}}{\Gamma \frac{p+q}{n}};$$

$$\therefore \left(\frac{p}{q} \right) = \frac{1}{n} \frac{\Gamma \frac{p}{n} \Gamma \frac{q}{n}}{\Gamma \frac{p+q}{n}} \dots \dots (13.).$$

By equation (1.)

$$\Gamma \frac{p+q}{n} = \left(\frac{p+q}{n} - 1 \right) \Gamma \left(\frac{p+q}{n} - 1 \right);$$

$$\therefore \left(\frac{p}{q} \right) = \frac{\Gamma \frac{p}{n} \Gamma \frac{q}{n}}{(p+q-n) \Gamma \left(\frac{p+q}{n} - 1 \right)} \dots \dots (14.).$$

The first of these formulæ is to be used when $p+q$ is $< n$, and the second when it is $> n$.

From the preceding formulæ may be deduced a great variety of others, applicable to particular cases of definite integration.

By equation (6.)

$$\left[p, \frac{1}{2} \right] = \frac{\Gamma p \Gamma \frac{1}{2}}{\Gamma \left(p + \frac{1}{2} \right)} \quad \left[\left(p + \frac{1}{2} \right), \frac{1}{2} \right] = \frac{\Gamma \left(p + \frac{1}{2} \right) \Gamma \frac{1}{2}}{\Gamma (p+1)}.$$

Also by equation (1.) $\Gamma (p+1) = p \Gamma p$;

$$\therefore \left[p, \frac{1}{2} \right] \left[\left(p + \frac{1}{2} \right), \frac{1}{2} \right] = \frac{\left(\Gamma \frac{1}{2} \right)^2}{p} = \frac{\pi}{p};$$

Definite
Integrals.

∴ by equation (12.)

$$\left(\frac{2p}{1}\right)\left(\frac{2p+1}{1}\right) = \frac{\pi}{4p}, \text{ taking } n=2;$$

or

$$\int_0^1 \frac{x^{2p-1}}{(1-x^2)^{\frac{1}{2}}} \cdot \int_0^1 \frac{x^{2p}}{(1-x^2)^{\frac{1}{2}}} = \frac{\pi}{4p}.$$

This theorem, which is commonly proved by a direct but more laborious method,* is a particular case of a more general theorem.

If n be any positive integer,

$$\left[p, \frac{1}{n}\right] = \frac{\Gamma p \Gamma \frac{1}{n}}{\Gamma\left(p + \frac{1}{n}\right)}, \left[\left(p + \frac{1}{n}\right), \frac{1}{n}\right] = \frac{\Gamma\left(p + \frac{1}{n}\right) \Gamma \frac{1}{n}}{\Gamma\left(p + \frac{2}{n}\right)} \&c. = \&c. \left[\left(p + \frac{n-1}{n}\right), \frac{1}{n}\right] = \frac{\Gamma\left(p + \frac{n-1}{n}\right) \Gamma \frac{1}{n}}{\Gamma(p+1)};$$

$$\begin{aligned} \therefore \left[p, \frac{1}{n}\right] \times \left[\left(p + \frac{1}{n}\right), \frac{1}{n}\right] \times \dots \times \left[\left(p + \frac{n-1}{n}\right), \frac{1}{n}\right] &= \frac{\left(\Gamma \frac{1}{n}\right)^n}{p} \\ \therefore \left(\frac{pn}{1}\right) \times \left(\frac{pn+1}{1}\right) \times \dots \times \left(\frac{np+n-1}{1}\right) &= \frac{\left(\Gamma \frac{1}{n}\right)^n}{n^n p}. \end{aligned} \quad \dots\dots\dots (15.);$$

Let $\log \frac{1}{x} = u^2; \therefore x = e^{-u^2}; \therefore dx = -2e^{-u^2} u du;$

also the limits 0 and 1 of x correspond to $u = \infty$ $u = 0$;

$$\begin{aligned} \therefore \int_0^1 \left\{ \log \frac{1}{x} \right\}^{p-1} dx &= -2 \int_{\infty}^0 u^{2p-1} e^{-u^2} du = 2 \int_0^{\infty} u^{2p-1} e^{-u^2} du; \\ \therefore \int_0^{\infty} u^{2p-1} e^{-u^2} du &= \frac{1}{2} \Gamma p. \end{aligned}$$

Let $p = \frac{1}{2}; \therefore \int_0^{\infty} e^{-u^2} du = \frac{1}{2} \Gamma \frac{1}{2} = \frac{1}{2} \sqrt{\pi} \dots\dots\dots (16.).$

Tables of the values of the functions gamma for all values of the exponent p having been calculated by Legendre, and the values of the parenthetical functions $[p, q]$ having been made to depend in every case upon those of the functions gamma, by the equations (6.), (13.), (14.), it will be sufficient here to refer the values of the latter to those of the former. The values of the parenthetical functions may, nevertheless, be approximated to, independently, and formulæ for that purpose are given by Legendre, (*Intégrales Eulériennes*, chap. iv.)

The following is the formula by which the tables of Legendre for the functions gamma were calculated.

$\Gamma(1+x)$ representing, when x is a positive integer, the product $1.2.3\dots x$; it is shown by Euler (*Cal. Diff.* p. 465), that

$$\log \Gamma(1+x) = x \log x + \frac{1}{2} \log (2\pi x) - x + \frac{A'}{1.2x} - \frac{B'}{3.4x^3} + \frac{C'}{5.6x^5} - \dots\dots$$

where $A' B' C'$ are the numbers known as the numbers of Bernoulli

Let $\log R = \frac{A'}{1.2x} - \frac{B'}{3.4x^3} + \frac{C'}{5.6x^5} \&c.;$

$$\therefore \log \Gamma(1+x) = \log x^x + \log (2\pi x)^{\frac{1}{2}} - \log e^x + \log R;$$

$$\therefore \Gamma(1+x) = \left(\frac{x}{e}\right)^x (2\pi x)^{\frac{1}{2}} R \dots\dots\dots (17.).$$

The first member of this equation has hitherto been taken to represent the product $1.2.3\dots x$, it has been taken, therefore, to represent only those values of the function gamma in which the exponent is an integer; the second member of this equation is, however, continuous. The whole equation is therefore continuous. It serves to determine the value of $\Gamma(1+x)$, and therefore, of Γx , which equals $\frac{\Gamma(1+x)}{x}$ for all values of x .

It is a remarkable property of the series which determines the value of $\log R$, that it is convergent only up to a certain term determined by the value of x ; and that after this term is attained, it becomes divergent. This feature of the calculation, to which particular attention must be paid, results from the irregular values of the Bernoullian numbers, which diminish to the third, and afterwards increase. A limit beyond which the number of the term of incipient divergence of necessity lies may be determined as follows. Let it be the $(n+1)$ th and let B_n and B_{n+1} be the n th and $(n+1)$ th numbers of Bernoulli.

$$\therefore \frac{B_n}{2n(2n-1)x^{2n-1}} < \frac{B_{n+1}}{(2n+2)(2n+1)x^{2n+1}}; \therefore \frac{B_{n+1}}{B^n} > \frac{(2n+2)(2n+1)}{(2n)(2n-1)} x^2.$$

* See Lacroix, *Traité des Différences*, p. 394 and 403.

Definite
Integrals.

Definite
Integrals.

But Euler has shown (*Cal. Diff.* p. 429.) that $\frac{(2n+2)(2n+1)}{4\pi^2}$ is a limit to which the first member of this inequality continually approaches, as n increases, but which it never exceeds;

 Definite
Integrals.

$$\therefore \frac{(2n+2)(2n+1)}{4\pi^2} > \frac{(2n+2)(2n+1)}{(2n)(2n-1)} x^2; \therefore 2n(2n-1) > 4\pi^2 x^2, \text{ and } n > \pi x.$$

SECTION VII.

THE SUM OF A SERIES OF PERIODICAL QUANTITIES MAY SOMETIMES BE DERIVED FROM A DEFINITE INTEGRAL.

Let α and p be any two real quantities, of which the latter is less than unity.

$$\frac{1}{1-p\varepsilon^{\alpha}\sqrt{-1}} = 1 + \sum_1^{\infty} p^n \varepsilon^{n\alpha} \sqrt{-1} \quad \frac{1}{1-p\varepsilon^{-\alpha}\sqrt{-1}} = 1 + \sum_1^{\infty} p^n \varepsilon^{-n\alpha} \sqrt{-1}.$$

Adding, and subtracting, and substituting $\cos \alpha$ and $\sin \alpha$ for these values in terms of $\varepsilon^{\sqrt{-1}}$ and $\varepsilon^{-\alpha}\sqrt{-1}$,

$$\frac{1-p\cos \alpha}{1-2p\cos \alpha+p^2} = 1 + \sum_1^{\infty} p^n \cos n\alpha \dots \dots (1).$$

$$\frac{p\sin \alpha}{1-2p\cos \alpha+p^2} = \sum_1^{\infty} p^n \sin n\alpha \dots \dots (2).$$

The first of these equations may be written thus,

$$\frac{1-p^2}{1-2p\cos \alpha+p^2} = 1 + 2\sum_1^{\infty} p^n \cos n\alpha \dots \dots (3).$$

Let k be any real or imaginary quantity, multiply both sides of this equation by $\varepsilon^{-k\alpha}$, and integrate in respect to α from $\alpha=0$ to $\alpha=x$;

$$\therefore \int_0^x \frac{(1-p^2)\varepsilon^{-k\alpha} d\alpha}{1-2p\cos \alpha+p^2} = \frac{1-\varepsilon^{-kx}}{k} + 2\varepsilon^{-kx} \sum_1^{\infty} \frac{np^n \sin nx}{k^2+n^2} - 2\varepsilon^{-kx} \sum_1^{\infty} \frac{kp^n \cos nx}{k^2+n^2} + 2 \sum_1^{\infty} \frac{kp^n}{k^2+n^2}.$$

Let $p=1-\rho$. The integer in the first member of this equation will then become

$$\int_0^x \frac{\rho(2+\rho)\varepsilon^{-k\alpha} d\alpha}{4(1-\rho)\sin^2 \frac{\alpha}{2} + \rho^2}.$$

Now, suppose ρ to be infinitely diminished, so that p may approach indefinitely to unity, all those elements of the integral which correspond to values of α , which are not also indefinitely small, will then manifestly be evanescent; the whole finite value of the integral will thus be made up of those elements for which α is indefinitely small, and it is immaterial within what limits we take it, provided we include these. Instead, then, of taking it between the limits 0 and x , let us take it from 0 to α^t , making at the same time α and ρ indefinitely small. The integral then becomes

$$\int_0^{\alpha^t} \frac{2\rho d\alpha}{\rho^2 + \alpha^2} = \pi.$$

Making, then, in the preceding equation, $p=1$, and supposing, $x>0$ and $<2\pi$,

$$\pi = \frac{1-\varepsilon^{-kx}}{k} + 2\varepsilon^{-kx} \sum_1^{\infty} \frac{n \sin nx}{k^2+n^2} - 2\varepsilon^{-kx} \sum_1^{\infty} \frac{k \cos nx}{k^2+n^2} + 2 \sum_1^{\infty} \frac{k}{k^2+n^2} \dots \dots (4.)$$

If the superior limit x be taken $=2\pi$, the quantity $\sin \frac{\alpha}{2}$ will become infinitely small immediately about the superior as well as the inferior limit, and there will be two periods immediately about these limits through which the integral will not be evanescent. The value of the integral for this last period may be determined like that of the first to equal $\pi\varepsilon^{-2k\pi}$, and taking $p=1$ $x=2\pi$, equation (4.) will become

$$\pi + \pi\varepsilon^{-2k\pi} = \frac{1-\varepsilon^{-2k\pi}}{k} - 2\varepsilon^{-2k\pi} \sum_1^{\infty} \frac{k}{k^2+n^2} + 2 \sum_1^{\infty} \frac{k}{k^2+n^2},$$

or

$$\frac{1}{k} + 2 \sum_1^{\infty} \frac{k}{k^2+n^2} = \left(\frac{\varepsilon^{k\pi} + \varepsilon^{-k\pi}}{\varepsilon^{k\pi} - \varepsilon^{-k\pi}} \right) \pi \dots \dots (5).$$

Adding this equation to equation (4.), we obtain

$$\varepsilon^{-kx} + \frac{2k\pi\varepsilon^{-k\pi}}{\varepsilon^{-k\pi} - \varepsilon^{k\pi}} = 2\varepsilon^{-kx} \sum_1^{\infty} \frac{nk \sin nx}{k^2+n^2} - 2\varepsilon^{-kx} \sum_1^{\infty} \frac{k^2 \cos nx}{k^2+n^2}.$$

Changing the sign of k ,

$$\varepsilon^{kx} + \frac{2k\pi\varepsilon^{k\pi}}{\varepsilon^{-k\pi} - \varepsilon^{k\pi}} = -2\varepsilon^{kx} \sum_1^{\infty} \frac{nk \sin nx}{k^2+n^2} - 2\varepsilon^{kx} \sum_1^{\infty} \frac{k^2 \cos nx}{k^2+n^2};$$

whence multiplying the first equation by ε^{kx} , and the second by ε^{-kx} , and subtracting and adding the two equations respectively,

Definite
Integrals.

$$\left. \begin{aligned} \frac{\pi}{2} \left\{ \frac{\varepsilon^{k(\pi-x)} - \varepsilon^{-k(\pi-x)}}{\varepsilon^{k\pi} - \varepsilon^{-k\pi}} \right\} &= \sum_0^\infty \frac{nk \sin nx}{k^2 + n^2} \\ \frac{\pi}{2k} \left\{ \frac{\varepsilon^{k(\pi-x)} + \varepsilon^{-k(\pi-x)}}{\varepsilon^{k\pi} - \varepsilon^{-k\pi}} \right\} &= \sum_0^\infty \frac{\cos nx}{k^2 + n^2} \end{aligned} \right\} \dots\dots\dots (6.).$$

Definite
Integrals.

These remarkable formulæ, first given by Poisson in the *Journal de l'Ecole Polytechnique** for 1823, may be considered as embracing all the results which analysts have hitherto obtained by different means with respect to the summation of series of the sines and cosines of multiple arcs, and of sines ascending inversely by the squares of the natural numbers.

It must be observed that they suppose the value of x to be included *between* the limits 0 and 2π , and that they do not necessarily exist at those limits, although the second in point of fact *does* by reason of the equation (5).

k may represent any quantity, real or imaginary: let it become $k\sqrt{-1}$;

$$\therefore \left. \begin{aligned} \frac{\pi}{2} \left\{ \frac{\sin k(\pi-x)}{\sin k\pi} \right\} &= \sum_0^\infty \frac{nk \sin nx}{n^2 - k^2} \\ \frac{\pi}{2k} \left\{ \frac{\cos k(\pi-x)}{\sin k\pi} \right\} &= \sum_0^\infty \frac{\cos nx}{n^2 - k^2} \end{aligned} \right\} \dots\dots\dots (7.).$$

THE ROOTS OF CERTAIN TRANSCENDENTAL EQUATIONS MAY BE EXPRESSED BY DEFINITE INTEGRALS.

Let $\phi\alpha$ be any function of α less than unity independently of the sign, and let x represent any real quantity;

$$\therefore -\log. (1 - \varepsilon^{x\sqrt{-1}}\phi\alpha) = \varepsilon^{x\sqrt{-1}}\phi\alpha + \frac{1}{2} \varepsilon^{2x\sqrt{-1}}\phi^2\alpha + \frac{1}{3} \varepsilon^{3x\sqrt{-1}}\phi^3\alpha + \dots$$

Now n and n' being any two different whole numbers, we have

$$\int_{-\pi}^{\pi} \varepsilon^{(n'-n)x\sqrt{-1}} dx = \int_{-\pi}^{\pi} \cos(n'-n)x dx + \sqrt{-1} \int_{-\pi}^{\pi} \sin(n'-n)x dx = 0.$$

But if n and n' represent the *same* whole number, this integral, instead of vanishing, evidently becomes 2π .

If, therefore, we assume n to be any positive whole number, and multiply the preceding equation first by $\varepsilon^{-nx\sqrt{-1}}$ and then by $\varepsilon^{nx\sqrt{-1}}$, and integrate, we shall obtain the equations

$$\int_{-\pi}^{\pi} \varepsilon^{-nx\sqrt{-1}} \log. (1 - \varepsilon^{x\sqrt{-1}}\phi\alpha) dx = \frac{2\pi}{n} \phi^n \alpha \quad \int_{-\pi}^{\pi} \varepsilon^{nx\sqrt{-1}} \log. (1 - \varepsilon^{x\sqrt{-1}}\phi\alpha) dx = 0.$$

Now let $F\alpha$ represent any other function of α ; then representing $\frac{dF\alpha}{d\alpha}$ by $F'\alpha$, multiplying the above equations by $\frac{1}{1.2.3\dots(n-1)} F'\alpha$; then differentiating them $(n-1)$ times in respect to α , and writing $fx\alpha$ for $F'\alpha \log. (1 - \varepsilon^{x\sqrt{-1}}\phi\alpha)$, we have

$$\begin{aligned} \frac{1}{1.2.3\dots(n-1)} \int_{-\pi}^{\pi} \varepsilon^{-nx\sqrt{-1}} \cdot \frac{d^{n-1}fx\alpha}{d\alpha^{n-1}} dx &= \frac{2\pi}{1.2.3\dots n} \frac{d^{n-1}\{F'\alpha \cdot (\phi\alpha)^n\}}{d\alpha^{n-1}} \\ \frac{1}{1.2.3\dots(n-1)} \int_{-\pi}^{\pi} \varepsilon^{nx\sqrt{-1}} \cdot \frac{d^{n-1}fx\alpha}{d\alpha^{n-1}} dx &= 0. \end{aligned}$$

Giving to n in these equations every value from 1 to ∞ , then adding them, observing that the terms under the sign of integration, the factors $\varepsilon^{-nx\sqrt{-1}}$ and $\varepsilon^{nx\sqrt{-1}}$ being omitted, will in the first represent the expansion of $f(x_1\alpha + \varepsilon^{x\sqrt{-1}})$, and in the second $f(x_1\alpha + \varepsilon^{-x\sqrt{-1}})$, and, finally, writing for these quantities their assumed values,

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} \varepsilon^{-x\sqrt{-1}} F'(\alpha + \varepsilon^{-x\sqrt{-1}}) \cdot \log. \{1 - \varepsilon^{x\sqrt{-1}}\phi(\alpha + \varepsilon^{-x\sqrt{-1}})\} dx &= \frac{dF\alpha}{d\alpha} \cdot \phi\alpha \\ &+ \frac{1}{1.2} \frac{d\left\{\frac{dF\alpha}{d\alpha}(\phi\alpha)^2\right\}}{d\alpha} + \frac{1}{1.2.3} \frac{d^3\left\{\frac{dF\alpha}{d\alpha}(\phi\alpha)^3\right\}}{d\alpha^3} + \dots \end{aligned} \dots\dots\dots (8.),$$

$$\int_{-\pi}^{\pi} \varepsilon^{x\sqrt{-1}} \cdot F'(\alpha + \varepsilon^{x\sqrt{-1}}) \cdot \log. \{1 - \varepsilon^{x\sqrt{-1}}\phi(\alpha + \varepsilon^{x\sqrt{-1}})\} dx = 0 \dots\dots\dots (9.).$$

But by Lagrange's theorem, if y be determined by the relation $y = \alpha + \phi y$, the second number of the first of the above equations will be represented by $Fy - F\alpha$. Subject to this condition then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \varepsilon^{-x\sqrt{-1}} \cdot F'(\alpha + \varepsilon^{-x\sqrt{-1}}) \cdot \log. \{1 - \varepsilon^{x\sqrt{-1}}\phi(\alpha + \varepsilon^{-x\sqrt{-1}})\} dx = Fy - F\alpha \dots\dots\dots (10.).$$

In the particular case in which $F\alpha = \alpha$, this last equation becomes

$$y = \alpha - \frac{1}{2\pi} \int_{-\pi}^{\pi} \varepsilon^{-x\sqrt{-1}} \cdot \log. \{1 - \varepsilon^{x\sqrt{-1}}\phi(\alpha + \varepsilon^{-x\sqrt{-1}})\} dx +$$

* Vol. xii. cahier 19, p. 418.

† This formula, representing under a finite form the root of the equation $y = \alpha + \phi y$, was first given by M. Perseval, (*Mém. Inst.*, tome i.)

Definite
Integrals.

Performing by parts the general integration, of which the definite integral is indicated in equation (10.), and observing that

Definite
Integrals.

$$F'(\alpha + \varepsilon^{-x} \sqrt{-1}) = \frac{dF(\alpha + \varepsilon^{-x} \sqrt{-1})}{d(\alpha + \varepsilon^{-x} \sqrt{-1})} = \frac{dF(\alpha + \varepsilon^{-x} \sqrt{-1})}{dx} \cdot \varepsilon^x \sqrt{-1} \cdot \sqrt{-1}$$

it becomes

$$\sqrt{-1} F(\alpha + \varepsilon^{-x} \sqrt{-1}) \log. \{1 - \varepsilon^x \sqrt{-1} \phi(\alpha + \varepsilon^{-x} \sqrt{-1})\} - \int \frac{\{\varepsilon^x \sqrt{-1} \cdot \phi(\alpha + \varepsilon^{-x} \sqrt{-1}) - \phi'(\alpha + \varepsilon^{-x} \sqrt{-1})\} F(\alpha + \varepsilon^{-x} \sqrt{-1})}{1 - \varepsilon^x \sqrt{-1} \phi(\alpha + \varepsilon^{-x} \sqrt{-1})} dx.$$

Now the quantity not included under the sign of integration being supposed developable in a converging series, according to the powers of $\varepsilon^x \sqrt{-1}$ and $\varepsilon^{-x} \sqrt{-1}$, it follows that it will assume the same* value at the limits $x = \pm \pi$, and that it will therefore disappear from the definite integral;

$$\therefore \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\{\varepsilon^x \sqrt{-1} \cdot \phi(\alpha + \varepsilon^{-x} \sqrt{-1}) - \phi'(\alpha + \varepsilon^{-x} \sqrt{-1})\} F(\alpha + \varepsilon^{-x} \sqrt{-1})}{1 - \varepsilon^x \sqrt{-1} \phi(\alpha + \varepsilon^{-x} \sqrt{-1})} dx = Fy - F\alpha.$$

Now since $F(\alpha + \varepsilon^{-x} \sqrt{-1})$ may be expanded in a converging series, according to the powers of $\varepsilon^{-x} \sqrt{-1}$, and that every term of that series will vanish when integrated between the limits $x = \pm \pi$, except the first term, we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} F(\alpha + \varepsilon^{-x} \sqrt{-1}) dx = F\alpha.$$

Adding this to the preceding equation,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\{1 - \phi'(\alpha + \varepsilon^{-x} \sqrt{-1})\} F(\alpha + \varepsilon^{-x} \sqrt{-1})}{1 - \varepsilon^x \sqrt{-1} \phi(\alpha + \varepsilon^{-x} \sqrt{-1})} dx = Fy \dots \dots (11.).$$

Transforming equation (9.) in the same manner, it will become

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\{1 + \varepsilon^{2x} \sqrt{-1} \cdot \phi'(\alpha + \varepsilon^x \sqrt{-1})\} F(\alpha + \varepsilon^x \sqrt{-1})}{1 - \varepsilon^x \sqrt{-1} \phi(\alpha + \varepsilon^x \sqrt{-1})} dx = F\alpha \dots \dots (12.).$$

Let P be a quantity independent of α and x , and less than unity, we may then assume $\phi y = P$, and $\therefore y = \alpha + P$.

By equation (11.)

$$\therefore \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F(\alpha + \varepsilon^{-x} \sqrt{-1})}{1 - P \varepsilon^x \sqrt{-1}} dx = F(\alpha + P).$$

By equation (12.)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F(\alpha + \varepsilon^x \sqrt{-1})}{1 - P \varepsilon^x \sqrt{-1}} dx = F\alpha.$$

Now

$$\begin{aligned} \int_{-\pi}^{\pi} \frac{F(\alpha + \varepsilon^{-x} \sqrt{-1})}{1 - P \varepsilon^x \sqrt{-1}} dx &= \int_0^{\pi} \left\{ \frac{F(\alpha + \varepsilon^{-x} \sqrt{-1})}{1 - P \varepsilon^x \sqrt{-1}} + \frac{F(\alpha + \varepsilon^x \sqrt{-1})}{1 - P \varepsilon^{-x} \sqrt{-1}} \right\} dx, \\ \int_{-\pi}^{\pi} \frac{F(\alpha + \varepsilon^x \sqrt{-1})}{1 - P \varepsilon^x \sqrt{-1}} dx &= \int_0^{\pi} \left\{ \frac{F(\alpha + \varepsilon^x \sqrt{-1})}{1 - P \varepsilon^x \sqrt{-1}} + \frac{F(\alpha + \varepsilon^{-x} \sqrt{-1})}{1 - P \varepsilon^{-x} \sqrt{-1}} \right\} dx; \end{aligned}$$

therefore, adding and subtracting the preceding equations,

$$\begin{aligned} \frac{1}{\pi} \int_0^{\pi} \{F(\alpha + \varepsilon^{-x} \sqrt{-1}) \pm F(\alpha + \varepsilon^x \sqrt{-1})\} \left\{ \frac{1}{1 - P \varepsilon^x \sqrt{-1}} \pm \frac{1}{1 - P \varepsilon^{-x} \sqrt{-1}} \right\} dx &= F(\alpha + P) \pm F\alpha; \\ \therefore \frac{1}{\pi} \int_0^{\pi} \frac{\{F(\alpha + \varepsilon^{-x} \sqrt{-1}) + F(\alpha + \varepsilon^x \sqrt{-1})\} \{1 - P \cos x\}}{1 - 2P \cos x + P^2} dx &= F(\alpha + P) + F\alpha \\ \frac{P \sqrt{-1}}{\pi} \int_0^{\pi} \frac{\{F(\alpha + \varepsilon^{-x} \sqrt{-1}) - F(\alpha + \varepsilon^x \sqrt{-1})\} \sin x}{1 - 2P \cos x + P^2} dx &= F(\alpha + P) - F\alpha \end{aligned} \dots \dots (13.).$$

These results are of an exceedingly general form, for the function F may be any whatever; but it must not be forgotten, that although their form is finite, yet, being deduced from series which are infinite, they are not necessarily true if these series be divergent. The conditions on which the convergency of the series depend must not, therefore, be neglected; $\phi\alpha$ must be less than unity, and the value assigned to α must in no case be such as to give to the differential coefficients of $F\alpha$ an infinite value, or the second member of equation (10.) cannot be taken to represent the series in equation (8.).†

Making $P=0$ in the first of equations (13.),

$$\frac{1}{\pi} \int_0^{\pi} \{F(\alpha + \varepsilon^{-x} \sqrt{-1}) + F(\alpha + \varepsilon^x \sqrt{-1})\} dx = 2F\alpha.$$

Multiplying this equation by $\frac{1}{2}$, and subtracting it from the equation from which it has been deduced, there results the following:

* The reader will observe, that $\varepsilon^{\pm nx} \sqrt{-1}$ may be put under the form $\cos nx \pm \sqrt{-1} \sin nx$.

† For examples of the cases of exception the reader is referred to the paper in which Poisson has demonstrated these remarkable formulæ, *Journal de l'Ecole Polytechnique*, vol. xii. p. 483, &c.

Definite
Integrals.

$$\frac{1-P^2}{2\pi} \int_0^\pi \frac{F(\alpha + \varepsilon^{-x\sqrt{-1}}) + F(\alpha + \varepsilon^{x\sqrt{-1}})}{1-2P \cos x + P^2} dx = F(\alpha + P) \dots \dots (14.).$$

Definite
Integrals.

As an example of the application of equations (13.), let us assume $F\alpha = \alpha^A$, and let $\alpha = 1$. If we limit the values of A to those which are positive, this supposition will include none of the cases of exception. Also

$$F(1 + \varepsilon^{x\sqrt{-1}}) = 1 + (\varepsilon^{x\sqrt{-1}})^A = (1 + \cos x + \sqrt{-1} \sin x)^A = 2^A \cdot \cos^A \frac{x}{2} \cdot \left\{ \cos \frac{Ax}{2} + \sqrt{-1} \sin \frac{Ax}{2} \right\};$$

$$\therefore F(1 + \varepsilon^{-x\sqrt{-1}}) + F(1 + \varepsilon^{x\sqrt{-1}}) = 2^{A+1} \cdot \cos^A \frac{x}{2} \cdot \cos \frac{Ax}{2}$$

$$\{F(1 + \varepsilon^{-x\sqrt{-1}}) - F(1 + \varepsilon^{x\sqrt{-1}})\} \sqrt{-1} = 2^{A+1} \cdot \cos^A \frac{x}{2} \cdot \sin \frac{Ax}{2}.$$

By the substitution of which, equation (14.) becomes, dividing by 2^A and writing $2x$ for x ,

$$\left. \begin{aligned} \frac{2(1-P^2)}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos^A x \cdot \cos Ax}{1-2P \cos 2x + P^2} dx &= \left(\frac{1+P}{2} \right)^A \\ \text{and by the second of equations (13.)} \\ \frac{4P}{\pi} \int_0^{\frac{\pi}{2}} \frac{\cos^A x \sin Ax \sin 2x}{1-2P \cos 2x + P^2} dx &= \left(\frac{1+P}{2} \right)^A - \frac{1}{2^A} \end{aligned} \right\} \dots \dots (15.).$$

From these formulæ may be obtained an infinity of others by the method of differentiation under the sign of integration. Thus differentiating in respect to A , and then making $A=0$,

$$\begin{aligned} \frac{2(1-P^2)}{\pi} \int_0^{\frac{\pi}{2}} \frac{\log_e \cos x}{1-2P \cos 2x + P^2} dx &= \log_e \left(\frac{1+P}{2} \right), \\ \frac{4P}{\pi} \int_0^{\frac{\pi}{2}} \frac{x \sin 2x}{1-2P \cos 2x + P^2} dx &= \log_e (1+P). \end{aligned}$$

If we differentiate equations (15.) n times in respect to $P=0$ they will become

$$\begin{aligned} \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \cos^A x \cdot \cos Ax \cdot \cos 2nx \, dx &= \frac{A(A-1)(A-2) \dots A-n+1}{1 \cdot 2 \cdot 3 \dots n} \cdot \frac{1}{2^A} \\ \frac{4}{\pi} \int_0^{\frac{\pi}{2}} \cos^A x \cdot \sin Ax \cdot \sin 2nx \, dx &= \frac{A(A-1)(A-2) \dots A-n+1}{1 \cdot 2 \cdot 3 \dots n} \cdot \frac{1}{2^A} \end{aligned}$$

Taking the sum and difference of these equations,

$$\begin{aligned} \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos^A x \cdot \cos (A-2n)x \, dx &= \frac{A(A-1)(A-2) \dots (A-n+1)}{1 \cdot 2 \cdot 3 \dots n} \cdot \frac{1}{2^A} \\ \int_0^{\frac{\pi}{2}} \cos^A x \cdot \cos (A+2n)x \, dx &= 0. \end{aligned}$$

As another application of equations (13.) let us assume $F\alpha = \varepsilon^{A \cos x}$, A being a constant quantity, the factor $\varepsilon^{A \cos x}$ will be common to both sides of these equations, and the second of them and equation (14.) will give

$$\left. \begin{aligned} \frac{1-P^2}{\pi} \int_0^\pi \frac{\varepsilon^{A \cos x} \cos (A \sin x)}{1-2P \cos x + P^2} dx &= \varepsilon^{AP} \\ \frac{2P}{\pi} \int_0^\pi \frac{\varepsilon^{A \cos x} \sin (A \sin x) \sin x}{1-2P \cos x + P^2} dx &= \varepsilon^{AP} - 1 \end{aligned} \right\} \dots \dots (16.).$$

If we assume $P=0$, the first equation gives us

$$\frac{1}{\pi} \int_0^\pi \varepsilon^{A \cos x} \cdot \cos (A \sin x) \cdot dx = 1 \dots \dots (17.).$$

Equalling the coefficients of the same powers (n) of P in the expansions of the two members of equations (16.),

$$\left. \begin{aligned} \frac{2}{\pi} \int_0^\pi \varepsilon^{A \cos x} \cdot \cos (A \sin x) \cdot \cos nx \, dx &= \frac{a^n}{1 \cdot 2 \cdot 3 \dots n} \\ \frac{2}{\pi} \int_0^\pi \varepsilon^{A \sin x} \cdot \sin (A \sin x) \cdot \sin nx \, dx &= \frac{a^n}{1 \cdot 2 \cdot 3 \dots n} \end{aligned} \right\} \dots \dots (18.),$$

where n is any whole positive number.

If we assume the integrals to be taken in two distinct parts, the one extending between the limits 0 and $\frac{1}{2}\pi$, and the other from $\frac{1}{2}\pi$ to π , and in the first part substitute for x , $\frac{\pi}{2} - x$, and in the other part for x , $\frac{\pi}{2} + x$, then will the values of both parts of the integral remain unaltered, provided the limits of the integration of both be made 0 and $\frac{\pi}{2}$.

* Let Φx be any function of x ;

$$\therefore \int_0^{2a} \Phi x \, dx = \int_0^a \Phi x \, dx + \int_a^{2a} \Phi x \, dx.$$

In the first of these two integrals,

Definite Integrals.

If, moreover, we assume $n=2i$, or $2i-1$, according as it is even or odd, we shall obtain the four following equations:

Definite Integrals.

$$\begin{aligned} \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\varepsilon^{A \sin x} + \varepsilon^{-A \sin x}) \cos(A \cos x) \cos 2ix \, dx &= \frac{(-1)^i A^{2i}}{1.2.3 \dots (2i)} \\ \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\varepsilon^{A \sin x} - \varepsilon^{-A \sin x}) \cos(A \cos x) \sin(2i-1)x \, dx &= \frac{(-1)^{i-1} A^{2i-1}}{1.2.3 \dots (2i-1)} \\ \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\varepsilon^{-A \sin x} - \varepsilon^{A \sin x}) \sin(A \cos x) \sin 2ix \, dx &= \frac{(-1)^i A^{2i}}{1.2.3 \dots (2i)} \\ \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (\varepsilon^{-A \sin x} + \varepsilon^{A \sin x}) \sin(A \cos x) \cos(2i-1)x \, dx &= \frac{(-1)^{i-1} A^{2i-1}}{1.2.3 \dots (2i-1)} \end{aligned}$$

As an example of the application of the general equation (11.), let us assume $\phi x = A\varepsilon^x$, A being a given constant, y will then be determined by the equation $y = \alpha + A\varepsilon^y$, and it will be expanded in a converging series by Lagrange's theorem, if $A\varepsilon^x$ be less than 1, independently of the sign. Supposing this, then, to be the case, assuming $Fx = \alpha$, and observing that $\phi'y = \phi y$ and $y = \alpha + \phi y$, so that $\alpha + \varepsilon^{-x\sqrt{-1}} = \alpha + \phi'(\alpha + \varepsilon^{-x\sqrt{-1}})$,

we have by equation (11.),

$$y = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - \varepsilon^{-x\sqrt{-1}})(\alpha + \varepsilon^{-x\sqrt{-1}})}{1 - A\varepsilon^x \varepsilon^{\cos x} \varepsilon^{(x-\sin x)\sqrt{-1}}} dx.$$

Now since the value of $y - \alpha$ is, by Lagrange's theorem, a converging series, ascending by powers of A , it follows that the coefficient of α in the expansion of the second member of the above equation, is unity, or that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - \varepsilon^{-x\sqrt{-1}}}{1 - A\varepsilon^x \varepsilon^{\cos x} \varepsilon^{(x-\sin x)\sqrt{-1}}} dx = 1.$$

The value of y becomes, therefore,

$$y = \alpha + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - \varepsilon^{-x\sqrt{-1}}) \varepsilon^{-x\sqrt{-1}}}{1 - A\varepsilon^x \varepsilon^{\cos x} \varepsilon^{(x-\sin x)\sqrt{-1}}} dx;$$

or observing that $1 - \varepsilon^{-x\sqrt{-1}} = 1 - \cos x + \sqrt{-1} \sin x = 2 \sin \frac{x}{2} \left\{ \sin \frac{x}{2} + \sqrt{-1} \cos \frac{x}{2} \right\}$

$$= 2\sqrt{-1} \sin \frac{x}{2} \left\{ \cos \frac{x}{2} - \sqrt{-1} \sin \frac{x}{2} \right\} = 2\sqrt{-1} \cdot \varepsilon^{-\frac{x}{2}\sqrt{-1}} \sin \frac{x}{2}$$

$$\frac{\sqrt{-1}}{\pi} \int_{-\pi}^{\pi} \frac{\varepsilon^{-\frac{x}{2}\sqrt{-1}} \sin \frac{x}{2}}{1 - A\varepsilon^x \varepsilon^{\cos x} \varepsilon^{(x-\sin x)\sqrt{-1}}} dx = 1$$

$$y = \alpha + \frac{\sqrt{-1}}{\pi} \int_{-\pi}^{\pi} \frac{\varepsilon^{-\frac{x}{2}\sqrt{-1}} \sin \frac{x}{2}}{1 - A\varepsilon^x \varepsilon^{\cos x} \varepsilon^{(x-\sin x)\sqrt{-1}}} dx$$

The imaginary quantities may be made to disappear from this equation by combining the elements of the integral, which correspond to positive and negative values of x , and integrating the whole, thus combined, from 0 to π . The following will be the result in the case in which $\alpha=0$,

$$\int y = \frac{2}{\pi} \int_0^{\pi} \left\{ \frac{\sin \frac{3}{2}x - A\varepsilon^{\cos x} \sin \left(\frac{5}{2}x - \sin x \right)}{1 - 2A\varepsilon^{\cos x} \cos(x - \sin x) + A^2 \varepsilon^{2 \cos x}} \right\} \sin \frac{x}{2} \, dx.$$

SECTION VIII.

The sum of a series of periodical quantities has in the preceding section been expressed by a definite integral, and in a particular case the integration has been performed, and the function itself which expresses this sum has been determined.

let x be assumed $= a - x_1$.

Now

$$\frac{d \int \Phi x \, dx}{dx_1} = - \frac{d \int \Phi x \, dx}{dx} \cdot \frac{dx_1}{dx} = - \frac{d \int \Phi x \, dx}{dx} = - \Phi x = - \Phi(a - x_1)$$

$$\therefore \int \Phi x \, dx = - \int \Phi(a - x_1) \, dx_1;$$

$$\therefore \int_0^a \Phi x \, dx = - \int_a^0 \Phi(a - x_1) \, dx_1 = \int_0^a \Phi(a - x_1) \, dx_1.$$

Similarly,

$$\int_a^{2a} \Phi x \, dx = \int_0^a \Phi(a + x_1) \, dx_1$$

Definite
Integrals.

The converse of this proposition is now to be discussed. IT IS PROPOSED TO EXPRESS THE VALUE OF A FUNCTION BETWEEN CERTAIN LIMITS OF THE VARIABLE ON WHICH IT DEPENDS, BY THE SUM OF A SERIES OF PERIODICAL INTEGRALS TAKEN BETWEEN THOSE LIMITS. Definite Integrals.

Let fx be an arbitrary function, whose values, between the limits fl and $f-l$, it is required to express by a series of periodical quantities.

In equation (3.) of the last section, if $\frac{\pi(x-x_1)}{l}$ be substituted for x , both sides multiplied by $\frac{fx_1}{l}$, and the equation integrated in respect to x_1 , between the limits l and $-l$, we shall obtain the following equation:

$$\int_{-l}^l \frac{(1-p^n) \cdot fx_1 \cdot dx_1}{2l \left\{ 1 - 2p \cos \frac{\pi(x-x_1)}{l} + p^2 \right\}} = \frac{1}{2l} \int_{-l}^l fx_1 dx_1 + \frac{1}{l} \sum_1^n p^n \left\{ \int_{-l}^l \cos \frac{n\pi(x-x_1)}{l} fx_1 dx_1 \right\}.$$

n being a whole positive number, and its values extending, under the symbol Σ , from 1 to infinity.

Now, suppose p to be indefinitely near to unity, the first member of this equation will then equal fx ; for, on this supposition, all the elements of the integral will be evanescent, except those for which $\cos \frac{\pi(x-x_1)}{l}$ approximates indefinitely to unity, or x_1 indefinitely to x . Let $x_1 = x + z$, and $1-p = \rho$. For those values of x_1 for which z is indefinitely small the integral becomes then

$$fx \cdot \int_{-(l-x)}^{(l-x)} \frac{\rho dz}{\rho^2 l^2 + \pi^2 z^2} \dots \dots (1.).$$

Moreover, since the integral in this expression is evanescent for all values of z , except such as are infinitely small, it is immaterial between what limits we take it, other than such as are infinitely small, and we may take it without altering its value between the limits $+\infty$ and $-\infty$, in which case it will become equal to unity, and the proposed integral will become fx .

Now this integral, thus taken, includes all the elements of the integral in the first member of the preceding equation, except those which are evanescent. Making $p = 1$;

$$\therefore fx = \frac{1}{2l} \int_{-l}^l fx_1 dx_1 + \frac{1}{l} \sum_1^n \left\{ \int_{-l}^l \cos \frac{n\pi(x-x_1)}{l} fx_1 \cdot dx_1 \right\} \dots \dots (2.).$$

This is the theorem of POISSON.*

It has been stated to be immaterial between what limits the integral (1.) is taken, provided they be not infinitely small, and the value of that integral is taken on the supposition that they are not. If, then, l differ from x only by an exceeding small quantity, the theorem fails.

In this case the value of the integral is made up of those of its elements which lie immediately about both its limits, all the rest being evanescent. We may then suppose this whole integral to be the sum of two others. Commencing respectively from its limits, and extending each to infinity, it will be represented thus,

$$fl \int_{(l-x)}^{\infty} \frac{\rho dz}{\rho^2 l^2 + \pi^2 z^2} + f-l \int_{-\infty}^{-(l-x)} \frac{-\rho dz}{\rho^2 l^2 + \pi^2 z^2}.$$

whence, since each of the integrals equals $\frac{1}{2}$, we obtain,

$$\frac{1}{2} \{fl + f-l\} = \frac{1}{2l} \int_{-l}^l fx_1 dx_1 + \frac{1}{l} \sum (-1)^n \left\{ \int_{-l}^l \cos \frac{n\pi x_1}{l} fx_1 dx_1 \right\} \dots \dots (3.).$$

Comparing this equation with equation (2.) it is manifest that the latter fails when $l = x$, except the form of the function be such that $fl = f-l$.

The function fx is entirely arbitrary, it is not even subject to the law of continuity: thus we may suppose fx to have certain finite values during that portion of its transition from $f-l$ to fl , which lies between the limits $f\lambda$ and $f\lambda'$, and to be evanescent without those limits, the integral being thus to be taken between the limits λ and λ' instead of l and $-l$. But it is important to observe, that for the values $x = \lambda$ and $x = \lambda'$ the theorem will fail, the actual value of fx being one half of that which is given by it. In the demonstration of it, x , about which the finite value of the integral is found, is supposed to lie between the limits of integration, the integral includes, therefore, elements which lie in both directions, positively and negatively from x , and which constitute each one half of the whole integral; but when x does not lie between these limits, but coincides with one of them, the integral embraces only the elements which lie in one direction from it, the others being without the limits of integration, and therefore evanescent. Thus the integral is in this particular case one half the general integral.

Let us now suppose that fx vanishes for all negative values of x , including $-l$, but that it does not vanish for the positive values between 0 and l . The theorem then becomes

$$fx = \frac{1}{2l} \int_0^l fx_1 dx_1 + \frac{1}{l} \sum_1^n \left\{ \int_0^l \cos \frac{n\pi(x-x_1)}{l} fx_1 dx_1 \right\}.$$

Replacing x by $-x$ we have

$$0 = \frac{1}{2l} \int_0^l fx_1 dx_1 + \frac{1}{l} \sum_1^n \left\{ \int_0^l \cos \frac{n\pi(x+x_1)}{l} fx_1 dx_1 \right\}.$$

* *Théorie Mathématique de la Chaleur*, p. 188.

Definite
Integrals.

Adding these equations, and subtracting them,

$$fx = \frac{1}{l} \int_0^l f x_1 dx_1 + \frac{2}{l} \sum \left\{ \int_0^l \cos \frac{n\pi x_1}{l} f x_1 dx_1 \right\} \cos \frac{n\pi x}{l} \dots \dots (4.),$$

$$fx = \frac{2}{l} \sum \left\{ \int_0^l \sin \frac{n\pi x_1}{l} f x_1 dx_1 \right\} \sin \frac{n\pi x}{l} \dots \dots (5.).$$

 These two equations constitute the theorem of LAGRANGE, given in the third volume of the *Anciens Mém. de Turin*.

 In equation (2.) the limits l and $-l$ may be as great as we please. Let them be infinite. The second member of that equation will then represent the function fx for all real values of x from $-\infty$ to $+\infty$. Moreover, the first term of it will vanish by reason of the infinite divisor l , and the second term may be written under the form

$$\frac{1}{\pi} \sum_1 \int_{-l}^l \cos \left(\frac{n\pi}{l} \right) (x - x_1) \cdot f x_1 \cdot \left(\frac{\pi}{l} \right) dx_1.$$

 If, therefore, we write the infinitesimal quantity $\left(\frac{\pi}{l} \right) = d\alpha$, and $n \frac{\pi}{l} = \alpha$, the symbol $\sum_1$ will become $\int_0^\infty$, and we shall have

$$fx = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{+\infty} \cos \alpha (x - x_1) f x_1 d\alpha dx_1 \dots \dots (6.).$$

which is FOURIER'S celebrated theorem. (See CALCULUS OF FUNCTIONS, art. 225.)

 This theorem being in point of fact embraced under the preceding theorem of Poisson, the method of its application is the same. Thus to apply it to a given function fx of x , whose value it is required to determine between certain given limits, it must be converted into a discontinuous function, evanescent beyond certain other arbitrary limits including these, and the integration $\int_{-\infty}^{+\infty}$ then made in respect to these arbitrary limits instead of $-\infty$ and $+\infty$: and, in general, the integration is not in any case to be made in respect to these infinite limits, unless the arbitrary limits spoken of above, beyond which the function is evanescent, be themselves infinite.

The following example may serve as a verification of the theorem:

 Let $fx = x$, and let x be supposed to be evanescent beyond the limits λ and λ' , therefore, by the theorem of Fourier,

$$x = \frac{1}{\pi} \int_0^\infty \int_\lambda^{\lambda'} x_1 \cos \alpha (x - x_1) d\alpha dx_1.$$

$$\text{Now } \int_\lambda^{\lambda'} x_1 \cos \alpha (x - x_1) dx_1 = -\frac{\lambda'}{\alpha} \sin \alpha (x - \lambda') + \frac{\lambda}{\alpha} \sin \alpha (x - \lambda) + \frac{1}{\alpha^2} \{ \cos \alpha (x - \lambda') - 1 \} - \frac{1}{\alpha^2} \{ \cos \alpha (x - \lambda) - 1 \};$$

also

$$\int_0^\infty \frac{\sin \mu \alpha}{\alpha} d\alpha = \frac{1}{2} \pi;$$

 and multiplying by $d\mu$, and integrating in respect to μ , under the sign of integration in respect to α , we have, taking the integral from 0 to μ ,

$$\int_0^\infty \frac{\cos \mu \alpha - 1}{\alpha^2} d\alpha = -\frac{1}{2} \pi \alpha.$$

 Applying then these formulæ to the preceding integration $(x - \lambda)$, and $(x - \lambda')$ being substituted successively for μ , we get

$$\frac{1}{\pi} \int_0^\infty \int_\lambda^{\lambda'} x_1 \cos \alpha (x - x_1) dx_1 d\alpha = \frac{1}{2} \lambda' + \frac{1}{2} \lambda + \frac{1}{2} (x - \lambda') + \frac{1}{2} (x - \lambda).$$

 Now the second member of this equation reduces itself to x , and thus the theorem is verified.

SECTION IX.

ON THE VALUES OF CERTAIN DEFINITE INTEGRALS DETERMINED BY THE METHOD OF INTEGRATION OR DIFFERENTIATION IN RESPECT TO AN ARBITRARY CONSTANT UNDER THE SIGN OF INTEGRATION.

 Let u be any function of x , involving the arbitrary constant α ;

$$\therefore \frac{d \int u dx}{dx} = u; \therefore \frac{d^2 \int u dx}{dx d\alpha} = \frac{du}{d\alpha}; \therefore \frac{d^3 \int u dx}{d\alpha^2 dx} = \frac{d^2 u}{d\alpha^2}; \therefore \frac{d \int u dx}{d\alpha} = \int \frac{du}{d\alpha} dx \dots \dots (1.);$$

 that is, the integral $\int u dx$ is differentiated in respect to α , when u is differentiated under the sign of integration in respect to α . In the last equation, for u substitute $\int u d\alpha$;

$$\therefore \frac{d \int \{ \int u d\alpha \} dx}{d\alpha} = \int u dx.$$

Integrating

$$\int \{ \int u d\alpha \} dx = \int \{ \int u dx \} d\alpha \dots \dots (2.),$$

 Definite
Integrals.

Definite Integrals. that is, the integral $\int u dx$ is integrated in respect to α when u is integrated under the sign of integration in respect to α . *Definite Integrals.* The above processes of integration and differentiation proved of the general integral are necessarily true of the definite integral, and proved of the first orders of integration and differentiation, they are necessarily true of every other.

By means of them we may, from the known integral of one function between certain limits, determine the integrals of an infinite number of others between the same limits,

Thus, let $\int_a^b u dx = A$, A being given, and let u_n , A_n , $u^{(n)}$, $A^{(n)}$ represent respectively the n th integrals, and the n th differential coefficients of u and A in respect to the arbitrary constant α ; then will

$$\int_a^b u_n dx = A_n, \text{ and } \int_a^b u^{(n)} dx = A^{(n)}.$$

EXAMPLE 1.

$$\int_0^a (a^2 - x^2)^{-\frac{1}{2}} dx = \frac{a\pi}{2}.$$

Differentiating with respect to a n times,

$$\int_0^a (a^2 - x^2)^{-\frac{2n+1}{2}} dx = -\frac{\pi}{2(2n-1)a^{2n-1}}.$$

EXAMPLE 2.

$$\int_0^1 x^{m-1} dx = \frac{1}{m}.$$

Integrating with respect to m under the sign of integration,

$$\int_0^1 \left\{ \frac{x^m}{\log x} + C \right\} = \log m;$$

whence since when $m=1$ $\log m=0$; $\therefore C = \frac{-1}{\log x}$; $\therefore \int_0^1 \frac{x^m - 1}{\log x} = \log m$.

Another application of this method of differentiating and integrating under the sign of integration is the following.

Let $\int_a^b u dx$ be assumed $= A$, and let this equation be n times differentiated in respect to the arbitrary constant

α . It may occur that, between two or more of the resulting equations, the integrals $\int_a^b \frac{d^n u}{da^n} dx$, &c., which enter into these, may eliminate. A differential equation will then be obtained in A on the solution of which the determination of the value of the proposed integral will have been made to depend.

EXAMPLE 3. To determine the definite integral $\int_0^\infty \epsilon^{-x^2 - \frac{a^2}{x^2}} dx$ assume it $= A$, then differentiating with respect to a ,

$$\frac{dA}{da} = -2 \int_0^\infty \epsilon^{-x^2 - \frac{a^2}{x^2}} \cdot \frac{a}{x^2} dx = -2 \int_0^\infty \epsilon^{-\left(\frac{a}{x}\right)^2 - \left(\frac{a}{x}\right)^2} \cdot d\left(\frac{a}{x}\right),$$

or

$$\frac{dA}{da} = -2A; \therefore A = C \epsilon^{-2a}.$$

To determine C , let $a=0$; $\therefore C = A_0 = \int_0^\infty \epsilon^{-x^2} dx = \frac{1}{2} \sqrt{\pi}$ (equation (16.), Section VI.);

$$\therefore \int_0^\infty \epsilon^{-x^2 - \frac{a^2}{x^2}} dx = \frac{1}{2} \sqrt{\pi} \epsilon^{-2a}.$$

EXAMPLE 4. To determine the definite integral

$$\int_0^\infty \epsilon^{-x^n - \frac{ba^n}{x^n}} dx,$$

under which form is included the preceding. Assume it $= A$: differentiating twice with respect to a ,

$$\frac{d^2 A}{da^2} = -n(n-1)ba^{n-2} \int_0^\infty \epsilon^{-x^n - \frac{ba^n}{x^n}} \cdot \frac{dx}{x^n} + n^2 b^2 a^{2n-2} \int_0^\infty \epsilon^{-x^n - \frac{ba^n}{x^n}} \cdot \frac{dx}{x^{2n}}.$$

But integrating by parts we obtain

$$\int_0^\infty \epsilon^{-x^n - \frac{ba^n}{x^n}} \cdot \frac{dx}{x^{2n}} = \frac{1}{nba^n x^{n-1}} \cdot \epsilon^{-x^n - \frac{ba^n}{x^n}} + \frac{1}{ba^n} \int_0^\infty \epsilon^{-x^n - \frac{ba^n}{x^n}} dx + \frac{n-1}{nba^n} \int_0^\infty \epsilon^{-x^n - \frac{ba^n}{x^n}} \cdot \frac{dx}{x^n}.$$

Also the term without the integral sign vanishes for both of the limiting values of x , 0 and ∞ ;

Definite Integrals.

Definite Integrals.

$$\therefore \int_0^{\infty} \epsilon^{-x^n - \frac{ba^n}{x^n}} \frac{dx}{x^{2n}} = \frac{1}{ba^n} \int_0^{\infty} \epsilon^{-x^n - \frac{ba^n}{x^n}} dx + \frac{n-1}{nba^n} \int_0^{\infty} \epsilon^{-x^n - \frac{ba^n}{x^n}} \cdot \frac{dx}{x^n}.$$

$$\frac{d^2 A}{da^2} = n^2 ba^{n-2} \cdot A \dots \dots (3.).$$

 $\therefore$ eliminating,

On the integration of which differential equation the value of the proposed integral is made to depend.

 To reduce it to an equation of the first order, let A be assumed $= \epsilon^{fRda}$;

$$\therefore \frac{d^2 A}{da^2} = \left(\frac{dR}{da} + R^2 \right) \epsilon^{fRda};$$

 $\therefore$ by equation (3.)

$$\frac{dR}{da} + R^2 = n^2 ba^{n-2} \dots \dots (4.);$$

 which is the equation well known by the name of Riccati's equation, integrable under a finite form by the ordinary methods, when $n-2 = \frac{-4i}{2i \pm 1}$, where i is any positive integer. The value of the proposed definite integral

 may therefore always be determined whenever n is included under the form $\frac{2}{1 \pm 2i}$.

 For a yet more general integral, of which this is a particular case, the reader may see Euler, *Calcul Integral*, chapter ix. and x.

EXAMPLE 5. To determine the values of the integrals

$$\int_0^{\infty} \epsilon^{-bx} \cdot \cos ax \cdot x^{n-1} dx, \text{ and } \int_0^{\infty} \epsilon^{-bx} \cdot \sin ax \cdot x^n \cdot ax,$$

let the first integral be represented by A, and the second by B;

$$\therefore \frac{dA}{da} = - \int_0^{\infty} \epsilon^{-bx} \cdot \sin ax \cdot x^n \cdot dx$$

$$\frac{dB}{da} = \int_0^{\infty} \epsilon^{-bx} \cdot \cos ax \cdot x^n \cdot dx.$$

And integrating by parts,

$$\int \epsilon^{-bx} \sin ax x^n dx = -\frac{1}{b} \epsilon^{-bx} \sin ax x^n + \frac{a}{b} \int \epsilon^{-bx} \cos ax x^n dx + \frac{n}{b} \int \epsilon^{-bx} \sin ax x^{n-1} dx,$$

$$\int \epsilon^{-bx} \cos ax x^n dx = -\frac{1}{b} \epsilon^{-bx} \cos ax x^n - \frac{a}{b} \int \epsilon^{-bx} \sin ax x^n dx + \frac{n}{b} \int \epsilon^{-bx} \cos ax x^{n-1} dx.$$

 Now the exponents b and n being supposed positive, the first terms of the second members of each of these equations vanish for the values of x , 0, and ∞ . They become, therefore, by substitution,

$$\left. \begin{aligned} \frac{dA}{da} &= \frac{a}{b} \frac{dB}{da} + \frac{n}{b} B \\ \frac{dB}{da} &= \frac{a}{b} \frac{dA}{da} + \frac{n}{b} A \end{aligned} \right\} \dots \dots (5.).$$

Eliminating B between these equations we have

$$\frac{d(a^2 + b^2) \frac{dA}{da}}{da} + 2na \frac{dA}{da} + (n + n^2) A = 0$$

 Assuming in this equation $A = (b^2 + a^2)^m R$, and $\tan^{-1} \frac{a}{b} = \alpha$, we shall obtain ultimately, by substitution and reduction,

$$b^2 \frac{d^2 R}{d\alpha^2} + (2n + 4m) ba \frac{dR}{d\alpha} + \{ (n + n^2 + 2m) b^2 + (n + 2m)(1 + n + 2m) a^2 \} R = 0;$$

 in which equation, if we assume $m = -\frac{1}{2}n$, we shall obtain

$$\frac{d^2 R}{d\alpha^2} + n^2 R = 0 \dots \dots (6.);$$

whence by the known methods

$$R = C_1 \cos n\alpha + C_2 \sin n\alpha;$$

$$\therefore A = \frac{C_1 \cos n\alpha + C_2 \sin n\alpha}{(a^2 + b^2)^{\frac{n}{2}}}.$$

 To determine C_1 and C_2 let α be made $= 0$ in the values of A and $\frac{dA}{da}$;

Definite
Integrals.

$$\therefore A_0 = C_1 b^{-n}, \quad \left(\frac{dA}{da} \right)_0 = C_2 n b^{-n}.$$

Definite
Integrals.

Now

$$A_0 = \int_0^{\infty} e^{-bx} x^{n-1} dx,$$

$$\left(\frac{dA}{da} \right)_0 = 0;$$

$$\therefore C_1 = b^n \int_0^{\infty} e^{-bx} x^{n-1} dx = \int_0^{\infty} e^{-bx} (bx)^{n-1} dbx = - \int_0^{\infty} \left\{ \log \frac{1}{bx} \right\}^{n-1} dbx = \int_0^{\infty} \left\{ \log \frac{1}{bx} \right\}^{n-1} dbx = \Gamma n C_2 = 0;$$

$$\therefore \int_0^{\infty} e^{-bx} \cos ax \cdot x^{n-1} dx = \frac{\cos \left\{ n \tan^{-1} \frac{a}{b} \right\}}{(a^2 + b^2)^{\frac{n}{2}}} \Gamma n \dots \dots (7.).$$

Eliminating $\frac{dB}{da}$ between equations (5.), and substituting for A and $\frac{dA}{da}$, we obtain

$$\int_0^{\infty} e^{-bx} \sin ax \cdot x^{n-1} dx = \frac{\sin \left\{ n \tan^{-1} \frac{a}{b} \right\}}{(a^2 + b^2)^{\frac{n}{2}}} \Gamma n \dots \dots (8.).$$

Let $b=0$, and $n=1$;

$$\therefore \int_0^{\infty} \cos ax \, dx = \frac{\cos \frac{\pi}{2}}{a} \Gamma 1 = 0$$

$$\int_0^{\infty} \sin ax \, dx = \frac{\sin \frac{\pi}{2}}{a} \Gamma 1 = \frac{1}{a}.$$

These values of $\int_0^a \cos ax \, dx$ and $\int_0^a \sin ax \, dx$ are to be considered as limits of the values of $\int_0^{\infty} e^{-bx} \cos ax \cdot x^{n-1} dx$ and $\int_0^{\infty} e^{-bx} \sin ax \cdot x^{n-1} dx$. The integrals of periodical functions, such as $\cos ax$ and $\sin ax$, which extend to infinity, can in no case have any other interpretation than as the limits of other integrals whose elements continually diminish, and in their infinite state vanish.

EXAMPLE 6. To determine a differential equation on the solution of which the integral

$$\int_0^{\infty} \frac{\cos ax \, dx}{A + Bx^2 + Cx^4 + \dots + x^{2n}}$$

may be made to depend.

Assume this integral to be represented by u ; and let the denominator be supposed not to vanish for any real value of x , so that the integral may not become infinite between the values of x , 0, and ∞ . Differentiating the proposed integral twice, four times, &c., $2n$ times in respect to a , and multiplying successively by $+A$, $-B$, $+C$, $-D$, &c., the denominator will, by the addition of the resulting equations, disappear, and we shall obtain the equation

$$Au - B \frac{d^2 u}{da^2} + C \frac{d^4 u}{da^4} - D \frac{d^6 u}{da^6} + \dots \pm \frac{d^{2n} u}{da^{2n}} = \int_0^{\infty} \cos ax \, dx.$$

But it has been shown that

$$\int_0^{\infty} \cos ax \, dx = 0;$$

$$\therefore Au - B \frac{d^2 u}{da^2} + C \frac{d^4 u}{da^4} - D \frac{d^6 u}{da^6} + \dots \pm \frac{d^{2n} u}{da^{2n}} = 0 \dots \dots (9.).$$

On the solution of which differential equation by the known rules the value of the proposed definite integral is made to depend.

EXAMPLE 7.

$$\text{Let } A = \int_0^{\pi} \frac{\sin^p x \, dx}{(1 - 2a \cos x + a^2)^n};$$

in which a is an arbitrary constant, and p and n any quantities whatever, positive or negative, with this condition, that $p+1$ is positive, without which the value of the integral would be infinite. Differentiating twice with respect to a ,

$$\frac{dA}{da} = 2n \int_0^{\pi} \frac{(\cos x - a) \sin^p x \, dx}{(1 - 2a \cos x + a^2)^{n+1}}$$

$$\frac{d^2 A}{da^2} = 4n(n+1) \int_0^{\pi} \frac{(\cos x - a)^2 \sin^p x \, dx}{(1 - 2a \cos x + a^2)^{n+2}} - 2n \int_0^{\pi} \frac{\sin^p x \, dx}{(1 - 2a \cos x + a^2)^{n+1}}.$$

Now, integrating by parts, we have

Definite
Integrals.

$$\int \frac{\cos x \sin^p x dx}{(1-2a \cos x + a^2)^{n+1}} = \frac{1}{p+1} \frac{\sin^{p+1} x}{(1-2a \cos x + a^2)^{n+1}} + \frac{2(n+1)a}{p+1} \int \frac{\sin^{p+2} x dx}{(1-2a \cos x + a^2)^{n+2}};$$

Definite
Integrals.

and since the first term of the second member vanishes at the limits 0 and π ,

$$\therefore \frac{dA}{da} = \frac{4n(n+1)a}{p+1} \int_0^\pi \frac{\sin^{p+2} x dx}{(1-2a \cos x + a^2)^{n+2}} - 2na \int_0^\pi \frac{\sin^p x dx}{(1-2a \cos x + a^2)^{n+1}}.$$

Multiplying this value of $\frac{dA}{da}$ by $\frac{p+1}{a}$, and adding it to that of $\frac{d^2A}{da^2}$, we have

$$\frac{d^2A}{da^2} + \frac{p+1}{a} \frac{dA}{da} = 2n(2n-p) \int_0^\pi \frac{\sin^p x dx}{(1-2a \cos x + a^2)^{n+1}};$$

also adding the first value of $\frac{dA}{da}$ multiplied by a , to that of A multiplied by n

$$a \frac{dA}{da} + nA = n(1-a^2) \int_0^\pi \frac{\sin^p x dx}{(1-2a \cos x + a^2)^{n+1}}.$$

Eliminating, therefore, the integrals in the two last equations,

$$(1-a^2) \frac{d^2A}{da^2} + \left\{ \frac{p+1}{a} - (4n-p+1)a \right\} \frac{dA}{da} - 2n(2n-p)A = 0. \dots \dots (10.).$$

On the solution of which differential equation depends the value of the definite integral under consideration. We shall here confine ourselves to the discussion of a case in which this equation presents itself under a form readily integrable, referring the reader for the discussion of other integrable cases to the paper of Poisson, in the tenth volume of the *Journal de l'Ecole Polytechnique*.

Let $p=2n$; $\therefore \frac{d^2A}{da^2} + \frac{2n+1}{a} \frac{dA}{da} = 0$, whose integral is $A = C_1 + C_2 a^{-2n}$.

In the determination of the arbitrary constants C_1 and C_2 , two cases present themselves. First, that in which $a < 1$, and secondly, that in which $a > 1$. In the first case the proposed integral may be developed in a converging series, ascending by whole and positive powers of a ; in the second case the series will descend by the powers of $a-2n, -2n-1, -2n-2$, &c., in which no power can equal zero. From this it follows that in the first case no term or terms of the expanded integral can equal $C_2 a^{-2n}$, or $C_2 a^{-p}$ - p being essentially negative or a fraction; and that in the second case no term can equal $C_1 a^0$; so that in the first case C_2 must equal 0, and in the second, C_1 must equal 0;

$$\therefore \text{in the first case, } \int_0^\pi \frac{\sin^{2n} x dx}{(1-2a \cos x + a^2)^{2n}} = C_1; \text{ and in the second case, } \int_0^\pi \frac{\sin^{2n} x dx}{(1-2a \cos x + a^2)^{2n}} = C_2 a^{-2n}.$$

Also making in the first integral $a=0$, we have $\int_0^\pi \sin^{2n} x dx = C_1$; and making in the second $a = \infty$, $\int_0^\pi \sin^{2n} x dx = C_2$.

$$\text{In the first case therefore } \int_0^\pi \frac{\sin^{2n} x dx}{(1-2a \cos x + a^2)^{2n}} = \int_0^\pi \sin^{2n} x dx = \frac{1.3 \dots (2n-1)}{2.4 \dots 2n} \pi.$$

$$\text{The second case } \int_0^\pi \frac{\sin^{2n} x dx}{(1-2a \cos x + a^2)^{2n}} = a^{-2n} \int_0^\pi \sin^{2n} x dx = \frac{1.3 \dots (2n-1)}{2.4 \dots 2n} a^{-2n} \pi.$$

SECTION X.

ULTRA ELLIPTIC FUNCTIONS.

Elliptic functions, properly so called, and ultra elliptic functions, are included in common under the formula

$$\int_0^{x_1} \frac{fx dx}{(x-\alpha)\sqrt{\phi x}}.$$

where fx and ϕx are rational functions of x .

If the dimensions of ϕx be not less than 3, this formula represents an ULTRA ELLIPTIC FUNCTION.

An ultra elliptic function is said to be of the n th class when the dimensions of ϕx are represented by the formulæ $2n+1$, or $2n+2$.

In the formula

$$C \frac{1}{x} \frac{\Psi x fx}{x-\alpha} - \Psi_\alpha . fx + C \dots \dots (1.),$$

which, by ABEL'S THEOREM (Section II., page 499), represents the sum $\sum \pm \psi x$ of any number of elliptic or ultra elliptic functions; the second term may vanish, fx being of the form $(x-\alpha)f_1x$, so that $fx=0$. Moreover, the first term may vanish, $\frac{1}{x}$ not entering in the first power into the expansion of $\frac{\Psi fx}{x-\alpha}$ according to the powers of $\left(\frac{1}{x}\right)$.

Thus the two first terms of the formula may vanish, and the theorem of Abel reduce itself to

$$\sum \pm \psi x = C; \quad 3 \text{ z } 2$$

Definite
Integrals.

or the second term only may vanish, and the equation become

$$\Sigma \pm \psi x = C \frac{1}{x} \frac{\Psi x f x}{x-a} + C;$$

Definite
Integrals.or neither of the terms of the equation may vanish, but fx may represent unity, in which case

$$\Sigma \pm \psi x = C \frac{1}{x} \frac{\Psi x}{x-a} - \Psi a + C.$$

All elliptic and ultra elliptic functions are included under some one of these three cases, or are reducible to others included under them; and, as they are included under the first, the second, or the third case, they are said to be of the FIRST, the SECOND, or the THIRD ORDERS; a division analogous to that of ordinary elliptic functions, and, indeed, including it.

To determine under what circumstances the term $C \frac{1}{x} \frac{\Psi x f x}{x-a}$ is evanescent.

$$\Psi x = \frac{1}{\sqrt{\phi x}} \log. \left\{ \frac{1 + \frac{\theta_2 x (\phi_2 x)^{\frac{1}{2}}}{\theta_1 x (\phi_1 x)^{\frac{1}{2}}}}{1 - \frac{\theta_2 x (\phi_2 x)^{\frac{1}{2}}}{\theta_1 x (\phi_1 x)^{\frac{1}{2}}}} \right\}.$$

This equation may be expanded under the form

$$\Psi x = \frac{2}{\phi_2 x} \sum_1^{\infty} \frac{1}{2N-1} \left(\frac{\theta_2 x}{\theta_1 x} \right)^{(2N-1)} \cdot \left(\frac{\phi_2 x}{\phi_1 x} \right)^N$$

the values of N being taken from unity to infinity.

$$\text{Now } \frac{\theta_2 x}{\theta_1 x} = \frac{x^{n_2} \left\{ l_2 + k_2 \left(\frac{1}{x} \right) + \dots + a_2 \left(\frac{1}{x} \right)^{n_2} \right\}}{x^{n_1} \left\{ l_1 + k_1 \left(\frac{1}{x} \right) + \dots + a_1 \left(\frac{1}{x} \right)^{n_1} \right\}} = \left(\frac{1}{x} \right)^{n_1 - n_2} \odot \frac{1}{x},$$

where $\odot \frac{1}{x}$ is a function ascending by powers of $\frac{1}{x}$ from $\left(\frac{1}{x} \right)^0$ and n_1, n_2 are the dimensions of $\theta_1 x, \theta_2 x$.

Similarly, if m_1, m_2 be the dimensions of $\phi_1 x, \phi_2 x$, $\frac{\phi_2 x}{\phi_1 x} = \left(\frac{1}{x} \right)^{m_1 - m_2} \Phi \frac{1}{x}$,

where $\Phi \frac{1}{x}$ is a function ascending by powers of $\frac{1}{x}$ from $\left(\frac{1}{x} \right)^0$;

$$\therefore \left(\frac{\theta_2 x}{\theta_1 x} \right)^{2N-1} \left(\frac{\phi_2 x}{\phi_1 x} \right)^N = \left(\frac{1}{x} \right)^{(2N-1)(n_1 - n_2) + N(m_1 - m_2)} \left\{ \odot \frac{1}{x} \right\} \left\{ \Phi \frac{1}{x} \right\}.$$

Similarly it may be shown that

$$\frac{fx}{(x-a)\phi_2 x} = \left(\frac{1}{x} \right)^{m_2 - f + 1} f_1 \frac{1}{x},$$

where $f_1 \frac{1}{x}$ is a function ascending by powers of $\left(\frac{1}{x} \right)$ from $\left(\frac{1}{x} \right)^0$, and where f represents the dimensions of fx in x ;

$$\therefore C \frac{1}{x} \frac{\Psi x f x}{x-a} = 2 \sum_1^{\infty} \frac{1}{2N-1} \left\{ F \frac{1}{x} \right\} \left(\frac{1}{x} \right)^{(2N-1)(n_1 - n_2) + N(m_1 - m_2) + m_2 - f + 1},$$

where $F \frac{1}{x}$ is a function ascending by powers of $\frac{1}{x}$ from $\left(\frac{1}{x} \right)^0$.

Hence, therefore, it is manifest that the term $C \frac{1}{x} \frac{\Psi x f x}{x-a}$ is evanescent, if, for all positive integral values of N ,

$$(2N-1)(n_1 - n_2) + N(m_1 - m_2) + m_2 - f + 1 > 1;$$

or

$$N \{ 2(n_1 - n_2) + (m_1 - m_2) \} + m_2 + n_2 - (n_1 + f) > 0.$$

And if this inequality do not obtain for all positive integral values of N , then the first term of the formula (1.) is not evanescent.

Now let us suppose that m_1 and n_1 are taken greater than m_2 and n_2 , or equal to them, then, its first term being positive, the inequality will be satisfied for all values of N , if it is satisfied for its first value, unity, or if

$$2(n_1 - n_2) - m_1 - m_2 + m_2 + n_2 - n_1 - f > 0;$$

or if

$$n_1 - n_2 + m_1 - f > 0 \dots \dots (2.).$$

Also, since m_1 and n_1 are greater than m_2 and n_2 , the dimensions of the first member of the equation

$$(\theta_1 x)^2 (\phi_1 x) - (\theta_2 x)^2 (\phi_2 x)^2 (\phi_2 x) = P(x - x_1)$$

Definite
Integrals.

are those of its first term ;

$$\therefore 2n_1 + m_1 = \mu \dots \dots \dots (3.).$$

 Let the difference of the dimensions of the two terms of the first member of the equation be δ

$$\therefore 2n_1 + m_1 = \delta + 2n_2 + m_2.$$

 Eliminating m_1 and m_2 , observing that $m_1 + m_2 = m$, we have

$$\mu = n_1 + n_2 + \frac{m + \delta}{2} \dots \dots \dots (4.).$$

 Eliminating m_1 by means of equation (3.), the inequality (2.) becomes $-(n_1 + n_2) + \mu - f >$, and eliminating $-(n_1 + n_2) + \mu$ by equation (4.), it becomes

$$\frac{m + \delta}{2} - f > 0 \dots \dots \dots (5.).$$

 It is on this inequality, then, that depends ultimately the evanescence of the first term of the formula (1.), provided m_1 and n_1 be assumed greater than m_2 and n_2 .

 Now, fx may, or it may not, be of the form $(x - \alpha) f_1 x$, where $f_1 x$ is a rational function of x .

 If it be of that form, the second term of the formula (1.) will vanish since $f\alpha = 0$.

 Moreover, the evanescence of the first term depends upon the condition expressed by the inequality (5.); or, since we may assume the arbitrary functions $\theta_1 x$ and $\theta_2 x$ of such dimensions that δ may equal 0 or 1, according as m is odd or even (see equation (4.)), it depends upon the inequality

$$\frac{\left(m + \frac{1}{2}\right) \pm \frac{1}{2}}{2} - f > 0 \dots \dots \dots (6.).$$

 fx being of the form $(x - \alpha) f_1 x$, and this inequality obtaining, the two first terms of the formula (1.) will vanish, and the function will be of the FIRST ORDER.

 fx being of the form $(x - \alpha) f_1 x$, and this inequality not obtaining, the second term only of the formula will vanish, and the function will be of the SECOND ORDER.

 Now let us suppose that fx is not of the form $(x - \alpha) f_1 x$.

 Since fx is a rational function of x , therefore $fx - f\alpha$ is of the form $(x - \alpha) f_1 x$.

Let

$$fx - f\alpha = (x - \alpha) f_1 x;$$

$$\therefore fx = f\alpha + (x - \alpha) f_1 x;$$

$$\therefore \phi x = f\alpha \int \frac{dx}{(x - \alpha)\sqrt{\phi x}} + \int \frac{f_1 x \cdot dx}{\sqrt{\phi x}} \dots \dots \dots (7.),$$

 where the integral $\int \frac{dx}{(x - \alpha)\sqrt{\phi x}}$ belongs to the THIRD ORDER of functions, that in which fx equals unity; and the

 integral $\int \frac{f_1 x dx}{\sqrt{\phi x}}$, being of the class in which fx is of the form $(x - \alpha) f_1 x$, has been shown to be the FIRST or

SECOND ORDER. All ultra elliptic functions are therefore reducible to one of these three orders of functions.

 From equation (7.) it appears that when $f\alpha = 0$ any function of the form $\int \frac{f_1 x dx}{(x - \alpha)\sqrt{\phi x}}$ may be made to de-

 pend upon another of the form $\int \frac{f_1 x \cdot dx}{\sqrt{\phi x}}$.

 Now, any integral included under the last of these forms, in which the dimensions of $f_0 x$ are greater than $m - 1$, m being the dimensions of ϕx , may be made to depend upon another or others in which the dimensions of $f_1 x$ are less than $m - 1$. For let M be a multinomial of m' dimensions;

$$\therefore \frac{d\{M\sqrt{\phi x}\}}{dx} = \frac{\left(\frac{dM}{dx}\right)\phi x + \frac{1}{2}\left(\frac{d\phi x}{dx}\right)}{\sqrt{\phi x}};$$

 of which the numerator is of $m + m' - 1$ dimensions. Now the coefficients of the multinomial M which enter into this expression are $m' + 1$ in number; these may therefore be so taken as to cause $m' + 1$ of its terms to vanish and to give it the form

$$\frac{d\{M\sqrt{\phi x}\}}{dx} \{x^{m+m'-1} - A_{m-2}x_{m-2} - A_{m-3}x_{m-3} - \dots - A_1x - A_0\} \frac{1}{\sqrt{\phi x}}.$$

Let

$$\int \frac{x^m dx}{\sqrt{\phi x}} = \psi_m \text{ and } m + m' - 1 = \lambda;$$

$$\therefore M\sqrt{\phi x} = \psi_\lambda - A_{m-2}\psi_{m-2} - A_{m-3}\psi_{m-3} - \dots - A_1\psi_1 - A_0\psi_0;$$

$$\therefore \psi_\lambda = M\sqrt{\phi x} + A_{m-2}\psi_{m-2} + A_{m-3}\psi_{m-3} + \dots + A_1\psi_1 + A_0\psi_0;$$

 that is, a function ψ_λ , or $\int \frac{x^\lambda dx}{\sqrt{\phi x}}$, in which λ is equal to or greater than $(m - 1)$, the dimensions of ϕx diminished

 Definite
Integrals.

Definite
Integrals.

by unity, may be made to depend upon other similar functions $\psi_{m-2}, \psi_{m-3}, \dots, \psi_1, \psi_0$, in which λ is less than $m-1$. Now any integral of the form $\int \frac{f_1 x}{\sqrt{\phi x}}$, in which $f_1 x$ is a rational function of x , may be supposed to be

Definite
Integrals.

made up of terms of the form $\int \frac{x^\lambda dx}{\sqrt{\phi x}}$. Any such function then, which may be made to depend upon others in which the dimensions of fx are less than those of ϕx diminished by unity.

Now if any function ψx be of the n th class of ultra elliptic functions, $2n+1$ or $2n+2$ are the dimensions of ϕx in that function, according as these are odd or even. Every such function may then be made to depend upon others of the form $\int \frac{f_1 x dx}{\sqrt{\phi x}}$, in which the dimensions of $f_1 x$ are less than $2n$ or $2n+1$, and since $f\alpha=0$, it is a function of the first or second order. The inequality (6.) becomes then, in either case,

$$\frac{2n+2}{2} - f > 0,$$

or

$$n+1 > f.$$

and on this condition it depends whether a function of the n th class, in which $f\alpha=0$, be of the first or second order of functions.

The least number of terms which can compose the sum $\sum \pm \psi x$ appears by equation (4.) to be that for which n_1 and n_2 are each zero, or $\theta_1 x$ and $\theta_2 x$ reduce themselves to constants, and for which δ is zero or unity, according as m is even or odd. This least value is therefore $\frac{m}{2}$ or $\frac{m+1}{2}$. Greater than this μ may be of any value, the

proper dimensions being given to $\theta_1 x$ and $\theta_2 x$. The equation $(\theta_1 x)^2 (\phi_1 x) - (\theta_2 x)^2 (\phi_2 x) = \mathbf{P}(x-x_1 \dots \dots \dots 8.)$

determines the n_1+1 coefficients of $\theta_1 x$, and the n_2+1 coefficients of $\theta_2 x$ in terms of the quantities $x_1, x_2, \dots, x_\mu$ by μ equations obtained by equating the coefficients of like powers of x , there remain then $\mu - n_1 - n_2 - 2$ equations between the μ quantities $x_1, x_2, \dots, x_\mu$ themselves, $n_1 + n_2 + 2$ of which are therefore indeterminate. By equation (4.) it appears that the number of equations of condition may be represented by $\frac{m+\delta}{2} - 2$.

As an application of the theorem of Abel to the theory of ultra elliptic functions, let us take the following example of a function of the second class.

$$\text{Let } \psi x = \int \frac{dx}{\sqrt{1-x^5}}. \text{ Now } 1-x^5 = \left(1+x \frac{\sqrt{5}-1}{2} + x^2\right) \left(1-x \frac{\sqrt{5}-1}{2} + x^2\right) (1-x).$$

$$\text{Assume } \therefore \phi_1 x = \left(1-x \frac{\sqrt{5}-1}{2} + x^2\right) (1-x) = 1-x \frac{\sqrt{5}+1}{2} + x^2 \frac{\sqrt{5}+1}{2} - x^3, \text{ and } \phi_2 x = 1+x \frac{\sqrt{5}+1}{2} + x^2,$$

let

$$\theta_1 x = a+x \quad \theta_2 x = b+cx.$$

In this case $fx=(x-\alpha)$; $\therefore f\alpha=0$; also $m=5$ $f=1$; therefore by the inequality (5.) it appears that

$\mathbf{C} \frac{f_1 x \psi x}{x-\alpha}$ is evanescent. Thus the function is of the first order, and $\sum \pm \phi x = C$.

To determine the conditions necessary to this relation of the functions ψx we have by (8.)

$$(a+x) \left(1-x \frac{\sqrt{5}+1}{2} + x^2 \frac{\sqrt{5}-1}{2} - x^3\right) - (b+cx)^2 \left(1+x \frac{\sqrt{5}+1}{2} + x^2\right) \\ = (x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5) \dots \dots \dots (9.)$$

Equating the coefficients on both sides of this equation, we shall obtain five equations of condition, three of which will determine the constants a and b , and c , in terms of x_1, x_2, x_3, x_4, x_5 , and there will remain two equations of condition between these last five quantities. Three of these five quantities are, therefore, indeterminate; let them be $\frac{1}{2}, 1, -2$;

$$\therefore (a+x)^2 \left(1-x \frac{\sqrt{5}+1}{2} + x^2 \frac{\sqrt{5}-1}{2} - x^3\right) - (b+cx)^2 \left(1+x \frac{\sqrt{5}+1}{2} + x^2\right) = (x-x_1)(x-x_2) \left(x-\frac{1}{2}\right) (x-1)(x+2).$$

It is required from this equation to determine the values of x_1, x_2 in terms of the given values of x_3, x_4, x_5 , whence will be known all the functions which enter into the sum $\sum \psi x$, and also the constants a, b, c ; thus the functions $\theta_1 x$ and $\theta_2 x$ will be determined, and from these the signs with which the different terms enter into that sum. Assuming in the above equation $x=1$, there results the condition $b+c=0$. Hence, substituting for c , and dividing by $1-x$, we obtain the equation

$$(a+x)^2 \left(1-x \frac{\sqrt{5}-1}{2} + x^2\right) - b^2 \left(1+x \frac{\sqrt{5}+1}{2} + x^2\right) = (x-x_1)(x-x_2) \left(x-\frac{1}{2}\right) (x+2).$$

Assuming in this equation $x=\frac{1}{2}$, and $x=-2$,

Definite
Integrals.

$$\frac{1}{2} b^2 (6 + \sqrt{5}) - \left(a + \frac{1}{2}\right)^2 (6 - \sqrt{5}) = 0, \quad 3 b^2 (4 - \sqrt{5}) - (a - 2)^2 (4 + \sqrt{5}) = 0.$$

Definite
Integrals.

From these equations we obtain $a = 20.45345 \dots$, and $b = 20.03253 \dots$. Moreover, equating the coefficients of x^3 and x^0 , there result the equations $b^2 + \frac{\sqrt{5}-1}{2} - 2a = x_1 + x_2 + \frac{1}{2} - 2$ and $b^2 - a^2 = x_1 x_2$; whence we have $x_1 + x_2 = -440.08338$, &c., $x_1 x_2 = -17.04939$, &c.; $\therefore x_1 = 0.03873 \dots$, $x_2 = -440.12211 \dots$.

All the values of x entering into the sum $\sum \pm \psi x$ are now known. It remains only to determine the signs of the several terms of this sum. These are determined (see p. 499) by the sign of the fraction $\frac{\theta_1 x}{\theta_2 x}$, or in this case by the sign of $\frac{a+x}{b(1-x)}$ for each particular value of x ; or substituting the values of a and b by the sign of $\frac{20.45345, \&c. + x}{(20.03253, \&c.) (1-x)}$.

Giving to x the three supposed values of $x_1, x_2, x_3, \frac{1}{2}, 1, -2$, this expression is *positive*. It is also positive for the value x_1 of x , but when x_2 is substituted it becomes negative. All the terms of the sum, therefore, are positive, except that involving this value of x ;

$$\therefore \psi(0.03873, \&c.) - \psi(-440.12211 \dots) + \psi \frac{1}{2} + \psi 1 + \psi - 2 = C.$$

Thus the value of the constant C is determined, which satisfies the equation $\psi x_1 - \psi x_2 + \psi x_3 + \psi x_4 + \psi x_5 = C$, for all values of x_1, x_2, x_3, x_4, x_5 , subject to two equations of condition, which equations of condition are determined by equating the coefficients of like powers of x in the equation (9.).

The value of C may be verified by making any other assumption with respect to the indeterminate quantities x_2, x_3, x_4 than that made above, provided only that this assumption give the same signs to the corresponding integrals. If we take these three quantities each equal to zero, the two following values will be obtained for C ; these are *numerically* the same value with the preceding:

$$\begin{aligned} & -\psi(-2.18399 \dots) - \psi(0.56596 \dots) + \psi 0 + \psi 0 + \psi 0, \\ \text{and} \quad & -\psi(-4.52014 \dots) + \psi(-0.71592 \dots) + \psi 0 + \psi 0 + \psi 0. \end{aligned}$$

It is a remarkable fact, observed by Legendre, who has made these calculations,* that this value of C is represented by the transcendent $-\frac{1}{2} \psi - \frac{1}{0}$, and from the laborious solution of a great number of similar examples he has arrived at the conclusion that, however the conditions of the case may be varied in respect to the sum of five functions of the form $\int \frac{dx}{\sqrt{1-x^5}}$, the constant C will be represented numerically by the transcendent $\left(1 \pm \frac{1}{2}\right) \psi\left(-\frac{1}{0}\right)$, or by a transcendent of two terms of which this quantity is one, and $\psi 1$ the other. He has carried the verification of this remarkable fact to ten places of decimals. The following are some of many examples:

$$\begin{aligned} & \psi(0.36491 \dots) - \psi(-3.83087 \dots) - \psi(-2) - \psi(-1) - \psi\left(\frac{1}{2}\right) = \psi(1) - \frac{3}{2} \psi\left(-\frac{1}{0}\right) \\ & -\psi(0.36491 \dots) - \psi(-3.83087 \dots) - \psi(-2) - \psi(-1) + \psi\left(\frac{1}{2}\right) = \psi(1) - \frac{3}{2} \psi\left(-\frac{1}{0}\right) \\ & -\psi(0.13189 \dots) + \psi(-0.87733 \dots) - \psi(-1) - \psi(-2) - \psi\left(\frac{1}{2}\right) = -\frac{1}{2} \psi\left(-\frac{1}{0}\right) \\ & -\psi(-0.20080 \dots) - \psi(-6.88759 \dots) + \psi(-1) + \psi 0 + \psi 0 = -\frac{1}{2} \psi\left(-\frac{1}{0}\right). \end{aligned}$$

Now $\psi 1 = -\cos \frac{\pi}{5} \psi\left(-\frac{1}{0}\right)$. All the values of C may, therefore, be embraced under the formula

$$-\left(1 \pm \frac{1}{2}\right) \cos \frac{\pi}{5} \cdot \psi\left(-\frac{1}{0}\right).$$

NOTE I.

Note on equation (e), page 502.

In this equation it is assumed that, for the three arbitrary positive values of x the positive sign only is to be taken in the sum $\sum \pm \psi x$, and therefore that for these three values, the signs of $\frac{\theta_1 x}{\theta_2 x}$ or $\frac{a_1 + b_1 x}{a_2 + b_2 x}$ are the same. It is

* *Calcul des Fonctions Elliptiques*, third supplement.

Definite
Integrals.

the object of the following note to discuss the circumstances under which this assumption may be made.

Subtracting the two first and the fifth of the equations (page 502) which determined the coefficients a_1, b_1, a_2, b_2 , we have

$$2a_2(a_2+b_2)k'^2 = \pm P_4 \cos \phi - P_4(1 - \sin \phi) \text{ and } 2a_2(a_2-b_2)k'^2 = \pm P_4 \cos \phi - P_4(1 + \sin \phi);$$

$$\therefore \frac{a_2+b_2}{a_2-b_2} = \frac{\pm P_4 \cos \phi - P_4(1 - \sin \phi)}{\pm P_4 \cos \phi - P_4(1 + \sin \phi)}; \quad \therefore \frac{a_2}{b_2} = \frac{\pm 2P_4 \cos \phi - P_4(1 + \sin \phi) - P_4(1 - \sin \phi)}{P_4(1 + \sin \phi) - P_4(1 - \sin \phi)}.$$

Proceeding similarly in respect to the third and sixth, and fourth and sixth equations,

$$\frac{b_1}{a_1} = k \cdot \frac{\pm 2P_4 \Delta \phi - P_4(1 + k \sin \phi) - P_4(1 - k \sin \phi)}{P_4(1 + k \sin \phi) - P_4(1 - k \sin \phi)}.$$

The sign of $\frac{\theta_1 x}{\theta_2 x}$ changes then for different values of x , as does the sign of

$$1+k \cdot \frac{\pm 2P_4 \Delta \phi - P_4(1 + k \sin \phi) - P_4(1 - k \sin \phi)}{P_4(1 + k \sin \phi) - P_4(1 - k \sin \phi)} \cdot x$$

$$1+k \cdot \frac{P_4(1 + \sin \phi) - P_4(1 - \sin \phi)}{\pm 2P_4 \cos \phi - P_4(1 + \sin \phi) - P_4(1 - \sin \phi)},$$

or as $\frac{(1-kx)P_4(1+k \sin \phi) - (1+kx)P_4(1-k \sin \phi) \pm 2kxP_4 \Delta \phi}{-(1-x)P_4(1+\sin \phi) - (1+x)P_4(1-\sin \phi) \pm 2P_4 \cos \phi}$. Hence giving to x any one of its values, $\sin \phi_1$,

as $\frac{\Delta^2 \phi_1 P_3(1+k \sin \phi) - \Delta^2 \phi_1 P_3(1-k \sin \phi) \pm 2k \sin \phi_1 P_4 \Delta \phi}{-\cos^2 \phi_1 P_3(1+\sin \phi) - \cos^2 \phi_1 P_3(1-\sin \phi) \pm 2P_4 \cos \phi},$

or $\left(\frac{k \Delta \phi_1}{\cos \phi_1} \right) \frac{\Delta \phi_1 \{ \Sigma_3 \sin \phi + k^2 P_3 \sin \phi \} \pm P_3 \Delta \phi}{-\cos \phi_1 \{ 1 + k^2 \Sigma_3 P_2 \sin \phi \} \pm P_3 \cos \phi}.$

If in this expression ϕ_1 be replaced in succession by ϕ_2, ϕ_3, ϕ_4 , the symbols P_3, Σ_3 , applying in each case to the three remaining terms, its sign will determine that of the corresponding term in the sum $\Sigma \pm Fk\phi$. Now it is shown, page 502, that the positive sign is to be taken in the last term $\pm P_3 \Delta \phi$ in the numerator of the expression. If, therefore, $\phi_1, \phi_2, \phi_3, \phi_4$ be positive, the numerator is essentially positive, whichever of these quantities be taken; and if in the term $\pm k^2 P_4 \cos \phi$ of the equation of condition (d), page 502, the negative sign be taken, the denominator is essentially negative for all values of ϕ . This negative sign being then taken in equation (d), and the values of ϕ being assumed to be positive, there are no changes of sign in the sum $\Sigma \pm \psi x$.

NOTE 2.

Note on Equation (6.), Section VI.

Assume $x = \varepsilon^{-z}$; $\therefore \Gamma p = \int_0^1 \left\{ \log. \frac{1}{x} \right\}^{p-1} dx = \int_0^\infty \varepsilon^{-z} z^{p-1} dz$;

$$\therefore \Gamma p \cdot \Gamma q = \int_0^\infty \varepsilon^{-z} z^{p-1} dz \cdot \int_0^\infty \varepsilon^{-u} u^{q-1} du = \int_0^\infty \int_0^\infty \varepsilon^{-z-u} z^{p-1} u^{q-1} dz du.$$

Now if the variables u and z of any double integral $\int \int V du dz$ be changed into x and y , it becomes

$$\int \int V \left\{ \left(\frac{du}{dx} \right) \left(\frac{dz}{dy} \right) - \left(\frac{du}{dy} \right) \left(\frac{dz}{dx} \right) \right\} dx dy.$$

Assume then $u+z=y, u=xy$; $\therefore z=(1-x)y$, and $\left(\frac{du}{dx} \right) \left(\frac{dz}{dy} \right) - \left(\frac{du}{dy} \right) \left(\frac{dz}{dx} \right) = y$; $\therefore \int \int V du dz = \int \int Vy dx dy$.

Making these substitutions in the preceding equation, and observing that the values 0 and ∞ of u correspond to the values 0 and ∞ of y , and the values 0 and ∞ of z to the value 1 and 0 of x , we have

$$\Gamma p \cdot \Gamma q = \int_0^\infty \int_0^1 \varepsilon^{-y} (xy)^{p-1} (1-x)^{q-1} y^p dx dy = \int_0^\infty \int_0^1 \varepsilon^{-y} y^{p+q-1} x^{q-1} (1-x)^{p-1} dx dy$$

$$= \int_0^\infty \varepsilon^{-y} y^{p+q-1} dy \cdot \int_0^1 x^{q-1} (1-x)^{p-1} dx = \Gamma(p+q) \cdot [q, p]; \quad \therefore [q, p] = \frac{\Gamma p \cdot \Gamma q}{\Gamma(p+q)}.$$

This demonstration applies to every case in which p and q are positive quantities, or imaginary quantities of which the real part is positive. It is nearly that given by Jacobi, in *Crelle's Journal*, vol. ii. page 307, which moreover differs but little from the method of Poisson, *Journal de l'Ecole Polytechnique*, vol. xii. p. 477. The method of reduction by means of double integrals appears first to have been used by Laplace.

Definite
Integrals.

MORAL AND METAPHYSICAL PHILOSOPHY.

Moral and
Metaphy-
sical Phi-
losophy.

Introduction.

EVERY man is surrounded by a world with which he converses by means of his senses. Every man is connected with other men by necessary or by voluntary relationships. Every man exercises a power over things which in bulk and ordinary force are far superior to him, and is influenced by things which are far inferior to him.

A man may ask, What is the order and constitution of the world which I see lying before me? Such an inquiry gives rise to *Physical Philosophy*.

A man may ask, What is the meaning of these bonds by which I am connected with my fellow-men? Some of them it is obvious cannot be broken. I cannot cease by any act of mine to be a brother or a son; but I can act as if I were not in either of these relations. Under what conditions must I live to be in conformity with this position in which I find myself? How may it become a practical contradiction? Such inquiries give rise to *Moral Philosophy*.

A man may ask, What connection subsists between me and the world around me? Am I subject to the same laws which govern it, or to different laws? Are the powers which I exercise akin to those which these things exercise, or have I some peculiar power of my own? If so, in what right and under what conditions do I possess this power? Such questions give rise to *Metaphysical Philosophy*.

Moral Philosophy then commences with an observation of our actual position as men; Metaphysical Philosophy with a comparison of our position with that of the things around us. Both are distinguished from Physical Philosophy, in that both refer primarily to man. But the moral philosopher assumes a position distinct from that of the physical student; the metaphysical philosopher is always seeking to establish such a position. The habits of feeling, and the external circumstances which lead men to reflect on morals, are widely different from those which lead them to become metaphysicians. A desire to find the foundation of our practical life is the great impulse to the moralist. A desire to find the foundation and the limit of our faculties is the great impulse to the metaphysician. All the primary elements of Moral Philosophy are found embodied in laws, manners, political institutions; these give birth in later times to speculations, schools, and systems. The metaphysician endeavours at first to interpret the difficulties which he finds in himself by speculations about the external world, and feels by degrees the necessity of studying the order of human society. But these lines of thought are continually intersecting each other; it is impossible to follow the track of any great moral question without entering into the region of pure Metaphysics. It is impossible even to understand the meaning of the problems in Metaphysics until we have some acquaintance with the grounds of Morality.

We wish to assist our readers in tracing the progress of human thought upon these momentous subjects.

The history of Moral and Metaphysical Philosophy is profoundly interesting; nor will one who engages earnestly in the study find himself at any loss for documents. What he chiefly wants, we conceive, at any rate what it seems most within the power of an Encyclopædia to supply, are some hints as to the right method of using his materials. He may be glad of an outline map, which though very rudely sketched, and only indicating here and there a mountain and a river, may yet give him such a general impression of the country as he could scarcely have obtained from a more minute survey, and may enable him with greater interest to examine particular localities in the company of more learned and critical travellers.

Moral and
Metaphy-
sical Phi-
losophy.

HEBREW PERIOD.

FROM B. C. 1920 TO B. C. 600.

Preliminary Observations.

The early records of the world are contained in the Hebrew Scriptures. These Scriptures appear to trace with much order and clearness the progress of human thought during the period which they embrace. The first conclusion of every one would be, that they were indispensable documents in any sketch of moral and metaphysical inquiries. But his second thought might probably be, that the character which they assume as records of a Divine revelation makes them unsuitable for this purpose. This opinion he will find confirmed by two sets of persons, who take very opposite views of these records. The first affirm that the subjects of which they treat are different from those of the ordinary historian and philosopher; the second affirm that they are merely a part of ordinary history, and that the apparent peculiarities in their style of thought and composition are either not really characteristic of them, or may be accounted for by mere accidents of age or locality. Either of these hypotheses would make these records useless to our work. The first, obviously, by showing that they have no connection with our subject; the second, as certainly, by depriving them of all authority, and showing that the pretensions upon which they are based are spurious. To the common historian they might be serviceable, even upon this last supposition, as stating actual facts, though referring them to a false principle. The moral historian they would only mislead, for every step in the progress of human thought must be erroneously marked, and wrongly connected with that which preceded and followed it. But possibly some reflection may lead us back to our first conclusion. We may notice that the first view is open to this fatal objection,—that it contradicts the writings which it proposes to magnify. Those writings take the utmost pains to convince us, that the persons whose actions they record were like all other men; were engaged in the actual business of life, were sons, fathers, brothers, shepherds, warriors, statesmen. Their feelings,

Hebrew
period.
Moral ap-
prehensions
and inqui-
ries, how
connected
with a re-
velation.

Moral and
Metaphysical
Philosophy.

thoughts, deeds, are essentially human; the whole history must be rewritten in order to deprive them of that character.

But there is an objection equally destructive of the second hypothesis. It is at war with *facts*. An obscure Syrian tribe, as Voltaire called it, has exercised an amazing influence over the destinies of mankind; over all the feelings, thoughts, and deeds of men; even to the most remote corners of the universe. This is a fact which must be accounted for; you do not surely account for it by saying that there was nothing peculiar in the condition of this tribe. By all the contrivances which you make use of to diminish the wonder of the narrative, you increase the wonder and difficulty of this problem.

This argument would suggest itself to the ordinary reader. Another suggests itself still more strongly to the moral and metaphysical student. The attempt to reduce the Hebrew records to the law of ordinary histories, presumes that those histories are clear and intelligible. This *he* knows not to be the case. A thousand puzzling and perplexing phenomena occur to him at every step of his progress; he feels that he wants some help to the course of thought indicated by the mythology, the poetry, the formal speculations of the people of Greece or Rome. It is, he is persuaded, because the moral and metaphysical peculiarities of these nations have been less studied than the mere events which befell them, that their history has ever been supposed to be the solution of riddles, and not the riddle itself. This thought suggests to him a method of reconciling those two hypotheses, to either of which separately he had found it impossible to assent. If these records should themselves be the key to the solution of the puzzles he meets with elsewhere, it *must* be true, as the first affirm, that these records are in all respects very remarkable and peculiar; it *must* be true, as the last affirm, that they have the closest connection with all other records relating to the life and affairs of men. With this suggestion to guide him, he is led to reconsider the opinion, that the history of a revelation cannot be the history of human thoughts, feelings, and exercises. He perceives that it is true to this extent; philosophy must be the search after wisdom; it must be therefore the formal opposite of wisdom *communicated*, that is to say, of a revelation. But he knows also, that in all common education we communicate information, in order to excite the desires and faculties which receive information. He does not decide the question whether this is the only method of learning; it certainly is the most common. If so, a revelation, though it may not be philosophy, may be the first source of it; and the history of a revelation may contain, not only a series of communications, but also an accurate record of the awakening of thoughts, feelings, and faculties, by those communications. There is nothing impossible or contradictory in this notion; if a revelation be possible, this would seem the most reasonable form of it. On turning to the Jewish records, he finds that it is the actual form of revelation therein exhibited; he finds these records purporting to be the history of the education of a nation by means of a revelation.

We take them to be what they pretend to be. In doing so, we adopt the common faith of our country; we reconcile two opinions, either of which is itself at variance with facts; and we find, as we think we shall show hereafter, the clearest exposition of the history of philosophy in those ages and countries of which

this revelation does not treat. With these arguments to justify it, the assumption we think is not unwarrantable. Every historian must take something for granted, if he adopt a method at all. *He* seems to take least for granted who does not frame an hypothesis, but adopts one; who does not set it up against other hypotheses, but explains them by means of it; and who subjects it to the surest and severest of all tests, by boasting that it will enable him to discover an order in that which is otherwise unconnected, and remove difficulties otherwise inexplicable.

Moral and
Metaphysical
Philosophy.

With the first part of the Mosaic records we have no immediate concern. They are indeed profoundly interesting to the student of philosophy; the questions of which they treat recur at every step of his progress. But it would be an abuse of language to say that these documents, however closely connected with human inquiries, do themselves describe the progress of human inquiry. We reserve the consideration of them, therefore, for a future stage of our history, when we shall have occasion to compare the views of the origin and order of the world; of the early state of man; of his position in reference to nature and to God, which are developed in them, with those which prevailed in other countries and ages. The necessity of such a preface to the history of the chosen nation will be best understood when we understand the object for which they were chosen; nor is it a needless, though a very obvious remark, that these records presume the facts of the subsequent history, and were written for the benefit of a people who believed themselves to be separated from the rest of the world, and adopted into covenant with the God of Abraham, Isaac, and Jacob.

One observation, however, must be deduced from the history of the antediluvian world, in order that the narrative which follows it may be intelligible.

The historian represents mankind to have been divided into two classes: one following nature and inclination, rushing prematurely into political society, inventing mechanical arts, neglecting family relationships; the other abiding in those relationships, and in faith in an unseen Protector and Lord. The confusion of these two races, and the general lawlessness and violence which followed it, is marked as the immediate occasion of the Deluge. The race is preserved in a single family. After the Deluge we observe another step in human progress. The covenant entered into with Noah, as representative of mankind, marks the commencement of a national period; the terms of the covenant exhibit men in their relations to God, to nature, and to each other. The animal world is given up to their use and government, subject to a restriction upon the eating of blood. Men are declared to be mutually responsible for each other's lives, and the feeling of confidence in an unseen Being, who has established these relationships, is called forth and upheld by the promise that He will not henceforth let loose the waters for the destruction of mankind. The historian interrupts the course of his narrative to relate an event illustrating the connection between the new form of society which this covenant established, and the family relationships, for which the former dispensation had been the witness. One of the sons of Noah exposing the drunkenness

Early
history.

Moral and
Metaphy-
sical Phi-
losophy.

of his father, a curse is pronounced upon his descendants, which the Jews, who knew the character of the people on whom it had fallen, were well able to understand as a solemn declaration that every race among whom the parental relation is not respected, becomes inevitably "a servant of servants."

Under this promise, and with this warning, the descendants of Noah go forth to colonize the earth, after their families, after their tongues, in their lands, after their nations; words clearly indicating that the establishment of national societies, each having a distinct language, and preserving their family relationships, was the purpose contemplated in this dispensation, and was not, as has been strangely imagined, the effect of that rebellion against the Divine order, of which the first recorded manifestation took place on the plain of Shinar. That event is so simple in itself, and so simply told, so exactly suitable to the age of the world in which the scene is laid, that no wise man could ever have fancied it to be a mere allegory, if other men had not as perversely overlooked the principles which the fact involves in controversies about the fact itself. That the ground of all human society is laid in faith on an unseen Being, that by this faith man is raised above his own animal nature and the animal natures which surround him, that by this he is preserved from the fear of the grand objects and operations of nature, which are continually passing before his eyes in the heaven above and the earth beneath;—this is the principle of that covenant of which we have been speaking, the principle of the whole Bible, the principle on which we shall find all history, and the history of man's inquiries respecting himself especially, illustrating and establishing.

The transaction of the tower of Babel is the practical rejection of this principle. A portion of the descendants of Noah, in their journeys through Mesopotamia, lose their confidence in the promise which was the bond of their connection with an unseen Lord, fall into a slavish dread of the powers of nature, and attempt to provide against the evils with which these powers threaten them by brick walls. The result is a dissolution of society, indicated by the loss of that which is the great bond of human fellowship—communion of language.

This event has always been connected with the establishment of that Babel or Babylonian monarchy which is spoken of throughout the Scriptures as one, under all its changes of dynasty, because it preserved the same identical features, because it was originally established and continued to rest upon that principle against which the Jewish nation was the abiding witness. The disbelief of a fatherly government; the acknowledgment of power and not righteousness, as the idea of Godhead; the consequent worship of the powers of nature; the reverence for mere power in man;—these are the characteristics of that system, of which the building of the tower of Babel was at once the commencing act, and the permanent symbol. Under such a system of tyranny and superstition, language in its proper sense does not exist; words are merely the utterances of animal necessities, no bonds of feeling or society; nor, without that Divine interference which is recorded in the next passage of the book of Genesis, have we reason to suppose that man would long have had any instrument for expressing even those wants which are common to him with the brutes; far less that he would ever have found any method of communicating his deeper

wants, or that he would ever have been conscious of them. This interference is the commencement of Hebrew history, and, in one sense, of the moral and metaphysical history of mankind.

Moral and
Metaphy-
sical Phi-
losophy.

Family Period of Hebrew History, from B. c. 1921 to B. c. 1706.

A descendant of Shem, born in that country in which the foundations of the Babel monarchy had been laid, is awakened, the historian tells us, to a consciousness of his relation to an invisible Lord. The same Being who makes him aware of this mysterious connection inspires him also with confidence in his promise and protection.

Grounds of
relative
morality,
how first
apprehend-
ed by the
Jews.

The feeling of his own relationship to this Being is accompanied by a sense of the subjection of nature to Him. He has power to bestow the soil of the earth on whom he will,—it pleases Him to bestow a portion of it upon Abram. But the feeling of being a favoured man, though strong, is not the strongest in the mind of this lonely shepherd; far deeper is the feeling that he is a man,—in him all the families of the earth are to be blessed. This feeling is not one of a vague benevolence; it is connected with the thought of a distant posterity, over which he feels that the same Being who cares for him will preside; in whose fate he feels that he is and shall be interested. These truths, thus simply revealed to his heart, become the foundation of his life, and the key to the system of practical education to which he is submitted. Every outward event of his life, however trifling, is used by his Divine instructor to bring him into the consciousness of some principle. He goes down into Egypt with his wife, tells a lie for her protection, and finds that it has nearly proved her ruin. A separation takes place between him and one of his kindred, who is intended to be the founder of another race; it is made the means of kindling in him a fresh assurance of the distinct and glorious office which is destined for his seed. Again, he is brought into a wider sphere; a predatory war breaks out among some of the Canaanitish nations, in which his nephew is taken captive; he acquires a sense of the power which man possesses who is not a slave of outward accident and appearances, but is depending on an unseen Protector; an assurance that strength does not lie in multitudes; a new apprehension of the importance of his own position among the selfish and slavish people of Canaan. Yet he is not allowed to form an exaggerated notion of his own dignity. He is not alone in his worship of God; nor is the dignity, which he feels belongs to him as the destined head of a nation, the highest that God bestows on man. As he returns from the fight, he is met by a mysterious personage, who blesses him in the name of the Lord of heaven and earth; assumes to be his priest, and in that character receives gifts from the father of the faithful.

Hitherto this man has been brought into a lively sense of his connection with an unseen Lord, and of the favourable regard of that Being. This was the first part of his discipline. It was the ground of his practical relations to his fellow-men. It enabled him to be an honest shepherd, an affectionate kinsman, a good neighbour, a brave warrior, a humble and reverend man. But now began a second stage of his education, in which he was to be brought by the same practical method into the perception of deeper truths. A numerous seed had been promised him, yet he has no

Moral and
Metaphy-
sical Phi-
losophy.

son, not even the hope, or the apparent possibility of one; yet the Divine promises have in other respects been fulfilled to him; he is the possessor of numerous flocks and herds. The question, who shall be the heir of these, arises within him; he is told that a son of his own shall inherit them; he believes, and his faith is counted unto him for righteousness. Then, first the idea of a righteous Being, of One who must be true, however appearances and possibilities seem to be at variance with His word, is fully and practically presented to the mind and heart of Abram. He believes in this Being, and rises into a participation of his character,—feels that he is righteous, because the Being in whom he trusts is righteous. On this truth the moral life of the patriarch stands; on this truth the nation of which he is the destined founder is to be built. A vision of the future condition of his descendants, of their bondage, and of their deliverance, is the reward of his faith; and the mysterious sacrifice which accompanies it prepares him for the future discovery of another important element of national life. From this time the perception of a law or order which man cannot transgress with impunity, is more and more brought out in the mind of Abram, while the former acknowledgment of a merciful Protector is wonderfully interwoven with this new faith. The quarrels of his wife with the concubine she had given him, lead him into a clearer understanding of the nature of family relationships; yet he sees that the innocent woman is not forgotten by his Lord, and that her posterity, though not the heirs of the blessing promised to him, are still cared for. And now the habit of moral distinction, which had been beginning to be cultivated in him, was further strengthened by an institution which connected together his family life with his relation to God, and both with the prospect of a race to come out of his loins. The institution of circumcision is witness of the Divine covenant with Abraham and his race; the everlasting sign of a distinction between man and that which belongs to man; between man living according to that which is below himself, or man looking to that which is above himself; between man in union with God, and man separated from Him. The mere sign was permitted, nay intended to be diffused over many nations, partly as a witness that the *fact* was true of all, though the knowledge of the fact, and its practical consequences, belonged only to one, partly as a proof that it was the sign of a covenant and not a charm, and that, except in connection with the idea of that covenant, it was unmeaning.

As soon as the patriarch had been made conscious of his own spiritual position as a man, a remarkable vision to him in the plain of Mamre leads him into another and higher step in his education. He learns that the Divine Being is not merely his protector who gives him flocks and herds, and has promised him greater things than these, but that he has in him the capacity for fellowship with this Being: he learns at the same time that only by an act of condescension on the part of God can he be raised to enjoy this fellowship. If we consider the total effect of that vision on the mind of the simple shepherd of Mesopotamia, we shall see now, by the simplest method, the three feelings of the reality of God; of his connection and sympathy with man, and of an awful mystery which made it impossible to apply the rules and standards of number and time to Him, were gradually brought out, and how at the moment,

when the trust, and gratitude, and awe, the sense of fellowship, and the sense of distance, had reached their highest meeting-point, the announcement of the intended judgment upon Sodom concentrated them all in adoration of a righteous Being; yet how that very adoration called forth all the human sympathies of Abraham, and gave birth to the most confident, and humble, and awful intercession. The grand idea of retribution was one which the mind of the patriarch was now able to receive. All the meaning of righteous punishment without caprice, punishment in order to support order, punishment for the preservation of the universe—certain, discriminating, merciful, now dawned upon his mind, and became a part of its substance.

This is properly the moral stage of the education of Abraham. The third and concluding one is indicated by the long-expected birth of one who should be heir of the promises.

Both the former parts of his discipline, that which had instructed him in his relation to a Divine Being, and that which had brought him into some perception of His character, were necessary before he could fully understand his own position—could feel that, as a man, he was destined to exhibit the image of those relationships and of that character in his own life. He could now estimate the dignity of the parental office; could feel that a birth is the most solemn of all events; could understand how confidence in a Being above ourselves is essential to the right performing of every social duty; he could understand also how all the powers and energies of life must be referred to the same Being, and lastly, that these principles were the foundations of the state of which he was to be the head.

As soon as Isaac is born, the son of the concubine is cast forth, to be hereafter the head of a race, retaining much of the patriarchal character, and yet formally contrasted with the Jewish commonwealth. The members of it feel themselves brothers, fathers, husbands, and so far they carry about the blessings of circumcision with them. They are not merely given up to brute nature, but that grand feeling of being *persons*, of being distinct from their *nature*, of being united to a Being above themselves, has not dawned upon them, and without this men must continue in hordes, and can never enter into organized society.

This is the first offshoot from the Abrahamic stock, and is distinguished by many important characteristics from any of those tribes which existed previously to the call of the patriarch.*

As the Hebrew family becomes more and more distinct from these branches of it, the moral idea upon which it rests becomes more and more evident, and Abraham himself is initiated into higher and deeper principles. The last and grandest point of his education is that respecting the meaning and nature of sacrifice. We do not mean that he had not yet performed sacrifices, or that the act had not (as in the case of the solemn transaction, when he was informed of the destinies of his descendants) been invested with a great moral significance; what we mean is, that the *principle* of sacrifice had not yet been expounded to him by some

* It must not be forgotten, however, that the Moabites and Ammonites, the descendants of Lot, are never confounded with the other nations, but are described throughout Scripture as having some moral features in common with those of the Jews, and on that very account as more capable of being contrasted with them.

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

great practical lesson; it was indeed evidently reserved as the climax of his education; that without which it would not have been complete, and which in its turn must be interpreted by the previous steps of his discipline. The Being who had inspired him with such confidence in Him as his personal protector, and as the faithful and righteous Being, calls upon him to take the child of promise and to sacrifice him upon a hill many days' journey from his tent. To decide whether this suggestion was from a good or an evil source may have been difficult, or even impossible to the mind of the patriarch, in spite of all his previous training, but of this he felt certain, that absolute submission was due to that Being whom he served. And the absolute confidence which he reposed in Him, made him feel that he would not be suffered to obey any wrong impulse, and that the reason why the thought had been presented to him would, in due time, make itself manifest. He goes, therefore, as directed, simply and ignorantly confiding in the wisdom and goodness of the Being from whom he had received this child, to whom he had committed himself, and who, he believed, had mighty destinies in store for them both. The child-like submission receives its reward, his son is spared, and a new light is shed into his mind respecting his own character and relations, and respecting the God whom he worships.

The mighty mystery of a sacrifice of the will, as that which the perfect and Divine will requires of us; of the fruit of that sacrifice being the highest of all blessings, perfect sympathy between the creature and his Lord; lastly, the dim apprehension which after ages were to interpret, of the truth, that He who requires the sacrifice himself sets the example of it; that in this, as in all things, man is only his image:—these were the blessings with which Abraham was blest, and which he was permitted to transmit to his posterity, to be incorporated into their institutions, to become a part of their daily life, to preserve them from the abomination of human sacrifices, and to open their minds gradually to the apprehension of the great voluntary sacrifice.

Here the moral education of Abraham may be said to be completed, though the two remaining transactions recorded of him exhibit him as a witness for some great principles of moral order,—reverence for the dead, fidelity in transactions with men of a different race and faith, and the sanctity of marriage. We think it must be allowed that the simplest and most literal view of this history gives it an immense importance in the eyes of the moral historian; if we reflect ever so carelessly upon the ideas which it has developed, we must see that they have been those upon which the life of man and the life of society has turned. If we reflect ever so little upon the method by which those ideas became a part of the patriarch's mind, we shall see how wonderfully it connects together the ordinary blessings and transactions of life with his highest speculations; if we reflect ever so little on the moral effect of these discoveries, we must see that they tend to form a real manly character, and to make each man a true servant of his fellow-man. What is most remarkable in the history is, the absence of all glaring incidents, all oriental extravagance; the facts of the patriarch's life are ordinary facts, with nothing heroic or portentous about them, all their value lies in their significance; the events are only striking from the principles which they involve.

The history of the remaining patriarchs is interesting

and intelligible upon precisely the same grounds as that of Abraham. They are another chapter in the family records of the world; another step in man's education, by means of ordinary relations and every-day facts, and their own follies and dangers; into a perception of their relation to an invisible and permanent Being; into a knowledge of the social order in which He has placed them, and into contentment with this will and submission to that order. Throughout, the principle of faith or confidence in an unseen Being, grounded henceforth upon His express covenant with the Abrahamic family, is marked out as the basis of their family life. Those who possessed this had the grand element of humanity, that which raises man above sensual and brutish instincts; those who wanted it, became, in the main, the slaves of those instincts; though noble and generous feelings, the indications that they had a right to a better state than any to which they aspired, dawned through their ignorance and degradation. On the other hand, a disposition to craft and dishonesty beset those who felt that they had a higher wisdom and perceptions unshared by the men around them. The first are left to accomplish their wishes and work out the principle of self-will by becoming powerful men and founders of societies. The latter are trained by a severe and humbling discipline into a feeling of the reasonableness of their faith, and its inconsistency with all trick and deception. As the circle of relationships becomes larger, other truths are brought out in the same practical method; the mischiefs of polygamy, the conditions under which primogeniture is sacred, and on which it may for a higher end be superseded; the distinction between those who, retaining the sign of the covenant and remaining heirs of its privileges, mistook their position and sought for a selfish dignity; and those who understood that their great glory was to be heirs of a blessing intended for mankind, and to be witnesses for the God of Abraham.

It must often have excited the surprise, and perhaps the contempt, of the ordinary annalist, that the story of Joseph and his brethren, the favourite of peasants and children, should occupy a larger space in the records of the world's early history than all the politics of the Pharaohs and of Egypt. The historian of philosophy, on the contrary, feels that this family narrative develops principles which are really more important to mankind, and especially more important in records of the family age of the world, than the most minute chronological narrative of the dynasties and revolutions of Egypt could have been. But at the same time he will be far from admitting that the few facts contained in the book of Genesis respecting the condition of Egypt, and the influence exercised over it by a young Hebrew slave, are not in a political point of view of the highest worth. They reveal some most interesting truths respecting the progress of society, and the influence of moral wisdom and insight upon the institutions and destinies of a nation.

In the history of Joseph we see facts that are indeed most wonderful, but yet none which, in the ordinary sense of the word, are portentous. They are most strictly supernatural events, events indicating a supernatural Author, but they are events which rather exhibit the moral system and order of the world, than any departure from it. The feud with which the adventures of the youth begin, his purchase by the Ishmaelitic merchants, (the carriers of spices from Arabia to Egypt,) his rise in the house of Potiphar, the circumstances of

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

his disgrace and imprisonment, if more romantic than the facts in the lives of the simple shepherds of Palestine, are in nowise strange or incredible events. It is still the principle contained in them, the exhibition of inward and spiritual wisdom, triumphing over accidents, with the constant witness borne by them to those eternal truths which the education of Abraham had developed, that has made them so precious to all countries and ages. The facts which follow, understood simply and literally, have the same, and a still higher value. Nothing would be, *à priori*, more likely, and yet there is no fact which we are more glad to know from positive testimony, than that dreams were one of the means by which the minds of men, not possessed of an actual revelation, were awakened to the consciousness of spiritual faculties and to the existence of a spiritual world; they have that effect upon the peasant of our own day. However much they may have been used as instruments of superstition, men would be far more superstitious, because more sensual, without them. Again, the dignified position which the young heir of the covenant assumes, his calm assertion of a knowledge which those around him did not possess, yet the utter absence of all pretension in that assertion; his feeling that he is a witness for the God of Abraham, and that this God is the God of the whole earth, and watches over the welfare of all people, and reveals himself to those who are not within the scope of His covenant; in one word, the sense of a high vocation, and of that vocation being for the good of mankind, is the most striking and beautiful testimony that can be imagined to the effects of this Hebrew revelation in cultivating and expanding the heart and faculties of man.

Not the least memorable passage in the history is that which exhibits the Hebrew youth as educating the King of Egypt to habits of reflection and foresight, teaching him by a simple method to understand that there is a Divine order and system in the world, and that the providence of God is the ground of providence in man. This elementary lesson in political wisdom was a more effectual method of communicating moral and theological wisdom to the heart of Pharaoh than any direct indoctrination; and it explains how that method which we have observed in the Divine teaching of the chosen family became the model for all those instructions which they imparted to mankind, and which were pledges of the fulfilment of the promise, that in them all the families of the earth should be blessed.

Interval between the First and Second Period of the Hebrew History.

Egypt.
Influence of
these first
moral ap-
prehensions
on political
institu-
tions.

In that version of the story of Prometheus, which Æschylus has made immortal: Io, the human animal who had been admitted to a fatal commerce with the Divinity, after many a weary journey through Scythian deserts, persecuted by the gadfly, and unable to escape from the pipe of the shepherd Argas, is told by the great patron and benefactor of mankind that in Egypt she shall bring forth a son to Jupiter. But he declares that twelve more generations shall pass away before the child is born, who shall restore form and repose to the degraded wanderer, and deliver the tortured prophet from his rock.

We can find no explanation of this mythus so rational as that which supposes the foundation of civil society in Greece, (attributed to Hercules,) as a corresponding

and countervailing power to the dead laws of nature, (impersonated in the tyrant Zeus,) to be that event which gives freedom to the long crushed spirit of foresight and wisdom, and rescues humanity from the restless and half-brutal condition of pastoral or nomadic society. According to this interpretation, the first part of the prophecy will of course intimate, that an imperfect attempt at political order, one not sufficient to give the energies and intelligencies of men their proper play, had been previously attempted in another country, and that this country was Egypt. We would not rest upon what may possibly be a mistaken view of this story, though it is strikingly confirmed by another version of it, given by the sophist Protagoras, in the dialogue of Plato, which bears his name, if there were not so many facts and traditions to show that it was the prevailing opinion in Greece. But how is this opinion consistent with the belief that the monarchies of Asia were the first in the world, a belief which sacred and profane history seem equally to support? the answer has been in part given already. These Asiatic monarchies began, according to the Scripture report of their origin, with rebellion against the first principles of moral order; they discarded the idea of fatherly government, trampled on the relations of family life, were connected with the worship of nature. Here was that true enthroning of Zeus, the mere lord of nature, in opposition to the Lord of man, which the Pythagorean poet perceived to be hinted at in the traditions of his country. The spirit of the earth, at once glorified and degraded by its union with the Lord, might be truly said to seek refuge from such tyrannies amidst hordes of wandering shepherds; and the spirit of wisdom and moral insight to be cast forth, incapable indeed of destruction, but conscious of subjection, bound and tormented.

It is quite possible then that moral culture and political order may have had their first birth in Egypt; and if this be once admitted, we believe it will be difficult to find a more intelligible account of that birth than the one which is contained in the simple narrative of Moses. The family of Abraham, as we have seen, was the formal witness against societies based upon mere combination, without reference to human relationships, and to a Divine father. A youth deeply imbued in the principles in which, through three generations, his family had been instructed by a Divine teacher, comes into Egypt in a period of approaching dearth. We need not suppose that the people at this time were mere barbarians; they had evidently traces of an older faith lingering among them, an appointed priesthood, who were the guardians of this faith, and (possibly) an hereditary monarchy. But we cannot believe that they had escaped the fate of all societies prematurely founded; or that the government of the Pharaohs was much less tyrannical, or their worship much less natural, than that which prevailed in Mesopotamia. By the providence of the Hebrew counsellor the monarch becomes possessed of stores, to which the people are compelled by their wants to resort. All their disposable property being exchanged for corn, they finally give up their lands; the race of independent proprietors ceases; Pharaoh becomes the general owner of Egypt. This revolution being thus quietly effected, Joseph is able to establish a system, resembling in its leading features the feudal system of modern Europe. This observation will only sound strange to those who have identified the accidents of that system with its essence, and have traced its origin to ideas of military

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

subordination. On the contrary, we affirm that family ideas were at the root of this system; the military part of it was only a graft upon a previous stock. In this sense we are bold to say that a feudal system is the cradle of all institutions. They may be strangled in their cradle; feudality may become all in all; and then a nation, in the proper sense of the word, will never be developed; but a nation must have some such commencement, or it will never acquire any healthy, vigorous life. The institutions of Joseph were not those which were intended for his own nation; they had had the discipline of a long family period, which had prepared them in the appointed hour to obtain, in the truest sense, an organized polity; but they were the only institutions which could have given rise to a moral and political order in Egypt: these must have been the features of Epaphus, if indeed he was born there.

We are by no means anxious to disprove those hypotheses respecting the migrations of priests across the land, or down the rivers, in which some modern mythologists delight—*Valeant quantum valebunt*; but since they leave one portion of the problem which they profess to solve untouched; since they explain merely whence certain notions may have been derived, not in the least how they became a part of the national mind, and as this is the question in which the moral and metaphysical student is interested, we do not see why the profoundest respect for the inventors of these theories should prevent us from believing a story transmitted to us on sufficient authority, and perfectly consistent with itself, which does extricate us from this embarrassment.

It should not be forgotten that views, however learned or interesting, of the manners, the government, or the literature of a people, or of an age, do not really satisfy the mind, unless they teach us how each bore upon the other; how each, in fact, involved the other. Even the exquisite chapter in Mr. Thirlwall's *History of Greece* upon the heroic age seems to us, we confess, to prove that the greatest ability, diligence, and command of resources are not sufficient to create a complete and living picture when the facts of it are distributed according to a formal and artificial method. The history of Joseph gives us very little help in understanding the details of Egyptian art, or worship, or policy, but we think that, by a few very simple touches, it supplies exactly those hints which we need to give order and coherency to facts with which every body is acquainted, but which probably dwell very loosely and confusedly in our minds. Taking the history of Joseph's adventures, of the influence which he acquired over Pharaoh, of the order which he established, of the descent of the Abrahamic family into Egypt, as we find them recorded in the book of Genesis, we can explain with tolerable clearness the position and circumstances of that people, the degree of cultivation which it was possible for them to reach, and the steps of their moral and political degeneracy.

The people constituting one large family, of which the king is the superintending father, and the priests the appointed instructors, acquire the notion of a fatherly government, which is strengthened by intercourse with the singular race which had been admitted to occupy a portion of their land; partly by positive traditions respecting their own education which the members of that race communicate, much more by the silent witness which is borne to this principle by the very existence of families so connected and upheld. The sense of a

superiority to nature which accompanies the cultivation of human relationships gives the first impulse to thought and intelligence among the people; and that intelligence is exercised, as it would of course be, in reference to their peculiar occupations—in discovering the arts of which an agricultural people are in need. There is no occasion to resort to the fantastic notions which the national vanity of Josephus has invented, and to suppose that the children of Abraham taught astronomy and geology to the Egyptians. They did what was much more valuable, they led them into that position in which men are able to seek out for themselves, at least that portion of knowledge which raises them above mere savage dependence upon chance supplies of daily food; enables them to sow lands in expectation of an harvest; to feel that there is an order in the world; to consider the past, and to look forward to the future. The priests, as we have said, would necessarily be the depositories of this knowledge, because it is a knowledge which only becomes possible when truths relating to the connection of man with his Maker, and consequently his superiority to the world around him, have been diffused in a country.

An agricultural people who have reached this point of civilization were not incapable of any of the conceptions which tradition leads us to ascribe to the early Egyptians. A perception of beauty implies a much greater moral and political progress; when the wonder inspired by the objects of outward nature is modified by a still higher admiration of the human form as the seat of intelligence or affection. But mere massive architecture seems to be especially connected with those first astronomical observations, which are awakened in the minds of a people devoted to tillage. The contrast between the simplicity of their daily occupations, and the grandeur of those objects which seem to regulate them, favours the development of that habit of mind which has created obelisks and pyramids. A notion of utility is connected more or less closely with all these buildings; but it cannot be said to be the motive of their erection. The religious feeling, the sense of awe and worship, which, as we have shown, is excited before any agricultural or astronomical knowledge is possible, was no doubt the primary impulse; one, however, which would never have left such memorials of itself, unless the government and habits of the country had lost much of that patriarchal character which we have attributed to them. This alteration would be accompanied or preceded by another, which the experience of history enables us without difficulty to set down. How soon in a country so wonderfully fertile as Egypt the victory over nature is changed into a deeper subjection to nature, how soon the absence of a curse on the soil moves a curse to the tillers of it, we need not tell any moderately thoughtful student. One of the first and most natural indications of this degeneracy is the tendency to consider men in reference to the services which they perform, rather than according to a law of human relationship. Institutions, of which territorial divisions are the basis, take place of those of which family life is the basis. In an agricultural country thus corrupted, a system of castes becomes inevitable. Again, family life ceasing to be the groundwork of institutions, the worship loses its original character. We need not inquire how pure it may have been in its best estate. Possibly there may have been at all times a confused notion that each relation of the family had

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

some distinct divinity for its prototype; at all times there may have been among the Egyptians a reverence for brother, sister, and mother gods; perplexing and interfering with, though still terminating in, the acknowledgment of a Father. All we contend for is, that the idea of a being related to man will have preponderated in their worship over the idea of mere gods of nature, and that in proportion as they become sensualized, the latter feeling will have gained the ascendancy, that these family gods would have become hopelessly enshrined in those objects of sensible utility over which they were at first piously supposed to preside; that there will soon have been castes of gods corresponding to the castes among men; that at last the objects and the powers will have become identical, and that thus, though we will not presume to fix at what precise period animal worship became prevalent in Egypt, there was no abomination of that worship which would not in due time naturally and inevitably be invented. The priesthood, in the mean time, who should have been the great means of resisting this downward tendering of the national mind, are carried along with it, and retain their power by ministering to it.

The awful and mysterious ideas of an elder worship remain among them to impart greater significance and fearfulness to the objects which that worship had once taught them to rule. The reverence for the priesthood is upheld no longer by their being the guides of the people to a wisdom which lifts them above themselves, and above nature, but by the notion that they possess some means of making the powers of nature hostile or propitious to them. It was the inevitable accompaniment of these changes, that the king, from the father of a set of families which composed the nation, should become the head of its castes. Still he would not be a despot in the Asiatic sense of the word; a feeling of his connection with the people over whom he ruled, that he was, as his taste or disposition might prompt him, the first warrior, or soothsayer, or handicraftsman of his land, would widely distinguish him from those rulers of the East, who looked upon their subjects merely as instruments of personal lust or ambition. Such rulers as were the Pharaohs would not indeed improve the cultivation of their people, but all skill and genius that were in the country they would call forth and employ. What were the limits of cultivation compatible with its moral and political condition we have considered already, and we may add, that the moral degradation of Egypt, though it retarded the progress of that cultivation, and reduced the body of the people nearly to the condition of brutes, may yet very well have permitted the continuance, even through many ages, of that degree of knowledge and art, which are expressed in its existing remains; in that portion of them, we mean, which cannot be proved to be the work of a period when the Greeks had far more than repaid to Egypt all that they are alleged to have borrowed from her. We reserve any remarks which we may have to make on the indications furnished by pictorial writing, of national progress in intellect, till we come to speak of the introduction of ordinary letters, a subject which will very soon engage our attention.

It is a question of very little importance whether the king who knew not Joseph was an invader or usurper, and of a different tribe from the monarchs who had preceded him. At all events it is probable that the thought, so natural to a monarch of some rude sagacity or cunning,

of enslaving or extirpating a singular race, dwelling in the heart of his country, and possessing inextinguishable privileges, was connected with other revolutionary schemes, for which, however, the minds of the people had been long preparing. The abolition of the institutions of Joseph, the complete substitution of a *caste* system for a feudal system, were probably the great achievements of his policy. He may have been a popular, as he was no doubt a magnificent monarch; originating, perhaps, or prosecuting with more vigour than any of his predecessors, the works for which Egypt has been, and will be for ever remembered. But it was he, and such as he, who caused Egypt to be without a history, to drag on weary ages of existence, in which the chronologist vainly attempts by unwearying diligence to detect a few uncertain landmarks. In destroying the family institutions of his country, he destroyed the seeds of its moral life, which might have sprung up and brought forth fruit in a healthy vigorous national society. He converted the priests into observers and contrivers of portents to answer the purposes of the politician; the politician into a mere plotter how to make a set of human machines work to the greatest advantage; and the people into mere slaves and brutes. He gave full sway to all those natural influences which are continually degrading the heart and spirit of man, till night and darkness become the only symbols by which the condition of Egypt could be adequately represented.

LEGAL PERIOD.

FROM B. C. 1491 TO B. C. 1271.

The Jewish historian is silent respecting the condition of his own people during the years of their sojourn in Egypt. There is enough, however, in his future narrative to warrant the opinion, which we might have formed without his authority, that they sank with the nation which their ancestor had been the means of raising. It would have been a violation of the design of their great Teacher, that they should have been more than a collection of families. They could communicate to another race the first principles of political order, because those principles are involved in the relationships for which they were the appointed witnesses. But these principles were not intended in their own race to generate institutions till the foundation of them had been laid in a more august revelation than any with which they had yet been favoured.

In this period then, previous to the commencement of their slavery, we may presume that they fell almost into the condition of an Arab horde; preserving, no doubt, accurate genealogies, the sign of their covenant, and in some families traditions of the knowledge and of the promises which had been imparted to their ancestors; but gradually losing their faith, and sinking, by the same steps as the Egyptians around them, into animal and natural worship. If we suppose them to have taken any interest in the speculations of the Egyptians respecting the arts of life, and the motions of the stars, we might have another way of accounting for their loss of patriarchal simplicity, and of that moral wisdom for which it was the great guarantee. But it is more consistent with reason and analogy to consider them as mere herdsmen, separated by customs and traditions from the people among whom they dwelt, acquiring only those notions and habits which insensibly pervade

Moral and
Metaphy-
sical Phi-
losophy.

Grounds of
positive
morality,
how first
appre-
hended by
the Jews.

Moral and
Metaphy-
sical Phi-
losophy.

the minds of all men, where there is no strong influence to counteract them, and daily exhibiting more of that sensuality and division, the growth of which among the sons of Jacob is so accurately traced by their honest historian. We know that this people did become an orderly national society. If we could obtain an accurate record of their passage into it, we might expect to find a key to many questions which will hereafter occupy our attention. Such a record should illustrate the law of national life; its connection with those relationships in which it has its birth; the distinct ground upon which it rests; how it is connected with morality; how it affects intellectual culture; how it promotes or retards the establishment of those wider relations which exist between men as the members of one race. The Scripture history, if it fulfil the purpose at which it seemed to be aiming in its account of the patriarchal family, ought to give us the hint of these discoveries, and to endure the test of its principles being applied to the exposition of puzzles which present themselves concerning the political and moral position of men generally.

This assistance, we believe, it will be willing to render; this test, we think, it will not decline.

The vulgar fancy of Josephus has crowded the history of the deliverer and legislator of his country with events, in which no person, who really feels the importance of the work which that hero performed, can compel himself to take the slightest interest. What can it signify whether or no he was the general of Pharaoh in his wars against the Ethiopians? or who could for a moment feel that the dreary pages which record his exploits have told him as much of the character of the man as the few simple words which describe his first feelings respecting the bondage of his country; his first consciousness of a vocation to be their deliverer; his efforts of rude and inconsiderate patriotism; his cruel disappointment at the discovery that the slaves whom he hoped to rescue from outward tyranny wore their chains upon their hearts? Or what narrative of his transactions after his exile in the land of Midian could teach us so well as his own clear and beautiful outline, the course which his thoughts must have taken, while for forty years he was tending his flocks in the deserts of a strange land? In that silence and solitude the traditions of his country would rise up before him in a new and startling shape. Hitherto they would have been connected with thoughts of the wonderful honour put upon his ancestors, of the degradation of their descendants, of the hope which still hung over their destinies. Now he would be led to connect these manifestations of an invisible Being with strange questionings respecting *himself*. The feeling that, as a Hebrew, he had some relation to the God of Abraham, of Isaac, and of Jacob; the feeling that such an hereditary, and as it were vicarious connection, was not sufficient to uphold one whose countrymen had lost all feelings of fellowship and kindred with their ancestors, and who was himself cut off from those countrymen, would force itself upon his heart, and would awaken in him the most awful of all human anxieties. "I am a distinct solitary person; I have a being of my own; of this being, no time, or accident, or circumstance can deprive me; this being no relations to my fellow-creatures can comprehend." With these thoughts would come obscure dreams of a higher relation perceived, not realized; a longing for the simplicity of those early fathers, of

whom he had been told, who walked daily in the cheerful sense of friendship with an unseen Being, without any of the restless strivings from which he was suffering. In his mind, the thought of a presiding, and ever watchful Parent, upholding his children in their path, directing them whither they should go, giving them promises, increasing their flocks and herds, and still more wonderfully allowing them to beget children in their own image, were mixed with perplexing notions of natural gods—the powers of the world, superintending the different creatures that minister to man; providing him possibly with the means of sustenance; unrelated and indifferent to the man himself. To harmonize these conflicting feelings would be impossible; his heart sighed after and secretly acknowledged such a God as his father worshipped, but his understanding led him to a very different conception; and the awe of his mind seemed to require some mere absolute Being, a Lord of man, not one who merely cared for him and provided for his wants. Of such a Being, the very contradictions in his heart seemed to speak. What was this sense of wrong, of solitude within him, but the intimation that there was such a Being, that there was a right, and a righteousness? At this anxious stage of thought he would recur again to his reflections and the bonds which connected him with his countrymen. The same conscience which recognised a superior judge, recognised his relation to other men; reminded him of acts inconsistent with his position towards them. There was a mysterious connection between these thoughts; where was it? what was the meaning and ground of it? A hope would begin to dawn through his miserable self-questionings; perhaps there might be one answer to them all; perhaps it might be vouchsafed to him; perhaps he might yet be the destined deliverer of his nation. This hope would assume a softer and quieter character, and yet not be quelled, when, after tending his flocks all day, he looked upon his wife and children, and saw in them the very pledges of Divine protection, which his fathers had esteemed most precious. It would be deepened and solemnized when he returned to the back parts of the desert, and beheld again that awful scenery which had nourished, though it could not awaken, the thoughts within him. To all these thoughts, the necessary consecration of every man destined to do any great work for mankind, the awful revelation in the bush, at the sight of which, he took the shoes from off his feet, was an answer.

The dreadful name, "I am that I am," was that name which for forty weary years he had been seeking. Here was the declaration of an absolute Being, the ground of his own being. But the accompanying words, which declared that He who had this awful name was the God of Abraham, Isaac, and Jacob, that He remembered his covenant; that He was the enemy of oppression and wrong; that He was a deliverer, and was about to manifest himself in all these characters, was exactly that reconciling discovery which he had longed for. The absolute personal Being whom he needed to sustain that feeling of personal distinctness, otherwise intolerable, was not another from the Being whom his fathers had worshipped; was still the head of all human relationships. This was that Being of strict and absolute righteousness, whom in the sense of his own evil he had felt to be so dreadful, and yet desired to know; in knowing whom he felt that he was evil no longer; for he rejoiced in His good, sympathized in

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

His good, and desired to see it prevail over all that resisted it in others and in himself.

Here was the deliverance from all those natural, sensual notions, into which the member of a degraded race, even if he had not been learned in all the wisdom of the Egyptians, must necessarily have fallen. The absolute Being was not a natural being; not removed from man, as a star is removed from man, by a hopeless absence of sympathy, but the perfect archetype of that righteousness which man perceives, recognises, and may exhibit in his own acts. "And these truths," the patriot would exclaim, "do not belong merely to me, though revealed to me; they belong to my nation; they shall be practically manifested in its deliverance from bondage; manifested through us to the whole universe."

On the basis of this revelation the Jewish polity stood. The idea of an absolute Being was the first principle of it; the idea that He was a righteous Being, the enemy of oppression and evil, the second; and these views were changed into practical realities by a third principle,—that this Being was the God of their fathers, that He had selected them from all the nations of the earth to be His witnesses, that He was the upholder of their family life. How these principles were expressed in their constitution, how they permeated every part of the system, our readers will understand just in proportion as they are attentive students of the books in which the nature of these institutions is declared, and of those in which their practical working is exhibited. We neither contend for nor against the notion which Michaelis and others have so ingeniously elaborated, that each law, and custom, and institution of the Jews had some reference to their position as an Oriental nation. In proportion as we believe the system to have been a wise one, grounded on deep principles, and constructed by a Divine architect, we must be inclined to believe that there was a skilful provision for the circumstances of the particular people to whom it was given, which of course would include the geographical position, the climate of their country, and their relation to the people around them. As little are we inclined to enter into the controversy between our learned countryman Spencer and the Dutch divines, whether any portion of these institutions may have been adopted and purified from some previously existing in the nation to which the Hebrews had been in bondage. If our view of the character of Joseph's instructions be a correct one, it will readily be admitted that they were *seeds* of institutions, not in the strict sense of the word institutions themselves, and that they would have given birth to various laws, customs, and systems, mixed of course with mythological notions, which for the very sake of removing those notions from the minds of his people the Divine legislator may have presented in their true and uncorrupted form in his own economy. There is nothing, we say, at all impossible or improbable in either of these speculations; the only complaint which the historian of philosophy is inclined to make against them is, that they have rather tended to obscure the groundwork of the Mosaic institutions, and to leave on men's minds the impression, that it requires little investigation in comparison with those subtle arrangements which concerned the outward life of the Jewish nation; or those which were to restrain them from the adoption of certain evil customs in the surrounding nations. A habit of looking rather at the

negative than at the positive parts of the Jewish system, its provisions rather than its principles, has prevailed to a great extent among our divines and biblical critics. The vague phrases that it was intended to provide against idolatry, and for the recognition of the Divine Unity, have hindered students from steadily examining these institutions, and considering what information they give us respecting the grounds of national and social life. Our conviction is, that there can be no clear understanding of the principles of political order, as they existed in Greece and Rome, and as they do exist in modern Europe, until the constitution of this Divine commonwealth is meditated upon in an honest and humble spirit.

Our limits will only permit us to notice just so many of its peculiarities, and of the facts which record its establishment, as are necessary to throw light upon the connection between the moral speculations and the social condition of man.

Passing by, therefore, the hint which is given in the very commencement of this narrative of the distinction between the priest and the lawgiver; passing over that series of remarkable signs by which the dominion of an unseen God of righteousness over the different powers of nature was established, the tendency to natural worship in the minds of the Egyptians condemned, and the legerdemain of their magicians confounded, events most important as asserting the principles on which the Jewish nation was to rest; passing by for the present even the last and most terrible of these signs, which bore most closely upon the ideas of primogeniture existing in the old world, we proceed to notice the three leading peculiarities of this system—the *TRIBE*, the *CODE*, the *TABERNACLE*.

The *tribe* system is the link between the family and the national life of the Hebrews. There were not a set of lares and penates presiding over the hearths and households of the Hebrews, and a Jupiter Capitolinus who ruled in the citadel; the God of their fathers was the God of their nation. There was no caste system, separating them according to their occupations; the distinctions were all genealogical; the land, when they possessed it, was to adapt itself to distinctions previously established, not to be the cause of any. The family life was bound up with the national by the festivals, most of all by that of the Passover, the celebration of a glorious common deliverance, to the sacrifice attending which every family contributed its quota. It was connected again with the institution and administration of the laws by the selection of a set of elders, the heads and representatives of the different tribes, to be the magistrates of the land. It was connected with the ecclesiastical body, in that one whole tribe was formally constituted the priesthood of the nation; that the priests succeeded by a law of hereditary succession, and were considered, in their devotion to the immediate service of the tabernacle, as substitutes for the first born of every family. By this means the idea of primogeniture was upheld, while there was a perpetual witness against the abuses to which it sometimes leads. The early history of the Jews had warned them, by the instances of Ishmael, Esau, and Reuben, that the principle of primogeniture, though grand and sacred, exists for certain moral ends, for the sake of which it may be sacrificed. The new institution reminding them what that moral end was, kept up the balance of feeling, the absence of which has in different nations by turns given a dangerously exclusive pre-

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

dominance to feudal notions, or expelled them altogether. These are a few hints respecting this side of the Jewish constitution, which our readers may find an interest in following out for themselves through all the details of it.

In doing so, we would entreat them never to lose sight of the great moral and metaphysical lesson which the harmony of their family and national life taught the Jews, and for the sake of which we have alluded to it here; namely, of the essential distinction, and the necessary union between relative and absolute morality. Into how many questions the neglect of this distinction, or the want of this union, has led men; how essential the practical recognition of it is to the theoretical apprehension of it; how this recognition is connected with political institutions and theological opinions, our subsequent history will abundantly testify.

The prescriptive Code of the Jews was not given them till the institutions of which we have been speaking had been formally established. They had recognised the God of their fathers as their deliverer, their protector, and their friend, before they were called upon by any solemn act to recognise Him as the I AM. Yet, as we have shown already, on this recognition national society depends; without it, there was no permanence for the family life which it protected. The same grand truth, which Moses needed to call forth in him the feeling of his own personality, and to prevent that sense from overwhelming him, was needed to give being to the Jewish nation. Family life exists in the feeling of connection and association; national life, in the feeling that each man is a distinct person.

After an awful preparation of three days, amidst thunders and lightnings, and the sight of a devouring flame, those words were proclaimed which every Jew felt were addressed to him; which withdrew him from relationships, friendships, sympathies, into the secret chamber of his own spirit, obliged him to feel that he was a distinct living person, obliged him to recognise something in himself as at war with an order, which he at the same moment confessed to be right. Yet along with the awakening of this dreadful sense of personality was a deeper and stronger feeling, that he was the member of a commonwealth, that every act which he or any other man did affected the commonwealth, that the protection to which he as a single person must look, was that which overshadowed the whole society. "I am the Lord thy God which brought thee out of Egypt," were the first words that every Hebrew heard from the holy mount, and which, before even the sense of evil had been awakened in him, reminded him that he personally shared in the blessings of his nation; assured him that he personally was observed and protected by the unseen head of it.

The history of the Jewish nation and of the world is a commentary on this code. Whatever causes have broken up commonwealths, or made society untenable, or turned men into brutes, are therein set forth and predicted. The loss of faith in an unseen protector and guardian, the idolatry of sense, the loss of reverence for the name which speaks of the reality and personality of Him whom we worship; the loss of an institution which teaches men to view Him as an object of fixed contemplation, the ultimate resting-place of the spirit, and not merely as ever acting for them, and which connects that instruction with the rest of all creatures from their labours, thus reminding them that, both in their

toils and in their repose, they are His image—loss of reverence for parents leading to incessant emigration; loss of reverence for life, for marriage, for property, for character; lastly, which is the source of all these, the grasping, self-seeking, money-getting spirit;—what more wonderful index can we possess to the history of the decline and fall of empires? The idea of this code necessarily excludes all prohibition of mere excess. It is an important part of the statutes of a nation, to provide against acts of excess, as occasion shows them to be detrimental to the community; but it would be a violation of all principles of order and distinction to introduce them into a system of universal prohibition. In the Jewish economy, there is a strict and admirable discrimination between the laws and the statutes of the nation, between the positive denunciations against fixed evil and the provision against emergencies and overt acts; between the rules which had reference to the actual condition of society in a particular nation and the principles which are the groundwork of society everywhere. But a still more important distinction is rigidly observed between the literal code and the righteousness which is the ground of the code; which can be contemplated only in the acts and life of a person; which is positive, not negative; which is attained through a knowledge of the character of the Being we worship. A recognition of his authority is implied in national order, the recognition of His righteousness is connected with a higher universal human order.

The third part of the Jewish economy has a reference to this higher righteousness and higher order, and recognises man as capable of holding communion with the unseen ruler of his nation. THE TABERNACLE was the grand witness and practical manifestation of this truth, involved alike in the family and national life of the Jews, the crowning glory of both; for which both existed. The first part of the Jewish system expressed the idea of fatherly care and government; the second, the idea of a righteous Being formally distinguishing between good and evil; the third portion, the idea of communion with this Being; the last idea including, though not superseding, both the others, and constantly reminding the Jews that the end of the Divine education and law was to exalt them to the contemplation and knowledge of himself. With this grand theological idea was connected, as we have already intimated, a great human principle. It was the universal side of the Jewish polity, it was the link which bound the particular nation to the human species, which enabled them to understand the promise made to Abraham, that in his seed all the nations of the earth should be blessed, which led him to regard his own separation as a means to an end, not an end in itself.

By keeping these observations in remembrance, we are able to understand the relation of the parts of the Jewish ecclesiastical system to each other, to the civil government, to the members of the nation, and to mankind; an inquiry more important than it may first appear to the study of the moral and metaphysical history of the Jews and of other nations. The system may be referred to these three heads: 1. the tabernacle itself; 2. the priests; 3. the sacrifices. The building which was prepared and set up with so many awful ceremonies, and which attended the Jews in all their wanderings through the wilderness, brought continually before them the idea of an awful and mysterious presence; not floating in the air, not diffused through the

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

forms of nature, but a real and personal Being, connected primarily with them; only, in the second place, with the rest of creation. While man was thus elevated in the scale of creatures, by feeling that the Maker of all things was his friend, the idea of absolute truth and righteousness presented in the law, seeming to crush his aspirations, really gave them a surer confidence and a higher aim. The possibility of becoming like this Being by contemplating Him; the hope kept alive by so many national promises of a more complete manifestation; the thought of a deep abyss of being, which man could never fathom, but on which he might repose; these thoughts, meditations, hopes, were kept alive by the presence and by the order of the mysterious tabernacle, and were communicated, as we shall see hereafter, through it, and through the more magnificent, permanent edifice which afterwards received the ark of the covenant, to the mind and heart of the Jewish prophet.

The *ministers* of this awful sanctuary were distinguished, though by no means separated, from the elders who represented its family constitution, from the chief who promulgated and interpreted its laws. The office of the whole body was distinguished by garments for glory and for beauty, betokening, as the words inscribed upon their foreheads implied, that they were a dedicated race, the witnesses to the whole nation of the fact of a connection between them and a perfect Being, even when they were not performing the duties by which that connection was expressed and realized. Yet they were in no sense separated, either from the nation corporately, or from its particular members. They represented the dignity, the purity, the glory of the nation, that which it really was, not what by forgetfulness of their privileges its members might become.

They took cognizance of the most practical wants and sufferings of the people; they were the divinely appointed physicians of the land, educating the suffering Jew to feel that his bodily health was watched over by a Divine guardian, to understand that disease itself is under a law, and that the knowledge of the law leads to the remedy. In the case of leprosy, higher truths were indicated, and a practical apprehension of them imparted to the people. Even disease was made a means of teaching them the lesson which the whole Jewish polity was devised to illustrate—that communion between men is grounded upon their common relation to an invisible Ruler, and that the impurity of one man may defile the whole congregation. Another branch of the priest's office also, that of defining the clean and unclean food, enabled them to act as the practical instructors of a rude and ignorant people. Without supposing any deeper reason for the prohibitions than that which Michaelis has assigned, and even being ready to admit that in some cases that reason will be difficult to discover, we contend that mere arbitrary appointments of this kind must have been most useful in cultivating habits of distinction in those who were subject to them, in making them feel that they were in the midst of an order, even when they were not able to interpret or reconcile the different portions of it.

But the grand vocation of the priest, that which really brought him into connection with all the members of the commonwealth, that which gave the meaning to all his other functions, was to offer *Sacrifices*; those daily sacrifices which each man brought to the door of the tabernacle; that more august annual sacrifice which the high priest, the Coryphæus of the body, offered on that

day when he went alone into the holiest of the holy. We have pointed out the idea of communion as that to which the tabernacle and the office of the priest alike pointed, which being absent, institutions become a dreary weight, law an intolerable oppression. To this idea we must refer sacrifices, or we have no clue to their meaning in the Divine commonwealth, or in those whose system we shall have hereafter to examine.

That the nation, and every member of it, is united to its Divine King and head, that as long as each member abides in the covenant, retains his allegiance to the head of the nation, and his fellowship with the other members of it, he enjoys this connection; that by every transgression of the law which defines his position in reference to his God, and to his brethren, he practically forfeits it; that something must be given up before the connection can be re-established: these are the elementary principles in the idea of sacrifices, as it is expounded to us in the Jewish Scriptures, and as it is illustrated even in the most corrupt practices of heathen nations. The question which we shall find at issue between them is this: Is the hinderance to this intercourse some capricious distaste in the mind of God towards the suppliant, or is it the impossibility of fellowship between two beings, in whom there exists a dissimilarity of feeling and character, and whose relation to each other has by some act on one or other side been disturbed? If the former notion subsist, the value of the offering would of course regulate the petitioner's chance of success. A hecatomb will be worth a hundred times as much as a single ox; a child or a beautiful woman, in countries where human creatures are revered, will often be thought a more hopeful offering than either. The other noble idea of sacrifice is that which is expressed in the Hebrew institutions. The Jew must not, dare not, fancy that a more costly offering than the appointed one will do him greater service. If he brought to the door of the tabernacle of the congregation some precious sacrifice, devised after his own conceit, and not according to the provisions of the law, the priest was bound to reject him, and to reprove him, or he virtually renounced his own office. Every thing in the matter and mode of the offering was rigidly prescribed; it was the *will* of the offerer only which the law permitted and required to be free. Thus the idea of an unchangeable Being lay at the groundwork of an institution which, in every other country, left on the mind the most lamentable sense of fluctuation and caprice. Thus he was taught to feel on the one hand that he was in a fixed relation to the King and Father of the nation, and yet that, as this King and Father was a moral Being, all the blessings of that relation are forfeited by an immoral state of mind. Thus he was taught by what kind of acts that immoral state was produced, and that relation violated; by acts, namely, of self-will, separating him from God and from his brethren. Thus he was taught what effect that self-will produces, how it enslaves a man to the animal nature, above which he had been raised, by being admitted into fellowship with his Creator. The animal which he offers, teaches him what it is that he has to sacrifice; the voluntary will which is required of him, teaches him what it is that has need to be restored; the priest, who alone can offer the sacrifice, teaches him that the connection between him and the invisible Being is upheld in the person of some mediator; and, together with the great act of atonement, prepares him for the idea of a

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

Being who, by the offering of his own body and the entire sacrifice of his own will, shall be the bond to unite together the whole race with the perfect and absolute will. So wonderfully did the idea of moral order in the universe, of God, as the founder and upholder of that order, of man, as a rebel against that order, and yet capable, by a mysterious process, of being brought to apprehend it, evolve itself through institutions apparently the most purely national and exclusive.

We can but hint at a subject closely connected with sacrifices, that of purifications, as a part of this great system of education, and one which illustrates some of the most difficult passages in the moral and metaphysical history of mankind.

By following out the method we have been endeavouring to develop, and especially by keeping in mind the threefold principle of the Jewish commonwealth, answering to its patriarchal, legal, and ecclesiastical institutions, our readers will be able, without difficulty, to trace the connection between this or any other part of the scheme which we have left unnoticed, and its general design. It remains that we should say some words on the intellectual development of the Jewish nation, corresponding with these steps in its moral and social progress. Moses tells us, that the deliverance from Pharaoh was celebrated in a triumphal song. This is the first instance of Hebrew *poetry* in the Scripture; for if the prophecy of Jacob respecting his descendants be rhythmical, (a doubtful question,) that form must have been imparted to it by the historian, in order to preserve the connection which the mind of a Jew, and even of most pagans, recognised as so close between prophecy and poetry. This fact is worthy of our notice; for it is an observation which has been often made, but scarcely enough dwelt upon by those who have endeavoured to ascertain the meaning and functions of poetry, that it never existed apart from some definite form of political society, and that the periods in which the bonds of political union have been most strongly realized, are those in which poetry has manifested itself in its greatest power. To say that poetry has never coexisted with savage life, or with any condition of society in which the selfish feelings have predominated over the social, would be a trite common-place. But it is a great question whether any kind of social feeling except that to which we give the name of NATIONAL, does naturally express itself in this way. In the communion of mere family life, the soul is unconscious of its own workings; the man becomes, in the strictest sense of the word, *attached* to the members of his own circle, and forgets himself in them. A result not altogether dissimilar takes place in the universal communion of the church, which, when it is strongly realized, often absorbs the conscious faculties of a man altogether, in the contemplation of the Being who is the centre of it, or of the different members of it, whom he is to love even as himself. It is exactly in that communion between men, which exists in a political body circumscribed by definite limits, feeling itself in a certain sense at war with the rest of the world, that each man vividly and consciously realizes his relation to the whole, and rises through the sense of that relation to the dignity of a personal being; and of these feelings poetry seems to be the legitimate and intended expression. The crisis of deliverance is that which gives occasion to its first utterance; one person, who most strongly feels that he is a member of the nation, expresses by words the excited feeling of his own spirit, and those

words married to music are felt by the whole society as their own. That the first poetry would inevitably be song, that it will be a long time before it is even conceived possible to separate it from music, we need not tell any reader. The explanation of the fact we find in the connection of poetry with national life, and in those peculiar conditions of national life which we have been endeavouring in this chapter to set forth.

If we have been right in our remarks respecting the feelings which give birth to poetry, it will not be strange that it should be contemporaneous with the commencement of orderly and coherent *language*. After the formal notions of language have become parts of our habits of thought; after we have been used to separate altogether the beholder from that which he contemplates; objects from the feelings with which they are viewed, acts from their agents, qualities from that in which they inhere, it is delightful to find the connection restored to us in poetry; to find that we are in the midst of a living world; to see deeds performing by actual persons; dead names restored to life by being united with their attributes. Not that poetry leads to any confusion of these great distinctions, but rather enables us to realize them more vividly; and above all, that grand distinction between persons and things, which is half lost when we treat of both alike as mere subjects of narrative and speculation, is felt in its power when we find ourselves actually in the midst of our fellow-men, sympathizing with them, and especially rejoicing to see how they exercise a government over nature. From these considerations we may understand why poetry, which restores language when it has been corrupted, must also be almost identical with it, in its infancy, why the common notion may be no extravagant one, that the Greek language was created by the Homeric poems. Mr. Thirlwall has endeavoured to solve the difficulties respecting the introduction of LETTERS into Greece by supposing that they too may have been first employed in the poet's service. That written characters will appear very shortly after the language which they are to transmit from generation to generation has acquired a distinct order and formation, we cannot doubt. Both, we believe, accompany the commencement of a national period; but as there seems no reason why poems should not be preserved, for a time at least, by recitation, we do not see the need of letters for the sake of; but we do perceive the indispensable necessity of letters for the sake of that which is the most characteristic element of national life—its laws. We are thus led to adopt, in preference to Mr. Thirlwall's hypothesis, a much older and less ingenious one,—that written characters, as distinguished from that writing which is the imitation of natural objects, first come into use in every country where formal codes or charters are grafted upon its institutions and common law. By this observation our readers will see that we adopt as literal the declarations of Moses respecting the giving of the law, and the method of its preservation.

And it seems perfectly consistent with the whole idea of this economy, that the deliverance of men from the thralldom to nature and natural worship, the worst part of their Egyptian thralldom, and without which the other parts of it could never have existed, should have been accompanied by that introduction of letters, as well as of laws. Neither were necessary in the first patriarchal age. Then there was no explicit code; then heaps of stones were sufficient to establish contracts and preserve traditions. When this patriarchal system had given

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

birth to the feudal institutions of Egypt, the want of some other method of communicating the will of the sovereign or the dogmas of the priests would have been felt. Natural objects have still the dominion over the heart and mind. Language is not felt as an expression of human thought, but as a description of natural objects. The invention of a system of hieroglyphics is the natural result. It may have afterwards been appropriated to the service of the priest, and been invested with a mysterious dignity, when ordinary writing had been introduced from other countries; but it never can have been invented at a period after ordinary characters were in use. Nor can we conceive that the soul of the Israelite could ever have enjoyed that emancipation from the tyranny of sense which was intended for him when he was brought under the law of an invisible and personal Being, if he had not been taught that the words which were the utterance of his spirit were not dependent on the forms around him; that nature was the image of the thoughts which his words expressed, not they the image of nature; and if to sustain these all-important instructions he had not been provided with such means of formally distinguishing those words, as made the use of natural symbols unnecessary. Upon this observation we would ground another. Song, we have seen, was the first expression of the Jewish spirit after its deliverance from captivity. But the earliest records of the Jews are not, as in the case of the Greeks, poetical. Poetry belongs, in its highest form, and its most mature efforts, to the third stage of their history. The difference is very interesting, and is connected with the difference between the Hebrews and Greeks generally. The peculiar Divine education of the Hebrews, instead of superseding the regular progress of the human spirit, gave the most orderly direction to that progress. But the natural overflow of the human spirit in poetry, at the very infancy of its political existence, is not favourable, as Greek history will show us, to the consistency of society, or to the permanence of moral feeling. Unless law and history in a nation precede the continuous exercises of the poetical faculty, there is little hope that any fixed national order will control the struggle of the individual intellect and will. It was therefore most accordant with the Divine scheme, that the teaching of the nation, when that teaching for the first time took a literary form, should be through the means of a history, possessing indeed much of the charm of poetry, by its simplicity and clearness, but still formally distinguished from it, not merely by the absence of rhythm, but also by its strict chronological method, and the want of a central object.

The third step in the Hebrew intellectual development of this period is connected with their ecclesiastical life. Architecture here, as every where, had its source in the idea of communion with an invisible Being, and presented that idea outwardly. The account which is given of the fabrication of the tabernacle expresses with great clearness the principle of all these records: "See, I have called by name Bezaleel, the son of Uri, the son of Hur, of the tribe of Judah; and I have filled him with the Spirit of God, in wisdom and in understanding, and in knowledge, and in all manner of workmanship, to devise cunning works, to work in gold, and in silver, and in brass, and in cutting of stones, to set them, and in carving of timber, to work in all manner of workmanship. And I, behold, I have given, with him Aholiab, the son of Ahisamach, of the tribe of Dan; and in

the hearts of all that are wise hearted I have put wisdom, that they make all I have commanded thee." The revelation of certain great truths to the legislator of the people, the selection of a particular man to set forth those truths in sensible forms, which forms are intended as the means of educating the whole people into the perception of the truths;—this is exactly the order which the Scripture is uniformly presenting to us. The Lord of man unveils himself to the faith and wonder of a man whom he destines to be the moral guide and teacher of his people. But outward nature is the expression and mirror of those truths. Certain men here and there are endowed with the gift of perceiving the correspondence between nature and the mysteries in the heart of man; these are of a lower grade than the former in the moral scale, but still they have a glorious vocation. They also are the teachers of mankind, converting the very senses which are wont to be its tyrants into the means of its deliverance. We shall find hereafter how strikingly this view of the relations of men to each other in a state illustrates the doctrine of Plato on the same subject. We shall not here enter into any observations respecting the nature of Hebrew architecture; its contrast to the mere massive temples of Egypt, or to the graceful and definite temples of Greece. Such remarks, if we have leisure for them, will occur best when we notice the final achievement of the Jewish spirit in this direction, in the age of Solomon. We merely notice the subject at this period in order to show how the same Divine power which gave the Hebrews a polity, and made that polity a means of awakening them to a perception of the three great moral principles in which all society is based, tribe relations, positive law, communion with God, show forth their intellects, also, in three directions corresponding to these. First, in poetry and music, expressing their sense of common relationship, and giving form, and meaning, and harmony to language; next, in letters, arbitrary devices, in which man exults because they are arbitrary, determined by his will, not by nature, to convey the laws by which he is bound, and the thoughts which he utters from generation to generation; lastly, architecture, by which he compels nature itself to give him symbols which may represent the deepest speculations and principles of his mind.

Such is a very short view of the Divine education of the Jewish nation, at this most important crisis of its life. Our readers will not fail to perceive how it is connected with that earlier period of family education; nor, we think, while they contemplate the relation of one to the other, the gradual and wonderful growth of one out of the other, by processes so analogous to the subtlest and most mysterious processes of nature; finally, the integrity and coherency of the whole system; will they refrain from asking themselves whether the supposition, that the historian who records it was a fabulist, does not involve the belief of more portents and miracles than the most child-like acknowledgment of his veracity?

The nature of our work does not permit us to notice any further particulars respecting the history of the Jewish people during this period.

The journeys through the wilderness do indeed illustrate the deepest moral truths, and are most instructive, as an illustration of the experimental method by which the members of the chosen nation were educated, to understand the tendencies of their minds, the meaning of

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

the law of God, and the purpose of his discipline, and when we come to study the history of the people at a later period, we shall find how the records of these journeys at once gave the impulse and supplied the materials to the reflections and meditations of their seers. Equally important is the history of their entrance into Canaan; of their victory over the slavish, degraded, natural people of the land, through the might of their confidence in an unseen, personal God, and the faithful records of their own degradation and misery, when they forgot to worship their deliverer and friend, and became the servants of their senses.

But these records are only the illustrations and working out of principles, of which we have already spoken, and therefore, though most important in a history of humanity, can hardly be said to belong to a history of philosophy.

*Interval between the Second and Third Hebrew Period.
The Phœnicians.*

The Phœ-
nicians.
Influence of
the appre-
hension of
positive
morality
imparted to
the He-
brews on
political in-
stitutions.

It is an opinion which was prevalent in former times, and has gained strength from some recent discoveries, that the Edomites had attained a remarkable civilization at the time when the Hebrews were bondsmen in Egypt.

Supposing there to be any sufficient evidence that the excavated buildings of Arabia Petraea were really the work of so early a period as this, they would be the symptoms of a people who had attained a very unnatural and precocious knowledge of the mechanical arts, unsustained by that moral cultivation which alone can make the history of a nation interesting, if it be not more correct to say, which alone can give it a history at all. The inferences which have been drawn from the book of Job respecting this people, are at best of a most doubtful character; for that book was at all events intended to be, and must be leavened with Hebrew wisdom. It may be intended as the history of a man who retains the old patriarchal faith without being a partaker of the blessing of the law and civil polity of the Jews, and if so, the scene of it would very appropriately be laid in the country of the descendants of Esau.

But before we found any conclusions upon it respecting their moral state, we must assume both the date of the book and the object of it, and then, contrary to all principle and analogy, must speak of it as an Edomite, instead of a Jewish work, and this too without attempting (as who dares attempt?) to exclude it from the canon of Scripture. At any rate, it would be inconsistent with our idea to recognise any nation as spiritually important which has not made its importance felt by exercising some direct influence on the character and condition of mankind.

There is, however, a most interesting question connected with this part of our subject. All traditions seem to prove that Phœnicia must have arisen into importance between the time in which the Jews invaded Palestine and the reign of David. How far may the formation of its civilization and power be traced to the influence of Jewish ideas? Is there any thing to warrant the supposition that the Jews, in their second or legal state, may have acted upon the conquered nations of Palestine, or upon their immediate neighbours, in any thing like the same manner that their fathers in the patriarchal stage appear to have acted upon the condition of Egypt? It falls distinctly within our purpose

to throw out some hints upon this subject, because it is connected with the influence of moral principles upon the order of society, and because it will remove many difficulties out of our way when we enter upon the study of Greek philosophy.

It is certain that the Greeks attributed a part of their own cultivation to Egypt, a part to Phœnicia. It is not necessary to inquire whether the stories respecting their early communication with either country be partly true or wholly fabulous. They prove at least this much, that the Greeks of later times discovered in a certain portion of their own national ideas so close a resemblance to those which they found embodied in the traditions of the priests of Thebes or Memphis, and in another portion of them so close a resemblance to those which were recognised by the merchants of Tyre, that they had plausible reasons for supposing them to be derived from those two sources respectively.

Now we think that some modern theories have considerably embarrassed the question, Which portion of the Greek feeling and mythology most corresponded to that of the agricultural, and which, to that of the trading people? It has been maintained that the two divinities of Phœnicia, Baal and Astarte, represented, one the productive and the other the passive powers of nature. So refined a notion as this, it seems to us, can never have been the foundation of any national worship; it is evidently a late philosophical exposition of a popular belief, which must have had a very different and a much more living root. We do not deny that there is a stage in society in which the worship of outward nature, in the strict sense of the word, becomes the characteristic of a people. We shall have to examine by and by such a stage, and to consider the meaning of its phenomena. We shall find them altogether different from those of Phœnicia, and we shall find that they assume a much more grand and startling shape than that of investigations and reflections respecting the productive and passive powers of things. Men do not get their notions of these from the outward world, but from themselves. Baal and Astarte we believe to have been neither more nor less than the Lord and Lady of Phœnicia, who might be worshipped in a hundred different natural forms, but the idea of whom was suggested, not by the sensible world, but by the forms of national life. Here then would be the great difference between the Egyptian and Phœnician. The one would worship family gods, fathers, mothers, brothers, sisters, husbands; the other would worship national gods, kings, queens, legislators, champions, heroes. The first worship, as we have shown, is connected with the cultivation of the soil, and gives birth to rude and massive forms of art; the second will be found in union with a commercial spirit, and with an approach to the more definite and precise architecture. The tendency of the former will be to mere grossness; of the other to mere sensuality. The intellectual degradation will be greater in the first, the moral degradation in the second. If these observations be correct, there can be little difficulty in determining which portion of the Greek character and mythology had a counterpart in Egypt, and which in Phœnicia; and we think that a careful examination of the legends respectively given to each country would lead to the same result with our own *à priori* speculations; we think that both would show, that whereas the ideas of family life, which existed in the Greek mind and were embodied in its forms of worship, would enable the travelled Greek to feel a

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

sympathy with that early form of society of which he found traces in the African nation; his hero worship, his notions of art and letters, all that strictly belong to him as a political being, would very naturally, though, perhaps, quite falsely, have been supposed to be brought to him by the restless traders of Asia. Here then would seem to be the answer to our first question. We have explained the difference between the position of the Jews at this time and in the patriarchal ages. We have seen that while a permanence was given by institutions to all their early teaching, great moral principles hidden and implied in that teaching had been brought out into clear manifestation, and formed the ground of the Jewish national life. It was not merely in the father of the family that they beheld a type of their unseen guide and friend; in the legislator, in the priest, in the conquering leader, and finally in the judge, they had the witness of his presence and his power, the living type of his order and government. The last-named officer especially marks at once the stage at which the Hebrew nation had arrived, and the difference between them and all other people. He was the champion, the hero, who was raised up at some particular emergency to restore the fallen commonwealth; but his heroism was not his main characteristic; it never enabled him to become the founder of a dynasty, or even to exercise the kind of power which the able tyrants of the Greek colonies acquired in a later age. His name indicated that he was the representative and expounder of the law, that he recognised right and wrong as permanent distinctions, which it is the highest honour of the ruler of a state to perceive and enforce, which he never can create or set aside. This was the great and glorious distinction of the Hebrew—the lock of hair, with which when he parted he became weak as another man. While he retained it, he was the unconscious means of communicating to tribes, which had no participation in his higher privileges, notions of order and law, which raised them entirely out of the condition of savages, even where they did not issue in any high moral cultivation. Of such notions the Phœnician worship seems to have been the expression. Conceptions of a Divine judge and law-giver and king would dawn upon them through the forms of their old traditional religion; would incorporate themselves with its polytheism, and would give that polytheism a humanized character. As these ideas obtained greater power and prevalence, society would become organized, and would in its turn give to the moral conceptions more shape and definiteness. Civilization in the pupil would seem to proceed far more rapidly than in the master, for the sense of their own powers, which men acquire as soon as they feel themselves members of an organized community, would not be checked by the feeling of submission to a prescribed system and a pre-established law. They would arise speedily into the condition of a kingdom without any of that tedious preparation by which the Hebrews were taught to understand what a kingdom means. They would soon have their seaports, their fleets, and their colonies; in every country they would leave vestiges of themselves. They would expand the minds of those to whom they came with notions of distant worlds, of the power which man possesses over the elements, with a dim feeling of the dominion which mind is meant to possess over matter. And little will it have been suspected by those who received this lore, or those to whom they have transmitted it, that all this feeling

of spiritual dignity and human grandeur were first imparted them by an obscure tribe, subject to the severest rules, and practising the most inexplicable rites. Little will it have been understood that in those rules and in those ceremonies lay the secret principle and true conditions of that power over nature which the Phœnicians seemed to wield so mightily, and the Israelites to exercise so feebly; that while the Phœnician acquired some notions of a Divine order from the regularity of human society, which consequently changed with all its fluctuations, the Hebrew was able to discern a fixed and permanent ground for all institutions and government, in an unchanging personal Being; that while the absence of all domestic ties seemed to give freedom to the movements of the Phœnician, the Hebrew, by preserving the strictness of those ties, was delivered from the restless spirit of emigration to which the other was the perpetual victim. Finally, that while the Phœnician was able to establish a few colonies, and to give to a portion of them forms of government which lasted amidst much conflict for a few centuries, till they were buried under corruptions which were inherent in their constitution from the first, the Jewish commonwealth has exercised an influence over mankind from generation to generation; an influence which is felt and acknowledged the more in each country in proportion as its inhabitants are orderly, peaceful, and happy.

Moral and
Metaphy-
sical Phi-
losophy.

PROPHETIC PERIOD.

FROM B. C. 1271 TO B. C. 600.

If it is a probable supposition that Jewish ideas in the second or national stage of their history exercised an influence over the conquered tribes of Palestine, analogous to that which the patriarchal wisdom exercised over Egypt, it is quite certain that the Jews of the period immediately preceding the rise of Samuel suffered from the sensual idolatry by which they were surrounded, as their fathers in the time of Moses had suffered from the animal worship of their African taskmasters. The form of degeneracy is exactly what we might have expected. It had reached the vitals of national life when the priests became sensual and infidel, when the sons of Eli introduced into the worship of the tabernacle the abominations which were practised in the temples and orgies of Palestine. As every page of Hebrew history and of every other history assures us, such proceedings were a sure prognostic that a crisis was approaching, and that if society was to be upheld there would occur some tremendous visitation through which it would pass into a higher state.

The birth of a boy, not in a miraculous manner, but in answer to the prayers of his mother, who immediately dedicates him to the Lord, is recorded with great solemnity and minuteness by the inspired historian as the index of the commencement of a new era. This boy, we are told, was in very infancy made conscious of a spiritual presence, unseen by the eye, but not removed from communion with man. He was appointed by his invisible teacher to warn the high priest that the sacred office was about to pass from his family, and that defeat and slavery were in store for his nation. And by indications, of which this is a specimen, the historian informs us that Samuel was understood by all Israel to be a **PROPHET**. We have thus clearly and simply pointed out to us the meaning and object of a function which

Grounds of
absolute or
universal
morality as
first appre-
hended by
the Jews.

Moral and
Metaphy-
sical Phi-
losophy.

had hitherto existed in connection with other offices, but which now first became the characteristic of a distinct and recognised class. The prophet, as such, was neither a legislator nor a civil officer, nor a minister of the tabernacle, though he might by accident be any one of the three. He interpreted the movements of society, set forth the meaning and spirit of the Divine law, showed how the ignorance and transgressions of rulers, priests, and people lead to national ruin. He instructed his countrymen in the fact, that there is a great moral order in the world, which men may violate, but violate to their own ruin. Thus he carried men into a region above positive law, into the deep and inward principles of law, into their connections with the heart of man, into their effect upon his actions. That society is not a machine moving by an impulse communicated to it at creation, or receiving its direction from the chance impulses of human will, but that its primary order was devised, its continual progress superintended by a living person, from whom all law had proceeded, in whose character the truth of all law is embodied; who rules over all the operations of nature, but communicates to men a knowledge of his character, that they may become conformed to it, and of his purposes that they may be his vicegerents in executing them; these were the grand foundation of the prophet's light and teaching. But this high form of personal and political morality would have been worth nothing if it had not been grounded upon a revelation distinct from all that had preceded, though harmonious with them and rising out of them. The belief of a Divine Word dwelling with man and teaching him the ground of his life, in knowing whom only he could rise to the height intended for him, and become properly a man, the ideal of that wisdom and goodness, of which the Jew had seen the indications in the fatherly discipline of the God of Abraham, of which he had heard the announcement in the awful name spoken to Moses in the burning bush, and repeated in the thunders of Sinai, was that which raised Samuel to the dignity of a prophet, and which constituted all his successors the living examples as well as the formal teachers of the people. How these great politicians and profound moralists were taught to express their thoughts, and why they took the form of poetry, are subjects for future consideration. The instance of Samuel and some of the earlier prophets who left no written records enable us to ascertain the essential characteristics of the office before we examine its accidental peculiarities. But another great change was also in preparation, another great step in the political and moral development of the Jewish nation, of which the great crisis to which we have adverted was to be the precursor, of which the institution of the prophetic order was the necessary accompaniment. If our readers have entered into our view of the moral and political ends for which the Hebrew commonwealth was instituted, they will be at no loss to understand the necessity of asserting for a time the naked majesty of law, stamped with the awful sanction of a Divine revelation. Without such a discipline, the grand idea of a theocracy, which is the basis of all political ideas, could never have been received into the mind of that people or any other; neither natural order nor personal morality could have had any practical existence. For national order without positive law is a contradiction of terms. Morality which is not grounded in the belief of absolute righteousness is a phantom of opinion and fashion. But the remarks which we have

VOL. II.

made in our last section may enable our readers to see how important it was to the political system of the Jews, no less than to the ideas which that political system embodied, and which through its means became a portion of the national mind, that at some period the people should be taught to attach the feeling of permanence to the person who administered its law, as well as to the law itself. In no other way could they have learned to feel themselves one people with their forefathers and their posterity. Formal law, however deeply engraved on tablets, or written on parchments, can never convey this impression. It gives each man the sense of his own personal responsibility; it does not communicate to him the sense of union and fellowship. This sense was partially created in the Jewish mind by patriarchal institutions and memorial feasts. Those institutions resting on a recognition of a God of Abraham, Isaac, and Jacob, preserved a recollection of the *past* of each age. But that recollection is imperfect and dream-like, till it is upheld by other institutions which connect both past and present with the future. Such an institution is an hereditary monarchy. An hereditary priesthood had already been established; for from the first it seems to have been intended by the Divine lawgiver to put that strength and permanence on the ecclesiastical institutions of the commonwealth, which its civil institutions were only to acquire by degrees.

A breach in the straight line of sacerdotal succession, the rule being broken for the sake of the principle for which it was established, was one of the indications of the great political crisis which Samuel had foretold. Another, which proclaimed that it had arrived, was the loss of the ark, the symbol of the Divine presence, of national unity. For a period after its restoration Samuel succeeded to the office of judge; his character we have already seen was invested with a dignity which had not belonged to his predecessors; he was, emphatically, the prophet-judge. The two offices were closely connected. The power of distinguishing the boundaries of right and wrong, which appertains to the interpreter of the law, implies, according to the Hebrew idea, that discernment of the mind and will of the lawgiver which is the especial privilege of the seer. Samuel was therefore the perfect judge, fulfilling the meaning of the institution, just as it was about to pass into a higher one. His sons, we are told, did not walk in his steps, and then the wish for a monarchy first arises in the minds of the people. The feeling which dictated this desire is very clearly manifested. The ill conduct of their judges, and the decay of their own faith, led them to regard the majesty of law as a dream, the awful name of an invisible lawgiver as powerless. They longed for a leader of their armies; for one who should not administer the law, but be himself the law. With so slavish a state of mind, reverence for God, the feeling of national unity, the regard for national institutions, was incompatible. The principle of idolatry was in the heart of the people; it must soon have found its expression in all the outward forms of idolatrous worship. The presence of the visible ruler would destroy all faith in an invisible righteousness. Power, not justice, would be the object of their fear. The appointed moral and political teacher of the nation expounds to them the sinfulness of their wish, tells them its consequences, and yet permits them to gratify it. For though itself the offspring of caprice and sensuality, it was the index of a real want in their minds which their Divine teacher had himself placed

Moral and
Metaphy-
sical Phi-
losophy.

4 c

Moral and
Metaphysical
Philosophy.

there; for which he had intended in due time to provide the satisfaction. But first it was necessary that the people should be taught by a terrible discipline what a monarchy created by self-will and upheld by self-will really is; how real that law is which they had supposed only to be an imaginary terror; how it derives its reality from the reality of the awful and invisible lawgiver; how wretched and precarious personal and social life must be where that name is not recognised, and feared, and delighted in. The history of Saul is a most important chapter in moral and political philosophy. From first to last it is an illustration by appalling facts of the connection between an arbitrary government and an indifference to high principles in the people. A slavish habit of reverence for mere power producing national cowardice; this temper in the king producing caprice, vulgar artifice, indifference to great moral and political distinctions, and unconsciousness of any vocation or duties. The two great crimes in the early part of his reign, that of acting as the mere selfish invader of a neighbouring territory, instead of feeling himself the minister of God's justice upon a tyrannical king and guilty nation; and that great civil atrocity of confounding the priestly with the kingly office, are texts on which the political student should meditate long and carefully; on which he will find the history of the ancient and modern world is the instructive commentary. The lesson is not less profitable because it is obvious from the history that Saul had no deep-laid plot to establish an arbitrary government. Carelessness of old landmarks, a habit of thoughtlessness, and indistinction, and self-will, without the least settled intention of evil, were sufficient to make him an oppressor, and destroy the moral feeling of his subjects. But during this self-willed reign the connection of moral and political principles was asserted positively as well as negatively by an exhibition of the blessings of faith, as well as of the miseries of disobedience. In the personal history of David we have the counterpart of the life of Abram. As the one was trained by a singular course of discipline to apprehend principles which were to fit him for being the founder of a family; the other, by a still more detailed and remarkable discipline, was prepared for his future position as a ruler of the nation. Every step in his various and adventurous career was contrived to call forth in him a sense of his personal insufficiency, confidence in the continual succour of a Divine protector; a feeling that he had received from Him a vocation to a mighty work which could only be carried on by His help; a desire to be the image and representative of this Being, in the care which he exercised over His creatures. Thus he was prepared to be the first of a line of sovereigns reigning by Divine covenant and institution, upholding the permanence and unity of the nation, acknowledging the Divine law as the will of all their proceedings, communing with the righteous and invisible Lord, and presenting Him to the people, continually upholding their hopes that He would manifest himself to them yet far more perfectly; would establish a righteous and everlasting kingdom over the whole earth. Such is the idea of a Jewish sovereign, partially exhibited in each king, just so far as he was a faithful and understanding man.

David is always held out to us as the most complete, though still a very imperfect realization of it. In him we find the link between the characters of the prophet and the king; the one exhibits that same moral order

of the nation in acts, which the other expounds in words. The very existence of the king is a prophecy, and the prophet lives to interpret the relations and duties of the king. He is a link between those civil and ecclesiastical functions which must never be confounded, never separated. He cultivates in the nation the feeling of that connection with an invisible Being, which is implied in all the acts of the king, without their contradictions. He is the recluse and the statesman; in one he lives in the most awful and mysterious contemplation, and yet there is no one of his contemplations which does not bear upon the most ordinary transactions of life. In David we have the essentially moral, and the essentially political character, harmonized in one, yet not so lost in each other, but that when the king departs from his right position and breaks the laws which he is bound to obey, the separate function of the prophet shall come forth in another man to rebuke him.

And now that the nation realized the full sense of its unity in the person of its monarch, now that it could look back with delight upon its history, feeling itself the inheritor of the covenant with Abraham, Isaac, and Jacob; with delight upon its law, seeing it honoured and upheld in the person of a living sovereign, and with delight upon its prospects; for the future was pregnant with a race of monarchs, and with One in whom all their glories should be concentrated, and their dominion extended over the earth; poetry came forth with mightier power than at their first deliverance from bondage to be the expression of their thanksgiving, their conflicts, and their confidence. The king himself, he who gives the one heart to the nation, is also the Coryphæus through whom that heart makes its feelings known. That the whole of the book of Psalms is not Davidian is true, but popular language happily expresses the feeling of the collection by giving his name to the whole of it.

The idea of an invisible righteous Ruler in whom the nation subsists, in whom each member of it feels and realizes his own dignity, against whom all selfish tyrants, all the powers of the world, are striving, who upholds each one who trusts in Him against their enemies within and without; in whom the weakest may find a refuge, apart from whom the strength of the strong is of no avail; whose righteousness shall at last be demonstrated to be mightier than all the energies which oppose it; for the continual acknowledgment of whom his image is presented to the nation in every anointed religious king; of whose permanence the prolongation of a dynasty through a number of dying persons is a perpetual witness; who shall finally be shown forth perfectly to man in a perfect humanity;—this may be said to constitute Hebrew poetry. The conflict of naked power with righteousness, of the visible with the invisible, of confusion with order, of the devilish with the Divine, of death with life; this is its subject. And because this is the subject of all human anxieties this book has been that in which living and suffering men in all ages have found a language which they have felt to be a mysterious anticipation of and provision for their own especial wants, and in which they have gradually understood that the Divine voice is never so truly and distinctly heard as when it speaks through human experience and sympathies.

The reign which followed that of David is closely connected with it, yet it is in some sort the commencement of a new era. The idea of a righteous king was

Moral and
Metaphysical
Philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

that which, amidst all the infirmities of his character, and partly by means of those infirmities, David was appointed to present. No mere feudal monarch, yet asserting the principle of hereditary succession; no head of castes, but the spring and fountain of all the moral, and intellectual, and practical energies of his people, who gave the impulse and direction to their devotions, imparted consistency and meaning to their political order, diffused a sense of justice as the foundation of power over the land, led their armies to victory. It was agreeable to what we might almost call the natural order of things; though we shall have written much in vain if we have not led our readers to feel that human events do not follow a natural, but a moral order; that in the son of this great prince, supposing him endowed with his father's principles, peaceful magnificence should be more remarkable than the higher and more active qualities which he inherited. In place of a warrior poet, possessed with the idea of a righteous Being, and seeking to present that idea in our actions, we have in Solomon the great statesman, meditating on the principles of human actions and feelings, and the relations of the different parts of the system of things to each other and to man, and to their Divine Author—full himself of the idea of wisdom, and holding it his highest glory to hold communion with that wisdom, and to set it forth in his conduct. The books of Proverbs and Ecclesiastes may be spoken of as that which answers most formally in the inspired volume to what is elsewhere called philosophical speculation. In the very beginning of the book, the great problem which is expressed in the word philosophy is announced as the subject of it. It is the search after wisdom and understanding; not in any sense after the wisdom and understanding which are seen in nature; on this we are told that the monarch had meditated, but his thoughts upon it could have had no place in the history of the Divine revelations, and of the education of man into a knowledge of himself and of God; but after the sources of that political wisdom and equity by means of which the character of God is revealed, and the thoughts of man awakened; a search therefore after the foundation of our personal and our social life. Two powers are introduced as contesting for the heart of man, the harlot sense, the phantoms and attractions of the outward world, and the Divine word or wisdom. No abstract notion of propriety or excellence, no formal rules of law, but an actual person, with whom it is possible to hold true and real fellowship, is claiming for its own the man whom a thousand seductive influences are seeking to destroy. If he embraces the one alternative, he becomes a thoughtful, reasonable man, he understands himself, he feels his connection with the beings around him, he enters into sympathy with the order and harmony in which he is placed, he enters into communion with the Being who created the earth and the fountains of waters, who gave to the outward world its form and its beauty, but whose delight has been with the sons of men, who created them for converse with himself. If he takes the other course, he sinks into slavery and death; all personal consciousness, all moral dignity, at last all intelligence, are buried in sensuality, cowardice, superstition. The preface to the book is the key to its contents. The idea of a power upholding the moral order of the world, and leading man by different steps into the apprehension of it, and of a power of nature or death acting on man to degrade and sensualize him;

of two orders of men, one taking the upward, one the downward course of life; freedom and happiness being the privilege of the first; disorder in the faculties and affections, and the final destruction of both being the consequence of the other, may be traced through the whole of it. It is brought out in the most practical manner—in reference to the highest and lowest transactions of man, to the throne of the king, and the scales of the shopkeeper. In every event and act of life the contest between light and darkness is shown to be going forward. Every step may be one which exalts or depresses the character, which weakens its capacities for good, or cultivates and ripens them. It is a book throughout of action and business, continually affronting the notions of those who think that there is one philosophy for the schools and one for the hourly friction of life; showing that the highest principles of philosophy are those which are most real and most directly applicable to practice. It would be manifestly absurd to deduce a metaphysical or moral system, in the ordinary sense of the word, from this book. But, perhaps, we shall find that some portions of the Greek philosophy have scarcely suffered less from the attempt to reduce them into a mere set of dead dogmas than this Hebrew philosophy would. That which is truly required in philosophy, that it should keep one end in view, that it should regard that end as real, yet as one of search or discovery to each new student; that it should lay down a method for that search, that it should show how facts and phenomena of the most various and apparently contradictory kind explain themselves when they are referred to this one principle; these characteristics are all found in the book of Proverbs. Its other great peculiarity is the antithesis of its different sentences. This is no mere accident of a style of writing, it is essential to the idea which the writer seeks to develop, and it is connected with some of those observations of contraries which we shall see hereafter exercised the minds of all the Greek philosophers, from Pythagoras downwards. We have no hesitation, therefore, in setting this down as a Divine type of the future labours of men in this direction. In comparing it with the book of Psalms and with the Prophets, we may discover the formal difference between poetry and philosophy, and may see at the same time how nearly their objects are connected, how inevitable it is that they should often be confounded. The poet contemplates every thing as moving, acting, living; he feels the necessity of permanent forms and principles, not for their own sakes, but as the only condition of their life. The philosopher, as such, inquires primarily after the ground and substance of things, and regards their life as the essential condition or attribute of their substance. Each, therefore, has one vast province assigned to him, not by accident, but by Him who is the all-perfect reason and the founder of all order, and each by the same law has his own prescribed mode of expression, which by degrees he discovers for himself. Verse, which, as we have seen, is the first garb of human thoughts, after they have thrown off the loose drapery of conversation, that which in union with music expresses the sense of a living harmony and orderly succession in things, is soon found a most inapt vehicle for describing those fixed forms and accurate distinctions with which the philosopher is concerned. He therefore finds out a method of discourse in which the conversational manner of the historian is combined with the dogmatic dignity of the lawgiver. He is

Moral and
Metaphy-
sical Phi-
losophy.

the inventor of those rigid principles of discourse which are again expanded into a style, and mixed with some poetical elements by the later historian, and which suggest the formal limitations as the poets suggest the living laws of grammar. The idea of philosophy must impregnate all poetry which arises after the commencement of civilization, or it is in danger of becoming the reflex of temporary forms and passions; but poetry never can be written for the purpose of expounding philosophical principles without losing its proper nature. On the other hand, the living feelings of poetry must impregnate the mind of the philosopher, lest, instead of observing substantial principles, he should only take notice of the dead shapes of things; but the philosopher cannot aim at the vital forms and expressions of poetry without forgetting his own true function.

The book of Ecclesiastes, the other great specimen of sacred philosophy, is also not poetical. In the Proverbs, wisdom is brought before us as a personal object, wooing the man to its embraces, delivering him from all his seducers, satisfying him with its own perfect love. It is addressed, as we have seen, to a man entering upon life who has as yet had no experience of good or evil influences.

In the book of Ecclesiastes we have the mental biography of a man who has merely exulted in knowledge as a power, which was not to deliver him from the world, but to make him the master of it. In the very first verses of the book, he tells us the subject of it—it is vanity; the world and all its proceedings regarded out of their connection with God; man attempting to recreate by his wisdom that which he finds established—making himself the centre. The maxim of the book is this, you can find no centre of things in the world around you, you yourself are not the centre of the world; you must refer yourself and all things that you see to a Being above you, or all is vanity and vexation. When we come to investigate the problems of Greek philosophy we shall find how useful this experimental book is in the exposition of them. Many of the forms of thought prevalent in the schools of Greece have their germ here. We have not the theory, but we have the feeling and experience which were its parents. The sense of restlessness and confusion in all things continually encountered by perceptions of a mysterious order; the alternate attempt to glorify the senses and the intellect at the expense of each other; the generalization of the rules of life from experience which seem satisfactory, and then, as new facts present themselves, give place to their contraries; the attempt to find a key to the problems of nature and society, and the constant discovery that there is something in them which baffles our speculations. These are hints which every student will at once acknowledge to be the heads or titles to long and weary chapters in the history of Moral and Metaphysical Philosophy. In fact, the more we study the dogmas of sects, in different periods of the world, and attempt, as Mr. Hume has done, in reference to later Greek speculations, to find out the facts of experience and the prominent feelings which gave the character to each, the more we shall acknowledge the truth of this book, and its utility in translating mere notions of the understanding into the real language of the heart.

Among the poetical books, the book of Job is the one which most nearly approximates to the philosophical character. On this account partly many modern writers sup-

pose that it belongs to a much later period of Jewish history than the time of Moses, or even of David. The usual answer to such an argument, that it might please the Divine wisdom to give that form to even the earliest writings is clearly inadmissible, at least by those who recognise the principle of a Divine education, and perceive throughout all these books the traces of a wonderful order in that education; and as either hypothesis must depend merely upon internal evidence, the authority of the Jews being, it is believed, not decisive either way, there can be no reason for retaining an opinion respecting the early date of this book, which was probably itself a departure from some older one. Be that as it may, the book of Job presents us in a dramatical form with a history of a man, the specimen of his own nation, the specimen of humanity, struggling under an overpowering weight of calamity to discover the cause of human misery, how its existence is compatible with the mercy of God; the sense of inward evil, fighting against the feeling that in some sense he is a righteous man, surviving amidst patient hope, doubt, and despondency.

The reader is taught in a preface or prologue that he is about to view the conflict between the spirit of good and evil, each wrestling for the heart of a man. The man is then introduced in his lowest depth of suffering, still holding fast his faith in a God of justice and righteousness. In his misery three friends come to him, who set before him the vision of a God of mere power, delighting to baffle and confound his creatures, and between whom and them there exists no bond or sympathy.

In the vigour with which Job repels this view of the character of the Being whom he worships consists the grandeur of his arguments, in his attempts to reconcile this idea with actual appearances, and in his uncertainty whether to call the evil or the good within him, *himself*, consist their confusion. He had acknowledged the righteousness of God, but he had not bowed before His wisdom. God answering him out of the whirlwind overwhelms him with a view of the mysterious order, disposition, and harmony of the natural world, brings him into an awful sense of his insignificance, and into a still more awful understanding that in himself, separate from the Being who had thus vouchsafed to converse with him, he was unrighteous and unclean. From this state he is restored to all his former peace, joy, and prosperity. This is the Jewish drama, and it seems to us to present the very idea of a drama. To identify it with other specimens of that species of composition would be monstrous; equally monstrous, we conceive, to reject the notion which has so universally prevailed, that it has some relation to other dramas. The nature of that relation our past remarks will have made sufficiently evident. It is as it were the essential archetypal form of dramatic writing; as if we should suppose that a general, by selecting a few of his men to go through certain evolutions for the purpose of illustrating his general system of tactics, gave the first hint for the game of chess.

Of course it is not for the purpose of explaining the nature of poetry merely as a branch of art that we make these remarks. But the history of poetry, and in a less degree than the other fine arts, is so interwoven with that of man's moral and metaphysical speculations that we can only give meagre and doctrinal views of the one, without frequently taking notice of the other.

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

The book of Job in itself awakens some of the most anxious questionings of the human heart, and yet we think its importance is not appreciated unless it be viewed as the Hebrew form of that drama which in Greece was the great connecting link between the active and speculative life of man. That which has been called the Epical poetry of the Hebrews we regard precisely in the same light, not to be tried by any inferior standard, but the ideal by which the nature of all the specimens of a similar kind of composition existing elsewhere may best be ascertained.

That mysterious person, the unseen guide and teacher of man, whom Solomon had contemplated as the source of all wisdom to kings and rulers, to labourers and handicraftsmen; this awful person, as Mr. Coleridge has remarked, "is the hero of the Hebrew Epic." How all passing events, all convulsions of empires, all changes of dynasty indicate his presiding government, illustrate the eternal laws of justice and righteousness, by which he is ruling; how he has chosen one nation to be the centre of the universe, that its kings might exhibit the pattern of his justice, and wisdom, and condescension to man; its priests, that holiness and purity of character to the likeness of which he would raise man, the whole order civil and ecclesiastical, the deep inwoven harmony of that scheme according to which all the ages of the world had been disposed; how the violations and transgressions of this order, by the perverse will of men, their refusal to be the witnesses for God to the nations, the lust and pride of rulers, the selfishness of priests, the ignorance of the people, were preparing the way for the dissolution of their commonwealth; how yet the purpose of that commonwealth would be fulfilled, because it stood not on the caprices of men, not in the faith of a peculiar people, but had its foundation in the "fact" of an union between God and man, in the person of the Mediator, in the certainty that He was the bond of all society; how this Being would finally be manifested to man; the completeness of his union demonstrated; how by transcendent arts, yet beheld only in the distance, reflected in imperfect human mirrors, the full meaning of this incarnation would be accomplished, and the foundation laid of an universal human fellowship, grounded upon the fellowship between man and God:—these are the principles by uniting together political facts with political science, yet appealing to the hearts, and making themselves intelligible to the faith of the poorest sufferer, of the most humble wayfarer—because they set forth the character of One, with whom he, as much as the ruler of his land, was permitted to converse—which found their expression in the Hebrew Epical Poetry.

Each inspired prophet appears at some peculiar national crisis to expound the principles of the Divine government in reference to that crisis, and thus to bring out into manifestation a scheme of moral order so necessarily applicable to all times, that in each new age, each particular event, however limited by the words of the prophet to the period and nation for which he wrote, should be supposed to be some one then in progress or preparation. Every new prophet in this practical way was appointed to present some new sign or phrase of the Divine character, as the one most needful at that time to be contemplated, some new principle of that scheme which can only be seen to be complete and circular when it is beheld by the light of a future dispensation, and by that light only, perhaps, when the eye has been purged from all the films of mortality.

Interval between the third Hebrew Period and the Commencement of Greek Philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

We spoke at the commencement of our discourses of those monarchies, built upon the worship of power and of nature, the commencement of which is referred by the Hebrew historian to the Tower of Babel, and against the principle of which the family life and national order of the Hebrews were a formal and continual protest. What mutations these monarchies underwent, the deeds of their monarchs, the changes of their dynasties through many centuries, are not recorded in any authentic document, and would not be worthy of repetition if they were.

Till some principle gave vitality to their dead tyrannies there could not be enough of social order or moral life among them to produce records. Nearly all the prophecies are connected with these monarchies and their relation to the Jewish people, but it is obvious that the Chaldean dynasty, whenever it may have been established, was very different in kind and character from those which had preceded it. A race of magicians, the study of astronomy, magnificence and decoration in buildings, distribution of provinces under Satraps: all these are indications of a different stage of society, a character of humanity which it is impossible that any of those earlier savage despotisms could have possessed. To what were such habits, studies, and institutions owing? and do they correspond to the third stage of Jewish education, as the changes in Egypt did to the first, and those in Phœnicia to the second? Every thing we see in the Jewish mind at this time was leading to the idea of a mysterious Being, presiding over human society, directing the movements of the external world, ruling and disciplining the hearts of men, and binding them together. There was an universality in the views into which the Jews were now led, which, though it had grown out of their former relations, was very different from them. The parody of this universalism, that false image of it, which a nation not agricultural like the Egyptian, with no facilities for commerce like the Phœnicians, living under a clear and beautiful atmosphere, used to absolute government, and educated in notions of conquest might exhibit, is exactly that which we know to have been the worship and the government of the Chaldeans. Far removed from the animal idolatry of the Egyptians, almost as far from the human worship of the Phœnicians, the Chaldean saw in the powers of the world, and especially in the stars over his head, the images of that all comprehending and all observing superintendence which the Jews were taught to attribute to the unseen God of righteousness, the God of their fathers. These mighty dynasties, who ruled the world below, exercising so great an influence over all the operations of nature, must likewise have some mysterious connection with the destinies of a being, who, even while he was humbling himself below the powers of the world, could not help feeling himself more important than all. Hence the hint for a science of astrology, built up partly on an observation of natural phenomena, partly on principles connected with the heart and conscience of man—projected outwards to meet these phenomena—partly upon that recognition of an absolute Divine supremacy which the Jewish faith, acting upon old Chaldean notions, had called forth. Such worship will always be connected with grander conceptions and occasionally with a deeper prostration of mind than either of the other forms can be. It will

Influence
of Hebrew
wisdom in
Chaldea.

Moral and
Metaphy-
sical Phi-
losophy.

be compatible with wide schemes of conquest, with magnificent cities, with buildings, not suggested by notions of utility, but claiming admiration for their own sakes. But in those forms of art which refer to the countenance and proportions and life of man it will be deficient; it will have (in this respect resembling the Egyptian) a sacerdotal, not a popular literature; all those energies which lead to self-reflection will not so much be crushed by outward power as never awakened. The principal qualities developed will be a sort of political sagacity or cunning; the principal moral virtues will be magnificence and clemency. These are something like the characteristics noted in the Scripture as belonging to those Asiatic monarchs, of which the Chaldean may be called the first specimen; the first, that is to say, in which the savage features of the Babel system were softened and subdued by a more mild and human expression. The thoughtless and ignorant Jew who travelled into these lands, and admired their fine palaces and august natural temples, little knew how much the neighbourhood of his own despised tribe tended to produce the cultivation which raised those palaces, and inspired those grand conceptions; little knew when, on going back to his own people, he imported the Chaldean speculations and the Chaldean horses and chariots, how much he was labouring to undermine the principles of that cultivation, and to destroy the only principle which could ever uphold it, or give it further extension. The neighbouring idolaters of Syria, the intercourse with Egypt, too frequently resorted to as a protection against the older Assyrian monarchy, had done much to sensualize and to animalize the Jewish people. Family life had decayed through popular grossness, through neglect of the patriarchal institutions—of the memorial of feasts and holy traditions which connected them with the past—national law and order had been undermined by the commercial spirit. There wanted but this Chaldean imposture, the substitution of grand notions of nature for a belief in God, to complete the overthrow of the national character. The corruption advanced gradually, amidst warnings and exhortations, embodying the deepest political wisdom ever expressed in human language. At times, through the instrumentality of a righteous king, sweeping away the forms of superstition, and restoring the forms of the individual worship, and drawing them to their proper centre, it was arrested. But when this Chaldean element had been introduced, the complaint was near the vitals of the state. The destruction of groves and high places might remove the temptation to Phœnician idolatry; this worship of nature was inward, consummate apostasy. A subjection to the people in whom it had reached its height; an experience of that all-crushing despotism which always had coexisted, and always must coexist with that system of faith which recognises no real bond of sympathy between the worshipper and his God, was the method which the Divine wisdom chose for the discipline and restoration of the fallen nation. During that period of captivity the prophet Ezekiel felt, with no exaggerated nationality, that the glory of the earth was departed, that no one formal, practical witness for righteousness, and truth, and order was left upon the earth, and that nothing but a faith which, nourished by the forms of his country's worship, enabled him to look beyond them, could withstand the dreadful conclusion, that mankind would be henceforth left without a remembrance or

a hope to choose its own delusions, and be the victim of its own ignorance.

The Scripture history, however, which has traced so far the progress of human education, does not leave us to fancy how these melancholy dreams were refuted, and the faith which the holy man was enabled to maintain in spite of them confirmed. A book, the removal of which from the canon, according to the peremptory sentence of some modern critics, will at all events leave a most inexplicable chasm in a history which has hitherto been continuous, and which, with this addition, and that of the two or three prophets after the captivity, offers an intelligible link between the records of the chosen people and those of other nations subjected to a different discipline—this book shows how that knowledge, of which the Hebrews were the depositories, influenced the mind of Chaldea. All such influences must of course have acted directly upon the monarch, and through him upon the people. The discovery of a wisdom speaking to the heart of man, which surpassed and overreached the superstitious guesses drawn from the observation of phenomena; the grand idea of a Being ordering the course of empires, not by the motion of the stars, but with reference to a moral end; the idea of a Being of foresight and order, interested in man more than in nature; an object of his trust, and able to protect him against the powers of nature and man—these, with a severe and harmonizing discipline, are said to have wrought upon the mind of Nebuchadnezzar, to have brought him to honour and fear the God of righteousness and not of nature, and to have diffused a mild and gracious spirit through his government.

Although it is not the proper business of this Treatise to defend the genuineness of these documents, or of those which sketch the history of Asiatic monarchies, from the Chaldean, through the Medo-persic of Darius, and down to the Persian of Cyrus, that branch of the Alexandrian monarchy which was established in the persons of the Seleucidæ; yet we may be permitted to remark, that if the forger of these histories and prophecies lived, as is supposed, in the reign of Antiochus Epiphanes, he possessed a most miraculous faculty of clothing his invention with the forms of an age in which he was not living, and a most unhappy talent of making it inapposite to the period in which he wrote. The Syrian monarch was Hellenized throughout. Greek worship, Greek taste pervaded his mind and kingdom, and these it was his ambition to set up in Judea. But the scenery of the book of Daniel is throughout Oriental. The spirit of Asiatic monarchy breathes throughout it. To suppose the heart of the Maccabees stirred up by such a composition is reasonable enough, if it were found among their sacred books. But the man who should have set down to write it for the purpose of exciting them must have been a madman as well as a liar, incapable by his dishonesty of patriotism, no less incapable of arousing it in others, by his ignorance of the feelings and circumstances to which he addressed himself. Whether either of these suppositions are consistent with the method displayed in the book, a method as fully recognised in the objections of Porphyry as by any Christian panegyrist, and whether consequently the difficulties from internal evidence, which has been chiefly resorted to, are not greater for rejecting the book than for retaining it, we leave for our reader's consideration. But if we admit its truth, we are authorized to assert that, as a fact which reason and experience

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphysical
Philosophy.

would have warranted us in expecting, that as Jewish influence gave her early feudal institutions to Egypt, gave that impulse to Phœnicia which converted it into the first mercantile nation of the world, so it also was the means of converting the Babel monarchies of the East into those comparatively humanized and orderly despotisms which Aristotle did not consider unworthy of being treated as organic polities.

Thus, then, we have taken a view of the progress of Moral Philosophy in that nation in which we believe it to have been cultivated by a Divine Instructor. We have seen in what way family life, political institutions, experimental history, and the direct teaching of ecclesiastical ordinances trained the chosen people to a knowledge which raised them above nature, enabled them to realize their position as men, brought them to a sense of the law under which man exists, of the tendencies which dispose him to violate that law, of the power by which he is preserved in obedience to it, of the consequences which result from its violation. The feeling of being brought into a system of moral order different altogether from that natural order to which he saw all things around him subjected, yet one not less real, not less permanent, not less, but more emphatically, the expression of the will of a personal Being—who was leading him up, through the framework of outward laws and institutions, into the apprehension of a still more deep and mysterious order, in a still more universal sense belonging to man; the manifestation of which would in some remarkable way be connected with the manifestation of himself—was wrought not by violent inculcation but by gradual processes of experimental discipline into the heart of the Jew. The grand and awful question respecting the nature of evil, its relation to man, its relation to nature, its relation to God, was made known to him by the same living process. Habits of reflection and meditation were, as we have seen, not the ornaments of his character but parts of its very substance. His practical life was a supernatural life, he could not act without thoughts, which carried him above nature, above himself; he could not fight battles, or rule kingdoms, or pronounce the most trifling judgments, but under a sense of mysterious government and fellowship. Yet he was a man of ordinary vulgar passions, not different in any one respect from his fellows, teaching them as much by the sensuality into which he constantly sank, as by the heights of wisdom and moral grandeur to which he was afterwards raised, the mystery of manhood, that debasement is not its law, that dignity is not its nature, that a man may be brought into such conformity, such sympathy with his law, that he shall feel dignity to be his rightful inheritance, debasement his natural bondage; that this state is only possible when he beholds his dignity in a Being above himself, united to himself and to his fellows; when he can disclaim every thing selfish as not his own, as the relics of an imaginary and false condition out of which he has been delivered.

GREEK PERIOD.

Preliminary Observations.

Commence-
ment of
Metaphysical
Philosophy.

If we would distinguish the subject on which we are about to enter from that which we have been examining hitherto, we must not content ourselves with saying that the Hebrews were the subjects of a Divine education,

and that the Greeks were left to follow the workings of their own minds.

Moral and
Metaphysical
Philosophy.

The evidence of all history, had we no higher witness, would lead us to believe that the spirit of man is never awakened to any earnest or continuous inquiries but by an influence superior to itself. From the Hebrew Scriptures we learn that all human society is so constructed as to be an instrument by which the Divine Teacher acts upon man. Human relationships, laws, political events, are (as we have seen) the agencies whereby man is led to seek after a knowledge of his Maker and of himself. Wherever these exist, we may be sure that this higher influence is at work, whereby their existence is recognised and felt to be important, there we may be sure it is strongly at work. We are bound, therefore, to suppose that it was not wanting in Greece, and the earnestness and continuity of the speculations in which her sons engaged add irresistible force to this conviction. It is not in the absence of Divine cultivation, but expressly in the absence of that kind of cultivation which takes place by means of a Divine revelation, that we seek for the difference in the two nations. The mind of the one is led to seek after some centre to which all that it beholds without, all that it feels within, may be referred; the other has a centre given with which it is taught to connect every observation and every conscious act; the one gropes in all directions after its object, the other is restricted to meditations on the relationships of man; the one ardently desires to understand the meaning of the powers which it finds in itself, whence they proceed, and to whom they belong, and under what limitations they exist; in the other the sense of power is either merged in the feeling of responsibility or else realized only in the outward acts which manifest it. We have here indicated the characteristics of the Greek philosopher—he is seeking for a centre. He is seeking this centre in nature, in the instruments by which man studies nature, in his own thoughts, in institutions. He is seeking with an ultimate reference to the discovery of the nature and limitations of his own faculties. Now here, it will be seen, is that kind of inquiry which, in the outset of our sketch, we called *metaphysical*, and which we distinguished from *moral* by this characteristic, that it was suggested by the desire to discover the foundation of our *powers*, not the foundation of our *relationships*. The Hebrew and the Greek we consider the originating points of two great lines of thought. We shall find these lines continually approaching, moral discoveries implying metaphysical truths, metaphysical theories seeking a ground in morals: at Alexandria we shall find them attempting to form a *speculative* union; in Rome we shall see an unsuccessful, though very interesting, experiment to combine them in *practice*.

Before the history of ancient philosophy closes we shall endeavour to show where they did find an actual meeting point; and then we shall be able to trace with a greater feeling of security, and with a better hope, the course of modern philosophy, seeing more clearly both the point from which it starts, when it is moving with retrograde steps, when it is advancing, and what is its ultimate goal.

Having made these remarks, it is scarcely necessary to explain why we do not assign a separate division of our work to the philosophy of India and the other Oriental nations. We have proposed to ourselves the task of tracing the progress of thought and inquiry. Now the sacred books and poems of India may contain

Source of
Greek Phi-
losophy—
India.

Moral and
Metaphy-
sical Phi-
losophy.

vast materials for thought, the results of many speculations, but no one will contend that they afford us the least clue to a discovery of the method which the human spirit has followed in its search after foundations of things. We hold it better, therefore, even for the purpose of illustrating these remains or deposits of an unknown antiquity, that we should not be tempted by them to leave a path in which we can trace a regular series of clear and definite footmarks. They will be understood far better when they are contemplated as the chaotic mass out of which more harmonious speculations have been disposed, than where they are studied in their own rudeness and incoherency.

To the purely physical philosopher the cosmogonies of India must always have an interest because they are obviously attempts to explain the meaning of observed phenomena. They cannot be neglected by any writer on theology, for they exhibit one of the forms in which the feeling of a superior power and the necessity of worship find expression. The moralist is struck by observing how thoughts of justice and goodness insinuate themselves into this physico-theology, and give it all that it has of consistency. The metaphysician observes with a more melancholy interest the hopeless attempts of the spirit of man to disentangle itself from the confusions of sense. But neither of the last can feel that these cosmogonies directly belong to his province, for they neither indicate a search after the law of our connection with other men, nor after the nature of the faculties which we find in ourselves.

It is, however, a question of some interest whether the Greek philosophy is at all derived from these Oriental sources. On the doctrine, which was formerly much more prevalent than it is now, that the Greeks were indebted for part of their knowledge to Hebrew seers, we need not say many words. Were the notion built upon any alleged facts, it would become us to consider these facts, and only reject the conclusions from them, if we found that they were not proven, or that they were neutralized by more powerful facts on the other side. But as no one has pretended to do more than affirm the possibility of an intercourse between certain Greek philosophers and certain of the inspired writers, or the possibility of their attaining to a secondhand knowledge of the inspired books, we have only to encounter the theory which has given these affirmations plausibility, that unless we admit them there is no way of explaining how the Greeks became possessed of their wisdom consistently with our reverence for the Jewish Scriptures. Now we stated by anticipation our objections to this theory, when we contended, first, that these Hebrew books themselves most clearly and consistently explain how the Greeks might acquire all the thoughts which they did acquire without any external teaching; and, secondly, that we can best understand these thoughts and those which were imparted to the Hebrews by supposing them to have started from different points, and to have proceeded in a different direction. The other controversy, whether the Greek speculations can be traced to an Indian, or if not to an Indian, to any Eastern source, must be settled by an examination of Greek mythology, for through this, and not immediately the foreign influence, must have reached the Greek philosophy. Such an examination is on other accounts of importance; for though mythology in itself, and the period which is called the mythical or heroical, cannot, for the reason we have stated, be considered as belonging

to the history of human inquiries, yet the influence which it exerted upon the later development of the national mind ought not by any means to be overlooked.

The mythology of Greece has three stages, answering to those we have taken notice of in the Hebrew education, and which we have seen respectively paralleled in Egypt, Phœnicia, and Chaldea. Professor Müller, with evident truth, has remarked, "that we may trace in the mythology of Homer clear indications of an earlier worship." This he supposes to have been the worship of natural objects. For general reasons, which we have stated at large already, we object to this doctrine, and we do not think it is borne out by the evidence to which he himself appeals. In the phrases "Father of gods and men," and all those phrases and stories which ascribe family relationship to the gods, whether among themselves or with men, and not in the epithets "Cloud-compeller," "Thunderer," &c., we discover the leading characteristics of the older and simpler faith. Respecting the second stage there is no controversy. It is clearly the national or political stage. The council chamber of the Argive chiefs is transfigured into the council chamber of Olympus; the gods are kings and chieftains; they acknowledge laws and enforce them. This is Professor Müller's own opinion, though by what means such a worship could have been superinduced upon a mere natural worship he does not undertake to explain.

In neither of these stages of religion can we trace the slightest resemblance between Indian and Greek ideas. In India there is the constant want of a human moral foundation; in early Greece there is scarcely the slightest attempt at speculation upon the origin of world or the principles of things.

If we take one of the clear harmonious pictures in Homer, and compare it with the monstrous conceptions of the Indian cosmogonies, we shall be at once convinced that no possible resemblance can have existed between the spirit of the two nations at this period, or the sentiments in which it expresses itself. The Greek himself, as we have already hinted, was more successful in discovering a connection between his own religion and that of other nations. It may be quite true that the family mythology sprung up in the mind of the Pelagian herdsman without any external culture. Still it had its counterpart in the faith of the Egyptian herdsman. It may be that the Greek citizen had no need of being taught by others to worship Divine legislators and monarchs. Still he might find a similar worship among the political Phœnicians, only there was this remarkable difference, that he did not stand still at the Egyptian point of cultivation, losing that which he had attained in animal grossness; that on the other hand he did not commence at the point of civilization from which the Phœnician thoughts took their rise, and thus sink into the mere cunning commercial sensualist, but that the national branches grow out of a fair and family trunk, which contained indeed within it the seeds of its own decay, but contained also a living and healthful sap. To this difference we consider it as principally owing that the Greek was able in this second stage to clothe his thoughts in manly and lasting poetry. But as these two steps of Greek mythology are obviously in no degree connected with the mythology of the East, so likewise they produced no direct effect upon the philosophy of the nation. We cannot determine, therefore, till we have

Moral and
Metaphy-
sical Phi-
losophy.

Mythology

Moral and
Metaphy-
sical Phi-
losophy.

Dionysus.

considered its third stage, whether this philosophy was not impregnated with Indian ideas.

It has been often remarked that the name Dionysus does not occur in Homer. Now, if we consider only the dignity which this Dionysus afterwards attained as the patron and inspirer of the Greek drama, we shall not want any further proof that his introduction into the Greek pantheon was the symbol of a most important era in Greek feelings. And the more we meditate upon his character and offices the more we shall be struck by the new region of thought to which this new divinity belongs. We do not mean that there is no indication in Homer of the kind of feeling out of which this worship must have grown. His heroes are constantly spoken of as inspired with courage and counsel by some Divine protector or protectress; and the idea of constant communication between the Olympian gods and men here on earth, denoted by visible signs, but really carried on between the heart of the man and the Deity, is the very soul of the poem. But the acknowledgment of a Being who is emphatically the inspirer, the lifegiver,—who, therefore, becomes connected with the most secret emotions and energies of the human spirit, and whom the worshipper inevitably associates with all the movements and activities in the world which he sees around him,—this carries us out of the circle of local gods dwelling on a certain Thessalian hill into strange indefinite apprehensions of a presence to be felt and realized rather than beheld or understood. Every one knows in what grotesque forms this awful feeling enshrined itself; the god becoming identified with that which was the aptest outward symbol of internal life and joyousness, and the mind of the worshipper disporting itself in all fantastic combinations and contrasts, as if it were struggling with a thought that overmastered it, and found its revenge in inventing these ludicrous natural types of the conqueror. Much, doubtless, of all that is deepest and strongest and most contradictory in man may be found embodied in the history of Bacchus and the Satyrs. Yet this was not all. The Greek himself felt that there was something in him which no stories could tell, no pictures embody. He had need of *Mysteries*, and these mysteries spoke not to the initiated only, but to every one who had heard their name, and knew that they existed, of something about him, above him, within him, which language could not interpret.

Mysteries.

We have no thoughts of entering into the inquiry what these mysteries signified, an inquiry which would scarcely have been mooted, or at least not pursued with so much theoretical exclusiveness, if their real influence upon the mind of the people had been understood. We can easily adopt all the explanations that have been given of them, and acknowledge them all to be true, and yet after all believe that the power which they exercised arose mainly from their witnessing to the truth that there is something most nearly and intimately concerning us which is above all explanation. Consistently with this idea the mysteries may have spoken of a unity in which all separate things begin or terminate, of a Divine power which could not be expressed in images, of a Divine element in man, of the secret operations in the womb of the earth. Whatever a man found too large for his thoughts to comprehend, this, whether it were in God, in himself, in nature, became the mystery which he connected with the rites and symbols to which he was introduced, or of which he heard dim

rumours. Indeed, it is perhaps the very difficulty to know where precisely this mystery was—in nature, in God, or in his ownself—which created his anxiety; only this he knew, that it was somewhere, and that not to own it was to deny himself.

How this Dionysiac worship and these mysteries were connected with the feeling that gave birth to *cosmogonies* we may not at once be able to discover. But it would seem that the characteristic of the state of mind which this faith expressed is a feeling of the latent powers in man and in all nature. It would seem, also, that the period during which this change in the national mind took place, was one in which pictures of human political life, such as Homer presents us with, must have lost their charm, because there was nothing in the actual circumstances of men with which they could be compared. In the feeling and literature of such an age there will have been inevitably a bias towards physical theology. In such a time the works attributed to Hesiod may have been produced; for though the *Weeks and Days* and *The Theogony* may not have proceeded from the same author, the common name given to them seems to show that they belong to the same period of moral cultivation. Here, then, we find a point of coincidence between the habits of thought in Greece and in the East. But the resemblance is not surely sufficient to prove that the one was derived from the other; it is not greater than that which we have observed between the two earlier states of feeling and those prevailing in Egypt and Phœnicia. In fact, it is precisely the same kind of resemblance; in each case the Greek, exhibiting a steady and remarkable growth, and passing for a time into those conditions of thought and feeling which became the fixed and hopeless prison-houses of the people with whom we compare him.

From the moment political feelings were reawakened in Greece, that bias towards physico-theology, of which we have spoken, received a mighty counteraction. Then we hear of the Homeric rhapsodies being read, admired, collated; then the old local mythology recovered some of its former life, and the new idea which had threatened to exclude it became blended with it. Yet that idea was by no means lost. It had even acquired new power from the political life which seemed to threaten it with destruction. The mysteries were honoured and accounted most precious by the statesman.

Again, the idea which was contained in the worship of Dionysus gave birth to the deepest thoughts and the noblest literature. It informed and possessed the mind of the poet, enabling him to realize, and making him eager to express, thoughts which belonged to his kind even more than to himself. It worked also in the mind of the philosopher, making him feel that there was something in himself which needed explanation; leading him to connect the world within with the world without, and to hope that, by some means or other, by studying the phenomena of nature, by applying the new found sciences of mathematics and astronomy, or by exploring the recesses of language, he might find the solution of the enigma.

Ionic School.

About the time that Judea sank into a province of Ionian the Great Babylonian Empire, the philosophical spirit school first began to manifest itself among the Greek colo-

Moral and
Metaphy-
sical Phi-
losophy.

Cosmogonies.

Moral and
Metaphy-
sical Phi-
losophy.

nists on the Western coast of Asia, and in the South of Italy.

Many circumstances may diminish our surprise at the fact that in Asia, rather than in its proper home, the Greek genius should have put forth its first buds. The Greek colonist was on the very soil which was hallowed by the traditions of those elder days when European wisdom, exerted in a righteous cause, triumphed over the pride and luxury of Asia. Seeing himself constantly surrounded by men whom he felt to be his inferiors in sagacity and individual prowess, who formed only the elements of a massive Oriental despotism, he would be inclined to exercise all the wit which was his hereditary endowment, and which the exertion of maintaining a position in a foreign land must have ripened. He had already cultivated his native resources by commerce and by travelling. He kept alive in his new country those tribe distinctions which united the family feeling to the national. His political talents were called forth in the effort to preserve the municipal government in each city, or to maintain its combinations and its rivalries with its neighbours. The conflicts through which the mother-country was working its way to settled freedom had terminated here in the rule of particular tyrants, men promoted by their shrewdness, activity, and political wisdom; who rather called forth these qualities in others by their example than discouraged them by their jealousy. The genial Asiatic climate, which gave such sweetness and softness to the dialect of these colonists, imparted also a portion of its influence to their character. It quelled the excessive restlessness without extinguishing the energy of their minds, and thus disposed them more readily to exert that energy in inquiry and contemplation.

On the coast of the Ægean the influence of the Ionian tribe was predominant; the colonies in Southern Italy were almost exclusively Dorian. There the genius for order and legislation which belonged to the tribe had already displayed itself, and the names of Charondas and Zaleucus are surrounded with an obscure traditional dignity like that which belongs to Lycurgus.

Yet it seems that the philosophical spirit could not exert itself in either country, till the vivacious and volatile temper of the one tribe had acted upon the sober and severe wisdom of the other. We shall hear presently how an Ionian travelled into Italy, and what a new character was imparted to his inquiries by the disposition towards order and fixed institutions which he found prevailing there. And it is equally true that the commencement of directly philosophical investigation among the Asiatic Greeks was preceded by the appearance of a set of men who were evidently political sages, whose thoughts were all more or less connected with practical human life, and who embodied them in terse sentences, such as the Spartan delighted in, but were very far from the free habits of the rival tribe. And we think our readers will discover proofs in the short sketch which we shall give of Ionian philosophy, (the subject which deference to common practice, as well as a desire to exhibit the gradual movement of civilization from the East to the West, compels us first to notice,) that a human element did enter most strongly into it even when it seemed exclusively to consist of speculations on nature. The portion of it which relates to man may seem insignificant when compared with that which treats of the external world; but if we look carefully, we shall find that this is really the

centre of the whole, and that, when we refer every thing else to it, each system becomes internally coherent as well as connected with the rest.

Thales is by universal consent the father of Greek philosophy. The place of his birth seems well ascertained, there is great doubt about the date of it; but all authorities would seem to place him in the period during which the Ionian cities were in their most flourishing condition, and his own city of Miletus conspicuous among them. Some traditions would represent this circumstance as a matter of indifference to the philosopher, others speak of him as taking the deepest and most practical interest in the affairs of the colonies. These reports are easily reconciled; a meditative man will always seem a recluse to the world, even if his thoughts have the most direct application to its business. The mere fact of *Thales* being included among the seven Sages of Greece, would prove that he was not merely devoted to physical speculation, for their fame is entirely moral and political. The sayings attributed to *Thales* are so expressive of the spirit which informed all Greek philosophy, that they may seem to have been dramatically assigned to him as the traditional founder of it. But the sentences, "Know thyself," "All things are full of the gods," will at any rate prove that his countrymen supposed the foundation of his philosophy to be in some sense spiritual. Still more strikingly is that feeling manifested in his reported maxim, "That all things are open to the gods, even our thoughts;" for it is an absurd supposition of Bayle, that he uttered such sentiments merely to disguise his real opinions. Very probably he never uttered them at all; if he did not, they are indications of his character; if he did, it is a gross anachronism to fancy them insincere. *Pythagoras* concealed truths which he thought too sacred for the multitude, but only the later schools adopted any contrivances for the purpose of hiding their atheism.

Nor do we think these opinions of the Greeks at all inconsistent with the speculations which they attribute to *Thales*. Water is the first principle of all things; this was the central doctrine of his system. Now, if we may believe Aristotle, this thought was suggested to him not so much by contemplating the illimitable ocean, out of which, as old cosmogonists taught, all things had at first proceeded, as by noticing the obvious fact that moisture is found in all living things, and that if it were absent they would cease to be. It is evident from the same authority that he connected this observation with facts which he noticed in himself; nay, we think it may quite reasonably be inferred that Aristotle felt this analogy to be really the most important point in the theory. Just what this moisture is to the ground, necessary to its being what it is, yet not the ground itself, just such a thing do I find in myself, something without which this body would not be what it is; a seed of life within me, something which, if it were withdrawn, this, which now seems so firm and solid, so active and vigorous, would become a mere dry, useless husk, ready to fall into pieces. Such we suppose to have been the sense, the feeling, contained in the opinion of *Thales*. We do not mean that he looked upon water as a symbol only of his soul: to imagine this would be to anticipate a discovery which was the result of a series of gradual guesses and experiments, hereafter to be recorded. *Thales*, no doubt, believed this humour or moisture to be, as he said, the essence and principle of all things. He believed it in some sort a principle

Moral and
Metaphy-
sical Phi-
losophy.
Thales.

Moral and
Metaphy-
sical Phi-
losophy.

to which he might refer himself. But the thought which lay behind this notion, and in vain sought to compress itself into it, is the one which it really concerns us to take notice of. For thus we are able to perceive in this early philosophy a most interesting step in self-consciousness, and we obtain a valuable help towards understanding how the successive theories respecting the outward world were connected with the progress of apprehension respecting the world within.

Anaximan-
der.

Anaximander is the next in order of the Ionian philosophers, but the interval between him and Thales seems too wide to admit the notion of their being connected together as master and disciple. He was born, however, in the same city, and the speculations of his predecessor produced, we imagine, more influence upon him than some historians, observing the marked opposition between them, not only as to their outward form but as to their principle, have been willing to admit. *Anaximander* appears to have advanced in astronomical and geographical knowledge far beyond Thales; it would seem also that he was the first of the Greeks who recorded his speculations in writing. From these two circumstances we might, perhaps, have guessed that his philosophy would have acquired a new character, that the simple observations of Thales would give place to more extended observations; but that, on the other hand, the lively consciousness of what was passing within him would be lost to a great degree in the attempt to impart it through logical forms to others. Such are in fact the characteristic differences in these two early thinkers. They offer one of the simplest and most primitive illustrations of the difference between thought employed in the solution of actual difficulties, and thought employed for the construction of a theory. A fact so simple, so obvious to every child, as that every thing requires moisture for its life, appears to this observer on a large scale as pregnant with no meaning; yet seeing that an hypothesis has been suggested for explaining the facts of the universe, he too is urged to reflect, and to try if he cannot produce another more comprehensive and rational. He looks round upon the world, and sees that there are going on upon its surface millions of facts like that to which Thales attached such importance. He cannot find which of these facts is the most potent and generative, but he can at least find a name for that to which all other things owe their origin: he calls it ἀρχή, beginning. This is something; but he can do more, he can find out a name for the whole multitude of powers, principles, influences, facts that are at work in this universe; he calls it τὸ ἄπειρον, the Infinite. The person who invented such a phrase might well feel that he had reached the Hercules' pillars of philosophy. What could be said after him which he had not comprehended in his formula? We believe, however,—it is justice to *Anaximander* to believe—that there was something in his heart and mind which he himself could not comprehend in it; that this word—infinite—brought with it thoughts upon which he meditated in silence and awe, and could not express to others or to himself. The intellect which imposes upon itself words for realities, generalizations for principles, be it remembered, is still the intellect of a man; it would be falling into his own delusion to suppose that he felt no more than he could put into phrases.

Anaximander is commonly said to have made earth his centre, as Thales did water. This language, without being wholly erroneous, may easily mislead us; the earth and the water are not opposed in these two sys-

tems, as the (so called) solid element to the fluid; for we have seen that, according to Thales, water meant, not an extended separate mass, but the invisible moisture in all things. On the other hand, the earth, in *Anaximander's* mind, must have been contemplated as including both seas and land, and all the phenomena which characterise either of them. For this reason it was the natural expression of his notion of the infinite and indefinite, redeemed that notion out of the circle of mere abstractions, and formed a link between his speculations and those of his predecessors and his successors.

The doctrine of *Anaximenes*, the third philosopher of this school whose name has been preserved, exhibits a very interesting step in the progress of internal reflection. That within us which is not visible, or audible, or tangible, does not present itself to him as a mere quickening or vivifying principle; he feels it to be the ruling thing, the great controlling power which all that is visible, audible, and tangible is meant to obey. The necessity of moisture to all things is not the fact which lays hold of his imagination; still less could it be satisfied with a mere theory or index to a great number of facts. He is struck by the dominion which air exercises over all nature, how secret and invisible a power it is, yet how universally and inevitably acknowledged. The importance which he attaches to this observation is not the unconscious effect of an internal feeling; he says, plainly, that the air rules over all things, as the soul, being air, rules in man. These words which Plutarch gives us are very interesting. They strikingly indicate a period when a man is struggling to discover something in nature which answers to something in himself, and does not know which is the ground of the other. The thought of what he is himself dawned upon him while he was noticing the outward fact; that would therefore seem to him the first and most important; but again there is something in him which would not permit him to consider it the first. He must assume himself as the ground of the comparison; here there is another important advance in discovery. The philosopher is beginning to suspect that the outward object is symbolical. He has not realized the conviction, but it hovers about him and is ready for a future development. One other observation it is necessary to make respecting *Anaximenes*. The theology of Thales was by no means physico-theology: it seems scarcely to have connected itself at all with his speculations on the principles of things. He acknowledges that the gods saw the very thoughts of men, and that satisfied him. What they had to do with moisture he did not care to inquire. The logical formulas of *Anaximander* must have been far more fatal to the recognition of direct personal powers. The Infinite must have included them as well as all other things. In *Anaximenes* we discover the first attempt to combine the ideas of an elder faith with his reflections on nature. The ruling power of air, or in air, why should it not be he who in ancient times was called Jove? The thought may have caused a pious pleasure to the inquirer; it seemed to give philosophy greater fellowship with humanity. But we may find hereafter that whenever it has been reproduced, it has been equally mischievous to theology and morals, to metaphysics and physics.

The last and greatest of the Ionian school, properly so called, is *Heraclitus* of Ephesus. Of all the Greek philosophers, except Socrates, this man seems to us to possess the most marked individual character. That

Moral and
Metaphy-
sical Phi-
losophy.

Anaxi-
menes.

Moral and
Metaphysical
Philosophy.

vulgar notion of him as the crying philosopher must not be wholly discarded as if it meant nothing, or had no connection with the history of his speculations. The thoughts which come forth in his system are like fragments torn from his own personal being, and not torn from it without such effort and violence as must needs have drawn many a sigh from the sufferer. Neither is that other notion of him, as "the obscure or dark" man, an unfounded one. The fire that was in his heart and brain, and of which all the world around him presented the image, no doubt emitted much smoke which did confuse and stifle, not, perhaps, to his displeasure, the careless gazers and passers by; but there was something within him which neither his tears nor his smoke at all adequately represent. If Anaximenes discovered that he had within him a power and principle which ruled over all the acts and functions of his bodily frame, and which had no so living type as the air, Heraclitus found that there was a life within him which he could not call his own, while yet it was in the very highest sense *himself*, so that without it he would have been a poor, helpless, isolated creature—an universal life, which connected him with his fellow-men, and with the absolute source and original fountain of life. The individual life (said he) is really a life at all only so far as it has communion with this universal life; separated from that it exists indeed, but potentially, not actually. "Like as the coals which, when they come into contact with the fire by connection with a different nature from their own, become fiery, but when separated from it burn out, thus the portion from the encompassing whole, which hath taken up its tabernacle in these our bodies, by separation therefrom becomes well nigh irrational, but by the communion of natures through its various passages is made homogeneous with the universal. Now this common and Divine reason, by participation whereof we become reasonable, Heraclitus calls the criterion of truth; whence that which appeareth to all in common is the thing to be believed, for it is received by this common and Divine reason, but that which happens to any single person alone is not to be believed, for the opposite cause." Such is the profound fact which was revealed to the mind of this Ionian philosopher, and which, as the extract we have made partly shows, he saw expressed in the relations of fire to the different objects of the universe. It was inevitable that an Ionian of this age should seek in the forms of outward nature an expression for the truths of his own being. Heraclitus had no wish and no power to invent an entirely new method of speculation; nay, that which he inherited from his predecessors seems remarkably suitable and almost necessary for a mind of his character. Far more imaginative and poetical than any of them, he contemplates the outward object which symbolized the operations of his mind with a delight only the keener for being mixed with a certain under consciousness which they did not possess, that it was an image and not an archetype. All the secret workings and manifestations of fire suggested to him new thoughts and reflections which may sometimes have carried him very far from the original principle which gave occasion to the analogy. But in general we shall find that the different opinions attributed to Heraclitus by others, or which are found expressed in his own fragments, dispose themselves harmoniously enough about this centre. If there is a universal life in things which alone constitutes

them what they are, then these things looked at in themselves are visions, phantoms only; the phenomena of the universe exhibit merely an endless flux and reflux. But contemplate the meanest organized body as organized, and it acquires a meaning, a reality; it becomes a part of the universal harmony. The soul again which beholds these phenomena, what a poor paltry thing it is, a victim of delusions and impostures at every moment! Miserable creature, only to be wept over or despised! But this soul, sharing the universal life, and conversing with it, how glorious it becomes! capable of knowing the name of Jupiter, which is its only business and object.

Our philosopher was therefore almost necessarily an aristocrat, in the Greek sense of the word. For order and law (says he) we should struggle to the death; the individual powers which democracy worships are utterly contemptible. In all the utterances of Heraclitus there seems to be a singular consistency, and what we know of his life is a luminous commentary upon it. The sense of a harmony existing beneath a perpetual conflict of powers, and making that very conflict the means of their preservation, pervaded his being, gave the tone to all his thoughts, and realized itself to him in all the inner forms and outward images of nature.

These four men, THALES, ANAXIMANDER, ANAXIMENES, and HERACLITUS, seem to be clearly distinguishable from all those who have been commonly classed with them as members of the same school. They were, in the strictest sense, Ionians; they succeeded each other at no very considerable intervals, and there seems to be a connection between their opinions, which is not the less striking because it is often a connection of opposition and contrast. To find the law of his own being by meditation on the external world, this, as we suppose, was the object which each proposed to himself. On the whole, each set about the accomplishment of his object with much sincerity and singleness of aim. We cannot therefore help regarding their inquiries with a very deep and affectionate interest, or doubt that they exercised a very important influence on the Greek mind. Nor is it impossible that the thoughts and strivings of these simple and earnest men might teach something to men of a more enlightened generation, who can study the laws of nature and of mind without perceiving that they have any relation to each other, or that they themselves have any important connection with either.

Pythagorean School.

We now come to the second of the Greek schools, which had an Italian birthplace, an Ionian parent, and, in many important respects, a Dorian character. Pythagorean school.

In order to understand the peculiarities of this school and its influence upon Greece, two remarks ought to be especially borne in mind. The first is that the particular doctrines to which it gave birth are referred by the most trustworthy authorities to Pythagoreans rather than to Pythagoras. The persons whom Plato and Aristotle seem habitually to recognise as the propounders and defenders of this system, were men living at a distance from all the circumstances which acted upon their master; not Italian colonists, not members of any particular guild or corporation, but schoolmen subjected to the ordinary influences of Greek society. Our second remark is, that in spite of this practice of theirs no phrase ever escapes their lips which can lead us to sup- Pythagoras.

Moral and
Metaphysical
Philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

pose that they considered Philolaus, Lysis, Eurytus, or Archytas as really entitled to the fame which the voice of all Greece assigned to Pythagoras himself. It seems to have been admitted universally, and most by the wisest men, that there was something in him which he had not transmitted to his successors, something which, whether capable of definition or not, had left a permanent impression upon the character and literature of his country.

These considerations seem to us very important helps in understanding the true nature of this philosophy. Within it we believe are certain elements of thought which came forth from the soul of a living man, and were by him communicated in their power to other men, not in digests or argumentative treatises, but in pregnant maxims, in allusions to passing objects and scenes, in explanations of difficulties as they arose, in a discipline suggested rather than imposed and practised before suggested. These thoughts are what the Greeks from age to age felt of Pythagoras; when they spoke of him they meant these; what husks of opinion formed about them afterwards, when they had been dried in the schools for use or exportation, is a question certainly not without interest, but chiefly interesting to those who have had some taste of the genuine fruit.

Pythagoras was born at Samos, in what precise year may be doubtful, certainly at a time when the Greek colonists had become conscious of a strong impulse towards philosophical inquiries. Under what master he studied has been a subject of great controversy; if we might venture to choose one guess out of many, which may all be false or all true, we should take that which assigns his early training to Anaximander. That philosopher, as we have seen, carried his mathematical studies further than any of his predecessors; his geometry in a great degree determined the nature of his theory. A youthful pupil of earnest character and high imagination coming into contact with such a thinker, would be likely to experience a great conflict of feelings. The science, new not only to himself, but in some measure to all around him, would seem to him strange, wonderful, sublime; the doctrine appended to it would repel him as cold, vague, and unsatisfactory. He would begin, we may fancy, to meditate on his teacher's favourite phrases, "the infinite—*τὸ ἀπειρον*—this forsooth is the sum of all things in the universe. A conclusion how unlike that which geometry would have suggested! that leads us to the idea of limitation, distinctness, in each thing and in all things. And is not such limitation, such distinctness, that which constitutes their perfection? Surely it is this, the *πέρας*—the ultimate limit, and not the limitless, which the wise man is to seek after.

Geometry.

Arithmetic.

Again, Anaximander talked of an *ἀρχή*. Here, indeed, he has profited by his science. Mathematics did teach him the necessity of this. He found every line starting from a point, every series beginning from a fixed number. But why forsake the teacher? Why go abroad to look for your beginning when you have it in the very instrument which you carry with you? You Ionian philosophers are groping after unity in the world about you. But where did you get that idea? Was it from their numbers, from this geometry? Surely it is there that we find not the mere shadow of unity, but unity itself." Such thoughts we may fancy began to work in the mind of Pythagoras while he was yet in Ionia, and to excite in him a discontent with previous methods of inquiry, and a hope that he might discover a better. With these feelings we may suppose him to set out on his travels,

not, however, till he has carried away another (perhaps a greater) benefit from his Asiatic education. The rhapsodists, who used to sing the legends of earlier days to the Greeks of that region in which Troy stood, and in which Homer or a number of Homers lived, these were perhaps silent. But not only their words were preserved, and by this time at least committed to enduring characters, but what was perhaps even more important, the melodies in which they had spoken still lived in the hearts of the people. The impression of the Dorian and Lydian measures on the young Greek must have been very deep; it might be effaced afterwards in some by the passion for abstract speculation, in others it would give speculation itself a richer and more poetical character. All the thoughts of Pythagoras respecting the mystery of number seem to have combined themselves from the very first with musical feelings and associations. Was not music itself an illustration, the highest illustration, of this mystery? Whence came that strange disposition of thoughts and words into verse, whence the fascination of melody and tune? whence, if number be not the secret law, the moving soul of the universe?

All these apprehensions and imaginations might have dwelt in his mind and produced little fruit, or no better fruit than a crude philosophical system. But Pythagoras, as we said, became a traveller. The reports of the regions through which he journeyed are all uncertain. This at least we may conjecture with tolerable confidence, that he was brought into contact with human beings in a variety of different positions and circumstances, and that he began to think more deeply of their nature and destiny. And then with what new interest, in what a new light, would the number-mystery present itself to him! This surely was the very problem which all legislators had been seeking to solve,—in what way a number of apparently separate units might be able to feel themselves really a unity. All society, all government was but the working out of this problem. Away then once and for ever with all Ionian experiments after a physical unity; here was the true field for examination and discovery. But what is it in man which has the capacity for association and organization? It is not resemblance in feature which produces it; it is not contiguity in space which produces it. It is that wonderful thing which inhabits this animal frame—which can transport itself beyond all limits of space and time. I, Pythagoras, can carry myself back to the age of Achilles and Agamemnon; doubtless in some condition or other I actually lived in their time. I can project myself for-wards into ages that shall come; doubtless under some condition or other I shall live in those ages. But in what condition? This soul, which can thus look before and after, can shrink and shrivel itself into an incapacity of contemplating aught but the present moment, of what depths of degeneracy it is capable! what a beast it may become! And if something lower than itself, why not something higher? And if something higher or lower, why may there not be a law accurately determining its elevations and descents? Each soul has its particular evil tastes, bringing it to the likeness of different creatures beneath itself; why may it not be under a necessity of abiding in the condition of that thing to which it has adapted and reduced itself?

Such thoughts Pythagoras may, or may not, have borrowed from Egyptian priests. Doubtless to a man in his posture of mind every old tradition, every relic

Moral and
Metaphy-
sical Phi-
losophy.

Music.

Metem-
psychosis.

Moral and
Metaphy-
sical Phi-
losophy.

Law.

of national faith, will have been precious; still there was nothing in the doctrine of a metempsychosis which might not easily and naturally have grown out of his reflections upon that which makes men human, and enables them as human beings to associate with each other. To another and deeper discovery he was no doubt led by studying the governments of different countries, especially those which had received the Dorian impress. He found everywhere in these communities that the bond of connection was the recognition of a power superior to man, a righteous law-giving power. No human legislature ever dared to dispense with this recognition, no society could cohere without it. Deep and awful idea! The union of men presuming a still deeper ground. Is not this ground the *πέρας*, the ultimate unity, after which we are seeking?

Thus gradually we suppose the idea of limitation, which Pythagoras had acquired from geometry, and which had been brought out in his mind in opposition to the notion of an all-comprehending infinite or indefinite; and the idea of beginning and succession which he had acquired from arithmetic, and which had come out in his mind in opposition to the notion of a mere external ground of things, fused and softened as they both were by the sense of a music dwelling deep in the heart of the world, may have become associated with practical thoughts respecting the nature of the human soul, and the bonds by which souls are related to each other, and finally, with still more awful contemplations respecting the being of God.

But it followed from the very nature of these thoughts, no less than from the character of the mind in which they were formed, and from the processes by which he had arrived at them, that they could not be satisfactorily expressed in the most luminous spoken or written discourse. Pythagoras felt that his principles could only be embodied in a society of living men. In the bonds by which they were held together in their dealings with each other, and with men without; in the silence and fear with which they acknowledged an invisible ruler was his inmost meaning to be expressed. Thus would the proportions and relations of the universe be manifested on their highest ground; thus would the mystical harmony be felt and acknowledged; thus would the dignity of the human soul, its capacity for growth and perfectionment, be proved; thus would the nature of distributive justice, the geometry and arithmetic of politics be practically realized; thus would the idea of God be taken out of the cloudy formless region in which philosophers had caused it to dwell, and be felt as the foundation of social life. Such, we apprehend, was the feeling of Pythagoras; and if so, we may go a little further than Mr. Thirlwall; we may admit that the corporation or order which he established in the South of Italy is not only for the ordinary historian, but even for the historian of philosophy, a more interesting subject of contemplation than the mere details of his doctrinal scheme. Instead of feeling the difficulty which some feel to connect Pythagoras the thinker with Pythagoras the politician, we own we find it a hard matter to conceive how they should be separated. We do not know what his philosophy would mean, or how it would be generically distinguished from the Ionic on the one side and the Eleatic on the other, if it were not for this very circumstance, that Pythagoras sought for that unity in a fellowship of living persons, which the former had hoped to find by observations upon the sensible world, the latter

by investigating the demands and constitution of the mind itself. Looked at from this point of view, the speculations of Pythagoras and the practical experiments in which they terminated, seem to us full of the profoundest interest. We would entreat our readers not hastily to throw aside the consideration of them, because they are told that Pythagoras was a fantastical dreamer; that he could not distinguish a geometrical point from a numerical unit; that he talked wildly about tetrads; that he fancied himself a supernaturally inspired person; and that he pretended to give to his society the mystical and divine foundation. Let them rather consider whether there be not some fantasies and confusions in our own minds upon these apparently simple subjects, which a kind-hearted consideration of the truths which were in the heart of Pythagoras, and of the imperfect and erroneous methods of sensible exposition which they sometimes found for themselves might remove. Let us not say that he was a slave to certain idle notions about number, till we have studied the history of human thought with some attention, till we have understood how much all men (ourselves included) are subject to delusions of this nature, until we have satisfied ourselves that this was not one of the earliest efforts to escape from the delusion, instead of to perpetuate and propagate it. Before we cast any very heavy blame upon him for fancying that he might sometimes lawfully use the great gifts which had been bestowed upon him to startle men with a sense of his own dignity, rather than to instruct them in the true conditions of theirs, let us observe how constantly human vanity, under the most favourable circumstances, has been liable to this temptation; how much more it is likely to assail those who are establishing or presiding over a society, than those who live only in their studies; how far more really disgraceful and mischievous after all that habit of mind is which leads a man to look upon his wisdom and power as self-derived, as ministering only to self, than that which acknowledges the fact of communications from above, and only falls into mistakes and conceits respecting the method of them. Before we condemn Pythagoras for believing that a human community has a mystical groundwork, let us ask what other groundwork it can have, if it is not built upon the sand, and destined to instant destruction.

Eleatic School.

We have now briefly examined two of the Greek schools. We have found the Ionians seeking for unity in external nature, the Pythagoreans in the constitution of human society. We have seen how the latter was led by his speculations upon man into the recognition of a higher and more transcendent unity in God. It was this part of his philosophy which seems to have attracted the attention of *Xenophanes*. The countryman of Pythagoras in both senses, an Italian emigrant from the Ionian colonies, this remarkable man felt the influence of the thoughts which each region had generated. He felt that there was a world full of wonders lying about him, and he endeavoured in the true Ionian spirit to understand this world. But the grand truth of Pythagoras, that man wants another ground to rest upon than the world, had presented itself to him with no less irresistible evidence. There must be God as well as a world, and the world and God are not one. The passage from one of these truths to another was not, how-

Moral and
Metaphy-
sical Phi-
losophy.

Eleatic
School.

Xenophanes

Moral and
Metaphy-
sical Phi-
losophy.

ever, bridged over by this inquirer as it had been by Pythagoras. He did not ascend from nature to those instruments by which man takes hold of nature, and from these to man himself, and from man to God. The two ultimate points of the series were at once brought into connection and contrast in his mind. There is something which I see with these eyes, hear with these ears, there is something which I do not see, or hear, or taste, but which emphatically is. That which I see, and taste, and handle is continually changing the aspects which it presents to my eye, the impression it makes on my ear or my palate. Can I affirm of this in any sense that it is? Is it anything more than that which appears, and of which the appearance yesterday can by no means be identified with the appearance to-day? Can I affirm of any thing surely that *it is* but just of this one thing of which I have no evidence, but simply this necessity of affirming it. Such are the first hints of a principle which has produced more real effect upon philosophy from that age to our own, and which in different ways has been working more secretly at the heart of all philosophy, than any of which it is possible for us to speak.

Xenophanes, when he had once made the discovery that the most real is not that of which his senses gave him information, nay that, strictly speaking, his senses gave him no information of what is real, announced the fact to the world in poems, of which we still possess some fragments; but it was not the speculative side of this discovery which most interested him. That Being whom my mind thus affirms, what is he? Into this solemn inquiry Xenophanes entered with an intelligent earnestness and freedom, and yet, in general, with an awe which inspire us with a great respect for his character. Starting from his one immovable position, that the senses give us no information about God, he began with a bold and rude, but certainly not an impious hand, to tear in pieces the fables of paganism. All the notions of the gods in Homer are connected with sensible forms and appearances; these, therefore, are essentially false and degrading. They are multitudinous, because it is from the impressions on the senses which reflect a multitude of objects, and so many phantoms of the same object, that the notion of them has been derived. And even if these appearances were not sensible, how can any conceptions of men about God be otherwise than false? Must not they be derived from themselves, or from things on a level with and lower than themselves? If an ox could worship, would he not impute to his god horns and hoofs? Away with all conceptions then. And is your Pythagorean better than any other? What do you mean by saying that God is either finite or infinite? the finite and the infinite are both alike thoughts of our own. Thus reasoned Xenophanes, full of earnest faith in the reality of that which he was, nevertheless, reducing so very near to a mere negation. He had learned a precious truth,—Man can feel a ground of being deeper than his own being; man can be certain that God is—but *what* he is? of this he could not be certain, only what he is not. In this mournful conclusion was Xenophanes obliged to rest, and it spreads a mournfulness over all that he thought, even when he is most earnest and true. For he was a real man, no mere philosopher, who can come at this same conclusion, and repose in it, because a conclusion is all that he wants, besides the glory and self-satisfaction of being the concluder; he had a heart which delighted in the truth which had

been revealed to him, and, therefore, did not delight in the miserable void into which, partly through a real love of that truth, partly through hurry and impatience of all other truth which men before him had acknowledged, he was unhappily led.

A healthier thinker than Xenophanes, yet one in whom it is not perhaps possible to feel the same interest, took up his course of inquiry; this was *Parmenides*. The question respecting the nature of God, which had so occupied the philosopher of whom we have just been speaking, does not seem at all in the same degree to have agitated him. His mind rested on the principle of Xenophanes, that what the senses present to us are appearances; that only that which the mind affirms without the aid of the senses actually is. What is this then that the mind affirms? Xenophanes had said "God." Parmenides said "Unity," or "The One." My senses tell me thousands of things, and yet has not every man who thinks and feels ever been groping after some one root and ground of all these things? This, verily, is what man wants, and this is affirmed to be by that within him which fights against apparitions and phantoms. Plurality is merely one of these apparitions, deceitful, transitory; nothing abides but unity, this is permanent, eternal. Such a belief could not dawn upon the mind of a Greek philosopher without imparting to him a feeling of deep wonder. He had seen a succession of Ionians questioning all nature to tell them of this unity. He had seen Pythagoras evoking it out of the relations of number, and actually constructing a human society to illustrate it. And now it seemed that this unity was a condition of the human mind itself—it had been seeking high and low, it had been seeking for that in all other things which really dwelt only with itself. The confused look of a child gazing upon a new world, is but a faint emblem of the surprise with which such a thought must have possessed the mind of an earnest seeker on whom it has just burst. Yet he cannot doubt that it is a true thought. It makes so much of all that had before been perplexed in his mind intelligible, it accounts so well for the thoughts which have revealed themselves to other men. But what consequences follow? Faith in the things about us becomes impossible; we live in a shadow world; we do not, in fact, behold any thing. For these distinctions of things, this apparent multitude of objects, exists not; it speaks to our fancy only. The unity which the mind beholds and demands, this only has substance.

Every one must see how this doctrine of *Parmenides* *Zeno* laid him open to the jests of witty men, such as grow upon the surface of all lands, and of which Greece and her colonies were certainly not less productive than others. These wits believed no doubt that they were opposing self-evident facts to mere dreams. But Parmenides had a friend and disciple who was not willing to leave them in undisturbed possession of this opinion. Zeno of Elea was convinced that there was not only positive falsehood, but direct absurdity in that doctrine which experience seemed so irresistibly to establish, and he boldly undertook to make this absurdity palpable to the popular mind. In a series of arguments, some of which are still preserved to us, he endeavoured to show that space cannot exist, that we cannot suppose a plurality of objects without attributing self-destructive qualities to them; that if there be a number of real existences, this number must be both finite and infinite; lastly, that the notion of movement involves a contra-

Moral and
Metaphy-
sical Phi-
losophy.

*Parme-
nides.*

Zeno.

Moral and
Metaphy-
sical Phi-
losophy.

diction. Our readers would not, perhaps, be much interested in these early specimens of Greek subtlety; nay, they would be inclined to denounce them as the exploits of a mere word-conjuror. But assuredly Zeno deserves no such name. He was both in action and speculation a brave man, and we owe to him a most important practical discovery. In fact, he occupies a peculiarly important position as a thinker, which it is for the advantage of our future studies in Greek philosophy that we should understand.

Words.

Every philosophy must have an instrument or organ to work with if it would make itself intelligible. Some external object served this purpose for the Ionic philosopher; lines and number for the Pythagorean. But what was the instrument of the Eleatic philosopher? He seems to have ascended into a region of such pure metaphysic, and so entirely to have rejected all common and sensible analogies, that one does not at first see how he can ever impart his doctrine to others, or at least suggest any successful method for pursuing investigations in his own direction. And this difficulty seems to have been felt by Xenophanes, and to a certain extent by Parmenides. Precious and pregnant as are the hints which each of them present us with, it seems likely that they will be obliged to stop at the point which they have already obtained, and to leave no race of successors. But Zeno has found the solution of the puzzle; he has found that *words* bear to this philosophy the relation which sensible objects and numbers bear respectively to the other two. The language in which we discourse with each other must needs embody the law and principle of our own mental workings, and it was exactly this principle which the Eleatics were dealing with. Zeno had only an imperfect consciousness of this truth, but he acted upon it, and it bore useful fruits for him and for us. There was no falseness in his use of words. He felt that they did affirm and embody truths, and he employed them for the purpose of elucidating truths. And herein he was surely as honest as those wits who set themselves to confute Parmenides and philosophers of his class by an appeal to experience. For they too profit by our belief in words. They awaken our consciousness to the fact, that the words which we speak bear on them an impress and image of the external world; and it is this consciousness which they rest upon as their real defence against the philosophers who set at nought the evidence of the external world. Zeno awakens our consciousness to the fact, that the words which we utter express something to which there is no counterpart in the external world, and he rests upon this consciousness to oppose the conclusions of those who set at nought the witness of their own minds. Both appeals are in themselves fair, and carry conviction with them. But the one merely convinces us of a fact which we took for granted previously; the other obliges us to perceive truths lying very near our inmost being, which were yet almost entirely hidden from us.

Logic.

But it is the practical discovery which was the direct result of this search after an organ or instrument for the Parmenidean philosophy that obliges us to regard Zeno with most admiration and gratitude. Mathematical science we owe, according to the best historical evidence, to the East. And it entirely accords with the calm, contemplative, and yet sensual character of the Orientals that this should have been their contribution to human knowledge. But the science of *Logic*, the science which declares not what are conditions to which external things

are subject, but the conditions under which we ourselves speak and judge—this was of purely Greek invention. No other people had ever the subtlety to conceive the possibility of such a science, far less to ascertain its distinct province and its appointed work. Though logic, in a formal and narrow sense, is considered as the antagonist of poetry, yet only a most imaginative and poetical nation could have discovered the meaning and necessity of logic, and have given it the statue-like perfection which it has attained in Greek hands. Now Zeno is believed, on the best grounds, to be the inventor of logic. He first was led clearly to perceive that the mind has a distinct law regulating its own affirmations, and he consequently was first stimulated to inquire what this law may be. How much we owe to him from this achievement we shall understand better as we advance. Our principal object here has been to point out the connection between it and the Eleatic philosophy, of which Zeno was the accomplished and able defender; to indicate the kind of influence which that philosophy exercised upon the mind of Greece; to show how important a place it fills in the history of human inquiry; and to excite our readers' interest in the future development of the doctrine which they have beheld in its first germ. In so very rapid a sketch as ours it is clearly impossible to do more than notice what seems to us the living and central peculiarity of each system as it rises up before us. But the real germinant principle is often hard to discover amidst the multitude of mere notions and opinions with which it has environed itself. An historian will distrust his own sagacity in detecting it, and will rejoice greatly if he can find it any where in its rude and primitive shape. Little value as it may seem to have, nay, as it may actually have, while it remains in this condition, yet he deems it not wise to wait till the animalcule has become a perfect insect, or till the insect has died, before he commences his examination of it. We offer this apology for noticing the anti-Socratic schools, briefly indeed, but yet at a length which many may think disproportionate to the time that we shall be able to bestow upon their successors. We are convinced that our readers whom, as we have remarked, we wish not to furnish with a history, but to put in a right method of procuring one for themselves, will have a clear or confused understanding of the palmy period of Greek philosophy between the age of Socrates and of Aristotle, as well as of the age of senility and decrepitude which followed, exactly in proportion as they study or pass over the years of its infancy. Let them not hope to understand Plato or Aristotle, or even Epicurus, Zeno of Cittium, and Carneades, if they have begun with despising Heraclitus, Pythagoras, and Parmenides. Nay, we might go further and say, that we should greatly doubt the pretensions of any one professing to have a clear understanding of Hobbes, Locke, Leibnitz, or Kant, who could discover nothing but confusion and barrenness in these early inquiries.

The Sophists.

Hitherto our attention has been directed to the comparatively silent and solitary labours of thinking men in provinces remote from either of the two Grecian capitals. But from the conclusion of the Persian War the thought and speculation as well as the active energies of Greece had been continually seeking for themselves some centre of light and attraction. We have already observed that

Moral and
Metaphy-
sical Phi-
losophy.

Sophists.

Moral and
Metaphy-
sical Phi-
losophy.

The
Sophists.

the two leading tribes of Greece respectively embodied those two great qualities of the human spirit—respect for order and law; consciousness of individual powers, and a desire to exercise them. As in the speculation of particular men neither of these bore much apparent fruit till it had been acted upon by the other, so was it also to a great degree in the actions and movements of Greek society. Till Athens had produced men whose principal aim was to give her something of the order and compactness of the Lacedemonian polity, the great intellectual energies of her sons never really displayed themselves. On the other hand, till Sparta had been in some degree brought into friendly contact with Athenians, and had experienced some of the lively impulses by which they were continually actuated, it never became really conscious of its own position as the witness for the stable power and order by which these impulses are counteracted. The crisis when these two feelings and principles began to put themselves in conscious opposition to each other—that crisis in which their mutual dependence upon each other became also most apparent, and began to be most realized by thoughtful men—is the period to which we always look back as the most interesting one of Greek history. Then did Greek poetry come forth in the form which seems to have been most exactly fitted to express that feeling of an overruling system, and of a power in man to strive against it even if he must ultimately succumb, with which the Greek mind was now exercised. Then the sculptor and painter, and in a less degree the architect, was teaching his countrymen to feel that nothing which they saw could have meaning or reality if there were not a deeper meaning or more perfect truth beneath, which the highest human work fails to embody, which the very meanest proves to exist. Then, lastly, which is the point most immediately concerning our present inquiry, experiments of practical legislation and political organization began to combine themselves in a thousand strange and perplexing ways with political, moral, and metaphysical theories. The workmen in this particular branch of human labour were not poets or artists, but a race which has been known to all future ages by the name of Sophists.

It has been a very common fancy that some resemblance exists between this body and the schoolmen who appeared in Europe during the middle ages. We confess that if we were seeking for two classes of men, the most directly contrasted in all the important features of their characters, we should have selected precisely these two. That they both engaged in philosophical inquiries,—both attached considerable importance to logic—and that some members of each class took an interest in philology—and that to men living in a circle of influences entirely different from that to which either was exposed, the notions of each may appear about equally grotesque—this is undoubted. There is just enough of these superficial resemblances to make the difference in all other respects the more striking. The schoolmen of western Europe, if they ever forgot that truth was the end which they were pursuing through all labyrinths,—and this faith seems not entirely to have deserted any of them,—lost sight of their object through mere simple delight in tracing the windings of the road; the sophist of Greece disbelieved in truth altogether, and proposed to himself another object, that of exercising power over the minds of men. The error of the schoolman was, that he set too much store by all instrumental

VOL. II.

arts, as if they were precious for their own sakes. The sin of the sophist was, that he estimated every thing by its tendency to promote a low, grovelling, secular end. The great misfortune of the schoolman was that he mixed too little in human society, understood too little the actual workings of human life. The misery of the sophist was that he lived in a perpetual round of bustle and excitement, caring for no thought but so far as he was able to translate it into talk, perpetually confounding together the realms of speculation and action.

This, in fact, is the distinguishing characteristic of the sophist. He must always be talking and doing, or teaching others to talk and do. For this purpose he avails himself of all modes in which the human feeling had expressed itself in all different periods, sympathizing with none, able to convert all into an argument that man's life is merely a shifting mode, that nothing is, but every thing seems. Now he is a great mythologist; knows all that poets have imagined; has a thorough insight into the popular faith; he can tell you how this and that notion had its origin, he can adopt it into his own system, or laugh at it as he finds most convenient. Now he is a natural philosopher; the thoughts which a Thales or an Heraclitus actually worked out become part of the implements of his trade; he can bring these theories skilfully along side of the old cosmogonies, and so at his pleasure prove the ancients to have been profound naturalists, or show that all their conceptions about the divinities meant nothing. Again, he is versed in all the ideas of Pythagoras about order and society; he can parade his acquaintance with the doctrine of transmigration, or with the mysteries of the numerical scale, that men not possessed of this knowledge may feel how absurd it is for them to pretend to any perception of right or wrong. But most of all he has studied in the school of Zeno, and can tell how little the evidence of experience is worth, how little reality there is in things, what secret powers lie hid in words.

What can a man do after a king? what can a sophist hope to accomplish after these Greeks? Protagoras, Prodicus, Gorgias, Hippias, may be fairly said to have carried the trade to the furthest limits of which it is capable. They were not, as is sometimes imagined, mere pretenders to talent or erudition; they were men of more than Greek sagacity; some of them even appear to have possessed creative powers; all possessed large and various information. Whatever mere intellect without a moral meaning or purpose can do they accomplished. Nay, they may be said to have demonstrated how very much it can do, and what is the nature of its doings. They proved that it has the power to weaken the sinews of a nation, to destroy, as no wars or tumults can, its innermost heart. We should beware of despising their gifts, which were in all respects most remarkable. Nor could the possessors of them, even by the art of the most consummate genius that the world has ever seen, have been made the objects of our scorn as well as our indignation, if that consummate genius had not exhibited them side by side with a man of earnest and single aim, and compelled us to feel that true power is in *this*, and that all other powers must shrivel and look helpless when brought into collision with it.

Physical Philosophers.

The readers of Cudworth's *Intellectual System of the*

4 B

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphysical
Philosophy.

Universe will no doubt expect that we should assign a prominent place in this part of our sketch to the name of Democritus. In that work his theory appears as a central point in Greek philosophy. Plato and Aristotle seem to be mainly employed in illustrating or undermining it. Our excellent and learned countryman, it must be remembered, had proposed to himself a specific object, that of relieving the Cartesian philosophy from the imputation of atheism; and with a view to that object he may have exhibited the different schools in their proper proportions and relations. But when his book is taken as a manual or guide to Greek speculation generally, it cannot fail, we think, to confuse the student's judgment respecting the documents which he is to examine, greatly to exaggerate the importance of a particular class of inquiries, and to throw another and far more important class almost wholly into the shade. In a sketch of moral and metaphysical inquiries we conceive that neither Democritus nor his predecessor Anaxagoras have, strictly speaking, any title to be mentioned; we notice them both here, because they throw some light upon the character and objects of the man who was in a remarkable manner the centre of all the systems which came into being before or after his time.

Anaxagoras.

The men of whom we speak were emphatically and exclusively naturalists; the first we have met with who deserve that description. Thales, Heraclitus, Pythagoras, Zeno, started, as we have seen, with a desire to interpret their own thoughts, perceptions, powers; nature mingled itself in their investigations, but never formed the staple of them. Accordingly they were politicians, feeling the deepest interest in the social condition of their countrymen; though colonists, yet most truly and heartily Greeks. With the philosopher of Clazomenæ the case was altogether different. Living in Athens in the most stirring times of the republic, the friend and tutor of Pericles, he seems to have cared nothing for men. Life is better than annihilation, (this we are told was one of his favourite sayings,) because, while we are alive, we can contemplate the sun and the stars. The idea of a moral world, a world of affections, relations, sympathies, never presented itself to him as a reality. That of which his senses told him, was the only one in which he could believe. Not that he was merely a slave to the judgment of his senses; he was a Greek, actuated by a spirit of earnest inquiry, incapable of resting satisfied with mere surfaces and phenomena. But as he had no sympathy with the feelings and doings of men, so neither had he the intuition into the life of things which seems to wait upon that sympathy, and to be very rarely separated from it. It was not the energies and movements of the animal or vegetable world that occupied him, but merely the amalgamation of its parts. "Nothing, properly speaking, is born or dies; birth is the composition of elements, death the dissolution of them;" this was the observation of which Anaxagoras had made, and by help of which he believed that he could interpret the universe. While working out with sincere diligence the meaning of the only fact into which he had gained any insight, he threw out many valuable hints respecting the laws of cohesion and affinity, and the doctrine of a vacuum, for which later science has willingly confessed its obligations to him. It must be owned also that he did not refuse to acknowledge another principle different from the one with which he started, when he found that the problem of the world could not be solved without it. Little as he seemed to

value his position in the city of political wisdom, he could not help seeing how intellect was there every day asserting its dominion over the elements of matter. A similar power he was obliged to own, a power different in kind from the atoms of which the world is composed; a mind, an intelligence must have given them their first impulse and order. Whether this were its last and only work, or whether it ever after mingled with, and sustained, and regulated the force which it had communicated to these atoms, or which they gained from their union with each other, on this point Anaxagoras expressed himself indistinctly and inconsistently. It would be most unjust to blame him for his confusion, far rather we ought to hail it as an evidence of his honesty in recognising facts whether he could adjust them to his theory or no. It is quite another question whether we can acknowledge him for a martyr, because he was imprisoned as an atheist, and afterwards banished to Lampsacus. That the charge was formally false we cheerfully admit. One who refers the origin of the world to a mind cannot be an atheist. Let it also be admitted that such a man may connect much deeper thoughts with his speculation than we can at all measure or conjecture. But to all intents and purposes, so far as the Athenians were concerned, the accusation was true. The gods whom they worshipped were gods related to themselves, superintending their hearths and their city, their helpers, judges, lords. To substitute for such beings a mere *opifex mundi* was to introduce all the practical consequences of atheism, to rob them of a precious truth, a truth which they felt to be, and which *was*, the stability of the state and of the life of each man in it. If we confound this crime with that which was committed many years after against Socrates, we must confound two men whose feelings and objects were as nearly as possible the direct antipodes of each other. We may grant that the sublime physical conceptions of Anaxagoras assisted in forming the mind and the eloquence of the greatest statesman of Athens; but let it never be conceded that his persecution bore the slightest resemblance to that of her greatest moral teacher.

The atmosphere of Athens had, as we have seen, an unacknowledged influence upon the mind of Anaxagoras. From this influence Democritus was entirely removed. Though he visited many lands merely as a traveller and observer, he seems never even for a moment to have shared in any of the feelings and practical interests of the period in which he lived. The desire of offspring and the love of country, said this Thracian, are evil passions; they prevent us from feeling that our home and nation is the world. From this flight of cosmopolitism we may easily conclude that Democritus, if he had energy to become a speculator at all, would travel in the same road with Anaxagoras, and would proceed further in it than he ever did. This was actually the case. The feeling of Anaxagoras, that decomposition and composition are the only laws of the universe, pervaded his mind. The notion of a moving power or intelligence, not being suggested to him by any thing in his daily life, took no hold of it. But to supply the void which the want of that thought left, he attached a kind of vague and awful meaning to space. In place of the affinities and cohesions of his master he conceived that the atoms move in a vast void, and that herein is the true essence or origin of things. Democritus, the prince of materialists, had a kind of mysticism of his own. He used the phrases of the Eleatics;

Moral and
Metaphysical
Philosophy.

Democritus.

Moral and
Metaphy-
sical Phi-
losophy.

he talked of an eternal truth and of the ideas of things. But the eternal, invisible truth, which he perceived was *space*, his ideas were the infinite atoms which are and have been for ever circulating in it. We shall be compelled to notice this theory again when we speak of Epicurus and of the noble Roman poem in which it is worked out. Properly speaking, the illustration and history of it, as well as of that high form of dynamical philosophy which was contained in the poems of Empedocles of Agrigentum, belong to the student of physics. It is interesting to us as one of the many forms of opinion which were acting upon the Greek intellect, not directly, but in combination with the practical speculations of the sophist.

Socrates.

Athens
during the
Peloponne-
sian War.

Thucydides does not once interrupt the stately march of his narrative to notice the theoretical disputes of the period of which he wrote. Perhaps the modesty which is so beautiful a characteristic of his mind withheld him from touching upon a topic foreign to the habits and feelings of a soldier. Perhaps he was secretly conscious that his facts, his speeches, and his quiet reflections would really throw more light upon it than many laboured disquisitions. One circumstance must have struck all his readers—that if other historians record instances of crime as great as any that appear in his pages, no one exhibits to us so many self-conscious, self-justifying criminals. Evidently this peculiarity is not to be accounted for by the temper or talents of the historian; Thucydides is not an ambitious, professional analyst of motives like Tacitus; he would not have introduced men explaining their evil acts and reducing them to principles, if he could, by any other means, have given a faithful picture of the time. Another peculiarity of the Greek mind, at this crisis, may be seen in the very surface of his history. Whatever might be the alleged causes of the Peloponnesian war, whatever tribe jealousies may have fomented it, the people of Greece were divided from each other on a theory or a principle. Aristocracy and democracy were, more than Sparta and Athens, the watchwords of the combatants; these abstractions were, to young Athenians especially, in place of the love of soil and country which had achieved such wonders at Marathon and Plataea. Such facts prove that the mind of the nation was sophisticated.

Aristo-
phanes.

Against this evil condition of society few powerful influences seem to have been at work. By far the most efficient it would appear was the comic poet Aristophanes. This remarkable man, uniting the genius and poetical spirit which generally belong to an earlier and more unconscious age with the distinct purpose and worldly observation of a later one, had set himself steadfastly to the work of making the new revolutionized Athens look ridiculous in its own eyes. No statesman, warrior, poet, philosopher, who, as he thought, was lending his hand towards the destruction of old notions and customs, or towards the cultivation of the fashionable tone of thought and feeling, escaped his vigilance. Mob flatterers and worshippers of all kinds, but most of all those who flattered it with appeals to its too acute understanding, he regarded as abominations to whom the lover of his country should grant no quarter. That satire would be the most powerful instrument for working upon the feeling of Athenians; that wretches who had learned to laugh down all grave admonitions would ill abide

laughter far keener than their own, were at least plausible conclusions. We must not therefore hastily conclude that he was wanting in strong moral indignation against crimes which he would appear only to treat as ridiculous; still less that his affectionate feeling for the past and his sound political doctrines were merely the fruits of his discontent with particular men of his day, and not rather the causes of it.

Yet neither reason nor evidence warrant us in believing that the success of Aristophanes was proportioned to his zeal or to his genius. He may have abated some of the nuisances which were infesting Athens. He may have diminished the race of sycophants, have made the vulgar kinds of mob-persuasion less effectual, have even done something to abate the litigious spirit of his fellow-citizens. But he can have helped very little to root out that which he felt to be the real cancer of the nation's being, that which fed upon the hearts not of the worst but of the best and noblest and most promising of the Athenian youth. No one could apply any sound remedy to this evil who merely despised the age into which he was born and mourned over one that was departed, who merely saw the effects of the sophistical poison, without understanding its nature and the constitutions on which it was working. He only could hope really to reform the young men of Athens who could heartily and affectionately sympathize with them, who did not merely express his contempt or indignation for their favourite teachers, but who was ready to follow them through all their windings and subtleties, who, without for a moment forgetting the great purpose of leading them back to realities, could yet grapple fearlessly with the most shadowy and impalpable abstractions. A man of this kind would have great and sore difficulties to encounter, through which nothing but the clear perception of his object could possibly lead him unhurt. His inward conflicts, before he could be fitted for his task, must be severe; of his outward, the greatest perhaps would be this, that his purpose would be infallibly misconstrued by those who were aiming, with very different instruments indeed, to resist the same evils. It would be perfectly inevitable that he would pass with them for one, perhaps the subtlest and most mischievous, of the sophistical class. Because he sought to make men feel that there was no resting place in any of their theories or opinions, he would be suspected of universal scepticism; because he led them to feel that even they were not without a ground to stand upon, if they would seek for it, he would be accused of undermining the ground on which their forefathers stood; because he endeavoured to look through the clouds which had been drawn up from the earth, into the serene heaven that lay behind them, it would be fancied that he wished to make these clouds his permanent habitation, that he invoked their protection and did them homage.

Such a man was Socrates, and this was his fate. He was hated by sophists, and ridiculed as the worst of them. He treated the diseases of his country according to a method exactly the opposite of that which Aristophanes adopted, and therefore he was denounced by Aristophanes as the great promoter of them. We have now to consider what his method was, how it affected his own age, and what traces it has left of itself for subsequent generations.

Socrates was the son of a statuary and a midwife. He was born in a little burgh of Attica. When he came to Athens we know not with any exactness; probably

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

Education
of Socrates.

about the commencement of the Peloponnesian war, when Pericles was still living and Anaxagoras teaching. That he frequented the school of the latter has often been asserted, and is at all events no improbable surmise. Nor is it at all unlikely that he may have entered with considerable ardour into the studies of his master, and may have carried away from him many valuable hints respecting mathematical and physical science. But no one who considers his character as it is presented to us in the writings of either of his disciples, or even in the play of his great satirist, can doubt that even from the first he must have looked upon these acquirements as subsidiary to another and higher design. The main effect of all physical speculations upon his mind was, it would seem, to awaken the inquiry, "What are all these to me? Let atoms be connected by what law of affinity they will, let them whirl at random through space or be guided by a directing intelligence, still this question remains, What am I?" We have traced the rise of such a thought in the mind of a Jewish sage who was destined to be the deliverer and lawgiver of his people, and to hear from a divine voice a direct answer to his inquiry. What a reality and meaning does it give to that history and that revelation to find the same feeling awakened under circumstances so different in a person not partaking the blessings of a divine covenant, though happily not removed beyond the reach of a divine influence! Not in a lonely desert, but in a crowded city; not to the member of a rude tribe, but to a man dwelling among the most intellectual of all people at the moment when their civilization had reached its highest pitch; not to one led by exile and want of other sympathy into meditations upon his human and domestic relations, but to one who was estranged from them, as well by the condition of society around him as by his own peculiar position, did this wonderful thought present itself. In the midst of the gossip of the Agora and the bustle of the Piræus, where every person had to tell of some new speculation, or was preparing for some new enterprise, amidst crowds who were rushing to the popular assembly, or the judgment-house, or the theatre, did this man begin earnestly and with his whole soul to dwell upon the thought that he himself was something, and that to know what he was was more important to him than to know ought else that man could teach him.

His
Demon.

This question Socrates was led to ask himself, and let it not be thought that it was one to which he gained no reply. The deep and satisfying answer, indeed, was wanting, but he learned a lesson which, whether the language in which he expressed it were correct or not, was in itself most true, and proved its truth in his life. He found that he had a companion and teacher within him reproving him when he had erred, warning him of evils about to be done, approving, censuring, judging his thoughts, and words, and deeds. Xenophon asks, with plain, soldier-like honesty, whether his accusers could really believe that he told a lie about this matter; and hints that it would shake his faith in all reality to suppose that a false impression on the mind of a man so clear-sighted and free from superstition could produce the effects which he himself had witnessed. We frankly own that in this XIXth Century after Christ we are just as unable to disbelieve in the Demon of Socrates as his own disciple was. Any notion of a particular impression appertaining to him, of which all men may not be partakers, we should of course reject; nor does there

appear any clear proof that Socrates himself claimed any such privilege. When he spoke of his Demon, he was asserting the fact of a conscience—asserting it not as a name or a notion, but as an actual reality, and asserting it in language perhaps more honest and less contradictory than those are wont to use who describe it *merely* as a faculty belonging to the person on whose conduct it pronounces so absolute a verdict.

That Socrates, however he may have sometimes expressed himself, did not really mean to describe this as a special gift vouchsafed to himself, we gather from the simple fact, that the whole labour of his life was to interest others in that inquiry which had led him to his first apprehension of it. What charms he used to draw a circle about him whose thoughts he could lead in this or any other direction might seem at first inexplicable. Most of the sophists were men advanced in age and reputation when he first appeared in Athens. They promised to fit men for being politicians, orators, generals, and offered very plausible evidence to prove that they could do what they professed. He promised nothing. He was come, he said, to exercise his mother's profession on behalf of those who had thoughts of which they wished to be delivered. You could not understand what line he took; whether he was a philologist like Prodicus, or a professor of statesmanship like Protagoras; he seemed to be all things by turns, and nothing definitively or constantly. Personal gracefulness and beauty were great recommendations among the Athenians; he had large projecting eyes like those of a bull, a flattened and upturned nose, a protuberant stomach; he wore a tattered cloak, and was seldom seen with sandals. Nevertheless, the youth of Athens began to flock about him; even noble young men believed that he had something to teach them, and that by some means or other he would be able to impart to them the art of governing better than the more regular doctors. It is impossible to say that some of the causes which we have mentioned as likely to alienate his countrymen may not themselves have contributed to this result. The Athenians liked a humorist, and a humorist Socrates, by his outward negligences, as well as by the whole tone of his discourse, showed himself to be. Moreover he had a most hearty genial way of interesting himself in whatever interested those with whom he was mixing; as little of solemn quackery as was ever found in the composition of any man. Add to this that he was a thorough, genuine Greek; Greek in all the habits of his mind, Greek in his taste for society, Greek in wit and argument, Greek in a brave unflinching love for his own land, Greek in making freedom (to a much greater degree than is usually observed or acknowledged) the passion and end of his life. But all these circumstances together could not have availed to counteract the many disadvantages under which he laboured, if he had not possessed the real magnet which must draw the hearts of young men after it, be they never so reluctant, a knowledge of the thing which they are really wanting, and which they have been toiling in vain to find.

Political power was, as we have seen, the one prize which the sophists proposed to themselves and held out to their pupils as the reward of all the trouble which they bestowed upon abstract speculations. Now, though there were different roads to this end, and though each teacher believed himself, and induced his disciples to believe, that his was the shortest, yet one method was common to them all; all sought to acquire power by

Moral and
Metaphy-
sical Phi-
losophy.

His cha-
racter and
influence.

His Dia-
logues.

Moral and
Metaphysical
Philosophy.

means of words. The mastery over words was the great art which the Athenian youth was to cultivate; his own feelings, and an observation of what was passing every day in his city, told him that there was a charm and fascination in these which the physical force of an Oriental tyrant might vainly try to compete with. It seems to have been the first observation of Socrates when he began earnestly to meditate on the condition of his countrymen, that in this case, as in most others, the tyrants were slaves, that those who wished to rule the world by the help of words were themselves in the most ignominious bondage to words. The wish to break this spell seems to have taken strong possession of his mind. But the wish would have been ineffectual, and would only have interfered with the main feeling of his life, if he had not been able to connect the study of words with that deep question respecting his own being of which we spoke just now. As he reflected, he began more and more clearly to perceive that words, besides being the instruments by which we govern others, are means by which we may become acquainted with ourselves. In trying really to understand a word, to find out what was the *bonâ fide* meaning which he himself gave to it, he found that he gained more insight into his own ignorance, and at the same time that he acquired more real knowledge than by all other studies together. In this work he knew that he was really honest, he was feeling for a ground, he was breaking through a thousand trickeries and self-deceptions. If then he was to deliver his countrymen from that miserable shallowness into which they had been betrayed by the ambition of wisdom and depth,—if he was to lead them out of the multitude of systems about morality into any firm feeling that there was a morality,—above all, if he were to rescue them from the worship of power,—this must be his means. He must not stop to canvass the wisdom of this proposition or that. He must not denounce with great moral indignation some that struck him as very mischievous or outrageous. He must not candidly and generously concede the truth and wisdom of those which seemed to him plausible or reasonable. But in every case he must lead his disciples to inquire what they actually meant by the words of the propositions which they were using, and must consider no time wasted which they honestly spent in this labour; no perplexities or contradictions dangerous which started out of their own minds in the course of it. No doubt this would be a most irritating, vexatious course of proceeding. No doubt an opponent who had adopted a certain proposition, and was provided with abundance of arguments in defence of it, would be tortured beyond measure by finding himself not fairly encountered upon those arguments, but led back into a question which he had assumed, forced to give an account of a word which he fancied every one was agreed upon, and not permitted after all to bring any of his own resources into play. It was most perplexing for a disciple who had come expecting that a certain doctrine would be either established or refuted, and, perhaps, that the ingenious arguments on both sides of the question might serve his purpose in a popular assembly, to find that he got no decision either way, and moreover that he himself had been talking all his life in a language which he did not understand, and using words as if they were algebraic characters. But yet in some way or other the sophist was taught that he was in the presence of one stronger than himself. He might chafe and fret, and

complain that he had been treated with great unfairness. But he could not say that his opponent had not got the better of him in his own word fighting; he could not say that all the scepticism which he had brought into play against the common thoughts and feelings of his countrymen and of mankind had not been made to tell with tenfold force upon himself; he could not help owning and feeling that there was one in conflict with him who had some other end than the mere exercise or display of power, and yet who did possess a power before which his own quailed. On the other hand, the disciple, amidst all his bewilderment, will have gone away with a feeling that he (perhaps for the first time in his life) had actually learned something, and with a conviction that if there be not something better than the attainment of dominion over other men's minds, there is at least a most important and indispensable preliminary to it, unless we would have our own the sport of every deceiver.

The infinite humour and vivacity of Socrates must of course have been of the greatest service in such dialogues as these. But oftentimes his opponents will have fancied that he was merely indulging his humour when he was in fact following out his principle. The practice of confessing his uncertainty or his ignorance upon any subject that was presented to him, which formed in their eyes the chief element of his "irony," was not always or generally affected. We make no doubt that he often entered upon a discussion without knowing whither it would lead, actually, as he professed, hoping to be a learner by the result of it. He was certain not of a particular conclusion, but that his method was a sound one, and that it would conduct each person who followed it to clearness and truth. It is probable that his discoveries respecting himself and his fellow-creatures were the practical fruits of this method. For instance it was by repeated experiments that he convinced himself of the immense importance of the habit of recollection; how the mind that wants it is at the mercy of all accidents; how the mind that possesses it is continually realizing its own possessions, receiving them as if they were then for the first time bestowed. Upon this principle the greatest part of his moral discipline depended. The necessity of removing the impediments to recollection, of leading the mind away from mere sensible images and impressions into an examination of its own treasures, was the purpose and ground of it. But this principle was redeemed from any Brahminical tendency by his constant use of words and sensible images as the means whereby a man feels his way into the principles and grounds of his being. It is in trying to understand all common things, what the carpenter does with his wood, the shoemaker with his leather, the mason with his stones, it is by really getting to know what we mean when we talk of all these things, that a man learns to understand himself. It is not therefore an escape from common life, from daily business, that the withdrawal or recollection of Socrates pointed at. It is the habit of seeking out in every thing that which it really is, and not merely its shapes, and appearances, and accidents, which the man is to cultivate, and which is ultimately to fit him for perceiving that which is deepest and truest in himself. Now it is the faith attained by repeated proofs and trials that man has that in him which does desire to find out the truth of things; and again that he has an inclination to be constantly conversing with the mere images of things, and that

Moral and
Metaphysical
Philosophy.

His irony.
Recollection.

Moral and
Metaphy-
sical Phi-
losophy.

A moral
foundation.

Know-
ledge.

Reverence.

just so far as the first of these tendencies is kept uppermost, and subordinates the other to it, he is in his honest sound position, and that just so far as the lower tendency is uppermost, he becomes a mere shadow-pursuer and shadow-fighter, which is the soul of Socrates's doctrine. It was not adopted as a scheme to supplant any other scheme; he stumbled upon it as a fact which he could no more gainsay than any one for which he had the evidence of his senses, a fact which *was*, let it be explained as it would, and must be recognised in all our dealings with ourselves or with other men. Hence followed the principle which undermined all the sophistical conclusions,—there is a right and there is a wrong. You want to know who invented these distinctions. You want to know what the worth of all the old maxims, laws, religions is which assume them. You say that they must have been invented some time or other by the strongest or the cunningest, and that what the strong and the cunning invented once they can now pull down. I answer you, man *is* something, and to be that which he *is*, is right, and not to be that which he *is*, is wrong. Most men are living as if they were something else than they are; it is the business of the wise man to know what he is, and by all means to seek to be that and nothing else. That which is crooked then cannot be made straight. To perceive, not to create, to walk in a line, not to devise one, is our business. From this statement it will be seen in what sense knowledge seemed to Socrates the basis of morality. Those who suppose that he meant to exalt the human faculties and to make them the grounds of virtue and of truth, do not merely mistake, but invert his meaning. To destroy the worship of power, and especially of intellectual power, may be said to have been the purpose of his life. And in nothing did he show this more than in his doctrine respecting the relation of knowledge to morality. As the outward eye sees certain objects, and is good for nothing except as it sees them, so the inward eye perceives certain objects and is good for nothing except as it sees them. The objects are there. It is the whole blessing of the man to behold them, as he beholds them he is like them, but they *are*—not the variable functions of his mind, but the eternal, unchangeable principles and grounds of it. A notice of Socrates is only an occasion for indicating this faith; in speaking of his great disciple, we must strive to expound it. But it is necessary to show that his whole philosophy proceeds upon the recognition of something which it does not give, and consequently drives us to those ancient laws, to that faith and reverence which Aristophanes accused him of subverting. It is pleasant to compare the actual effects of the methods adopted by the philosopher and by the satirist in reference to this very point. The comedian, anxious to preserve respect for the mythology and the notions of his fathers generally, did nevertheless, because his mind never rose above the familiar irreverent level of Greek Pagan feeling, take such liberties with the gods of his country as must have tended to lower them still further in the estimation of a sceptical age. The moral teacher, feeling the want of a communication from heaven to meet the wants and demands of which the mind by self-inquiry becomes conscious, listened with deep attention to all the ancient traditions of Greece, and probably of other lands, professed his disregard of all attempts to explain them into mere natural theories; never pretended that he had discovered any truths which had superseded them or even

made practical attention to them on his own part needless; never permitted himself to spend any of his overflowing humour (a humour not inferior to that of Aristophanes himself) upon them; never condemned any of them, except when he felt that they robbed the objects of fellowship and worship of the awe and sympathy which was their due, when the imitation of their characters would have made his own dishonourable and base. Even then it was not the faith of his country so much as the Homeric representations of it which he censured.

This view of the character of Socrates will seem unsatisfactory to those who have been in the habit of imputing it to him, either as a merit or a crime, that he set himself in opposition to the steadfast and deliberative faith of his countrymen. It is a notion less prevalent indeed in these days than it was formerly, but not yet obsolete, nor always we think supplanted (when it has been abandoned) by a truer opinion, that Socrates proclaimed the doctrine of monotheism, and died a martyr for it. The supporters of this hypothesis find infinite difficulties to overcome in the records of Plato and Xenophon, and are obliged to acknowledge either that the story has been perverted by cowardice, or that their hero was guilty of unaccountable inconsistencies. In the remarks which we have made respecting Anaxagoras, we have indicated our feeling upon this subject. We do not think that a mere negative monotheist, a mere denier of Pagan stories and traditions, would be worthy of the esteem which Socrates by nearly every part of his conduct extorts from us. The glory of unbelief belongs to Democritus, to Protagoras, to Epicurus. Some notion of a divinity still hung about all these; if the not acknowledgment of many gods be the same with the recognition of one, they are the great witnesses for truth in Greece. But if to seek after God with the continual desire of fresh light, yet with a determination to abandon none that he had,—if the continual asking for a Being of purity and truth, in whom his spirit might rest, rather than the violent rejection of any of the fragmentary notions of such a Being which might be scattered among the forms of ancient Paganism,—if even a feeling that he had no right to dispense with the aids of ancient worship, except so far as his conscience told him that they must proceed from falsehood, because they led to immorality,—if these be the feelings which would have endeared a man, placed as he was, to a Jewish sage, then none of the praise bestowed upon Socrates, though it may often have taken a wrong form, has been misapplied or exaggerated. A Christian has no occasion to fear that in doing honour to him, he is dishonouring revelation or setting up some miserable substitute for it. Other philosophies may affect to be a provision for all the wants of humanity, that of Socrates is expressly a discovery of the want, and a cry to heaven for the satisfaction of it.

As it is the internal life of Socrates with which we are chiefly concerned, we must not dwell on any of the well-known circumstances of his outward history. Undoubtedly Athens has a right to plead in mitigation of her guilt towards him the fact that his name was associated with those of the two citizens who, by two opposite lines of policy, had wrought her the greatest mischief—Alcibiades and Critias. It cannot be denied that this circumstance may have unconsciously had weight with men not habitually indifferent to justice, and that the abundant evidence which proves to us that Socrates had

Moral and
Metaphy-
sical Phi-
losophy.

Mono-
theism.

His pu-
nishment.

Moral and
Metaphysical
Philosophy.

done his utmost to prevent the genius of the unhappy victim of mob popularity from proving his own and his country's ruin, and had incurred the bitter animosity of the tyrant, may not have availed to remove the prejudice in his own day. But after all fair allowance for this bias, it is impossible not to believe that young men, who knew that he had restrained them from corruptions to which they surrendered themselves when his influence was withdrawn, were the great abettors of the charge that he corrupted the youth; that men who dreaded and hated the notion of an ever-present conscience, were those who eagerly voted him guilty of introducing new demons; and that those who convicted him of not honouring the gods whom the city honoured, were inwardly conscious that they honoured no gods at all. The crime showed how little that national liberty, which the Athenians had just recovered, could avail without the higher liberty of which Socrates had spoken; it excused, though it could not warrant, the despair to which some of his disciples yielded respecting the condition of their countrymen; it has proved to all subsequent generations that every really honest and wise man, even if he has not the highest and most offensive wisdom, must in general choose between the indifference and the hatred of those for whose sake he has lived.

The
Socrates of
the *Clouds*.

The hints which we have thrown out may, we think, enable our readers to reconcile the three documents which we possess concerning the life of Socrates. If we look first at the Aristophanic portrait, we shall find that it is indeed a broad and extravagant caricature, but still drawn by a consummate artist who, even in distorting the expression of his original, shows that he had studied it. We could not consistently bestow this praise upon him if he had, as some of his commentators pretend, represented Socrates as a natural philosopher. But the name of the play of which he is the hero is really almost the only excuse for such a notion. And who that knows anything of the genius of Aristophanes, or of the delicacy of the Athenian taste, will suspect him of perpetrating, or his audience of tolerating, the wretched conceit that a man worships the clouds, because he is fond of gazing at the stars? Far rather are the airy nymphs whom the philosopher is said to have substituted for the gods of his country, the patronesses of those attempts to catch the thin, delicate, evanescent meanings and shadows of the meanings of words which might so plausibly be imputed to one who estimated philology highly for its own sake, and found it so indispensable a weapon in his warfare with the sophists. The basket too in which the philosopher is found hanging between heaven and earth, because he wishes to mingle his thoughts with the congenial air, indicates no sort of apprehension on the part of the poet that Socrates looked upon himself as a mere particle of the general life of the world, and desired to be reunited with his native element; (thoughts which would have very well become Anaxagoras;) but on the contrary, points to that doctrine of the withdrawal and disengagement of the spirit from the appearances and phantasms of the world, which we have spoken of as forming so capital an article in the moral creed of Socrates, and of which his idea respecting the condition of the soul after death is only the expansion and fulfilment. The maps and geometrical instruments which the old Athenian found in the phrontisterium partly prove that illustrations from subjects with which the education of the Athenian youth made them familiar, were frequently in the

philosopher's mouth, and partly seem intended as a joke at the Socratic attempt to reduce morality to a science. The dialogue respecting the cause of thunder is evidently intended far more as a caricature of the philosopher's method of discourse than as an exposition of any of his particular opinions; the chief object being to leave an impression on the hearer's mind that Socrates substituted some special demon of his own (which the poet, to keep his metaphors consistent, and to strike an oblique blow at the really physical speculators, calls *Διων*) for Jupiter. It is necessary to make these remarks in justification of Aristophanes, for if in these parts of his play he has wished to represent Socrates as a naturalist, the whole plot of it is absurd and inappropriate. Why should Strepsiades go to a natural philosopher that he may learn how to cheat his creditors? or how should such a teacher give Pheidippides lessons in beating his father? But the most remarkable feature in the whole play, the contest between the just and the unjust principle,—this is at once decisive as to the meaning of Aristophanes, and shows us how conspicuous a place the truth of a conflict within us between a power of good and of evil, an uplooking and a downlooking mind, must have held in all the discourses of Socrates. It is this truth of which his ingenious satirist has laid hold, and which, with the quick, intuitive discernment which might be expected from an Athenian, and such an Athenian, he has perceived to be the most characteristic and important peculiarity of the system he was ridiculing.

The one point in the life of Socrates of which Aristophanes shows himself to have been utterly ignorant is, the object of it, and this is the one point upon which *Xenophon* is anxious to give us information. This worthy disciple is too much upon the stilts, he is too anxious to show us Socrates in his dignity, and therefore we miss the hearty humorist, who may be seen, though disguised, in the comedian's picture. It was natural that a soldier should be more struck with the positive conclusions at which Socrates arrived upon direct practical matters, than with his method of arriving at them. It was equally natural that the professed apologist should be eager to exhibit his master in the way that would be most intelligible to plain persons, who had been puzzled with reports of his strange argumentations, and who had fancied that some great mischief must lurk in them. But if we bear these facts in mind, and look upon *Xenophon* as rather the expounder of the Socratic discipline than of the man himself, or of his principles, we shall probably be much more struck with the agreements than with the differences between him and the other biographers. Constant homage to an invisible guide and teacher, the distinction between the principle in man that looks upward and that which gravitates to the earth, the recognition of all restraints upon the animal nature as means for the enfranchisement of the true man, we shall find in every page of the *Memorabilia* as the actuating convictions of the philosopher's life. Standing alone, *Xenophon* would be unsatisfactory, nay, even misleading. His Socrates would be almost as much a mere bundle of fine qualities or true opinions, as his *Cyrus*. But he is most useful in giving clearness and steadiness to the apprehensions which we derive from other, and, on the whole, better sources. We see clearly in him that Socrates did from first to last keep a moral end before him. We see that he was, to all intents and purposes, a practical man.

Moral and
Metaphysical
Philosophy.

The So-
crates of
the *Memo-
rabilia*.

Moral and
Metaphy-
sical Phi-
losophy.

The
Platonic
Socrates.

And this discovery, instead of making it more difficult to interpret those accounts of him which some think inconsistent with it, really makes those accounts more intelligible and more consistent with themselves, than we should otherwise have thought them.

In the Socrates of PLATO we find both the Aristophanic and the Xenophontic Socrates the mere humorist and debater, and the mere moralist, uniting to form the real man. It has often been said that the brilliant imagination of this philosopher created a hero between whom and the actual Socrates there were, perhaps, very few points of resemblance. Certainly it would be a hopeless task to vindicate Plato from the charge of a brilliant, and more than a brilliant, imagination. But two meanings may be given to this word. If it signifies a contempt of reason and probability, the gift, we apprehend, must belong in a much lower degree to Plato than to those who conceive it possible for a person living in the very city wherein Socrates had been for years walking and talking, to have palmed upon his countrymen a false or fantastic image of him. If, on the contrary, by imagination we understand the power of giving to that which would be otherwise a mere shadow, substance and life, it must surely be a most serviceable ally to him who would collect and harmonize the remembrances of an actual character no less than to him who would call into being one that never existed. Strong affection may supersede the necessity of such a faculty in a mere biographer, or rather, perhaps, may awaken it. But one who has not only to describe the thoughts, words, and acts of a friend, but to show how they bore upon the state of his country, and how they will bear upon men's speculations and lives for ages to come, has need that no ordinary measure of this faculty should be imparted to him. Now this is the work of Plato. It was Socrates, as the guide into a particular line and course of thought, that he proposed to exhibit. But in order to do this, it was absolutely necessary that he should be brought livingly before us, that we should see not his opinions, but himself; that we should be able to trace the workings of his mind, to see how he acted upon others and they upon him. By any other means Plato would have been unable to give us the true Socrates, and without presenting us the true Socrates, he could never have brought out with any clearness and distinctness the different sides of his own philosophy.

Influence of Socrates upon the subsequent Developments of Greek Philosophy.

Practical
character
of the
Socratic
philosophy.

But there are prejudices upon this subject which will not be overcome, even if we should succeed in persuading our readers that the Platonic dialogues contain a faithful picture of Socrates himself. The question will still be asked, Is the Platonic *system* the work of the master or the disciple? And it will not be answered by the obvious remark, that the seed of a doctrine may be sown by one thinker, and may only fructify in the speculations of another. For it is the old notion that Socrates brought philosophy from the clouds to dwell with men, and it is the equally common notion that Plato carried philosophy into a more fantastic region than it had ever sought for itself before. The one is regarded as characteristically the teacher of practical wisdom, the other as characteristically the idealist. What relations can

have existed between men so opposed in their objects, and in the habits of their minds?

Some writers, we have seen already, have found it possible to escape from this difficulty by denying that the traditional phrase respecting Socrates has any propriety. Instead of bringing philosophy from the clouds, they agree with Aristophanes that he made them its ordinary habitation; his grand object was to mystify his hearers; to the old disputes respecting the origin of things, the dignity of number, the perception of unity, he added a host of new puzzles of purely native invention. Such a person, therefore, might well be the parent of the ideal philosophy; it was but a consistent form of folly created out of the primeval nonsense to which Socrates had contributed the most important, that is to say, the most absurd, elements. From all the advantages of this explanation, however, and from the popularity which always attends the impugnors of an established reputation, we have effectually excluded ourselves by our previous remarks. We have admitted the truth of the notion which for so many ages has been current; we have endeavoured to show that Socrates did bring his philosophy to bear upon the actual condition of society, and that it had a substantiality to which none that preceded could make any pretensions. If the liveliest interest in the commonest, the most mundane transactions of his fellow-creatures,—if intercourse the most familiar with all classes, high and low,—if courage displayed, not in word fighting, but in actual battles with swords and spears,—if the pursuit of a direct practical end,—entitle a man to be considered something better than a dreamer, Socrates abundantly vindicated himself from that disgrace.

But were his inquiries then less deep or less mysterious than those which had occupied the schools of Asia or of Italy? We believe not. We believe that they were more human and practical, and *therefore* more deep and mysterious. Hitherto the questions which had engaged the students of Greece had related to the powers of man. Thales and his successors had questioned nature to tell him what they were; Pythagoras had studied their workings and combinations in human society. Parmenides had discovered that they have a wonderful region of their own, apart from nature, apart from fellowship. Then came the Sophists amalgamating all these doctrines; adding nothing to the knowledge of man's powers, or of the conditions under which they exist, but converting them into objects of worship, and into instruments for destroying all the faith and order of the world. This is evidently a crisis in our history. Philosophy was setting itself up against all that is dearest and most sacred to man; this any ordinary observer could discover. Socrates was permitted to see that it was also compassing its own destruction, that the idolaters of man's spiritual powers were making the exercise of those powers impossible. Philosophy, or inquiry, presumes that something *is*; the Sophists of that age, as of every subsequent age, taught that the thoughts and powers of men *originated* that which they took an account of. Laws, virtue, gods were all their offspring. Then, said Socrates, philosophy is a lie, the powers of man are a lie, and you who are worshipping these powers are worshipping a lie. For in the very constitution of these powers that is assumed to exist, which you affirm them to have created; we are only thinking beings in so far forth as we acknowledge objects of thought which preceded and are distinct from the thought itself. We

Moral and
Metaphy-
sical Phi-
losophy.

Mysterious
character
of the
Socratic
philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

repeat these assertions, because it depends entirely upon our understanding them whether we understand Socrates, whether we see the connection between him and his great disciple, nay, whether we are able to take one firm step in manly investigation ourselves. Socrates then was the person who was destined to perform the task which we prophesied must at some time or other be performed, of connecting metaphysical with moral inquiries; of proving that a man, in order to understand his powers, must understand his position, in order to exercise his thoughts rightly upon nature or upon the relations of society, or upon his own spiritual properties, must arrive by some means at the knowledge of what he is. This is that firm practical ground upon which Socrates placed philosophy, abundantly justifying the opinion of mankind that he had given it a honest, less airy character than belonged to it heretofore; this is that deep invisible ground upon which he placed philosophy, abundantly justifying the indignation of those wits who complained that he forced men to more painful exercises of thought, and involved them in profounder mysteries than all his predecessors. A few words respecting the schools which owed their origin to the disciples of Socrates will illustrate these remarks; will show how great a revolution he had accomplished; will explain how far it is possible to separate the practical from the mysterious in philosophy, and will enable us to establish the assertion that Plato was guilty of no fraud, but evinced his singular honesty, his insight into principles, his skill in the choice of means, and his admirable modesty, by making Socrates the expounder of truths which, in their fulness at least, he himself had been the first to unfold.

Socratic
schools.

The Cyre-
naic school.
Aristippus,
Theodorus,
Hegesias,
Anniceris.

The immediate outgrowths of the Socratic philosophy and discipline were three schools, ordinarily distinguished as the *Cyrenaic*, *Cynic*, and *Megaric*. These may be said to be the parents of the most conspicuous theories with which later Greece was occupied. The Cyrenaic doctrine, having mingled with a tributary stream flowing from the physics of Democritus, terminated in Epicurism. The Cynic combined with the Megarian to constitute Stoicism. The Megarian moreover contributed one element to the important speculations which had their home at a much later period at Alexandria. It is interesting, therefore, to trace the leading thoughts of each, and to show how they originated with Socrates.

Aristippus of Cyrene seems to have been a man of a singularly easy, happy temperament. Never having been troubled with any internal conflicts, and having probably a healthy frame and a good digestion, he found little in the world to annoy him. Pleasures excited him not, pains passed lightly over him. Few men, one would have thought, would have had less sympathy with Socrates, who was a hard fighter all his life long with himself and with the world. Nevertheless this earnest thinker had charms even for Aristippus. Socrates said that we are not to yield to circumstances, but are the masters of them; and the light spirit which no circumstance affected or oppressed found an interpretation for the maxim in his own experience. The perturbations and restlessness of the thoughtless, unrecollected man were frequent topics for the pity and warnings of Socrates; could there be a more natural inference than that freedom from annoyance, a dismissal of all careful and turbulent anxieties, is the great end of philosophy? In addition to these, the well-known common-places of his master's discourse, Aristippus could no doubt quote

authentic fragments of his conversations, in which he had seemed to assume pleasure as the end of life, and to adjust his other maxims to this conclusion. He could tell, we may be sure, of cases in which Socrates, addressing himself to his own lazy, voluptuous habits of mind, and reprehending them, had yet seemed to make it his object to prove, not that they were leading to a wrong end, but that they were ill-chosen means for accomplishing that end. Aristippus, therefore, easily persuaded himself that he had a good title to call himself a Socratic, nay, that he was the best and most complete interpreter of the Socratic views, when he announced the great discovery, that pleasure and pain are the ultimate principles of human life; that the pursuit of the one and the avoidance of the other is and must be the business of every man. Whatever honour belongs to the first formal promulgator of a doctrine which has occupied so prominent a place in the philosophy of all ages as this, must in all justice be given to Aristippus. That in which he is distinguished from later and less practical reasoners of the same class is in the distinct and honest assertion, that the momentary, concrete gratification, and not the complex notion of happiness, is and must be the object of men's desires and labours. It was easy for Aristippus to adjust some other portions of the Socratic creed to this doctrine. If the choice of what is agreeable, and the rejection of what is disagreeable, be the great virtue of the human soul, how conveniently might the language of Socrates respecting the connection of virtue with reason and knowledge be pressed into the service of the new sect. Of course it is the intellectual faculty which prefers and discards, and why should not these acts of judgment be the same with those acts of reason, that perception of what is and what is not, to which the master had so constantly referred? And as for the apparent self-restraint and bodily privations of Socrates, these were in no real contradiction to the Cyrenaic theory, which admits, of course, all varieties of taste, and may well permit one man to seek mental pleasure at the expense of corporeal, another corporeal by the loss of mental. This school underwent several changes. In the hands of Theodorus pleasure and pain ceased to be real outward objects, and self-seeking and self-glorification became the defined, acknowledged ends of the wise man. In Hegesias the hope of attaining pleasure is exchanged for a mere invention of contrivances to avoid pain. Anniceris seems to have taken off the rough edges of the doctrine, and to have prepared the way for its merging in the more general notions about happiness which were matured by Epicurus.

The Cynic school—as it presents itself in the persons of Antisthenes, its founder, and Diogenes, its only very notorious disciple—is the formal opposite of the Cyrenaic. Yet they added one to the numerous illustrations of the old maxim, which Mr. Coleridge has observed to be of all maxims the most pregnant for the philosopher and the philosophical historian, “Extremes meet.” Both in fact started from the same Socratic maxim; both may probably have alleged the same discourses in vindication of their system. The wise man should not submit to circumstances, but rule them, said Aristippus; his whole business is to arrange his circumstances that they may produce the maximum of pleasure, and the minimum of pain. A man is to be superior to his circumstances, said Antisthenes, and therefore he is by all means to overcome his sensibility

Moral and
Metaphy-
sical Phi-
losophy.

The Cynic
school.
Antisthenes
and Dio-
genes.

Moral and
Metaphy-
sical Phi-
losophy.

to pleasure or pain, and endeavour to live solely within himself, cultivating that nobler part of him which is not affected by outward impulses and impressions. If the first could allege passages from the discourses of Socrates in support of his theory, the latter could more confidently appeal to the whole course of his life, to his habitual endeavours after a victory over mere sensations. The Cynics were in fact more disciplinarians than doctrinists. They had a hard dogmatism of their own, but they were much more ambitious to show their own indifference to passing accidents, than to discover principles and reasons for such an indifference. Of the two professors of the school Antisthenes seems to have been the honester, Diogenes the more original. The first was hard and narrow, but apparently sincere; the second was an ostentatious coxcomb, from whose proud and insolent spirit were emitted now and then sparks of what might have been genius, if it had been accompanied with simplicity of character and a true purpose.

The Megarian school.
Euclides,
Eubulides,
Diodorus,
Stilpo.

Euclides of Megara was unquestionably a more sagacious and subtle man than any of those we have named. He was attracted to Socrates by no hope, either of obtaining a theory respecting life, or of discovering a scheme of self-culture, but by his unrivalled skill in disputation. Had Euclides lived thirty years earlier, he would have been an Eleatic or else a Sophist. But in nothing is the effect of the Socratic teaching, and the change it had wrought upon the minds of his countrymen, more remarkable than in the moral tone which it imparted to the thoughts of those who would otherwise have been metaphysicians merely. To argue was the taste and the vocation of the Megarian school, but their arguments were all irresistibly drawn to the question, "What is the good?" In pursuing this inquiry, they were naturally led to those pregnant positions of Socrates respecting evil, as a departure from, and rebellion against what *is*, which constituted, as we have seen, the ultimate, and, in one way, the most characteristic part of his philosophy. This principle, in fact, disjoined from all the living processes by which Socrates had arrived at it, and by which he sought to make other men conscious of it, and exhibited in naked opposition to all other ideas of virtue or goodness, constituted the Megarian doctrine. All their labours were employed in disproving the obvious and apparently irresistible opinion, that those things whereof the senses give us information, are the most real and certain. We have heard how Zeno defended the doctrine of his friend and master Parmenides by showing the utter instability of all sensible presumptions and conclusions. The Megarian school adopted the same method. The difference lay in the characters of the respective periods; the purpose of Zeno was to support the metaphysical idea of oneness,—of the Megarian, to support the moral idea of absolute, unchangeable Being.

The history of this school is melancholy and instructive. Euclides, though the bias of his mind was to disputation, felt the grandeur of the moral truth which he had learned from Socrates. In Eubulides positive faith was superseded by delight in his own subtlety, and in the confutation of antagonist arguments. The mere forms of the understanding, apart from all vital principles or results, seem to have been the objects of admiration and reverence to Diodorus Cronos. Lastly, Stilpo seems to have lost the characteristic idea of the Megarian school altogether, while he carried its charac-

teristic infirmity to its greatest height. Not to establish the existence of objective truth, but to show how an intellect may be formed which shall be most impassive to influences from without, and least disturbed by affections from within, was his problem. One of his pupils was Zeno of Cittium, the author of Stoicism.

Moral and
Metaphy-
sical Phi-
losophy.

Plato.

Plato conversed with both Aristippus and Antisthenes. With Euclides he enjoyed a closer intimacy than with either of them; for to Megara he and other disciples fled after the death of Socrates, when it seemed less safe to dwell in Athens. It would be rash to say that the direction of his own thoughts was determined by his observation of these three men, for it is a notion apparently well supported by internal evidence that his *Phædrus* and his *Laches* were written in the lifetime of his master. Yet it seems impossible to doubt that he had very early noticed the tendency in his different fellow-disciples to adopt certain sentences which fell from their teacher's lips, and from these to form systems and schools, and that he had considered very deeply whether there were no course by which he might escape from the like temptation. If Socrates had compounded his creed out of the different systems then prevalent in Greece, it could surprise no one that the elements thus artificially put together should reassert their independence, and in some new shape, perhaps, be claimed as the property of the minds to which they were severally most adapted. But every thing that he had seen of his master made this supposition impossible. The very secret of the power which Socrates exercised over men of tempers and intellects so dissimilar arose from his having been emphatically original. Whether he had studied the doctrines of other schools or no, it is evident that every thought which he uttered came fresh and living from himself, or, rather, was the united fruit of his own reflections and of those of the persons with whom he conversed. It was evident that he had been able to minister to other minds, because he knew so well what was passing in his own, and had sought out every principle as the solution of an actual difficulty. But it is fair to suppose that every philosopher is in some sort an inquirer into the working of his own mind, nay that his philosophy, so far as it is sincere, is an exhibition of his own mind. How then was Socrates, who was so remarkably *himself*, preserved from that narrowness and exclusiveness, into which Aristippus and Antisthenes, both sincere men in their way, had obviously fallen? Plato could only answer the question by supposing that it was the healthy habit of always connecting his own thoughts with outward circumstances, and with the puzzles of the age in which he was living, which prevented the Socratic doctrines in their owner's hands from ever stagnating into a mere theory. The obvious resource for making his philosophy complete and general, and suited to all times, was to strip it of those accidental features which had adapted it so happily to a particular crisis. Plato was convinced, by reflection and experience, that precisely the opposite course was the safe one. The poetry of Homer could be read and enjoyed in the age of Pericles, not because it stood aloof from all temporary and local accidents, but because it was enveloped in them. It was exactly when men were presented to them as they were in an entirely different state of manners, that they were able to realize

Relation
between
these
schools and
Plato.

Moral and
Metaphy-
sical Phi-
losophy.

them as their brethren and their countrymen. Reasons will no doubt occur in multitudes to the reader, why the analogy of poetry is inapplicable to philosophy; it is sufficient for our purpose that they did not weigh with Plato. No one knew so well as he—no one felt so strongly—the essential difference between poetry and philosophy; he even was betrayed into exaggerations in his attempt practically to assert it. But he was convinced that it did not consist in this, that the poet obtains immortality for thoughts which he utters by adapting himself to the feelings and the temper of the age in which he lives, and the philosopher by divesting himself of them all. He thought he could see that the abandonment of all living and practical sympathies, the attempt to divorce himself from human interests, gives to the philosopher that narrow and bounded character, from which he hopes by these means to deliver himself. If, then, Grecian wisdom was not to retrograde from the point to which Socrates had brought it, or if it was ever to become useful in other countries and periods, Plato concluded that it must not resolve itself into speculations or declamations about this or that scheme of life, this or that principle of action or pursuit, but must be content to exhibit itself in the conversations of actual living men, not of some imaginary day, but of that day, talking about the actual matters of which they did talk when they met in the streets or at their feasts; not taking the least pains to forget the men among whom they were living, or the transactions that were occupying them, or to adopt any more universal mode of thought and speech than that which was common among them.

Why he
selected the
Dialogue.

The DIALOGUE of Plato is not then, as some have represented it, an artistical invention, in which the philosopher sacrificed his severe judgment to his imagination, or to a desire of reputation for dramatic skill with his contemporaries, or with posterity, or to the ambition of presenting his truths in an agreeable form. It is most evident that he regarded it as a necessary mean for the elucidation of the truths with which he believed himself to be possessed; and that he is not at all more anxious to impress any one principle upon his readers than this, that in the Dialogue, rightly used, we have the right induction to all principles. It is strange, indeed, that Plato should be accused of sacrificing the interest of his disciples to a selfish desire of fame, by that method which he has adopted specifically for the purpose of leading them onwards step by step in self-inquiry; or that he should be supposed to have used this as a means of conciliating their favour, when, in fact, it has caused more conscious vexation and irritation to every superficial student of subsequent days, as it did to every superficial talker of that day, than any which his genius could have devised. A mere artist endeavours to carry us at once into noble contemplations, which make us conscious of our own greatness and dignity. It is Plato's continual desire that we should feel our own way into these contemplations, that we should understand that if we would really have them become parts of ourselves, we must ascend into them through very rugged and thorny paths, that we should be aware how many frivolous difficulties we shall be suggesting to ourselves and must clear away if we would really see any thing as it is. His Dialogues are literally an *education*, explaining to us how we are to deal with our own minds, how far we are to humour them, how we are to resist them, how they are to enter-

tain the glimpses of light which sometimes fall upon them, how they are to make their way through the complications and darkness in which they so often feel themselves lost. Nowhere but in the sacred oracles do we find an author so cognizant of our own perplexities, so little anxious to hide them from us; nay, so anxious to awaken us to the consciousness of them, in order that we may be delivered from them. Herein lies the art of Plato. Most consummate art it is we admit; superior in the depth of insight which must have led to it, and in the influence which it exerts to that which is displayed in almost any human composition. Still it is not art, in the sense commonly given to that word; it has no independent purpose of pleasing. It does not work underground, leaving the ordinary man to feel its effects simply, and the thoughtful man to judge of its character by its effects. On the contrary, it most anxiously draws your attention to its own methods and contrivances: that you should enter into them and understand all the springs and valves that are at work, is as much the writer's ambition, as that you should accept any one of the final results. Indeed, he does not acknowledge the results as yours, till in the region of your own inner being you have gone through the processes which lead to it.

This is so primary and characteristic a principle with Plato, that we must be permitted to fix our readers' attention upon it, as that through the understanding of which they must attain to all his other principles. Plato above all men must be studied in Plato. A hearty and sympathizing acquaintance with one Dialogue will do more to initiate a student into what is blunderingly called his system, than the reports of all philosophical critics and historians. There you find no digests of doctrine, no collections of ready manufactured notions, to be adopted and carried away. Every one is alive and at work. The actors too are not, as in our best Dialogues, in those of Berkeley for instance, personages with significant names; they are real Phædruses, Gorgias, and Protagoras, discoursing in a place which is actually ascertained to us by an accurate and vivid description, about some passing question in the folds of which are found to be contained all the deepest and highest principles of our being. These are drawn forth not violently by any predetermination that such and such facts shall give forth such and such a moral, but by the ordinary accidents of conversation, amidst explanations and contradictions, the confusion of disciples, the anger of doctors, clumsy attempts at reconciliation by good-natured by-standers. The dialogue is often a *Siris*. Like Berkeley's admirable treatise, it may be bound here on earth to no worthier a stake than the properties and virtues of tar-water. Oftentimes the starting point may be one far less worthy than this, the lying speech of some sophist in support of some mischievous and vulgar paradox. Yet the chain is unwound with a skill of which our modest countryman would have cheerfully confessed that his was but a feeble copy, till its highest link is felt to be about the throne of Him whose name it was the privilege of Berkeley to utter, the honesty of Plato to declare unutterable.

Thus far we have described Plato as reasserting the entire principle of Socrates against those who had dis- membered it. But a notion has gone forth, and has received support from a highly able and eloquent French commentator of our day, that Plato was an Eclectic; in

Moral and
Metaphy-
sical Phi-
losophy.

Character
of his
Dialogues.

His sup-
posed
Eclecti-
cism.

Moral and
Metaphy-
sical Phi-
losophy.

other words, that his object on every occasion was to set in opposition two imperfect principles, and, either by merely showing their inadequacy, to suggest the hint, or, by clear exposition, to develop the form of a third idea which should include them both. This is the most plausible shape which the theory has taken. Another and common way in which it is stated is, that Plato framed to himself the notion of a philosophy which, taking its start from the doctrine of Socrates, should adopt into itself all the other Greek philosophies, whether metaphysical or moral, and that accordingly we do find in him not only an attempt to harmonize the doctrines of the schools, which took their name from Socrates, but also of those which preceded him. In both these statements there is, as it seems to us, much truth: yet truth put into a form which is exceedingly likely to mislead a reader, utterly to pervert his notions respecting the real object of Plato, and to deprive him of the benefit which a meditation on these remarkable works might procure him. We suspect that, in considering these theories, we may both arrive at a clearer apprehension of Plato's meaning, and gain some light which will profit us in all our future inquiries.

Moral dis-
tinction the
primary
purpose of
Plato.

One main object of Plato in using the dialogue as his mode of communicating truth, was, that he might discover the meaning that was latent in words which had been disguised by conventional usage, and might lead the inquirer to recognise this meaning as that which had been implied in them from their very origin, and had been really floating about in the minds of those who had given them quite a different signification. Hereby he was carrying out the method which Socrates, as we have seen, had been throughout his life maturing, and to which we have traced the success of all his experiments in moral science. For this practice was grounded upon a faith which it ripened day by day into certainty, that there is in every man that which apprehends and recognises truth; that the truth is continually near him; and again, that his view of it is continually interrupted and distorted by the phantoms which are presented to his senses. In drawing forth this truth out of the mind of the student and teaching him to realize it as his own, consisted, as Plato believed, the great duty of the Socratic teacher; to this all his labours were to be bent; so far as he did this work faithfully, he might hope to be rewarded with greater illumination. Never, however, was it to be forgotten that the discipline was a moral as well as an intellectual one, nay, that it was primarily and essentially moral; that he must resist the attractions and bribery of sense in order to escape her impositions. Now the process we have described leads to a result which often looks very like the result of *Eclecticism*. An opinion seems to be rejected as false, an opinion that is set in opposition to it is shown also to be unsatisfactory, and then at last a truth is seen, or suspected to be hidden somewhere, which both alike had been aiming ineffectually to reach. The reader of Plato's Dialogues will be encountered again and again with instances of this sort. But let him beware of hurrying to the conclusion, that the reconciliation of these opinions, or the construction of another opinion which shall be more comprehensive than both, was the aim of the teacher. If he will quietly accompany him along the road, he will find that in such conversations as these, *distinction* is much more his object than accommodation. To distinguish between those images which the mind shapes for itself out of

the objects of sense when it is sense-ridden and sense-possessed, and that sound meaning and reality which it is capable of perceiving when it has sought to purge itself of its natural and habitual delusions, to teach it the art of rejecting as well as choosing, and to put it in the posture for either one act or the other; this is the intention of Plato. It may be that he has done more to introduce harmony and unity into moral speculations than any philosopher who has ever lived; we fully believe that he has. But he begins with cultivating in us the habit of moral distinction. He begins with leading us to feel that truth and falsehood are radical ultimate contradictions which cannot be accounted for or resolved into any others. To see that which is, as it verily is, this is the highest privilege of the best and wisest men; to see things as they are not, confused, sensualized, corrupted, this is the misery and curse of the thoughtless, slavish victim of inclination. To open that inward eye by which the reality of things is discerned in other men, is the vocation and privilege of him who has himself served an apprenticeship to truth, and feels that he is her servant. Such, we conceive, is the object of one large class of the Platonic Dialogues, which are the induction or vestibule to the rest. In these Plato is distinctly and emphatically Socratic. They must, indeed, differ in a most important respect from the actual conversations of Socrates, in that the end must always have been much more present to the mind of the writer, than it could have been to that of the speaker. Socrates argued with a sophist, and in the course of his argument traced out a line of thought which led his hearers to that ground of reality which his opponent was keeping out of sight, and presuming not to exist. Plato makes the refutation of sophists, and every turn and winding of the discourse, a mean of arriving at the principle which is always before him as the object of human inquiries. In Socrates the strongest feeling seems to have been, "I am certain there is something which is not appearance or phantasy, which man did not shape out for himself, but which will remain when all phantasies have disappeared, which is, and which I must recognise if I would be any thing but a phantom or shadow myself." This was the conclusion of a practical working mind. By earnest meditation upon this conclusion, Plato comes to feel that if there be an unseen reality in all things, a truth, a substance in things, of which the eye sees only the shape and the colour, there must be a truth and substance which has none of those sensible adjuncts, which is in itself, and the beholding of which is the function and highest attainment of the purified spirit. Now the outward shell of this opinion so closely resembles the doctrine of Euclides that we cannot wonder that some critics, in their desire to reduce the philosophy of Plato into fragments, should have pronounced several of the earlier Dialogues to be not in fact his, but the production of the Megarian school. All in which they found this substance, this *το ον*, put forward as the end of human investigations, they naturally connected with a system which had the assertion that good and being are identical for its prominent characteristic. Those who agree with us in the view we have taken will at once see the plausibility of the critic's notion, and its utter untenableness. In no part of Plato's works is the distinction between him and the Megarians so conspicuous as in this where he is asserting their own principle. For by adhering closely to the method of Socrates, by making

Moral and
Metaphy-
sical Phi-
losophy.

Purely
Socratic
Dialogues.

These Dia-
logues not
Megarian.

Moral and
Metaphy-
sical Phi-
losophy.

his Dialogues not the declaration of a truth, but a mental exercise to arrive at it, he has not only divested the doctrine of all its dryness and prickliness, but he has shown how it is connected with those other more obvious notions to which the Megarian set it in rude opposition. Pleasure is not the good, they said; self-denial is not the good; being is the good. Yes, said Plato, but there is a being in pleasure, there is a reality in it as well as a falsehood in it. Whatever man has found an expression for in language, whatever man has pursued as an object in life, there is in that a truth, a substance, which may be distinguished from the lying phantom which surrounds and counterfeits it. And so far as a man does this, so far does he put himself into the right condition of mind for arriving ultimately at the perception of that truth, that being which is encompassed with no accidents, but which actually is. But then, in order to attain or to cultivate this state of mind, there must be a discipline, a curbing, and contradiction of the lower nature, and therefore this too is a good.

Opinions
reconciled.

Without, then, any purpose of combining opinions, nay, while resolutely maintaining boundaries, and using a most subtle test for the discrimination of the true from the apparent, Plato had actually reduced the three doctrines which assumed the name of Socratic into a certain relation and harmony. It now became him to consider how far this same doctrine and method might be applied to the earlier philosophers of Greece; how far his master had been anticipated by Xenophanes, Heraclitus, Parmenides, or Pythagoras; how far he had thrown back a light upon them which might make their speculations more intelligible and consistent with each other. Here commences, in our judgment, the second class of the Platonic Dialogues, that in which the link between Moral and Metaphysical Philosophy, between the doctrine of being which Socrates had asserted, and the question respecting unity, which had been the great occupation of the Greek mind, is illustrated and developed.

Second
class of
Dialogues.

Our readers will not have forgotten that the leader of the Eleatic school, Xenophanes, was in one respect distinguished from his successors. His language at first sight seems remarkably to accord with that of Socrates. That which he supposed to be the true object of man's contemplation was God, or "The Being." Yet, while doing justice to the course of thought by which he arrived at this conclusion, we were obliged to admit that he was essentially a destructive thinker; that he reduced his Being to a mere negation of human qualities and attributes; and that Parmenides found a much happier expression for the results of his inquiries when he said that they simply led to the affirmation of oneness. How then did the doctrine of Socrates differ from that of Xenophanes? It was separated from it by a whole heaven. The Being of Xenophanes was altogether exclusive, the Being of Socrates was altogether inclusive. If the language of men contained such words as "just," "merciful," "good," if it attributed these names to certain acts, then, whether these words or no had been understood, whether they had been rightly applied or no, there was a reality corresponding to them, there was a "justice," a "mercy," a "goodness," and all these centred and united themselves in The Being. No sophist could embarrass him with the question, "Seeing man also uses the words unjust, unmerciful, bad, why should not these also have

their appropriate archetypes? and why may not these, as much as the others, dwell in that permanent and all containing substance?" For it was assumed in the very hypothesis that all these are departures from that which is, that they are intrinsically falsehoods. Now it was by reflection upon this difference, so delicate yet so vital, so strikingly marking the man who was fighting against all popular opinions and faith from the man who was finding out substance and life in all, that Plato seems to have gained his first insight into that doctrine of Ideas which constitutes the most native and peculiar portion of his philosophy, that which may not wrongly be called its purely Platonic portion. We are perfectly willing to admit the assertion which the other disciples of Socrates seem to have made with no little vehemence, and which Aristotle has adopted from them, that no such principle as this was enunciated by Socrates in any of his discourses. Yet we believe as undoubtedly, that by his steady adoption of the Socratic method Plato arrived at this principle, and that they failed in apprehending it only because they neglected that method. In endeavouring to make this remark clear, we shall also perhaps be able to give our readers such insight as a treatise like this may hope to give into the subject itself.

The Greek word for *appearance* and for *opinion* is the same. An opinion is that which *seems* to each man. Now the whole of the education and discipline of Socrates had been to lead his disciples away from appearances to realities. And just so far as he did this he felt that he was leading them from opinions to knowledge. His experiments upon others convinced him, his own heart told him, that there is in us a thirst after knowledge, that with less than knowledge we cannot be satisfied. These at least are Socratic assertions; no one pretends that these were palmed upon him by Plato. But how could this be? The essence, the being of a thing, or of a person, seems shut up in that thing or person. I may acknowledge that it is there, but how can it ever come within the region of my perceptions? Must it not be after all some shape or image or phantom of this thing which I take account of, and not the very thing itself? Supposing this were admitted, the Socratic philosophy falls to the ground. And what falls with it? Not a scheme or a system, but the faith that truth is any how cognizable by man, the faith that he is not the necessary dupe of shadows and impostures; the fact that he is a moral being. It was not a doubtful question whether these results would follow from such a determination as this; they had followed; the practice and education of Socrates had been nothing less or more than a deliverance from them. The sophists had turned the world into a shadow world, in which they could safely practise their juggleries. It was that great assertion that something is, and that what is only may be known, which had discomfited them, and made a mock of their subtleties. Yet unless this great practical puzzle could in some manner be resolved, the conclusions of Socrates, however ascertained to be sound by the reason and moral feeling of every one who fairly worked them out, would be liable to continual assaults on the side of the understanding. On this point, too, Plato was not left to his own conjectures. The sophists had stolen their armour from the real, the honest philosophers of Greece. They had not dared to grapple with this difficulty, and the proofs which they had left of its existence in their speculations

Moral and
Metaphy-
sical Phi-
losophy.

Ideas.

Practical
difficulty in
the Socratic
doctrine.

Difference
between
Xeno-
phanes and
Socrates.

Moral and
Metaphy-
sical Phi-
losophy.

Heraclai-
tans and
Eleatics.

had been eagerly laid hold of by those who knew so well how to suck the poison out of every flower.

We have intimated that a faithful and affectionate study of the strange, earnest thoughts which occupied Heraclitus might be profitable to any man. But his sayings concerning the endless vicissitude of things, and the falsehood of all human conjectures, made far more impression upon the Greek mind than his deeper thoughts respecting the universal light in man, and the power he possesses of conversing with that which is universal. These latter sentiments had only connected themselves with the vague pantheistic notions which were now gaining ascendancy, (notions probably very far indeed from the mind of Heraclitus himself, who thought it the ultimate wisdom to know the name of Jupiter;) the former, in the hands of Protagoras, had become a system which excluded all feeling of constancy and permanence or order. One man has one notion of the things which he beholds or meditates upon; one man another. Any one of these notions may be as right as another, and that we cannot have more than such notions, that we cannot arrive any more nearly to the truth of things, is a proposition not so much to be proved as to be taken for granted. Meantime the Eleatic "fixedness," which was the formal opposite of the Heraclitean "flux," served the purpose of the deceiver equally well. To be able to deny the fact of plurality, and so embarrass the minds of men respecting the objects of their worship, was just as convenient a line of policy as to upset their faith in their own convictions. Such observations might have led another man to despair of all inquiry; they only gave Plato a stronger moral interest in prosecuting those upon which he had entered.

Notions
and ideas.

All the notions, you say, which the mind forms respecting that which the bodily eye sees, or that which its own inward eye sees, are confused, fluctuating, contradictory. My notion of the flower is not the very flower; my notion of what is just is not the very just. Most true, Heraclitus; most true, Protagoras. But these notions are indexes, guiding-posts to that which is not false, or confused, or contradictory. This notion of the flower and of justice proves that there is a very flower—a very justice. Again, the mind is capable of beholding the Being, the One. But of this Being, of this One, all the notions, imaginations, premonitions of the sensual understanding offer most miserable and counterfeit resemblances. True, Xenophanes and Parmenides; yet there is that in this Being, this One, which does and must answer to these notions that which they are trying, however vainly, however awkwardly, to express. If, then, we connect the results of these inquiries, which start from such opposite points, what follows? There are forms permanent and unchangeable in which that which is manifests itself as it is; in which we behold it as it is. Are these forms, then, in the beholder, or in that which he beholds? We answer; the region of pure being, that in which the inner mind dwells, may be (one might expect that it would be) under some corresponding law to that of sensible phenomena. At all events there could be no *a priori* presumption against the doctrine that as a sound cannot, by the very nature of language and of things, be referred only to that whence it proceeds, but likewise involves the supposition of an ear which receives it, so there may be such a presentation of that which actually is, of the substance or essence of each thing, as can neither be understood merely in

reference to that thing, nor merely in reference to that whereunto it is made, but must by its nature be spoken of as appertaining at one and the same moment to both. But then this presentation cannot by its very nature be fluctuating or variable, it must be permanent and substantial, or it cannot make known that which is permanent and substantial; it must be the very opposite to that which is its parallel in the world of sense. Are we to say of such ideas or forms that they are eternal as well as substantial? To answer "yes" would perhaps startle no one, if these ideas or forms had merely reference to justice, goodness, or even beauty. But when we speak of the actual flower and tree that we behold as having a primary form or idea, is there not something dangerous in a doctrine which would represent such forms as eternal? The reply is, that this statement would do but very partial justice to Plato. For if in the minutest thing he believes that there is a reality, and therefore in some sense an archetypal form or idea, yet he believes also, just as firmly, that every idea has its ground and termination in one higher than itself, and that there is a supreme idea, the foundation and consummation of all these, even the idea of the absolute and perfect Being, in whose mind they all dwelt, and in whose eternity alone they can be thought or dreamed of to be eternal. This remark may also relieve the doctrine of another objection. These ideas, being by their very nature substantial, must be substantially in him who perceives them. It is only seeking to remove the difficulty a step further from us, and falling into a contradiction and absurdity in the attempt, to suppose that there are indeed forms or ideas of things, but that we have only notions or conceptions of these ideas. The idea itself must be considered as with us and in us; the notion which we form about that whereof it is the idea, when we begin to use our senses, to compare and to reflect, must not be identified with the idea, but is a witness and proof of its presence, and that we are feeling after it; to realize or to possess the idea is to have the science or the knowledge of the thing. But then this assertion, that these ideas are substantially with us, must be taken in connection with what has been said before, and it will be seen at once that, instead of affirming the ground and root of our knowledge to be in ourselves, this is the very falsehood which Plato was seeking to overturn. These ideas are the witnesses in our inmost being that there is something beyond us and above us; to enter into the idea of any thing is to abdicate our own pretensions to be authors or creators, to become mere acknowledgers of that which is. And to enter into that deepest and ultimate idea, which is the ground of our being, must be in the deepest sense an abdication of our own notions and imaginations, an act of submission to, and reception of, the Truth.

Here, then, we find Plato most consistently carrying out the principle which it had been the vocation of his master from first to last to assert. Here we see how perfectly harmonious the Socratic doctrine, that knowledge is the end of life, is with that humility and confession of ignorance which are at the root of all the Socratic discipline and culture. Here we see the harmony between knowledge and being; how necessarily a certain state of character and affections is presupposed in every act of knowledge. Here, lastly, we see how truly Plato did reconcile those two forms of philosophy, one of which had dealt with the objects of our knowledge, one with our acts of perception—how truly he did

Moral and
Metaphy-
sical Phi-
losophy.

Ideas sub-
stantial.

Ideas how
in us.

Plato no
theorist.

Moral and
Metaphy-
sical Phi-
losophy.

discover a truth, one side of which each had dimly perceived, yet how little this was the result of any project for harmonizing opposite theories,—how much rather it was the effect of resolutely pursuing a principle which supersedes theories altogether, and so far as it is faithfully acted upon, delivers us from our bondage to them. Not to frame a comprehensive system which shall include nature and society, man and God, as its different elements, or in its different compartments, and which therefore necessarily leads the system-builder to consider himself above them all, but to demonstrate the utter impossibility of such a system, to cut up the notion and dream of it by the roots, this is the work and the glory of Plato. He who is attempting the construction of such a Babel must understand not merely that he will not find the model of it in Plato, but that before he advances one step he must undo every thing that Plato has done, must disprove all his conclusions, and prove the falsehood of the process by which he has arrived at them. Those commentators who can find in Plato nothing but the most exquisite ridicule of all the system-makers in his own and in past days have, it is true, understood him imperfectly. That ridicule would not be so delightful and satisfying as it is, so thoroughly genial and consolatory to every earnest student, if he did not feel that it was the handmaid of the most severe demonstration, that it was only another aspect of the most generous and noble sympathy with every thing that is honest and humble, and practical and true. The kind and experienced teacher smiles at our useless waste of time in attempting to build, but it is that he may urge us to the more profitable occupation of seeking after the foundation of that which is built. The one is the employment of those who desire to be gods, the second of those who believe that the highest blessing of which man is capable is to know God.

His sup-
posed clas-
sification.

But if it be true that Plato is almost free from that propensity for theories which has beset most philosophers, nay, that his principle, consistently followed out, is positively incompatible with them, how is it that he is so commonly supposed to have mapped out the domain of human knowledge into the three provinces of Dialectics, Ethics, and Physics? In considering this question we shall perhaps discover the purpose of a third class of his Dialogues, and be able, moreover, to contemplate his so called Eclecticism under yet another aspect.

His dialectics and
ethics inseparable.

The critics who have discovered this classification in Plato evidently cannot mean that there is one portion of the Dialogues which does, and one portion which does not treat of dialectics. They must be aware that every dialogue exhibits the dialectic method of Plato, that every one is making with more or less success some new trial or application of it. Neither, we think, can they pretend, without doing violence to the purpose and language of their author, that these dialectical dialogues are not also ethical. Not only is a moral purpose conspicuous throughout them, but, as we have said before, the development of the method by which the truth is perceived and ascertained is inseparably interwoven with a moral culture. The disengagement of the mind from sensible impressions and sensible fascinations is the joint effect of restraint upon inclination and of the art by which the apparent is distinguished from the real. Without the feeling of this connection and intertwining of the ethical with the intellectual discipline, the most beautiful Dialogues are unintelligible; nay, the desire to separate that which Plato has believed inseparable, is

perhaps the main cause of the narrow and partial views which have prevailed as to the object and construction of his works. Look, for instance, at the *Phædrus*. Lysias had proposed a certain thesis respecting Love, and had defended it with abundance of ingenuity in studiously balanced sentences, the aptest clothing for a sophistical purpose and a sophistical method. Socrates shows, first, how easy it is to meet his arguments with a counter series as ingenious and as artificially expressed. Then he discovers the radical defect of both sets of arguments, that they were dealing with a word, the meaning of which had not been ascertained. In the attempt to discover what this word signifies, to separate the true from the apparent meanings of it, he unveils the whole principles of his dialectics. But there is combined with this exposition the most distinct declaration and warning, (assuming a form which, however unfit for us, was most appropriate to the evil condition of Greek society, and proves the purity of the writer, who, in the midst of such a society, could maintain so elevated a standard,) that only by restraining the grosser appetites can we be in a state for apprehending the true nature of Love. In the conclusion of the Dialogue, the two principles are harmonized in a splendid mythus, wherein the disciple is taught that only he who governs himself, who has his lower nature in subjection, can be fit for the highest exercise of his faculties, for the contemplation of that which verily and indeed is. We cannot reduce this Dialogue under any of the partial names and descriptions that have been given of it. We cannot consider it a mere attack upon the rhetoricians, or a mere development of the Socratic method for testing the meaning of words. Neither does it seem to us a treatise on pure love, or on the idea of beauty. All these subjects may be hinted at, and even most valuably illustrated. But the reader of the *Phædrus* must be contented to feel how they sustain each other, and to let them form themselves into a whole in his mind, without being eager to give the absolute supremacy to any one of them. He will then find, we believe, that this Dialogue is one of the most conspicuous, we might say the type dialogue, of that class which teaches us how to make substance or being the end of our inquiries and meditations; but he certainly will not be able to discover whether it is more ethical or dialectical. The *Gorgias* is another almost equally striking instance of the same kind. Formal critics have determined that this too shall be merely an attack upon rhetoric, or else that it shall have the merely moral object of explaining the nature and purpose of punishment. It must strike a person, who only hears of this discussion, that a work which could suggest such opposite interpretations must be most incoherent and rhapsodical. The more he reads it the more he will be struck with the sequency of its thoughts, with the natural and easy manner in which one grows out of another. And he will find, we believe, upon reflection, that as the intellectual purpose of the sophist was inseparably combined with the moral, as the pursuit of political power for an end was inseparably united to the cultivation of a treacherous art as the means, so it was impossible to introduce a sounder intellectual discipline among the youth of Athens, without leading them at the same time to perceive that the true purpose of their lives was not the acquisition of dominion or the escape from suffering and punishment, but the attainment, even through suffering, punishment, and disgrace, a deliverance from the moral evil

Moral and
Metaphy-
sical Phi-
losophy.

The *Phæ-
drus*.

The *Gor-
gias*.

Moral and
Metaphy-
sical Phi-
losophy.

which obstructed their search after truth, and made power a curse to them. Here, again, he distinguishes the true from the apparent, not by the help of ethics without dialectics, or of dialectics without ethics, but of both conjointly. How the case stands in reference to physics we shall have to explain shortly. But thus much we may affirm now, that whensoever his opponents have engaged in physical speculations, Plato is not unwilling in those Dialogues, which have most distinctly a moral purpose, to cope with them, and that he never in such wise divides these two provinces as to suggest the thought that the principles by which either is governed may not most usefully or injuriously affect the other.

The *Theætetus*.

Of this fact we could easily convince our readers, if we could afford space for an analysis of the *Theætetus*. But we must do no more than commend that exquisite specimen of Platonic wisdom to their careful study, and proceed to show how the notion that Plato established a formal division in the subjects of human thought may have originated, and may be reconciled with our belief in his hatred of theories and consistency of purpose.

Search
after unity.

We have often observed that the founders of the different schools in Greece had been led by the circumstances of their position, or by the peculiar tendencies of their own minds, to choose for themselves a distinct sphere of observation. It was no forethought or wish to be a natural philosopher that drew Thales into his course of speculation. It was merely that the outward facts submitted to his senses were those which struck him as most likely to contain the solution of the problem which he found within. Accordingly, many political and ethical theories may have been mixed by him, as we know they were by Heraclitus, with the physical doctrine from which he started. In like manner Pythagoras was led by a series of scarcely known or acknowledged influences gradually to desert the maxims of his native soil, and to make Society, or the State, the subject of his inquiries. To these we know he added a scheme of nature which he tried to reduce under the same law as the facts which related to the order and government of men. Parmenides and Zeno again entered into their purely metaphysical region, not from any resolution of choosing one province of thought more than another, but merely by reflecting upon the two preceding systems, and finding that they were inadequate. They too sought to make their own system universal, and to make the scheme of the political world and of nature, in some sense dependent on the laws which concern the region of pure mind. Plato then found the fact already established for him, that there actually are these three lines in which the thoughts of men, when they are strongly exercised, naturally run. He had not to create any artificial distinctions; the natural distinctions had been discovered by the experience of his predecessors. What remained for him? To follow them into each of these regions, to inquire how far any of them had discovered the unity of which he was in search, to consider whether what they had looked for in nature, in society, in the mind of man, may not be implied indeed in each of these, yet have its foundations beneath them all. This, we believe, was the final and consummate effort of the Platonic philosophy. As there is a set of Dialogues which seem to us designed merely to unfold the Socratic doctrine of Being, another expressly intended to develop the principle of Ideas, as necessary to the support of the former, and as solving a problem which the Heracleitans and Eleatics

had shown to exist, so we believe there is a third in which Plato reflects upon his master's discoveries and his own, and exhibits them in direct application to the three subjects of nature, of society, and of knowledge. Every one will recognise in the *Timæus* an attempt to discover a unity for the external universe; in the *Republic*, an attempt to discover the meaning of political unity. And, therefore, without contradicting our previous assertion that, in all his Dialogues, Plato is evolving a dialectical method, we may also admit that there are some in which he proposes himself the direct and formal aim of showing how this method is a guide to the true unity in knowledge. Nor must it be forgotten that Plato was enabled by his position as a reviewer of past systems, was obliged by his position as an expounder of Socratics, to reverse the order which we have followed in tracing the rise of these schools historically. He not only might begin with those principles which the Eleatics had expounded, and descend to the natural speculations of the Ionians, but he could not follow any other course if he was to make the Socratic doctrine of Being, and the Socratic method of distinguishing the real from the fantastic, his guiding stars through the whole journey. Here, then, we must begin our notice of his experiments in search of unity. We are eager to introduce our readers to that which we consider the crowning labour of his life, the end at which he was obscurely aiming through the whole of it—his POLITY; but we must first refer to his discussions respecting the conditions and meaning of SCIENCE. A few words will then be sufficient for the less important, though by no means uninteresting, question—how far his views respecting the PHYSICAL WORLD were in conformity or disagreement with his other principles.

There was one great and obvious difference between the position of the Parmenideans and that of the Pythagoreans or Ionians. No one could doubt that they had a real subject for their inquiries, let those inquiries be as idle as they might. But the Eleatic had to produce both the dream and the interpretation; to maintain that there was a region of a pure mind, as well as to show what was transacted there. Plato, therefore, had also two tasks. He had to show that their assumption was sound, before he ventured to inquire how far it was able to carry them. For this purpose he adopted a method which has greatly puzzled many of his readers. In the Dialogue entitled *Parmenides* he introduces Socrates, full of youthful vivacity, broaching the doctrine of ideas in opposition to the antiplurality doctrine of Parmenides and Zeno. The aged philosopher treats his antagonist with most graceful courtesy, allows him to put forth one explanation after another of his scheme, shows him that all are untenable, then encourages him to hope that, after a more severe philosophical training, he will understand himself better, and finally proceeds to establish his own doctrine of the One in a series of annihilating propositions, wherein he shows that it must exist, and that it cannot exist under any conditions or limitations with which the understanding is acquainted. Probably no one but Plato ever ventured upon such an experiment as this—the experiment we mean of showing that his own principle was untenable except so far as it is connected with and grounded upon the principle of another philosopher who did not recognise it. But this Dialogue of the *Parmenides* does not merely serve the purpose of establishing the truth that the mind witnesses of something which is not under sensible

Moral and
Metaphy-
sical Phi-
losophy.

Unity in
science.

The *Parmenides*.

Moral and
Metaphy-
sical Phi-
losophy.

The
Sophist.

laws, but it also prepares our minds to feel the want which the Eleatic doctrine could not satisfy. A conviction of the purely negative character of the method grows upon us as we read, and while we assent to its conclusions we feel an increasing moral interest in seeking for some higher point of view from which we may contemplate it, and that imperfect substitute which the young Socrates had proposed for it. In the *Sophist* this wish is to a great degree realized. There we have an Eleatic stranger discussing with the young Theætetus the meaning of the word Sophist, and the qualities of the animal which it denoted, how far it belongs to the same genus with the Philosopher, or how they are distinguished. In the course of this dialogue we arrive at the conclusion that the great object of the sophist is to set up a universal science. By universal science he means merely a capacity of talking upon all manner of subjects, of framing a set of images of that which is, and passing these off for substances and realities. But here a difficulty arises: Parmenides, whose opinions the Eleatic stranger might be supposed to favour, has told us that we are not to inquire respecting that which is not; that the conception is in fact an impossible one. How then will our definition of the sophist practically avail us? How shall we be able ever to pursue him on to that ground whereunto we have had the clearest evidence that he has betaken himself? The Eleatic stranger finds himself obliged then, much as he fears the guilt of parricide, to inquire into the soundness of this doctrine of his honoured countryman and teacher. By degrees the fallacy unveils itself. We find that in refusing to recognise the notion of not-being, he was in fact shutting his eyes to something which is. The contradiction may appear startling, but we do not escape it by refusing to look it in the face. We have actually stumbled upon an instance in which that which is adverse to reality must be treated to all intents and purposes as real. And when we look a little further into the use of language, we see more and more the impossibility of giving that definite rigid exclusiveness to the word or notion of being which it must bear in the system of Parmenides. We find that we cannot by any means identify our notion of sameness or of difference with our notion of being. And yet the notion of being enters into both of these; there is a sameness and a difference between things. Whither does all this lead? It leads to the conclusion which, as is so constantly the case in Plato, is carried more directly home to our understanding than it is expressed in words, that Parmenides is after all dwelling in a region of words. For all that he seems to have sounded the very inmost depth of thought, and though he has actually discovered that there are depths which words do not reach, yet he himself is at last only setting up one notion against another notion. It is not Being, but the notion of being which he has been investigating, and which he has necessarily investigated most imperfectly. And now, then, the vision of a new kind of science wholly unlike that of the sophist, yet in one sense as universal as his, wholly different from that exclusive dogmatic notional philosophy of Parmenides, yet like his, having unity for its condition and ultimate ground, opens upon us. This is the science of Dialectics, the science expressly appertaining to the philosopher. It is that which rejects no form of thought or language as unfit for its investigation, but searches out the idea of each, and ascertains what notions and phantasms are inconsistent with it, and attach themselves to

it, and have sought to make themselves part of it. Being it looks upon as the object of its search, but being connected with life, connected with power, not a dry abstract notion, the mere negation of other notions. This must be the science of sciences; not because it reduces all forms of thought to one, or because it includes all existing sciences, but because it discovers in all forms of thought an ultimate ground of unity beneath them all, because it assigns to each science its specific object, and its relation to every other.

Every science is seeking after a foundation. It rests upon the faith that there is a law for the facts which it inquires into; what that law is, is the subject of its inquiry. The science should explain to us how the mind proceeds in the search after these laws through whatever set of facts we may be looking for them. But its own specific object is the deep ground of all laws, the being from whom they derive their life and potency. This dialectics then is the search after premises. Parmenides and Zeno had gone no higher than to the notion of a Logic which, taking the premises of the mind for granted, should affirm what conclusions will legitimately and necessarily flow from it. This was in fact their universal science, as Rhetoric was that of the sophist. The first overthrew all facts, setting up a law in the mind in opposition to them. The second confounded law and fact, using certain laws of the mind to overturn admitted facts, or the contradictions of facts, to disprove the existence of law. The principle of seeking for laws in and through facts is the principle by which Socrates upheld the faith and morality of men against the invasions of Rhetoric, and by which Plato upheld the possibility of discovering science against the invasions of Logic. Here is that inductive science which two thousand years after had to maintain the same battle against the same enemies, when it for the first time clearly and efficiently asserted itself as the only guide to a knowledge of the physical world. The Socratic doctrine of being, makes us feel that it is necessary. The Platonic doctrine of ideas makes us understand that it is possible. The demand in the mind for a unity which shall be not negative, but positive, not the amalgamation of parts, but that which preceded all distinction into parts, and remains unaffected by it; this makes us confident that it is real.

The connection between the Platonic dialectics and the Platonic politics is indicated by several passages in the dialogues, which are intended to unfold the character of each. Thus in the *Sophist* the Eleatic stranger proposes to examine the meaning of three words, *sophist*, *philosopher*, *politician*. The last is reserved for a separate discussion, and actually forms the subject of one dialogue. But it is obvious that Plato intended us to feel how strong was the relation between the three persons denoted by the words, and how impossible it would be to investigate the last without a previous knowledge of the other two. We have had occasion to remark already, that while the sophist sought to make his rhetoric a substitute for all knowledge, his real object was to make it subservient to political ends. The young Athenian, under his teaching, began more and more to feel that the rhetorician and the statesman were convertible terms, that the efficient and practical ruler of men was the efficient master of words. On the other hand, it is obvious that the logic of Parmenides and Zeno must have made politics a mere set of abstract propositions, to which an earnest and enthusiastic man might impart a practical meaning and life, but which would in fact lose

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphysical
Philosophy.

all the qualities which had endeared them to him as a logician, in proportion as they received this impregnation. Zeno himself was probably a specimen of this class, but we have no reason to suppose that he left any successors; his disciples must either have devoted themselves merely to inquiries respecting entity and unity, or have become politicians of the rhetorical kind. Now it is obvious that the dialectics of Plato, by their very nature and definition, could not pretend to be themselves politics. They necessarily assumed facts, not indeed as premises to start from, but as the raw materials, in the heart of which premises lay hid. The laws and conditions of society were to be investigated by this dialectic, could not be investigated without it; but they were presumed to exist, and they were perfectly distinct from the faculty and science by which they were discovered. Plato then was in a condition to do justice to Pythagoras as he had done justice to Parmenides. He could believe that there might be much in the discoveries of the one respecting the order of society, as there had been much in the discoveries of the other respecting the conditions and requirements of the pure mind, and he prepared himself to use the philosophy which he had ascertained to be true in one region for an investigation of the observations which had been made and the doctrines which had been broached respecting the other.

But this is not all. The reader of Plato's *Republic* will discover, not, perhaps, without some surprise, that questions respecting the nature and objects of dialectics occupy a considerable place in this political treatise. Hence it has been inferred that the title of the work is altogether misleading; that it is not the purpose of Plato to teach us the conditions and the relations of a state, but merely to raise a platform upon which to carry forward with more interest, and probably with greater success, the philosophical inquiries which he has commenced in his other Dialogues. The *Republic*, according to these commentators, is a kind of laboratory which he has built for the purpose of pursuing his experiments into the nature of the individual man. On many accounts this hypothesis will be likely to find acceptance in this day, even if it had not been supported by talents and erudition of the first order. We are not anxious to refute it, but we are anxious that our readers should study the *Republic* with a free spirit, and that they should not be hindered by a theory from perceiving how it illustrates a subject which has occupied us so much already, and must occupy us at every step of our future progress; we mean the relations between the mind of man and the constitution of society. To Plato we believe was committed the task of expressing the deepest wants and necessities of our being, and of discovering or prophesying the kind of satisfaction that must be provided for them. We make no apology for dwelling so long upon his name in this rapid sketch of moral and metaphysical inquiries, because we are satisfied that if we put our readers in a right course for studying his works and those of Aristotle, and the Jewish Scriptures, they will be able to trace with little assistance from us the progress of these inquiries in modern Europe, whether in the age of the fathers, of the schoolmen, or in the period since the Reformation; whereas the most able and elaborate discourses upon the ethics and metaphysics of later times without this preparation can we think avail them very little. It must, therefore, be a great point with us to ascertain whether Plato is silent upon those wants which belong to our social being and

position; whether he thought them of a purely secondary and accidental character; or whether he found them so imbedded in the constitution of man that he could not investigate the law of each man's internal life without also investigating that by which he is related to his fellows. The supporters of the former opinions (as the readers of Schleiermacher's Introduction to the *Republic* will perceive) have felt that they needed and have not failed to exercise the most admirable ingenuity in working out their conclusion. Those who adopt the latter may content themselves with beseeching attention to the express language of Plato, and with briefly recapitulating the heads of his dialogue.

We find ourselves at the beginning of the *Republic* in a circle of Athenians met to witness a religious ceremony lately introduced from Thrace. The most prominent and interesting person in the group is old Cephalus, a cheerful and benignant octogenarian, with whom Socrates begins a discourse on the comforts of old age, and the advantage or disadvantage of riches. A remark of Cephalus leads to the inquiry, whether the definition of justice given by Simonides, that it consists in speaking the truth, and giving to each man that which is his due, is satisfactory or no? Cephalus being called away to perform a sacrifice, his part in the dialogue devolves upon Polemarchus. But we have not advanced further in the discussion than to a general consent of the parties that the mere external acts indicated in the words of Simonides cannot satisfy the idea of justice; and again, that its obligations cannot be affected by our position to each other as friends or enemies; when the sophist Thrasymachus breaks in with a vehement assertion that the whole notion of justice is a fraud practised by the strong man upon the weaker. The consequence, that the life of an unjust man would be in itself desirable, is one from which he does not shrink, and to the examination of this doctrine the greater part of the first book is devoted.

In the beginning of the second, we find young Glaucon and Adeimantus professing themselves dissatisfied with the manner in which Thrasymachus has defended his cause, and with his haste in abandoning it. They have no sympathy with his views, they are convinced in their feelings that justice is intrinsically good, but the arguments by which it is proved to be so seem to them inconclusive. Is it not true that society has put honour on a certain course of conduct which ministers to its own security and advantage? Has it not succeeded in helping out the weakness of its own sanctions by religious feelings and terrors? Are we not brought into the world under this twofold set of impressions in favour of what is called justice, and against injustice? Can we suppose a man retaining the idea and practice of justice in opposition to laws and his fellow-men, and without the imagination of some divine sanction or patronage? The question Socrates allows to be difficult; perhaps it cannot be at once answered. But might we not arrive at some solution of it if we examined it upon a larger scale? The question presumes an existing state of society; it supposes this notion of justice, be it a fiction or reality, to be necessary to the support of a state. Shall we inquire then how it becomes necessary to a state; what justice in a state is? Then possibly we may know better what justice in an individual is, and whether it can or cannot be maintained under the disadvantages imagined by Glaucon. Now we would remark in passing, that those who consider the political

Moral and
Metaphysical
Philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

part of the *Republic* merely accidental and subordinate, look upon this introduction and upon the manner in which the subject of a state is introduced as the great staff of their theory. They maintain that it is altogether so much easier and more consistent to believe that the subject started in the outset is really the main and central one of the book, and that all subjects which may occur by the way are merely intended for the illustration of it, that no student of Plato who really understands that teacher's method will tolerate any other supposition. We admit at once that this is the object of Plato, that the *Republic* is an inquiry into the nature and meaning of justice; that if this end be lost sight of, it would be quite impossible to understand the connection of its different parts. But then we say that the most natural and obvious construction of Plato's words leads to the belief that the idea of political justice was in his mind inseparable from that of individual justice; that this was precisely the quality which he perceived to be the meeting point between the spheres of ethics and politics, say, rather, which proved that these are two concentric spheres. The clear establishment of this relation is, we conceive, the great purpose of these two first books. They are a lesson to all future reasoners with sophistical men, that they can only maintain the moral ground safely when they are content to follow their opponents on to their own political ground, when, instead of contending that there is an order which the individual is obliged to follow, supposing society to have no existence, they will be at the pains to prove that there are eternal principles involved in the constitution of society itself, to which its individual members will conform themselves, not because they are content to sacrifice their own distinct personality, but because they have no other way of asserting it. Whether the ethical writers, from Hobbes to our own day, would have fared worse or better if they had attended to the admonition of the most experienced and subtle of all the antagonists of sophistry, we may have occasion to inquire hereafter. At all events, those who believe that the main purpose of the Platonic Dialogues is to discover and develop a method of thought, must, we think, be strangely hampered by a theory which compels them to suppose that on this occasion the result arrived at is the only important consideration; and that the processes for attaining it are altogether artificial, and yet withal most clumsy and cumbrous.

But to return; starting with the hope that the proposed scheme for seeking after justice in a state will lead to a solution of our difficulties, Plato proceeds at once to the formation of a society. Now any one who looks in the arrangements which he here sets forth for the outlines of some imaginary perfect commonwealth, will unquestionably be much disappointed. So far from there being any thing mystical or Utopian in his primary conception of the society, every thing is as terrestrial and common-place as the merest materialist could desire. Men meet together and find that each is not sufficient to provide for his own animal wants. Different necessities arise, different capacities discover themselves among the persons associated adapted to these necessities; hence division of labour, distinct occupations and professions. By and by comes the desire for an excess of the good things which earth produces; hence tumults and external wars. Now we feel the necessity for a set of guardians or watch-dogs of the state—what manner of persons must these guardians

be? Clearly they are in danger of becoming wolves instead of watch-dogs. How is this to be prevented? How are we to produce in them those internal qualities which are so obviously necessary for the welfare of the whole community? We say that these arrangements are very unlike what might be expected from the builder of an imaginary commonwealth. No assumption of any advantageous position; no previous theory about the wants and feelings of the persons composing the society; above all no hope, by outward contrivances and dispositions, to avert the occurrence of crime. How can this be accounted for? Does not such apparent carelessness about external contrivances rather favour the notion that the society is only a scaffolding? We answer, if it were so we should be utterly unable to account for the scaffolding not being more elaborately constructed. If Plato had the liberty of forming his own plan and choosing his own materials; if it were a matter of utter indifference to him how far these are consistent with the nature of things provided they did but help him to a discovery which will afterwards be good and entire on its own ground, whatever steps may have led us to it, any attempt to conform to dry ordinary facts as they meet us in the world would be idle, and, as an offence against art, censurable. If, on the contrary, his object were not to frame a society after an ideal in his own mind, but, as we have supposed all along, to investigate the conditions of political unity—the idea involved in the very existence of society by departing from which it has become confused and incoherent—if this be the view of the treatise, which is most consistent with the rest of Plato's philosophy, and which most clearly exhibits its relation to the other Dialogues, then it can be no matter of wonder to us that Plato should carefully abstain from any exercises of imagination while he is setting before us the bare and naked elements of which society is composed, that he should take pains to convince us that he is not a creator but a searcher; not one of those poets to whom he can assign no place in his commonwealth, but one of those investigators of the truth as it actually is whom he would put at the head of it.

The *Republic* of Plato then assumes selfish desires to exist, and the evil results of them to have occurred. And it is an examination of this question,—under what conditions can we suppose it permanently to cohere in spite of the tendencies to decomposition which manifestly discover themselves within it? When, however, we direct our attention to the means of preserving any body from decay, we may either consider what positive precautions are necessary in order to resist the progress of its corruption; or, on the other hand, by what means it is possible to call forth and invigorate those principles of life which it must have within it in order that it may be at all, though their presence may only make itself manifest through the power which opposes them. Plato again and again gives us to understand that it is with the development of these principles of life, and not with the outward regulations for the repression of evil, that he concerns himself in this dialogue. It is on this account that the *Republic* has assumed to many persons the appearance simply of a treatise on education, nay, of little more than a censure upon the existing Greek education. These persons remind us that he has scarcely given the first rude hint of his society before he tells us that the minds of the guardians of the state must not be corrupted by those false ideas of the gods which occur in the poems of Homer. That a great part of the third

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy

book is occupied with the consideration of two particular branches of education, gymnastics and music, and even with a minute and elaborate inquiry respecting the kinds of music which serve or frustrate the ends of education. That the subject is renewed in the sixth and seventh books, where the use of all the sciences in forming the mind of a statesman is carefully investigated. All this is true, yet we are persuaded that the value of this portion of the work will be far more appreciated by those who consider it subordinate to the great object of discovering the principle which lies at the foundation of society, which connects it with the processes of the individual mind, and which gives consistency and harmony to both of them, than by those who determine that it shall be the sole and independent purpose of the dialogue. Nothing is more entirely consistent with the purpose which we have attributed to Plato than his placing a right view of the character of the gods at the threshold of his education; for the idea of the good, and of this as connected with order, is that which underlies his whole scheme. To make men feel that this *is*, that this exists, to teach them how to distinguish between the anomalies of society and its principles, is his great endeavour. Above all things, therefore, we must see that our models are not defaced with our own corruptions.

For precisely the same reason the subject of music becomes invested with so much importance. To develop the sense of order and harmony in the minds of the members of the commonwealth, is the secret for making it really that which it pretends to be. It is a part of this order that the feeling of it should be first communicated to the guiding, guarding minds of the society, that from them it should diffuse itself through the whole; and by this means a new and most important element of our *Republic* is brought to light. Among the guardians will be some of a higher order than the rest. They will be those on whom the education has produced its complete mellowing effect; distinguished from the others in this, that whereas *they* have chiefly derived a more braced and masculine tone of character from the union of gymnastic exercises with the higher forms of music, these have imbibed the very essence of the music, have acquired that perfectly harmonized temper, that sense of wholeness which makes them the true ideals and representatives of the entire community. These are obviously our *magistrates*. But how shall we persuade our people that these different qualities exist, and that they constitute a fitness for the different offices in the state? We must tell them a story, says Plato, in order that we may bring home this conviction to their minds. We must inform them that they were all made originally out of the earth, which, on that account, they are to love as their common mother; reckoning themselves brethren in consequence of their relation to her. That however it pleased the gods to introduce different materials into their composition, making some of gold, some of silver, some of inferior metals; that it is important that these should not be confounded, but should be kept distinct and applied to distinct uses, in order that the society may receive benefit from each of them. Such is the parable by which our author teaches us that he looks upon himself not as the contriver of some imaginary scheme, but only as following out the intentions of Providence in the institution of society.

All this time we seem to have been forgetting our original question respecting justice. But we find in the

fourth book that the arrangements of the state, and even our long discussion upon music, have been preparing the way for a more clear developement of this idea. We have discovered three classes in society, and we have seen that each of these classes embodies a certain characteristic quality, which through it becomes the quality of the whole fellowship. The class of magistrates expresses to us the very idea of wisdom, superintending, distinguishing, arranging; the class of guardians, the very idea of fortitude, sustaining, amalgamating, preserving; the inferior classes, while they keep their position, the very idea of temperance, self-restraining, and submitting. Without any of these it is obvious that a society could not exist, and the permanence of its existence depends upon the degree in which the qualities of each class interpenetrate the rest. But then does not this imply the existence of still another quality—of some principle of power which fuses together all the classes and all these qualities—which belongs not primarily or particularly to one class, but must by its very nature be predicated of the whole? This is evidently that musical principle which we have been seeking by all our education to instil. But what shall be the name of it? Is not this that Justice which we have been trying to understand the meaning of? Do we not translate the rude outward notion of Simonides into a real practical, satisfactory idea, when instead of making justice consist in giving every man his due and in speaking the truth, we describe it as that which determines the true relation of all things and persons to each other, the very law and harmony of the world. Yet may we not go still a little deeper? Our first object was to discover the nature and effect of justice, not in society, but in the individual. At every step of our progress we seem to have found proof that these two considerations are inseparable; that the law of society must be the law of the individual. Now, perhaps, we are in a condition to explain this fact more particularly. Our inquiry has brought to light three classes as the necessary constituents of the state; a class of magistrates, a class of guardians, a class occupied in supplying the animal wants of the whole body. Whence the necessity for this distribution of society? Is it not that there is a similar distribution of parts in the man himself? Is there not in him a reason, energy, or will, cupidity or an animal nature? And if these are not to exist in perpetual discord, the man in perpetual misery, must there not be that in him which preserves each of these apprehensions or tendencies in their proper relation to the rest, giving the supremacy to reason, preserving the strength and purity of will, subjecting cupidity? Is not justice then necessary to each of us?

This point being ascertained, Socrates is willing to finish the dialogue. But Glaucon and Adeimantus remind him that justice being according to him the principle of harmony or unity in the commonwealth he is bound to explain the other conditions of this unity. For if any inevitable circumstances make this union impracticable, justice itself is impracticable, and if for the commonwealth, then, according to the whole tenour of the argument, for the individual also; so that we should be obliged at last to acquiesce in a conclusion not very unlike that sophistical one which we have been labouring to confute. The four next books, then, from the beginning of the fifth to the end of the ninth, are occupied with these questions: first, in what sense are family relationships compatible with the unity of a commonwealth? secondly, how far can it consist with the individual sel-

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

fishness and ambition? thirdly, how can it consist with that law of decay and degeneracy to which all societies seem to have been subject?

What Plato's statements are upon the first of these subjects we have no need to inform any reader. Those who know scarcely any thing else of him have heard that he has somewhere spoken of the two sexes as intended to perform exactly the same duties and exercises, and that he has connected with this doctrine another (which indeed in a logical mind will generally be inseparable from it) of a community in wives and children. They have heard also that these notions actually enter into the composition of his perfect commonwealth; and that he wishes to supersede all those existing relations of father and child, wife and husband, which we have described in the beginning of our sketch as lying at the foundation of all moral apprehensions and all political order. The question then naturally suggests itself, not whether we are prepared to offer any justification for this part of his speculations, but how, while such a huge and hideous blot exists in them, we can venture to speak of them as important; above all, can devote so much time to the examination of them? Many readers and admirers of Plato have dwelt with much satisfaction on the fact, that in the *Laws* (a later work undoubtedly than the *Republic*) he appears to have changed his views, and to recognise the sanctity of human relationships as they exist. We confess that we do not regard the passages referred to as a recantation. Even if Plato considered them so himself, (of which there is no proof,) we feel convinced that he would have relapsed into his former opinion, if he had again devoted himself to the task of studying the idea of a commonwealth. The *Laws*, it seems to us, are intended to explain the conditions under which any particular nation exists; whence proceeds the coercive power by which the evils of its members are restrained; how it is to be preserved as a distinct community. For this end Plato perceived the importance of distinct relationships; he could not help seeing that they lie at the very foundation of national life; that with the loss of them it would perish. The *Republic*, on the other hand, is not an inquiry respecting the conditions of a particular state. Phrases may occur in it again and again which seem to define this as its object; but others far more pregnant in their meaning, and oftentimes uttered unconsciously, show that another and grander aim was present to the mind of the writer, and was haunting him when he could not realize it. He felt that there should be some body which expresses not the law of a confined, definite, national life, but the law of society itself, the principle of its unity. He felt that such a body as this is implied in the existence of every national community, but yet transcends it, and is not subject to its limitations. We could easily produce proofs of this feeling from every book of the *Republic*, but we know none in which it comes forth more strikingly than in that fifth book of which we are now speaking. The idea of a universal Greek society is there formally put forth, yet it is evident that this does not satisfy the mind of Plato; he has the dream of something still more comprehensive: a feeble sophist would have tried to express the dream in big words; he is content to suggest the nearest practical approximation to an expression of it that his circumstances made possible. But with this universal society Plato does not see how distinct relationships are compatible. Perfect community seems the very law of its being; whatsoever interferes with

this seems to frustrate its intention. Here then we see at once the ignorance and knowledge of Plato. How such an universal society as this could grow out of a national community, out of a family, and could preserve uninjured, in harmony with itself, both those holy institutions which had been its cradle, this he did not know; this wisdom was reserved for the shepherds of Palestine. To them it was only communicated by degrees, and their chief duty consisted in keeping that which had been divinely given them in the sure confidence that more would be added. But this *was* permitted to the sage of Greece; he was allowed to feel the necessity of a universal community to the life of man. He was permitted to feel that it was a great living truth implied in the existence of society, though yet undeveloped. To such insight and honesty of purpose, rejecting no light that has been vouchsafed, it is granted that even the crudities and ignorances into which it falls in the search after truth shall be for the benefit of future generations, nay for the practical correction and exposure of these very crudities when they are reproduced by men of a different spirit. The fifth book of the *Republic* is a curious anticipation of every scheme of universal society which has been propounded by religious fanatics or political theorists from the propagation of Christianity to the present day. It remains a standing practical testimony from the wisest man in the ancient world, that this is the only consistent law, and must be the ultimate law of every such society, whensoever it attempts to exist alone, as a merely spiritual or cosmical family. Rejecting then with indignation the *errare mehercule malo* of the Roman academician, and follow Plato only so far as he loved truth, we may yet find a worth even in this unfortunate passage of his writings.

The portion of the *Republic* comprehended in the sixth and seventh books is second to no part of it in interest. The difficulty to be solved in it is the compatibility of such a state as we have described with the selfish notions of men. Plato does not blink the question. He at once declares his conviction that such a state could only be administered by philosophers; and he then goes on to explain what he means by a philosopher; why it is that the persons generally bearing that name are unfit to be practical politicians; what their real relation as to the rest of their countrymen is. Here then we have the full exposition and developement of that doctrine which we found lying at the root of Socratic teaching; that the selfish self-seeking principle leading men to animal gratifications is the source of disorder and confusion in the life of man, not really the moving spring of it; that there is in man something higher which is not satisfied with itself, but which seeks after converse with the good. The philosopher is the man who is holding this converse; whose mind is fixed on the true end and meaning of things, upon the substance—the reality of them. The rest are following images and shadows, but still in the pursuit of these are confessing their want of a good, and are blindly feeling after it. The philosopher, if he descends to the pursuit of their shadows, becomes worthy of their contempt, for there is a perpetual contradiction between the higher aims of which he is conscious and the grovelling course he has actually taken. If, on the other hand, he steadily keeps his own idea in sight, he is necessarily unintelligible to them, and on that account they despise him. But suppose, having worked his own way out of the mine in which they are dwelling, and no longer receiving light

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

through the little crannies which transmit it broken and confused, and lead men utterly astray as to the fountain from which it has flowed, he has come out into the open sunlight, and by it seen all objects as they are, he neither glorifies himself by living apart from them, nor yet submits to confuse his light with their darkness, but goes down amongst them that he may lead them by the same track which he has himself trodden into the clear day. Would he not then be filling his function as a philosopher and yet be most truly a politician? If the question occurs to you, what is this upward road? Plato is ready to consider it with you. There is a certain education recognised among men; they teach arithmetic, geometry, as well as the gymnastics and music, we spoke of before, and they evidently attach a high value to these studies. Are they wrong? surely not. They are wrong only in this that, throughout their whole lives, they are seeking shadows instead of substances, and that they have made all these sciences helpful to their low ambition. Arithmetic and geometry have been resorted to merely for secular commercial ends; they might be made the means of purifying the mind to a perception of the truth of things. What is the appropriate function of each of these sciences, with a view to this object, he carefully inquires: and this inquiry brings out the necessity of that grand, deeper science of dialectics, which directly leads to the contemplation of truth as truth, of good as good, in its pure essence. Now a nation thus guided and educated comes into the condition of such a republic as we have described. Its wisest, deepest minded men will become its magistrates; the community will have one end; that principle of justice, which assigns to each his proper place, imparts a sense of proportion and harmony to all, will be diffused through it and actuate it. Thus, then, the existence of ambition and selfishness does not upset the idea of our republic, does not prove that it is not implied in the nature of society, does not show that it may not be at some time or some where realized.

We come next to that law of decay in societies which most speculators have recognised, and which the Pythagorean philosophers fancied they could express in certain numerical ratios. Plato has given a very valuable turn to the inquiry by connecting it with the cardinal doctrine of the *Republic*, that the life of men and the life of states explain each other. In conformity with this doctrine, he maintains that there is a democratical, an oligarchical, and a tyrannical form of character answering to those respective forms of government. This form of character is obviously a departure from some true and original model. The same may be shown of the governments; and it is possible in each case to trace the process of degeneration, and to show how that which takes place in society, and that which takes place in the individual, react upon each other. In this part of the dialogue Plato proves that his faculty of close, lively, practical observation had not been impaired but strengthened by his converse with transcendent realities. It would be hard to find a passage in any ancient work on which a modern statesman might more profitably meditate, or in which he would be more sure to find hints explaining to him the facts of his daily experience, than the eighth and ninth books of the *Republic*. The result of the investigation is the same as in the former case. This law of degeneracy exists in the commonwealths of the earth, just because they have not understood and steadfastly contemplated that original model,

that perfect idea of a commonwealth, which is also the original model and perfect idea of a human character. It is a contradiction and absurdity then to allege the fact of this degeneracy as a proof that no such model is to be found. But after all these inquiries does the thought still linger about the mind, where is it to be found? Plato answers, (book ix. p. fin.) ΑΛΛ' ἐν οὐρανῷ ἴσως παράδειγμα ἀνάκειται τῷ βουλομένῳ ὁρᾶν καὶ ὁρῶντι ἑαυτὸν κατοικίειν. Is it wonderful that such words should have suggested to some of the Christian fathers the recollection of those words in the *Epistle to the Hebrews*, which describe the hopes of the head of the covenanted people, Εξεδέχετο γὰρ τὴν τοῦς θεμελίους ἔχουσαν πόλιν ἧς τεχνίτης καὶ δημιουργὸς ὁ Θεός: or those which describe this hope as accomplished, Ἡ πολιτεία ἡμῶν ἐν τοῖς οὐρανοῖς?

There is still one subject upon which it is needful to say a few words, especially as Plato has devoted his last book to the full exposition of it, we mean his opinion respecting the imitative arts generally, and especially of poetry, so far as it is included among those arts. It is evident that our author attached great importance to these opinions, and yet that he was never wholly satisfied with them. He touches upon the question almost as soon as he has sketched the first outline of his society; he recurs to it again when his task seems completed, partly as if he felt there was no security for the reception of his idea while any doubt overhung this point; partly as if there was something in it which he had not fully penetrated. Again, in the introduction to the *Timæus*, he offers a kind of apology to the poets for his severity, and appears to think that they may have an important vocation, though he does not clearly understand what it is. It is observable that the grounds upon which he places his arguments in the third and in the tenth books, are not precisely the same. In both, indeed, he dwells much upon the fact that the poet must adapt himself to the opinions of mankind respecting actions and character, otherwise they will not acknowledge the verisimilitude of his picture; hence he must need pervert the truth of things, and can never exalt those minds to which he accommodates himself. But in the last he appears to see a peculiar mischief in poetry from its tendency to destroy the harmony of character, to weaken self-control, and thus to undermine the justice and order of the commonwealth by the honour which it bestows upon all excited and passionate feelings. The latter argument might lead one to suspect that Plato was at least in part determined to these views by the circumstances of his own age. The exaltation of passion, the want of balance and harmony in characters, the preference of weak, earthly creatures to calm and stately ideals, were the great characteristics of the Euripidean, as distinguished from the Sophoclean drama. Add to this the influence of a poetical age (an influence felt most when that age had departed) in fostering the worship of mere creative power, and the notion of the mind of man being the origin of all that is, which lay, as we have seen, at the root of all Greek sophistry, and which it had been the great aim of Socrates throughout his life to combat. Still Plato's attack upon Homer, and his eagerness to disprove the common opinion that the Greeks were really indebted to him for much of their organization and cultivation, are proofs, we think, that he was not merely affected by these temporary considerations. We leave it then as a hint for our reader's reflection, whether this reluctant condemnation of poetry by one who had been

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphysical
Philosophy.

himself a poet in the formal sense of the word, and in the best sense continued a poet always, may not be explained in the same way as we explained just now his theory respecting relationship. In an early part of our sketch we remarked that poetry seemed to belong primarily and almost exclusively to national life. The sense of national union gives the first impulse to it; when that sense is weakened it withers, with its revival it starts to life again; without it men would never become conscious of their own powers, their own affections, their own wants; and in the consciousness of these consists the joy and freedom of their life as the citizens of a state. By calling this forth in the Greeks, Homer may be said to have made them a nation, a nation full of life, full of turbulence. But is there nothing higher than this mere consciousness of power? Is there no higher condition of society than this of being citizens of a state? The *Republic* is an answer to the question. It teaches that far beyond this consciousness of power lies the contemplation of truth and goodness, and the assimilation of the soul to these. It shows that far beyond the mere feeling of energy to dare, to act, to revenge, lies the perception of order and harmony, and intimate fellowship with a Being above us, and the beings around us. It teaches that there is a universal society, of which this contemplation and assimilation are the ground, this perception of order and harmony the life, of which this fellowship is the result and the realization. With this community, says Plato, poetry hath little to do. Praise of the gods, eulogies of great men, these are the only fields for its exercise. Strictly speaking, we think he is right, that is to say, if it were possible for us, as it was necessary for him, to separate (how important it is to distinguish we hope we have explained) the national life from the universal life, the national society from the universal, poetry, which is the soul of the first, would, except in the cases named by him, be excluded from the other. If we would connect all the vital energies of which poetry is the expression, with those deeper insights, that perfect moral state and moral life which belong to the higher region, we must also understand, and by understanding realize for ourselves at least, and so far as is permitted us for mankind, the law by which the universal and the national society sustain each other.

The *Republic* ends nobly with a discussion on immortality, which has been less popular than that in the *Phædo*, because the scenery of it is less solemn and affecting, but which for its own merits seems entitled to even more attention. We are far indeed from thinking so lightly as some have done of these arguments from reminiscence, and from the law of interchange between light and darkness, death and life, which occur in the dying conversation of Socrates. On the contrary, they seem to us pregnant with the deepest meaning. But we cannot help thinking that when Plato had once realized in his own mind the connection between the life of the individual and the life of society, he felt he had a stronger ground to fix his hope of immortality upon—that he had found the point where the witness in the heart meets the demands of the reason. The sense of belonging to a community, stretching behind and before, outlasting the deaths of generations of men, is an evidence to each man of his individual immortality, which you may be quite unable to translate into syllogisms, but which happily supersedes the necessity of them. Plato only went to the roots of this feeling, when,

having shown that the existence of the individual and of society are alike based upon the idea of justice, and are alike sustained by the contemplation of that which is true and permanent, and alike die a moral death when they contradict the principles of their being, he affirmed that the accident of physical death can as little change the condition of one as of the other, and that as each has lived here must be his life hereafter.*

We have dwelt so long, for reasons which we have explained already, upon this great summary of the ethical, metaphysical, and political philosophy of Greece, that we can afford time but for one remark which is necessary in order to show how the doctrine of Plato is connected with that of the great predecessor whose labours we suppose that it was his intention to review. That there is a Pythagorean character in the *Republic*, the book on music, the passages on geometry and arithmetic, and certain mystical sentences respecting the law of decay in a state, which have defied the skill of commentators, abundantly prove. But if we look well at the work, we shall find that the whole of it may in one sense be called Pythagorean. For the discovery of the musical law which gives internal wholeness to a state, as distinguished from that external law by which its parts are prevented from falling asunder, is in fact the object of the treatise. Wherein then does he differ from Pythagoras? Precisely in this; that while he gives music and arithmetic their due honour as instruments for cultivating in man the feeling of his own position and relations, he does not deduce that position and those relations from any combinations of notes or series of numbers. He makes justice—a moral principle—the music of his commonwealth. And this is the more remarkable and the more honourable, because it is evident that he felt the temptation to be a cabbalist, and never divested himself of the belief (perhaps no deep thinker was ever able quite to divest himself of it) that there is something profoundly and mysteriously interwoven with the life of man in the relations of lines, of numbers, and

* As the *Republic*, like so many other of the Platonic Dialogues, closes with a mythus, and as the passage in the third book on Lying brings the whole subject of the use which Plato thought it lawful to make of fables and legends directly before us, it may be as well to make one remark on this subject. Throughout this dialogue, even more than in his other writings, it is evident that dearly as he loved truth for its own sake, and firmly as he believed it could be contemplated in its pure essence, he yet felt that there was no criterion of truth so sure as that it governed practice and was the law of life. To substitute a pure idealism for the faith of his country was never his object or his dream. He hated such attempts, not more for their hardness and cruelty than for their utter inconsistency with his whole doctrine. He left them to men who did not believe that ideas were substantial, who thought they were mere creations of the mind and had nothing to do with living acts. While then he was very jealous of all those stories which evidently hindered men from acknowledging goodness and truth as the ultimate ends of their existence, he was equally certain that somehow or other all great principles must have an investiture of facts, and cannot be fully or satisfactorily presented to man except in facts. And if no such series of facts embodying and revealing truths were within his reach, rather than leave it to be fancied that his truths are bare naked conceptions of his mind, he will invent a clothing for them: it is the least evil of the two. But it is an evil; it exposed him to fearful contradictions; it often puts his love for truth in the greatest jeopardy. Then what pretence have those to the name of Platonists who wish to believe that there is no series of facts containing a revelation of supersensual and transcendent truths, who think it an *a priori* probability that the deep want of such facts which Plato experienced has not been satisfied; who are determined even by the most violent treatment of historical evidence to prove that whenever a supposed fact manifests a principle, it must be a fable?

Moral and
Metaphysical
Philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

of sounds. It was a great merit thus to keep the practical ground so steadily, and never to forget that this is really the highest ground. By doing so he was enabled to perform the same service in one sphere which he had already performed in another, to discover the political principle which Pythagoras had been seeking for amidst the influences that connect us with nature, as he had discerned the scientific principle which Parmenides had been groping after amidst the forms of our own minds.

It is a great satisfaction to us that our duty, as historians of moral and metaphysical inquiries, does not call upon us, or even permit us, to say many words on the subject of Plato's *Physics*. Still the *Timæus* is so curiously connected with the *Republic* by the exquisite introduction to it, in which Critias tells the story of the submerged state, so like in all respects to that which Socrates had described the day before, (here we have the doctrine of reminiscence obviously brought into play, and a new evidence that our philosopher considered the *Republic* as no work of imagination, but the discovery of a truth implied and forgotten in the constitution of all societies,) and so much importance has been attached both in early and later times to this dialogue, as if it contained the very heart of Platonism, that we cannot venture entirely to pass it over. With respect to the link between the *Politics* and the *Physics* of Plato, we would not speak confidently. He may have perceived a closer relation between the moral *κοσμος*, which he had been investigating in the *Republic*, and the material universe which *Timæus* creates, than we are able to trace. But this we think is evident, that he did enter upon his new task with a kind of consciousness that it behoved him to fill up a gap in his speculations and to complete his review of the ancient philosophy, and at the same time with a secret apprehension that the light which had hitherto guided him might forsake him in this region. It is strange at all events that, while undertaking to develop a subject so important in Greek eyes as the creation and organization of nature, he should make Socrates a merely listener. To a faithful student of Plato it must still seem more strange that he should on this occasion utterly desert his customary method, that the dialogue form should be merely used to throw a graceful dramatical veil over the introduction, and that in the expository part it should be exchanged for the haranguing style to which Plato was in general so averse. And yet it is to this cause more than any other that the *Timæus* owes its reputation among those who undertake to furnish summaries and synopses of Platonic doctrine. Elsewhere they found him balancing opinions, often refusing to pronounce a verdict upon their respective merits—most unnecessarily tedious (as they think) in tracing the road to a conclusion, most unaccountably and ill-naturedly forgetful of the duty of clothing it in precise, available, transferable formulas. Here on the contrary, though his language may be more obscure than it is in other places, though there may be more allusions to ill-understood portions of Greek speculation than in all the rest, though, lastly, his teachings refer to a question upon which we all believe that he could have only very partial illumination; still the manifest convenience of catching so Protean a philosopher for one moment in a rigid definite state, has outweighed all these considerations, and has made the Platonic cosmogony the grand storehouse from which diligent redactors have been wont to collect their notions of the mind and the works of Plato. Nay it has even been a plausible and popular

theory, ingeniously accounting for the uncertainty of the other Dialogues, that they were only intended as a vestibule, to the inner oracle of the *Timæus*. Having sufficiently explained our views respecting these so-called uncertainties, and having endeavoured to show how much the method of Plato is part and parcel of Plato himself, we must needs regard this particular work with very different feelings. Not pretending to behold with indifference the splendid theory which it develops, aware how closely that theory is connected with some of those which exercised the strongest influence upon the minds of men, especially in the first ages of the Christian Church, and being very willing to accept for Plato the compliments which natural philosophers have paid him for his intuition of truths hereafter to be established, we must yet confess that the *Timæus* seems to us chiefly valuable because it illustrates the worth of the principle from which it is so signal a departure. In every other dialogue Plato is teaching us how to discover a universal law in any particular fact which falls under our notice; here we have huge hypotheses to begin with, and all facts fitted and disposed according to them. For whatever there is of truth in these hypotheses he is indebted to his previous studies in another direction. Having arrived by his own sure course of upward investigation at the doctrine of ideas, he was able to see that the world must be created according to an idea. But having attained this point, his light forsook him, he was not able to apply his dialectic to the elimination of this idea, from the names or facts in which it was imbedded. He had simply to trust to his imagination to construct a theory. Whereas in other cases he is a philosopher seeking for light, and when he could not perceive the track of it, showing where it ought to be, and from what unrisen sun it must flow, here he is a presumptuous theologian, assuming himself capable of declaring that which must be revealed, and thereby losing the right way to that which may be discovered. Bacon does him no injustice in respect to his *Physics* when he says that he confused and corrupted them with theology, when he implicitly includes him among the giants who piled hill on hill in hopes of reaching heaven. Would that our countryman, for the honour of his own character, for the sake of the ages which were to follow him, had been as willing to recognise the truth of Plato, as he was acute in detecting his falsehood; as honest in acknowledging him for a guide, as he was right in pointing him out for a beacon. He would then have seen that the *Timæus* was in contradiction to the principle of induction, because it was inconsistent with the principle of Plato. He would have seen that in one solitary instance the Greek sage was betrayed by that ambition of completeness and circularity (which far more than the desire of fame deserves to be called the last infirmity of noble minds) into the examination of a subject on which he only could dogmatize, and could dogmatize only by forsaking his own method. He would have confessed that the *Novum Organum* was but the extension of that method to a new class of subjects. He would have taught his disciples that the course of investigation which promised them such new discoveries in the world of sense, which was grounded upon the great principle that man is but a seeker after that which really exists, which is prosperous in proportion as they endeavour simply to behold that which is, and not to darken it by the mists of his own conceptions, had been ages before marked out as the only one by which they may safely hope to become acquainted

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

with the truths of their own being. He would solemnly have conjured them to remember that a heathen, uninstructed by that revelation which deals directly with these transcendent truths and lays them open to every peasant, had yet perceived that they must be the most precious which a man can know, and that only in knowing them he is truly a man. He would have told them, that if ever that study, in which the heathen sage forgot his usual wisdom, should become the only one in which Christians care to be proficient, if ever *ex reseratione viarum sensus et accensione majore luminis nature aliquid incredulitatis et noctis animis nostris erga divina mysteria oboriatur*—or there should grow up a feeling towards these mysteries which is worse than unbelief, or else another form of it, a stupid acquiescence in them without the acknowledgment that they answer to any cravings in the heart, any necessities of the reason, any predictions of the imagination, then for the sake of the age and country upon which such a disease had fallen, for the sake of all that should follow it, for the sake of physical knowledge itself, which can never long flourish apart from moral light, it would be most desirable that men should resume the study of the Athenian philosopher, should realize the wants of their minds by observing those which he experienced in his should consider in what way we can find an adequate provision for both.

*Aristotle.**

Aristotle.
External
difference
between
him and
Plato.

A student passing from the works of Plato to those of Aristotle is struck first of all with the entire absence of that dramatic form and that dramatic feeling with which he has become familiar. The living human beings with whom he has conversed have passed away. Protagoras, and Prodicus, and Hippias, are no longer lounging upon their couches amidst groups of admiring pupils; we have no walks along the wall of the city, no readings besides the Ilissus, no lively symposia giving occasion to high discourses about love, no Critias recalling the stories he had heard in the days of his youth, before he became a tyrant of ancient and glorious republics; above all, no Socrates forming a centre to these various groups, while yet he stands out clear and distinct in his individual character, showing that the most subtle of dialecticians may be the most thoroughly humorous and humane of men. Some little sorrow for the loss of so many clear and beautiful pictures will be felt by perhaps every one. But by far the greater portion of readers will believe that they have an ample compensation in the precision and philosophical dignity of the treatise for the richness and variety of the dialogue. To hear solemn questions treated solemnly, to hear opinions calmly discussed without the interruption of personalities; above all, to have a profound and considerate judge, able, and not unwilling, to pronounce a positive decision upon the evidence before him; this they think a great advantage, and this, and far more than this, they expect, not wrongly, to find in Aristotle.

Still we are of opinion that a person who is able to render justice to the method of the master, will, on the

* It may be as well to remind the reader that two elaborate lives of Socrates and Plato appeared many years ago in this ENCYCLOPEDIA, and that one of Aristotle has lately been added to them. The author of this article has therefore no excuse for dwelling upon any particulars of their history, not directly tending to the elucidation of their doctrines.

whole, be the most likely to appreciate the disciple; at all events we shall not understand either well if we content ourselves with a vague notion that one was a consummate artist, the other a profound practical philosopher. That Plato did not adopt his dialogue form for any artistical purpose, but simply because it was necessary for the developement of his idea of science, we have contended already. And we feel it equally necessary, in order that we may claim for Aristotle the true and very noble position which of right belongs to him, not to let it be supposed that his pretensions to be either practical or profound rest upon his want of those qualities, and his abandonment of that method by which Plato is distinguished. In common parlance we are wont to consider those most practical whose studies are most connected with real, living, passing questions. Now it was the actual opposition of sophists, which drove Socrates and Plato to seek for principles not yet recognized, lest they should lose those which they had. Aristotle had the advantage of being able calmly to examine sophistical arguments, because it was the hour in the school for that particular subject to be lectured upon. It was a question of life and death in Plato's day, whether we have something permanent to rest on or no; for men in every town of Greece were abusing the name of Heraclitus in support of the doctrine of a perpetual flux. Aristotle could label this question physical or metaphysical, and patiently balance it against some opposite theory. The Parmenideans forced Plato to investigate the nature and conditions of science, for they threatened it with a hopeless stationariness. Aristotle is under no such alarm; he has merely to make out a system of analytics. It was because the Body of Socrates was about to pass through death, that he was led to consider the meaning, and nature, and enduring properties of the Soul. Aristotle begins his treatise on the same subject, with inquiring whether the subject is to be considered in reference to any particular person at all or abstractedly, and whether we are to speak of it physically or dialectically. Without determining which of the two courses we have indicated is the best, we think it must be a violence upon ordinary usage to say that the latter is the more practical. Neither is the quality of depth precisely that one which we conceive ought to be predicated of Aristotle, when it is our object to contrast him with his predecessor. It was the necessary consequence of Plato's situation and of the task which had been committed to him, that he was always seeking for principles—for a foundation to rest upon. The most simple every-day facts interested and puzzled him; nothing that human beings were interested in was beneath his attention; but then it was the meaning of these things, the truth implied in them, which he was continually inquiring after. He found the commonest word that men speak, the commonest act that men do, unintelligible, except by the light which comes from another region than that in which they are habitually dwelling. Of this feeling there are no traces in Aristotle. To collect all possible facts, to arrange and classify them, was his ambition and perhaps his appointed function: no one is less tempted to suspect any deep meaning in facts, or to grope after it. In like manner to get words pressed and settled into a definition is his highest aim; the thought that there is a life in words, that they are connected with the life in us, and may lead at all to the interpretation of its marvels, never was admitted into his mind, or at least never tarried there. In this disposition there may be a comfort and an advantage; but it certainly

Moral and
Metaphy-
sical Phi-
losophy.

This differ-
ence no
proof of his
intrinsic
superiority.

Moral and
Metaphysical
Philosophy.

Aristotle
incapable
of under-
standing
Socrates.

is not that upon which persons who are careful in the use of language could bestow the epithet *profound*.

Another prejudice in reference to these great men it is necessary to remove, or we shall not understand their relative positions. It is often fancied, and Aristotle seems not altogether anxious to do away the impression, that Plato's disciple forsook him when he forsook his master; that the later philosophy is in some important respects a return to the simple faith of Socrates. If what we have just said be correct, this notion must be not only wide of the truth but in direct contradiction to it. The personality of Plato was precisely that quality of his mind and of his writings which he had inherited from Socrates. That he so seldom deviated into abstractions; that he preserved so strongly the feeling, we are actual men, wrestling against evil tendencies within, and evil powers without, capable of being educated, and of educating each other into a longing after, and perception of, the perfect goodness and truth, this he owed to Socrates. His own especial work was to connect this personal struggle with the orderly developement of principles. It was precisely then with the Socrates in Plato that Aristotle was incapable of feeling sympathy. That he had a general reverence for his good sense, that he recognized him as the useful and victorious opposer of what was mischievous and unphilosophical, and that he sincerely believed him not to have held certain offensive opinions of his disciple; this we can easily imagine. But that he the least admired the Socratic method, or that all his wisdom could avail to teach him into what conclusions that method must necessarily lead one who habitually followed it, we cannot believe.

His
alleged
jealousy of
Plato.

Though these remarks seem for a moment derogatory to the fame of this wonderful man, they will be found upon reflection to relieve his character from some unjust imputations, to set his actual merits in a clearer light, and to explain the kind of influence which he has exerted, and must always exert over mankind. There are passages in his works which, in the opinion of some over watchful and sensitive critics, indicate a personal jealousy and dislike of Plato. They remark that he does not introduce his comments upon him in a manly, philosophical spirit, but generally with some of those affected phrases of reluctance which display often more than the strongest vituperation the ill will that is lurking within. Possibly far less meaning would have been seen in these passages, if the gossiping anecdote-mongers of later Greece had not illustrated them by stories of dissensions between the master and the pupil, which, though obviously derived from a very vulgar invention or a memory generally treacherous, because always trivial, still unconsciously influence our minds when we have once heard them, and prevent us from fairly looking at the evidence which gives them their only plausibility. Separated from these stories, the quotations we think prove no more than that Aristotle felt a certain irritation and displeasure when he perceived there was something in the words of Plato, which his large intellect and immense information did not enable him to comprehend. To be continually haunted with a consciousness of this kind, in all definable qualities I am equal, nay superior to my predecessor, I have reduced subjects into far greater order; I analyze far more perfectly, I have a far greater store of facts at my command, and yet there is in him something quite *undefinable*, which seems to make an incredible difference between us; this may have been no doubt very vexatious

even to an honest and great mind. For it was not merely the personal humiliation of such a reflection which would be grievous to him, it would jar against the strongest feeling of his mind that nothing ought to be incapable of definition, and that whatsoever does defy it can scarcely be of any great worth. While then it is no doubt possible that petty quarrels may have been stirred up between two such men by admirers and flatterers, who were equally incapable of understanding either, we have no need of that supposition to account for the sneers and taunts (if such they must be called) which now and then displease us in Aristotle. We shall fully explain these to ourselves when we consider the characters of the men, and connect them with the characters of their times. This last point is of more importance than we are at first inclined to fancy. When we think of Aristotle as the disciple of Plato, we group them together as if they belonged to the same portion of Greek history. But when we acknowledge Plato in his true character, as the expounder of Socrates, we feel at once that the age which bore the name of Pericles, but may really be said to date from the Persian war, was the one which he was appointed to adorn. This was an age of individual energy, when pregnant events were transacted in insignificant localities; when the lowest party-contests were developing the most universal and permanent principles. That which was followed was an age of concentrated, organised power, of successful assaults upon freedom, of grand conceptions, of extensive conquests, of what has been well called *material* sublimity. Of such an age Aristotle was, we believe, a more perfect representative than Alexander.

In conformity with these remarks it will not be difficult to show wherein the peculiarity of the Stagirite philosophy consisted; how it grew out of the Platonic, how far they are contradictory, how far one occupies a space which the other had left void. We have seen by what steps Plato was led into his high estimate of Dialectics. He watched his master maintaining a safe moral position against the attacks of the sophists. To assert realities against appearances and counterfeits was his single aim. Keeping this steadily before him, he almost unconsciously wrought out a method entirely different from that of previous philosophers. As Plato reflected upon the end which Socrates had proposed to himself, he perceived the full practical meaning of that truth which in terms had been asserted by Xenophanes and others, that Being is the object of all our inquiries. He saw at the same time how necessary it was to connect the end with the method; for till that method had been practised, Being had been a word, a notion, a negation; not an object to be really beheld and striven after. Hence the immense importance of bringing that method forward; of presenting it substantively, as it were, to his pupils; not allowing them *merely* to contemplate it as leading to certain results, but as the safe and universal means of arriving at any results. We have alluded to a class of dialogues having this purpose; and these, or something answering to them, (*approaching*, it is possible, the nature of ordinary exposition, though we can scarcely believe that Plato ever abandoned the dialogue as his vehicle of instruction,) must have been the peculiar study of the academy, as such, and expressly of the more advanced disciples.*

* In the *Life of Aristotle*, published in this *ENCYCLOPEDIA*, it has been shown, we think most satisfactorily, that the *acroa-*

Moral and
Metaphysical
Philosophy

They
belong to
different
periods.

Connection
between
the two
philosophies.

Moral and
Metaphy-
sical Phi-
losophy.

Hence there will have grown up among these pupils a feeling respecting dialectics which Plato would have been anxious to discourage, and yet which his own works continually tended to foster. Seeing it used as a key to unlock the secrets of social life, of moral life, of physical life, and seeing, likewise, the pains which their master took that they should examine the wards of the key, it was most natural that they should think a much fuller and more systematic developement of this all-important science was desirable and even necessary. It would occur to them that there was something like confusion and irregularity in the proceedings of their great teacher. Had he not strangely mixed together inquiries respecting the grounds of morality with statements respecting the nature of science? Surely it would have been much better, much more orderly, that these questions should have been kept distinct and referred to particular heads. And were there not also some indications of narrowness in Plato, which a more accurate habit of distinction would have delivered him from? Had not his aversion to some of the usual abuses of rhetoric led him to undervalue the whole art, when it was undoubtedly capable, like every other, of being reduced to strict laws, and must deserve to be contemplated without any reference to its accidental results? The same might be said of his doctrine respecting poetry; the same, still more strongly, of his ill-concealed indifference to physical speculations. If all these subjects could be directly looked at in themselves as distinct branches of human culture, how much increase of knowledge might be expected in each, how much increase of clearness respecting the capacities and limitations of the human intellect! Such thoughts, we suppose, may have been at work in the minds of many who frequented the school of Plato. In few they will have borne any fruit, in most of these few the unripe or blighted fruit of some feeble theories, professing to universalize the system of Plato, really proving that their authors knew nothing either of Plato or of themselves. But there was one who was able to make the thoughts of the rest intelligible. To him Platonism will have appeared a needful preparation for a complete and circular philosophy. Its unsystematic character, its imaginative flights, its disregard of certain provinces of thought, will have seemed to him indications of rudeness and infancy. And he will have conceived the thought of assigning to each study its true position, that one which Plato declared and proved to be so important, occupying the first place, being exhibited in its full proportions, and determining the character and treatment of the rest. Dialectics, then, was in some sense the centre of both philosophies. Nor would it be correct to say that Aristotle consciously altered the signification which the word dialectics had borne in the discourses of his predecessor. He only wished to give the study more distinctness and prominence, to exhibit the processes and operations of which it treated apart from any particular applications and results. But in fulfilling this desire, the character of the pursuit became inevitably changed. The feeling of Plato was, There are certain objects presented to my mind; they may be sensible objects,

as trees; they may be objects for the understanding merely, as names; but objects they are still; things thrown in my way, and I must know what they mean; I must find out the truth of them. For this end I must have dialectics. The object has vanished from before the eyes and mind of Aristotle; he has begun to devote the whole energy of his mind to the contemplation of dialectics in themselves. What is the consequence? The sense of requiring them as the means of escape from the impositions which intercept our views of things as they are, becomes more and more weakened, till at last it disappears altogether. That principle which it had been the business of Plato's life to assert against Protagoras and his school, that the mind is not its own standard, that the aspects under which objects present themselves to us, do not constitute our knowledge of them, but that we may arrive at an acquaintance with them as they are in themselves; this principle, which had given his dialectics all their meaning, is no longer felt with any potency by his disciple. On the contrary, it is precisely the aspects under which we see and judge of things that he proposes to investigate. He wants to know what are the rules and conditions under which the mind, by its own constitution, considers and discourses. He makes the mind a centre, referring every thing to itself, just as those did with whom Plato contended. But he differed from them in this, that their intention was knavish, his most honest. They set up the doctrine that all things are merely as they seem to us, for the purpose of unsettling all faith, and proving the judgment of each individual to be a lawful standard. He sought to convince men that all is not unstable and fluctuating, by showing them that there is a fixed rule to which human judgments must conform, which limits the exercises of individual taste and caprice, which tests and reduces to order those appearances which the sophist pretended were infinite.

From this statement it will be easily apparent that the definition of dialectics in Plato and Aristotle may be almost the same, and yet that the whole scope and object of the science indicated by this common definition will be different. One as much as the other could say, Dialectics is that science which discovers the difference between the false and the true. But the false in Plato is the semblance which any object presents to the sensualized mind, the true the very substance and meaning of that object. The false in Aristotle is a wrong *affirmation* concerning any matter whereof the mind takes cognizance; the true, a right *affirmation* concerning the same matter. Hence the dialectics of the one treats of the way whereby we obtain to a clear and vital perception of things; the dialectics of the other treats of the way in which we discourse of things. Words to the one are the means whereby we ascend to an apprehension of realities of which there are no sensible exponents. Words to the other are the formulas wherein we set forth our notions and judgments. The one desires to ascertain of what hidden meaning the word is an index; the other desires to prevent the word from transgressing certain boundaries which he has fixed for it. Hence it happened that the sense and leading maxim of Plato's philosophy became not only more distasteful, but positively more unintelligible to his wisest disciple, than to many who had never studied in the academy, or who had set themselves in direct opposition to it. When Aristotle had matured his system of dialectics, there was something in it so perfect and satis-

Moral and
Metaphy-
sical Phi-
losophy.

Connected
with reali-
ties in
Plato, in
Aristotle
not.

Dialectics
of each.

matic treatises of Aristotle differed from the *exoteric*, not in the abstruseness or mysteriousness of their subject-matter, but in this, that the one formed part of a course or system, while the others were casual discussions or lectures on a particular thesis. The remark in the text is an extension and adaptation of this doctrine to the case of Plato.

Moral and
Metaphy-
sical Phi-
losophy.

Ideas.

Logical
treatises.

Their com-
pleteness.

factory, that he could not even dream of any thing lying out of its circle, and incapable of being brought under its rules. He felt that he had discovered all the forms under which it is possible to set down any proposition in words, and what there could be besides this, what opening there could be for another region entirely out of the government of these forms, he had no conception. At any rate, if there were such a one, it must be a vague uninhabited world. To suppose it peopled with other and those most real and distinct forms was the extravagance of philosophical delirium. Accordingly, when he speaks of the doctrine of substantial ideas, of ideas, that is to say, which are the grounds of all our forms of thought, and consequently cannot be subject to them, he is reduced to the strange, and for so consummate a logician most disagreeable, necessity of begging the whole question, of arguing that, since these ideas ought to be included under some of the ascertained conditions of logic, and by the hypothesis are not included under any, they must be fictitious.

As we proceed we shall have occasion to notice how this primary difference affected the views of these philosophers upon all questions which came under their notice. At present we speak of it, in order to show that the methods having a perfectly distinct object, do not of necessity interfere with each other; that the Platonic doctrine is not absurd, because Aristotle could find no place for it in his system; that the labours of Aristotle are not useless or ill directed, because they do not supply, as he fancied they did, any satisfaction to the inquiries which Plato had awakened.

Every orderly examination of Aristotle must then, we conceive, take its start from his treatises on logic. That these are not the most interesting of the works which he has bequeathed to the world we may easily admit, but, unless something is understood of their nature and purpose, it is scarcely possible to understand the character, the value, and the necessary limitations of his opinions on ethics, on politics, on rhetoric, on poetry. We shall presently quote the opinion of an eminent writer on physical science, to prove that a just estimate of Aristotle's labours in that department depends upon our knowledge of the importance which he attached to the forms of logic. And the settlement of the long debated question which falls more within the province of this article, what precise meaning he attached to the word Metaphysics, or what portion of his thoughts his disciples referred to under that name, can, we think, be hoped for only from a previous understanding of the connection in his mind between the idea of science and that of dialectics.

In the Berlin edition of Aristotle, the *Categories* occupy the first place. Some doubt has been entertained respecting the genuineness of this treatise, which modern inquiries appear to have removed. It would in many cases afford a reasonable ground of suspicion against a work, that it exactly filled up a gap in a set of acknowledged works by the same author, so that with it they form a complete system of instruction upon the subject which they treat. But it is a set-off against this consideration, that roundness and completeness are the great characteristics of Aristotle, and that it is less hard to imagine how a perfect series of his logical writings can have come down to us, than to believe any pupil capable of supplying a void that he had left in it. The difficulty, too, of supposing one man to possess the full mastery of this subject, which is indicated by the succes-

sive works which he has left, is diminished when we remember that the original conception of the study was not his, but Zeno's. How naturally that conception arose in a mind which had once entered into the great principle of Xenophanes and Parmenides, that the mind has laws of its own, and is independent on the appearances and determinations of the senses, we have explained already. What more than this discovery, and the application of it in confuting sensible conclusions, may be owing to Zeno, we do not know. It is not impossible that some of the sophists, while they turned the art to the worst purposes, may have yet done something for the refinement and improvement of it. In the school of Euclides not only the practice but the principles of logic must have been studied and elucidated. With these materials to work upon, it seems nowise incredible that one trained in the school of Plato to the greatest subtlety and precision of thought, and possessing in himself a comprehension and a diligence quite unparalleled, should have been able so to produce a design and an edifice which after ages have found it scarcely possible in the minutest particular to alter or amend. He had not to raise a science from its foundations, but *lateritium invenit marmoream reliquit*, may perhaps be said of him with as strict truth as of almost any architect that the world has seen.

Now the work on the *Categories* seems to be a most fitting vestibule to this building. On entering it we feel at once that the purpose to which it is consecrated is altogether different from that which Plato has been teaching us to regard as all important, and we feel that it is a true purpose still. There is a way of penetrating into the nature and essence of things, whether those which present an outward image to my senses, or those, equally real, which merely utter themselves to my mind. With this way Aristotle does not concern himself. But it is equally certain that our mind does form notions and conceptions about the things belonging to both these kinds which it contemplates, and it may be that these conceptions themselves are subject to certain rules. They may be defined and classified; there may be a general set of conceptions to which all particular conceptions will refer themselves. This Aristotle affirms to be the case. Under the ten notions of Substance, Quantity, Quality, Relation, Time, Place, Position, Possession, Action, Passion, he says you may reduce all your notions. Now a Platonist is very likely to ask, But what do these words, Substance, Quantity, &c., themselves signify? "How do I know what Substance is, better than I know what a man or a horse is?" "Quantity better than I know what three cubits long is?" And these are questions which, as we shall find hereafter, had need to be asked, and were asked with effect and advantage when the Aristotelian province of thought had endeavoured to bring all other provinces within it. But for any further purpose than for destroying this pretension they are impertinent. When I study an actual man or an actual horse, the substance is doubtless the *or* unknown quantity which I am inquiring after to assume that I know it, is to stop all investigation. But I understand the *name* substance as well as I understand the *name* man or horse. And who told you that, because there is a science of *things*, there is not a science of *names*? that there are not laws of dependence and affinity among them? and that conformity or nonconformity to these laws is not exactly what we mean by coherent or incoherent discourse? There is no alter

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

native between the assertion, that the desire so deeply implanted in us of arrangement and classification is a mere disease, or the belief that it arises from the sense of certain limitations and conditions to which our minds themselves are subject, and is another name for the wish to understand what they are.

The rules of grammar, the terminology of every art and science, the very attempt to be intelligible, presume these. And there is no safety from the efforts of men to invent divisions and schemes of thought, no safety for the great principles and laws, which these dividers and schemers are continually narrowing and stifling, but in the clear and steady perception of certain necessary boundaries not imposed upon us by our fellow-men, but by the nature of our own understandings. Let then the reader carefully consider this work on the *Categories*. Let him ask himself whether it has not the effect of clearing his mind, and that in no ordinary degree, respecting his own modes of speech; whether it does not lead him to feel more than he did before, that his words, winged though they may be, can take no chance flight, but must move along an appointed preordained path; and, therefore, whether there be not a witness in himself that Aristotle has a distinct and reasonable end of his own, which it is very much for our interest to be acquainted with.

The treatise
Περὶ
Ἑρμηνείας.

From the investigation of these general forms under which we reduce all the notions that enter our minds, he proceeds in his treatise *Περὶ Ἑρμηνείας*, to inquire respecting the mode of our affirmations and denials. In this treatise he develops the nature and limitations of propositions, the meaning of contraries and contradictories, the force of affirmations and denials, in impossible, contingent, and necessary matter. This subject has been so fully treated in the article LOGIC in a former volume of this *ENCYCLOPÆDIA*, that we can have no excuse for travelling over the same ground. For the same reason we must say nothing respecting the two books of *Ἀναλυτικά Προτερα*, which develop the syllogistic principle and process. But we cannot sufficiently impress on our readers the remark which has been made already for another purpose in that article, that Aristotle is not in these books the mere inventor of an art, but the masterly expounder of the facts upon which that art rests, and but for which it would have no meaning. He does not teach us how to make propositions, but what propositions are, and necessarily must be, according to the conditions of the human intellect. He does not tell us how to make syllogisms, but how we do syllogize, when we do not violate the laws of our mind as much as we should violate the laws of our body, if we tried to walk upon our heads instead of upon our feet.

The first
Analytics.

The later
Analytics.

But Aristotle perceived that this analysis of our mental operations was not sufficient. He had told us how we *discourse*, but he had not told us how we *know*. Are these forms of Logic themselves knowledge? Is the syllogistic demonstration the same thing with science? Or, is one kind of it science? Or, are the results of it science? Or, are there certain premises assumed in it which also do or may belong to science? What are these? how do we get at them? Such are the important inquiries which occupy Aristotle in his *Ἀναλυτικά Ὑστερα*. We shall endeavour to seize a few of those points in the investigation which will best enable the reader to estimate the character of Aristotle's mind, and to see how he stands related as well to his great master as to the expounder of the inductive philosophy.

The treatise opens thus: Πᾶσα διδασκαλία καὶ πᾶσα μάθησις διανοητικὴ ἐκ προϋπαρχούσης γίνεται γνώσεως. φανερόν δὲ τοῦτο θεωροῦσιν ἐπὶ πασῶν. αἱ γὰρ μαθηματικαὶ τῶν ἐπιστημῶν διὰ τοῦτον τοῦ τρόπου παρά- γίνονται καὶ τῶν ἄλλων ἕκαστη τεχνῶν. ὁμοίως δὲ καὶ περὶ τοὺς λόγους οἱ τε διὰ συλλογισμῶν καὶ οἱ δι' ἐπαγωγῆς. ἀμφότεροι γὰρ διὰ προγινωσκομένων ποιοῦνται τὴν διδασκαλίαν οἱ μὲν λαμβάνοντες ὡς παρὰ ξυνιέντων οἱ δὲ δεικνύντες τὸ καθόλου διὰ τοῦ δήλου εἶναι τὸ καθ' ἕκαστον. ὡς δ' αὐτῶς καὶ οἱ Ῥητορικοὶ συμπίθουσιν. ἡ γὰρ διὰ παραδειγμάτων ὅπερ ἐστὶ ἐπαγωγή ἢ δι' ἐνθυμημάτων ὅπερ ἐστὶ συλλογισμός. There are two very important words in the opening clause of this sentence which we imagine were carefully distinguished in the school from which Aristotle came, διδασκαλία and μάθησις. We feel confident, also, that the last being, by the force of its name, the method of learning and acquisition, would have uniformly taken precedence of the other, which points to the communication of knowledge. That the order is here changed, that διδασκαλία is put foremost as if it included the other within itself, is a very significant circumstance, which is an explanation of much that follows. Plato, it is well known, had a profound reverence for mathematics. He was wont to say, "Let no man undisciplined in geometry enter the halls of philosophy." Now we cannot account for this admiration unless we suppose him to have perceived in the mathematical process something akin to his own method. But this resemblance certainly does not lie in that which we are wont to call the mathematical demonstration, it does not lie in the machinery of axioms, definitions, hypotheses, propositions. This machinery, valuable as it is, has scarcely a Socratic element in it. What remains? Plato, we conceive, would have answered, Exactly that which is the essence of mathematics, exactly the μάθησις. For this demonstration is but the διδασκαλία, a necessary and inseparable accident of the science, but implying the presence of something else and unmeaning without it. The μάθησις is the process whereby in any particular triangle I arrive at this as one of the laws which belong to it as a triangle, that its three angles are equal to two right angles; the διδασκαλία is the formal verbal enunciation of that law and the confirmation of it by certain deductions from previously admitted premises. In this sense mathematics would seem to him not indeed the science but the best preparation for it, because it recognises certain permanent forms and principles existing in visible objects, which we are capable of entering into and discovering. Now Aristotle, as we have often hinted, perceives only the forms and laws of our mind. That, consequently, which was the subordinate and accessory part of mathematics in the judgment of his master, became in his judgment the whole. He looks upon mathematics simply in reference to its demonstrations; the characteristic of mathematics is that it deals with necessary matter. If you ask why necessary, the only answer you can get is, that it starts ultimately from some self-evident propositions. So that, after all, the proposition, a mode of our own mind, becomes the ultimate ground of all things. Or if you will find out a Hercules' pillar beyond these, you have the *Categories*. Pent within these limits, it becomes difficult to grasp the meaning of science. Aristotle feels the difficulty, and with his usual honesty does not evade it. He acknowledges that it may puzzle us to tell whether science is only that which we obtain after a series of satisfactory syllogisms, or whether the premises assumed in those

Moral and
Metaphy-
sical Phi-
losophy.

Aristotle's
idea of
science.

Moral and
Metaphy-
sical Phi-
losophy.

Aristotle's
doctrine
respecting
sensible
percep-
tions.

sylogisms must not of themselves be entitled to the same character. And he can only say, that we must deal fairly with facts, and that what we are bound to assume as known, we do actually in some sort know. Ultimate knowledge then, as well as primary knowledge, the most perfect truth which the philosophers can attain, as well as the point from which he starts, is still a proposition. All knowledge seems to be included under the two forms,—knowledge that it is so, knowledge why it is so. Neither of these can of course include the knowledge at which Plato is aiming, knowledge which is correlative with being—a knowledge not about things, or persons, but of them.

But if these forms of our mind be the ground of our knowledge, we cannot at once arrive at them. What are the steps? Here comes in the Aristotelian doctrine about sensible objects. Our perception of these is not strictly to be called knowledge; that word, whether used about premises or conclusions, the data of a syllogism or its results, has still reference to what is universal. Our sensible apprehensions refer only to particulars, to individuals. But from these, which are the most evident to us, we come to the more general by a process of induction. What this induction is, and how entirely it differs from that process which bears the same name in the writings of Bacon, the reader will perceive the more he studies the different writings of Aristotle. He will find, first, that the sensible *phenomenon* is taken for granted as a safe starting point. That phenomena are not principles, Aristotle believed as strongly as we could. But to suspect phenomena, to suppose that they need sifting and probing in order that we may know what the fact is which they denote, this is no part of his system. The sensible impression was to him satisfactory, not indeed to rest in, but as a true beginning; all the difference between those who acquiesce in it and the most consummate philosopher lay in the use which the latter made of his power of generalizing and syllogizing. It is in this way that Aristotle has become the parent of all the modern schools of sensible philosophy, which schools have, nevertheless, drawn their very best and most convincing arguments from the errors into which Aristotle and the Aristotelians were led by the adoption of their own hypothesis. The first book of Locke may be justly said to be an elaborate and satisfactory exposure of the notions into which the Stagirite school is driven, by its determination to recognise no foundation of truth but sense and experience. They were too learned and thoughtful not to perceive that universal forms are in some way or other demanded by the mind; and because they would not acknowledge them as the grounds of our mind, they are forced to seek for them in the mind, and thus to conceive them in the shape of propositions. Hence the fancy of an innate notion that whatever is, is; that the whole is greater than its part, &c.; a fancy which Mr. Locke has confuted amidst such shouts of triumph from his admirers, while he was, in fact, conspiring with Aristotle to disparage the principle which delivers us from such fallacies. But we are anticipating a future page of our sketch.

The *Topics*. These *Αναλυτικά Υστερα* deal, it will be seen, entirely with demonstrative reasoning. The *Topics*, a much longer work, refer wholly to probable reasoning. On this subject Aristotle, it seems to us, is much more at home than on the other. We have intimated our suspicion that he never did possess or could possess the

idea of science. It lay altogether out of his province; when he tried to grapple with it he necessarily brought it within conditions and forms which robbed it of its very essence. But no one ever did so much as he to give a scientific form and semblance to those subjects which are by their nature not scientific. No one ever did so much to give our thoughts precision and clearness respecting all the wavering and fluctuating matters that fall within the domain of opinion and of ordinary conduct. When we look at the *Topics*, we are brought to confess (not without a certain reluctance) that there is no method of persuasion or human discourse so loose and random but that it may be subjected to analysis, and shown to involve certain inevitable limitations. Undoubtedly there is some justification for our discontent with these resolute reductions of all our thoughts and arguments to a system; we feel that the vital power is not there; that what really brings men into consent with either a falsehood or a truth, the energy and conviction of him who utters it, is taken no account of, and that the habit of contemplating arguments without reference either to the truth of that which they are meant to establish, or to the moral influence which they exert, is somewhat hardening and deadening to the mind. But we shall find that the work of a thorough master in this line, like Aristotle, will on the whole produce a good effect. He so entirely understands himself and what he is able to do, that we cannot commit the mistake which inferior writers often draw us into, of fancying that nothing is wanted but what he tells us. His exquisite dissections teach us what more blundering dissections might not; that there is something which cannot be dissected. To understand Aristotle rightly, the *Topics* should be read together with the three books on Rhetoric. We cannot speak of that work now because so much of the moral and political wisdom of Aristotle is presumed in it. Nevertheless it is closely connected with this work on probable arguments. The *Topics* are to it what the six books of Euclid are to a treatise on practical mechanics. Now, though it may seem strange to talk of a geometry expressly for a rhetorician, and though we have already shown in what sense geometry was very alien from the habits of Aristotle's mind, yet we cannot help seeing that just so far as a person who has the faculty of persuasion can be taught that his is not a mere craft, but has maxims and laws to govern it, he will become at any rate less mischievous, and may become a sincere, true-minded man. In another point of view the *Topics* give that roundness and completeness to the logical system of Aristotle which we said that he evidently desired, and had in so remarkable a manner realized.

The last treatise of Aristotle directly upon a logical subject, is that *On Sophistical Proofs*. From this short work we should pass in a complete criticism upon his works to his voluminous writings on physics. In these we have no direct interest. To show, however, how the logical ideas of Aristotle affected his views in this department, we shall take the liberty of extracting a passage from Mr. Whewell's *History of the Inductive Sciences*, vol. i. sec. 2. "The Aristotelian Physical Philosophy."

"The principal treatises of Aristotle are, the eight books of *Physical Lectures*, the four books *Of the Heavens*, the two books *Of Production and Destruction*, for the book *Of the World*, is now universally acknowledged to be spurious, and the *Meteorologics*, though full of physical explanations of natural phenomena, does

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphysical
Philosophy.

not exhibit the doctrines and reasonings of the school in so general a form; the same may be said of the *Mechanical Problems*. The treatises on the various subjects of natural history, *On Animals*, *On the Parts of Animals*, *On Plants*, *On Physiognomies*, *On Colours*, *On Sounds*, contain an extraordinary accumulation of facts, and manifest a wonderful power of systematizing; but are not works which expound principles, and therefore do not require to be here considered.

Extract
from
Whewell
on the
inductive
sciences.

"The *Physical Lectures* are the works concerning which the well known anecdote is related by Simplicius, a Greek commentator of the VIth Century, as well as by Plutarch. It is said that Alexander the Great wrote to his former tutor to this effect: 'You have not done well in publishing these *Lectures*, for how shall we, your pupils, excel other men if you make that public to all which we learned from you?' To this Aristotle is said to have replied: 'My *Lectures* are published and not published; they will be intelligible to those who heard them and to none beside.' This may very easily be a story inscribed and circulated among those who found the works beyond their comprehension; and it cannot be denied that to make out the meaning and reasoning of every part would be a task very laborious and difficult, if not impossible. But we may follow the import of a large portion of the work with sufficient clearness to apprehend the character and principles of the reasoning, and this is what I shall endeavour to do.

"The author's introductory statement of his view of the nature of philosophy falls in very closely with what has been said, that he takes his facts and his generalizations as they are implied in the structure of language. 'We must in all cases proceed,' he says, 'from what is known to what is unknown.' This will not be denied; but we can hardly follow him in his inference. He adds, 'We must proceed, therefore, from universal to particular. And something of this,' he pursues, 'may be seen in language; for names signify things in a general and indefinite manner, as *circles*, and by defining we unfold them into particulars.' He illustrates this by saying, 'thus children at first call all men *father*, and all women *mother*, but afterwards distinguish.'

"In accordance with this view he endeavours to settle several of the great questions concerning the universe which had been started among subtle and speculative men, by unfolding the meaning of the words and phrases which are applied to the most general notions of things and relations. We have already noticed this method. A few examples will illustrate it further: whether there was or was not a *void* or place without matter, had already been debated among rival sets of philosophers. The antagonist arguments were briefly these: there must be a void because a body cannot move into a space except it is empty, and therefore without a void there could be no motion; and, on the other hand, there is no void, for the intervals between bodies are filled with air, and air is something. These opinions had even been supported by reference to experiment. On the one hand, Anaxagoras and his school had shown, that air when confined resisted compression, by squeezing a blown bladder, and pressing down an inverted vessel in the water; on the other hand, it was alleged that a vessel full of fine ashes held as much water as if the ashes were not there, which could only be explained by supposing void spaces between the ashes. Aristotle decides that there is no void on such arguments as this:—In a void there could be no difference of up and down; for as in nothing there

are no differences, so there are none in a privation or negation; but a void is merely a privation or negation of matter; therefore, in a void, bodies could not move up and down, which it is in their nature to do. It is easily seen that such a mode of reasoning elevates the familiar forms of language and the intellectual connections of terms to a supremacy over facts; making truth depend upon whether terms are or are not primitive, and whether we say that bodies fall naturally. In such a philosophy every new result of observation would be compelled to conform to the usual combinations of phrases as they had been associated by the modes of apprehension previously familiar.

"It is not intended here to intimate that the common modes of apprehension, which are the basis of common language, are limited and casual. They imply, on the contrary, universal and necessary conditions of our perceptions and conceptions; thus all things are necessarily apprehended as existing in time and space, and as connected by relations of cause and effect; and, so far as the Aristotelian philosophy reasons from these assumptions, it has a real foundation, though even in this case the conclusions are often insecure. We have one example of this reasoning in the eighth book, where he says that there never was a time in which change and motion did not exist; 'for if all things were at rest, the first motion must have been produced by some change in some of these things; that is, there must have been a change before the first change;' and again, 'How can *before* and *after* apply where time is not? or how can time be when motion is not?' 'If,' he adds, 'time is a mensuration of motion, and if time be eternal, motion must be eternal.' But we have sometimes principles introduced of a more arbitrary character, and, besides the general relations of thought, the inventions of previous speculators are taken for granted; such, for instance, as the then commonly received opinions concerning the frame of the world. From the assertion that motion is eternal, proved in the manner just stated, Aristotle proceeds by a curious train of reasoning to identify this eternal motion with the diurnal motion of the heavens. 'There must,' he says, 'be something which is the first moved;' this follows from the relation of causes and effects. Again, 'Motion must go on constantly, and therefore must be either continuous or successive. Now, what is continuous is more properly said to take place *constantly*, than what is successive. Also the continuous is better; but we always suppose that which is better to take place in nature, if it be possible.' We see here the vague judgment of *better* and *worse* introduced, as that of *natural* and *unnatural* was before into physical reasonings.

"I proceed with Aristotle's argument, 'We have now, therefore, to show that there may be an infinite, single, continuous motion, and that this is circular.' This is, in fact, proved, as may readily be conceived, from the consideration that a body may go on habitually revolving regularly in a circle. And thus we have a demonstration, on the principles of this philosophy, that there is and must be a first mover, revolving eternally with a uniform circular motion.

"Though this kind of philosophy may appear too trifling to deserve being dwelt upon, it is important for our purpose so far to exemplify it that we may afterwards advance, conscious that we have done it no injustice.

"I will now pass from the doctrines relating to the

Moral and
Metaphysical
Philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

motions of the heavens to those which concern the material elements of the universe. And here it may be remarked, that the tendency (of which we are here tracing the developement) to extract speculative opinions from the relations of words must be very natural to man; for the very widely accepted doctrine of the four elements, which appears to be founded upon the opposition of the adjectives *hot* and *cold*, *wet* and *dry*, is much older than Aristotle, and was probably one of the earliest of philosophical dogmas. The great master of this philosophy, however, puts the opinion in a more systematic manner than his predecessors.

"We seek," he says, "the principles of sensible things, that is, of tangible bodies. We must take, therefore, not all the contrarieties of quality, but those only which have reference to the touch. Thus black and white, sweet and bitter, do not differ as tangible qualities, and therefore must be rejected from our consideration."

"Now the contrarieties of quality which refer to the touch are these, hot, cold; dry, wet; heavy, light; hard, soft; unctuous, meagre; rough, smooth; dense, rare." He then proceeds to reject all but the four first of these, for various reasons; heavy and light because they are not active and passive qualities; the others because they are combinations of the four first, which, therefore, he infers to be the four elementary qualities.

"Now in four things there are six combinations of two; but the combinations of two opposites, as hot and cold, must be rejected; we have, therefore, four elementary combinations which agree with the four apparently elementary bodies, air is hot, wet, (for steam is air,) water is cold and wet, and earth is cold and dry."

"It may be remarked that this disposition to assume that some common elementary quality must exist in the cases in which we habitually apply a common adjective, as it begun before the reign of the Aristotelian philosophy, so also survived its influence. Not to mention other uses, it would be difficult to free Bacon's *Inquisitio in Naturam calidi*, 'Examination of the Nature of Heat,' from the charge of confounding together very different classes of phenomena under the cover of the word *hot*.

"The rectification of these opinions concerning the elementary composition of bodies belongs to an advanced period in the history of physical knowledge, even after the revival of its progress."

"The Aristotelian doctrines concerning motion are still founded upon the same mode of reasoning from adjectives; but in this case the result follows, not only from the opposition of the words, but also from the distinction of their being *absolutely* and *relatively* true. 'Former writers,' says Aristotle, 'have considered heavy and light *relatively* only, taking cases where both things have weight, but one is lighter than the other; and they imagine that in this way they defined what was *absolutely* (απλως) heavy and light.' We now know that things which rise by their lightness do so only because they are pressed upwards by heavier surrounding bodies; and this assumption of absolute levity, which is evidently gratuitous, or rather merely nominal, entirely vitiated the whole of the succeeding reasoning. The inference was that fire must be absolutely light, since it tends to take its place above the three other elements; earth absolutely heavy, since it tends to take its place below fire, air, and water. The philosopher argued also with great acuteness that air, which tends

to take its place below fire and above water, must do so *by its nature*, and not in virtue of any combination of heavy and light elements. For if air was composed of two parts, which give fire its levity, joined into other parts which produce gravity, we might assume a quantity of air so large that it should be lighter than a similar quantity of fire, having more of the light parts. It thus follows that each of the four elements tends to take its own place, fire being the highest, air the next, water the next, and earth the lowest. The whole of this train of errors arises from fallacies which have a verbal origin; from considering light as opposite to heavy, and from considering levity as a quality of a body, instead of as the effect of surrounding bodies."

These remarks of Mr. Whewell's show how necessarily the method of Aristotle obstructed the true observation and interpretation of nature. To us they are important chiefly from the connection which has always been felt to exist between the physical treatises and those which by Aristotle himself or some disciple are classed under the name of Metaphysical. These latter writings have been the subject of much dispute. Some have supposed that they owe their name to an accidental juxtaposition with the books on nature, among which it was evident that they could not conveniently be classed; some to a feeling in the mind of Aristotle, that he was ascending into a higher region, above and beyond that in which he had been dwelling previously.

One of his early commentators appears to adopt both explanations of the title. "We," he says, "make our beginning from those things which in nature are the last, seeing that these are the better known to us. For this reason, then, he discourses to us first concerning physical matters, for these are last in nature, but to us first. But this present subject is first in nature, but to us last, for the imperishable things are older than the perishable, and the ungenerated than the generated. Aristotle then discourses to us first concerning those things that are moved without an order, (in his book on *Meteors*;) then again concerning those things which are moved according to an order or system, (in the work on the *Heavens*, concerning the stars and the spheres;) and finally in this treatise he discourses to us concerning those things which are in all cases unmovable. Now this is theology, for such a study befits the gods. For this reason the work is inscribed *After-the-Physics*. (Μετα τα φυσικά,) seeing that he first discoursed to us concerning the physical things, then consecutively concerning this; it is proper, therefore, to read it after the physical treatises; this the title shows." Another of the Scholiasts is more decisive: "This work is entitled *Μετα τα φυσικά*, not in reference to the character of the book, but to the order in reading it, for he treats concerning physical *principles*." These hints, if considered in connection with the books themselves, will, we conceive, explain the origin of the two theories, and in a great degree reconcile them. Without pretending to ascertain the proper succession of treatises which are comprised in this division of the works, there can be no doubt, we imagine, that the review of previous Greek philosophers is the intended introduction to the rest. This review is prefaced by the remark, that there is a kind of knowledge to which the word wisdom (σοφία) is especially applicable; it is conversant with principles, (αρχαί.) These principles may be looked for in four ways. Either in the subject-matter, (η ὕλη και το ὑποκείμενον,) or in that which produces motion, (ὅθεν ἡ αρχη της κινήσεως,)

Moral and
Metaphy-
sical Phi-
losophy.

The Meta-
physics.

Σχολία ὑπὸ
Ἀσκληπιοῦ
ἀπὸ φωνῆς
Ἀμμωνίου
τοῦ Ἐρμίου.

Moral and
Metaphy-
sical Phi-
losophy.

or in the substance, (*ἡ οὐσία καὶ τὸ τι ἦν εἶναι*), or in its purpose, (*τὸ οὐ ἐνεκα*.) Aristotle believes that one or other of these courses of inquiry was followed by each school of Greek thinkers, and was considered by that school as the only and all-sufficient method. Thus the Ionic philosophers studied the matter of things in hopes of discovering a primary element to which all other things might be referred. Those of this class who selected fire as their element were naturally led by the effects which they observed resulting from that power to speculate upon the meaning and mystery of Motion. Hence a new kind of inquiry was started, which proceeded, however, much in the spirit of those respecting Elements till Anaxagoras discovered the necessity of an Intelligence to set physical agents in movement. As, however, he had only recourse to this ultimate principle when other instruments failed him, the Atomic theory, which furnished a more plausible explanation of the facts of nature than his Homæomeriæ, easily supplanted them. Between this theory and that which affirmed Numbers to be the first principles of things, Aristotle appears to detect a connection, one not well supported by chronology. That doctrine of numbers he considers the first form of the inquiry after the Essence or Substance of things. The archetypal Ideas of the Platonists, who regarded numbers as a kind of intervening powers between sensible things and pure essences, is the second and higher form of it. The inquiry respecting the object or purpose of things had not, he imagines, been pursued distinctly by any class of his predecessors, but it had entered somewhat confusedly into the speculations of them all.

How far this able and interesting sketch of previous systems is satisfactory we need not now stop to inquire. We have given our reasons, which might, we think, be confirmed by many unconscious testimonies from Aristotle himself, for regarding Socrates as the key-stone of Greek speculations. If Aristotle be right, he was only an accident or interloper among them—of right, therefore, only mentioned in a parenthesis as chiefly devoting himself to ethical questions, Plato's intellectual descent being traced not through him to the Italian and Heraclitan schools. We might derive much evidence in favour of our view from the remarks respecting the mixture of the fourth class of inquiries which had reference to the intention or utility of things with the three which were suggested by physical observation. We might show, from the tone of the whole treatise, that Aristotle did not sympathize enough with his predecessors to understand the complicated thoughts which were at work in them, and was therefore naturally tempted to find classifications where no classification existed. But we do not refer to this book as a help in understanding Aristotle's predecessors, but Aristotle himself. It removes, we think, almost entirely the difficulty which has been raised respecting the use of the word Metaphysics in application to this and to the twelve books which follow it. It is evident that he is here tracing the method by which, as he supposes, men were led to discover in Physics something beyond Physics, which is their necessary ground. To say that all these inquirers discovered something which was distinct from physics was far from his meaning. It is one of his great objects to show that the first seekers after a principle really went no further than the mere outside facts which they proposed to investigate; still they, as much as the others, were in search of an *αρχή*. Their

inquiries, as much as those of their successors, (some of whom, as the Atomists, were as purely physical as they had been,) must be classed among the *τα μετὰ τὰ φυσικά*; for they groped at these if they did not find them. The simplest explanation, then, which has been given of the expression is in one way the truest. But that explanation by no means excludes, nay it involves, the belief that Aristotle is finding out the nature, and defining the objects of a science not merely coming next in order to physics, but, in kind, transcending them; and, we conceive, there is no difficulty in understanding why this word may have now first been consecrated to such a purpose, though the inquiries which we have called metaphysical had been proceeding for so long a time in Greece. Not only has it happened in a thousand cases that the name for a study has not been invented, or at least, not authoritatively stamped and circulated till it has performed its greatest achievements, and the ardour for it has declined—not only is Aristotle peculiarly the definer of boundaries, the systematizer of thoughts, previously living and at work—but also it might be inferred as much from the common view of Greek philosophical history which we have taken, as from his more ingenious view, that it would have been impossible, before his time, to have comprehended, under one such word as this, the subjects which are here expressed by it. Aristotle is confessedly feeling his way to a distinction which, according to our notion, they could not recognise, which, in his notion, at all events, they had not recognised. How questions regarding nature are connected with man, or what common ground there is for both was, we have suggested, the great puzzle of those philosophers, who busied themselves, as Aristotle says, with the *ὑλη*. That they should have formally separated them, no one would expect. An entire confusion of speculation with practice was the characteristic of the Sophists. They endeavoured to turn practice into speculation. Socrates endeavoured to make all speculation practical. Plato showed how the practical ground necessarily involved an absolute truth which the speculative eye only could behold. It remained then for our philosopher to make this partition, to assign one region to the philosopher as a philosopher, another to men as men, and as a necessary consequence, to make the region of philosophy a region in some way connected with nature, though passing beyond it.

We are very anxious that this point should be understood, because the history of subsequent philosophical inquiry in a great degree turns upon it. The metaphysics of Aristotle are troublesome reading, partly from the frequent repetitions which occur in them, partly from the difficulty of discovering a sequence in the books. Nevertheless they should be read by any student who wishes to investigate the questions which have occupied men in later times. We shall illustrate our previous remarks by tracing a very rude outline of the subjects which are discussed in them, and recording some of the solutions Aristotle has given of the difficulties which he starts.

A kind of appendix which follows the first book contains a proof that causes are not infinite, that there is consequently a possibility of carrying on that inquiry in which past philosophers had engaged. The same short book contains some important remarks upon the manner in which the search was to be conducted, upon the contributions to truth which each school may have made, upon the advantages which a philosopher may derive

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

from attending even to popular notions, upon the dislike which some have to exact mathematical reasoning, and the determination of others to have nothing else, and upon the proper limitation of mathematical accuracy to things without matter. We have here also the clear announcement of a principle which the student of Aristotle would have need to keep constantly in recollection, θεωρητικῆς μὲν τέλος ἀλήθεια, πρακτικῆς δ' ἔργον. He adds an explanation, which still further illustrates his meaning, and makes the difference between him and his master more conspicuous, that the practical man has nothing to do with the eternal or the absolute, but only with the relative. This book ends with a promise of an inquiry into the meaning of the word nature, which is not however fulfilled in that which is commonly placed next to it.

This second book is a collection of doubts or questions to be hereafter resolved. The first doubt is, whether it is the business of one science, or of more, to inquire into all kinds of causes or principles. This question involves the very subject of the whole treatise. So many different subjects seem to be included in that province to which the general name σοφία has been given—matters purely belonging to the senses, the causes of motion, the nature of being, the reason and purpose of things—how is it possible to suppose a single science dealing with principles apparently not admitting either of analogy or contrast? Secondly, are we to look upon the most comprehensive genera to which individual things can be referred, or upon the atoms of which they consist as their principles? The third question is connected with this, is there any thing besides individual things? If not, how can they be known, for are not individual things infinite, and is not knowledge of that which is one and universal? Fourthly, are the principles of things perishable, and of things imperishable, the same? Fifthly, (which is the great question of all,) are Being and Unity the essences of things that are and not distinguishable from them, or are we to seek for the το ον and the το ἐν as if they had each a distinct nature? Sixthly, are numbers, bodies, planes, and points, substances or not? Such are the general controversies of which we are to hope for some settlement in the books that follow.

The third book may be considered an answer to the first question. There is a Science which contemplates Existence as Existence, and whatever appertains to it in reference to it, not like other sciences, merely the attributes of certain particular existences. There are, he says, certain things peculiar to Being as Being, and these are things concerning which it is the philosopher's function to investigate the truth. The dialectician and the sophist resemble indeed the philosopher; being is the common subject-matter to all three. They discourse concerning the subjects which are in a peculiar sense his property. The dialectical δύναμις differs from the philosophical in its nature; the sophistical in the intention of him who uses it (τον βιον τῇ προαίρεσει). ἐστι δὲ ἡ διαλεκτικὴ πειραστικὴ περὶ ὧν ἡ φιλοσοφία γνωριστικὴ ἢ δὲ σοφιστικὴ φαινόμενη, οὐσα δ' οὐν. This is an important passage as illustrating the difference between the Platonic and the Aristotelian dialectics. According to Plato dialectics was the human science, the science of the philosopher as such. Σοφία in its essence belongs to God. Aristotle, we see, discovers a difference between dialectics and the highest philosophy, and inevitably; for, as his dialectic treats only of the forms of

human thought, as it deals with knowledge merely in the sense of the powers and means of knowing possessed by us, there must be another science concerning the objects of knowledge as such. But then, in this science also, the objects cease to be objects, they become *subjects* for man's contemplation, they become Metaphysics or Ontology. The hint respecting the *Sophist* contained in this sentence should also be compared with the elaborate exhibition of his character and functions in the dialogue between the Eleatic stranger and Theætetus. It connects itself with the inquiry, whether mathematical axioms are subjects of inquiry for the Ontologist or highest philosopher. The answer is in the affirmative. Those axioms were assumed by the mathematician. The student of physics sometimes meddles with them, but rashly and presumptuously; they are first principles, or *αρχαί*, and as such are cognizable only by the person whose office we are defining. Now as it was the especial delight of the sophist to deny axioms, to say that the same thing could be and not be, this becomes the natural place for settling his pretensions. Consistently with their characteristic difference, Aristotle represents him, not in Plato's manner, as one who invents counterfeit images of that which is, but as one who attributes accidents to accidents instead of to substances. We can scarcely conceive two portraits of the same person so correct and felicitous, and yet expressing so thoroughly the manner and principle of their respective artists.

The fourth book is a book of definitions. We can only convey a notion of its importance by giving a list of the words defined. They are Principle, (*αρχή*), Cause, Element, Nature, Necessity, Unity, Being, Substance, Sameness, Opposition, First and Last, Power, Quantity, Quality, Relation, Perfection, Limitation, *Τὸ καθ' ὃ* (*secundum quid*); [the meaning is easily understood from Touchstone's words in *As You Like It*, "In respect that it is of the country it is a good life, but in respect that it is not of the court it is a vile life, &c."] Disposition, Habit, Passion, Privation, Inclusion, Derivation, Part, Whole, *κολόβον*, (the mutilated,) Kind, Falsehood, Accident. The explanations of some of these words will, it is obvious, have been repeated from the *Categories* and the *Analytics*. Some of them will be better understood from the arguments in the subsequent books than they could be from any formal definitions; still they are worthy to be read and reflected on. The one on "Nature" is perhaps the most important, but to his notions of this word the entire treatise is the only satisfactory clue.

The fifth book explains Aristotle's view of the difference between physical and ontological science. They agree in this, that they are theoretic not practical or poetic. They differ in this, that the physical deals with that which has a capacity for change or movement, and with that which is embedded in matter. The primary philosophy deals with the unchangeable and with that which is separate from matter. Mathematical science lies between them, resembling ontological science in the first characteristic, physical in the second. If there be such a science as Theology, it must be a part, and the highest part, of the primary Philosophy. The *το θεῖον* (the divine nature) must be pre-eminently *των ἀκινήτων καὶ χωριστῶν*. The condition of those things whereof physical science treats, is that they are susceptible of accidents; not that there can be a science of accidents; as such they exclude science altogether.

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

But there may be, and there are, principles and causes of those things which admit of accidents, which are not themselves in any sense accidents, but are necessary. Pursuing these, physical science still retains its formal distinction from that higher science which deals with existences and essences as such. There is another sense in which the phrase *τοον* had been used, especially by the Platonists, which it is necessary to distinguish from our notion of it. It had been confounded with truth; not being with falsehood. Now truth, according to Aristotle, is not in the things but in the mind. Affirmation combines, negation separates, falsehood separates that which should be combined, combines that which should be separate. But the existences with which we are dealing are simple or uncompounded. Here again we have one of the capital and vital points of difference between the two philosophies.

The sixth book contains some of the most important distinctions and differences in the whole treatise. It begins with repeating a remark previously made, that Ontology has a twofold object. It deals first with that of which other things are predicated; secondly, with that which is predicated of it. Now substance is that of which other things are predicated. What then is to be included among substances? Are walking, sitting, being in health, substances? No, all these imply a subject, to which they must be considered as referring. In this way we get rid of the notion of an essential good, as well as an essential warmth or whiteness, &c. All these alike are considered as qualities of some subject, and what that subject is must be sought in each individual which offers itself to our observation.

But does not substance when thus considered necessarily connect itself with body? Here is one of our great puzzles. Some would have the *boundaries* of body to be substances; some would have substances which are in no sense cognizable by the senses; some would make the One the primary substance, and suppose different sets of substances, such as numbers, magnitudes, souls, to be generated from this. Aristotle's opinion is this. To every subject belongs, first, *ύλη*, which we must translate matter; secondly, *μορφη*, or form. The matter of a thing is not merely an accident of it, it is its necessary condition. But this matter is not its essence; something else is implied in it, something which it presents or makes manifest. Applying this principle to the questions which occupied the third class of philosophers, mentioned in the introductory book,—the Platonists namely, and the Pythagoreans,—it appears that this form is the true *είδος* of which they dreamed. It is the essential thing in each thing; it is not substance such as we behold it, but that in virtue of which substance is possible, without which it is inconceivable. But it does not exist apart from each particular substance; it is that which enters into the definition of every subject, and without which the definition would be no definition; obviously, therefore, it must be viewed in that subject, and cannot be contemplated as a distinct, peculiar essence. Tested by this rule it is obvious also that all notions of an ideal form of hollowness or of pugnosedness (we use Aristotle's favourite illustration) must be out of the question; these cannot be, primarily at least, subjects for a definition; they presuppose something whereof they are properties, and in that, and that only, can you look for an *είδος*. All notion again of Being as distinct from the particular person who, or the thing which is, falls to the ground.

Socrates and the being of Socrates are identical; the *αυτοεκαστον*, of which he had talked, is nothing else but this *είδος*, or form inherent in the thing itself.

The mode in which this same principle is applied to another class of inquiries, those which relate to the genesis or first origin of things, requires a more minute examination.

In considering any production, we find, first, something whence it has been generated; secondly, something by which it has been generated; thirdly, the result or the thing itself. There are three modes of production—natural, artificial, automatic. In natural productions we discern at once a matter; nay, in the largest sense, Nature itself may be defined that out of which things are produced. Every thing that becomes has a nature, which is only another way of saying that it has a *ύλη*; and that in each thing which might not have been is this *ύλη*. Now the result formed out of this matter or nature is any given substance—a vegetable, a beast, a man. But what is the producing, generating cause in each case? Clearly something akin in kind to the result. A man generates a man. Then we suppose in the resulting thing a productive force distinct from the matter upon which it works. And this is our *είδος*. And it is the combination of this *είδος* with the *ύλη* which both produces a substance and constitutes it. Look now at artificial productions. Here the *είδος* is still the producing power. It is in the soul. The art of the physician, the plan of the architect, is that *είδος* which produces actual health or an actual house. Here, however, a distinction arises. In these artificial productions is supposed a *νοησις* and a *ποιησις*. The *νοησις* is the perception and internal entertainment of the form; the *ποιησις* the creation out of the given matter. But we mentioned a third mode of production not strictly natural or artificial, but by the action of the thing itself. For instance, a cure may take place by the application of warmth; a body may become warm by rubbing; this warmth then in the body is either itself a portion of health or something is consequent upon it like itself, which is a portion of health. Evidently this implies the previous presence either of nature or of an artificer. Evidently also there is a necessity that this kind of generative influence should combine with another. There must be a productive power; there must be something out of which it is produced. In every case there will be an *ύλη* and an *είδος*. Upon this principle, that which is generated is the whole substance, consisting of matter and form. But the form, properly speaking, is not generated. It is reproduced in each particular subject in combination with a certain matter, and it becomes a new and peculiar form in virtue of that combination. There is necessary then to every production a certain form and a certain matter; and all the qualities appertaining to this substance which is produced, *must* inhere (not actually, but potentially) in the substance producing, and may belong to the form when they are produced.

It remains to consider how this doctrine bears upon the inquiries of those philosophers who busied themselves with the search after a primary element; (those who sought for the *το ού ενεκα* are reserved for another discussion.) But before we can enter upon this subject several of the doubts in our second book must be resolved. First, as to the meaning of the words Part and Whole. The first and most obvious signification of part has relation to quantity; but this has

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

nothing to do with our subject. What we want to know is the connection of the idea of part with substance. Assuming the division of substance into *ύλη* and *εἶδος*, we should say that in a brass statue the brass formed part of the statue, as the complex of form and matter, but not of the statue considered as a Form. Now as Form is the proper subject of a definition; as it can be described in itself, and as that which is material cannot be so described, it comes to pass that in certain cases we necessarily speak of the parts as constituting the whole, and in other cases not. We define a circle without reference to its parts. We define a syllable by the letters or elements which compose it; for the parts of the circle are material parts, the parts of the syllable are formal, logical parts. Of course, if you look upon a syllable as composed of certain letters in wax, or even of sounds in the air, its divisions become material and do not fall within the scope of a logical definition. Again, in a material division you assume the whole as preceding the part. On the contrary, logically and formally, the part precedes the whole. For instance, if you define the life of an animal you will describe it by some of its functions. None of its other functions can be performed without sensation. This particular faculty of sensation, therefore, will precede and be assumed in the existence of the whole animal. This principle holds equally in reference to æsthetic matter, (that which the senses take account of,) as in noetic, (the figures of mathematics.) Generally, therefore, it may be affirmed that the question as to the priority of part and whole depends entirely upon the distinction between matter and form being admitted, and that it is not possible to settle if that distinction be disregarded. At the same time Aristotle admits the difficulty of defining simply with reference to form, and not to the complex substance, which consists of it and of matter together. He acknowledged that the attempt to divide matter from substance and look upon things sensible as not sensible, has led to all the Pythagorean and Platonical inventions, which he regards with so much dislike.

Another question, in which these philosophers are also involved, follows immediately upon this. How are substances connected with *kinds*? If there be certain types after which all sensible things are formed, these types would seem to be universals, and those things with which the senses converse particulars. All possible differences and properties which can be discovered in the most marked individual of any kind must then upon this showing be included in those primitive, universal forms; but, according to logic, precisely the opposite is the case. The genus is divested of the difference which goes to the composition of the species, and of the properties which go to the composition of the individual. Your genera can never be types of the individual. By their very nature they are deficient in all that characterises him. The *εἶδος* then which forms the essential in each thing which makes it be that which it is, must be looked upon as individualized by the *ύλη* with which it is connected, as being apart from the modifications which it thus undergoes only a logical existence, the highest genus to which it is ultimately referred being pre-eminently that which can be only contemplated by and in the mind. Such we take to be the meaning of Aristotle, and from it the doctrine seems to follow very closely with which he winds up this book, and which applies the meaning of it to those who had dealt mainly with the *ύλη*. Any fact or thing being given, I have no

further occasion to trouble myself about the *ὅτι*. This the sense, or something corresponding to sense, supplies. I am not to ask what is the musical man when I see a musical man. He is that which I behold and nothing else. My business is with the *διότι*, the cause. Why is he this or that? And the answer is in the *εἶδος*, the form or constitution. This is the ultimate reason of that which each thing is. Consequently I do not get nearer the cause or reason of things by reducing them into their natural elements. The analysis may be physically proper or useful; but it does not lead me to that of which I am in search. Every thing which is, and which I can either behold with my senses or my mind, is not the A or the B whereof it is composed, but is something else; the synthesis of the A and the B involves the presence of a form or existence, which cannot be found in either of them separately. So that find out as many primary elements as you will, you do not thereby find an *αρχή*.

Our main business then is to discover the meaning of this *εἶδος*, and the relation which exists between it and the *ύλη*. The seventh book takes up this subject, and carries forward a hint which was given in the last—one which is, perhaps, the most pregnant of all the hints in Aristotle's writings, and that which has most effect upon his whole philosophy. The *εἶδος* or *μορφή* is an energy, the *ύλη* is a *δύναμις* or potentiality, implying and requiring the action or co-operation of the energy to produce a result. *Οὐσία*, as we said before, is the synthesis of these; omitting the *ἐνεργεία* you come merely to certain material elements and combinations which do not in any way give you the actual things you are examining. The difficulty is respecting those things which appear to have no *ἐνεργεία* in themselves, as a house. Must the substances of these be considered as something distinct from them and external to them? The answer is this:—Do you mean to ask whether the material house, that is to say the stones and cement, is a substance? Certainly not; you have excluded the very notion of substance by the mode of your question. But do you mean to ask whether that house is actually something? You assume it by your very question. You cannot define any thing without treating it as a substance, satisfy yourself as you will about the reason it is so; there is something then not distinct from the house, but implied in it, which is a form or *εἶδος*. The test that there is, is your own attempt to define it.

Proceeding upon this principle of an energy and a *δύναμις* in each substance, he shows how needful it is in any inquiry after causes to keep the three questions in sight. By cause do you mean potentiality, *δύναμις*? By cause do you mean moving power? By cause do you mean the form? (the two last *may* always be the same; still the inquiry after the constitution of each thing is distinct from the inquiry after its productive force.) Do you mean, lastly, the *οὐ ἐνεκα*? (This also may be the same question with the last, though differently stated; that is to say, the constitution of each thing may determine the purpose or object of it.)

But this use of the word potentiality (*δύναμις*) suggests another doubt. Has not every subject a potentiality for contraries? Must not we say that every healthy body is potentially sick? that water is potentially both wine and vinegar? The answer is that this absence or deprivation of qualities is an accident of these qualities, and not itself a quality. For a dead body to become alive, it must pass into a certain *ύλη*, which has therefore

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

the potentiality of life; for vinegar to become wine it must pass into water; nothing similar happens in the opposite case. Finally, he applies this principle to the solution of that difficulty respecting unity at which he had so often hinted, and on which he had expressed his opinion with sufficient plainness already. If the definition of a man be that he is a biped animal, how comes it that each of these "animal" "biped" do not constitute a separate entity? What, in short, is the ground of our conception of each thing and person as one? Aristotle intimates that this question is hopeless and unanswerable if put in this form, for then by each variety of your definition you create a new puzzle. He would rather then assume the unity of each thing as a fact or datum of the understanding, and account for its being reduced into different constituent elements in a definition. And this is accounted for by the necessary co-presence of matter and form in each thing, and from the matter being merely a potentiality, and the form an energy. In this way the dream of a unity distinct from the individual thing is got rid of, as the dream of a substance distinct from each particular thing had been got rid of before.

The eighth book is still occupied with the subject of potentialities and energies. Aristotle inquires into the different senses of the word *δυναμις*; what we mean when we say that a thing can or cannot be. There is a use of this language in geometry, which is metaphorical, and not to our present purpose. We say that a thing can or cannot be, meaning that it is or it is not. But all *δυναμεις*, in their proper sense, have reference to some primary *δυναμις*, which is the cause or beginning of a change in some other thing as another thing. Warmth, for instance, is a *δυναμις*. It is so equally in that which warms, as in that which is warmed. The *δυναμις* in the thing warmed hath reference or relation to the corresponding primary *δυναμις* in the thing warming, and this necessary implication of that which answers to itself in something besides itself, is its characteristic nature. To every capacity there will of course correspond a certain incapacity, which may be understood either as the absence of a faculty of communication, the absence of a faculty of reception, or, again, merely as a negative want, or as a positive state involving that want.

Δυναμεις are divided into rational and irrational; the rational those which subsist in the reasonable soul, the irrational those that are merely physical. All art and knowledge are of the first class. Now, if we look for a radical distinction between them, we may find it in this way. Warmth, an irrational *δυναμις*, has the power only of producing warmth; the art of the physician has the power of producing either health or sickness. Generally, therefore, the one kind of power can produce contrary changes, the other only a certain change; and these contrary changes are wrought by the rational powers with and upon the irrational powers.

This description of *δυναμεις* might seem in some points to trench upon that notion of energies which Aristotle had given us in his last book. He proceeds therefore to distinguish them. The Megarian sect, in conformity with their general rule of reducing every idea into that of being, and of excluding all distinct notion of production and becoming, had identified Power or Capacity with energy. Where there is no building, said they, there is no builder. Apply this, says Aristotle, to arts, and the man who has studied the longest ceases

to have the art as soon as he ceases to exercise it. Apply it to things without reason or life, and there is nothing sweet, or warm, or cold except at the moment when it is tasted or felt; an argument not, perhaps, very destructive of the proposition in our minds, but which was very effective as against the Megarians, who had the greatest horror of the Protagorean doctrine. It is then possible for a thing to have the capacity of being and not to be, and to have the capacity of not being and to be; that of which it is the capacity takes place where something is superadded to it, which is Energy. Energy is analogous to motion. You cannot predicate either motion or energy of things which are not; the moment energy or motion is added to them they are; but many things which are not have a possible or potential existence.* At the same time energy is not to be confounded with motion. The difference does not lie where we might suspect, in that motion belongs to that which is irrational, energy to the rational, for learning is referred to the head of motion, sight to the head of energy. The difference is in this, that every motion is incomplete, tending towards an end, but not including the end in itself, that energy has an end in itself, and that it does not involve a pause or a termination. Learning, building, walking, all imply a termination. Seeing, thinking, being happy, imply no termination; these are energies.

Upon this showing, energy, in the order of reason and of substance, precedes *δυναμις*; in the order of time not always. It precedes in the order of reason because the first of all capacities or possibilities is the capacity or possibility of energizing. A man who has the faculty of building, is one who has in him the capacity of using his energy in the art of building. In time it is otherwise. The primary energizing power of course precedes, even in this sense, that which receives the impression of it, Form being older than Matter. But if you take the case of any particular person or thing, we say that its capacity of being that person or thing precedes its being such actually or energetically. But then, though this is the case in each particular thing, there is always a foregone energy presumed in some other thing to which it owes its existence. And thus the principle is asserted which we shall find afterwards turned to practical account in the Ethics, that the exercise of any particular energy precedes the habitual use of the faculty appertaining to that energy; that it is by playing on the harp we become harp players. Several important ethical doctrines are in fact developed in the course of this inquiry, but of these we shall take distinct notice hereafter. One pregnant notion more directly bearing upon our present subject occurs in the course of it. *Δυναμις* had been defined that which is the cause of change in some other thing as another thing. But this notion wants a resting place, unless you believe that there is some primary *δυναμις* presupposed in all others, which is the beginning of motion. This is *φύσις*, and thus

* "Ὁὐκ ἔστι δὲ", adds Aristotle, "ὅτι οὐκ ἐντελεχία ἐστίν." Met. 8. iii. 36. This passage, perhaps, determines as clearly as any we could produce the meaning of the word *ἐντελεχία*, which is so important in some branches of the Aristotelian philosophy. It is the opposite to *potentiality*, yet would be ill translated by that which we often oppose to potentiality, *actuality*. *Εἶδος* expresses the substance of each thing viewed in repose,—its form or constitution; *ἐνέργεια* its substance, considered as active and generative; *ἐντελεχία* seems to be the synthesis or harmony of these two ideas. The *effectio* of Cicero therefore represents the most important side of it, but not the whole.

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphysical
Philosophy.

we have arrived at the most complete notion of it which we can expect. And this first and primary cause of all change, Energy still precedes and surpasses. Aristotle proceeds to show wherein energy is better and more glorious than *δύναμις*; as, for instance, because *δύναμις* is the same of contraries. The capacity of health and sickness is the same; of stillness and movement; of being raised up and of falling down. But one of these must be good, and therefore the energy which determines which of these contraries shall have effect must be better than the faculty or capacity. Two consequences follow. The energy is that which makes things be evil which have only the possibility or potentiality of evil in them. Secondly, in those things which are primary and eternal there is no evil, no fault, no decay; the capacity for these lies in nature. The importance of these two axioms will be felt by every moral and theological student. Another more doubtful proposition, one which has been extensively applied in another direction, is added respecting *discovery*. It is that discovery means the bringing things into energy which exist potentially; because knowing is an energy.

This book concludes with another reference to the relation between truth and being, falsehood and not being. Truth and falsehood being the accordance or discordance of our judgment with the actual state of things, there are three cases which may fall under our notice. First, things always united and inseparable, or things always separable and never united. Respecting these the judgment must be uniform; the same will be truth in all cases, falsehood in all cases. Secondly, things which may be either separated or united. Here comes in the possibility of that being true to-day which is false to-morrow, true under one aspect which is false under another. Thirdly, things perfectly simple, things admitting neither of division nor combination. To these the words true and false do not apply, but merely knowledge and ignorance. You either know such things or you do not. Respecting these there is no mistake, no deception possible; but merely the presence or absence of knowledge. All substances which exist in energy merely and *εν δυνάμει*, are of this character; they are analogous to the sensible objects whose existence you ascertain by touch or sight; the want of touch or sight, not a false opinion, excluding them from you.

In the ninth book we come again upon the question of unity. The name One is used, he says, in four ways. It means that which is *continuous by nature, a whole, an individual thing, that which is predicated of a whole*. The general sense of the indivisible is common to all these. And again unity in any of these senses we may attribute to some particular substance which is inseparable in place, in form, in thought, as well as to some actually indivisible whole. The fundamental notion of unity he conceives to be that of a measure to quantities; without such a measure quantity is inconceivable. There may be something actually indivisible, there may be that which is indivisible to our senses, an actual unity in form, and a supposititious unity in matter. Each will bear the name, because each will be used as a measure. The need of such a measure, he asserts, in opposition to the Protagorean notion of man being the measure of all things, which he treats as a silly truism, putting on the form of a paradox, and producing the effects of a falsehood.

The existence of a distinct absolute unity is denied on

precisely the same ground as the existence of a distinct absolute substance. The one is always some one thing or nature. In colours, if you suppose them all to originate from white, white is the one. In voices, the elementary vowel, and so in all other cases. Of course, then, the Ionic attempt to discover some matter, such as air or fire, which shall be unity, is as unreasonable as the Parmenidean, Pythagorean, and Platonic to invest unity itself with a formal and separate character.

"The one" is the undivided, or the indivisible; this is the primary notion of it, to which all others may be reduced. "The many" then will mean the divided or the divisible; from which, as more cognizable by the senses, the one will be inferred. The question occurs next, how the one and the many are opposed to each other; whether the "many" and the "few" are not equally opposed, and whether, if this be the case, unity is not merely an element of plurality. This question introduces a discussion respecting the different modes of opposition; the opposition of contradiction, of things in relation, of privation, of strict contrariety, *κατ' ἀντιφασιν, τῶν πρὸς τι, κατὰ στερησιν, κατ' ἐναντιωσιν*. Possibly there has been some confusion of different lectures or reports in this part of the book; for in the lengthened explanation we seem to lose sight of the original subject. Our readers cannot fail to have remarked how much the idea of a "law of opposition" in things entered into all Greek speculations, so as to seem to many the foundation of them. Aristotle contemplates the subject from the logical side; the forms of opposition which he discovers in our minds determine his view of the actual opposition which exists in nature. And in this way his remarks on this point, though apparently irrelevant, throw considerable light on his doctrine respecting unity. What our understanding wants in order to explain to itself the existence of multitude, this he called "the One." Unity was therefore, in his mind, identical with Singleness.

In the next book, which is for the most part a recapitulation of puzzles and solutions already given, (not, however, to be passed over on that account, for Aristotle's repetitions of himself, or the reports of his different pupils, generally clear away many difficulties, and here, especially, the remarks on the nature and limitations of the primary philosophy and his confutation of the two cardinal sophisms of Protagoras are, in many respects, more complete than those in the third book,) we shall notice merely his analysis of Motion and his remarks on the idea of the Infinite. Motion is neither an energy nor yet merely a potency; but it must be contemplated, alternately, as each. A lump of brass is potentially a statue; the energy which is to make it one is in the mind of the sculptor. The motion, *i.e.* the transition from its condition as brass to its condition as a statue, is not found in the brass, neither is it found in the mind; it is that which gives the potentiality of the brass its meaning and connects it with the energy. Or to express this in a formula, "Motion is the entelechy (the perfecting power or principle) of the potential as potential." He admits the difficulty of finding an expression for this idea, but he shows, by an examination of previous attempts, that his own, however awkward, is the only one which is satisfactory.

On the subject of the Infinite, which had so much exercised the minds of previous Greek speculators, and had been resorted to as an ultimate solution of so many difficulties, he aims at no precision of language. By its

Moral and
Metaphysical
Philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

very nature it excludes precision. To bring it into a scheme, or regard it as a helpful definition of nature or the universe is, in his judgment, absurd; it can only be looked upon as marking the *ne plus ultra* to which human thoughts and inquiries can reach, or, at least, have already reached. The limitations by which alone you are able to deal with the subjects that fall under human cognizance, it excludes by its very name. His opinion on this point is characteristic of his mind, and it has an important bearing upon the history of metaphysics. Scarcely any more interesting question occupied the Greek mind than that which was at issue between the schools of Pythagoras and Xenophanes, whether it is more true and reverential to speak of God as the *το πτερας*, or as the Infinite. Aristotle's concluding remark on the subject of the Infinite should be quoted for the casual light which the latter clause of it throws on his idea of Time, an idea which the student of modern philosophy has so much need to reflect on:—*το δ' άπειρον ου ταυτον εν μεγέθει και κινήσει και χρόνω ως μία τις φύσις, άλλα τὸ ὑπερον λέγεται κατὰ τὸ πρότερον, οἷον κινήσις κατὰ τὸ μέγεθος ἐφ' οὗ κινεῖται ἢ αλλοιούται ἢ αὐξεται, χρόνος δὲ δια τὴν κίνησιν.*

Book eleventh opens with an attempt to ascertain the number of Causes or first principles. There is in every sensible substance a capacity of change: these changes are four: changes in respect of substance, of quality, of quantity, and of place. Generation and corruption are the names for the first kind of change, growth and decay for the second, alteration (*αλλοιώσεις*) for the third, transference (*φορα*) for the fourth. These are changes into contraries. But contraries themselves do not change; there must be something which undergoes the change from one to the other of them. This something must be *matter*. Being does not arise from not being; but being in potency is changed into being in energy; and being in potency is matter. Assuming this, there are three *ἀρχαί* or first principles; the two contraries Form and Privation, (*στέρησις*), and the Matter which passes from one to the other.

But Aristotle says that this enunciation is strictly applicable only to material things; in these the element (*στοιχείον*) and the principle (*αρχή*) are the same. You have heat, the *εἶδος*, cold, the *στέρησις*, that which has the potency of being either *ύλη*, the resulting substance, the flesh, bone, or whatever it may be. But in those things which are apprehended by the mind, another idea intrudes itself. Besides, the two opposites, health and sickness, and the matter, which is susceptible of both, you have the health-making art of the physician. The principle in this use of it acquires a double meaning which does not belong to the element. It must be contemplated both as the stationary Form and the moving Power. There must in a sense be four *ἀρχαί*, though only three elements. These conclusions have been, the reader will perceive, partly anticipated, but it is needful to repeat them here; for here is the link between Aristotle's Metaphysics and his Theology; this is the road, or at least one step of the road, by which he arrives at the conception of a First Moving Cause. To the unfolding of this conception the greater part of this remarkable book is devoted. We can but give our readers the results of an argument which Aristotle evidently felt to be the summing up of his metaphysical series. There must be an eternal, immovable Substance, which is at the same time the source of all movement. The primary notion of this substance is that it is an

Energy. The notion of potentiality is excluded from it, for the highest form of being is incompatible with the mere capacity of being. And seeing matter and potentiality are convertible terms, it must be immaterial. There is no refuge from the notion that all things proceeded from darkness and nothingness, except in this belief. Energy being anterior to mere potency, eternity must be predicated of the chaos or night out of which things are supposed to be formed, in a different sense from that in which it is affirmed of the Primeval Being.

We must attribute a continual negative existence to them, but a continual operative existence can only be attributed to the first cause. We want them to account for Corruption and Decay. We want the other to account for actual Existence and Life. Matter is in no sense a cause either to itself or to any other thing; and to a first cause we necessarily attribute self Causation. Other things impart motion, having first received it; this must be its own Mover. The next step in the apprehension of this being we obtain by the consideration of our own intellect and volition. There is an object of cognition or thing to be known, an object of volition or thing to be desired. By these respectively the intellect and the volition in us are set in motion. That which appears good to each understanding or will actuates that; that which primarily actuates must be that which is good. But in this we discover the union of the faculty of knowledge with that which is to be known, the union of the faculty of will with that which is to be desired. Thus self-contemplation and self-delight must be the essentials of Deity.* By other processes of reasoning he arrives at the conclusion that the Being must be without parts, (*αμερής*), without passions, (*απαθής*), our readers will perceive the change of gender, and will easily believe that *τὸ θεῖον* is the more common antecedent than *ὁ θεός*, and subject to no vicissitude. (*ἀναλλοίωτον*.)

We then approach the grand question, whether there is one such cause, or many. In nothing is the difference between Aristotle and his master more remarkable than here. We have seen with what tenderness Plato treated the mythology of his countrymen, not from cowardice, but because he felt that it contained a latent truth for which no philosophical abstractions or generalities could offer a substitute. Aristotle having satisfied himself that the argument was in favour of a one cause, sweeps away all notions which interfered with it, considers the gods whom his country worshipped as derived from certain astrological notions, and as setting forth the secondary sensible substances which proceed from the first immaterial cause.

With equal decision he denounces (upon this new ground) the different philosophical schemes which had been substituted for the old cosmogonies; attributing to them these two common vices, that they had acknowledged an antithesis and contradiction in things, but had not taken account of that third element, matter, which is the only explanation of the evil and disorder in

* We must quote the fine passage in which this argument is summed up. *Ἡ δὲ νόσις ἢ καὶ αὐτὴν τοῦ καὶ αὐτὸ ἀρίστου καὶ ἡ μέλιστα τοῦ μέλιστα. αὐτὸν δὲ νοῖ ὁ νοῦς κατὰ μεταλήψιν τοῦ νοητοῦ νοητός γὰρ γίνεταί διγγάνων καὶ νοῦν, ὥστε ταῦτον νοῦς καὶ νοητόν. τὸ γὰρ δεκτικὸν τοῦ νοητοῦ καὶ τῆς οὐσίας νοῦς. ἐνεργεῖ δὲ ἔχων. ὥστ' ἐκείνο μάλλον τοῦτο ὁ δοκεῖ ὁ νοῦς εἶον ἔχειν καὶ ἡ διαφορά τὸ ἥδιστον καὶ ἀρίστον. εἰ οὖν οὕτως εἴ ἔχει ὡς ἡμεῖς ποτέ, ὁ θεὸς αὐτὸν, θαυμαστόν· εἰ δὲ μάλλον, ἔστι θαυμασιώτερον. ἔχει δὲ ὧδ'. καὶ ζωὴ δὲ γε ὑπάρχει. ἡ γὰρ, νοῦ ἐνεργεία ζωή, ἐκεῖνος δὲ ἡ ἐνέργεια· ἐνεργεία δὲ ἢ καὶ αὐτὴν ἐκείνου ζωὴ ἀρίστη καὶ αἰδώς. Met. xi. 7.*

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

the universe, and that they had substituted many original principles for the one.

In the two last books their doctrines (respecting ideas and numbers) are again discussed at great length, and with Aristotle's wonted ingenuity. It cannot be expected that we should go over arguments to which we have so often adverted, and which we are less anxious to present fully and formally than to fit our readers for studying them in the places where they occur. But we may take this opportunity of remarking, that the continual renewal of these discussions with the Platonists and Pythagoreans is very important in helping us to determine the nature and connection of these particular treatises as well as the character of Aristotle's whole mind and system. It is evident that he felt the refutation of these opinions, and the substitution of something else for them, to be in a manner the business of his life. At all events, it was the needful preliminary to all his more positive proceedings; while his mind was haunted with these notions, the system, physical, metaphysical, or moral, which he proposed to rear, had, it seemed to him, a dubious and infirm foundation. We look, therefore, upon the metaphysical treatises (whether capable or not of being reduced into a formal sequence and unity) as having this subject for their centre. To show what ideas are not, and what they are; to establish the doctrine, that the *εἶδος* is not distinct from the particular individual substance—existing apart and connecting it with some other higher substance, but merely its inherent form; to connect the *εἶδος*, which is the constituent principle of each thing, with the *ἐνέργεια*, whereby it is called into existence, and thus to make the same answer satisfactorily to the two Greek inquiries respecting the nature of being and origin of matter; to explain the nature and conditions of the *ύλη*, and by depriving it of all intrinsic substantial properties, and reducing it into a mere potency, practically to get rid of the old Ionic investigations; then finally to hint at a principle of which his moral writings are the full exposition, that the final cause or the *οὐ ἐνεκα*, is also connected with the *εἶδος* and *ἐνέργεια*; that the good or purpose of each class of substances is known when we know what its nature and proper energy are; this is the object at which he is aiming most consistently amidst all his windings and recapitulations in the books of metaphysics. But this object is connected, on the one side with *Logic*, on the other (as the scholiast is so anxious to inform us) with *Theology*. Though we have not seen our way to adopt Ritter's method of identifying the logical treatises with the metaphysical, (a plan inconsistent with the very words of the third book,) though, as it seems to us, we should sacrifice by such a course much insight into the habits of the philosopher's mind, and the growth of his opinions, which we obtain by seeing them distinctly, and yet acknowledging the most intimate connection between them, we believe Aristotle to be primarily and at heart a logician; to have become thoroughly enamoured of the forms of logic, and convinced that they supplied a satisfactory exposition of the facts of the world; and then gradually to have worked out in his mind an ontological system, which gave the rationale of those forms and interpreted their relation to different phenomena. Now, if it be true, as we have maintained, that the *mathematician* has another set of laws, discovered to him in the course of his inquiries, from those with which the logician is conversant, we need not be surprised either that the arguments of Aristotle against Ideas should

be so constantly mixed with allusions to the Pythagorean study of Lines or Numbers, or that that study should actually have been the *base* of the principle which he is endeavouring to subvert.

It is on all accounts a more important inquiry how *theology* became interwoven with either set of speculations. We think it cannot be denied that the recognition of an absolute being, of an absolute good, was that which gave life to the whole doctrine of Plato, and without which it is unmeaning; that on the contrary, it is merely the crowning result, or at least, the necessary postulate of Aristotle's philosophy. In strict consistency with this difference, it was a Being to satisfy the wants of man which Plato sighed for; it was a first cause of things to which Aristotle did homage. The first would part with no indication or symbol of the truth that God has held intercourse with men, has made himself known to them; the second was content with seeking in nature and logic for demonstrations of his attributes and his unity. When we use personal language to describe the God of whom Plato speaks, we feel that we are using that which suits best with his feelings and his principles, even when, through reverence or ignorance, he forbears to use it himself. When we use personal language to describe the Deity of Aristotle, we feel that it is improper and unsuitable, even if, through deference to ordinary notions, or the difficulty of inventing any other, he resorts to it himself.

The light which the metaphysical treatises throw upon this point makes them an important introduction to Aristotle's ethical system.

We might have concluded from his *Dialectics*, that he utterly rejected the Platonical doctrine of Ideas as a scientific exposition. But it would not follow that he should discard that belief in some ideal of excellence which had impregnated all mythologies, and had never been banished from the hearts of men. But the Aristotelian conception of God as a ground of nature, simply, leads us at once to feel that no recognition of his perfection can have the least connection in his mind with a scheme of practical life and conduct. We feel that it is not with Plato, or any philosopher who had attempted to give the rationale of men's dreams on this matter, he will feel a want of sympathy, but that he actually has not discovered in himself, and does not recognise in his brethren, the want which all ages had been contriving in so many forms to express. And then it becomes an interesting question, what will be the groundwork of the Aristotelian ethics answering to that *Theology* which is so obviously the foundation of the Platonic?

The answer to this question brings us to a very important treatise of Aristotle which embodies more of what has been commonly called metaphysics in our day, especially here and in Scotland, than the works professedly bearing that title. We mean the three books *Περὶ Ψυχῆς*. The first of these books is occupied as usual with an examination of previous theories on the subject. He despatches very elaborately the different notions respecting the soul which Democritus, Empedocles, or the Pythagoreans had encouraged. He shows why we can never be satisfied with calling it motion, or the principle of motion, the primary element or number. He then proceeds in the second book to develope his own doctrine. The soul belongs to the category of entities. It has then, of course, a matter and a form; the matter here, as elsewhere, coincides with its *δύναμις*; the form is (*εἰτελεχεια*.)

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

The soul is neither of these separately, but the result of both. There goes to the forming of sight the energy of vision, and the faculty of vision, and there is, in addition to both, an organ, an actual eye. What is true of this sense is true of the whole substance of which it may be said to form a part. The soul, possessing both its energy and its faculty distinct from the organ through which both are manifested, does yet require such an organ. The soul is not a body, but neither is it without a body. Generally it is the distinction of a living creature, (*ζωον*), that it has a *ψυχη*. But all living creatures have not a soul exercising the same *δυνάμεις*. We may define all the faculties which can exist in any living creature to be these: first, the faculty of receiving nourishment; (*θρεπτική*;) secondly, the faculty of sensation; (*αἰσθητική*;) thirdly, the faculty of motion in place; (*κινητική*;) fourthly, the faculty of impulse or desire; (*ορεκτική*;) fifthly, the faculty of intelligence, (*διανοητική*.) The threptic faculty is the lowest of these, and is present in all cases. The soul, therefore, as endued with this one faculty, may be attributed to vegetables. Wherever any of the higher faculties are present, there all the lower will exist also. Under each of these heads a very interesting discussion arises respecting the character and limits of the particular faculty. The question, for instance, under the first head is, whether the life in each plant or thing must be considered as the active or only the passive instrument in self-sustentation? Under the head of sensation many more complicated points arise, and Aristotle enters into the whole theory of the subject, examining the operation of each sense in detail. This, it may be remembered, is the discussion which is carried on with so much liveliness and profundity in one part of the *Theætetus*. The opinions there attributed to Protagoras (and so far as the doctrine of sensation goes, apart from its moral consequences, not denied by Plato) is nearly the same as that maintained by Aristotle. Sensation is neither in the organ of sense nor in the object, but is generated between both, and is the effect of the medium through which they hold communion with each other. The question as to the motive faculty involves us at once in a consideration of that which is higher than itself. Movement must depend upon impulse. This will be true in all creatures. And in spite of the effect of the appearances which are produced upon or by means of the senses, in generating impulses or desires, we must not impute a governing power to sensation; we must rather think that the nature of the faculty of impulse determines how these shall influence it, than that it is determined by them. It would seem then, that each creature has a nature, which is expressly seen in this faculty of impulse. Wherein then does man differ from other creatures? Neither, it would seem, in the absence of this impulsive faculty, nor in its being less properly his nature than it is that of other animals, but rather in his having the dianoetic faculty to direct it and act with it. In the coincidence and conspiracy then of these two faculties will consist the true nature of man. Thus, then, the soul may be considered as containing three portions, logically not materially separate, one absolutely without reason, the other rational, another participant of reason.

Ethics

In this psychological system we discover the root of the Aristotelian ethics of which we were in search: they begin in psychology and terminate in Politics.

Aristotle informs us, very early in the *Nichomachean* treatise, that he looks upon politics as the *ἀρχιτεκτονική*

VOL. II.

ἐπιστήμη, and ethics as an introduction to it. The two, therefore, are not identified in his mind as they are in Plato's; it is quite possible, nay necessary, to treat of them distinctly. According to his uniform method, he seeks for the grounds of combination and society in the nature of man. He cannot tolerate Plato's simplicity in admitting outward necessities and accidents to be the *occasions* of it; for this simplicity necessarily involves another proceeding which seemed to him not simple but pregnant with all Plato's idealism, that of supposing some higher Unity than that which is expressed in the character of any particular society to be involved in the constitution of society itself. A principle of equality and adjustment is that which seems to him to pervade all things, to be in a manner a law of the universe, and to be especially the secret of human order and government. The like principle, taking a different form, is the mainspring of his ethical system. Virtue lies in a mean, in a sense it may be said to be a mean, so that, on the one hand, government, which is also a mean, is naturally occupied in sustaining the virtue of particular men, and on the other, this virtue is itself the great conservation of government. This observation ought to be made, as without it the connection between these two spheres, which is as much acknowledged by Aristotle as by his master, will not be apparent. Many difficulties also will present themselves to the reader as insurmountable, if he looks at the ethics as an entire system, and do not remember that a directing educating power is for practical purposes presumed to reside in a governing body, the functions and nature of which have not yet been defined. But we are not to suppose that Virtue, or the attainment of this mean, is in Aristotle's judgment the formal object at which either the life of each particular man, or society at large, is aiming. When once the notion of an absolute good, which "those dear" and troublesome men, the Platonists, had introduced, was taken out of the way, there remained one obvious and generally admitted end of all human desires and searchings. HAPPINESS is emphatically the human *τέλος*. But if human, then the definition of this happiness must be sought in that which is peculiarly the characteristic of the human class. It cannot exist in any of those powers or faculties which are common to it with other classes, not therefore in the threptic or the æsthetic powers merely and chiefly. And any how, it must be in some exercises or energies that it will consist, for in these the soul or life of every creature makes itself manifest; it must be then in the energies of our best and highest nature, exercised not at intervals, but through a whole life, a life possessing so much of external prosperity as shall permit them a free scope. But all energies must have a certain direction; the right direction of its energies constitutes the *virtue* of each class. What then will be specifically the human virtue? It must of course be in the man, and, according to our psychology, the *ορεξίς* (the impulsive faculty) is the constitutive faculty of the human soul, though its excellence consists in its subjection to the dianoetic faculty. Virtue then will imply the presence and the harmony of both these; still it will be found most positively and characteristically in the former. It must be then a habit. But of what kind? To what does it point? What is its aim, seeing that an absolute good, or an ideal, is out of the question, and that happiness cannot be the aim, because it is the very nature of happiness which we are now resolving into its elements? We are

Moral and
Metaphy-
sical Phi-
losophy.

4 K

Moral and
Metaphysical
Philosophy.

not, Aristotle says, to trouble ourselves about scientific accuracy in our definitions; our purpose is purely practical; we want to form an actual man of a certain character, not a theoretic man. Well, then, practically speaking, excess is in every case that to which you attribute mischief and derangement. There is an excess called Timidity, and an excess called Fool-hardiness, an excess called Prodigality, and an excess called Narrowness or Avarice. But the extremes suppose a mean. This is the end at which our habit aims. Virtue generally lies in this. But we are aiming at action; and actions are not general, but specific; how then shall we arrive at the notion of specific virtues? Their species will be determined by their distinct objects. Certain tendencies and habits will be conversant with external pleasures. Certain others with passions of the mind itself; in each case it will be found that the practical purpose defines the virtue. But though a general description may be given both of the excesses and the means which correspond to them, a description which will be really applicable in all cases, it must ever be remembered that the excess itself may be different for each man, actually different according to his actual circumstances, different in its effects and influence upon him according to his greater proneness to one side or the other. For instance, liberality of course will be practically a different quality in the rich man and the poor man, and the temptation to profusion will be that which is to be most resisted by one, to meanness by another. Virtue will be therefore in a *mean*, that is one to us, and not one that can be absolutely and invariably ascertained by rule.

Hence it follows that this habit supposes the exercise of a faculty of choice or predetermination. But what is predetermination? Is it the same as the act of willing? Clearly not; that has reference to ends, this to the choice of means and the attainment of ends. But it implies a right end, and a right determination of the will to that end. It may be called *ορεξις βουλευτική*, (the reader will observe how steadily his psychological axiom which we have spoken of is kept in view throughout the scheme.) But to what cases will this will or counsel refer, and how far is it dependent upon ourselves? Clearly we do not consult about things absolute or eternal, nor about things within the sphere of accident. What remain are all such things as are done by us or with our concurrence. Now of such some may be doubtless taken from under our control by actual violence practised upon us; such cases give rise to various questions of casuistry, as to the course which a virtuous man will take, whether he will submit to do wrong or to die, each of which must be determined on its own merits. With respect to ordinary cases, the doubt arises, whether inclination is not itself a force upon the will and on the reason both. Such a notion Aristotle disposes of, first, by the remark that an influence upon the impulse or will cannot by any reasonable man be confounded with a force by which its operations are hindered; and secondly, by admitting that an incapacity for any particular action may doubtless be produced in any man by these influences, but that this incapacity is itself the result of a previous habit which need not have been thus formed. Habits then are in our own power, actions not always. Having settled these foundations, the particular ethical virtues come next under his consideration. Here lies a field for the exercise of his always acute and often delicate habits of obser-

vation. It is alien from the temper of mind which Shakspeare has wrought into us, to contemplate any character as the mere developement of any single specific quality. We do not like to hear of a man as the Magnificent, or the Magnanimous, or the Modest, or the Temperate, or the Just. But, doubtless, there was something in this which suited well with Greek habits. The Aristides, Themistocles, Cimon, if they had not any of them a distinct purpose of realizing a particular form of character, yet drop more readily into certain moulds than the traditionary characters in the story either of ancient Rome or modern Europe. How a similar tendency was revived at one period in Christian society, and how its revival was connected with a scholastic reverence for Aristotle, we may have to notice hereafter.

Among these virtues it behoves us especially to take notice of two, because they throw some light upon the entire system, and upon ethical inquiries generally. The first is Justice. (*δικαιοσύνη*.) Is not this virtue itself an abstract of all the virtues? We have seen how Plato answers the question in his *Republic*. Aristotle treats it differently, yet so as to make us see how much he had felt the influence of his master's ideas, even when he rejected them. In one sense (he says) *δικαιοσύνη* may indeed be said to be a complex of virtues. For as it is the habit which mainly disposes to allegiance to the laws, and as laws prohibit excesses of all kinds, and encourage virtues of all kinds, this will have respect to them all. But yet there is such an offence as Overreaching, and there must be a specific virtue answering to this. But the specific virtue will bear relation to the general. Inequality, in matters appertaining to property, will be the evil. Evenness or equality will be the virtue. This evenness or equality implies, on each side, an excess, a more and a less; a more and a less, however, in reference to given persons. The conservation of the right proportion or relation of things to persons, and the restoration of the balance when it has been violated, is then that at which this virtue especially aims. Take away the restriction to property, and this virtue would seem to be in a remarkable manner the very Virtue of Virtues; so emphatically is it the preserver of the mean. But that very restriction makes it more difficult to tell how far this virtue belongs to the individual, and how far to the state, so that *δικαιοσύνη*, though bearing a much more limited meaning, becomes, to our author as to his master, a kind of debateable ground between the two regions. At all events *δικαιοσύνη* must be looked upon as characteristically the ethical virtue of a statesman.

The other virtue we must speak of is *σωφροσύνη*. As this is opposed to *ακολασία*, and involves the general notion of an even habit of mind not overcome or disturbed by sensual desires, it might seem to include within it *εγκράτεια*, or self-restraint. But as in the psychology of Aristotle the soul consists of a rational, an irrational, and a quasi-rational part, the quality which implies a control of the irrational or merely animal nature, will not necessarily be the same with that which concerns the quasi-rational, that is, the passions and affections to which the will is subject. As the name *σωφροσύνη* is given to the first, the name *εγκράτεια* is used for the second. Hence a curious consequence. This self-government seems something distinct not only from the peculiar *αρετή* which has reference to sensual desires, but even from *αρετή* itself. That has its chief seat in that nature or impulsive faculty over which *this* is sup-

Moral and
Metaphysical
Philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

posed to rule. We are somewhat puzzled, therefore, after going through our catalogue of virtues, to find a book opening with the remark, that there are three moral states to be avoided, *κακία, ἀκρασία, θηριότης*, and three good states corresponding to these, *ἀρετή, ἐγκράτεια*, and a certain divine excellence as much transcending ordinary humanity as *θηριότης* sinks below it. This result will appear inevitable to any one who reflects upon the system; that which is conservative of virtue must in some way be distinct from virtue, but we must acknowledge also that it considerably impairs the symmetry of a design otherwise singularly complete.

Before, however, Aristotle touched upon this conservative power, which, of course, is connected with the purely ethical part of man, it was needful for him to expound more distinctly than he had done in his psychology the nature of these dianoetic faculties to which such important functions are committed. He begins with reaffirming the position so often insisted upon, that the *ορεξις*, and the *διανοία*, must co-operate in order that any good moral act may be the result, or, as he expresses it, with neatness and in more strict accordance with his own notion of the *διανοία*; that which is affirmed or denied by the one should be that which is pursued or avoided by the other. Now the soul, he says, may affirm or deny truly in five ways, by *τέχνη*, by *ἐπιστήμη*, by *φρονησις*, by *σοφία*, by *νοῦς*. *Τέχνη* is what in modern language would be called the creative power or faculty, the poetic organ in its highest and lowest sense. *Ἐπιστήμη* is the converse of this. It deals with that which cannot be otherwise, it does not fashion anew but perceives; what it deals with are universals not particulars. Aristotelian science is, as we have seen already, conversant with conclusions not premises; but there must be some faculty which deals with premises, a tact, intuition, or spiritual sense, this is *νοῦς*. The sphere of this faculty would seem to be very limited, for as it is bounded on one side by *ἐπιστήμη*, it is bounded on the other by *σοφία*. This faculty, we were told in the *Metaphysics*, was conversant with *ἀρχαί* or principles; it might therefore seem to cover the whole ground which is assigned to *νοῦς*. But that which affirms things to be so and so without a reasoning process, is undoubtedly distinct from that faculty which, through long and winding labyrinths, searches for causes. Now, when the *νοῦς* is said to deal with premises, the first kind of operation is indicated; when the *σοφία*, the second. *Φρονησις*, the last of the five, is different, and yet has something of the character of the preceding. Its sphere is with the altering and the alterable, like *τέχνη*, but it is not productive or creative, but perceptive and distinguishing. So far it resembles *ἐπιστήμη*, differing from it wholly in its subject-matter. It has a quickness of tact like the *νοῦς*, but this is merely the result of practice and experience. It is, therefore, like *σοφία*, a laborious investigating faculty. Yet its end is not speculation, but practice. The *φρονιμος* or practical experimental man, therefore, is contrasted with the *σοφος*, or the meditative speculative man; though it is not denied that *σοφία* may assist and be usefully connected with *φρονησις*. From this analysis it is evident that this last quality is especially that which, in combination with a right *ἦθος*, or a proper condition of the impulsive faculty, produces virtue. The doctrine of Socrates, that virtue is *φρονησις*, (a doctrine by the way which is somewhat carelessly stated, for the real Socratic doctrine treats virtue as *ἐπιστήμη*, the knowledge of what is absolutely good, prudence being only a guardian

faculty to preserve the soul when seeking that knowledge from the seductions and confusions of sense,) this doctrine is said to form only one side of the truth; *φρονησις* is not virtue, though virtue cannot exist without it.

We have seen that *δικαιοσύνη* does to a certain extent occupy the same position in the Aristotelian and in the Platonic system as a link between morals and politics. But Aristotle could not help perceiving that this quality, under the conditions which he had imposed upon it, explained but very imperfectly the connection of human society with the life of the individual. This dry and hard principle of distribution, commutation, and rectification, could never be substituted for the music of Plato's *Commonwealth*. Reflecting on this difference, it seems to have struck him, that in the idea of Friendship we have that which fills up the void, and that Friendship together with justice constitute the social law. Regarded in this light, Friendship occupies the most important place in a system of ethics, which is always looking onwards to Politics. And we cannot wonder that Aristotle should have devoted two elaborate books to the consideration of it. Any one who is acquainted with the traditions and with the mythology of Greece must be aware how much the Greek mind was occupied with this subject. Here, as elsewhere, physical and moral thoughts became intertwined, and the same language was used to explain the law of sympathy between the skies and earth, and that between man and man. Aristotle is careful to disengage himself from these ambiguous phrases, which he had not perhaps imagination enough to perceive were more than metaphors, and fixes his mind upon friendship as one of the essential conditions of our nature to which the very existence of communion must be referred.

This being assumed, he has no hesitation in setting aside many popular notions of friendship as giving a wholly inadequate view of its nature. The doctrine which refers friendship either to Utility or to Pleasure as its ultimate foundation, he rejects not with sentimental indignation, but as being at variance with facts and reason. The transitoriness of such friendships, and their dependence upon accidents, are arguments as much to the practical man as to the philosopher, that the essence of the quality is not to be discovered in them. The friendship of good men for each other must then be that from which we are to deduce the nature of friendship itself. Here, and here only, we learn the conditions, or even the possibility of friendship, for, properly speaking, it is not possible, except upon the supposition that one man can really delight in another, and love him as himself. A politician seriously reflecting on the existence of society, must feel that a principle is at work among men which can be only defined in these terms; that all the imperfect appearances which it presents in the world, so far as they are imperfect, make its meaning less intelligible; that, supposing selfishness absolute, it could not exist at all, and that the highest form in which it exhibits itself is the test of its character. These important conclusions are stated again and again, and with the greatest precision, by Aristotle. On the strength of them he affirms, that the idea of equality or proportion is as much discoverable in friendship as in justice; only that in justice the worth of each object is the first consideration, its fitness to us the second, in friendship, fitness or suitableness the first, worth the subordinate. On the same ground he maintains, that friendship is to be seen in its true operation, not in

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

clubs, or societies, or partnerships, where men associate for some specific object, (though here also Justice is required as its assessor and its complement, every kind of society implying a law to regulate it, as well as a motive to form it, a principle of government as well as of concord,) but in a *polity* of which all these must be considered as portions. An inquiry, therefore, into the different kinds of government becomes connected with an inquiry into the law of friendship.

In this passage we discover how much Aristotle surpassed Plato in his apprehensions respecting the nature of relationships, while he fell so far short of him in every thing that concerns the absolute. He discovers in the relationships of father to son, of husband to wife, of brother to sister, three primary forms as it were of friendship; and the grounds of the three kinds of government to which all others may be reduced, Monarchy, Aristocracy, Timocracy, and of which the three corruptions are, tyranny, oligarchy, democracy. Under each of these true forms of government friendship and justice will be found existing and mutually sustaining each other. Friendship, however, will take its peculiar form from the form of the society. It will be the friendship of patronage and of reverence in a monarchy. It will have the conjugal model in aristocracy, one party being respected as the superior in worth, and retaining that respect only while he asserts dominion on that ground. The fraternal type of equality will be preserved in all friendships under a timocracy. On the other hand, in the depraved forms friendship will be depraved and weak; and in a tyranny, which he regards as the worst of all, because the corruption of the best, both it and justice will disappear, because subjects are there regarded as animals, and as such incapable of these human qualities. The existence of this law of sympathy being then established as one of the two necessary conditions of human fellowship, and virtue being shown to be the necessary condition of friendship, Aristotle proceeds to solve a great many questions of very deep and practical interest. The most important of these turn upon the relation between friendship and self-love. In what sense is friendship a part of self-love? in what sense opposed to it? As the notion of pleasure or utility had already been separated from friendship, it is obvious that the vulgar notion of self-love must be separated from it also. Still ordinary language intimates that there must be some analogy between the two ideas, and it seems hard to arrive at any higher description than this, that the friend is loved as another self. May not the difficulty then be solved thus? may not self-love be itself distinguished from all associations of profit and loss? and may we not affirm that the wise and good man is the true self-lover, the person who alone is at one with himself, and can take pleasure in his own company? If this be so, it would not be correct to seek for the ground of friendship in self-love, it would be more correct to say that they mutually illustrate each other. Only the man who has the capacity of friendship will have himself for a friend, and only he who can enjoy and love himself, is capable of enjoying and loving another. These two books on friendship are certainly not the least profound in Aristotle's writings, and to the general reader they will be far the most delightful. From this subject we proceed, in the tenth book, to the question of Pleasure; what it is, how far, according to the doctrine of some philosophers, it is to be denounced as an evil, how far it is a

good, or connected with *the good*. Aristotle argues against many prevalent definitions of pleasure. He shows why it is neither a *κίνησις*, a mere movement, a *γενεσις*, the passage into a state, or an *αναπληρωσις*, the filling up of a want. He considers the universal longing of mankind a sufficient witness that pleasure is something real and worthy in itself, and not merely a means to some other end. An examination of the facts leads to the same conclusion. But it leads also to a refutation of the opinion, that pleasure can be made a distinct formal purpose of life. It is the flower or consummation of something else. The exercise of sight, the exercise of hearing, each brings its own appropriate pleasure after it. But the pleasure is connected with the energy or exercise, and cannot be severed from it. If then you would understand what pleasure is, and what are the highest pleasures, you must understand what energies are, and what are the highest energies. You cannot refer the last to the first, you must refer the first to the last. That energy, then, which is most appropriate to each creature brings the pleasure which is appropriate to that creature; "the energy of the soul, according to virtue," brings the highest pleasure to man. The pleasure which an act gives to him who performs it is the test of that act having become habitual to him, of his having acquired the character corresponding to that act. The man who delights in musical energies has become a musician. The man who delights in just acts is a just man. From this analysis of the nature of pleasure, the step is easy to a reconsideration of the meaning and nature of happiness, and so to a brief review of the whole treatise. Happiness he has found to be the end of man, and to consist in (not like pleasure, merely to be the effect of) the use of his highest energies. What then, on the whole, is the highest happiness? It is that of the contemplative man. If we can imagine what the life of the gods is, seeing it is absurd to attribute to them justice, because that has respect to contracts and conventions, temperance, because that implies temptations to which they cannot be exposed; and so of most of the other acts which preserve the mean for man, we must believe it to consist in contemplation. But then for the attainment of this celestial life in those who can attain it, there is need of early discipline and education. There is need that they should be trained to the avoidance of those extremes in which evil lies, and to the exercise of those virtues which are the only conditions of, and preparations for, the contemplative happiness, though it transcends them. And for the rest there must be a discipline to cultivate what capacities there are in them; or in case of resistance to such cultivation, to coerce and punish them. Here then is the field for the science of Politics. That science, Aristotle says, the sophists had resolved into a mere teaching how to talk and argue, but the foundations of it lie in ethical knowledge and ethical practice, and it must be worthless and rotten when these foundations are not discovered. This is the introduction to the book on Politics.

We can but give our readers a few hints to assist them in the study of this treatise, which, valuable and interesting as it is, can never bear the same relation to the other Aristotelian writings which the Platonic Republic bears to the other dialogues. Not, it will be evident enough from our former remarks, that the Politics of Aristotle are not most closely connected with his Ethics and his Ontology, but that they are connected with them rather as results and deductions, than as

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

being a principal and fundamental part of the design. The doctrine which has been set forth with so much diligence in the *Metaphysics*, of every substance presupposing a lordly energizing power, and a submissive receptive faculty or matter, re-appears again here in connection with the most obvious and outward facts. The relation of Male and Female is assumed as the first hint of the existence of society, and as containing the principle of it. The idea of fellowship implied in this relation involves another,—that of rule and subjection, which has its complete expression in the relation of Master and Slave. Compare these two relations with that relation in each man which has been explained in the *Psychology* and illustrated in the *Ethics*, between the reasoning power, the faculty participant of reason, and the mere animal nature, and you feel at once that the two explain each other, and set forth the condition under which society is meant to exist. Where the reason is developed and its magisterial authority acknowledged, the other faculty being in fellowship with and subordination to it, and the animal nature controlled and subjected, there you have as well the true condition of the individual man as the true condition of society, there the relations of husband and wife, and master and servant, will be preserved; they will not be arbitrary, but legal and orderly. In such a state of things only a polity is possible; yet, as this is the only true condition of each man, it is evident that a political state is his only proper and natural condition, every other must be anomalous. You find then the constituents of a polity in household, but a house is not therefore a miniature city; a city is not merely a collection of households; each has its own distinct nature and laws, though each alike has this characteristic—that in the human relations it consists, that in them you discover the end of its existence, and that all means and instruments are to be contemplated with reference to these. Economy is not primarily, but secondarily and accidentally, the management of the goods or property of the household, it is mainly the right ordering of the household itself. The slave is the connecting link between one branch of economy and the other; he must be considered as an instrument, and yet he must be treated as a man. A polity can be considered only as composed of freemen quite as much because a freeman only understands how to obey as because he only understands how to govern.

It follows almost necessarily from this view of the case, first, that the Platonic idea of unity should be as little heeded by Aristotle in his *Polity* as in his *Metaphysics*; that he should utterly abhor the attempt to embody that idea by abolishing distinct relationships, these being in his opinion the very foundation of society; that he should recognise all forms of government as good which have their ground in any actual relation, and all as evil which have become in any sense arbitrary; that he should therefore acknowledge, much as Plato did, three true forms and three departures from these; and that he should look on the democratical departure, the attempt to establish a society in which all should govern with at least as little complacency as the rest; that at the same time he should conceive that form from which this is a deviation, the form which treats all freemen as eligible to government, though not necessarily as participant of it, as his ideal. These seem to us the main principles of the book, which, being understood, the occasional difficulties and contradictions it presents will be less puzzling; its position, in

reference to the other parts of the philosophy, will be felt, and its value as a key to the political science of modern as well as ancient times will be appreciated.

Moral and
Metaphy-
sical Phi-
losophy.

The Later Sects.

We should have been glad to notice another treatise of Aristotle, which is valuable in many ways, and especially for affording an interesting practical illustration of his scientific method and some important applications of his *Ethics* and his *Politics*; we allude to the three books on *Rhetoric*; but we must deny ourselves the pleasure of dwelling any longer upon the masterpieces of Greek philosophy—the books which really teach us what Greece was, and what work it was appointed to do in the world,—that we may pass into the company of certain teachers who have earned a reputation far beyond their merits, and have even been able to palm themselves upon the world as the genuine representatives of the wisdom of a country which had lost all its capacity of producing great men before they were born into it. Every part of our story thus far has testified that the political life of Greece was the main pillar of its speculative life; every thing which we have read would prepare us to expect that when that political life was departed, its speculations would become withered and heartless; either this must have been so, or there can have been no fellowship between earlier and later Greece; the laws which apply to the one must be entirely inapplicable to the other. We do not believe this to have been the case; we do not believe that our anticipations respecting the state of the Greek mind after its institutions and its freedom should be gone are in the least confuted by the existence of those sects which persons who read of Greece with Roman eyes are wont to speak of as *the Greek schools*. Nevertheless we do not deny that these sects were permitted to exist for certain useful purposes, and it is worth while to ascertain, as far as we are able, what these purposes were.

We have spoken of philosophy throughout as an attempt to find an explanation of facts—the facts of society, the facts of the individual mind, the facts of nature. These facts of course remain, whether we are awakened to perceive them or not. Savages are held together by the same laws which bind together civilized men; they have the same hearts and consciences, they have the same sun and moon shining upon them. But because they have never felt themselves in an order, because the feelings of relationship and obedience and law have never been awakened within them, therefore their own thoughts are a dreary chaos of fears and hopes, and all the universe only a medley of incoherent images, partly pleasant, partly fearful. Suppose that sense of an order withdrawn from a nation in a high state of outward civilization, from a nation whose members had a gorgeous past to look back upon, with native quickness, with a sense of hereditary power, with the means of material enjoyment, with faculties refined to make use of them, and how will the result differ from that which we noticed in the other case? It will differ thus: in place of a language hardly competent to explain a single combination of human feeling, or of outward phenomena, only expressing a certain vague consciousness of surprise, doubt, and awe, will be a language containing forms for every variety of feeling, for every kind of description; in place of a few lively scattered impressions, a whole harvest of observations and reflections

Moral and
Metaphy-
sical Phi-
losophy.

gathered in by previous labourers; in place of an acquaintance with one set of very limited circumstances, a pleasure in contemplating and comparing infinite diversities of circumstances. With these differences, what remains in common?—A hollowness, vagueness, absence of all feeling of reality, the sincerest man owning little else but a sense of endless confusion, and mourning over it; the insincere making that sense of confusion his boast, and glorying because he can explain all things by declaring that they admit of no explanation. Such is emphatically an age of scepticism, an age which does not feel that it has any facts to rest upon, which does not really desire to know what any thing means, which believes that nothing has been done or proved in past time, which wishes to create all things new, but which having no feeling or sympathy with what is created, is utterly unable to do more than invent a few new combinations and recast certain forms of thought and language, out of which the meaning has departed. Greece, after the age of Alexander and Aristotle, was emphatically in this miserable condition. When the Sophists undertook to prove that the words Right and Wrong meant nothing, they were confuted by the actual existence of a government based upon the acknowledgment of right, and by a conscience in every man that this government could not subsist against his own and his neighbours' selfish and violent inclinations unless it were upheld by some divine power. The philosopher might be puzzled to reconcile this faith with the strange, disorderly, and unrighteous acts which tradition had attributed to the gods of his country, but the faith itself never lost its hold upon his mind; he assumed it as a datum with which other facts were to be reconciled, and which was to be in some sense the explanation of them. Now there was the phantom of a government based upon mere power; the sense of the necessity of a divine Ruler was gone; the majority continued to practise the rites, and maintain, through fear or indolence, the traditions which implied his existence; the speculative man could look without trembling at the question—whether these rites and traditions meant any thing, and could, as it suited his temper or convenience, practise them punctiliously or ridicule them, or do both at once. In such a languid state of society all really active and energetic thought was impossible, all that was done for a time was to ask questions which had been asked before by earnest men with the hope of an answer, for the purpose of showing that no answer could be given; and then, when pure unadulterated scepticism became too weary or too painful, came a series of attempts to seize hold of some one partial observation or consciousness, and to make all the universe merely an exposition and expansion of that.

Pyrrho.

The person who always stands as the type and representative of the earlier and more complete scepticism of this period is Pyrrho, and this is nearly all that can or need be said of him; for it would be a waste of time, at least in such a sketch as this, to attempt to harmonize the different reports of his sayings and doings. There is nothing recorded of him which marks him out as an original or generative thinker, but enough to give us warrant for believing that he had more character than those who surrounded him, and was therefore able to express in words, thoughts and feelings which belonged to them as much as to himself. Given—the Socratic method of disputation, the doctrine respecting contraries which pervaded all Greek philosophy, and was brought

out with the greatest clearness and precision by Aristotle, the great saying, that—a man knows only that he knows nothing—given, these old and stable elements, and combining them in due proportions with the insincere, vapouring, indifferent habits of Greece in his day, and it is not difficult to construct a Pyrrho who shall embody all the notions and have a right to use all the formulas which are attributed to him and his disciples. The old question between the Sophists and their opponents—the question upon which all Greek philosophy turns, whether any thing is, or whether *is* and *seems* are equivalent words, was now practically decided in favour of that conclusion which Plato and Aristotle had in their different ways used the whole energies of their minds to confound. Pyrrho merely adopted in terms the opinion which every one else in fact held, when he affirmed that nothing could be affirmed, that all was appearance and opinion, knowledge a dream and impossibility. Of course a man who endeavours to carry out such a theory as this, and to make it consistent at least with what he feels in himself and in others, must fall into innumerable extravagancies and contradictions. But nothing is so extravagant and contradictory to a person who has given up faith, as to suppose that any thing can be known; every monstrous hypothesis must be tolerated rather than this. It is an obvious remark, but one which it is important for us to make in connection with what was said of Words when we were speaking of Socrates and Plato, that language must have utterly lost its power and reality before this kind of scepticism could prevail. As long as men believe that the words which they speak are not mere sounds, or gusts of air, the words Being and Knowledge must be felt to be witnesses which no disputation can get rid of; but for a language to subsist when the polity and morality of the nation which spoke it are gone, is utterly impossible.

The first indication that the Greeks had grown tired of utter disbelief, and were seeking somewhere or somehow for a resting-place, is furnished by the appearance of *Epicurus*. Pyrrho had discovered that man has nothing to do but to doubt; still he was constrained to admit that he is also obliged in some certain sense to exist. His own doubts had, in fact, so little interfered with the practical exercise of his animal functions, that he is reported to have used them for nearly ninety years. All, then, is not quite vacancy: a man has something to do, even something to think about; he may exist, and consider how he can exist with least friction from without or from within. This problem Epicurus set himself to solve. Was it a new one, one which had never presented itself to the Greek mind before?—Proposed in this way, it was certainly new. Socrates endeavoured to find out how men must live, but he never taught them to make out a scheme of life for themselves. Aristippus taught, that the end of a man's life was to please himself, but he had been content to let his cheerful easy nature shape itself according to the circumstances which he found. Plato had sought for happiness in the knowledge of that which is good and true; his inquiry was of course quite obsolete. Lastly, Aristotle had made happiness the end of life; but then happiness was with him an energy of the soul, to be put forth in conformity with a system of existing relations, impossible, except on the belief of our being a body politic in which each exists for the sake of all. The simple question, therefore, how a man is to make himself comfortable, may be said to have been now for the

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

first time propounded as the staple question of philosophy; and being thus propounded, we need not wonder that it should for a time hold quite exclusive dominion.

A man must be brought into a strange condition of mind before he can believe that the universe and all that it contains exist only that they may tell him how he is to be comfortable, but when he has once arrived at this condition it will be wonderful indeed if his ears ever catch any sound which is not an echo of his demand, or some fragment of an answer to it. We must not wonder, then, that the Epicurean, rising up in a sceptical age, and starting with the great principle of scepticism,—that nothing is, but all things seem,—should have become the greatest and most imperious of Dogmatists. With the most perfect sincerity, while denying the existence of truth, and the possibility of knowledge, he affirms his own conclusions to be irrefragable. Do you deny that all men like pleasure and dislike pain? This is his kind of inquiry, which as it means simply, do you deny that all men like what they like and dislike what they dislike? certainly reduces an opponent to very considerable perplexity.

But while we admit that Epicurus did present philosophy in a new form to the world by his method of stating its principal problem, we can by no means allow him any originality in his method of solving it. Aristotle had shown most clearly that if an absolute good be not the end of practical life, happiness must be its end. Epicurus could say no more; he could only find out some new criterion of what happiness is, seeing that, for the reasons we have already mentioned, Aristotle's definition of it must needs be unsatisfactory. But in seeking for this criterion he had no resource but to adopt the old sophistical dogma; he could only say that we must refer every thing to the standard of our sensations. No doubt he may have refined considerably upon this thought, for, in the first place, he had a body of formulas provided him by Plato and Aristotle on the subject of sensations; and secondly, he lived in an age in which thought was less active, but in which all that contributes to mere gratification, whether bodily or mental, was far better understood. It is a question which has been much, and we think very unnecessarily debated, whether, by making sensation his standard of happiness, Epicurus did, or did not, mean to encourage what is formally called sensuality. The testimonies of antiquity respecting his personal character are various, and the most modern criticism seems rather inclined to revert to the vulgar opinion respecting it, rejecting, certainly with good reason, the absurd and fanatical panegyrics of some French and English writers in the last century. Upon the whole, we are inclined to believe that Epicurus was an apathetic, decorous, formal man, who was able, without any great difficulty, to cultivate a measured and even habit of mind, who may have occasionally indulged in sensual gratifications to prove that he thought them lawful, but who generally preferred, as a matter of taste, the exercises of the intellect to the more violent forms of self-indulgence. And this life would, it seems to us, be most consistent with his opinions. To avoid commotion, to make the stream of life flow on as easily and uninterruptedly as possible, was clearly the aim of his philosophy. For this end it was advisable to avoid the pursuit of wealth and honours, it was better to abstain from extremely vehement enjoyments, but it was absolutely necessary to get rid of all superstitious fears.

How these were produced was therefore an important question for the founder of such a system. The answer to it led him much further than he at first intended to go. Naturally, Epicurus cared only for moral questions, that is to say, for such questions as related to the management of human life. With dialectics, either in the Platonic or Aristotelian sense of the word, he had no proper concern; for what had he to do with a science which distinguishes the false from the true either in things or words? In physics he took even less interest; the fixed order of nature is a painful weight upon the mind of a man who aims at adjusting a scheme for himself, and who disbelieves in any actual order and government. Theology, so intimately blended with both of these in the earlier systems, would have been still more resolutely banished from his. But then all these subjects must pass under his review, because from all of them conclusions had been derived which affect man's serenity and cause him dreams. A study answering to logic must be used to explain the origin of opinions, and why some of them must, for want of better epithets, be called false and some true. A man has certain sensations, and certain images are presented to him from without. The sensations are to be trusted, the representations are to be trusted: only in the exercise of some power by which a judgment is formed from these sensations or representations does error arise. These are corrected by referring again to the sensations, the only ultimate standards. Here again the originality of Epicurus consists, it will be seen, wholly in his omissions.

The relation between the senses and the visible representations which are set before them had been treated by Plato and Aristotle, each in his own method, with the profoundest skill and discrimination. Epicurus had only to avail himself of such fragments from the intuitive observations of the one or the rigid analysis of the other, as were consistent with the rejection of their principle that there is another and a surer standard than that which the senses supply. In physics Epicurus was still an avowed copyist; and any one who studies the rest of his philosophy may perceive that, as he adopted the theory of Democritus, simply because it was the one which it was most comfortable to hold, so he was guided, in many changes which he introduced into it, simply by the wish to get rid of some distressing fact which interfered with his moral speculations, and made his scheme of life less practicable. The idea of mysterious powers in nature had been one which had at all times haunted the Greek mind, and to which the speculations of the philosophers bore as much witness as the fables of the mythologists. With this thought was connected another, still more oppressive to the mind of Epicurus, still more interfering with the calmness and apathy of the true sage. It seemed that these powers were organized, that there were the vestiges of a scheme in nature, that there was something in them which answered to man's own powers of art and contrivance, while it controlled them. From these painful impressions there was one joyful refuge. The notion of the world being composed of atoms which had met in empty space, had united and disposed themselves into the light forms or heavy masses of which our senses take cognizance, at once relieved Physics of its connection with Theology. And since one product of these invisible atoms was that which had been called the soul of man, a new light dawned upon the moral system from this natural philosophy, a new confirmation of the great principle that man

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

is a machine which may, like any other, be regulated and adjusted to produce certain desirable results. One fault, however, there is in the doctrine of Democritus. His descent of atoms in a direct line savoured too much of a determinate purpose, a fixed law. Suppose them to decline a little from the line—a very little—and this charm is broken, and, what is better still, you have a method of accounting for the existence of choice and freedom in man. If it be true that Epicurus really anticipated an actual physical discovery in this speculation, it is a new evidence that the divinity which he supposed took no interest in human thoughts or designs, does sometimes shape them into a strange resemblance of what is true, when they are shaping themselves into the most grotesque forms of falsehood. On the whole, it seems unnecessary to attribute to the founder of this philosophy any deep and malignant design of undermining the foundations of human belief or human conduct. He was an entirely different man from the sophists of the former age, far less acute, but also far less wicked. The worst that can be said of him is, that he exactly caught the impression of a wretched emasculated age; the best that can be said of him is, that he showed some skill in combining the notions of former philosophers into the only scheme of doctrine which could seem to the men of such an age plausible or possible. One thing should be noticed, that if Epicurism is ultimately destructive of moral habits, it is not to these, but to science, that it sets itself in direct opposition. The Epicurean is essentially the unscientific man—it would be more correct to say the hater of science; a fact the more striking, because in modern days, when physical science has established itself in the world by another agency than theirs, the disciples of this school have sometimes affected to take it under their patronage, and even to boast of themselves as the exclusive promoters of it. Perhaps we shall find, when we come to speak of them, that they have departed more in appearance than in reality from that which is, in truth, the fundamental principle of their sect.

The Stoics.

The second great effort against the Pyrrhonism of this age is the Stoic philosophy. In speaking of this system, as well as of the last, we must endeavour to detach ourselves as much as possible from Roman associations and representations. To do this entirely is out of the question, for till these philosophies were adopted by the living minds of Rome, they can scarcely be said to have found their meaning. Still it is important to consider the form of the statue as it came from Greek artists, now no longer able of themselves to impart animation to their works, before it was embraced by the Italian Pygmalion. If we reflect how deeply the feeling of an intercourse between men and a race superior to themselves had worked itself into the Greek character, what a number of fables, some beautiful, some impure, it had impregnated and procured credence for, how it sustained every form of polity and every system of laws, we may imagine what the effects must have been of its disappearance. If it be possible for a man, it certainly was not possible for a Greek to feel himself connected by any real bonds with his fellow-creatures around him, when he felt himself utterly separated from every being but them. But the sense of this isolation would affect different minds very differently. It drove the Epicurean to consider how he might make a world in which he should live comfortably, without distracting visions of the past and future, and the dread of those powers who no longer awakened

in him any feelings of sympathy. It drove Zeno to consider whether a man may not find enough in himself to satisfy him, though what is beyond him be ever so unfriendly. This again was no new problem, either for a practical man or a theorist to deal with. Again and again it had presented itself to the Greek sages; again and again experiments had been made to solve it, and the conditions under which it could and could not be solved had been profoundly investigated. Here then, as in the former case, we can expect no originality in conception; the sole interest lies in the thorough desolation of heart which led the philosophers of the Porch to venture once more upon this inquiry. We may trace in the productions which are attributed to Zeno a very clear indication of the feeling which was at work in his mind. He undertook, for instance, among other tasks, to answer Plato's *Republic*. The truth that man is a political being, which informs and pervades that book, was one which must have been particularly harassing to his mind, and which he felt must be got rid of before he could hope to assert his doctrine of a man's solitary individual dignity. Herein he showed a degree of insight and honesty which were wanting in his successors, who were anxious, it seems, to suppress his testimony against Plato. In other respects he was also superior to them. He appears to have carried out, with some consistency and steadiness in his life, the principle for which he was contending, really showing an indifference to outward circumstances, and maintaining, with less dogmatical affectation than many others, a creditable independence and uprightness of character. Probably his system was, on this very account, less compact and formal; at all events, we find that some of his immediate disciples adopted very opposite notions of it, one at least teaching a doctrine, as to its results, not very distinguishable from the Epicurean. The person who put an end to these diversities of opinion, and established the Stoic sect, was Chrysippus. He appears to have had a full measure of that taste for system building, and skill in the work, which belonged to all Greeks, and appeared most conspicuously in those who had least of practical earnestness and sincerity. It may be worth while to trace very briefly, how the first naked conception of a man striving to live in himself, uninfluenced by the men and things around him, worked itself out into a compact and tolerably consistent theory of the universe.

It was not likely that the Stoics would invent any division of studies different from that which use had so long sanctioned, or that they would be so negligent as not to extend their theory into the different departments of Morals, Logic, and Physics. They introduced, however, a novelty of expression. They spoke of virtue as being ethical, logical, and physical. It was not that these were three different kinds of virtue, but three different parts which composed it. Logic, they said, was the shell of the egg, ethics the white, physics the yolk. Much may be learnt from this language. First we learn that virtue, a certain state of mind or character in the individual man, is all that the Stoic is capable of conceiving. This is the ultimate idea upon which all ideas of truth, as well as of outward tangible forms, are dependent. It will be seen at once how easily such an opinion as this grew out of the Aristotelian doctrine, and yet how it may have appeared, in certain points of view, more unlike to that than to the Platonic. In the doctrine of Socrates developed by his great disciple, virtue in man is always in conformity to a standard out of himself, a participation in that which is absolutely good,

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

and inseparable from the possession of that which is absolutely true. This doctrine Aristotle rejected, striving to separate practical morality into one department, and the study of Being into another. Upon his system, virtue and happiness acquired the substantive and independent character which neither Socrates nor Plato could ever have assigned them. But the very separation which he had effected between the provinces of Ethics and Ontology, enabled him the more easily to follow the natural bent of his character, and to represent the contemplative life as the highest life—that which peculiarly appertains to the philosopher. The Stoic, in carrying out his conception respecting virtue, disconnecting it entirely from every dream of an absolute good, rejected with indignation his praise of *θεωρία*, and seemed to return to the older Socratic language by representing virtue as consisting in a conflict with appearances and deceptions. But the fact is that the Stoic took only the negative side of the Socratic doctrine; his virtue had nothing to converse with, nothing to behold but itself; the impediments which it was to clear out of its way, the temptations which it was to resist, were not those which dimmed the human vision and prevented it from beholding its proper object, but only those which made it less conscious of its own independence and glory. The Stoic ethics, therefore—borrowing all which was genuine and vital in them from the language of Socrates respecting the slavery of the undisciplined and sensual spirit, borrowing also and misapplying many of his phrases respecting the connection of virtue with science—so far as they were of native manufacture, consisted merely of pithy maxims of conduct, wire-drawn and minute, entering into the lowest details and frivolities, tending to emasculate the character under pretence of elevating it, and worthy of the censure that they were fitter for nurses than for philosophers. This was the white of the egg.

That Dialectics should have been nothing more than the shell, is a proof how very little of the real feeling of the Socratic philosophy had survived in this feeble imitation of it. In fact, the language to which we have just alluded, in which science was represented as connected, nay, almost identical, with virtue, meant, in the mouths of the Stoics, nothing more than this—that a virtuous man is a man of good taste, has a shrewd discernment of what should be accepted and of what should be rejected. Truth as reality, falsehood as a positive opposite to truth, they in nowise recognised. Dialectics, therefore, in the Platonic sense, which is the science of distinguishing between this truth and falsehood, they had no use for. All they understood was that in some way or other that which is desirable is preferred by some and not preferred by others, and that it was necessary, therefore, to inquire what it is in us which determines the fitness or unfitness of things, how we know when we get a right result from things and when we get a wrong one. In all essential respects their conclusions upon this subject were the same as the Epicurean, that is to say, the one has just as little belief as the other in any standard besides sense, in any difference between the real and the apparent. The difference of their moral scheme, however, made the Stoics unwilling to acknowledge this similarity; they wished to persuade themselves—to a certain extent they could persuade themselves honestly—that those who aimed at the attainment of virtue as their end, could not have the same fluctuating rule and measure of what was good, as those who aimed only at the production of certain pleasurable results. The Stoics

sought therefore for a science, though they could not reach it; and in their attempt to reach it they invented a number of acute verbal distinctions, which, with some mixture of grammar and rhetoric, constituted their dialectical virtue, as those rules of behaviour of which we spoke just now constituted their moral. Certainly this was very fitly compared to a shell containing no nourishment in itself, and so conveniently fragile as to afford an easy passage to the yolk within.

It may surprise our readers that the essential part of the Stoical egg, the whole of which is the type of Virtue, should be the physical part. But there is nothing really inconsistent in this notion, either with the origin of Stoicism or its after-developement. If the Epicurean undervalued physics because they spoke of something fixed and preordained, something therefore inconsistent with that adjustment and adaptation to circumstances and accidents which he considered his chief good, the Stoic, who wished to rise above circumstances, to attain a firm and independent position, as naturally delighted to contemplate an undeviating system. The perception of any real law and standard for man had forsaken all alike. But there are some who can never lose their deep feeling of the necessity for such a law; these therefore, in their despair of discovering it, will take refuge in the most exact type and counterpart of that which they are seeking, in the sequence of the operations of nature. The invariable attendant upon this feeling is, reverence for fate and necessity, with a proud and voluntary submission of ourselves to its dominion. Such a Fate became the god of the Stoic, strictly speaking his only god; but as he saw it imaged in the movements of the universe, and as he felt at times the need of something more real, more connected with himself than this abstraction, the World became the living form in which he contemplated the object of his worship. And since he found it expedient for strengthening some of his moral habits, and accordant with some of the maxims of his philosophy, not to reject old opinions, he easily persuaded himself to adopt the opinion of Aristotle, that the old legends did in fact represent processes in the material world. They might, therefore, without any violation of his philosophical dignity, be recognised and defended. Theology being thus identified with physics, it surely becomes no matter of surprise that the latter should be treated as the most inward and sacred part of morality.

One sect yet remains to be mentioned before we can complete our picture of Grecian philosophy in its decre-
The Aca-
demics.
pitude and decline. We have seen how entirely the very power of conceiving that which we have described as the central principle of Plato's philosophy had departed from his countrymen. All his language about Being was to them the merest dream; they could not even understand the elaborate arguments of Aristotle against his doctrine, nay, his own views respecting the Form and the Matter in each substance had become practically as unintelligible as the deeper speculations of his master. Under such circumstances we may easily conceive what a change must have taken place in the schools bearing their names. The Peripatetic, who worshipped the name of Aristotle, might still satisfactorily expound his Physics, his Logic, part of his Metaphysics, and whatever of his Ethics could be detached from the Politics; but the Academics found the incomprehensible part of their master's creed impregnating all his works; it could no more be detached from the *Phædrus* and the *Phædon* than from the *Republic*. There was, however, one circumstance in

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

their favour. For reasons which we have considered at quite sufficient length, the great principle of Plato is developed in dialogues, in which two propositions seem to be set up for the purpose of knocking each other down. Could any thing be more natural than the notion that Plato intended hereby to keep men's judgments in a perpetual equilibrium, to maintain, in short, a habit of entire interminable scepticism? The conclusion was most plausible; yet so much did there appear in the writings and in the whole purpose of Plato to refute it, so much did it seem the very object of his life to overthrow scepticism, that a long time elapsed before this plausible notion was able to establish itself. The Academy appears to have undergone many changes; what they were has been the subject of much controversy, but the language of Cicero leaves little doubt that it did at one time assert certain dogmatical propositions as the doctrine of Plato, and that it passed by slow degrees into that purely sceptical society from which he derived, or fancied he derived, his own opinions. The result is curious, but by no means inexplicable. Scepticism was, as we have said, the foundation of both the other sects, though they attempted to break loose from it. The Academy yielded submissively to the spirit of the times, and embodied it most consistently in his own *no views*. He discoursed eloquently upon all topics, carefully abstained from coming to a conclusion upon any; he could not say what was right, but he was satisfied that both Epicurean and Stoic were wrong, and this was quite sufficient for his purpose. Arguing was his vocation, a kind of arguing, however, which did not exclude an indulgence in all flights and flourishes of rhetoric, when the occasion might serve for their introduction. In fact, the disciples of that philosopher who wrote the *Gorgias* and the *Phædrus* became nothing else than the teachers of wiser and better men than themselves how to be Sophists and Rhetoricians. Such were the lees of Greek philosophy, from which one may conjecture in some degree how rich must have been its flavour, how full its body, when it was in its prime.

Roman Period.

If the view which we have taken of this last period be true, the trite saying that Rome conquered Greece with arms, and Greece Rome with wisdom, must be understood in a very modified sense. We do not mean that too much has been said of the effects which were produced upon the Roman mind and character by the Athenian teachers who came to Rome, or by those to whom they resorted in the schools of Athens. Undoubtedly the discovery of a new continent, which could be actually trodden with human footsteps and seen with human eyes, may have produced less wonder in the mind of a Spaniard in the XVth Century, than the apparition of the invisible world, which the Greek sages spoke of, upon the countrymen of Camillus and Regulus. The doubt we wish to express is, whether the Romans were merely conquerors with the sword, and whether they ever succumbed passively to the philosophy; whether they did not leave upon the thoughts and doctrines which they seemed to imbibe so simply and devoutly the impression of that character which several ages had been contributing to form. This question can only be answered by considering very shortly the remains of Roman philosophy, and the hints they supply respecting the state of feeling in the nation which produced it.

The poem of Lucretius naturally presents itself as the first and most marvellous outburst of that spirit which all the wisdom of the Censor and of the older Romans was unable to keep in check. Every thing at first sight tends to heighten the wonder which this poem produces in us. All the philosophical poems of that nation which had furnished the language of philosophy, and given birth to the most splendid poetry, have perished; only a few fragments remain of the verses in which Xenophanes and Parmenides conveyed their opinions; Empedocles of Agrigentum is but the shadow of a name. Nor have we any reason to believe that the productions of these men, though they were to all appearance men of genius and originality, deserve to be regretted, at least as works of art. How very strange, then, does it seem that the greatest effort of Roman genius should be a work written on a subject utterly alien from the habits of the Latin mind, by a young man struggling with a language which, for his purposes at least, was barren and uncouth; a language too which one would have thought could only have become poetical when it was used to speak of the actions of great men, and of people subdued to laws. It may seem to some even a more astonishing circumstance than any of these, that the theory which Lucretius undertook to defend and illustrate was, of all that Greece had produced from the days of Thales downwards, the hardest and most mechanical, one would have said the most flat and prosaic. It would be an easy and flattering way of meeting these thoughts which will naturally occur to any one who is reading the poem, to say that genius is only called forth by difficulties, and that if it had nothing to overcome it would not deserve its name. No doubt this is true, but still there is a fitness in the choice of subjects which we are generally able to recognise, and without which it is hardly possible that a work, even if it were written, could become a great national possession. With respect to the doctrine no thoughtful reader can believe that it was adopted, from a false notion either that it was particularly suited for poetry, or that great fame would be obtained by triumphing over its unsuitableness. Lucretius writes with the most entire conviction; his whole mind is evidently impregnated with his doctrine. And this was one necessary condition of his producing so great a poem; he felt what he said; a feeling of earnestness, which had not been in the mind of any Greek for at least two centuries, had got possession of his. He may have selected a miserable idol, but such as it was he rendered to it the most entire devoted worship. Now this earnestness he owed to his Roman education. The distinct aims and purposes of the Roman citizen, the active living world in which he dwelt, the feeling of order in all the movements of the society to which he belonged, presented the strongest contrast imaginable to the state of those who delighted in nothing else but either to tell or hear some new thing. At the same time, to a young Roman, who had contracted unconsciously this sense of order and reality, and who had been trained to contemplate the heroic models in the early history or the traditions of his country, who had not perhaps himself any strong impulses towards active life, or much of the legal and rhetorical abilities which were the qualification for it, the rage and contention of parties, the atrocities and the meannesses which, in the days of Marius, Cinna, and Sylla, he must have heard of every day, will have been most distracting. We may be able to perceive how much better the state of

Moral and
Metaphy-
sical Phi-
losophy.

Lucretius.

Moral and
Metaphy-
sical Phi-
losophy.

society in that age, with all its abominable crimes, was, than the dreary condition of Greece, even when its sleep was confused by those dreamy efforts of patriotism which the appearance of Flaminius and the Romans called forth. We may see that great social principles were struggling with each other in those conflicts of rival parties which could not have left such an impression upon history, if there had not been much good mixed with the apparently unbroken evil. But to a person of the time, such as we have supposed Lucretius to be, the misery and confusion will have presented themselves almost without relief; and then, little knowing how much he was indebted to the forms of his country's polity, and to the truths which lay underneath its false worship, for the disgust which such spectacles excited in him, he will easily have been led to question the worth of that faith and reverence by which a system of falsehood and cruelty seemed to be upheld. In such inquiries, he will probably have found most of the thoughtful youths about him engaged, with no great difference in the result, except that they could abide quietly in contempt of the popular opinions, while he required some positive substitute for them. Arriving with such feelings at Athens, what could be more natural than that the words of the clever Epicurean lecturer should take a hold of his soul which they never had obtained over the person who uttered them; that he should have welcomed them as the deliverance from an intolerable burden, as the discovery of a region of which he had been dreaming, but which he never believed to exist; nay, as the satisfaction of that love of order which his Roman discipline had imparted to him, and which the circumstances of Rome itself were continually affronting. A poem on the Nature of Things, written under such circumstances by a man possessing the vision and the faculty divine, might well embody some of the deepest, strongest, nay, truest feelings. The strongest patriotism, the greatest command of his native tongue, might be exhibited by a man apparently adopting all the habits and notions of a Greek. Under the guise of Epicurism, he might express a religious desire for a deliverance from the tyranny of powers whom he could not love; the Democritic concourse of atoms might convey to him his first real notion of any scheme or order in the universe: and such a poem might well become national, for it would express the very state of mind of the age in which it was produced as reflected in the person of the most genius; it would bring out the whole nature of the union which was effected in that age between Greek speculation and Roman life, and would show how the latter really asserted its dominion over the beggarly materials with which it had to work.

We have no excuse for dwelling at any length upon this noble poem, both because it is so well known, and because the illustration which it gives of Roman feeling at this crisis is the chief light which it throws upon the history of philosophy. In reference to that point, however, we would direct the attention of the reader to one circumstance. Epicurus, as we have seen, valued himself mainly upon the moral or human part of his system; the physical, which he borrowed from Democritus, was adopted only as a resource. Lucretius, on the contrary, at once fixed upon the atomic theory as the central part of his philosophy. Nothing can illustrate more strikingly the difference between him and his master. How to find an excuse for a voluptuous indolent temper, whether it were a sensual one or not; how

to arrange the world in conformity with it was the problem proposed to himself by the one; to recognise some kind of principle and connexion in things was the delight of the other; which, being their respective impulses, we may fairly say, that Lucretius differed more in spirit from Epicurus than either Zeno or Chrysippus did.

But these also were to have their disciples among the noble youth of Rome. The transformation which took place in the dry chips of doctrine which were delivered in such solemnity in the porch, when they were taken into such hands, was marvellous. If there were some of a more adventurous genius, who fled *ex fœce Romuli* to the study of the laws of the material universe, there were many more who found their great relief in contemplating the severe forms of the elder Romans, who either saw in the records of their country, or created out of the materials which they furnished, men of a stately character to whom wealth was indifferent, who loved their country above all things, and could sacrifice themselves, or whatever was dearest to them, for its sake. Between such simple men and the stiff solemn conscious Stoic a whole heaven would seem to intervene. Yet there was enough of external resemblance in the two characters to deceive those who felt that they wanted some knowledge which their fathers had not, who were unwillingly half ashamed of their old ignorance, and half afraid of their new philosophy, lest it should weaken their admiration and their patriotism. To be taught how they might upon rule and principle be that which their ancestors had been from some high and unattainable instinct, to be taught how they might be only better and more consistent Romans for this Greek infusion was most soothing and satisfactory.

That such feelings existed we have abundant evidence, but they did not, like the thoughts of the great Epicurean, find their principal expression in words. The lives of Cato and Brutus—the one more formal and severe, as of a person who felt that he was trying to support a character, the other more genial and free, like one who had really caught the spirit of the olden time,—both Roman aristocrats at heart, however they might speak the language of the schools—these are the true utterances of Roman stoicism, which have thrown a splendour around the doctrine that it could never have obtained either from its first teachers or from Seneca and the rhetoricians who afterwards talked of it in Latin.

Thus far we have seen the Romans only translating Cicero. into living words or living acts the dead formulas of the Greek schools. But Rome was also to have a formal philosophy of its own, and if not to make any new discoveries or to follow any course of thought which had not been previously marked out, at least to fulfil a useful office by giving a dignity to one particular department of thought, which, in the minds of even the greatest Greeks, had obtained only a secondary importance. We have spoken of Cicero as an Academic; and doubtless there was much in his character, in his political career, and in his rhetorical habits, which might have led us to predict that this was the sect to which he would be most naturally drawn. He appears to have had a singularly equitable, balancing, compromising nature. The circumstances of his age, the utter impossibility of adhering with steadiness to any one party, when parties were so constantly shifting their ground; his conscientious unwillingness not to take some part in political life, or to set up any immutable, unattainable standard for those who were engaged in it, confirmed all

Moral and
Metaphy-
sical Phi-
losophy.

The Ro-
man Stoics.

Moral and
Metaphy-
sical Phi-
losophy.

his original tendencies, and the profession of an advocate riveted and perfected them. Though certainly not the person to be fixed upon as exemplifying the highest form of the Roman character, he had in a remarkable degree the Roman temperament, and he seems especially formed to show us what the intellect of his nation was when at its greatest natural stretch, not raised by some extraordinary impulse of genius or devotion above itself. On this account it is that his letters, speeches, and dialogues present so perfect an image of his own age, and that some have thought a history of Rome might have been composed from them, if all other monuments were lost. But it must be observed, that merely practical wisdom is not able to express and embody itself in words till it has been mixed with an apparently incongruous element. If Cicero had not studied Greek philosophy, and been, so far as a most accomplished scholar and statesman can be so, a pedant, he would not have enabled us to understand himself or his countrymen as he has done. His philosophy was unquestionably important to him and to us; still it is amusing to hear him speak as if his habits of mind had been in any considerable degree moulded by its influence, when it is most evident that they were wrought into him by the influence of old forms and institutions, by the circumstances of his country, and the tone of the men who surrounded him. These contributed to the system of philosophy which he took under his patronage, far more than they received from it. He found the Academics treating philosophical questions in the same manner and with the same fairness as prosecutor and defendant were treated in the Roman courts, arriving at no settlement, as he could arrive at none in the disputes of factions, yet inclining to established notions in opposition to the dogmatical denials of the Epicurean, and to a moderate behaviour accommodated to circumstances, in opposition to the fixed rule of the Stoic, and he was irresistibly prepossessed in favour of their views. But that which gave those views favour in the mind of the Greek was their fitness for talk, a talk which might be carried on for ever without the least reference to life. That which endeared them to the Roman was their apparent suitableness for practice, their seeming to show the very point where the lines of philosophy and practice intersect each other. This was the point at which he was aiming; and in his own speculations, however they may seem merely derived from the Academics, he is continually bringing it before us. If there was one feeling in which the Greek academician was utterly deficient, it was the feeling of Duty, the feeling that there is a work which a man is sent into the world to do. This feeling is wanting in all the sects; each was framing a scheme of life, or aiming at some ideal of excellence; none was acknowledging a vocation. It would be very unjust to say that the sense of duty was absent from the minds of Plato and Aristotle; they had it unquestionably, or they would not have been what they were, or have done what they did. But it is true that it was not the prominent characteristic in either of them, or in any Greek. On the other hand, this feeling is the one which gives all the meaning and interest to the works of Cicero. You can always see that he is impatient of any subject which he does not think has a direct bearing upon human life; that when he writes upon such subjects, he has only the use of his left hand; that they never do really affect him at all; that he has no opinions upon them and does not care

to have any. He will therefore retail the opinions of all the philosophers so far as he knows them, or will allow some able representative of the different sects to state the views of each, and will seem to have no aversion to any thing but the dogmatism which each exhibits. Yet in the end you find that he has a set of firm convictions in his mind, which have remained undisturbed by all these controversies, and by his own nominal scepticism. He can see nothing but a difference of words between the Peripatetics and the Academics, and it is quite clear that he means to extend the observation to the original masters of those two schools. The whole subject of Being and of Ideas, about which Greek philosophy in its best days was wholly conversant, is an unknown world to him. Plato he looks upon chiefly as the most eloquent of men; from Aristotle he has gained good helps in the study of rhetoric. One cannot discover that he cared any thing about Physics; Logic he prized chiefly as a mental exercise, and in all disputes about the Nature of the Gods, so far as they bear upon either of these subjects, he seems to be neutral, and, if one tried him by modern rules, we might fancy, atheistic. But he is not so at all; he has a much stronger belief in a divine power, and a divine government, than many whose opportunities of knowledge are infinitely greater than his. But he attained this belief without any assistance from the Greeks, and he retained it not at all strengthened certainly, but not materially weakened by what he learnt through them. Without a Divine Being there can be no sanctity, no duties, no laws; this was the conclusion of his heart and reason both, and he felt that it was a deeper and securer one than any which arguments could furnish him with. This ground, therefore, he vindicated to himself; he brought out the idea of MORAL OBLIGATION with a distinctness with which it had not been presented before. Not that it would be easy to point out passages in his books in which the subject is discussed amply and satisfactorily; not that the student will not often have to complain of much looseness in his language upon it; not that he will not be sometimes puzzled to conceive how so much indifference to absolute truth can consist with a strong sense of moral duty: yet this seems to be the total effect of his books—the result which is left upon our minds both by their merits and their omissions. Though he has established nothing, he leaves us with the conviction that something is established, and that it is not something independent of us, but something with which we are concerned, and in conformity with which we are bound to act. His philosophical works, therefore, appear to have been unjustly exalted, and as unjustly disparaged. When he is used as an interpreter of the elder Greeks, when we try to understand Plato through his means, we confer on him a station which was never intended for him, and he will unquestionably lead us astray. On the contrary, when we regard him merely as the translator into eloquent Latin of what he had heard from his teachers in Athens, we degrade him just as unfairly. His philosophy has a substantive value; such language as his can never be a mere clothing for other men's conceptions, though it may not be a fitting expression for the very deepest ideas. And it should be observed, that in those works wherein he has adopted Plato's titles, and might seem to have followed him most closely, as in the *Laws* and the *Republic*, he has really drawn most upon his native resources, and established the truth of his own words, that he had learnt more

Moral and
Metaphy-
sical Phi-
losophy.

Moral and Metaphysical Philosophy. Moral and from the Twelve Tables, than from all the philosophers. This seems all that it is necessary to say in so brief a sketch as ours of Roman philosophy; for in the poem of Lucretius, the biographies of the last champions of the republic, and the conversations of Cicero, we have it in its purest form. It imparts richness and refinement to the *Georgics* of Virgil and the *Epistles* of Horace, and adds solemnity to the dirge which Tacitus sings over his fallen country. But the character which it assumed in these really remarkable men, as well as in those who adopted it merely as an excuse for uttering fine sentences, will be easily understood, if we form a right judgment of its earlier appearances. We pass on, therefore, to the last conspicuous form of ancient philosophy, which is the one that it assumed at Alexandria.

Alexandrian Period.

Under this head we do not propose to speak of the different Greek philosophers who may have settled in Egypt during the reigns of the Ptolemies, and have carried on the different physical, moral, and philological inquiries which had been suggested by the founders of their respective schools in Greece. These in themselves constitute no era in human thought; they are only important as parts of a stream which it would be very interesting, if it were possible, to trace, with all the different accessions that it received from various directions, till the time of its confluence with another from Palestine. No doubt the habits of the Greek mind underwent some modification after such a new and strange book as the Septuagint translation appeared among them, and so marvellous a people as the Jews began to hold familiar intercourse with them. But it is very difficult to ascertain what that modification was; very likely it was more striking in the humbler classes than in the philosophers. Their minds were well prepared to resist anything living, original, and true, even if it had not claimed the authority of a revelation. But the effect produced on the other party was very different. How far it is to be discovered in any of those books which form what we call the Apocrypha, we can do no more than conjecture, especially as the dates of those books are so uncertain. At all events it is so visible, and so generally recognised, in the writings of some who lived at the close of the Jewish age, that we need not waste our time in searching for earlier and more doubtful indications of it.

Josephus. It is hardly necessary to speak of the writings of Josephus as affording an instance of an attempt to combine Pagan with Hebrew notions. The experiment in his case is conducted so awkwardly, his philosophy was such an inconvenient addition to his better qualities, it evidently had so little other influence on his mind than the disastrous one of spoiling his simplicity and unfitting him to understand the records of his own land, that his case is interesting in this point of view, chiefly as indicating a prevailing habit of mind, and perhaps as exhibiting a form of it which we do not meet with elsewhere, that, namely, which resulted from the interweaving of Roman rather than Greek feelings with those he had derived from his birth. Philo is altogether a different person. In opening his works this observation at once suggests itself to us. Here is a man living in the age of Cicero, and possessing a mind certainly not naturally so large or rich as his to whom a whole world of thoughts have discovered themselves, of which the Roman's writings contain no intimation. And

what is more remarkable still, those parts of Plato's works which were to the cultivated and accomplished Roman mere blanks, are to this Jew full of life and meaning. His position in Alexandria was not more favourable for this purpose than was that of Cicero. The latter had all the advantage he could derive from the best and ablest teachers of Athens, the language of Plato presented no real obstacles to him, and if the schoolmen of Greece had become a set of shrivelled rhetoricians, there is reason to believe that the schoolmen of Grecian Egypt were not less so. All Philo's intuitions respecting the meaning of those deeper passages in Plato's writings which speak of absolute being, and of an idea or essential form, through which that being declares itself, he must have owed to some other help than any which he derived from his native genius or his accidental position.

We would have the student consider this fact seriously and steadily, for it will help him much in estimating the value of Philo's writings in themselves, and the kind of lesson which we are to derive from them. A certain degree of discredit attaches to his name as an allegorist and a mystic. At the same time his writings did exercise a great and remarkable influence over the first ages of the church, and no one, we are certain, can contemplate them with reference to the history of philosophy, without perceiving that they must have had this influence, and that it was not the effect of any spurious reputation for sanctity in the author, or of any accidental circumstances whatever. At the same time we admit fully that no person of simple, honest mind, whether he reverences the Hebrew Scriptures or not, can rise from the perusal of some treatises of Philo's, in which he tries to deduce different Gentile discoveries out of them, and to turn plain history into divine speculations, without a sense of annoyance and perhaps indignation. We feel there is something in it not honest and true; an impression strongly confirmed by the style, which is of that kind which may be called painfully expressive, every word being charged with some elaborate meaning, and there being, in even very elaborate and eloquent passages, a consciousness of eloquence which is most distressing, and which places Philo at a marvellous distance not merely from the prophets of his own race, but from those heathens who were the objects of his study and admiration. And yet if we are honest with ourselves, this conscientious displeasure will not hinder us from acknowledging that he has brought forth not merely great ideas but practical truths of the greatest moment to our lives, and that he has, in some way or other, imparted the conviction to us that these truths were contained in the scriptural narratives and stories from which he deduces them. These two impressions may seem very contradictory, but let the reader consider faithfully with himself whether they are not both made upon his mind, and if he finds it to be so, let him pause before he hastily or impatiently seeks to efface either. A little reflection may perhaps enable him to find that both ought to be retained, and that they may be reconciled.

One conclusion has been forced upon us by all our observations of moral and metaphysical inquiries hitherto. We have seen that in Greece they could not take their rise, where men had no sense of a government or social position; that the fact of that position was one which haunted the minds of those who seemed to be pursuing their inquiries in the most opposite direction, and greatly modified the character of them; that when these

Moral and Metaphysical Philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

inquiries were in their strongest and healthiest state, this fact became the centre towards which they all converged, that when the sense of it was lost the inquiries became feeble, narrow, unreal. This moral we have been obliged to repeat even to weariness, so directly does it result from every fresh portion of the evidence, and not least from that comparison of Greek inquiries with their supposed Roman copies, into which we have just been entering. Connected with this conclusion was another, that all brave and adventurous inquirers respecting those things which lie beyond the ken of our senses had received their encouragement and hope from traditions of men who had actually been favoured with communications from that invisible world. All the most thoughtful philosophers we have seen rather endured the puzzles and confusions of their traditions than parted with the faith which they conveyed to them, and never in their sober moments believed that the discoveries or intuitions of their minds could be a substitute for actual revelations, though they might constitute a capacity for receiving them. The loss of a clear conviction on this point in Aristotle was accompanied, we noticed, with the loss of a belief in absolute truth, at least as necessary for mankind. Still the belief lingered in his mind, that the contemplation of this truth was the very highest attainment of the philosopher. The next stage presented us with the loss of this feeling altogether, with an utter unbelief in all traditions, and also with a practical disbelief of any thing but that which the senses could measure. Which disbelief we found had no real place in the heart of a poet nominally Epicurean, because he was still, however unconsciously, possessed by the old habits and convictions of his country. All these discoveries would seem to favour the opinion which we took for granted at first, because it is the opinion which we all profess, that one nation did receive an actual communication from heaven, had an actual polity divinely contrived for the purpose of educating it into the truths which were contained in that communication, and that hence had arisen all the earliest and deepest searchings of the human spirit concerning itself and Him from whom it is derived. It would seem also to favour the opinion, which we also assumed, that at a certain period in the history of the world, after these inquiries under the direct influence of a divine revelation had reached a certain point, it was a part of the divine scheme that another set of inquiries, conducted without that influence, but not without the feeling of its necessity, or the recognition of a divine presence conveyed through many broken, obscure, and degenerating traditions, should have been carried on in some other country, specially prepared by the character of its people and the nature of its institutions for this purpose. Nor is it easy, as we observed, to avoid the belief that at some time or other these two schemes would find a point of coincidence. Yet that actual revelation should be made unnecessary by human discovery, or human inquiries be stifled by a divine revelation, is impossible, unless the history of both were falsified, for each had been manifestly demanding the other. The difficulty must have presented itself strongly to the mind of any man living in Philo's position; it must have been augmented greatly by the observation that for a long time there seemed to have been a dearth, as much of any powerful intuitions in the minds of men, as of any direct communications from above. The thought was surely not an unnatural one, that the former required to be

rekindled by light borrowed from the latter; that, on the other hand, the cold and dreary acknowledgment of a set of strange miraculous facts, might acquire an altogether new life if they were shown to be connected with thoughts and feelings which had deservedly won for men the reputation of sages. It is not necessary, surely, to conceive that any false spirit of accommodation, or any scepticism respecting the actual events which seemed instinct with so much meaning, moved him to this enterprise. It was one to which a man of an earnest, meditative spirit, living in such a time and in such circumstances, may have felt himself prompted by a very sacred impulse, and one which could hardly fail to produce some results most beneficial to mankind.

Accordingly we believe it was given to Philo to bring out into clearness, by help of the Jewish Scriptures, some principles of the Platonic philosophy which Plato himself could but partially apprehend. That which was a wish in Plato's mind—a wish and a kind of moral necessity—to connect the absolute being, the *to ov*, with personality, was involved in the very principles of the Jew's mind. If he ever used other language, as indeed he did too often, it was a wilful abandonment of his privilege, if not for the sake of pleasing Gentile ears, in the honest hope of making his countrymen less slaves to particular phrases by occasionally departing from them. It was given to Philo, in strict accordance with this difference, to connect the aspirations after wisdom in heathen men with those mysterious utterances respecting the Wisdom whose delight was with the sons of men, and the Word who spoke to the hearts of the prophets, and thus to establish the truth for which Plato was all his life contending, that the idea which the wise man beholds and tries to reach, is not a shadow cast from his own mind, but a substantial reality, existing distinct from himself, in which he is constituted. It was given to Plato, lastly, to show how those forms of virtue and loveliness, which Plato believed that the soul is meant to contemplate, are actually connected with that one fontal source of all loveliness, and have been exhibited to his creatures in various human models, who confessedly drew their light and life from him. In this way Philo may sometimes find a meaning for the word philosophy, which throws a light back upon all past Greek speculations.

But how then do we account for that want of simplicity which we spoke of just now? and on what ground do we affirm that with all his advantages he was a less true man, and therefore a less safe guide, than his Grecian master? The apparent contradiction may be explained in this way; Plato, we have seen, in all his loftiest discourses respecting the dignity of the philosopher, his distinction from the sensual herd, his right to govern them, never loses altogether sight of the truth that the philosopher is not of another race from his fellow-creatures, but only the one who is seeking to realize the humanity which belongs to all, that he is less exclusive, less self-seeking, less self-glorifying than others, that he values himself less upon particular gifts and properties, and seeks more after a common and universal life. We do not say that there are not many passages inconsistent with this feeling; we admit that it was the hardest thought for him to realize, and that he never did realize it thoroughly. Still the conviction lay very deep in his mind, and the *Republic*, rightly and simply interpreted, is throughout an endeavour, amidst great perplexities and contradictions, to express it.

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

Now this feeling ought surely to have been far more present to the mind of Philo. Every thing in the records which had been familiar to him from his childhood, should have taught him that his privilege was to belong to a race. Every member of that race was dear to Him who had entered into a covenant with it. The highest models of virtue held up to the imitation of Jews were not men who had thought great things of themselves, had proposed to themselves the attainment of some splendid character, had been chary of their reputations, but men who had identified themselves with their countrymen, had felt their sorrows and sins as their own, had wished for no better destiny than theirs; had not cared to be thought ambitious, rebellious, impious; had toiled on, in simple faith, entire dependence, and great suffering; thinking nothing of themselves, or if thinking, thinking only shame, desirous only to magnify that light which was shining and making itself manifest in them and through them. Of this Philo seems to have had but very indistinct glimpses. He himself is a conscious philosopher, striving consciously after a perfect form of character, and what he is, he imagines his brave and single-minded countrymen to have been; not plain simple herdsmen and warriors, serving God, and content to do any work he may set them upon; but men with the object constantly before them of making themselves heroes and saints. We do not mean that there may not be splendid passages in which he speaks of men living for others and not for themselves; but all this belongs to an ideal of character. It helps to make up the fine holy man and great philosopher. Hence results, we believe, that feeling of displeasure, and even disgust, with which honest minds turn away from Philo's commentaries upon the characters of the Old Testament. For this is the real cause of his disposition to allegorize. He cannot endure any thing so vulgar as sheep and oxen, as men fighting battles with actual swords and spears. He cannot see a meaning in these things, and therefore he must find a meaning for them. He cannot see God working in the actual affairs of earth, and therefore instead of seeing any thing as it is glorified and translated into the signification of something higher, he must separate essences from forms, till the first become vapours and the second husks. It is quite true that he did not act upon this notion in respect to the forms of his country's worship; those he prized, as containing the essence, as being some of the great means for cultivating the mind of the theosophist to the perfection for which it was destined. But while he was taking away the sense and feeling of reality from the things with which our senses converse, he was in fact destroying the principle of these forms; at all events limiting the mystery of them to sages, and undermining the great doctrine which they were proclaiming, that the highest life, the deepest truths, are the inheritance of men who live by the sweat of their brow.

Nothing could have been such a shock to the disciples of the Alexandrian school, as the announcement that a peasant of Galilee was He whom, in so many mysterious passages of their nation's records, they had been tracing as the source and spring of all the light which had been in the hearts of faithful men in the chosen nation, or even which had diffused itself through the world. Probably, not the proudest pharisaical ritualist, or the coldest moralist among the Sadducees, would have been so shocked at such a proclamation. For on the first of these, as he studied with

hard literal accuracy the Scriptural records, the thought may now and then have dawned that, as every past prophet or king, in whom any portion of the divine glory had revealed itself, had been cast low that he might be exalted high, this, in some great and transcendent way, should be the condition of him in whom it was to reveal itself perfectly. To the Sadducee, whether he studied in the Epicurean or Stoic school, (and probably there may have been some in that sect who had studied in each, the negative characteristic of indifference to the higher hopes of their countrymen, and not any particular positive opinions, being the symbol of their union,) it may now and then have occurred that no more perfect type of that moral quietness, which adapts itself to all circumstances, or of that moral grandeur, which rises above all, had ever presented itself to them in any records than that which they then saw before their eyes. However unable such thoughts may have been to break through the religious pride of the one, or the intellectual pride of the other, they may, one would think, have come to each occasionally. But the Alexandrian, conscious of the profoundest reverence for Scripture and philosophy, regarding the mere literalist and the mere moralist with equal contempt, but abhorring most of all every thing which vulgarized high truths, and brought them into contact with the hearts of simple men, will hardly have ventured to ask himself, even for a moment, whether any one Old Testament vision, whether any one Platonic aspiration, could have had its fulfilment unless the absolute Being revealed himself to his creatures in the likeness of a man; unless the perfect Idea of truth and goodness proved itself to be substantial and not imaginary; unless he showed that he could both exhibit himself to men, and abide in them; unless the Word could take flesh and dwell among men; unless, in their nature, he could overcome all moral evil, and turn physical evil from a curse into a blessing; unless he could establish that perfect commonwealth, of which Plato said, that the type of it was, perhaps, in the heavens, and which he hoped would one day be set up on earth. Through another entrance than that of Alexandrian wisdom, this faith was to find its way into the hearts of men, because it was emphatically a faith for men, and not for those who sought to separate themselves from men. In spite of that wisdom, and of all other powers and influences, it did establish itself in the world. The Christian church has become a fact which may be reasoned about as we will, but which remains a fact still. Whoso overlooks it in a record of philosophical inquiries, proclaims thereby that they have no connection with humanity; obliges himself to treat the deepest and most earnest speculations of men in the ancient world as dreams of which there is no interpretation, and leaves himself without a clue with which to trace the progress of thought in modern Europe.

PART II.

THE POST-CHRISTIAN PERIOD.

Although the space which lies before us seems far more extensive than that over which we have yet travelled, the greater part of the task which we proposed to ourselves at the commencement of this article is now accomplished. By taking up ethical and metaphysical inquiries at the point from which, as we believe, they started, not in some recent period in which they are

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphysical
Philosophy.

combined with all our own circumstances and habits, we have thought that we should be best able to make our reader understand what they are in themselves, and how they become connected with each other. We seem to have learnt from the history of the Jew that there are three stages in moral inquiries. The mystery of human relationships is the first subject of them; the mystery of positive law the second; the mystery of an order, grounded upon the character of an absolute being, the last. Now as we assumed the common faith of our countrymen respecting the divine and peculiar discipline of the Jew to be true at the beginning of our sketch, we shall assume the same common faith to be true, that this education was prolonged and carried on in the Christian church. In one case as much as the other we are confident that that assumption will be confirmed by the evidence of facts, and by the clear light which it throws upon the order and sequence of human speculations. Still we use it, for the present, as an hypothesis. In consistency with it, we believe the moral education of the modern world to begin where the Jewish education ended. We believe the Christian church to start from that idea of an absolute morality which was the final point to which the Jew was brought. We believe that the work of the first Christian age was especially to lay the foundation of society and of thought upon this deep, *Absolute* ground; that the object of the second Christian age was to raise on this foundation the edifice of *Positive* morality; that the work of the third age was to connect *Relative* morality with both of these. But we look upon the education of the Christian church as not merely, though primarily, the prolongation of the Jewish development. It should comprehend, also, if our view be a right one, that series of thoughts which we have seen occupying the mind of the Greek. Those thoughts (to which we found that the name of metaphysical was especially appropriate) turn upon these three great subjects—*Power, Being, Unity*. We have ascertained already that these have a close and inevitable connection with those other three of which we just spoke; that absolute morality connects itself with that question respecting unity, which may be said to be the central one in the Greek mind; that Law, in its largest signification, cannot be separated from the question respecting Being; that there is a certain link, as real, perhaps, though not so apparent, between that inquiry into the nature of our own Powers, which is the starting point of metaphysical investigation, and that meditation upon Relationships, which is the starting point of moral inquiries. We might therefore expect that the age which was most occupied with any one of these moral subjects would be also most occupied with the corresponding metaphysical one. In the remarks which we have yet to make, we shall devote ourselves entirely to the exposition and illustration of these statements. Our object will be simply to bring out the character of different periods. Persons and systems we shall allude to only as they may seem necessary for this end, or as they mark the commencement or termination of some era. We shall endeavour to set our hints as briefly and directly before the student as possible, because their worth, if they are worth any thing, must be proved by all his subsequent reading; to assist him in that reading, not to supply him with a substitute for it, being the object at which we have aimed throughout.

As we have spoken of three periods of human thought

in later history, we shall, of course, be required to define them with some accuracy. Fortunately, our division is one that is most uniformly recognised by the writers of general history: our first period extending from the coming of Christ, or rather from the reign of Vespasian to the reign of Charlemagne; the second, from that reign to the Reformation; the third, from the Reformation to the French Revolution. That since that time we have entered upon a new period will not, perhaps, be very strongly disputed, though, wherein consists its difference from that which preceded it, and what is its particular vocation, we may be better able to consider hereafter.

From the Coming of Christ to the Reign of Charlemagne.

It has been the fashion in some recent histories of philosophy to describe this period as one marked by the concurrence of Greek with Oriental ideas. That there is a truth in this language few will be inclined to deny; we especially should be contradicting all our previous remarks if we did not recognise it; but we must contend at the same time that it is vague language, and that it is often so used as to produce the greatest confusion and misapprehension. In some cases, learned and cautious authors have been betrayed by it into what seem, to plain persons, the most incredible hallucinations. Ritter, for instance, whose historical sagacity and research are indisputable, has made a digest of Indian speculations from the writings of Colebrook and other modern English Orientalists, for the avowed purpose of assisting us in understanding the education of Philo the Jew, and so in arriving at some general notion of the elements which entered into the philosophy of the age immediately succeeding his. Now seeing that Philo may have alluded, perhaps in about half-a-dozen passages of his writings, to the Gymnosophists of India, and that there is scarcely one of those writings which does not treat laboriously of the patriarchs, law-givers, and prophets of Judea; seeing that he attributes all his own trains of thought either to these or to those who wrote in his adopted language; seeing that the education of a Jew is one of the most marked and peculiar which it is possible to imagine, and the most unlike, in nearly all respects, to the education of a Hindoo; seeing, lastly, that Ritter furnishes us with no proofs whatever that the particular systems which he describes were the Hindoo systems of Philo's day, or that if they were he had the slightest acquaintance with them, it would be difficult to point out a more striking instance of an attempt to illustrate what is near and plain by what is distant and obscure. But we may derive some advantage from this extravagance. Ritter admits that the union of Oriental and Greek ideas, which he discovers in Philo, is the same in kind with that which is to be found in the writings of Christians and Pagans in the subsequent age. If, then, we feel at once that the phrase Oriental is affected and delusive as applied to the one, we shall be at least very cautious in using it when we are speaking of the other. At all events we shall be compelled to admit, on the plainest evidence, that the influence which proceeded from the ideas of which the Jew Oriental was the depository, were altogether different from those which can have been produced by the Persian or Indian Oriental. We shall admit that the effects of the former can be traced much

Moral and
Metaphysical
Philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

more visibly and clearly than those of the latter; and when we do find indications of both powers being at work, we shall believe that they were acting in very different directions towards most different ends.

It will be obvious from our last remark that we do not intend to deny the operation of certain Oriental ideas which were in no sense Jewish, upon the mind of this age; the proofs of that operation are we think irresistible, though their extent and importance may have been overrated. Neither do we deny that there is a point of resemblance between all systems of thought proceeding from the East, which does not exist between any of these systems and those which had their birth in Europe. Although the Greek speculations presumed a revelation, they never pretended to possess one; the Hindoo philosophers, even when they depart from the maxims of their sacred books, still acknowledge the existence of them. Even Greek *mythology* is mainly an ascent from manhood to godhead. Greek philosophy emphatically starts from human feelings and aspirations, and seeks out their proper object. Indian philosophy, as much as Indian mythology, is still occupied with speculations about divinity; would still be telling us how the supreme powers have been working in creation, manifestation, or destruction. Undoubtedly there is here a point of analogy with the Hebrew writings which it is important not to forget—important, however, mainly because it enables us to see more clearly the difference between that which purports to be a revelation of God to man and through man, and that which treats of the relations of God with the outward world, and with man only as a portion of that world. The former could meet that healthy portion of Greek inquiries which respected the life and the wants of man; the second might easily present itself as something fitted to complete those theories about the universe, which we have seen were some of them so splendid, and all of them so baseless.

These observations are necessary, not merely that we may understand the course of thought which the writers of the Christian church followed, but that we may not be misled respecting the feelings and inquiries of those who were endeavouring to cultivate and maintain the old Greek philosophy. Were we writing a history of the church, or of theology, it would behove us first to speak of the forms of thought which are to be discovered in Christian writings, and to look upon the different notions and systems which were afloat in the world, as bearing upon these and illustrating them. The student of philosophy will probably pursue the opposite method. He may wish to know first what men were doing who had no connection with the new faith, or were opposed to it; he will then think himself better able to judge whether the church was really meeting the wants of the period, or whether she was moving along in a region of her own, unconscious of them or despising them. Now in quarters which that Indian influence can scarcely have reached, or in which the traces of it are most difficult to discover, it will still be seen that the philosophical tone had undergone a remarkable change. The dry bones of Stoicism we found had been endued with a temporary life by the old republican spirit of Rome. When that spirit had departed, it might have been supposed that they would have crumbled at once. But this was not the case. Infinitely more genuine as Cato and Brutus were than Seneca, it is yet true, (and the affectations of the man only make the fact more remarkable,) that there are conceptions of a higher

kind in his books, than in those of his elder and better countrymen. Granting that they are conceptions merely, still they are indications of that being at work in men, which might find a real meaning for itself somewhere. A parasite in the age of Nero had thoughts respecting the assimilation of man to the divinity, respecting self-restraint and self-denial, which the honestest Roman in the age of Cicero had not. If we trace the progress of this same Stoicism, we find it again uttering, in the language of Zeno, feelings and aspirations of which he had known very little. Something more than the idea of a victory over circumstances—the idea of a victory over inclination and self—had dawned upon the mind of Epictetus. His conception of the philosopher's life may be as solitary, as little in the proper sense of the word human, as his master's. But the standard is evidently sublimer, and the sense of the difficulties to be reached in attaining very far more clear. Lastly, if, still following the same track, we look at the writings of the imperial sage Antoninus, it is difficult to say how far we have risen above the little frivolous maxims of conduct, "the hints for nurses," upon which the older teachers of the Porch prided themselves. Internal purity, practical affection and benevolence, contemplation of the divinity, nay, even humility itself, are not barely recommended in his writings or inculcated in formal precepts; it would seem there was an habitual aspiration of his mind after them.

And all these fine thoughts, high feelings, doctrines about the capacities of the human spirit, longings after emancipation, were awakened amidst the intense and loathsome corruption of the empire! Most strange whence such dreams should come, most strange that if they came they should have been accompanied with no longings for the improvement of the world's condition; that each sublime philosopher should have been content with thinking how he might get himself out of it! There were those who felt the contradiction, benevolent spirits who wanted something else to contemplate than visions of their own possible perfection, and who could not find it in the society around them. Such a person we conceive was *Plutarch*, who should be considered not so much a philosopher, as a kindly gentle antiquary looking out in the past history of his native land, as well as of that which had adopted it and the rest of the world, for specimens of courage in action or beauty in character, and then, in the same spirit, searching in the writings of philosophers, the sayings of poets, the fables of mythologists, for any thing which could sooth the spirit, inspire pleasant anticipations, make the contemplation of present evils less oppressive, and offer some prospect of real and permanent goodness. For this is common to him with the sterner Stoics of whom we spoke just now, and distinguishes both alike from the teachers of the preceding age, that the contemplation of some ideal of goodness seemed to them to be necessary for the life of men. So that a singular revolution had taken place. In that which we called the sceptical age of Greece, the age which produced the sects, all questions respecting the absolute and the ideal ceased as a matter of course. It was not proved, but taken for granted, that they meant nothing. In this age it was not proved, but taken for granted, that they meant almost every thing. An impulse which they could neither account for nor control was urging all the earnest men of the time, in whatever sect they might have been educated or whatever opinions they

Moral and
Metaphy-
sical Phi-
losophy.

Epictetus.

Antoninus.

Antiqua-
rianism.

Plutarch.

New
Stoicism.

Seneca.

Moral and
Metaphy-
sical Phi-
losophy.

might be inclined to patronise, in this direction. The only exceptions were those which must occur in every age and were nowise characteristic of this; a set of men who were entirely occupied in writing commentaries upon the letter of ancient philosophy, and another set who were devoted chiefly to medical inquiries, and who by degrees came to disbelieve in every thing for which they had not the evidence of their senses.

Miraculous
teachers.
Apollonius
of Tyana.

But neither the ideals of character, which Plutarch took delight in contemplating, nor his merely antiquarian philosophy, sufficed for the wants of this period. It was haunted with a continual sense of the need of some higher and more immediate inspiration, of some person in whom there should be a divine as well as a human element. Accordingly, about the close of the first Christian century, a strange person floats dimly before our imagination, to whom a hundred years after strange acts and qualities were attributed; who was supposed to combine, in a wonderful manner, the philosophical with the miraculous character. It is better to leave Apollonius of Tyana surrounded with the cloud in which it has pleased his worshippers to envelope him, and not to make any efforts to reduce him within the forms and dimensions of an historical personage. In that shape he best illustrates what necessities the time was conscious of, and what kind of methods it could devise for the satisfaction of them. The name and the legendary acts of Pythagoras appear to have furnished the investiture of the modern teacher; a circumstance which deserves to be noticed, not because any person possessing the least sense of justice would more think of attributing to Pythagoras the doings of Apollonius, than of making an ancient saint answerable for the conjuring tricks which a modern monk may have played with his relics; but because it shows how much men in that time were struggling to connect themselves with the past age, and to recover by that means, if by no other, the feeling of a union between themselves and the invisible powers of which the ancient men spoke. The particular philosopher selected on this occasion may indicate that the mystery of number and the idea of a society based upon a *unity*, which should not mean merely *singleness*, were haunting in some strange manner the mind of that generation.

The new
Platonic
school.

These thoughts, however, as we have seen in the former part of our sketch, had only their root in the speculations and practical experiments of Pythagoras; their full blossom in the Dialectics, the Physics, but above all, in the Polity of Plato. It cannot be surprising, then, that in a period which had already exhibited such tendencies, a new form of the Platonic philosophy should have speedily developed itself; and instead of appearing, like Stoicism and Scepticism, in a few detached thinkers, should have embodied itself in a school which lasted for several centuries, and which has always been felt to mark the philosophical character of a whole period.

Numenius.

Among the less known persons who seem to have laid the foundations of this school was Numenius, a Syrian. From what part of the East he derived that new element which was to mingle with Greek philosophy, may be judged from a saying of his, "that Plato was but Moses speaking Attic." In strict consistency with this observation, the great question which occupied him was, it would seem, that which occupied Philo above all others—the necessity of two beings; one God in himself, the other God in relation to men and to the world. The idea of some third power, as necessary

to connect both with the actual movements of matter, seems also to have presented itself to him. In these thoughts lie the germ of the new Platonic school, the founder of which, however, was not Numenius, but Ammonius Saccas of Alexandria, about whom a great controversy has been waged, whether he were or were not a person of that name who apostatized from Christianity to Paganism. The question is not of much importance to our purpose. We shall see presently what relation the ideas of modern Platonism bore to those of Christianity: whether they were offered as substitutes for the Christian mysteries, or worked themselves out in the minds of men as the legitimate conclusions from Plato's premises, the same question must still be asked, which of the two best meet and fulfil the necessities which Plato discovered in himself, and affirmed to exist in all men? For this was the object at which this philosophy proved itself to be aiming, as soon as it had acquired shape and consistency in the hands of the great disciple of Ammonius, Plotinus. This remarkable man saw clearly that Plato is continually pointing us to a ground out of and beyond ourselves. He felt, though with less clearness, for the habits of his mind were not scientific, that Plato has pointed out a method by which our thoughts may travel in the right direction towards that ground; but that more than this he has not done, and could not do. Plotinus perceived that the ascending series of human seekings must be met by a descending series of divine revelations or inspirations. While, therefore, it was the pursuit of the divine the ineffable unity, which was to be the object of all desires, and the satisfaction of all wants—while it was the privilege of the divine philosopher to enter into the immediate vision of Deity—all forms of mythology, the old Greek forms especially, but without prejudice to any that might be derived from the East, were to be pressed into his service. Necromancy, witchcraft, all methods of dealing with the unknown and the wonderful, were to be studied and resorted to by him. It is very true, nevertheless, that the philosopher did in a manner stand above all these things, fashioning and shaping them all into his great comprehensive theory; so that all the practical working religions of the world had existed for the sake of bringing forth a great metaphysical idea. That idea is of a primary essence, which is the One, a product of this which is the Reason, a third emanation from the reason, which is the Soul. It is an interesting question how far Plotinus could directly deduce this idea from the writings of Plato. In our sketch of that great man's speculations we made no formal allusion to his belief, or supposed belief, of a Trinity, such as that of which Plotinus spoke. We were much less anxious to set forth any particular results and conclusions at which he arrived, than to make our readers perceive by what steps he was led to acknowledge a Being in itself, a distinct utterance of that being implied in self-existence, and necessarily distinct from it, and some spiritual but most actual bond between that which is self-existent, and that which is uttered, if the idea of Unity is not to be sacrificed in the very effort of arriving at it. This principle, we say, is involved in all the scheme of Plato. He was led on by degrees into the scientific exposition and developement of the principle; but that which chiefly weighed upon his mind was a sense of its practical importance, as the only deliverance from the two opposing doctrines of an absolute being, which excluded

Moral and
Metaphy-
sical Phi-
losophy.

Ammonius
Saccas.

Plotinus.

The Tri-
nity of
Plotinus.

Moral and
Metaphy-
sical Phi-
losophy.

all knowledge or intuition, and of a perpetual flux and uncertainty in things. Hence it is in the applications of this truth to the controversies and sophisms of his time, for the purpose of proving that man may attain knowledge, and yet not have the standard of knowledge or truth in himself, that Plato's art chiefly consists, and that we become acquainted with his meaning. It followed that if Plato preserved his idea merely in the metaphysical form, it still retained the most substantive and practical value; and this made him the less reluctant so to leave it, and only now and then to intimate in some casual hint, or beautiful mythical story, that if it were a true metaphysical idea it must be more, that in some way or other there must be a point of union between it and that national theology with which he felt that all his speculations and discoveries did not enable him to dispense. The modesty and sincerity of Plato's mind are in nothing more apparent than in the two acknowledgments, which may be traced everywhere in his writings, that something more than philosophy is wanted, and that he himself had only a vocation to provide this.

Difference
between
Plato and
Plotinus.

But a time was come when the only test which could be given of the soundness of a metaphysical idea was its capacity of coalition and coincidence with moral and personal realities. Precisely the same necessity which drove Plotinus to incorporate old revelations or supposed revelations with his philosophy, drove him to seek a local habitation and a name for the principle which Plato had enunciated. In making the attempt, however, Plotinus forsook the footsteps of his honest master, and became an imitator of those parts of his writings in which he had most departed from his own principle. We have seen how in the *Timæus* the subtlest of European intellects fell into a kind of Asiatic vagueness and confusion. Plato, the cosmogonist and theogonist, is another man altogether from Plato the seeker of hidden truths in the facts which lay before him. And it is this side of Plato more than the other which is presented in the speculations of the new school. Mythology, therefore, which Plato received as a witness of something that must be, as the tradition of actual communications which had been lost or perverted, or as the attempt to supply a chasm, which nothing but some divine announcement could fully close up, was grossly adopted and defended with a passionate vehemence which he had never exhibited, and yet ever and anon explained into philosophical allegories in the very manner which he, in the *Phædrus*, had so evidently censured. Nor let any one hope to take away these phantastic portions of the Neoplatonic system, and yet to leave the rest of it consistent and entire. The school passed under the hands of many masters, all able and ingenious men, but this separation never could be effected. If any one thought of attempting it, he speedily discovered that the whole scheme must crumble, leaving nothing but those Platonic principles which demanded something from above to satisfy them, but did not pretend to supply it. It was a far more reasonable prayer of Plotinus, that the Emperor Gallienus would be so good as to establish a city on the Platonic model. The sage felt that he wanted something more than a compact theory, framed out of all the mythologies and philosophies of the earth—that some how or other his system, if it were a true one, ought to be able to embody itself in a living society. Doubtless the experiment was worth making; but the

Emperor, though exceedingly well disposed to all philosophical devices and inventions, declined the honour of undertaking it. When Julian determined to invest philosophy with the purple, the same fanatical zeal for the old gods still characterised him and all his teachers. It was felt to be even a more indispensable part of the system then than in the time of Plotinus; but he did not live long enough to establish a Platonic republic.

It would seem thus far, that the primary elements of the philosophy of this period were derived from Hebrew and Greek sources, and that any influence which may have been exerted upon it from any other quarter rather modified its outward shape than changed its essential character. We can easily understand how Indian theories about the creation of the world, Indian confusions between God and the world, may have mingled themselves with all the courses of thought to which this period gave birth; we can find no warrant in history or in reason for supposing that these theories and confusions gave the impulse or even the direction to any one of them. It is much easier to explain how the habits of thought which had for some time characterised the natives of Persia, should have acted strongly upon the minds of both Jews and Greeks. Accordingly, those who have treated of another leading feature of the philosophy of this period, its *Manichæism*, have had no hesitation in referring to the Persian doctrine of the conflict between a good and evil principle as the sole origin of it. We are very far from wishing to set at nought so general and long established a tradition as this; yet we think that the true nature of this important doctrine cannot be understood, unless we see how it was connected with inquiries which we have described as going on in regions subject to a set of influences entirely different from those which prevailed in Persia; and unless we consider what relation it bore to those controversies which we have spoken of as especially marking the temper of this age.

When we spoke of Socrates as the person who ascertained the meaning of those truths in practical life which previous Greek philosophy had been aiming to find, and which subsequent Greek philosophy was seeking to expound, we noticed especially, the way in which he was led to feel the existence of two principles within himself, one struggling upwards, the other gravitating towards the earth, and how all his moral discipline had reference to this discovery. That sense of a contradiction in things, which from the first had haunted the minds of Greek inquirers, was now transferred to its proper region; the observer's soul was felt to be, itself, the centre of this contradiction. But when Plato had pointed out the universality of those principles at which Socrates had arrived through personal experience, when he had shown how Being was the object after which the higher part in man is striving, and how it is mocked in its efforts by semblances and counterfeits, it became necessary for him to inquire how far, seeing that the true object has a reality wholly distinct from the mind of him who beholds it, a similar reality was to be presumed in that which is opposed to it. This question is discussed in a dialogue to which we very briefly alluded in a former page, called the *Sophist*. There, it may be remembered, this idea was gradually developed, that Not-Being must practically, and at whatever hazard of seeming logical contradiction, be contemplated as in some sense real. There is, perhaps, scarcely a more

Moral and
Metaphy-
sical Phi-
losophy.

Connected
with Greek
specula-
tions.

Moral and
Metaphy-
sical Phi-
losophy.

instructive or useful instance in the records of philosophical inquiry than this of the force by which an honest man, who follows a safe experimental course, is urged to conclusions which, to the mere intellect, seem most paradoxical, but which the conscience and reason affirm and recognise. Further than this, Plato seemed not to go; and, as usual, he did not translate his metaphysical idea with any definiteness and dogmatism into the language of morals and theology, though it was a moral feeling which had led him into it, and though he felt that, like every other absolute truth, it must in some way be connected with theology. It is a point not perhaps easily resolved, what relation there is between this part of his speculations and that which is presented to us in the *Phædo*. That he felt the body to be the prison-house and grave of the spirit, in the emancipation from which man's freedom and his full capacity of enjoying the vision of pure being begin, is undoubted. But that he therefore looked upon Matter as identical with that "Not-Being," of which he speaks in the other dialogue as, in fact, essentially evil, can by no means be inferred from his language. It was a subject on which he probably felt it unsafe to dogmatize, and on which, when he was viewing it from different sides, he may have expressed himself very differently. There is a much nearer approximation to this doctrine in Aristotle. As Being did not present itself to him in the same absolute way as to Plato, he was much less able to conceive any direct positive opposite to it. Form and matter were his two great contraries, and in the latter he certainly believes the capacity of evil to reside. Still he goes no further than to say, "without matter there could be no evil;" and, as we have remarked before, it is but in a modified sense that he attributes eternity to it.

How con-
nected
with Jew-
ish life.

The questions then with which Manichæism is conversant had presented themselves in every form to the Greek mind. The existence of two contending principles in man, those principles having each their counterpart and origin in that which is above man and distinct from him, and the connection of evil with matter,—these subjects had exercised and tormented the minds of men who had never heard of Arimanes and Oromasdes. That the *Jew* had been led to feel in himself, and to see in others, a mighty conflict, that he had felt all his good desires striving after a divine object, all his evil desires connected with enemies against whom he was to cry out for deliverance, it must be surely superfluous to remark. His whole life, personal and political, is the carrying on of this conflict, and the more his views expand, the more clear intuition he has of a perfect being and of a divine order, the more the sense of an opposing kingdom of disorder, and of an evil Will presiding over it, dawns upon him. If then we only suppose these two great lines of thought by some means to have been brought to a meeting point, so that neither could any longer be pursued separately, we should be able to account for questions about the absolute relations of good to evil, and about the way in which matter was connected with either, even if we had no help from Persian mythology or philosophy. But since we have admitted that they do enter into the history of Manichæism, and do in some measure define its nature, what place do we assign them? We answer, Manichæism is precisely the attempt to frame a theory about good and evil from the contemplation of the universe, without any practical experience of human conflict. Now this is the

characteristic which we remark in every Oriental theology but the Jewish—that it is exactly a speculation more or less ingenious, more or less suggested by actual recollections of a former revelation, or by the intuitions of men really aiming at truth; but unconnected with a human education, with a human life, not grounded upon a direct connection between God and man, but upon certain assumed relations of God with the world. The acknowledgment of two conflicting powers, both equally Creators, but one creating for good, the other for misery, is the natural and inevitable result of this method of thought; and this acknowledgment compounded with various conceptions, drawn sometimes from Judaism, sometimes from Greek speculations, sometimes from Christianity, about the different subordinate powers which these primary principles had called to their assistance, and about the final issue of the struggle, constitutes Manichæism.

We have thus noticed the principal topics which should engage the attention of one who is studying the philosophers of this period, endeavouring as much as possible to keep them distinct from the history of the Christian church. We must now as briefly as possible fill up our rude outline by noticing how at each one of these points it came into contact with thoughts and feelings which had been working from the most different directions.

And first, as we began with the revival of Stoicism, and the noble attempts which it made at a victory over accident and circumstance, over the fear of men and the dread of death, we may just allude to the practical life which was going on in the heart of the Roman empire, in the days of Seneca, Epictetus, and Antoninus. We are writing a sketch of philosophical inquiries, not a martyrology; but as these Stoics made it the test of a sound system that it should produce high actions, it cannot be out of place to remark that, by some means or other, men and women were enabled to evince a constancy in suffering, and a victory over ordinary inclinations and terrors, which these philosophers had scarcely dreamed of; that they were able to do this, not only without having the sense of being very great, heroic, and wonderful characters, but with a feeling of shame and humiliation, and thankfulness for a privilege to which they had no claim; that they were not obliged in order to compass this high standard of personal perfection to separate themselves from the rest of mankind, or to regard their welfare with no interest; but that, emphatically, they were members of a society, and looked upon their own acts as means of establishing that society, and spreading its influence over the putrid mass around them. And this we are bound to take notice of, not merely for its own sake, but because this high and transcendent form of morality was intimately and inseparably connected with the contemplation of an ideal of absolute goodness, and therefore throws as much light upon the forms of thought in this age, as upon its outward history.

For we noticed, secondly, a tendency in this time, which the writings of Plutarch especially develop, to look back at past models of greatness as some relief from the miserable spectacles of degeneracy by which the philosopher was surrounded. And we observed that this sympathy with past ages was nevertheless found to be inadequate; that some model, in whom the greatness and wisdom of past ages should reproduce themselves, one too in whom there should be something of mystery and divinity, must be found or created in that very time. The Christian church, taught by that very apostle who had been the great herald to the Gentiles that they

Moral and
Metaphy-
sical Phi-
losophy.

Christian
Stoicism.

Christian
antiqua-
rianism.

Moral and
Metaphy-
sical Phi-
losophy.

were not of a new but of an old stock, grafted into the privileges of the commonwealth of Israel, felt that all the heroes of the Jewish nation were of their own kindred, that they were set before them as examples of faith, struggling under all circumstances of prosperity and disaster; that they had not to take refuge in the contemplation of them from the sense of their own fall and decay, but could sympathize with them because they were called to exercise the same trust under still greater discouragements; lastly, that all these witnesses of the presence of God with men, and of his relation to them, had been leading up to one complete and consummate manifestation of him in the flesh of Man. If they had merely considered their position as heirs of Jewish privileges, or had merely been engaged in conflict with those who believed that the promises to the fathers had not been fulfilled, this fact of the incarnation would still have been the central one in all their thoughts and teachings. But very soon they discovered that it was this fact too which was their only resting point against Greek opposition, and the only satisfaction of Greek cravings. So deeply had the feeling of a divine humanity penetrated all the fables, as well as all the deeper speculations of Greece, so readily had it incorporated itself with those meditations which Philo owed to the Hebrew Scriptures, so strangely was it found to conspire with some of the rude and fantastic dreams respecting the creation, preservation, and destruction of the world, which had their home in other parts of the East, that when this fact was held forth by the Christians as one which they were commanded to promulgate, it naturally struck the philosophical Greeks as the new and grotesque form of an opinion which had been familiar to them from their infancy, though they never perhaps had so distinctly contemplated it before. Hence the obvious resource of setting up the theory or opinion of a divine humanity as the substitute for and contradiction of the assertion that the divine Word had taken flesh, and dwelt among men. The hint of this method of opposition having been once given, speedily a system began to shape itself out to which the religion of every region of the earth contributed its quota, a system which assumed the most enormous and contradictory forms, in which Judaism, Greek Pantheism, Hindooism were each by turns uppermost, which had no positive centre, but had for its negative centre the doctrine that Christ had *not* appeared in the flesh of man. This Gnosticism presenting itself to those who had never heard of philosophy, but had embraced Christianity as that which their consciences had need of, startled and staggered them; being presented to those who had a smattering of philosophy, destroyed their faith, and set them upon making the already heterogeneous compound still more shapeless by amalgamating with it some of the doctrines which they had learnt in the church. Philosophy thus appearing as the direct and formal antagonist of Christianity, it was natural that one class of the Christian fathers should turn away from it with some abhorrence, and should be kept in what was for them an useful ignorance of the fact, that in proclaiming the doctrine which was revealed to babes, they were also teaching the very one which sages were in need of. It was well that there should be such a class, for thus the fact was preserved in its simplicity, and it was manifested to the world that on this fact a polity was standing which all the polities of the world could not destroy, and for which all its theories could provide no substitute.

The incar-
nation the
centre of
Christian
faith and
teaching.

Gnosticism.

But it would have been equally strange if this had been the feeling of *all* those who were appointed to watch over the cradle of the infant church. It was surely to be expected that converts would enter it who had not merely found the fables of paganism incoherent and false, but who had been also through the sects of philosophy seeking rest, and finding none. Such men would surely have been doing great violence to all that their experience had taught them, would have been offending against real simplicity, while in the act of counterfeiting it, if they had passed by those metaphysical questions respecting Being and Unity which had occupied them in the schools, or had proclaimed that with these the truths which they had been taught at their Baptism had no connection. To have said so would, in their case, have been to have practised the grossest fraud on their Conscience and Reason. They felt that there *was* a connection, a very intimate connection which revealed itself more clearly every day, to those especially whose position at Alexandria brought them into contact with the speculations of Plotinus, Porphyry, and the new Platonic school. How could men, who were taught to declare that communion with Him whom they worshipped was the highest and most glorious privilege of the creature, turn away with indifference when they heard this very doctrine proclaimed in the schools of philosophy? How, when they heard these philosophers recurring to the ancient mysteries as witnesses of this privilege and means of obtaining it, could they forget that they had mysteries intrusted to them of which this was the meaning and the end? How, when they heard them speaking of a Trinity as not inconsistent with, but the very condition of an ineffable unity, could they turn away and say, "There is *no* relation between these thoughts and that awful name into which we are baptized?" Such language may sound simple to modern ears; in these Fathers it would have been the excess of affectation. Surely it was greater homage to their faith to say, "This communion which you philosophers talk of as meant for some of the sages of the earth, we declare is accessible to the poorest peasant, because it has pleased God to establish it in the person of one who was himself poor and suffering: these mysteries whereby we admit men to it, are indeed high and dreadful, but dreadful from their very simplicity; the difference between them and those to which you have recourse being precisely that which Plato pointed out between the darkness of the Sophist and the darkness of the pure philosopher, one dark from the clouds which he gathers round him to conceal the light, the other dark from the excess of light in which he dwells. Lastly, this Trinity, in which we too alone behold the perfect unity, is not with us a notion, not the description of certain qualities or attributes, but the name of a Living Being presiding over the cradle of every child as well as the final object of the faith, and knowledge, and love of every perfect man; nor this only, but also the foundation of a kingdom which is daily expanding its borders, despite of all resistance from the powers or the wisdom of the earth."

This last was a practical argument which did, no doubt, affect the minds of the Neoplatonists more than all that Clemens or that Origen could say to them. When they saw that a universal republic could not be set up on this theory of unity, but that a universal republic had been set upon the foundation of that unity which was proclaimed to the world by fishermen, when they saw that the patronage of Emperors had not availed to

Moral and
Metaphy-
sical Phi-
losophy.

Christian
Platonism.

The Chris-
tian asser-
tion of a
Trinity.

Moral and
Metaphy-
sical Phi-
losophy.

Athana-
sius.

keep on foot the old religion even in alliance with the new philosophy, but that Emperors were compelled to do homage to this unintelligible faith and fellowship, and to exalt the cross above the eagle, they must have felt at times that more than human logic and eloquence were contending with them. And soon they had to see the two systems brought to another and still more striking test. It might be hard to conjecture how the Christian church, when acknowledged as a power in the empire, should be able to furnish the same evidence of its stability which it had given in its days of weakness and obloquy. But such an occasion arose. A single man was called upon to maintain that to be a substantial truth, which Plotinus and his school recognised as a metaphysical idea. Against priests, against rulers, against apostatizing friends and violent enemies, against meek men who wondered that so much pains should be necessary to assert a mere dogma, against dogmatists who really cared for nothing but the logical form and therefore perverted that, did Athanasius maintain in synods, in palaces, in dens and caves of the earth, the principle upon which the Christian church rests. Strange spectacle for all men to witness! strange above all for those Platonic philosophers who could talk indeed of their Trinity, but who never dreamed that mankind could be interested in the assertion of it, or that they were to make any sacrifices for mankind. The warrior lived on through the days of Julian, lived to see the utmost done that could be done for Platonized paganism, and lived to see the principle, for which he had fought at such mighty disadvantage, recognised and triumphant.

The Chris-
tian doc-
trine of a
good and
evil prin-
ciple.

Augustine.

The relations of Christianity to Manichæism were to be brought out by a fighter of another kind than this, but not a less brave or desperate one. The Christian Fathers, who had hitherto been in contact with philosophy, were, as might be expected, the Greek—especially those of the Alexandrian school. The early Latin Fathers, seeing the speculations of Greece chiefly as they were expressed by Romans, or embodied in Roman practice, would be likely to deal with their outward effects rather than very carefully to investigate their nature; to attack the outward forms of Paganism rather than to inquire in what way the thoughts connected with those forms might have been preparing the way for Christianity. But when the Western Empire was exhibiting every day more certain symptoms of approaching dissolution, the Western Church began to assume a higher position, to understand more clearly its dignity, and to feel some deeper sense of its vocation as the power which was to establish a new form of society among the European nations. Augustine is the great representative of this feeling. It could not be a question with this great man whether he ought or ought not to take an interest in philosophical inquiries. Nothing less than an impious disregard of all the processes of his education, and a determination not to use the peculiar powers with which he had been gifted for the good of mankind, could have led him to neglect the subjects with which other men were occupied as not belonging to the province of a Christian theologian. The earliest direction of his thoughts, after he began to think earnestly and steadily, had been given by the Manichæans, and his personal experience wonderfully fitted him to enter into the very heart of their inquiries, and to understand that livingly, about which they theorized. Accordingly, various as were the questions which exercised him both after and before he became a

bishop, and large as was the sphere of his practical duties as well as of his thoughts, we may easily perceive that the question as to the nature of evil, its relation to matter, to man, and to God, was the central one in his mind from the very beginning of his life to the close of it. It has happened, unfortunately, that the controversy with Pelagius is that which, in modern times at least, has been regarded as his peculiar work, and that too little endeavour has been made to connect the writings which bear upon it with those which belong to the earlier periods of his history. From this partial conception of his character, some mistakes have originated; he has even been supposed to have darkened or denied some of the very principles which it was given him most clearly to perceive and elucidate. One might fairly say, that the Socratic doctrine of a living spirit in man, against which his nature is in constant conflict—a spirit striving upwards after the perfect good, as that nature is always dragging it down into an abyss of evil—of which no account can be given except that it is the contradiction of all good all order all unity—was the most prominent in his mind. This drove him to Christianity, because it alone taught him how He, who is the perfect good, has met the struggling spirit, has redeemed it out of its bondage, has endued it with strength to resist and to triumph. To bring out this principle in clearness and power, and to use it against Manichæism, by showing that the evil principle cannot be an original principle—cannot be a creator, but can only be contemplated as apostate and destructive, was his first great object. Next it was his aim to show how, in the personal conflict wherein man is engaged, the subjugation of the animal nature is necessary in order to the victory and deliverance of that spirit which is looking upwards, and how the degradation of the spirit to that animal nature has brought all corruption and degradation into it; and yet how different that degradation is from the essential pravity which the Manichæans attributed to matter; how entirely it has been the result of the conjunction of matter with a fallen will; how reasonable it is to believe that the restoration of that will is the precursor of a bodily deliverance. These great principles, which he had been all his life illustrating, were, in fact, the very truths which enabled him to resist Pelagius when he confounded the relations of nature and spirit, and supposed the possibility of some good in the creature, viewed in separation from the Creator. But they had, no doubt, to be presented in a somewhat different form from that which they had taken in the controversy with the Manichæans. That side of his faith, never absent at any time from the mind of Augustine, which concerned the dealings of the Creator with his creature, and the derivation of all good from Him, had now to be put forth with a prominence which the eagerness of controversy might easily make exclusive, nay, which might even lead Augustine himself to look back with fear on some of his former statements respecting the spirit in man, and its longing after the perfect good. These, nevertheless, just as truly as the other, expressed his real heart; and, without them, the other, even as confutations of the Pelagian doctrine, are practically ineffective.

We have shown then, we trust, that Manichæism, as well as Neoplatonism, the antiquarian philosophy of Plutarch, and the exalted Stoicism of Epictetus, had their counterparts, and their only satisfactory interpretation and realization in the Christian church. We have

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

Charges
against the
first Chris-
tian age.

False theo-
logy.

False mo-
rality.

no greater wish than that our statements should be thoroughly sifted, and that they should awaken the reader to pursue a course of study which will, if they are false, expose and refute them. Unquestionably they are not those which he will receive from the majority of modern writers who have treated upon the subject. Either he will be told that the Christian Fathers most wickedly and mischievously polluted the pure wells of Christian truth by mixing with them the fables of Platonism; or else that they followed an ideal perfection to the neglect of practical morality; or, lastly, that they made philosophy merely a branch of theology, and thus as well deprived it of its scientific form as took away all the energy and freedom of the human spirit in pursuing it. With respect to the first of these charges, we earnestly request our readers to consider whether what are called the fables of Platonism were or were not the questionings of the human spirit respecting the most awful subjects which can occupy it; respecting its own constitution; respecting the object for which it exists; respecting that which is, and which is to abide, and that which merely appears and is to pass away. What avails it that some men, busied in their books or in their merchandise, shut their eyes and say, there are no such questions; that they mean nothing; that only fools would dream of them? History declares that there are such questions; that they have tormented the minds, not of fools, but of the most sagacious, considerate, and, in the true sense, practical men that the world has ever seen. To set at nought this evidence is not to say we do not care about philosophy; it is to say we do not care about facts. But if these be real and not mock inquiries, they then refer to subjects respecting which a revelation should give information, or to subjects on which it may be expected to be silent? If to those on which it should be silent, then let us know on what subjects it ought to speak; if it has actually spoken upon them, then let us know how the stewards of this revelation could be simpler or wiser men for not setting forth what it has said. The accusation, in fact, proceeds on an entirely false assumption. It pretends that the early Fathers did not think that the revelation which God had given was adequate to the wants of his children, and, therefore, that they sought to help it out with human philosophy. The truth is, that they were putting contempt on human pride by declaring that the highest mysteries it had ever dreamed of were contained in the creed which they taught to their Catechumens.

The second complaint of the ancient Christians that they set up an heroic morality to the exclusion of the homely and practical morality, we have already answered by anticipation. We have admitted that it was not the work of this age expressly to inculcate the relative duties; that, from first to last, the most evident indications of Providence marked out for it a different (we do not say a higher) vocation. But unless it can be shown that those relative duties have another safer resting-place than that of those absolute principles which were in this period brought to light and practically exhibited, we shall be curious to know what proof of reverence for such duties these persons furnish who are impatient of any moment which the divine wisdom thought fit to occupy in preparing the ground for them. We are fully convinced that the history of a subsequent age, of which we shall have to speak by and by, furnishes the truest answer to these heartless murmurings, and teaches us

how long relative morality is likely to last when it looks up to nothing higher than itself.

But if neither theology nor morality suffered in the hands of the early teachers of the church, must we not at least concede some force to the third objection, that they dealt unkindly with philosophy? This question, like the two former, we believe, is best answered by history. We acknowledge that a scientific form of thought and a clear apprehension of our distinct faculties were wanting to this age, as readily as we acknowledge that a direct formal acknowledgment of the dignity and importance of human relationships was wanting to it. We acknowledge further that it was characterised by an ignorance of the laws of nature as great as that which existed among the earlier Greeks. Once more, we are firmly persuaded that scientific forms of thought, a knowledge of our own faculties, and an acquaintance with the principles of physics, are intended for man. But if we are further convinced that no acquaintance with the conditions of science, with our own faculties, with the outward world, could have been attained unless some satisfaction had first been provided for those questions which belong to the sphere of real knowledge—those questions respecting absolute permanent truth, which the Platonic philosophy raised, and which were exercising the minds of all men, however different their education, in that day—we are not, we think, testifying any indifference to that wherein this age was weak, when we say that it did not injure philosophy but sustained it, by its theological decrees and determinations. And if we are further convinced that when any one of the after ages has tried to undo or do again that which this age accomplished, its own work was lazily performed, or not performed at all, we are not, surely, looking spitefully upon the progress of the human species, but seeking to promote it, when we ask men to turn with reverence to the past, and to consider whether it is not a wiser and safer course to be thankful for the deposits which it has bequeathed us, than to be slandering it because it has not performed all tasks, and so left us to utter idleness.

It was in the reign of Justinian that the schools of Greek philosophy were finally suppressed. Many circumstances mark this out as a very important crisis in the history of the world, indicating wherein the period which was afterwards to commence would differ from that of which we have been speaking. A Greek emperor obtaining dominion, through the conquests of his generals, over Italy might seem to promise that the empire was once more to be consolidated, but that it would no longer have a Roman character. On the contrary, this very Emperor becomes the silencer of that philosophy which had been the distinguishing mark of Greek life for above a thousand years; and arranges, sanctions, and stamps with imperial authority, that law which had been as characteristically the badge of Roman greatness for about the same period. The effects of these proceedings might not be felt for a long time: the first was not in itself very important, for the schools of philosophy had worn themselves out, otherwise, no Justinian could have destroyed them. But both were significant, and, speedily, other more important events interpreted their meaning.

A right understanding of Mahometanism, and of the circumstances which assisted its propagation in some countries and retarded it in others, would be a great help in comparing this period with the one that follows

Moral and
Metaphy-
sical Phi-
losophy

False phi-
losophy.

Age of
Justinian.

Mahomet.

Moral and
Metaphy-
sical Phi-
losophy.

it. We observed that a tendency towards the contemplation of that which is divine and absolute, though in a certain sense characteristic of all men in this time, whether they were heathens or Christians, was very much more conspicuous in the Greek Fathers than in the Latin. The latter dwelt upon the fact of the Incarnation, and in doing so were led inevitably to teach that a transcendent kind of morality, not determined by formal precepts or by the ordinary social bonds but grounded upon assimilation to the character of God, had now become possible for men. Still it was with this fact as a part of divine history, and with the moral results from it, that they occupied themselves. On the other hand, it was the inclination of a very important section at least of the Greek church to dwell upon the consequences of this same fact as opening a vision and intuition of the Infinite and Eternal, as lifting men actually into that region which philosophers had dreamed of. Thus these two classes of men, starting from the same point, were drawn, one more to the human, one to the celestial side of the truth which they held in common. In the writings of each might be traced certain characteristic infirmities to which these respective habits of thought made them liable, and what were suppressed or only broke forth at intervals in the leaders and guides, were likely to become the prominent temptations of the bodies which they represented. In the Greek teacher might be traced little more than that impatience of common facts, and that dread of every thing anthropomorphic, which we observed in Philo. Among the people this inclination would easily give birth and countenance to heresies, of which the object should be wholly to set aside or to reduce into the merest vapour the idea of an actual divine humanity. However successfully these might be repressed by the arguments of doctors or the decrees of councils, the spirit of them had unquestionably entered deeply into the Greek mind and character. Now, seeing the morality of churchmen expressly depended upon the sense of a direct connection between themselves and their Creator; seeing that the other helps and appliances which later ages derive from a recognised standard of manners and a fixed form of Christian society were then wanting, one may easily conceive the instability and corruption under which the eastern Christian nations must have suffered when this faith was loosened or withdrawn. One can easily conceive how a form of faith, which asserted in the strongest terms the absoluteness of God; which utterly took away the idea of an incarnation; which furnished a code of precepts all plausible, many of them excellent; which came forth supported by a man really convinced of his own inspiration, at the head of a brave people who then for the first time felt themselves more than a horde—felt that they had really a principle to assert—must have made way among men who had lost their own faith already through an exclusive encouragement of that very habit of mind of which Islamism was the expression. It is equally manifest that those particular weaknesses which had appeared in the Latin church, weaknesses which laid its members open to the impostures of those who suggested a more rigid rule of life or who invented some human contrivance for encouraging devotion, did not make them liable to this particular infection. It is easy to say that they grovelled on the earth, and therefore were not liable to be misled by any hope of being raised to heaven. But such language would

Character
istics of the
Greek and
Latin
churches.

not be very appropriate or true. Augustine's contemplations were as lofty as those of any Greek. He as much aspired after communion with the perfect good; he even entered as deeply into philosophical speculations; and yet in him, as much as in any of the earlier Fathers, the Latin character is visible. His philosophy bears upon a great practical subject. It may ascend into the greatest heights, but it starts from the lowest ground, and is occupied with personal struggles. Another point too respecting him is worthy of note. He deals more than almost any one with individual life and experience. In some instances he may seem, to many, to have too much individualized his language so as to affirm that of particular persons which can only be affirmed of them as members of a society. And yet with this natural inclination he was brought into circumstances, especially in Africa among the Donatists, which compelled him more than most writers to enter into men's position as members of a fellowship and subjects of an ecclesiastical government; and the most elaborate of all his books is that on *The City of God*. While then the eastern nations, in their exclusive pursuit of that which is divine and transcendent, fell into Mahometanism, or hovered for some centuries on the verge of it, we may expect to find the Latin nations exposed to another class of very serious dangers from their too human propensities. But we may expect also to find, that whereas the one remained in a hopeless stationariness—even those who retained their Christian name acquiring a dull stupified character—the Latin people would shape out a course for themselves in many respects strange and new, but yet naturally arising out of that which the former age had traced and finally coinciding with it; that whereas Islamism, by excluding the belief of a direct intercourse between earth and heaven, lost the idea of an order and government of the world and necessarily became allied with despotism, the Christendom which rose up to oppose it would develop the ideas of Order, Constitution, and Law; and that by these the character of many other thoughts and speculations would be determined.

Moral and
Metaphy-
sical Phi-
losophy.

From Charlemagne to the Reformation.

In former portions of this *Encyclopædia* there have been elaborate discussions on the fortunes and writings of those who have acquired the name of the *Scholastic philosophers*. Even, therefore, if the limits of this sketch did not positively preclude such an attempt, we should have no excuse for entering at any length upon this subject. But we must carefully guard against the notion, that the space which we assign to this portion of philosophical history is any measure of the importance which we attach to it. Far from adopting an opinion, which was prevalent at least till very recently, that the questions which occupied the schools were trivial, senseless, and now wholly obsolete, we think it is difficult to overrate their intrinsic value, or the influence which they are exercising upon ourselves at the present moment. The persons who use the words Ontology or Nominalism and Realism with a sneer, little know how much those difficulties of which Ontology treats are besetting their own path; with what vehemence the controversy between Nominalism and Realism is carried on in their own minds and in the minds of all about them. We do not gain much by speaking contemptuously of our progenitors; we only contrive that we should suffer all the perplexities which they suffered

Scholastic
philosophy.

Moral and
Metaphy-
sical Phi-
losophy.

without the same consciousness of them which they had, and without their help in extricating ourselves from them. The mistake has been owing, we fancy, in a great measure, to a confused apprehension that the schools and the world have in all times, and had at this time especially, very little to do with each other. The fashion of scorning the active life of the middle ages is passing away; nay, is just at present giving place to a sentimental admiration. Men have discovered that something was done in this so-called dark time which we in our bright time could not well dispense with. But unless the speculative life of that period, besides obtaining the courteous treatment which it is likely to meet with under such a reaction, be viewed in connection with this practical life and shown to be inseparable from it, there is no chance, we think, of either being dealt with clearly and justly. A history which should do this would far more effectually expose the real evils of the middle ages and show whence those evils flowed, than all vehement party declamations against them which, being written without sympathy for the right, are very seldom successful in detecting the wrong. A very few hints as to some points which such a history might take notice of are all that we can furnish by way of comment upon these seven remarkable centuries.

When the
scholastic
age begins.

It has been a frequent subject of debate, whether the age of the schoolmen commences with the reign of Charlemagne, or not till rather more than two centuries later. Two remarkable men, it is admitted, arose within that period, to whom no thoughtful person would deny the name of philosophers—John the Irishman (Johannes Erigena) and Anselm; but it has been thought by many that there are characteristics in their writings which entirely distinguish them from the later schoolmen. This point, in which much of the character of this philosophy is of course involved, might, we think, be settled much more easily if the general history of the period between Charlemagne and Hildebrand were brought to bear upon it. It strikes one even at first sight that we are now contemplating quite a different world from the one which we were viewing, either in the time before or after the promulgation of Christianity. A race of men starting into social existence—these men coming under laws, submitting themselves to a new faith, beginning to understand what they were sent into the world for—how different a picture is this from that of the decaying Empire! Yet if we look again we are struck at least as much with the sense of continuance and permanence as with a sense of change. When we consider the materials out of which the new system was moulded, we find old habits of thought—traditions that had been dwelling for an unknown period in the woods of Germany—habits of subordination and temperance—a reverence for marriage—a military discipline—a worship of unseen powers with the strange institution of a week to commemorate them. If we look again at the power which was to work upon these materials, still every thing is ancient and venerable. Old Roman laws, Roman colonial government, the Roman sense of equity and obligation, the Roman language—these were some of the influences which were shaping the western chaos into a universe. And over all these, using them and subordinating them to itself, was a faith, which for eight centuries had existed in its present form, and was only the outgrowth and consummation of one which had been held by the Patriarchs in Mesopotamia. Besides these points of connection with the foregone time, there was a

VOL. II.

peculiar resemblance between these two first centuries and the kingdom which had been broken in pieces. A new western empire brought back many of the feelings which belonged to the military government of Rome. There was not yet merely a king of each nation, and an ecclesiastical head of Christendom. The king of one of the nations aspired to be the head of all, and to a great extent made good his claim.

Moral and
Metaphy-
sical Phi-
losophy.

It is commonly believed that Gregory the Great, the tamer of the Lombards, the establisher of a paternal government in Italy, the converter of the Saxons, was not a friend to letters. The charge has been disputed, and is, perhaps, much exaggerated. But were it true to the greatest extent, it ought not to detract seriously from the fame of this admirable man; it ought only to be taken as an indication of the circumstances attending the rise of society in the west, which it is important to reflect upon and to compare with others of a different kind. If, for instance, we contrast Gregory's alleged indifference to mental cultivation with the zeal for it displayed so short a time after by Charlemagne and Alfred, men certainly not more anxious than he to soften and regulate the characters of their subjects, we may gain much instruction respecting the feelings of this period.

Feelings of
Popes and
monarchs
respect-
ing the
methods of
improving
society.

Our first conclusion might be, that the ecclesiastic, as such, was the enemy of cultivation; the national monarch, as such, the friend of it. But this notion is refuted by all the subsequent history; nay, that history might even tempt us to assert the converse of such a proposition. The cause of the difference lies deeper. Gregory and Charlemagne alike felt that their subjects needed cultivation. They alike felt that this cultivation must be an ecclesiastical one; that the teachers of the nation must be those who had powers to act upon its mind. But the prelate thought that in the work of civilization the great instruments were those permanent truths which were delivered to men from heaven; the law-giver that men must be trained to understand themselves in order that they might know the necessity and blessing of obedience. These feelings never expressed themselves in such formal language as this; they might not be consciously present to the minds of either; but they were not the less real, and they found their utterance in acts. Gregory sent missionaries; Charlemagne founded schools. One sought to bring all nations into a universal society; the other laid a foundation for the cultivation of each nation.

To find out what relations existed between this universal society and the different national bodies which composed it was, it will be easily admitted, the great political conflict of the middle ages. We believe that the spiritual conflict was one closely connected with this. That there are certain longings and aspirations in our minds answering to the truths upon which the universal society rests, we have shown from the history of the Platonic philosophy through all its stages. But there is another state of mind which is connected with the sense of our being placed together under one order and government, of our being united in a distinct body under laws distinct from those by which surrounding nations are governed. Then comes the awakening of a sense of Personality; of that which is properly our Conscience; and, flowing out of this, a gradually growing Consciousness of certain forms and laws, to which our own understandings are subjected; and, later still, of certain Conceptions within us which must struggle onwards to birth. These are the particular character-

Political
and scho-
lastic ques-
tions in the
middle
ages.

Moral and
Metaphy-
sical Phi-
losophy.

istics which have accompanied the formation of every national society, or which may rather be said to be the formation of it. The problem, then, which all men were in one way or other seeking to solve in themselves during the eight centuries which we commonly call the middle ages, was what this sense of personality means, and how it was connected with or might be reconciled with those deep and transcendent truths which referred to the being and unity of God and his revelation of himself to man. And the specific inquiry into which the schools were led during the same period had respect to the forms of our understanding; how they are connected with or may be reconciled with those absolute truths, which seem to surpass our understandings, and yet, in some sense, to be meant for them.

The IXth
Century.

Transub-
stantiation.

Johannes
Erigena.

Berenga-
rius.

The IXth Century was a prophecy of all these conflicts. One great controversy which then came into existence may be said to contain and concentrate them all. Paschasius proclaimed the doctrine of transubstantiation. That sacrament which, in former ages, had been the witness and the embodying of a transcendent relation between men and their Creator, which by its very nature overreached the forms of human thought and expression, which was the perpetual declaration to men, that there is something that words cannot convey, was by this monk brought within the limits of a human conception. The high language of the Fathers, language intended to indicate the glory of a mystery which the understanding could not comprehend, was used to defend this dogma of the understanding; men were required to reduce their faith under the laws of their understandings, and then to call the acknowledgment of a transgression of those laws by the name of Faith. Assuredly the whole puzzle and contradiction of the time lay here. The germ of all political, personal, and scholastic confusions must have been in the century which produced Paschasius. Still it was only the germ. *Johannes Erigena*, the friend of Alfred, was invited by Charles the Bald to confute Paschasius, and he entered upon the task without fear, and for a long time without reproach. This great and accomplished man seems to occupy a position between the two ages of the world, to be the porch or vestibule of the schools, and yet to possess far more the spirit of the Fathers. He aspires to divide and arrange nature, to find out names for things; so far he is a schoolman. But his thoughts are Platonic—God is the beginning and end of all things. If, as has been thought, there is in his writings a latent tendency (of which he himself may have been unconscious) towards Pantheism, this may be an indication that the time for such large speculations was at an end, and that Johannes, though he escaped the errors of the time on which he was entering, was in some danger of those which belonged more properly to his predecessors.

But soon the character of the time became more determinate. The doctrine of Johannes was pronounced heretical; that of Paschasius was confirmed by Pontifical authority. The interesting and melancholy story of Berengarius, of his bold proclamations and defiances, of his retractations and humiliations, marks the new stage of feeling. The Pope did but utter forth the feeling of the time. His decree did but show that a dogmatical spirit had taken possession of the church, and that the perception of truths which could not be included in dogmas was daily becoming more difficult. But though difficult it was not impossible; and in this case, as in all other cases, it was through men who were brought into

contact with actual life, and were taught that philosophy either means nothing, or shows the way to truths which are good for others besides philosophers, that the tone of thought was raised. The struggles of high-minded priests in particular countries, in ours especially, against the tyranny of soldier monarchs, brought forth a sense of reality in their minds which raised them practically above the dogmatism of Paschasius, though they might not like Berengarius set themselves formally in opposition to it. *Anselm* was a man of this kind. His scholastic life was sustained by a political life, aiming at the same high ends, and calling for even greater sacrifices. Still more happily there were in his heart the deepest reverence and humility. His works, therefore, had all a real object, and turned upon questions which were occupying the church at large, at least as much as the schools; and they aim at distinguishing much more than at dogmatizing. Yet in them the scholastic habit discovers itself most clearly. Not to substitute understanding for faith, but at least to show how the understanding recognises that which faith receives, is his avowed object everywhere. Some of the great perplexities of the understanding, some of the logical meshes in which it afterwards became involved, he was ignorant of. But what he knew he sought to remove, perceiving with remarkable clearness that a different office was committed to the men of his age from that which had been committed to the Fathers, and yet that the two duties were harmonious, and that reverence for the past was a great help in fulfilling the work of the present.

It is generally understood that *Gregory VII.* felt very differently from his predecessors respecting the Berengarian controversy. It would seem that he not only loved Berenger as a man, but had a great sympathy with his opinions, and an honest dread of fixing the living words of Scripture into the dogma of transubstantiation. Such feelings would be perfectly consistent with the mind and character of this singular man. He evidently felt that the Bishop of Rome had a higher vocation than to dogmatize; that he was to assert the position of the church as an universal kingdom—real as any of the kingdoms of the earth, the only protectress both of invisible truths and of poor men's rights from royal and military ignorance and oppression. He was naturally disposed therefore to be impatient of those who would make him a mere censurer of opinions, and to have a respect for men of free and lofty courage like his own, who would break through logical fetters, and assert principles in their breadth and simplicity. Nevertheless it was the grievous error of this prelate, that he saw no way of claiming that position for the church which he felt was its right, except by inventing a title and dignity for himself which did not belong to him, and could only be maintained by attacking the rights and outraging the conscience of every national monarch in Europe. This was his falsehood, one which the sin of every previous bishop of Rome who had been asserting an unlawful pre-eminence over other bishops, or who had forgotten his glory as a servant in aspiring to be a master, helped to foster and sanctify; a falsehood closely connected in Hildebrand with high conceptions and great virtues, but which, being a falsehood, necessarily brought after it a long train of miseries. That which we have to speak of, was the growth and confirmation of that confusion of dogmas or decrees with permanent truths, which, as we believe, Gregory

Moral and
Metaphy-
sical Phi-
losophy.

Anselm.

The XIth
Century.

Moral and
Metaphy-
sical Phi-
losophy.

desired to discourage. The Pope having assumed the functions of an ordinary king or lawgiver, it inevitably came to pass that his laws were looked upon as the same in kind with those of any other monarch. The sense of a distinction between the great principles of which the church was witness, and the mere decrees which are enforced by civil authority, a distinction weakly upheld by former Popes, was now every day more and more obliterated; and hence confusions infinite and seemingly inextricable—in the World between the relations of the national monarch, and of him who assumed to be the universal monarch—in men's Hearts between the unchangeable laws of morality, and the different means which were suggested to enforce them—in the Schools between the realities of things and the names by which those things are generalized in our understanding. *Διός δ' ἐτελέετο βούλη;* through all these perplexities, clear distinctions were working themselves out; it was needful that men should learn in this way the nature of their political position, the grounds of personal righteousness, the forms and limitations of their own minds.

Realism
and No-
minalism
how con-
nected
with Plato
and Aris-
totle.

In noticing the philosophies of Plato and Aristotle, we have already distinguished between a science of realities and a science of names. We maintained that both were necessary; that Aristotle had no right to deny the principles of Plato which related to the one, that Platonists were not bound to deny the worth of Aristotle's positions which had reference to the other. At the same time we guarded against the notion that the Academic and Peripatetic doctrines could be reconciled, as Cicero tried to reconcile them, by saying that the difference was in words merely. We contended that the difference was an essential difference; that the minds of the two great teachers were of a directly opposite character; that whenever they attempted to deal with the same questions, they inevitably came into collision; that it was only by assigning to them distinct provinces, that we could recognise a sound meaning in each. Looking upon them as the originators of two great methods of thought, we deemed it needful to explain with greater minuteness than may seem suitable to such a sketch as this, what these two sciences are; showing how the science of realities developed itself in the different Dialogues of Plato, but most completely and consummately in the *Republic*; how the nominal science developed itself in the logical treatises of Aristotle, but most perfectly in his *Metaphysics*. We promised our readers that if they would be at the pains of following us in this portion of our discourse, they would be saved many troublesome and embarrassing explanations when they reached the portion of it at which we are now arrived. We have not therefore now to tell them what we mean by the words Nominalism and Realism. We have only briefly to explain how the sense of the opposition between them began to be felt in the middle ages. A translation of the *Organon* of Aristotle had been in the western schools since the time of Charlemagne. Such a book, so clear, coherent, and masterly, might well produce an effect on any age, by however much of other literature it was surrounded. But suppose it introduced at a period when men were drawn by various impulses such as we have described to turn their thoughts inwards upon the operations of their own minds—into nations possessing indeed some of the treasures of old Latin literature, but not having yet a history of their own to interpret them by, possessing also the inspired volume, but partly through

the same cause, partly through the dogmatic spirit which had taken hold of the rulers of the church, forced to look at it rather as a collection of sacred and important opinions, than as the free record of a divine revelation, and of a national life—and one can easily conceive how immensely precious *this* deposit must have seemed, how completely a gift of heaven, by which all other possessions were to be measured and appreciated. Such a sudden discovery to men of the modes and conditions of their own minds, such a strange revelation of the very thing which they were seeking to know! We have seen, however, how strong the old Platonic feeling, which had descended from the first period of the church, still was. The ideas of Being and Unity, the grand ideas of the Platonic philosophy, had become inseparably intertwined with the Christian mysteries. For, as we have observed, these ideas demanded some living personal mysteries to sustain them, and those which Plotinus and his school devised had been found inefficient and worthless. In teaching, therefore, the new science of Logic, the science which they learnt from the *Organon*, the school doctors still applied the principles of Plato. To these great universal ideas of Being and Unity, they attributed reality. Quite ignorant of the Platonic method, quite ignorant of the battles Plato had had to fight with Parmenides and his school, in order that the name or notion of being might not be substituted for Being itself, they taught it as a part of the science of logic, *the science of names and notions*, that universals have a life and reality of their own. Against this notion, Roscelin of Compiègne was apparently the first to protest. He had read an introduction to the *Organon* by Porphyry, in which the difference between the Peripatetics and Academics upon this subject of universals was pointed out. The truth of the Aristotelian doctrine came strongly home to his mind, and he began to assert strongly that Being and Unity are only the Attributes of substances, not Substances in themselves. But the ideas which these words represented had become united with the transcendent Truths of the Christian faith. It was a very difficult thing for any man so to sever them, that the denial of the one should not even in his own mind lead on to the denial of the other. It is very true that the logicians on the other side, just in so far forth as they were mere logicians, *implicitly* denied these realities. Still the *explicit* denial of them, or such a mode of expression respecting them as seemed to rob them of all meaning and substance, was reserved for the Nominalist. Happily Roscelin had a better opponent than any mere school doctor: Anselm, who, we have seen, had been bred in another age, and represented a different class of realists from the realists of the schools, undertook the controversy. But with the caution and charity befitting so good and great a man, while he replies to those statements respecting the Trinity which he had heard attributed to Roscelin, he carefully suggests a doubt whether this was actually his meaning; he fully allows the possibility of his having been betrayed into unfitting and inconsistent language, without really having a wrong intention; and he is only careful to clear from the confusions of philosophers a principle which he believes needful for every humble man.

It has been a subject of considerable controversy, Abelard, whether the celebrated successor of Roscelin in the Nominalist school, *Abelard*, was justly exposed to the same imputation of heresy with him. Without venturing to express any opinion upon this point, we may be

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

Bernard.

Aristotle
proscribed.

Hugo de
St. Victore.

Peter the
Lombard.

The XIIIth
Century.

Its charac-
teristics
those of
Aristotle.

permitted to warn our readers against being misled by hasty and dogmatical assertions on either side of the question. A thorough study of Abelard's philosophy (from which it would be by no means right to separate the romance of his life) would no doubt contribute very much to an illustration of all the topics connected with this memorable and interesting controversy. But vehement phrases, either imputing an intention to him of entering upon the theological ground, when he may have meant, if it had been possible, only to assert a logical position, or on the other hand, railing against those who opposed him as men of a savage turn of mind, enemies of freedom of thought, bigots, and zealots, can do little good, except to perplex our view of the history, and make the steady contemplation of its different aspects impossible. It is absurd to say that St. Bernard, the great antagonist of Abelard, excelled him only in his declamatory power, and in his orthodoxy. He was superior to him in every quality of heart and head, except merely in the logical faculty. He was by constitution, by principle, by habit, a Realist in the practical sense of the word—not skilled perhaps in the language of the schools, but what is better, able to sweep it away where it set itself in opposition to any great truth which belongs to mankind, or to any effort for its good. And it was this realist spirit, the spirit which prompted the Crusades as the outward act of the united church, and all zealous individual labours after a higher standard of life and excellence, which really held back the nominalism of the schools, and caused Aristotle for a time to be a proscribed name. The Popes saw that the *Organum* had produced a result which at least looked like heresy; and they had themselves abetted the confusion which made it difficult for men to distinguish, even in their own minds, between the denial of a substantial verity and of an imaginary abstraction. Still the spirit of inquiry continued. Some maintained the logical ground, following in the train of Roscelin or Abelard, or of their opponents. Some, weary of these conflicts, ascended to a higher level, and strengthening the piety and wisdom of Anselm with the dialectical skill which had been acquired in the late controversies, applied themselves to the practical study and elucidation of Christian theology. Hugo de St. Victore is the most conspicuous and best known of these, the one whom his contemporaries honoured with the name of the second Augustine. Another light of the XIIth Century was Peter the Lombard, who collected a series of propositions and questions from the Fathers, offering them without solutions of his own to the study and ingenuity of his contemporaries. His work had a great influence on the two centuries which followed.

Aristotle, we said, was proscribed by the ruling power of the church, and regarded with jealousy by some of its best members. But it was not in the power of decrees to change the character of a period; men exhibited that character in themselves, even while they opposed themselves to the outward indications of it. St. Bernard as much as Abelard belonged to a time in which thinkers were occupied with their own internal struggles primarily, and only with questions more strictly divine and transcendent in the second place. The disposition to organize, to form a rule and plan of life, to invent an order rather than to perceive one, were eminently characteristic of him, and of all good as well as evil men in this day. Now these are, as we explained when we were speaking of Aristotle, the marks of his

mind as distinguished from that of his master. In his Physics and Ethics, nay even in his Politics, we found him studying the forms of the mind itself, rather than entering into the contemplation of its objects. We found in him everywhere a tendency to arrange and systematize, to form a plan where Plato sought to discover a method. There was then a natural sympathy between this philosopher and the ages we are speaking of, which had already manifested itself very decidedly, and which was quite sure to prevail at last in spite of all opposition. Hitherto the *Organum* had been the only one of his works with which the schools had been acquainted, even through a translation. Now a new light was to break in from a strange quarter. During the time in which the mind of western Europe had been gradually forming itself under the influence of Christianity, the mind of the Arabian had also been awakened to reflection. The conquering nation, as it came into regions which Greek philosophy had occupied, found some link between what it heard of this philosophy and the deeper principles of the Koran. In Neoplatonism there were abundant points of resemblance with that idea of Oneness and Absoluteness which had driven out all others in the Mahometan system. With this form of speculation, therefore, their thoughtful men acknowledged an immediate sympathy, and the habit of mind, with the series of reflections thus generated, became the most permanently characteristic of the nation. Through the Neoplatonists, however, the Arabs became also acquainted with Aristotle, and the new world of observations respecting every part of outward nature which burst upon them through his writings seems to have inspired this people with an astonishment and admiration hardly second to that which the young Romans felt when they first discovered what men had been thinking and talking of for so many centuries in Athens. Here too there were points of sympathy with old Arabian habits, with the astrological studies to which their fathers had been devoted, and with that mathematical faculty which seems to be an Oriental endowment. The love of Aristotle, and the general impulse to cultivation among this people being favoured by a race of accomplished monarchs, translations and commentaries upon him became more and more numerous, and sects of philosophy began to appear connecting themselves more or less with the national faith, mingling more or less of the mystical feeling which was so remote from the character of Aristotle, but still all deriving their sanction from his name. It is natural to ascribe the importation of this feeling and knowledge into Europe to the Crusades. But the difficulty of course remains how to account for Arab translations and commentaries becoming effectual upon the Latin schools. The generally admitted explanation is curious and interesting. It was through Jews, the oldest teachers of the world, now its merchants, that the ruling philosophy of the middle ages was enabled to establish itself. A strong philosophical impulse had been felt among them also. Their teachers, in Spain especially, were most laborious in translating and explaining Aristotle, and the writings of the Greek philosopher passed first into Arabic, then into Hebrew, before they became in a Latin dress the guides of men's speculations for at least two centuries.

There are many circumstances which may suggest the opinion, that it was the character of Aristotle's ethical treatises which determined the judgment of the age and the church in his favour, and relieved his name from

Moral and
Metaphy-
sical Phi-
losophy.

Arabian
philosophy.

The Jews.

Ethical
tendencies
of this
period.

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy

Chivalry.

Religious
orders.

Aristotle
triumphant

Physics.

Roger
Bacon and
Raymond
Lully.

the stigma which the supposed heresy of Roscelin and Abelard had brought upon it. We have seen how certain habits or forms of character were Aristotle's substitutes for the absolute and real objects which Plato sets before us. Now it may be truly said, that the superstition of this age consisted in its tendency to deify certain forms of character, certain conceptions of the fancy, or certain modes of the intellect, and that its excellence and virtue was closely intertwined with this same superstition. The same circumstances which brought Aristotle's works into Europe, gave the great impulse to the spirit of chivalry, to the pursuit and worship that is of a noble idea, of an idea most important and necessary to man, but altogether incomplete, and being incomplete, obliged to connect itself with transitory circumstances, and to lean upon the reverence for a single class of society. These habits of mind, which, though brought into outward display by mailed warriors, dwelt also in the cell of the student, must have made Aristotle's brilliant pictures of the Magnanimous man, the Temperate man, the Gentle man, particularly delightful. His works must have been felt to be in some remarkable manner a prophecy of that Christian age. Nor was this the only point of attraction and sympathy. The growth of chivalry was not a more remarkable characteristic of the XIIIth Century, than the rise of the religious orders. As the purpose for which the church existed became less manifest, the notion of new religious bodies, and new schemes of discipline, began to act more strongly upon men's minds. In such bodies the grand ecclesiastical idea, that the end of our lives is the knowledge of God, is soon effaced, except in a few chosen spirits, by the doctrine, that the object of our lives is to attain happiness here or hereafter. With this doctrine that announced by Aristotle wonderfully accords. Moreover, the tone of his morality is peculiarly favourable to the notion, that a scheme of discipline is the great requisite for human life. The formation of habits is to be attained, he taught, by the constant observance of the influences which may incline us too much to the right or to the left; means in his system have a continual tendency to become ends. The earnest preachers of a new rule of life must have found such doctrines remarkably congenial to their minds, even if there had not been enough to fascinate them in Aristotle's love of systems partly at least arbitrary. The influence of the Franciscans and Dominicans, now the strongest and the most earnest which was at work, was therefore exerted energetically in his favour. It soon secured the prevalence and success of his whole philosophy. But there was also another part of his writings which took mighty hold of the Christian as it had of the Arabian schools. It was one of the happy circumstances in the development of modern Europe, that the zeal for physical inquiries which was the first excited in Greece, and which overlaid all others in India, did not arise till men had already been occupied with questions more nearly concerning themselves. Our almost countryman, Johannes Erigena, had indeed exhibited an inclination towards a class of studies which we may truly call national; but the spirit was checked, and did not reappear in any strength till four centuries after. Then arose another and perhaps greater Briton, the "Doctor Mirabilis," Roger Bacon, and mathematics, chemistry, and physics generally became as much the studies of Christians as they had already been of the Mahometans. In him we have the philosopher of experiment under

whatever disadvantages and ignorance it might be pursued; the opposing or balancing mind to his was Raymond Lully, the student of the secret powers of nature, the originator of the *Ars Magna*. But this impulse, whichever of these directions it might take, still found its great help and encouragement in the writings of Aristotle, and the books on the Heaven and the World, with the Natural History, were some of the first of those attributed to him which were studied and commented on.

These Ethics and Physics furnished then a new material for the mind of the age to work upon; and the opposition of those who cultivated exclusively the first to the laborious students of the other, illustrated in the imprisonment of Roger Bacon by Franciscan influence, forms one of its indications. But we have discovered already in the history of Greece, and above all in the writings of Aristotle, at how many points both moral and physical studies abut upon that to which he first gave the name of *metaphysical*. We have seen, in fact, that the meaning of that world only becomes intelligible when we consider it in its relations to other subjects, and again that Aristotle himself could not contemplate it apart from Theology. It cannot be surprising then that of all the writings of Aristotle, those which exercised really the most influence, and which became the interpreters of the rest, were the Ontological treatises. As we have given our readers an account of those books, both as to their matter and their form, we need hardly remark what a vast body of questions they must have suggested to the minds of men at all prepared for them by any previous discipline. Now Peter the Lombard had set the schools upon the questioning method. He had proposed, as we observed, a number of difficult points taken from the writings of the Fathers expressly for consideration and resolution. Such a hint once given, and being met by the new gift of Aristotle's metaphysics, was the exciting occasion of those mighty masses of ontological and theological puzzles and argumentations which constitute what is called "the jargon of the schools," by those who can tolerate nothing but the jargon of the world.

Albertus Magnus was the person who first appears to have availed himself of the new resources for the purpose of forming a complete system in which Metaphysics or Ontology was the key to the other portions. We have seen how the ideas of Form and Matter, the one as that which constitutes every substance what it is, the other as its condition and *sine quâ non*, lay at the foundation of the metaphysics of Aristotle, and determined his thoughts upon every other subject. We remarked how closely these ideas were connected with logic, so that the fact of any thing being capable of definition is with him the test of its having a form and being a substance. While, therefore, we admitted the great value and importance of this Aristotelian distinction, we could admit it only as a distinction in and for the mind. For the moment it becomes more than this, we get upon the Platonic ground which Aristotle believed to be merely imaginary. His forms become the ideas of Plato, and these ideas derive their meaning from the reality of one who is himself *THE BEING*, not merely a particular form, though it be the highest form, of being. Now the philosophy of Albertus, and of those who succeeded him, is, we conceive, subject to precisely the same criticism as that from which it is derived. If we want help in determining the conditions of our own mind, our own modes of

Albertus
Magnus.

Moral and
Metaphy-
sical Phi-
losophy.

thinking in relation to all subjects, we suspect that we can obtain it nowhere so well as from these schoolmen. And seeing that the whole subject of Christian theology, with all the lights which it throws upon other subjects, came under their notice, they clearly had an opportunity which it would be wrong to say that they neglected or abused, of applying the Aristotelian observations in a direction entirely new. At the same time they, as much as those who knew nothing but the *Organum*, were dealing simply with forms and notions of our mind; they were not really theologians, physicians, or metaphysicians, they only explain under what terms theology, physics, metaphysics, are conceived by our understandings.

Bonaventura.

This statement must be taken in a general sense, and not by any means as if it were meant to deny that most valuable hints respecting substantial truths, as well as respecting our conceptions of them, may be obtained from these writers. It is impossible that they could follow out their own course steadily and honestly without throwing light upon all that was delivered to them by a former age, or was to be the occupation of a subsequent one. Still less is it intended to insinuate that a mere intellectual theology expelled in this age spiritual meditations. Such a charge would be false respecting many teachers of the XIIIth Century; it would be especially inapplicable to Bonaventura, the seraphic doctor, whose great object was to teach how the soul might be united to God, and how the light which is contained in all arts comes from the primary light whereby He is revealed to men. Even in these spiritual and seraphic doctors, however, there was a tendency to contemplate all things as reduced under Subjects and Systems; and in them as well as in the Mystics who rose up a century later and were some of the instruments in undermining scholasticism, self-contemplation is still at the root, and the highest objects are viewed in their action and influence upon the beholder.

Thomas Aquinas.

The man who reduced these habits of thought into the most perfect order—who carried them into every region of study—who has always been recognised as emphatically *the* Schoolman, was Thomas Aquinas. Bonaventura had conceived the possibility of reducing all arts under theology. Aquinas may be said to have actually reduced them. If one man can ever be supposed capable of conceiving all possible forms of human thought, all possible difficulties which can be put into a logical shape, that man was unquestionably Aquinas. No one who looks into his writings can wonder that they should have been the text-book of philosophy and theology for generations. It is idle to say they became so, merely because they settled questions according to the decrees of the church. No one, probably, ever awakened so much scepticism of the logical kind as he; no one, apparently, had a more conscientious mind; no one was more unwilling to adopt a conclusion till he had thoroughly sifted it. If there were no puzzles in the mind but dialectical puzzles, or if those puzzles did not require something deeper than dialectics for their solution, Aquinas ought still to be our guide. But school philosophy itself was to furnish the hint, that they do require something else. A countryman of ours, Duns Scotus, discovered that there was one great blank in the system of Aquinas, perfect and circular as it seemed. We want, said he, a principle of individualization. We want to know, not merely how all things conspire to form one great whole, but how each

Duns Scotus.

thing becomes that very thing distinct from all others. Such, we take it, was the idea which was working in the mind of Duns Scotus; the thought which he was labouring to bring forth, and which did make itself felt as an important thought by a body of earnest disciples. By many of his assertions he seemed to be contending for a more realistic doctrine than that of Aquinas; to be condemning him for not imputing more substance to universals. But this, it seems to us, was a subordinate and accidental part of their differences; the other radical and characteristic. To us the contemplation of this difference is most interesting, for it illustrates an observation we have made before, that the scholastic philosophy was not, as some fancy, a thing wholly unconnected with real wants and controversies, but was only a private rehearsal of that which was acted openly on the great stage of society. The Thomists and the Scotists are but school types or images of that great battle between the universal and the individual, the united and the distinct, which was just at that time beginning to be the great battle of the world, and of the parties in which Italy and England were to be respectively the heads. The Neapolitan and the Northumbrian were well fitted to be the symbols of their two countries and avant-couriers of much that was about to happen in them. In both alike the age of Towns was beginning. But the splendid Cities of Italy were to form aristocratical Republics, the temporary deposits of a national spirit which was soon to subside in the universal government of the Church. The Towns of England were to be the cradles of a middle class; the great rival power to the aristocracy, the great helper to the national monarch in asserting his own position. Other indications were connected with these, more directly affecting the popular and the scholastic life in each country. In each a clear, orderly, native language began to manifest itself—the southern preserving more distinctly the traces of its Latin father, the northern of its Teutonic mother. The one language came forth in its fulness and power to be the utterance of Dante's wailings; the other to be the tongue of Wickliff's Bible. Theology was the material of the poem as well as the translation; but in one case it was the creative power availing itself of all the theological ideas of the age to express its sense of contradiction and evil; in the other it was the burdened conscience of man seeking to find some better interpreter of its wants and miseries than those ideas, imprisoned in their logical forms or exhibited in arbitrary inventions and impositions, could afford. The theological poetry of Dante, therefore, though opposing itself so fiercely to some of the corruptions of the times, and launching such thunders against some of the successors of St. Peter, was not to awaken, except in some few minds, any ardent longing for reformation. It is to be rather regarded as the first vehement utterance of a Spirit which soon found Painting a more suitable language than Poetry, and which sought to embody the mind and feelings of the church rather than to change them. But the individualizing principle, the sense that is of each man's own distinct wants and distinct evil which had appeared in England, and of which the acts of Wickliff and his followers were the expression, could not be satisfied with the formal recognition of its existence which Duns Scotus had vouchsafed. This principle, translated out of the Latin into the new English, meant the living soul of each man; and this could not be content till it had been acknowledged as

Moral and
Metaphy-
sical Phi-
losophy.

Links
between
scholastic
and popular
opinions.

Moral and Metaphysical Philosophy. an integral element in the being of the church, and till it had swept away every dialectical conception or authorized dogma which stood in its way.

The XIVth Century. The XIVth Century, therefore, which had been already exposed to these influences, and in which they were daily becoming more powerful, soon gave a shock to the school systems, whether they bore the name of

Occam. Scotus or Aquinas. A pupil of Scotus, William Occam, also an Englishman, began early in this century to reassert the nominalist doctrine of Abelard, with all the advantages which were derived from the information and experience of his age. Abelard had stood forth as the champion of Aristotle against Platonism. Aristotle was now a canonized dictator in the schools, and it was that very dictation which the nominalism of Occam was to subvert. For though in teaching that species and genera are only abstractions of our minds, that they had no entities corresponding to them, he might seem to be asserting the very doctrine of Aristotle, that substance is only predicable of each particular thing; yet Aristotle had in effect applied these forms of the understanding to the contemplation of every possible subject; and the schoolmen, his successors, had, as we have seen, carried those forms into regions to which Aristotle had no access. Whether the nominalism of Occam led him to scepticism about the articles of faith is a question of perhaps less difficult solution than that which relates to Abelard. His language being directed against forms which had worked themselves into union with every great mystery, must of course have at times an irreverent and sceptical appearance; but he seems to have recognised distinctly all the Christian truths as matters of faith, though he questioned the attempts of the understanding to deal with them. In another way he occupied a position singularly characteristic of the coming period. Though an ecclesiastic, he maintained the national rights of the French sovereign against the Pope. Now, as Occam may be regarded as the last philosopher of this age, as the one who in a certain sense wound up its history, (for from his day forth nominalism had decidedly the ascendancy, and no very strong man arose to maintain that logical formulas are real objects,) it may seem at first a melancholy reflection to compare him with Anselm, who arose at the commencement of it. One might be tempted to think that, exactly the thing which he essayed to do, to connect namely the faith of man with his understanding, had been, after so many centuries, found impracticable, and that the corresponding labour of his life, to maintain the principle of the universal kingdom in opposition to the tyranny of national monarchs, had been in like manner abandoned by wise ecclesiastics as unreasonable; nay, that precisely the opposite vocation, that of supporting national government, had now become that which they laboured to fulfil. These two reflections, which cannot well be separated from each other, are likely, we say, to present themselves to the student of this age, and there is some warrant for the melancholy feeling which they engender. Undoubtedly the understanding and faith of men had not been reconciled by the efforts of all the schoolmen; nay, the schism between them had now become wider than in the days of Anselm. Undoubtedly, also, the ecclesiastical government, instead of being hailed now as the protector of the weak against the strong, of the invisible against the visible, began to be regarded as the great instrument of oppression, as the power by which men were held down in a visible

bondage. So far the contemplation is gloomy. It seems as if nothing had been accomplished in this period, as if it had been one of degeneracy, not of progress. But if we look again, we shall see cause to abate these lamentations; nay, to exchange them for the language of thankfulness and hope. Through terrible conflicts, in spite of fearful sins, this age had been really effecting its work, and was to leave imperishable tokens for the generations to come. The first period after Christianity had left the form of a universal polity; had left ordinances, creeds, ecclesiastical institutions, the witnesses of this universal polity, the powers by which it was upheld, and by which men were enabled to possess and enjoy its benefits; it had left records of the oppositions through which transcendent and universal truths had been maintained and confirmed; it had left a literature connecting itself with the former literature of the world, and showing that what therein had been foretold or wished for had come to pass. If these deposits remained and remain to this day, is it not equally true that those middle ages have left their deposits? National societies grown up from infancy to manhood; the forms of law established; languages created and defined, new forms invented in which the conceptions of men could clothe themselves—forms of architecture, of poetry, and finally, of painting; last, and we are bound to say not least, the full power and dimensions of the logical faculty in man ascertained by a series of precious experiments determining what it can and what it cannot achieve. For let no one say, that the scholastic philosophy is obsolete in its effects, because the volumes which contain it are seldom read, and because it has been found to have failed in much that it hoped to do. Not the feeblest newspaper scribe, who writes praises of the XIXth Century, and talks about the discoveries of Bacon, and the vain squabbles by which men were distracted till his time, could cast even these empty phrases into a coherent and intelligible shape, if those schoolmen whom he abuses had not lived. As truly as we owe our laws and ecclesiastical buildings to the middle ages, so truly do we owe to them our forms of thought and language. We are very unhappy if we have not learnt much since that time, and we shall presently have to show in what direction that learning has been won. But in fixing the terms and conditions of human thought, we are bold to say, that men have only done any thing by going back to these schoolmen, and using the fresh light that may have fallen upon us to the more effectual consideration of the questions which they raised.

When one reflects on these facts, men may surely be well content, that what is called the revival of letters came when it did, and not four or five centuries earlier. Most sad would it have been for the world, if the western nations, instead of being left to work out a cultivation for themselves with only such helps from ancient lore as best suited the thoughts which were awakening in them, had been overlaid with heaps of books, in which their circumstances gave them no interest, which they could not interpret livingly, and which would therefore have crushed all sparks of native and original speculation. When that revival did come, the inhabitants of western Europe were in some way prepared for it—prepared at least, by their own sense of a national position, to enter into the national feelings, and the thoughts and inquiries accompanying them, whereof Grecian books are the exposition. But they were not prepared for the glare of new light bursting upon eyes long

Moral and Metaphysical Philosophy.

Termination of the school age.

The revival of letters.

Moral and
Metaphy-
sical Phi-
losophy.

Immediate
effects of
the new
learning.

The Refor-
mation.

used to turn only inwards; a sense of confusion mingled with their delight at the novel world of action which was opened to them; they knew not how to connect it with any thing that they had been taught to think of or believe; that belief became therefore daily less and less sound and sincere. Meantime a new order was rising into consequence, which had little sympathy in the old traditional feelings, nay which had been for a long time contending against them. The scholars of Europe disliked this class and its feelings; they would keep up the old forms and opinions to which they had ceased to attach any meaning, lest the unlettered men should prevail. These forms and opinions, used with this ignorance of their nature, this conscious indifference, would of course become more and more corrupted—the convenience of repressing by the sanctions of religion that which could not otherwise be kept down, suggesting new expedients, and no reverence for the sanctity of the thing to be profaned hindering the adoption of them. Nor was it difficult to make these impostures successful. For that which was most alive in those who composed the new class was the Conscience. The deep sense of internal evil, the feeling of a condemning law, which had connected itself closely with the whole theology of the middle ages, might be almost extinct in the lettered priests of the hierarchy, but had become deeper than ever in the minds of humbler men—deeper because, having broken loose from the forms and questionings of the schools, they were transferred to the region of each man's personal being. Having reached that point, these feelings could no longer be tampered with. All the old power and the new wisdom of Europe might be on one side, but reality was on the other. In such a conflict the learning of Leo, the policy and greatness of Charles, the physical forces, nay the moral forces of old habit and tradition which they had at their command, could not prevail, for that which alone gives life to all these was wanting. Yet, though it was a new class which was to assert its position in this struggle, though it seemed at first sight to be breaking through all established bounds, and to be removing ancient landmarks, it was really an old feeling that was coming forth, a feeling belonging to that which is permanent and eternal, and claiming that which was permanent and eternal to rest upon. It was not to maintain the powers of the understanding that Luther stood forth, but to declare the might of faith; not to dispute old creeds, but to claim them as the witnesses of a personal Being, and of man's connection with him; not to teach that the feeling of separation from this Being of which past divines and schoolmen had spoken, meant nothing but that it meant every thing; not to say that the sacraments which had been the possession of the Church for fifteen centuries had lost their power, but that they were the sure witnesses of man's reconciliation, and that only by accepting them as such could the burdened conscience be delivered. In like manner, it was not by teaching men that the old literature of which they had lately become possessed signified nothing and that their only work was to forget the past, but it was by opening to them the history of God's past dealings with the world, and showing them how these concerned all their common and daily life, that the great Reformer prevailed. And therefore it may truly be said, that he seems to have been appointed more to explain the meaning of the period which we have been considering, to justify its toils, to make them effectual for good, than even to give the

character to that on which we shall presently enter. It had been an age of *Law*, in which nations had been brought under the yoke of fixed and positive government, and in which individuals had sought to shape out a rule of life for themselves. Luther showed what law has to do with man, how fixed, absolute, irreversible it is! how impossible it is to evade its sanctions, or to substitute any inventions of ours in place of it! how the conscience of man answers to its decrees! It had been an age of discussions about Ontology, about Realities, and Names. And Luther's declaration of a personal being, in whom men might trust, and by trusting in whom they might be delivered from personal transgressions, was the practical Realism which satisfied the heart of man, which connected itself with the former questions about law, but left the understanding to pursue its own way in dealing with the names of things. It had been an age in which men had glorified certain human qualities, making these their ideals of human excellence, though they could only be attained by the members of one class, and needed outward ornaments to recommend them. He brought out the truth, that there is a perfect humanity for each man to behold and to claim as his own, a humanity in doing homage to which he does not worship himself, but God,—a humanity set forth in one who had entered into the feelings and the sorrows of every peasant. It had been an age in which men had sought to discover how universals are connected with individualities, and Luther rose up to carry forward the assertion of Wickliff with clearer light, that there must be an universal body, but that unless it recognises the distinctness of each particular man, it does not fulfil its own conditions.

It is no doubt true, and the fact requires as much to be noticed by the historian of philosophy as by the theologian, that the side of this last truth which had been most kept out of view during the middle ages, was the side which now thrust itself forward at times to the utter exclusion of the other. By a portion of the Reformers the distinctness of the individual man was felt to be all in all, the idea of a fixed and permanent constitution was thought of only as subordinate to this; and though this was not the case with Luther, it is true that his position, as well as that of his brethren, prevented him from asserting as he would have wished that there is a catholic body, though it do not derive its catholicity from the sovereignty of a local bishop. But this circumstance only shows more clearly how completely the Reformation was the winding up of that particular discipline to which western Europe had been subjected since the days of Charlemagne. Universal truths, the idea of a universal commonwealth, it had been the work of the first period to bring forth into clear manifestation. The truths which concern each man's own life, the national order by which that life is realized, had been gradually working themselves out in this second time. The Reformation so far brought the two ages into harmony, for it showed that each man truly lives only when he has claimed a higher life than his own, that each nation truly lives only when it acknowledges direct allegiance of its monarch to an invisible King. More than this might be wanted. The continuance of a number of nations in subjection to the Romish Pontiff, the continued recognition at least by *one* (and that the one which we spoke of as the first and ablest assertor of the individual and the national principle) of its relation to the elder church, even while it resisted the usurpation to

Moral and
Metaphy-
sical Phi-
losophy.

Luther
how con-
nected with
the age
which pre-
ceded him.

Moral and
Metaphy-
sical Phi-
losophy.

which that church had been subjected, may prove that something was wanted which this crisis did not develop. What it was we may discover as we proceed. But this at least seems certain, that we have arrived at a very notable crisis in moral and metaphysical history, the effects of which upon the age which followed it we must now begin to trace.

From the Reformation to the French Revolution.

Effects of
the break-
ing up of
the Feudal
system.

When alluding in an early part of this sketch to the form of society which was established through Hebrew influence in Egypt, and tracing in it some likeness to the feudal system of modern Europe, we remarked that the root of that system lay in family relationships; though all its outward characteristics were connected with notions of military subordination. In this opinion we believe most writers on the subject substantially agree, though they may not adopt the same language. For though the old notion of discovering this system among the German tribes has given place to the much juster and more reasonable doctrine which makes modern institutions an outgrowth from the old institutions of Rome, no one can doubt that there was a plea for that notion, and that some modification of it is necessary in order to account for the difference between Teutonic and Roman life, even when both had been brought under the same Christian influences. And therefore all considerate persons have attached much importance to that reverence for marriage and for the bonds of family life generally, which characterised our German ancestors; these they have regarded as constituting a capacity for the reception of Roman order and law. This being admitted, there seems little difficulty in understanding that the time when that which is properly called the system of feudalism began to crumble, the time when its legal and formal restrictions were relaxed, and when another order appeared in the nation which had to work out a system of government for itself, may have been the time when the original *principles* of this system—its properly Teutonic elements, were brought out with a new strength and distinctness. Every one must feel that our landed tenures, with the principle of primogeniture lying at the basis of them all, did in some way bear a more decided witness for a family principle than any institutions in the old world with which they might be compared, or from which they had been derived. Yet so long as the aristocracy formed the nation, so long as all, however little related to them by any real ties of kinsmanship, were connected with them as military dependants, so long as there was no class witnessing for some other than this family principle, and thus making that better understood,—it is evident that the real meaning of these institutions must have been latent, not developed. Other circumstances connected with the history of this age assisted in keeping this meaning out of sight. The reverence which the church seemed to pay to the institution of marriage, by reckoning it among the sacraments, was no practical compensation for the slight which it put upon that institution, by refusing it to the most revered members of the commonwealth. The separation of a portion of this body, not only from this but from all the relations of life, tended still further to weaken the respect for them. Chivalry, by carrying men to the pursuit of distant objects, and an ideal of character, had the same tendency; so that the influences which most determined the character of the period, however favour-

able to some exercises of virtue, and especially to reverence and loyalty, were clearly not favourable to the homely and household graces. While chivalry was the indication of a deep and sincere feeling, while the inspiration which prompted the establishment of the religious orders was alive, the evil might be less felt. But when a fashion had succeeded to a faith, indulgence to discipline, men could not fail to observe how much the pretence of something that was higher destroyed all plain and practical morality. The feeling was likely to be strongest in the new class—that which was on other grounds discontented with the tone of feeling in the church, which did not share in the visions of the aristocracy, and which, being removed from the temptations of wealth and foreign intercourse, was likely to practise a simpler and purer habit of life. These circumstances alone might seem to account for the manner in which the Reformation operated to substitute reverence for relationships for that homage to some human ideal which had distinguished the middle ages, and that pursuit of absolute excellence which had characterised the Fathers of the church. But such circumstances would not have produced such an effect unless the theological character of the Reformation had been moulded in conformity with them. It was so moulded in a remarkable degree. That truth to which the conscience betook itself, when it was pursued by the fearful image of a condemning law, was the belief of a relationship, established between God and his creatures, expressed in the baptismal Covenant, to be realized by the faith of each person. As the Bible was more and more appealed to by the reformers against the logic of the schools, the idea of this divine relationship as the ground of all human relationships—to be set forth and realized in them—became more steadfastly present to their minds. Grasping this principle they became less concerned about the undermeanings which men of a former age had perceived or suspected in the sacred writings. The plain letter was seen to contain a deep mystery, and through that mystery to expound the scheme of practical life, and the different relations and conditions of that practical life became, to an extent which persons who have formed certain conceptions of the Reformation for themselves would not easily believe, the subjects treated of in the most earnest exhortations, and even in the higher divinity of the time. Thus two curious and somewhat paradoxical results came to pass. The middle class, the class which represented the individualizing feeling, preserved the family feeling of which the aristocracy is the proper representative, and, in the countries which submitted to the Reformation, enabled the aristocracy, divested of its military dignities and fiefs, to assume its true position. And the Reformation itself, while it seemed to be lifting men out of the grovelling notions and practices of the day into that feeling of connection with the invisible world, which formed the basis of the older and sublimer morality, did in fact give a more human and common-place direction to men's thoughts, causing that the cultivation and the study of these relationships should be the centre of the moral studies and moral inquiries of all those ages which were directly subject to its influence.

But we have seen that there is a Greek as well as a Jewish, a metaphysical as well as a moral side to human inquiries. A habit of estimating highly the family relationships was one effect of the Reformation, but it was not the only one. Another has been much more noticed, and should be certainly not passed over; the sense of

Moral and
Metaphy-
sical Phi-
losophy.

Reverence
for rela-
tionships a
characteris-
tic of the
Reforma-
tion.

Moral and
Metaphy-
sical Phi-
losophy.

Power, of a power belonging to each man distinctly, and not to men acting merely as portions of a body. Those who teach that the powers of man woke at once from a deep slumber just at the beginning of the XVth Century, or somewhere in the course of the XIVth, do indeed use strange and preposterous language. For all the seven centuries during which the western people had been growing up these powers had been most wonderfully developing themselves. In the conflicts of political parties, in the conflicts of the schools, in splendid enterprises and lonely watchings, the human faculties had been acquiring a strength and an energy which no sudden revolution, if it were the most favourable which the imagination can dream of, ever could have imparted to them.

The
XVth
Century.

Vigorous
exercise of
the creative
faculty.

But it is true also, that the consciousness of these powers, the feeling that they were within, and must come out, was characteristic of the new age. They had been exerted before in ascertaining the conditions and limitations to which they were subject, exerted with the pleasure which always accompanies the feeling of duty, but not from a mere joyous irrepressible impulse. Set free from the bandages of logic, yet still with that sense of subjection to law which was derived from the logical age, exercised under the sense of a spiritual presence, without the cowardly dread of it; these faculties begun to assert themselves in the XVth Century with a gladness and freedom of which there was no previous, and perhaps there has been no subsequent example. In those countries which had effectually asserted a national position, and where theological controversies were so far settled, that they did not occupy the whole mind of thinking men, or require swords to settle them, this outburst of life and energy took especially the form of poetry. English poetry had from the first been connected with the feelings of Reformation and the rise of the new order; Chaucer and Wickliff expound each other. And now Protestantism manifestly gave the direction to the thoughts of those who exhibited least in their writings of its exclusive influence. The high feeling of an ideal of excellence which had descended from the former age, and which in that age had not been able to express itself in words, now came in to incorporate itself with the sense of a meaning and pregnancy in all the daily acts and common relations of life, and the union gave birth to dramas as completely embodying the genius of modern Europe, as those of Æschylus, Sophocles, and Aristophanes embody the genius of Greece. Throughout Europe the influence was felt. The peculiar genius of Cervantes did not hinder him from expressing the feeling which we have designated as characteristic of the time, only as was natural from his circumstances with more of an apparent opposition to the older form of thought. And he as well as Ariosto and Tasso were able to bring forth in their works the national spirit of their respective countries, just as Shakspeare, with all his universality, exhibits so strikingly the character and life of England.

How Aris-
totle and
Logic re-
gained a
portion of
the power
of which
the revival
of letters
and the
Reforma-

It is important to make these remarks, though they may seem foreign to our immediate subject, that we may understand the peculiar moral and metaphysical tendencies of the time on which we are entering. As yet those tendencies displayed themselves far more remarkably in the creative works of which we have spoken, and in theology, than in what are strictly called philosophical speculations. How closely these had identified themselves with theology in ancient Greece as much as

in Christendom, we have had continual occasion to observe. It was not likely that they would be severed now. The time probably when philosophy made most effort to establish itself on a ground of its own, was in the period immediately following the revival of letters, and preceding the Reformation. Then that habit of unbelief, which we saw everywhere growing up, especially in the cultivated classes, together with the impatience of scholastic fetters, drove men to seek among the sects of Greek philosophy which arose in the sceptical age, for amusement or sympathy. The new kind of practical doubts, far inferior in real depth to the doubts of the schools but more plausible and attractive which the history of ancient speculation supplied, took hold of superficial minds, and the Aristotelian forms were not found sufficient to subdue them. Then there arose up men of earnest and deep minds, such men as Picus of Mirandula and Ficinus, to revive the old Platonic ideas, and to show what witness they bore for those truths which men were ready to abandon. But the Reformation terminated this kind of controversy, or made it unimportant, by bringing real human interests into play, and connecting them with the direct facts of Christianity. Hereby, however, a new and unexpected result was produced. The reformers had been assertors of faith against logic, of the realities of the Bible against the formalities of the schools. But when they began to argue and to teach, they found that they too had need of the school wisdom. They found it impossible to distinguish false statements from true, to possess in any sense a theological nomenclature without resorting to Aristotle and his disciples. We have explained why it must have been impossible. Aristotle is emphatically the teacher of names; the ascertainment of the distinctions and boundaries—of the necessary forms and conditions—of the human intellect. The schoolmen did but show how questions relating to theology, and to every other subject with which they dealt, must be expressed according to these forms. In setting out a scheme of thoughts, either for controversy or instruction, it was not a matter of choice, but of necessity, that these forms should be adopted. There were no others; to be unscholastic in things pertaining to the schools, you must be incoherent and irrational. The main question now as heretofore was, how the scholastic forms and the realities which men had fled to as a refuge from their tyranny could live and dwell in harmony together. That they must coexist under some condition or other was certain; the history of modern Europe had demonstrated it. And that there would be some providential scheme for redressing the evils arising from the prevalence of either, was also to be expected, and has, we think, been proved by the different outbursts of religious feeling, for a time resisting all forms, and then subsiding into great formality in most of the nations. Whether there were any healthier and better means of reconciling that which is properly human, universal, and real, with that which is definite, scholastic, nominal, those utterances of actual wants which belong to man himself, with the distinctions which are needful for those who are to be teachers of others, or who are subjected to an intellectual discipline; and whether, if there be such an expedient, any nation has hit upon it—is a question which we shall not discuss here. Our object is to show how Aristotle, after being in a manner denounced by the reformers, was at last inevitably taken into their favour again; a circumstance not merely

Moral and
Metaphy-
sical Phi-
losophy.

tion at first
deprived
them.

Realities
and for-
mulas.

Moral and
Metaphy-
sical Phi-
losophy.

Old ques-
tions as-
sume a new
character
in this time.

affecting theology, but all other studies, and the progress of human inquiries generally.

The form of theological and philosophical questions in the first century after the Reformation was then derived from the previous age. It might also be said, that the questions themselves were not new. They had all been debated again and again in the schools. For instance, of the two great theological controversies, that relating to the Eucharist, and that to the decrees of God and their relation to the will of man, the first, as we have seen, began with the beginning of the scholastic age, and the latter had certainly been a topic for meditation in every part of it. But these controversies derived such a new character from the Reformation, that though they occupied men so much before, they became new and the signs of a new era. The sacramental question in the former age turned upon the attempt to reconcile the forms of our understanding with the need which we feel of admission into a region which is above our understanding. And this might, for a time, seem to be the difficulty of the reformers as much as of their predecessors. But if we look deeper we shall find that the feeling of a direct relationship established between the Creator and his creatures, a feeling not now surely for the first time realized but now for the first time occupying the central place in the thoughts of Christian men and Christian students, gave a new turn to this debate, and made the sentiment respecting it in the heart different even where the language was the same. A similar remark applies to the other great question, which belongs more strictly, as every one will acknowledge, to the sphere of *metaphysical* theology. The older theology starting from the highest ground, from the assumption of an all-creating, all-energizing will, sought to discover how the movements of the human will could harmonize with or resist this. In the time of the Reformation the strong assertion of the principle of faith, of a man's right to claim a distinct relation to his Maker, gave a kind of precedence to the acts and energies of man, and made the controversy turn upon the point, how the power, whereof a man becomes conscious, is modified, controlled, or created by the energy of the higher will. And thus this controversy became, in fact, the seed of all the metaphysical questions of the next three centuries, even of those which had the least of a theological character, as it might not, perhaps, be wrong to affirm that the sacramental question was the seed of all the great moral arguments which were carried on during the same period.

Pantheistic
philosophy
in the papal
countries.

But though there was this manifest direction of thought towards the subject of human powers and energies, it was scarcely to be expected that while those powers and energies were in their full exercise, questions about their nature would, as such, excite any great interest. The directly philosophical controversies of the XVIth Century, therefore, as distinguished from the theological, did not as yet distinctly assume the new impress. By far the most interesting of them were carried on in those countries which were still under pontifical dominion. They were struggles of the awakened spirit in that time to break loose from the tyranny of prescription and logic, and to assert a freedom for speculative men, which practical men in those countries had not been able, or, perhaps, had not sought, to achieve. The Protestant divines, who had found a practical reality, were, as we have said, content to let Aristotle reign in the schools; nay, were eager to main-

tain his authority, and by degrees, as their controversies with each other and the Romanists increased, became really attached to and enslaved by the logical forms which they at first only tolerated as needful for certain purposes. The free spirits who had been nursed in Italy under a clear sky, and amidst forms of beauty appealing to the imagination and the heart; who had felt the impulse given to the world by the ancient literature; who had caught the iconoclast spirit of the Reformation, though they entered little into its peculiar objects and proclamations, naturally wandered into quite another region. The dry intellectualism of Aristotle was to them intolerable. Could it be that the aspirations of their souls after the good and the beautiful, aspirations which they shared with the great men of old, with Plato, and Philo, and Plotinus, were all to be tied down in formulas? Could it be that the divine nature admitted either of such limitations, or of those sensual limitations which a corrupt priesthood had invented to please the vulgar? Could it be that the fair universe which lay out before them endured restrictions which even the minds of its ignorant beholders disdained? Such thoughts as these brooded over the spirits of many men in this time, but took possession of none so completely as of *Giordano Bruno*. They gave birth in him to a scheme in which noble thoughts respecting God, unsustained by a clear perception of the ground on which men may have intercourse with him, mingled far too closely with bright and clear visions concerning nature. The result appears to have been a Pantheism which was called Atheism at Rome; and for which he was burnt by men who had themselves and their own corruptions chiefly to blame for any aberrations into which a spirit, apparently longing after truth, was tempted in the pursuit of it.

Moral and
Metaphy-
sical Phi-
losophy.

Giordano
Bruno.

In many ways the history of Giordano Bruno is important, especially as illustrating the origin and symptoms of the like faith in later times. But it is most interesting to us in this place as an indication of the new feelings respecting *Nature* which were beginning to work in men's minds. We observed, that the physical speculations of the first Greek school were closely connected with the sense of power which was awakened in commercial colonists living under small organized governments wherein each man had an interest in the midst of the great Asiatic Empire. And though these inquiries seemed wholly turned towards nature, we observed that, by a careful consideration of the notions and schemes to which they gave birth, we might discover that the student was not really so much seeking for an explanation of natural phenomena, as for something analogous with, or the foundation of that which he had become less distinctly conscious of in himself. In this way we traced the very origin and meaning of metaphysical inquiries in their earliest form. They were attempts on the part of men to compare their powers with the powers which they saw at work in the world about them. Supposing this observation to be founded in a general truth, and not merely deduced from a particular experience, it ought to receive illustration from the history of the time at which we have now arrived. Strangely different as the position of the nations in modern Europe was from that of the Greek colonies in Asia or in Italy, there was this in common between them. The feeling of human power had been strangely and suddenly developed, and that power had especially been put forth in practical meditation upon civil affairs.

The
XVIIth
Century.
Physical
speculations
of this age.

Moral and
Metaphy-
sical Phi-
losophy.

How con-
nected with
its poli-
tical ten-
dencies.

Bacon.

His dis-
cipline.

Though the governments of the different nations had been for so long a time shaping themselves into form and consistency, it was now first that men began to think distinctly about governing, to consider it as an art. In every direction, under every variety of character, moral feeling, religious faith, this tendency and impulse might be seen to characterise the XVIth Century. The monarchs who ushered in this period were Henry VII., Louis XI., and Ferdinand of Spain. Machiavel was one of its early teachers; Calvin and Loyola, with the two most opposite sets of purposes which it is well possible to conceive, were each in their respective spheres devising schemes and laying down plans of government. Lastly, a complete political discipline was matured by the Guises and our Elizabethan statesmen. Much might be said of the effects of this temper on the morality of the time, especially in producing that prudential standard to which so many in their outward acts conformed themselves, who nevertheless did homage in their hearts to higher principles, and even lived for the sake of promoting them. All such observations would receive the greatest light from the history of a man whom we now advert to on quite another account as the founder of physical science in modern Europe. The student at first is somewhat puzzled in trying to connect *Francis Bacon*, the most astute and profound of political observers, with Francis Bacon, the author of the *Novum Organum*. But history seems to show very clearly the point of coincidence between the two characters, and to prove that the impulse to the observation of nature springs first of all from that sense of intellectual lordship whereby a man is able to feel that he has that in him of which nature may present many likenesses, but to which it can offer no parallel. Such a previous initiation into human affairs was also it would seem a necessary preparation for the particular work which Bacon was destined to perform. It was his calling, as we have hinted in former passages of our sketch, when speaking of the connection between his method and that of Plato, to deliver the study of nature from its subjection to the forms and conditions of our minds, to show how things may be known as they are in themselves, to prove that we are not bound to make ourselves the standards by which we judge of the objects which we behold. To effect such an emancipation as this—an emancipation, be it ever remembered, not from that which is in itself untrue, not from certain shapes and formulas invented and devised by men, but from those true and undoubted formulas by which the operations of our understandings are regulated—it was necessary that he should have an extensive school discipline. But it was surely as necessary that he should be taught by converse with men and their acts, that there is practically another kind of procedure than that which the schools follow; a way of entering into and becoming livingly acquainted with the feelings and conduct and purposes of beings, who, though of the same race, are in all individual tempers, circumstances, tempers, and habits different from us. Such a discipline, or, at all events, the character of mind which made such a discipline congenial, and induced him to enter into the studies that appertain to it, was a kind of needful induction to the belief that we can by some means feel our way into a clear, distinct, and living apprehension of the forms which the outward world presents to us, and that we can ascend to a knowledge of the actual laws, strange and mysterious as they may be, by which the state and movements of these forms are

determined. We have called this a belief; for beyond all doubt a faith in the possibility of attaining the end in this as in all other cases of real discovery precedes and sustains any attempts to ascertain the means of arriving at it. But for this faith something more was wanted than even that subtle and refined instinct which had been cultivated by Bacon's studies of men and converse with men. The recognition of an absolute ground of all human as well as all natural laws, the belief of his having actually made himself known to his creatures, and of its being possible for them to have a knowledge of him cleared from the fantasies and idols of their own imaginations and understandings, was the necessary foundation of all this great man's mind and speculations to whatever point they were tending and however at times one or both might be darkened by too close a familiarity with the corruptions and meannesses of the human world, or too passionate an addiction to the contemplation of the natural world. Nor should it ever be forgotten, that he owed all the clearness and distinctness of his mind to his freedom from that Pantheism to which the last great thinker of whom we were speaking so unhappily yielded. If he had in any wise confounded Nature with God, if he had not entertained the strongest practical feeling of men being connected with God through one who had taken their own nature, and not through the forms of the universe, it is absolutely impossible that he could have discovered that method of dealing with physics, which has made a physical science practicable. Pantheism—the *theory* whereof Paganism in its Greek and Indian forms is the legitimate and the only possible practical exposition—naturally disposes to that vague admiration and adoration of nature which was the great hinderance in the old world, and to a certain extent in the middle ages also, to the honest and steadfast examination of its wonders. To free men from this bondage, as well as from that materialism, partly the fruit of it, partly the reaction against it, which disposes us to look only at the surfaces of things or at the atoms which compose them, not to inquire after their secret powers and principles, was one great part of Bacon's duty. In fulfilling it, he did, as we have remarked before, fail in the homage that was due to his great predecessors. He never sufficiently showed in what sense Aristotle's principles, though inapplicable to realities, were nevertheless true and necessary. He never fairly acknowledged, or perhaps perceived that Plato, though he failed utterly in discovering the laws of nature, had asserted his own principle of knowing things in themselves, and traced out a method by which that knowledge is to be attained as clearly as Bacon himself. Both offences have been severely punished; crowds of sciolists have arisen, who, while abusing Bacon's name, to slander the two leaders of the old philosophy, have been, in fact, undermining his principles far more than theirs. Those who are really in earnest for his reputation, or for the cause of science itself, will endeavour to divest themselves of this bigotry, and will perhaps find themselves compelled to acknowledge that they cannot show their reverence for him better than by proving that the doctrines, which have now for two centuries been acting upon Europe usefully or mischievously, are not really in contradiction to those of men, who exercised so vast an influence over it during the sixteen previous centuries.

It will not be supposed by those who have gone along with our remarks on the analogy between the

Moral and
Metaphy-
sical Phi-
losophy.

His faith.

His free-
dom from
Pantheism.

His un-
fairness.

Moral and
Metaphy-
sical Phi-
losophy.

Effects of
Bacon's
writings
upon moral
and meta-
physical
studies.

Separation
of meta-
physics and
morals
from the-
ology.

philosophical movement of this age, and in the earliest age of Greece, that Bacon, by giving a new impulse and direction to physical studies, made those of which we are treating less popular or interesting. Following out that analogy, they would be prepared to expect that at this time the word metaphysical would acquire precisely that meaning which we have supposed to be its primitive one—in other words, that those who devoted themselves under the Baconian influence to inquiries respecting man, would propose themselves this question above all others for solution,—In what respect do the powers in man resemble or differ from those which we observe in nature? They will expect that those higher questions which were gradually developed out of this in the course of Greek speculation, those we mean which concern Being and Unity, would indeed be continually presenting themselves to men's minds, but would by no means form the principal or characteristic subject of their thoughts. They would be prepared also for a much more marked and sharp division between metaphysical and moral studies than had existed in any age subsequent to Christianity, perhaps we might say in any age since that of Socrates; lastly, they will easily believe that, as in those two former ages, men were occupied either with purely transcendent questions or with the relations of men's minds to that which is transcendent, theology and the revelation, which was believed to be the exposition of its truths, would much more directly connect itself with philosophy in them than in this. In the first case, philosophy would without any violence, as a mere matter of course, deduce itself from or refer itself to the principles of a higher science, because the subjects which it was contemplating in one aspect were those which that science was contemplating in another; in the latter case, on the contrary, theology would be rather brought in to supply gaps in the system which philosophy had tried to construct, the scriptures or the church appealed to as authorities against some of the conclusions at which philosophers had arrived. The datum upon which the earlier ages proceeded in all their thoughts and speculations was, that man is a supernatural being, treated as such by his Creator, even when he had himself abandoned his prerogatives, restored by a great redemption to his rightful position, enabled to claim it and to maintain it by a continual divine interference on his behalf. Now though this doctrine still remained as the foundation of the national faith and institutions in every part of Christendom, the men who were especially to govern the thoughts and speculations of the nations started from a different point. They were to contemplate man and nature side by side; not to assume any dignity in one above the other, but to try if they could find out whether they were subjected to the same laws or to different laws, whether they had the same constitution or a different constitution. Such we believe to be a fair description of the problem which was set before the men of the XVIIth and XVIIIth Centuries. We cannot regard without the liveliest interest their different attempts at the solution of it, or doubt that each one who had any honesty or insight, nay that one here and there who had not much of either, has contributed his quota of observation and experiment towards the settling of it. We do not believe that any one of these helps is useless or unnecessary. Nay, we do not doubt that even the speculations of these philosophers with which we have least sympathy may be turned to good account. We cannot for instance much

applaud the *destructive* tendencies for which some of them were conspicuous, because these would go to set aside facts which we believe others of them established. Neither can we in general rest satisfied with their *constructive* attempts, that is with the efforts of each to do more than the work which we conceive was set down for him—to provide a system, when all he had to do was to realize some fact. Yet we conceive that we could not well dispense with their destructive doctrines, seeing that these serve to determine the limits of their speculations and the exact amount and worth of that which they did see. Even less could we dispense with their crude attempts to form a complete and circular system; for these do not only supply most valuable admonitions to us of this day, but also testify to a want which these builders felt must be supplied in some way, whether they could supply it or no. Such are our convictions, and in this spirit we shall give a catalogue raisonnée of the most remarkable metaphysical writers in these centuries.

It is a different question how we are to treat those moral inquiries which were pursued sometimes separately, sometimes in conjunction with these respecting the nature of human powers and faculties. When we discover the same disposition to subvert the distinction between Right and Wrong which we noticed in the Greek Sophists, it seems hard to abstain from the same phrases of indignation which we permitted ourselves to use respecting them. And such, no doubt, it was most right for those to utter who heard these truths assailed, and observed, as Socrates did, the influence which the contempt of them was producing upon the youth of their land. They were not to wait for such persons to confute themselves, or for events to confute them. They were to assert strongly and clearly what they felt, for the good of their own age, and as a witness to those who should live afterwards. He who records a series of moral speculations may, however, take a different course. He need not go out of his way to invent an answer to those who have doubted or denied that there is any order in the moral world. He will find the answer written in sunbeams upon the history of the time. He has only to record facts, and it will be seen whether or no all the denials of all the philosophers that ever lived can destroy this order, or prevent it from making its strength manifest when it has been violated.

Two remarkable men born within a few years of each other, one in France, one in England, may be said to have given a character to all the period which followed them: these were *Descartes* and *Thomas Hobbes*. The former of these furnishes a remarkable illustration of the view which we have taken respecting the new direction of human thought. No one aspired more than Descartes to introduce a complete system of philosophy, to be a theologian, a physical philosopher, a moralist, and metaphysician in one; few persons probably entered upon the task with more of diligence and talent combined to assist them in accomplishing it. Yet irresistibly all his thoughts were drawn to the one question,—What is this mind of ours, how is it like or unlike the things with which it deals, and over which it exercises control? He might have a grand scheme or theory of nature, wonderful vortices, with most ingenious and not to his mind irreverent methods of connecting the world with its author. But underneath them all, Doctrine really giving them their value to himself, and in the main determining their influence upon after times, was his thought.

Moral and
Metaphy-
sical Phi-
losophy.

How the
specula-
tions of the
XVIIth
and
XVIIIth
Centuries
should be
treated.

Descartes.

Doctrine
concerning
thought.

Moral and
Metaphy-
sical Phi-
losophy.

The atomic
theory.

attempt at an answer to this one inquiry. "I am because I think; this *thought* makes the difference between me and all other things." The two suggestions which arose out of this, each having respect to one of the two kinds thus compared together, determined the other most important parts of his scheme. If other things are without thought, what can be positively affirmed of them? Simply that they are combinations of atoms. The qualities which are attributed to them indicate only their effects upon us; every thing that has not a soul can be dissected into its parts; so far Democritus was right. He was wrong only because he made the soul one of these atomic, divisible things. But what is the thought which is the essence of this soul? Whence comes it? Being its essence, it must certainly in its highest sense be within it. Thoughts, it is true, come into the mind. It receives thoughts and creates them. But there must be something below all this. There must be some thought which is innate in the mind itself. And here is a link between our minds and the absolute being. We have an innate thought that He is—a witness which to us is perfect and irresistible. Moreover a man thinks that he is free; how could he think so if he were not free? In the thought lies the freedom, even as in the thought lies the demonstration of the existence of God. Every one will surely feel that the man who proclaimed these positions was not merely uttering dreams. There is something in them which has a meaning, and which must be important. Let us wait and see how much of what he says will make itself good against men who have also their own words to utter. In the mean time let us be sure of this; Descartes has told us something about ourselves; he has discovered that we do think, and that this thinking is no trifle, but, whatever it be, a most solemn and mysterious process.

Thomas
Hobbes.

His psy-
chology.

When we speak of *Thomas Hobbes*, our thoughts turn, and rightly turn, to his political writings. These embody the character of the man, and without them all his other speculations would be unintelligible. Nevertheless, we ought to consider him also as occupied with the question respecting the relation between man and the rest of the universe, which we have spoken of as characteristic of this age. That which struck Descartes as constituting such a strange difference between man and all things else, that he could think, did not present itself to Hobbes as at all a remarkable peculiarity. Thought appeared to him simply in connection with that which we think about. There are certain things submitted to us with which we have to do, certain bodies which we can handle and dissect; these form the real subject upon which our minds exercise themselves, and these, and these only, are the province of the philosopher. This was very decidedly settling the question in the opposite way; it was determining that a man is not under any different law from the things about him; and this was precisely the point at which Hobbes was aiming. All nature is subject to certain necessary bonds and conditions; it cannot transgress its rules and boundaries; neither can man; it is merely a delusion to fancy that he has any more freedom than the rest of the universe. A series of previous powers and influences acting upon him determine him to be what he is, and to do what he shall do, and that thing which he calls a Will is just as much at the mercy of these influences as any other part of him. But then what does Society mean? How is it that men form commonwealths and mingle together with a sense of fealty to a common head and mutual

obligations to each other? Is not this different from some natural compulsion, such as mere bodies obey? How do they come into this state? said Hobbes. Look into the history of our own time and you shall see. What has been the result of our eighteen years of fighting in England? We began with a monarch, we have ended with one. Men have fought for ascendancy till they were tired, then by mutual compact they have stopped; they have felt that there must be a power over them, and that there was no help but to obey it. Do you want any other proof of the way in which society formed itself at first? A set of warring atoms called men struggled each against the other. Then came the notion of combination; two or three conjoined were a match for many separate ones. From this combination comes fellowship; men at last submit to one power through fear of each other, and he, by all influences that he can exercise over their fears and hopes, maintains his dominion. But it is mere power and necessity which rule just as much over these human creatures as over all others. Thus then we learn both what is to be done and what is to be known. It is most needful to know the nature of animals generally, and the ordinary laws of the universe which they obey: this is the subject of philosophy in its larger sense. It is most needful to know the nature of the particular animal you govern, to know all the influences of fear and hope which act upon him and determine him one way or another: this is psychology. It is most useful to know all the ways of turning these hopes and fears to account for the purpose of keeping him in subjection: this is politics. Pursuing these studies, and practising the art to which they lead us, we may avail ourselves of all helps—from sacred lore as well as profane; from the terror of superior power as well as the agency of ordinary compulsion; only we have to recollect that all these are means to an end. They are all used for the purpose of helping men to be in the only state in which it is tolerable for them to exist.

Such is *Thomas Hobbes*, the strongest of all dogmatists, to whom may well be applied the description given of the animal which has furnished him with the title of his greatest work: "*His scales are his pride, shut up together as with a close seal; one is so near to another that no air can come between them. They are joined one to another, they stick together that they cannot be sundered. The flakes of his flesh are joined together, they are firm in themselves, they cannot be moved. His heart is as firm as a stone, yea, as hard as a piece of the nether mill-stone.*" An example of a compacter style than his, with less of grace and freedom in it, our English language does not supply; nor perhaps has our English soil ever raised a more closely compacted man with less of affection or imagination. Yet we conceive there is much to be learnt from his writings. His stern demand for some object upon which the thoughts should exercise themselves, so that they should not merely be busy with their own workings, is in itself useful and important. His discovery that there is an Order or Government meant for man, that he, as much as any natural creature, nay more than any, must be in subjection to something, is surely a most important truth, which if not now for the first time discovered, it is at least valuable to receive a recognition of from such independent testimony. Nor ought we wholly to undervalue even that piece of imaginary history, his account of the way in which men fought themselves into subjection. Doubtless no such event is recorded, and all histories,

His poli-
tics.

His cha-
racter and
style.

Moral and
Metaphy-
sical Phi-
losophy.

so far as they are authentic, supply the strongest evidence that it never can have occurred. But is it not valuable even in this light as a horrible fantasy? Do we not feel that if no civilized nation has ever emerged out of this state of division and tumult, many may have plunged themselves into it, and may be plunging themselves into it now? Is it not well to know what man would be, if he were not connected by any bonds of fellowship with other men, what he may become if he break loose from that fellowship? We may get then, we think, so much of clear gain from these destructive speculations, and this also; the assertions that man has no Conscience and no Will, are found to be inevitably connected in the mind of the most consistent of thinkers with the doctrine that the power to which he is subjected on earth must be an absolute power, and that all spiritual influences which are exerted over him must act mainly upon his fears. It will very much assist the student in his subsequent investigations if he devote some time to a comparison of these remarkable thinkers, who foretell in a manner the habits of thought which were to prevail throughout this period. The obvious distinction between them is, that one affirmed man to have that in him which is *metaphysical*; that the other reduced him to the same rank as all other creatures, differenced only by the possession of a more cultivated organ of calculation, and a livelier sensibility to objects of terror or enjoyment. But our view of this opposition is very imperfect if we do not observe on what ground it rests. Descartes the Mathematician believes that Thought to be the most essential part of man which he finds necessary as the basis of a demonstration. Hobbes the Politician can find nothing in a man separated from his fellow-creatures but the lowest instincts; whatever is beyond this, therefore, he concludes must have come to him from without and not from within. Here are two lines of thought into which we see that men occupied with the inquiry respecting the nature of man as connected with other natures immediately diverged.

Relation of
Descartes
to Hobbes.

Politics and
mathema-
tics.

Of the one which our countryman traced out we shall have to speak presently. The other it is equally needful to work in its different windings; still more to feel clearly what was its starting point. The very existence of such a science as *mathematics* had ever been a warrant to men that there is something fixed and certain, not dependent upon the caprices and accidents of our minds. We have seen what hold it took of the mind of Pythagoras, and how necessary Plato considered it as an induction to all philosophy. But we may easily conceive what new dignity was added to this science now that nature had become the great subject of study, and now that logic had lost so much of its interest. We cannot wonder then, that when men began to study the nature of their own minds, and to compare them with the things around them, they should have been struck above all things with wonder at their possession of a science whereby they were able to take account of the forms and relations of that world against which they were measuring themselves. What was it? What did it mean? And what above all must I mean who have it; who possess this means of attaining certainty respecting the things about me? Surely this certainty must appertain still more inwardly and intimately to myself. This *thought* must be the surest of all things, its oracles the most infallible; no other demonstration can surpass its demonstrations. So reasoned Descartes, and the full effect and consciousness of his reasonings revealed them-

selves in the mind of a man of deeper reflection and greater consistency than himself, *Bernard Spinoza* the Jew. Perhaps there is no one of whom it is more right to speak cautiously than of this remarkable person. His contemporaries felt only the mischievousness of some of his propositions, and took no account either of the circumstances of his education, of the truth which seems to have been working in his heart, or of the confusions which they and he shared together. It is to be feared that in this day there may be a somewhat violent reaction in his favour; many hastily concluding that a man of so much genius, and so much confidence in his own conclusions, if he were not a demon, must be a great prophet. We conceive that a man trained as he was to the reverence for an absolute Being, between whom and his creatures the rabbins recognised no actual mediator, disgusted as he must have been with their frivolities, and cut off from all political fellowship, must have needs hailed with delight the doctrine which Descartes proclaimed, that there is a witness of this Being in our own selves, and that there is demonstration in this very witness. Nor can we wonder that a man whose conscience witnessed to him that this was so, should have been led on to believe that he wanted no further light respecting this Being than this, that he was the ground and subject of his own thoughts, or that advancing a step further he should have been tempted to look upon the acts and movements of the universe as so many acts or properties of this Being, or that, he should have been able to trace with curious coherency and in mathematical sequence how these different thoughts were derived from each other. In expressing such thoughts, he might easily use language which on some who heard it may have had all the effect of Atheism, and yet have seemed to himself at the most immeasurable distance from it; in considering the forms and essences of the world as divine attributes, he must have slid into Pantheism, while yet there may have been a line of separation in his own mind even between this doctrine and his own, and while by his clearness of thought he may have enabled other and happier students more fully to perceive wherein its evil consists. At all events we may make this remark, that if this Hebrew carried the Cartesian doctrine to its furthest point, he was by his national position united to his acquired philosophy, at the furthest possible remove from all those political feelings and influences which, as we have seen, governed the philosophy of Hobbes. Both were, in their own way, most coherent thinkers; the conclusions of both were about equally startling and extraordinary, though diametrically opposite. May we not hope that, in some way or other, a point of coincidence may be discovered between their two modes of thinking; that politics and mathematics, the objective and the subjective speculations, may not always refute each other, and yet both put forward such a front of necessity and demonstration, and that this principle, whatever it be, which reconciles them may not be quite so alarming as either, when contemplated in its separation and singleness?

If we adhered strictly to chronology we ought to Jacob have mentioned *Jacob Behmen* before Descartes or *Behmen*. Spinoza. This man seemed born to show what deep thoughts may work in the mind of a mechanic, and how, if awakened, they will ascend at once into the region of the permanent and the universal, overleaping all the ordinary forms and limitations of the intellect; yet how inevitably they will take their shape from the age in which they were produced. It is now ge-

Moral and
Metaphy-
sical Phi-
losophy.

Bernard
Spinoza.

Effects of
his Jewish
education.

Moral and
Metaphy-
sical Phi-
losophy.

nerally admitted by those who have looked into the writings of this celebrated shoemaker of Goerlitz, that he is not to be dismissed with the vague phrase of vituperation, "an obscure mystic;" that his obscurity was in a great degree the effect of unacquaintance with scientific language, but that, through it all, may be traced deep thoughts respecting God and man, by which philosophers might be greatly profited. But our main reason for referring to his writings is, that they illustrate the tendency to speculation respecting nature, to compare Nature with Man, and to connect both with the mysteries of Form and Number which we have seen was equally characteristic of the most learned men. By Descartes and Spinoza, mathematics were looked upon as the great instruments for expounding their thoughts; and as the witnesses for something unchangeable. Behmen could invent a mode of exposition for himself, and he wanted no testimony respecting the being of God. But to him the relations of forms and numbers became mysterious objects which he could see moving and operating wondrously in the divine, the natural, and the subterranean kingdoms.

The Cam-
bridge
Platonists

A somewhat similar tendency may be traced in a very different class of men from any of those we have been speaking of, those admirable and accomplished writers who have been called the *Cambridge Platonists*, Cudworth, John Smith, Worthington, Henry More and others. These men, though they lived in the same stormy period as Hobbes, had not caught the influence of the political feelings of their time, but rather had been driven by disgust with them in the opposite direction. In their desire to escape from the dogmatism of the times they had betaken themselves to the earlier Christian authors, and had learnt to feel (as they did) that Christianity was the satisfaction and complement of Platonism. But though they thus strove to cast themselves into another period, they really received a very strong impression from the one into which they were born. They looked upon Descartes as a kind of interpreter of Plato, connecting his thoughts about the immateriality and immortality of the soul with the discourse in the *Phædon*, and eagerly adopting his atomic theory respecting the rest of nature as a homage to that which is intelligent and voluntary; and the deepest thinkers among them seem to have been drawn by an irresistible impulse into Pythagorean and Talmudical speculations respecting numbers and forms. In spite of the name which they have received, they appear to us more properly Cartesians than Platonists, being far more busy about the soul than about its objects, and therefore in their ethical system sliding into the Aristotelian doctrine respecting the distinction between the Absolute and the Practical, and teaching how to form Habits rather than to rest in Principles.

England
after the
Restoration

But Descartes, it would seem, was not meant to exert any decisive or permanent influence over the English mind. The power of mathematics made itself practically felt here in the *Principia* of Newton, and left the field of metaphysics and morals to political influences. Those influences had very much changed their character since the time of Bacon. In his day men had entered upon the study of civil life with infinite delight, as upon a real pursuit in which they could hope to make continual progress, and in which some new varieties of character and conduct would continually be presenting themselves. The simulation and cunning which even wise and good men seemed to think lawful were the result of their

looking upon policy as an art, an interesting pursuit in which men were to propose to themselves certain ends, and to accomplish them by what means they could. But this habit of feeling was not sound and true enough to last; real questions arose, real principles were brought into play, with which mere prudence and skill could not deal. Then came that civil war in which, amidst all its crimes and evils, strong and noble passions were called forth on both sides; on the one a reverence for the person of the sovereign associating itself with reverence for the ancient Church of the land; on the other, respect for law connecting itself with the assertion of each man's distinct Personality and Responsibility. The war had terminated; the party which had asserted the majesty of conscience and law had violated Law and trampled upon Conscience. The party which had supported loyalty and the church had triumphed; but the monarch who represented these venerable principles in his heart despised them. The more their legal authority was recognised, the less they were felt as having a ground above all formal law. Meantime commerce was every day enlarging its borders. Its principle of barter and exchange was felt to be living though other principles were dead, and speedily this principle began to be recognised as the one which all things obey. In such a time, John Locke appeared, a man of great simplicity and honesty, who was bred in a Puritan school to feel little sympathy with the past, who was gifted with a keen eye to observe what in the notions of his time was loose and disjointed—the relics of other ages now ill understood—who would not profess more than he felt or knew upon any matter, and who, because many talked of things which he was sure they did not understand, rather suspected that this was the case with all that had ever professed to see more than he did, who had the power of making men perceive how much they believed already, who was nowise able nor much pretended to enlarge the sphere of their belief, or discover to them what they had not yet apprehended. It was his undertaking to remove Mysteries from the science of Government by resolving that relation between the king and the people, which had hitherto been expressed in religious forms into the plain notion of an Original Compact; to remove all difficulties upon the question how men should feel respecting the differences of Opinion in each other, or States respecting the differences of opinion in their subjects by the simple consideration that we are all liable to err; to take away puzzles out of the path of the Teacher, by showing that Education is to prepare men for the business of life; to put men in a simple way of Conducting their own Understandings, by warning them of the danger of being influenced too strongly by this opinion or by that; lastly, to set all matters clear respecting the Understanding itself, by showing that it receives its Ideas through the Senses.

In saying this we are describing (as may be judged from the character we have given of Locke) the effect of his opinions upon a time which had implicitly adopted them before he formally set them forth, rather than the intention of their author. He had no feeling that he was producing books which were to terminate all questions, and set men's minds at rest for ever. He merely announced what he had himself thought about different subjects, and why he did not see his way to adopt the more remote and recondite principles which some had dreamed of. Or if he did fancy that his doctrines on Government and Toleration were all that were needed, at

Moral and
Metaphy-
sical Phi-
losophy.

John
Locke.

His differ-
ent labours.

His cha-
racter and
intention.

Moral and
Metaphysical
Philosophy.

The effect
of his writ-
ings.

Innate
ideas.

Locke's
freedom
from sys-
tem.

least in his *Essay on the Human Understanding* he seems to have been fully aware that he was throwing out hints, not supplying a system. It is this honesty, and his indifference to inconsistencies, so long as he can make known what seemed to him on the whole true, which we believe to be the secret of his permanent fame, though it has very little to do with the popularity of his opinions. He is admired by a multitude of unthinking persons, because he is supposed to have swept away, or at least effectually undermined, all the feelings, notions, and habits of thinking which had connected themselves with the idea of philosophy from the earliest Greek ages to his day. He receives, and we hope will always receive, from sensible men a tribute of respect, not to be disturbed by the exaggerations of fanatical admirers or by a painful feeling that he has been the occasion, though not the cause, of much conceit, ignorance, and contempt, because they believe that he did remove some confusions which it was necessary to remove; that he affected less than most men to build when he was only pulling down; and that though he laid no deep foundations, he was in some degree the beginner of a series of experiments which it was needful should be made, and which were to terminate in the establishment of principles very different from any which he acknowledged. We will say a few words on each of these points.

After the notice which we have taken of the ideas of Plato in an earlier part of this article, it is quite superfluous to remark that the doctrine contained in the first book of Locke does not affect them in the least degree. Locke begins with assuming that Ideas are identical with Notions and Impressions, and, having made this assumption, proceeds to prove that they are derived to us through our senses. It had been Plato's great object to explain what he meant by an Idea, and how it differs from a Notion. Locke did not take the pains to understand him, perhaps it was not worth his while, but at all events he did not confute him. He does not indicate in any one syllable of his essay that he had a glimpse (we do not say of Plato's meaning, but) of the possibility that he should have a meaning. But while we admit this, we are not prepared to say, as some have said, that there was no opinion afloat to which Locke's refutation directly applied. We conceive the Cartesian doctrine of innate thoughts, though it may, by nice distinctions, be cleared of some of the absurdities which Locke ascribes to it, must ultimately expose the holders of it to the perplexities of which he takes notes. Of course Descartes would have denied that he meant by an innate idea a proposition clothed in words. But had he done anything to show how these ideas in the mind can be substantiated except in some form of words? If not, he was still open to Locke's argument, and he could only escape from it by betaking himself to a region with which his speculations do not intermeddle. There was then something for Locke to do in the way of destruction; and though he may have fancied his doings in this line much more extensive than they were, we ought to give him the full honour of that which he actually achieved. On the second point, the absence of system in Locke's writings, we need say nothing. It is admitted as much by his friends as his foes: the former affirming that he left something for his successors, Condillac and others, to complete; the latter urging it as one of his evil characteristics that he sanctioned so loose and unscientific a method in philosophy. Without wholly denying the last charge, we must yet allege this freedom from the ambition to be a

VOL. II.

Babel builder as one of the conspicuous proofs of his singleness and sincerity of purpose.

The last point is the one we are most anxious to press upon our readers' attention, that Locke did actually originate a series of the thoughts and experiments which have been in their issue of the greatest value. It may seem incredible that the proposition, "a man derives no knowledge except from his senses," should have any other than a negative importance. If we have *only* eyes, ears, and hands, we ought to be told so undoubtedly; but are we under any great obligation to a person who informs us that we *have* eyes, and ears, and hands? We answer, yes, we are. This communication, though it may not seem in itself very new, may be the first necessary step to truths which we shall not recognise with nearly the same facility. Such we believe to be the case. We believe, as we said before, that a new problem was set before men—to find out what their own faculties are, what they themselves are. We have seen two or three great attempts at an examination of this question. But they have taken in too large a sphere of thought; they have mingled themselves too much with the doctrines of a former age, and have rather indicated all the points hereafter to be discussed in this, than addressed themselves determinately to any one. It was needful to begin at the beginning; to see what the very lowest thing is that a man can do, in order that we may proceed steadily and gradually to find out what more he can do. May it not be also possible, that the order in which a man's faculties are to be trained and educated is precisely that in which, in the course of such experiments as these, they will reveal themselves to us? May it not be that men who were only perplexed as to the way of cultivating their own minds, or the minds of others, by being told that they had certain great and divine perceptions, may in this way be led to recognise one perception and intuition after another, till at last one is found to be there, which did indeed answer, and more than answer, to all that had been suspected of it? Supposing this to have been the case, we should surely see a very good reason not only for the existence of Locke's speculations, but why they should have been cast in that limited form which, in spite of all national prejudices and feelings of personal respect, we are bound to confess that they assumed.

We have mentioned by what circumstances in its France moral and political history England had been prepared to receive the doctrines of Locke, as well respecting the nature of government as the laws of the human understanding. There was not as yet the same impatience of mystery among the thoughtful men of France, though that nation was hasting on to a still more passionate abhorrence of it.

We noticed the establishment of the Jesuit body as one of the proofs of the strong impulse towards political organization and political studies which characterised the XVIth Century. This body strictly preserved the marks of its origin. It became the great teacher in the art of politics and in all other arts for the sake of this. To maintain the great universal polity of the church against the shocks it was receiving from the new life in the particular nations was unquestionably the object which was more or less consciously present to the minds of its founders. In their noble efforts to bring Pagan nations within this polity, they presented the fair side of their scheme. But it was a *scheme* still, a human invention for the sake of keeping up what the

4 P

Moral and
Metaphysical
Philosophy.

What posi-
tive truth
he taught.

Moral and
Metaphy-
sical Phi-
losophy.

Jesuits believed to be a divine institution. In Europe, therefore, the divine body which was to be upheld, and the human body which was to uphold it, became speedily confounded in men's minds. The Jesuit society was practically universal society which overlaid the national spirit in each country, but certainly never led men to dream that there was any thing higher or more celestial than that. And soon a system of morality grew up which was the natural fruit of this confusion. In describing this system it is common to say that provided good ends were sought all means were considered lawful. We do not think this phrase exactly describes the nature of the evil. The Jesuit body were guilty of any thing rather than an indifference about means, or a preference of ends to means. One would say that their great fault lay in the want of a clear perception of their own ends, and in the extraordinary devotion of their whole skill and subtilty to the invention of plans and projects. Give the followers of the Institute one clear and manly intuition of a high object, give them but the power of distinguishing between the church which they thought they were defending, and their own system, and we have no doubt that the means would have righted themselves, or to speak more truly, that none would ever have occurred to them but such as honest men might approve. And it was by keeping out of sight all high spiritual objects while they were training men with unrivalled skill and diligence to the use of instruments, by darkening their perceptions respecting the true nature and order of society, while they were suggesting all sage hints respecting the arrangements for carrying it on, that they introduced that mechanical and materialist philosophy of which the next age was to present the perfect form as well as the final results. At present there were (as we said) in France many brave and noble witnesses against this morality. Whether there may not be something in the *Provincial Letters* of that very spirit which they are attacking; whether they may not exhibit a too curious and skilful mode of dealing with points; whether the tone of scoffing which pervades them may not have helped to strengthen a temper already too manifest in the countrymen of Pascal, we do not wish curiously to inquire. At all events the deep meditative spirit which came forth in his other writings, and in those of the Port Royal School generally, and thence diffused itself among a noble and conspicuous part of the French philosophers, touching even, though not penetrating, some of the poets of Louis's court, assuredly helped to preserve the mind of France for a while from the corruption which on so many sides was threatening it. As metaphysicians, these eminent men were manifestly outgrowths of the school of Descartes, inclining to the mathematical rather than to the political, to the subjective rather than to the objective habit of mind. They rather took the polity of the church for granted, than much investigated its nature, abhorring politics as Jesuitical. It was likely that with this tendency they should bring forward into strong prominence the views of Augustine, and should rather perhaps dwell upon those which belonged to the period of the Pelagian controversy, than to those which are developed in his earlier writings. It was the tendency of the Jesuits to make human contrivances every thing, and practically to exclude the feeling of the divine will being the ground of life and law to man. If the Port Royalists sometimes too exclusively pressed the opposite doctrine, they must be judged leniently as men who had a fierce battle to fight, and were maintaining it under

Pascal and
the Port
Royal
writers.

great disadvantages. On purely ethical questions, their writings, with those of Malebranche and Fenelon, are, we conceive, of great value, but more to us than their own age. The reason is contained in a remark which has been made already. It was the particular province of this period, as we judged from certain indications at the beginning of it, to expound the nature and foundations of relative morality. But the position of the Port Royalists and even of Fenelon made them unfit to deal with this subject. They were led therefore almost inevitably into discussions which, though of a higher nature in themselves, were far less needed for the wants of the time, and, as we conceive, could in this time only be treated partially. This observation applies especially to the very important controversy between Fenelon and Bossuet on the question of the real object which the Christian moralist is to set before himself. "Selfishness," said Fenelon, "is the destruction of the human spirit. If it proposes to itself selfish aims either here or hereafter, it is not in its true state. Pure love, and the delight in him who is love, are its only legitimate objects." "By saying so," said Bossuet, "you destroy all the motives and excitements to virtue, all those hopes of a reward and fears of punishment hereafter, by which men are stimulated to act and exert themselves." The Pope of the day felt there was so much in Fenelon's doctrine which corresponded to the language of the earliest ages of the church, that it was long before he could determine to condemn him. But the spirit of the time was too strong for him. Fenelon might be asserting a truth—we believe he was asserting a great and deep truth—but it was one which had need to be connected by certain links with the life of man upon earth. If this morality were true as to the ultimate end, all our life here upon earth must be a discipline and preparation for it. How this was, how there could be a ladder connecting earth with heaven, the lowest duties of family life with the highest and most transcendent visions, Fenelon might occasionally dream; but his circumstances forbade him to develop the idea. His opponent was therefore able to take up what seemed to most men a safe practical ground, a ground which assuredly it was not safe to abandon, unless some firmer ground could be substituted for it. The most powerful member of the Gallican church declared Christianity to be a system of selfish morality, and after a feeble resistance the Romish Pontiff ratified the decision. The XVIIIth Century was to show what such principles acknowledged in words, and acted out in life, could do for the French and the Romish church, what sort of rock they would prove when the rain began to descend and the winds to blow.

But the Frenchmen and Englishmen were not the only philosophers of the XVIIIth Century. Germany, in the person of Leibnitz, was to give a very strong and clear prophecy of the works which it was destined afterwards to accomplish in this department. Though not speaking his native language, this philosopher had most of the characteristics which we have learnt of late years to associate with his countrymen: deep thought sustained by vast reading; a desire to find a common bond between sciences; a tendency to seek within us for the ground of that which is, rather than to contemplate the objects which are presented to us. In almost all these respects he is directly contrasted with Locke, in most of them he exhibits some likeness to Descartes. The Frenchman may be considered therefore the parent of his speculations, though he had seen clearly how vulnerable some

Moral and
Metaphy-
sical Phi-
losophy.

Bossuet
and Fe-
nelon.

Germany
in the
XVIIIth
Century.
Leibnitz.

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

His great
practical
position.

His theory.

of those propositions were which Locke had assailed. His discovery of a middle point between them, of a way in which the meaning of Descartes might be saved without the assertion of innate thoughts, is, we suspect, the real and permanent treasure which he has bequeathed to us. You say that there is nothing in our understanding which is not derived to it from the senses. True, said Leibnitz, but there is the *understanding itself*. The importance of this proposition, simple as it may seem, is, we believe, very great indeed. It was in one sense the strongest practical answer to Locke. It was the overthrow by an assertion, which appealed as directly to men's experience as his own did, of his theory so far as it opposed all inquiries respecting man himself, and led to the mere contemplation of his accidents and circumstances. It was a direct assertion that a man not only has and receives something, but *is* something. Looked at in another way, it was the second step in that series of experiments which Locke had commenced. A man has senses, said Locke; this was the last and lowest thing that could be said of him. He has something which *understands*—which stands below these senses, and can receive and deal with what they tell him; this was Leibnitz's supplement to that truth. But Leibnitz had other and higher aims than that of asserting such a proposition as this. Bayle had laid the foundation of French scepticism in his different writings, especially by bringing together the speculations of all past thinkers, and showing that they had arrived at no result, or that at all events the faith and the understanding had always been at variance. Leibnitz fancied that it was in his power to refute these assertions, that he could construct a *Theodicæa*, in which the doctrines of theology should be reconciled with philosophy. In making this attempt, he found himself led back by theology and philosophy alike to that question respecting Unity which occupied the Greeks and the earliest Christian age. A perfect self-subsisting Monad presented itself to him as the necessary condition and ground of all other being, as the reality from which all other realities are deduced. By this Being, this perfect monad, all things have been formed according to a pre-established harmony which is working itself out and manifesting itself through the imperfections of all individual creatures. This grand scheme must needs awaken, as it always has awakened, the greatest interest in thoughtful minds. They have felt that it does speak of things which they have need to acknowledge, that they cannot put it away from them with the words "This is nothing." At the same time they must feel, we think, that the scheme furnishes at best but the dry conditions, the logical form of a universe. As a dream, how much less beautiful it is than the Platonic Republic! How much less does it carry within itself the witness of being more than a dream. Surely it was not by such a theory that sceptics could be overthrown. They felt themselves in a dreary world of actual contradictions, a world in which they could grasp nothing that was real, in which shapes, and fantasies, and opinions were moving backwards and forwards in weary mazes, in which falsehoods and truths were so mingled together, that they lost their names and natures and became inseparable. What availed it to give such men the logical outline of a world, to show them that a harmony is conceivable? Let a man but have some glimpse of the reality, and assuredly the picture, even if it is but imperfectly drawn, will gratify him; but that any one should rise through the picture to the apprehension of the reality, where

every thing about him seems to contradict that apprehension, is indeed a vain expectation. Let us therefore give all honour to this deep-minded and enterprising German, acknowledging that he has realized the fact of our possessing an understanding, and giving him thanks for proclaiming it, but questioning how far that fact will go towards reconciling the confusions of the world, or how far it is possible to build up a system of theology and philosophy on the basis of it.

Among English thinkers, the one who seems at first sight to have received the strongest impression from the writings of Leibnitz, and to present the most striking parallel to him, is *Samuel Clarke*. He too attempted a demonstration of the being and attributes of God, and a solution of the puzzles respecting human freedom. He too brought a most vigorous understanding to bear upon this and all the other subjects which he handled. But the difference between the two thinkers is very remarkable. The demonstrations of Leibnitz have all a more or less direct reference to the demands and conditions of the intellect. The demonstrations of Clarke carry one into the outward world, and from its construction and formation would lead us to its Author. No doubt, the idea of an *à priori* necessity of a Divine Being is present to the minds of both; but the necessity in one case is much more derived from man's own constitution, and the need of certainty implied in it, in the other from the constitution of the universe. And this was the character which in the XVIIIth Century began more and more strongly to be impressed upon the English mind. As physical philosophy diffused itself, theology began to become physical also. The deeper men like Clarke, whose minds had taken their form and colour from mathematics, attempted *à priori* demonstrations; writers of feebler, though perhaps gentler character, contented themselves with deducing proofs of the divine existence, goodness, and wisdom from mere observations upon nature. Both alike assumed that man derived at least his primary knowledge of God from the external universe; both alike implicitly admitted the relations of God with man to be less close and intimate than his relations with nature. A revelation therefore was a supplement to the discoveries originally made in nature; nature told something, but not all; a Bible was sent to tell the rest, and to impart (what the outward world could not give) a knowledge of the laws which it had pleased God to establish respecting the conduct of his rational creatures. This was the only kind of language which was popular even among the divines of that day, who, of course, therefore thought very little of those mysteries which expressed the fact of a direct union between man and his Creator, and of those still more transcendent principles which are at the root of these.

It is obvious also that the question respecting the laws to which man is subjected—how far they resemble, how far they differ from those which other natures obey, must have received a new modification from this tone of feeling. In the time following the Reformation, the question was, How may the will of God and the will of man be reconciled? Hobbes changed it to—How can man's will be reconciled with his subjection to certain necessary natural laws? In this time, the distinction between the two questions was lost; they were supposed practically to mean the same—one being the more vulgar language of ancient theologians connected with some mysterious notions now obsolete; the other the language more suitable to a philosophical age. Along with this charge another should be noticed. The puzzle no longer

The
XVIIIth
Century.
England.
Samuel
Clarke.

Moral and
Metaphy-
sical Phi-
losophy.

Character-
istics of
English
literature.

Boling-
broke.

Shaftes-
bury.
Aristocra-
tical Plato-
nism.

referred to the effect either of this will or these laws upon the internal life of man, but simply upon his conduct. Could he in his ordinary acts be considered free? Could he abstain from committing murder, or do a kind service if he liked—or was he determined to one or other of these acts by some previous compulsion?

This point deserves to be noted because it is connected with so much else that is characteristic of this time. *Conduct* was now become any thing. All the literature of the age dealt with behaviour or manners—its poetry, clear, precise, definite, where more is *never* meant than meets the ear, unless it be a double entendre—as much as any other portion of it. These manners therefore were more and more felt to be the characteristic of man. By them, if by any thing, he is differenced from other creatures. So that the philosophy of the age began to turn much upon this point. ‘What do these manners or morals mean? What is the ground of them? Whither do they tend?’ This was the main question, not excluding the other respecting the powers, perceptions, or intellect which Locke had discussed, but generally taking precedence of it. The argument led to many curious results, very unexpected, one would think, by those who began it. Let us mention a few of them.

We noticed at the time of the Reformation how the old chivalrous Catholic morality of the upper classes was encountered by a more practical, personal, domestic morality in the middle classes. The distinction was not effaced afterwards. The Sidneys and the Brookes of the Elizabethan age maintained a form of character which, though different from that of the old knights, was legitimately derived from it, and the same came forth in direct conflict with the other style of character in the cavalier of the Civil Wars. In the time of Anne and the Georges this character still survived, but under a very different modification. The aristocrat strove to distinguish himself from the plebeian by being a philosopher, by seeing into the real sense or nonsense of things which the multitude humbly believed. As, however, there were of course various grades of wisdom in the persons who adopted this fashion, so there were two very distinct forms of feeling in those who set it. The first appears in *Bolingbroke*. It is that of a man with considerable sagacity and knowledge of things explaining the construction and order of the universe, why this is here, and that there; why it is better for the sun to shine in the morning, and the moon at night; why some men had better know nothing, and some men know every thing; all which is so very plain, that there needed no revelation whatever to explain it to the world, for it is in fact a part of natural philosophy and religion. This is the scheme which Pope rendered into such excellent verse in the *Essay on Man*, though the doctrine suffers much in his hands, from the want of the aristocratic nonchalance with which Bolingbroke announced it. The other and far nobler type of patrician morality was seen in *Shaftesbury*. The loyalty and religion of the chivalrous ages, so far as it was grounded upon any belief of an actual revelation or of mysterious powers, he considered accidents of superstitious times. But the loyalty to an idea of something pure and lovely he acknowledged as a genuine principle which the higher class of men are able to cherish in themselves, and which does not require helps and appliances from without to sustain it. Of such ideals we have often spoken before. After considering them in Plato, and ascertaining what he felt that they needed to give them a ground of reality and to connect them with man, we

need not consider what is said of them by one so vastly inferior both in insight and in science; still it is striking to notice how, in such an uncongenial atmosphere as that of the XVIIIth Century, buds from this old stem could still shoot themselves forth, much nipt indeed by frost and blight, but witnessing to the strength and solidity of the original tree, and not perhaps less to the nutritive qualities of the English soil in which it had planted itself.

But Shaftesbury did not really carry on the experiments of Locke; he only threw out a hint in favour of an old principle, which seemed at variance with them. A humbler man, still occupied with the same kind of investigation respecting the meaning of those manners, or morals, or proprieties, which seemed to distinguish the human race from the inferior animals, threw out a hint, which we may fairly note as the third fact respecting the powers and perceptions of men which was fairly ascertained and realized. *Francis Hutcheson*, an Irishman, naturalized in Scotland, affirmed that man has what he called a moral sense, a faculty or tact, that is to say, whereby he distinguishes that which is comely from that which is base, that which is harmonious in action or character from that which is discordant.

This fact Hutcheson observed and set it down as an important fact. We think it is so: whether it is a fact which ever can become the basis of a system, a fact which can explain the perplexities and anomalies of the world, as some of Hutcheson's Scotch successors have thought that it can, is another question altogether; our business is merely to record the observations which different thoughtful men have made, and to claim that they should not be passed over or made light of. On this ground we claim that Hutcheson's Moral Sense should be allowed a place beside Locke's Natural Senses and Leibnitz's Understanding. We do not determine the importance of the principle discovered, we only contend that it is one.

But this sense of the appropriate and the inappropriate, of the harmonious and the inharmonious, what has it to do with me and my life? Can it tell me what I should do or should not do? If it cannot, is there any thing in me that can? This question was grappled with by a man of far stronger and sterner understanding than Hutcheson. It might be supposed that *Bishop Butler* would have derived his answer to this question from some transcendent source, that he would have appealed to the Bible or to the Fathers, as testifying that there is something in man to which the moral law appeals. But this is far from being the case; no man understood more fully the kind of problems which persons in this day had to work out, and the kind of helps they were to use in solving them. He set himself to investigate the subject just as much in a way of pure experiment, with as little reference to revelation, or to any other authority, as Locke himself had done. And proceeding on this ground he arrived at the conclusion; man has not merely a moral sense, he has a CONSCIENCE, he has that which tells him not, this is fair, this is foul; but, thou oughtest to do this; thou oughtest to abstain from that. Here we conceive is a fourth fact obtained, in a way just as formal and legitimate as any of the others. We may believe that, from the mode in which Butler contemplated the subject, there was a necessary limitation in his statements which, if that mode be not taken into account, makes them more partial, and therefore less true. We may believe that he was obliged to describe the conscience as being simply *in man* when the very idea of it,

Moral and
Metaphy-
sical Phi-
losophy.

Hutcheson.
The moral
sense.

The con-
science.

Moral and
Metaphy-
sical Phi-
losophy.

even as expressed by himself, involves the necessity of its being not merely in him, of its being a connecting link with that which is above him. But we feel at the same time how inevitable this contradiction was, and how much the evidence for the fact gains by the existence of it. Starting from premises, and proceeding in a method which made the recognition of a conscience scarcely possible, Butler is driven by the mere evidence of facts to affirm its reality, and by the very awkwardness and inconvenience of some of his phrases to indicate a higher idea of this power or perception, not perhaps than he implicitly recognised, but than any which the line of argument he had taken up enabled him to express.

Social
affections.

But this was not the only fact which this remarkable man was permitted to add to those which had been realized by his predecessors. The Conscience, with its formula, Thou shalt, or thou shalt not, might seem to be expressly that which separates or individualizes. Is a man then intended to feel himself an individual creature? Is his association with his fellows merely an accident of his position, superinduced, as Hobbes had taught, by the fear of mutual destruction? Butler affirmed that the Social Affections were just as much involved in the constitution of a man as the feeling of self or self-love is; that you have no more evidence for affirming the existence of the one than of the other; that a man is drawn by as strong an impulse to his fellows as he is to assert his own distinct privileges and position; finally, that that Conscience which speaks with such distinctness and solemnity to each man's heart bears no stronger witness than this, that these social affections are true, and ought not to be violated. We think that the demonstration of this fact is quite as strong as the other, and therefore we should not hesitate to set it down as one of the principles which were added to that primary one of Locke, if we did not recollect that we had already announced a directly opposite theory from a person also of extraordinary vigour and acuteness, and that as yet we have shown no cause why one of these theories should be less tenable than the other. We must wait therefore awhile till we have some other test by which we can judge of their respective merits, or can ascertain whether there is any way of assigning some meaning to each which shall not clash with the other.

These three or four last discoveries might seem to be leading us in the opposite direction to that which we said was the bias of this age. They seem to show that man has something which is not merely natural, something different in kind as well as in degree from the other animals. Nay, they would seem to go further, to show that there is something in man not connected with his bodily structure, which is most important to him in all the practical operations of his life. The positions which were maintained about the same time by another philosopher appear at first sight to be of an entirely different character. *David Hartley*, a man of great honesty and simplicity of mind, and possessed of kindly affections and religious feeling, had entered with great earnestness into the doctrine of Locke respecting the derivation of our knowledge from outward objects through the senses; but he was convinced that this though a truth was not an ultimate truth. People may talk of receiving feelings and impressions through the senses. But how do they receive them? What are these impressions, and how are they derived to us? Hartley was struck with the wonderful connection that exists between the different impressions that are made upon us. Surely they are not casual and fortuitous, they must follow each other in an

orderly series, and would it not be possible to trace this series through all its links, to find out the key-note, and to see how each succeeding note vibrates in harmony with it? Such a scheme is surely possible, but if so, where are we to look for the key-note? On what part of us is it that objects make their impression? Most naturalists, most philosophers indeed of all kinds, would say, Upon the brain. Well then, by studying the nerves and fibres of this brain, we might convert this theory of vibrations into something more than a theory, we might actually trace in mathematical sequence how each impression, sensation, idea (for our readers must remember that the words in this philosophy are identical) arises, and how at last we arrive at the deepest and most complex ideas of all—those which refer to our own moral life and the being of God.

This Hartley has actually attempted—an attempt it seems to us undertaken with the worthiest feelings, as it is unquestionably carried out with the greatest ingenuity. Nor, we think, is it very easy for any one who reads his book to doubt that it has made some points good against all contradiction. One feels that the very conception of such a connection between man and the world, with which he is continually conversing, is almost a witness for its reality, whether Hartley has succeeded in tracing out the links of it or no. The beautiful fancy of a perfect adaptation between our bodies and the world that surrounds them, must, one would say, be more than a fancy; all power of taking in and delighting in the sights and sounds of nature witnesses its reality. The only point to be considered is, how far the theory will go. Are there not other facts which must just as much be taken account of as this fact of association? Is there not, for instance, this fact, that the man who has this brain is himself some one, that he must be before these associations could take place in him? Is there not also this other fact, that he actually exercises a dominion over that very nature which seems to be the origin of all these trains of association within him? But if so, this fact that he is, and that he has power over the inanimate world, are surely more closely connected with those moral truths, and with his relation to a Being greater than himself, than the fact of his outward associations ever can be. We believe, therefore, Hartley's book to be one of great usefulness and interest—to those especially who have some other solution of these deeper difficulties than any which he presents them with.

This conclusion might perhaps have suggested itself France. to some even in Hartley's own day. But there was a race of philosophers now rising to importance, who determined that any such weak attempt at the reconciliation of two kinds of facts which our English thinkers had in their different ways seemed to establish, was utterly ridiculous and impossible. We saw France a little while ago in a state of transition from a very high and mysterious philosophy into one which was to set all mystery at defiance. If we undertook to give a complete account of the sources of this change, and of all the accompanying changes in national feeling and character, we should have to record the melancholy too well known progress from the glittering glories of the early and middle life of Louis XIV. to the hypocrisy of his declining age, to the corruptions of the regency and the abominations of the next reign. But the direct origin of it must be referred to Descartes. The atomic theory of that philosopher survived all his other opinions, and it begat the belief among his countrymen that the only

Moral and
Metaphy-
sical Phi-
losophy.

Truth of
his princi-
ple—limits
of it.

David
Hartley.

Moral and
Metaphy-
sical Phi-
losophy.

Analysis
is morals
and meta-
physics.

Condillac,
Helvetius,

The po-
pular scep-
tics.

mode of dealing with physical substances was by analysis. Nothing could be known till it could be dissected. Acquaint yourself with the atoms of which it is composed, then, and not till then, you understand the thing; this was the maxim which established itself almost universally in natural philosophy. Now the notion which men in this age formed of natural science had, as we have seen, the most direct influence on their moral inquiries. It was quite inevitable then, that an analysis of human nature should be demanded as the one necessary thing for the purpose of understanding what man is. And it was equally inevitable that when this analysis was undertaken, the philosopher who could lay his finger upon the last and lowest thing in humanity, and show how every thing else might be resolved ultimately into that, would be accounted the *Magister Sententiarum*, the seraphic doctor of the new school. To such analysis then one class, and on the whole the ablest class, of French philosophers betook themselves. They did not in the spirit of Hartley undertake to show how one sensation may give birth to another, till at last one knows not what high results may come of them. But they said deliberately and dogmatically—all these feelings and powers that you have talked of, Understanding, Moral sense, Conscience, or what you please, are nothing else but sensations, they shall be nothing else, and whoever thinks them any thing else shall be denounced as a driveller or a knave or both. It is obvious that if these propositions be true, philosophy both began and was likely to end in the XVIIIth Century; nothing had been done in the five thousand years previous to that time, and the little that was required to do might be finished in a very short time. One does not see why, if Condillac and Helvetius really did what they pretended to do, any more books need be written in the world besides theirs. They had got to the bottom of things, they had found out the foundations of the universe. There men might rest as they could; and so indeed these philosophers would have thought if they had not found a whole world of notions and opinions based upon another hypothesis than theirs, and if they had not felt that these must be swept away in order that hereafter something might, perhaps, be built up on the firmer ground which they had discovered. In this business of demolition, however, the analysts were not found to be the best or most accomplished workmen; Voltaire, Diderot, and the encyclopædists generally did not accord them any very large measure of their respect. They had no wish to make men think so meanly of themselves as they were likely to do if they adopted the conclusion to which Helvetius especially would have led them. It was better on the whole that men should have a considerable notion of their own powers, and that they should believe these powers to have been held in check by priests and lawgivers, than that they should think those priests and lawgivers had given them credit for feelings and sympathies which they never possessed. The habit of feeling then encouraged by these men was rather that which had prevailed for some time in England, the feeling that every thing which was not at once intelligible was absurd; the rigid pursuit of any one doctrine or idea was disagreeable to them. With much respect for physical studies as a counter-weight and a counter-mystery also (for this they felt they had need of) to theology, the whole tendency of their minds was essentially unscientific; for however much knowledge of mathematics some of them may have possessed, they valued it for the sake

of its results merely, and not of its method, and such a value, whatever name it may put on, indicates a quite unscientific character. In all questions of morals, politics, art, and history, the most superficial conclusions were those which they prized most, the most loose and popular language was that which they affected. One would not therefore confound with them the men of whom we first spoke, though in the main adopting the same opinions, and anxious for the same results. They were apparently a simple, dogged kind of thinkers, more pleased with their own speculations and less caring to please the salons. Nor ought we to turn away from their doctrines respecting human nature with that utter scorn and indignation which some have expressed. That analysis is not to be the exclusive or the principal ruler in either physics or morals, that both studies will speedily be extinct if it become so, we fully believe. But that it is of some value in both seems just as evident, and we think it proved itself so in this instance. To say (as Condillac in effect said) every thing in a man can be reduced to sensations, practically you have no need to take any thing else into account, is, we conceive, to make our own lives and society monstrous and incoherent dreams. To say; separate a man from every relation under which he is created to exist, from every influence that is acting upon him, above, below, around him, from things human and divine, abstract him in short from every thing but the world that he beholds through his senses; then he will be merely a creature of sensation, and we believe a truth has been uttered, which, if when thus stated it sounds rather like a common place, has yet, we believe, need to be uttered, and may serve to put down much vague and silly declamation respecting the dignity of man and the grandeur of his nature. In like manner when Helvetius fancies that he can build up a society on the groundwork of the principle which has ever been the destruction of it, we may take leave to marvel and to doubt. But when by lively illustrations, and with much acuteness, he proves that this principle does lie deep in man, that it ever has been and ever is asserting its pre-dominance in him, nay when he pushes us on to the position that man, not in some manner raised above himself, is selfish and selfish only, we neither reject the proposition nor condemn it as self-evident, but are thankful to him for having given us much cause for fear, for shame, and for wonder. We believe then we must class among the discoveries of this time, this one of Condillac, that man not only receives his first impressions through the senses, but that in the last grovelling state to which we can conceive him here on earth reduced, he becomes merely a creature of sensation; and this of Helvetius, that there is yet in him a deeper and more awful tendency than this; to pure, absolute, unmitigated selfishness.

The scepticism in France was in one sense the deepest as well as the most universal that ever existed in a nation. In another, one would say it was by no means deep. The philosophers doubted of every thing, or rather assumed it as a matter of course that every thing was false which they could not understand. But they were never really exercised with doubts, they never fairly looked doubt in the face, and asked it what it was. This courage was reserved for those who thought and wrote in our language. Of all the men of the XVIIIth Century, the one perhaps whose heart was most free from scepticism, and whose understanding was most prone to it, was Bishop Berkeley. Other men said

Moral and
Metaphy-
sical Phi-
losophy.

Value of
the asser-
tions that
man is
merely a
creature of
sensation
and merely
selfish.

English
and French
scepticism.

Berkeley.

Moral and
Metaphy-
sical Phi-
losophy.

they had no assurance that any thing existed save that which their senses told them of. The thought presented itself strongly to his mind—and what assurance have I of that? How do I know that any of these things which I see, taste, or handle, actually are? A Cartesian might of course answer, “they are because I think them so.” But how can a believer in Locke find an answer? Certain impressions and notions are derived to me from these objects. Those notions and impressions indeed I have; them I can call my own. But is it not a delusion to say that these impressions which I have about things, these notions of them which are stamped upon my mind, prove the fact of outward existence? My senses report to me of a thing that it is green or white, and this greenness or whiteness, according to the oldest theories of vision, is just as much in the eye of the beholder as in the things themselves. But my senses cannot be shown to have any capacity of reporting to me a notion of existence. This surely is a thing which the eyes and the ears take no account of, and yet it ought to come from the senses, seeing it can come from no other quarter. Is it not then more natural, more consistent with the doctrine of Locke, however far it may seem from his actual conclusions, to say I impute this reality to the things around me; they have it not in themselves, they derive it from me? But if so, where is the ground of your denial of invisible things? The difficulty is not in believing them, but in believing the visible. Is it not rather the only way of understanding how those outward things get their reality to believe that we are real, and that our thoughts have to do with realities which we cannot see. In this strange way did Berkeley convert the objective philosophy of Locke into a purely subjective one; in this way did he make the very witness of our senses an argument against the reality of the things which they speak of. His book is one of the steps towards a new era in philosophy. We must therefore say a few words upon it. In the first place, we expect the reader at once and for ever to dismiss the notion from his mind that Berkeley was an unpractical man because he adopted this opinion. Every part of his life is a contradiction to such a notion. Instead of troubling himself only about the spiritual wants of his parish or his diocese, is he not anxious about the coughs and gout and rheumatisms of every poor man in it; and does not he invent a preparation of tar water for the purpose of curing them? Does this look like an idealist? Did ever any materialist give stronger evidence than he did of the most earnest interest in the physical and animal wants of humanity? Secondly, we require him entirely to reject the notion that Berkeley was merely starting an ingenious hypothesis for the sake of showing how much could be said on behalf of it. No one can read a page of his writings without believing in his entire simplicity and sincerity; and we should add, that few persons can have any great experience of the workings of their own minds, without feeling that he is actually stating a difficulty (and that in the clearest, truest manner) which has been presented to themselves. But thirdly, we do not require them to throw aside any belief which they really possess concerning the reality of the things which they see and handle. We do not even deny them the benefit of Dr. Johnson’s practical argument. They are, we conceive, fully entitled to use it, and to receive from it such confirmation as they want of a truth, which we believe it is very important that they should hold.

Idealism.

Berkeley’s
practical
wisdom and
sincerity.

The denial
of the visi-
ble no help
to a belief
in the in-
visible.

Fourthly, so far from agreeing with Berkeley that we gain any thing of faith in invisible realities by denying the reality of things visible, we cannot for ourselves understand how, by any consistent reasoning, a person who has adopted this theory can save himself from the disbelief of all substance, all reality in any thing but in the processes and operations of his own mind. By any consistent reasoning we say, for it is notorious that Berkeley was saved from this, that he did with the strongest faith acknowledge those truths of which his mind only could take cognizance to be above his mind and distinct from it. The result of these remarks is, that if we cannot affirm Berkeley to have made any positive addition to our store of realized facts respecting the perceptions and powers of man, yet that by fairly and honestly stating the difficulties of his own mind, with the purpose not of weakening men’s faith but of strengthening it, he has brought things to such a pass that some results will inevitably follow—some discoveries will be made, and perhaps a new method of thought which will remove some of the confusions, but not supersede the actual principles ascertained by the old one.

These hopes may at first seem rather baffled than confirmed by the labours of the next conspicuous workman in the philosophical field. This was *David Hume*—the name which we inevitably associate with the last possible, nay with an almost inconceivable and transcendent degree of scepticism. But here lay its advantage, it was inconceivable and transcendent. Hume lived amidst institutions which he had no wish to subvert; amidst a state of society which, whether good or evil, was about as good as he could wish or hope to make it. He consequently was not checked in any of his private speculations by the consideration of what would or would not serve particular aims and ends of destruction. He had a singularly quiet even temper, no strong dislike to any thing, but earnest feeling; no strong partiality for any thing, except perhaps for the Stuart family, partly because they were hateful to the Covenanters, partly it is to be hoped from some relics of national and traditional feeling. He had with this a clear and subtle understanding, which being undisturbed by those fears and hopes which in other men perplex its workings, could follow out its own conclusions whithersoever they might lead. Of inward questionings he knew so little, that he could look at every thing as a fair subject for questioning, without feeling that he was any wise concerned in the process or in the result. To this man it was given to see more clearly than any that had preceded him whither the course of inquiries which had been proceeding since the time of Bacon naturally led. Bacon had said, it is by a process of experiment that you attain to a knowledge of that which each thing is. From that time forth men had betaken themselves to experiments upon nature, upon themselves, upon the grounds of both. That the experiments were not without results we have taken pains to prove. That some of them seemed to point to a ground higher than any to which they reached we have intimated. Still it had been assumed, and the assumption gained fresh strength every day, that all which could be known was to be discovered thus; that for a man to talk of there being any thing beside that which he has experience of is vanity and delusion. In the primary proposition of Descartes respecting the certainty of our own thoughts, this feeling was latent, though that opinion seemed to lead him to a very different conclusion; when his notion of innate

Moral and
Metaphy-
sical Phi-
losophy.

David
Hume.

The wor-
ship of ex-
perience.

Moral and
Metaphy-
sical Phi-
losophy.

thoughts was exposed by Locke, the principle became more developed. Sensible experience became more and more acknowledged as the criterion of truth. The doctrines of Hutcheson, of Butler, and Berkeley, though they might affect some previous conclusions, still seemed to move in the same line. They all referred to certain experiences in man, and appeared to make these the tests of what is true. Hume then did but interpret a feeling that had been consciously or secretly at work before, when he said, "By this experience and by this alone we ascertain what is true." That which accords with the result of this experience we know to be real; that which dissents from it, at all events can never establish itself to us by any evidence.

Denial of
miracles.

The applications of this principle are tolerably well known. The two which are best known are indeed the key to all the rest, so that a short notice of them, and of the answers that have been made to them in England and Scotland, gives in fact a satisfactory view of Hume and his philosophical position. First, experience being the only standard by which we can try any fact which is submitted to us, it is impossible for man to believe in a miracle. No testimony can establish it. The testimony must itself be referred to the standard of experience. If it asserts something which lies beyond the bounds of experience, it cannot make itself good to us. It lies within the bounds of experience that the most respectable witness should be deceived, or even should utter a wilful falsehood. You are bound, therefore, to adopt that solution of the difficulty rather than the other which is self-contradictory. We need not repeat Paley's answer to this objection. It turns, as our readers are aware, upon three considerations: one the equivocal meaning of the word experience, which may be so extended as to exclude the belief of any fact to which we have seen nothing analogous; secondly, on the allegation that experience is outraged by the disbelief of testimony under certain circumstances and conditions; thirdly, on the probability that a divine being would on certain occasions violate the ordinary course of experience for the sake of making important truths evident to his creatures. All these arguments seem to us to have great value in themselves, but not to be adequate to the confutation of Hume. There may be a tendency in his doctrine to make each man's observation nearly the standard of his belief, and so to justify the indignant scepticism of the savage respecting the existence of snow. We are satisfied it has this tendency; but then show us some other standard less limited than this to which we may appeal.

Denial of
cause.

Again, it may be most true that the worth of testimony is necessarily shaken by this hypothesis, and that it drives us to conclusions the most painful to our own minds, and even, if you will, the most contradictory to probabilities. Still if there is nothing in our minds which answers to this testimony, is it not still a question between improbability and impossibility? The last appeal to the will of God carries us on to another ground, but then this is a ground which Hume's second proposition cuts from under our feet. For this is, that experience being the only measure to which we can refer the idea of a Cause is absolutely unattainable by us. A series of antecedents and consequents is all that our experience brings us acquainted with; to this series every thing must be referred. To impute efficiency to the first of a series is a mere gratuitous assumption. Here is the formula of Atheism, one not now for the first time promulgated; in a sense it may be said a very old formula

indeed, but brought out in this XVIIIth Century as the final result to which all human inquiries, thoughts, experiences, had been tending and in which they were to terminate. Against such a formula, Paley might try to bring his natural theology, feeling that all nature spoke of a designer. Unquestionably the feeling was of great worth. He said truly, that it was in him, and had been in all men. But he had to show against Hume, that he did not get the feeling from old religions, nay mainly from that very Christianity which he hoped to confirm by the help of it. He had to show that he really and honestly deduced the idea of a Will from that nature which seems to speak only of succession.

We are inclined therefore to think that the effort of Beattie, Hume's countryman, *Beattie*, against this scepticism was on the whole a more plausible one. He appealed to the Common Sense of mankind. He said there has been a general testimony in favour of these truths. They actually have been held; surely this is an argument that there cannot be any thing in human nature which formally repudiates them. Such a notion we conceive had a certain value in it, but more we think by what it suggests than by any thing that it actually makes known. It suggests the thought—What does this common sense of mankind mean? How do such high and extraordinary thoughts become in any sense the property of a race? How is it that common men can embrace them, and that it should require the wit of philosophers to show they are not tenable? This was a most important thought, one which might well set men on very earnest meditations, and which was connected with many movements that were about to take place. But Beattie's doctrine, looked at on another side, is rather a sop to inquiry than a satisfaction of it. It leaves the alleged impossibility just where it was, and only adds to it the additional marvel which Hume in his *Natural History of religion* had undertaken to clear up—how men could have been brought into the united acknowledgment of something of which they had no real witness. And there was a worse effect of this plausible doctrine, that it brought down thoughts, which either mean nothing, or have the deepest of all meanings, to the most superficial of all tests. The belief of principles respecting ourselves and God was to be supported upon the same ground as some custom or fashion which had lasted too long to be abolished.

Far more importance is to be attached to the doctrine of another Scotchman, *Reid*. His opinions under one form or another, through his own writings, or through his eminent disciple, Dugald Stewart, have obtained a considerable influence over the mind of England, the causes of which it is worth while to investigate. This may be briefly described as the school which endeavours to remove the difficulties Hume raised by enlarging the notion attached to the word Experience. Outward sensible experience seems to be all that Hume intends by the word. Inward Experience or Consciousness is that which Reid superadds to this as one of the grounds upon which our judgments are to be regulated. Now our readers must perceive at once that there is nothing new in this doctrine, that Descartes was at least as great a witness in favour of Consciousness and its authority as any Scotch schoolman of the XVIIIth Century could be. But philosophy, as we have often maintained, is either a reflection of the feeling of the time, or a prelude to some change which is about to take place in it.

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

Effects of
a religious
movement
upon phi-
losophy.

About the time in which Reid's system began to be proclaimed, a very strong reaction had taken place both in England and Scotland against that formal and natural theology which we spoke of as characterising the beginning of the XVIIIth Century. Other and deeper principles and feelings connected with this reaction we shall have hereafter to notice. One indication of it especially concerns us here. Dogmatic announcements of what it behoved men to do, or to think and profess, had formed the substance of the teaching hitherto. All reference to the feelings and conflicts within the heart had been omitted. The new tendency was, as might be expected, vehemently in the opposite direction; that which a man feels, knows, experiences in himself, was the subject of which all the more lively and popular divinity began to treat. A deep craving there was for that which should explain to men the workings in their own selves, which should deal with their own consciousnesses and consciences. Such cravings might seem to have very little connection with the speculations which were going on in the schools of philosophy. In one sense they had no connection with each other. Now more than at any former time there was a severance between religion and philosophy, and the new religious teachers especially were regarded by the philosophers with a pity and indifference which they fully returned. Nevertheless the strongest impulses that are at work in a time unconsciously affect all those who are living in it; and these Scotchmen, who perhaps less than any others would have liked to confess such an influence, gave the strongest tokens in the character of their doctrines, that they were subject to it.

But it must be confessed, that whether the appeal was made to internal or external experience, no position had been yet taken up from which Hume's conclusions could be effectually assailed. Every one of these attacks upon them was a witness that men could not yield to such complete and absolute scepticism. That religious movement of which we spoke was a declaration of the heart and spirit of men against it. Still it had a resting place in the understanding, and so many of all the notions and habits of the day in England and France seemed to be in accordance with it, that no one could say how soon it might come forth from that resting place, and make itself practically felt. It surely would be most satisfactory to see wherein its falsehood consists, and to see at the same time the ground of its plausibility. And if the experiment, (for an experiment it must be,) by which this point is ascertained, could be conducted by some perfectly fair unprejudiced person, living apart from the influences by which Hume or his adversaries were affected, not open to the suspicion of any religious or ecclesiastical bias, not substituting any ancient methods or notions for those of his own day, avoiding every thing which could be construed by the severest logician into an illusion of the fancy, we might regard it as a most providential circumstance. It is our belief, that a citizen of that nation which had already produced Leibnitz combined these qualities, effectually asserted the principle which was needed to confute Hume, and by doing so added the last to the series of facts of which Locke had announced the first. From what we have said of German philosophy as it was left by Leibnitz, it will be evident to our readers that it was proceeding in quite a different direction from the English and French philosophy of the same periods. Its character was, as we said, even more mathematical and subjective

VOL. II.

than that of Descartes had been. This tone had been preserved among the disciples of Leibnitz. A very eminent thinker, *Wolf* of Breslau, followed in his train. He did not adopt the grand scheme of Leibnitz concerning Monads and the Harmony of the world, and on Wolf that very account he gave a more decided emphasis to the other parts of his creed. The *Theodicaea* was, as we have seen, an attempt to reconcile the understanding with revelation. Wolf appears to have thought that all the principles of morals and metaphysics, nay the principles of all sciences, lie in the understanding, and may be deduced from it. He considered it possible to construct a complete scheme of ethics and philosophy from the data supplied by the mind itself. Probably no man had arisen since the middle ages who had so many features in common with the schoolmen. He differed from them, because he, like every philosopher since the Reformation, was consciously studying his own powers and perceptions, and not merely contemplating them as involved in the general science of Being. But in his resolute adherence to *a priori* dogmas rather than to the deductions of experience, he caught their likeness even while he wandered furthest from their positive conclusions.

The fact that a philosopher of this kind was the one Kant, who exercised most influence in Germany at the time when *Emmanuel Kant* appeared, ought to be well taken notice of by the English student. For he is apt to fancy that the critical philosophy of that great man was a scheme merely set up in opposition to the school of experience which prevailed in England, in Scotland, and in France. This is far from being the case. The studies of Kant were directed quite as much to the object of vindicating a domain for Experience, as of showing into what region its authority does not extend. We have seen that since Logic lost its supreme authority, another science, the one which is the great instrument in all physical studies, had been assuming importance in the speculations of one large and profound class of thinkers. Mathematics had been felt as at once the witness for our need of certainty, and as the means of attaining it. It was evident to a careful observer of the course of speculation during the last century and a half, that this science was the great staff of the *a priori* thinkers—of those we mean who sought for the grounds of knowledge in themselves—while the facts and appearances of nature were the foundation of all teaching in the other school. Kant's first important position then was a severe blow to the Cartesian and Leibnitzian school. Mathematics, he said, like logic, embody forms of our own minds; the one expresses the forms under which the outward world presents itself to us, the other the forms under which the understanding conceives and generalizes the sensible impressions that are submitted to it. Space and time are the forms under which outward things reveal themselves to us; the conditions and categories of logic those under which we judge and discourse of things. But yet it is obvious that mathematics and logic do connect themselves, and have been always felt to connect themselves, with different parts of our being. The line and method of thought into which they conduct us are most distinct from each other, oftentimes one would say most opposite. It is equally certain that the organ of mathematics must be distinguished from the organs of sense, to which it serves as a handmaid, and acts as a corrector. Is it not evident then that here are two distinct faculties—one the

Moral and
Metaphy-
sical Phi-
losophy.

Wolf

How he
asserted the
worth of
experience
against the
a priori
thinkers.

Moral and
Metaphy-
sical Phi-
losophy.

abstracting, generalizing, logical faculty, the other the properly mathematical faculty, that whereby we are able to take account of the conditions and forms under which the things exist that our senses report of? Would it not be well to restrict to the first the name of *Understanding*, and to find another name for the second, both being evidently powers or faculties of the mind? In asserting this distinction, Kant was asserting the freedom of experimental observation. He was contending that it should not be confined by mathematical fetters any more than by logical fetters; that we see things under certain conditions, just as we form our judgments under certain conditions; but that we are not to impute those conditions to the things that we see any more than we are to impute the conditions of our own judgments, our theories and classifications, to them as the schoolmen did. So far it will be admitted that Kant was strictly Baconian. He was in fact carrying out the very principle which Bacon asserted.

How he
asserted the
existence
of prin-
ciples an-
terior to
experience.

But while we have thus been defending the cause of experience against those who impugn its verdicts, what have we been compelled to say? We have been compelled to assume that there is some organ in us which has to do neither with the beholding of things nor with the merely abstracting and reducing them under heads, one which does not deal with facts submitted to experience, but affirms principles antecedent to experience. There is no distinction between the mathematical organ and the logical organ, unless we admit this. But if this be the case, if there be this capacity of acknowledging data which experience has not supplied, Hume's positions are at all events most unstable. Leaving all the results of them out of the question, treating the matter as dryly, hardly, formally as you can, we have been obliged to recognise some faculty in us which looks straightly and directly at principles not derived from experience, but at principles which surpass experience and regulate it. You say; but this has merely reference to certain mathematical forms, to the possibilities and conditions of things, not to realities. Most true; it has been our object to show this very thing, that this faculty of which we have been speaking does only deal with conditions and possibilities. But if even in things sensible, in things notoriously and confessedly submitted to the eyes and ears, we require some faculty to tell us principles antecedent to experience, who shall dare affirm that all the witnesses which men's minds have borne to principles and objects which are either nothing or most real, are false because they cannot be referred to the rules which experience supplies? You require us to bring the belief of a Cause, of our own Freedom, of our Immortality to some sensible standard. But there is no such thing as a sensible standard. The very facts of sense appeal to another standard. They must be tried by laws antecedent to experience. Which then is most wise, most accordant with the facts of experience, with the actual history of the world, to believe that these ideas of God, of our own freedom, of our immortality, have been established in the world by a succession of frauds and impostures upon the fears of men—to believe that they have been obtained by some laborious generalizations and deductions from outward nature, or from the laws of our own mind—or to believe that, as there is a faculty, which even in matters of sense affirms principles antecedent to sense, so there is a corresponding faculty affirming these great realities, a *Practical Reason*, as the other is a *Speculative Reason*?

The spe-
culative
and prac-
tical Rea-
son.

You say that all demonstrations of the being of God or immortality have been ineffectual. True, they have been so, and you see why they must have been so. It was not by the logical faculty that such truths ever could be arrived at. The plausibility of such arguments and demonstrations, whether deduced *à posteriori* from nature, or *à priori* from the human understanding, lay in this, that there was a higher witness of them present within all of us. We impute its testimony to the arguments, and so make them valid, when, in truth, according to any conditions of the understanding, they are not valid. Let them all be shaken to pieces, but let those who shake them explain whence the faith in them was derived, and say whether this faith be not a greater confirmation of them than the demonstrations would have been if they had been ever so solid and clear.

Such were the conclusions at which Kant arrived by a process of thought and reasoning certainly as severe, as little affected by any influences from the imagination, as little determined in one direction or another by reverence of authority or dread of consequences, as any that were ever followed by any thinker in the world. To adopt them in their fullest extent is not to avow ourselves Kantists, either in a sense which sets at nought the labours of previous inquirers, or in one which supposes him to have constructed a great philosophical system. We have shown our readers how we can recognise Locke as the assertor of an important truth, the remover of some confusions, and the originator of a line of inquiries, without being Lockites, nay while professing the strongest dissent from those who call themselves by that name, and while believing their master to have narrowed most grievously the limits of human belief. In like manner we could acknowledge that Leibnitz maintained a most important truth consequent upon and connected with Locke's, while yet the theory or system which he strove to ground upon it seems to us of comparatively little worth. And generally, in reference to all the thinkers of this period, we have contended that just so far as they were occupied in the search after human powers and perceptions they were proceeding in a profitable course; that just so far as they were attempting to set forth the objects upon which these powers were to exercise themselves, or to deduce any theory from the existence of these powers, they were going beyond their province, and attempting what they could not perform. At all events, therefore, we are consistent in applying this distinction to Kant. We hold that he alleged the most serious grounds for believing in the existence of faculties which had not been recognised by any of those inquirers who, since the time of Bacon, had devoted themselves to the examination of man's nature and to a comparison of it with other natures. We hold that this discovery, so far from setting at nought those which we owe to Leibnitz, Hutcheson, or Butler, illustrated and confirmed them, enabling us to see a warrant for their language which they could not see, and relieving it of many confusions and embarrassments. We think further, that Kant's principle throws back the greatest light on Berkeley's Idealism, Hume's Scepticism, Beattie's Common Sense, and Reid's Instinct; showing why Berkeley was driven to rest satisfied with the impressions on his own mind, and so to doubt the evidence of his senses, just because he did not feel that there was a power in that mind which had to do with invisible laws and realities, as those senses have to do with what is visible; how Hume was led to outrage the

Moral and
Metaphy-
sical Phi-
losophy.

Kant's
facts and
Kant's
system.

Kant con-
firms and
elucidates
other men's
discoveries
and opi-
nions.

Moral and
Metaphy-
sical Phi-
losophy.

conclusions of experience, because he could acknowledge nothing besides experience; how truly there must be a Common Sense in mankind respecting that which has not to do with outward experience, if the Practical Reason be indeed the great common inheritance of all, and yet how certainly that witness is not one lying on the surface of our minds, and capable of abetting all the fantastic conclusions which may have mingled themselves with deep truths through sensible idolatries, and ill understood experiences; but the deepest thing of all, testifying of that which is the most pure and necessary, and absolute; how certainly there must be high Instincts and Consciousnesses in the mind, but how they point to a ground above all instincts and consciousnesses to that which is most constant and permanent. But we hold at the same time that *Kantism*, or the attempt to build upon this doctrine of a practical and speculative reason, has inevitably led to the loss of all these good consequences, to the denial of that which other men had established, to the practical renewal in their worst form of all those confusions from which the Critical Philosophy, properly used, may be the deliverance. Our meaning is this. It was, as we have seen, part of Kant's object to set men free from the bondage to certain Forms which had taken the place of or else mingled themselves with those logical forms by which men had been enslaved in the scholastic ages. To accomplish this object, he was obliged to use language in which there lurks at least a possibility of much perplexity. He was obliged to say, "Space and Time are forms of our own mind."

Perversion
of his doc-
trine to the
denial of
other men's
facts and
the con-
fusion of
his own.

When he explained himself, it was quite evident that this assertion implied something very different from the corresponding one, that the forms of logic are forms of our own mind. By logic we compare different words and notions which are presented to us. The laws then which the understanding takes notice of are expressly and strictly its own laws; the laws which the reason takes notice of can only be considered as the laws which determine the relation between it and external things. We may therefore say, and we conceive ought to say, that Space and Time are the laws of nature just as much as that they are the forms of the speculative reason. We do not express the whole truth, except we resort to both these modes of speech. It does not lie in either, but it lies between them both. Space and time are not in nature as actual objects; to suppose this is to confuse all our experimental studies. But on the other hand, if we do not speak of them as being presented to us as well as existing in us, we shall unquestionably lose the very distinction which Kant realized, confound the forms of the reason with the forms of the understanding, and make them as tyrannical and dangerous as ever the others had been. Now to what extent Kant devised a nomenclature which could guard against this evil in respect to the speculative reason, and how far, if he did, his disciples have profited by it, we shall not stop to inquire. But the corresponding danger with respect to the Practical Reason many of them certainly have not avoided, and we believe while they consider Kant as their exclusive teacher they cannot avoid. As Kant said that the forms or laws of the external world were in the speculative reason, so he said that the ideas of freedom, of immortality, of God, are in the practical reason. In what sense such language may be used, without danger or irreverence, we endeavoured to explain when we were speaking of the Platonic philosophy. But it is quite obvious that the peril in the case of Kant is

much greater than in the case of Plato. Plato was not investigating the nature of a faculty. He was considering under what form truths present themselves to us, so that they may be at once our own and not our own, that we may recognise their self-existence, and yet feel that we possess them. The mischief to be guarded against was constantly present to the mind of the Greek. His doctrine of ideas was the strange, and to a person who had not felt the confusion, the incomprehensible and contradictory language by which he escaped from it. With Kant it was far otherwise. He was asserting distinctly and formally the existence of a certain faculty, and he was to define the nature and the limits of that faculty. How easily then might he slide into expressions which signified, if not to his own mind, at least to that of eager disciples who looked upon him as the great prophet of the world, that these truths are actually embodied in the reason; that all notion of them as objective truths presented to us has been once and for ever confuted; that all revelations pretending so to present them are merely the delivery, in confused forms suitable to vulgar apprehensions, of that which was already in us beforehand, which sages of old knew perfectly well, and which modern sages are to separate from all their inconvenient drapery, and exhibit so far as opportunity and prudence permit in their naked and original simplicity. The true answer to such notions, we conceive, will be found in a calm and clear explanation of Kant's doctrine, and of the steps by which he was led into it. We shall then see that any view of the reason which does not make it a Faculty of Apprehension and Converse, which does not suppose it to require objects, and to have its life only in the beholding of them, destroys its nature and definition. It will then be felt and understood, that the discoverer of such a faculty did not by discovering it make us better acquainted than we were before with the Objects upon which it is exercised; that he may not have been the least qualified to throw any light upon that matter, and yet that he may have been doing the greatest service by showing that those Objects are not addressing themselves to creatures with no capacity for comprehending them, but to a being whose deepest necessities are unsatisfied, whose deepest powers are unexercised till he can recognise them.

The metaphysical philosophy of this period (so far as it appertains to the nature of our perceptions or knowledge) had now reached the highest point which it was capable of reaching, and at that point had become mixed with the most important moral principles. It had also reached a point at which school philosophy became obviously connected with human life. The laws of the human understanding, and even the maxims of experience, may belong to learned men. The demonstrations about Being and Will had little to do with the multitude. But if Kant's doctrine be true, that there is that in every man, which not by slow processes of proof, but directly, testifies of the most absolute and essential verities, the whole race was interested in that as a fact which as a speculation belonged only to the few. Many other contemporaneous events indicated that a crisis was approaching when the deepest principles would be found to have the closest and most vital connection with the movements of men of whom philosophers knew nothing, and from whom, by their idolatry of their own peculiar gifts and methods of knowledge, they are often as widely separated as the worldly great by their idolatry of fortune and fashion. Something of this might be traced

Moral and
Metaphy-
sical Phi-
losophy.

Plato and
Kant.

Meeting
point
between
philosophy
and human
life.

Moral and
Metaphy-
sical Phi-
losophy.

Adam
Smith.

Burke's
vindication
of natural
society.

The Will.

Jonathan
Edwards.

in the kind of theories about this time coming into vogue in regions which German metaphysics had not penetrated. It was not strange to see the chosen friend of David Hume devoting himself to so purely experimental and objective a study as the *Wealth of Nations*. But it was somewhat prophetic of the approaching time, that he should set forth the feeling of *sympathy* as one of the leading principles of human nature, which accounted for that marvellous tendency to combine and associate discoverable among men, and for the action and reaction of huge masses upon each other—facts which would indeed soon require all the wits of all the philosophers to expound them. It was not an uninteresting or unimportant circumstance, that a friend of Adam Smith, nearly the greatest man of this age, *Edmund Burke*, began his remarkable career with a parody of Bolingbroke, teaching how the principles which philosophical aristocrats had advocated respecting natural *religion*, and its superior value to every thing revealed, might lead humbler folks to prefer natural *society* to that which gave titles and seignories. The step from the prophecy to the fulfilment, from the pretended vindication of natural society to the actual vindication of it by Jean Jacques Rousseau, is not long, but it is filled up with some facts which we must take notice of as most important in the illustration of philosophical opinions as well as of their relation to the history of mankind.

The great question of this age had turned upon the relation between human nature and other natures. Thus we must for the present express the question, though the language is not that which we might ourselves prefer, because, as we stated before, the point at issue really was whether man is or is not a *super-natural* creature. If we believe Kant's decision to be right, that man has a reason which is capable of conversing with that which is transcendent and absolute; we shall consider that, in reference to his capacities of *knowing* at least, this question was settled in the opposite way to that in which Hobbes had settled it. But there was another denial of Hobbes which had been felt by all men, by all religious men especially, to touch the vitals of truth even more nearly than this. He had denied man a *will*. He had taught that he is subject to the same kind of necessity as all other creatures. Hobbes believed himself—all his opponents believed—that in this principle his other conclusions were involved. Neither he nor they could understand what a scheme to redeem man could mean; what the idea of an intercourse between man and his Maker could mean if the fact were once established, "Man neither actually is nor is intended to be any thing but the obedient subject of certain motives or influences external to himself." But we have seen how strangely religious men had suffered the ideas of nature and God to mingle together in their speculations, and how deeply the habit of confounding them had worked itself into the mind of this age. It is not, therefore, matter of wonder that a man should have arisen, of strong logical power, and of great moral excellence, who proposed to himself the task of constructing a scheme of Christianity from which the idea of a Will should be formally banished. We use this language advisedly in speaking of *Jonathan Edwards* the American philosopher. For it is not true, as he and some of his admirers pretend, that he was merely reasserting in a more logical form the doctrine of Augustine, or at least the doctrine of Calvin. Between the principles of Augustine and those of Edwards there is a distance actually impassable;

nor do we believe that the Helvetian reformer would have protested much less vehemently than the African bishop against this supposed reproduction of his opinions. The redemption of the will out of its slavery to nature—including under the term nature all influences and motives whatsoever which prevent it from enjoying the perfect freedom of intercourse with, and subjection to the divine will—was the end which Augustine recognised in all the interferences of God on behalf of his creatures. Calvin may, no doubt, have sympathized more in that part of his writings which asserted the absolute power and dominion of the divine will over the human, than in that which spoke of its own cries for deliverance. But we do not think that even for a moment he conceived the subjection to a set of external motives, which Edwards spoke of as the proper condition and constitution of the human Will, was otherwise than the badge and proof of its bondage. We have a right then to say, that this theory, when connected with Christianity, was a new one; that it was an attempt to make the principles of Hobbes support a creed which he, on account of these principles, rejected. We are quite aware that Edwards rejected with indignation the idea of a natural law as governing human actions, and that his disciples believe him to have relieved the question of all its difficulties by introducing the phrase *Moral Necessity*. Our statement of his opinions will not, however, be disputed. He does treat man as properly and by constitution subject to a certain set of motives, let these motives be called by whatever name they may. Edwards does not believe that he is intended for freedom, or that freedom, in reference to him, means any thing. And this proposition his followers say that he made good by the most irresistible logic. We believe them just as we believe that by a train of logic equally irresistible Hume proved that there can be no Cause. Once admit that there is nothing above logic, once agree to deny the idea of a mystery which man owns, though his understanding cannot comprehend it, and we acknowledge that the belief both in a divine Cause and in a human Will must disappear. But it cannot disappear: that very Will which your logic has proved not to exist rushes in and tears its conclusions to atoms, and that Being of whom your logic can take no account demonstrates his own reality. Of such confutations of theories by facts we are about to speak; in the mean time let us do homage to the thorough honesty of Edwards's mind, an honesty which, though shown in following his premises to their actual conclusions, is shown also far more satisfactorily in surmounting these conclusions, and in penetrating into regions which they could have never led him. He speaks of Religious Affections as one who had them, and who could not therefore really assent to mere conclusions of the understanding, and he describes *The Being* as the end of all desires and contemplations in language familiar to Platonists, but which to the majority of his disciples must be utterly incomprehensible.

One would have thought that the events which were connected with the appearance of the Methodists in England, Scotland, and even in America, events which no doubt interested Edwards beyond all others, would have been enough to demolish his hypothesis; for surely their preaching was an appeal to the Will which lay deep in the peasants of these lands, and by that Will it was answered. That it could not have come forth except it had believed that it was summoned by a Will higher than its own, we are well con-

Moral and
Metaphy-
sical Phi-
losophy.

He did not
resemble
Augustine
or Calvin.

Moral and
Metaphy-
sical Phi-
losophy.

vinced. That it was the recognition of such will—such a *Spirit* which gave all the worth to this excitement, though, being mixed with confused human and natural notions, it may have given also the particular shape to its superstitions, we are also assured. This was the theological truth which upheld the newly developed sense of a Will in our countrymen, and prevented it from being destructive, as the belief of a Deliverer and Mediator was the theological principle which upheld the newly developed sense of an *Intelligence* at the time of the Reformation, and which prevented that from producing an immediate and universal scepticism. But under whatever condition, it *was* this sense which was stirring here and in all parts of Europe, and which was to prove what reality and might it has either for the highest good or the most fearful ruin.

Conclusion.

And now we are arrived at that great practical experiment which was to confute or establish the logic of Hobbes. Rousseau, the beginner of the new era, proclaimed that the savage state which Hobbes had denounced was the best for man. He proclaimed too the doctrine of a Social Contract with all the pomp and circumstance which Locke had been unable to impart to that theory. A whole nation was roused by such words from its slumber. The principle of savage life, the right of each individual man separate from his fellows, was asserted; governments were built upon this foundation, and fell to the ground; at last the strongest man prevailed and despotism followed. Were not the positions of Hobbes demonstrated? Assuredly, so far as this they were demonstrated. It was shown that man cannot exist out of society, that it is a dream to think of his existing out of it, that he must be held by some other bonds than by the imagination of a Social Compact. But in the mean time it *had* been proved that man has a *Reason* and a *Will*. A *reason* which, if it has not some absolute infinite object to satisfy it, will strive to make the finite infinite, to deal with facts of experience as if they were absolute truths, reconstructing a nation or a universe upon a theory—a *will* which must either acknowledge the mysterious government of a will higher than its own, or must work the confusion which Hobbes fancied in his dream that a human despotism could restrain. These were conditions which Hobbes had not taken account of. What result should they have upon our minds? Do they not seem to show that Society itself is a transcendent thing, that it cannot dispose itself according to any accidents and conventions, that its foundations lie where Plato fancied that they lie, and that every particular society which does not do homage to some universal society based upon a deeper ground than itself, is doomed to inevitable destruction? If this be so, then we may believe that those two lines of thought which began to diverge after the time of Bacon, those which we called the political and mathematical lines, have found their meeting point. Kant's discovery of a Reason in man is the termination of one line, and the discovery that a Society is necessary which deals with that Will and Reason as supernatural faculties, is the termination of the other. Still there was a third line of thought which has been running on beside these two through all these centuries. From the time of the Reformation we saw men beginning to feel the necessity of another and more human form of morality than that ideal morality which had belonged to the middle ages, or that transcendent form of Christian devotion which had characterised the age of the Fathers. We have seen men at a loss to understand what this

lower morality might be, resolving it into pure selfishness with Bossuet, treating it as mere Manners and Conduct with the Englishmen at the beginning of the XVIIIth Century. Yet it must be more than this, for Butler in that very time found that we had a Conscience, and he said also that we had *Social Affections*. May not these Social Affections be the link between the lower and the higher morality, not perhaps being separated by any wide chasm from those *religious* affections of which Edwards spoke? The reformers we remember discovered in our human relationships more than the symbols of a relationship between man and his Creator; they said that through the fulfilment of one he ascends to the knowledge of the other. If this were the case, we can better understand why the assertion of a solitary individual existence in France was connected with the denial of any relation to God, and why the Universal Society which the reason and will of man seem to require, must be one which is not merely Platonic, not merely based on Absolute Truths, but united with all Family Relationships, and with the distinct ordinances of National Life.

We have not scrupled to enter upon topics of this kind in a sketch of moral and metaphysical inquiries, because we know no other way in which we can so effectually prove that these inquiries have reference to realities and not to dreams, and that they have not been conducted in vain, but have led to results which are confirmed by as certain evidence, and are at least as important to the welfare of the human race as any that physical science can boast of. When we set down a number of warring opinions as the dogmas of particular thinkers or sects, we are easily led to believe that there is no moral Science, that all manner of questions have been discussed, and that nothing has been proved. But when we connect together the speculations of different periods, and mark how they bore upon the events transacted in these periods, we acquire a far more cheerful and hopeful feeling. Then it seems evident that no earnest thinker has lived in vain, that the conclusions of no age have been ultimately superseded or displaced by that of the one which has succeeded it. We find that in every time principles have been submitted to the severest discipline of opposing feelings, notions, and passions, of unwise and dishonest advocates, of cruel persecutors and dangerous patrons. We find that they have been overlaid with theories and schemes the most fantastical and incongruous—that they have been applied to cases to which they had no relation—that they have been distorted till they became curses instead of blessings, and yet we have seen them coming out of this ordeal only clearer, brighter, and stronger. This being the case, we cannot but account that doctrine a most narrow one, injurious to man, and blasphemous against God, whatever pretences of enlightenment or piety it may put in, which would disparage any portion of the world's history. Whether it assume the form of a reverence for the Past or the Present, whether it lead us to think that no light or wisdom has been vouchsafed to men in later days, or that all light and wisdom are with us, and that our fathers either dwelt in utter darkness, or were only catching a few glimpses of truth which we possess in clearness and fulness, we think it is equally at variance with the witness of our own consciences, the evidence of

Moral and
Metaphy-
sical Phi-
losophy.

Evidence
of a moral
and meta-
physical
science.

Moral and
Metaphy-
sical Phi-
losophy.

Character
of the age
on which
we have
entered.

Indications
of it in
Germany.

history, and the promises of revelation. Nay, we believe, in both cases, such an opinion operates against the reverence which it pretends to encourage. The feeling that we are hopelessly incapable of thinking and feeling for ourselves destroys all affectionate sympathy with those who came before us, causes us to regard them with fear and prostration of spirit, not with cordial admiration; as harsh pedagogues, not as friendly teachers. The habit of believing that the concentrated essence of spiritual and moral truths is contained in the speculations of our own day, makes those speculations an intolerable bondage, stifles all freedom and originality under the pretence of fostering them, and ties us down to transitory notions and modes, while perhaps we are in the very act of boasting that we have thrown off such modes, and have attained to the knowledge of that which is pure and permanent. Against these diseases no remedy, we believe, will be found so effectual as the patient study of history. Many indications seem to point out this as the particular work of the time upon which we have entered. Everywhere men will be found attempting upon some scheme or other to combine and reconcile opinions. Everywhere men will be heard talking about comprehension and universality, till others, disgusted by the hollowness of their language and their practical inconsistencies, strive to shut themselves up in some exclusive circle; but soon experience there the very same impulse, and repeat the experiment in some newer method. Everywhere, we think, it will be seen, that those who are labouring to devise theories and plans of their own exhibit a lamentable want of originality, and that those who spend most labour in studying and recording the doings and thoughts of others, kindle in themselves a true and native spirit. A glance at the progress of thought in the three principal nations of Europe since the French Revolution, will, we believe, confirm this opinion, and may help to show us in what kind of work a student of moral and metaphysical philosophy may now most profitably engage.

It is natural to turn first to Germany as to the country in which the philosophical spirit has most strongly manifested itself. About the time that the critical philosophy was promulgated, the creative powers of the Germans began to be awakened, and a race of eminent Poets and Critics arose in that country. How much this new literature was connected with the use of a native language and the growth of a national feeling we need not stop to explain. Any strong efforts of the imagination must, as we have often had occasion to remark in the course of this sketch, be national, and will generally be observed only at some great national crisis. But it is important to notice wherein this literature mainly differs from any previous European literature. It is characteristically historical and critical, reproducing old forms, bringing back the study of authors and the remembrance of traditions which had been lost, commenting upon the forms of poetry and art generally as manifestations of particular forms of feeling—in all respects acknowledging the past at least as a storehouse of facts and images. Whether this kind of writing may not ultimately have cultivated that idolatry of this age which it seems to be resisting; whether the great men of Germany may not have felt, that they who could look back upon past ages, and take note of the peculiar features in each, must needs be superior to all who went before them—that the faculty of criticism is the great faculty of all, and that he who can

use it is more wonderful than he who possessed the thoughts which are criticised—and whether this feeling may not have communicated itself in a tenfold degree to their disciples and successors, who, if they have none of their creative or their critical power, at least talk of each, till they work themselves into the sense of possessing it, are points upon which we are not qualified to express an opinion, though they are doubtless important for the consideration of Germans themselves. At all events, that effort at comprehension which we spoke of, accompanied with a zealous contemplation of the past, has belonged in a high degree to these German artists, and the systems of philosophy which have grown up during the period in which they flourished, seem to show that the inquiries with which we are more immediately concerned must eventually take the same direction. It is to the object and central point of those systems, not to any of their formal dogmas, that we shall refer. If the reader can allow for the difference between the periods in which they appeared, he may find in *Fichte* and *Schelling* parallels to some of the eminent doctors who divided the schools in the middle ages. The first a noble and earnest spirit, formed in a time of the greatest national excitement, and possessing the strongest national feeling, rose up to assert that the individual, the “I myself,” is the groundwork of all philosophy. He could not find in Kant, or in any other philosopher, a satisfactory recognition of this personality, without which he was sure that life is a dream, and therefore without the confession of which all philosophy must be a dream too. The intense necessity of this feeling we hope our readers will be as ready to recognise as they will the conviction in the mind of Schelling, that the attempt to ground a system upon it is hopeless. The need of an absolute ground of things was that which presented itself to him with as great force as the other thought had presented itself to Fichte, and this need he fancied in like manner that a system might satisfy. We are not obliged to inquire into its nature, seeing that it is said not to be satisfactory to its author. At any rate the want which each system expressed was felt and realized by innumerable minds, who, having felt it, probably became soon dissatisfied with the form wherein it had first made itself known to them. Another amiable and benignant philosopher complained of the harshness and dryness of Kant’s speculations. The heart (said Jacobi) demands satisfaction as well as the reason, and what is there in the critical philosophy which supplies it? Jacobi sought to supply it himself. We are not aware that he succeeded, though if he led others to feel the necessity, he surely did not live in vain, whatever his scheme may have done for them. But the German mind, while fully capable of sympathizing with all these vital principles, has unquestionably a craving after system and order for their own sake. When the tumult of national feelings had subsided, a scheme became popular, which aimed at the most extensive combination and comprehension, not starting from the lowest point, and gradually feeling its way upwards, but beginning from the highest ground, and reducing all the thoughts, speculations, intuitions, projects, expectations, sciences of men, to their proper compartments. This vast doctrine of Hegel alarmed the mind of a deep and earnest thinker, who knew from history how much such grand conceptions had in all ages done to stifle the thoughts and destroy the reality of the feelings which they so skilfully arranged. As we said in our notice of

Moral and
Metaphy-
sical Phi-
losophy.

Philosophy
of Kant
found to be
defective.

Principles
asserted—
attempts at
a system.

Moral and
Metaphy-
sical Phi-
losophy.

Plato, his method was in the most direct opposition to all such projects, and only by overthrowing that method can they ever effectually establish themselves. This seems to have been Schleiermacher's opinion. By developing with great clearness and power Plato's method, by distinguishing the faculties which he supposed to be occupied with the different subjects that Hegel's system endeavoured to bring into fellowship, and by cultivating a practical Christianity in the minds of his countrymen, he hoped to deliver them from the mischiefs of an all-embracing theory. Still we believe it is felt in Germany that something is wanting which Schleiermacher did not supply; something which should be a substitute for the Hegelian system, if that must be abandoned as dangerous; some scheme which shall begin as his professes to do from the highest ground, instead of ascending from the lowest. We have already expressed our opinion that Plato, though he always failed and sacrificed his own method when he attempted to construct such a scheme, yet had the deepest feeling that it was necessary. We looked upon the *Timæus* as his awkward endeavour to create a universal system—upon the *Republic*, as a wonderful attempt to discover what system is implied in the actual constitution of things. We ventured to intimate a doubt whether Schleiermacher, with all his knowledge of Plato, had not failed, partly through over ingenuity, partly owing to a disadvantageous position in discovering the meaning of the *Republic*. If he had seen this, and if with his study of the Gentile elements which enter into the composition of the Christian church he had meditated more deeply on its Jewish origin, we believe he might have been able not only to oppose Hegel, but to show that an actual Polity grounded upon an actual Revelation is the only effectual substitute for a circular System grounded upon a philosophical Notion. We cannot, therefore, help listening with the greatest interest to the reports which reach us concerning one of the philosophers of whom we have spoken already; who, it is said, is occupied with considering what there is in our minds which answers to and realizes those ideas which are presented to us in the Scriptures, and are embodied in the creeds of the early church. Not to provide a substitute for these, but to show how they concur with the needs and anticipations of the human spirit, is, we are told, the present purpose of Schelling's meditations. With a full sense of the hazards and temptations which may accompany such an inquiry, we yet believe that it is the most promising and the most truly rational upon which German students have yet entered, the one which gives most hope that philosophy will assert its true position, and will not intrude into a position that was never intended for it. A thinker, who has propounded a system and owned it to be inadequate without losing his hope, has passed through a discipline which promises well for his future inquiries, and we may venture to believe that they could not be better carried on in any part of Germany, than in a neighbourhood which is occupied with an earnest practical discussion between Protestants and Romanists.

Indications
in France.

Ultra-ma-
terialism.

If we turn to France, we shall find tokens of the same character, only the more remarkable for their opposition to the habits of thought which had for so long a time been established in that country. Some appearances there were for a long time after the peace of a tendency to carry on the speculations of the analytical and materialist philosophers further than they had been

carried even during the philosophical tyranny of the XVIIIth Century. But a vigorous, we might say a violent reaction has taken place. The Scotch philosophy of Reid first found its way into the French capital, and began to displace the more ultra Epicurean doctrines; then by degrees German metaphysics also came thither, and were eagerly studied. Translations of Plato, and histories of ancient philosophies, speedily followed, and of late it has been a great ambition with Frenchmen, under the guidance of M. Cousin, to combine together all the systems, or at least fragments from all the systems that have ever existed, and to frame out of them a complete psychological and ontological theory which shall account for the dogmas, as well as for the thoughts, feelings, wishes, and hopes that have been entertained by mankind. We have already expressed our opinion of such projects. We have said that in a great system of Ontology we suspect *Being* will cease to be; that in every comprehensive eclectic psychology you destroy the feeling in order to examine it. Nevertheless the indications that some principle of reconciliation, as well as some kind of system, is necessary, and that our business is rather to recognise the old, than to establish any thing new, are as strong in this case as in the last that we were examining. And the historical feeling in France has not merely wasted itself in attempts to conglomerate different speculations. It has taken another and a much healthier direction. Thoughtful men there have caught the spirit of Montesquieu, have inquired into the meaning and origin of institutions which their fathers despised, have acknowledged the existence of deep thoughts in men whom their fathers believed to be the dupes and slaves of designing priests, and have bowed to the necessity of a deeper wisdom than any which they could refer to particular men in linking together the thoughts of one generation to another, and directing the course and influences of outward events. That this critical spirit is liable to the same danger in France as in Germany—that men there are even more likely to exalt themselves, because they can appreciate the wisdom of their ancestors, as they once exalted themselves for scorning it—that a hollow fashion of reverence for antiquity is likely to take its turn with other fashions, and that there is much in these speculations themselves which leads us to fancy that we are on the highway to an almost perfect state of existence, because the thing or the abstraction called Civilization has been continually in progress, we fully acknowledge. Still we cannot help hoping that good results will ultimately come from a change so very remarkable and unexpected as this, which has taken place against the strongest resistance from all the worst and some of the most powerful elements in the French character.

The first signs that present themselves when we turn to our own country, would seem to betoken a remarkable difference between our state and that of our continental neighbours. A foreigner who should hear that Paley's *Moral Philosophy* was the recognised text-book in one of our Universities, and should inquire what the nature of that book was, would undoubtedly be surprised to learn, that though written by an eminent, and, in many respects, admirable divine, it made the good or evil of an action depend wholly upon its results, and taught in what ways we may arrive at right conduct by calculating those results. A book wholly overlooking or denying the facts which were established in the

Moral and
Metaphy-
sical Phi-
losophy.

Reaction.

Eclecti-
cism.

Philosophy
looking
backwards.

Indications
in England.

Moral and
Metaphy-
sical Phi-
losophy.

XVIIIth Century by different independent thinkers, to say nothing of the principles which were held sacred in earlier ages, would seem to have very little of the comprehensive historical character which has discovered itself in philosophy elsewhere. And indeed it has not. Yet we are by no means anxious to take advantage of the plea, that this book belongs rather to the past age than to the present, or to see in it no indication of the tone which we believe philosophy in the present day must assume if it is to maintain its ground.

Politics and
morals
united.

Paley's is a moral and political philosophy. It deals with morals as related to politics, as being by some link or other inseparably united to them. This we fancy is an important and characteristic difference between it and most of the writings of the XVIIIth Century, even those to which upon other grounds we should attach a much higher value. It is an instance, we think, of that remarkable clearness which accompanied in Paley what we fear must be called a superficial method, and even a superficial habit of mind, that he perceived the necessity of treating morals under its political aspects. It gives him a title to rank with Locke—a person much resembling him in character, though inferior to him in a certain transparency of intellect as the beginner of a new era of study, however differently from his conjectures it might be destined to terminate.

Metaphy-
sics sepa-
rated from
human
interests.

We are strengthened in this view of Paley by comparing him with *Dugald Stewart* and the philosophers of the Scotch school, who for a while held a divided empire with him. This school assumed in Stewart's hands a kind of eclectic character. It professed to deal with the mind upon the principles of Experiment and Induction, and was willing to pay all fitting respect to previous teachers. It took to itself somewhat more credit than it deserved for its principle of Induction, seeing that this was the principle upon which most of the inquirers since the time of Bacon had actually gone, and seeing that Mr. Stewart, though admiring experiment, would tolerate experiments only up to a certain point. As long as they took note of no principles antecedent to experience, external or internal, he admitted their validity. But without attempting to show that Kant had used a less severe method of induction than previous philosophers, without showing wherein his experiments had been fallacious, Mr. Stewart's faith fails him just as he reaches one particular crisis in philosophical inquiry, and he determines that to such strange doctrines as come from Germany he will give no ear. All this would have been very well borne in England, at least for a time, if these Scotch teachers had not laboured to keep their speculations free from any allusion to the actual life and business of men. They were philosophers, and philosophers merely. What had they to do with laws, and governments, and religions? Such matters lay entirely out of their cognizance; they would observe the most rigid silence upon them. Now seeing that they profess to be emphatically the practical philosophers, that they proclaim the schoolmen to be dreamers, and Plato and Kant to deserve no better name; this seems to our English tastes and apprehensions a little puzzling. What means a philosophy, and a practical philosophy too, which has no concern with any thing that we are thinking or caring about—which talks to us indeed about our own instincts and consciousnesses, but leaves us to fancy that these instincts and consciousnesses have no real objects in the heaven above or in the earth beneath. Such speculations might be very good for

creatures of another race, but they seem to be exceedingly unsuitable for toiling human beings.

We cannot wonder then that a philosophy which vigorously applied itself to human objects, should very soon have displaced one which seemed wholly fantastic. Paley was better for us than Stewart. But we could not stop where Paley stops. The scheme of utility was in him mixed with many incongruous elements, and he had followed it out very imperfectly to its consequences. Acknowledging expediency as the only guide in politics as well as in morals, he had yet defended institutions and revered creeds which had clearly not this basis, or are full of the most strange and incongruous materials if they have. Mr. Bentham could endure no such inconsistency. Starting from the one simple hypothesis, that Pleasure and Pain are the ultimate laws of the universe, he determined, not as former Epicureans had done, to invent a scheme of life for himself, but to model society upon that principle. As we have already examined this doctrine in its simple form, we need enter into no fresh discussion respecting this modification of it. Indeed, even if we had not that excuse, the task would now be unnecessary. The scheme of Mr. Bentham was brought forward as the last achievement of the philosophical spirit, the highest possible result of an advanced state of civilization. These phrases which worked most powerfully in its favour have of late been turned with deadly effect against it. Persons as utterly averse as he could be from ancient forms and institutions have attacked him on this express ground, that he has not kept up with the march of knowledge, that he adopted a narrow theory of human nature from a narrow induction, and wanted the capacity for acknowledging the discoveries of those who had taken in a more enlarged sphere of observation. But it has also been represented, we think very unjustly and very injuriously to his reputation, that we can abandon his moral and metaphysical theory, and yet maintain the political deductions from it. Now it seems to us, that one great and deserved cause of Mr. Bentham's popularity was his bold and honest assertion that these cannot be separated. We owe him, we conceive, very great gratitude for overturning the dilettantism which had become fashionable in philosophy, and for showing that the test of principles is their application to practice, and the test of practice its power of being referred back to principles. He was besides a powerful and consistent logician, who might be mistaken in premises, but who in the steps from his premises to his conclusion would very seldom stumble. We are bound, therefore, to consider that his utilitarian philosophy and his theories on practical legislation and politics must stand or fall together. He is, we think, the completest form of the utilitarian philosopher that has ever appeared, or is ever likely to appear. The influence he has exercised upon the times is not wonderful, when we consider either the plausible doctrine which he defended, the confusions in the minds of his opponents, or the real intellectual power and moral earnestness with which he assailed them. The rapid decline of his fame it requires some more words to account for.

The strong political bias of the English mind has been constantly urged as a reason why any deep philosophy must be always uncongenial to it. We have seen what confirmation this opinion seems to receive from the prevalence of Paley's doctrines, and the indifference with which Stewart, in spite of powerful patronage from a powerful body of Scotch friends or disciples, has been

Moral and
Metaphy-
sical Phi-
losophy.

First result
of the union
between
political
and moral
science.

Doctrine of
utility.

Final result
of this
union in a
search after
the deepest
principles.

Moral and
Metaphy-
sical Phi-
losophy.

Moral and
Metaphy-
sical Phi-
losophy.

An histo-
rical spirit
awakened.

regarded. But we have already hinted our belief that these indications are fallacious, and that wherever practical feelings are most alive, there ultimately, if not immediately, men will be driven to seek for the deepest foundations. In Greece we observed, that the most flourishing times of philosophy were the times of strong political feeling, and that it grew weak as those feelings decayed. We are not ashamed then to confess that we believe the doctrine of expediency which Paley sanctioned might have continued to prevail among us, if men had not been startled by observing to what practical conclusions that doctrine led, when it was fairly and consistently followed out by Bentham. To find plausible arguments against some of these conclusions was easy, to find a strong *vis inertiae* in the public mind which could withstand them was not impossible. But various experiences show, that neither the special pleading of clever men, nor the reluctance to move in comfortable men can be always trusted in an hour of need. Something more than this was wanted, and however imprudent many might deem it to examine the grounds of that which age and custom had made venerable, such an investigation was inevitable. The question was in what spirit should it be undertaken. If carried on at all, it must be carried on not timidly but boldly; the examination must be not partial but complete. But should it be pursued in a spirit of hatred or of love, of contempt or of reverence? The first method has been tried to the utmost by Mr. Bentham and his followers. If it is the most effectual, it should assuredly be persevered in. But a set of persons have arisen to say that it is not effectual; instead of going too far, you have not gone far enough. Instead of judging practices and institutions by tests which are too severe, by principles of human nature which are too deep, you have made use of the most weak tests, you have brought the most shallow principles to bear upon them. We demand a more rigid scrutiny: instead of a few flimsy inductions from a few facts which present themselves on the surface of society in the present day, we must look into history, and study the growth of our institutions, that we may know really what they mean. We must look into the same history to know what men have been thinking about in past ages, that we may see whether this doctrine of Pleasure and Pain is really that in which all other doctrines begin and terminate, or whether it is merely the deposit or scum that is left after all that is strong and vital in human thoughts and feelings has evaporated. To such inquiries men have addressed themselves; some beginning with an aversion from that which afterwards, by slow degrees, they were led to admire; some beginning with a confused sentiment of admiration, as well for institutions as for all the practices of those who administer them; but both agreeing at last in the confession, that the more their sympathies have been awakened for what was truly and permanently good, the more keenly they are able to distinguish between it and any evil innovations which fashion or selfishness may have grafted upon it; the more anxious to remove, though with a delicate and careful hand, all such unnatural excrescences. In this *Encyclopædia* at least, the plan and method of which were originally suggested by Mr. Coleridge, we should be most ungrateful not to own, that of all those who have assisted in imparting this feeling to the English mind, he was incomparably the first in genius, in knowledge, and in moral influence. We are anxious in this

place and in this connection to introduce his name, because the impression has been very widely diffused, both by his enemies and his admirers, that his mind was wholly cast in a foreign mould, and that all which Englishmen hailed or disliked as novel in his writings was already familiar to thoughtful men in Germany. For novelty, no one more eagerly disclaimed it on all occasions than he did. It was his greatest desire to revive and reproduce the old; he left it to the original writers in journals and magazines to supply the public with what is new. He was also most anxious to testify his obligations to the German writers, and that at a time when it was not a credit, but a disgrace to be suspected of any sympathy with them. But that a man who had been educated at an English public school and university, who had entered earnestly and passionately, yet, even in his most extravagant moments, with ardent patriotism, into the questions which agitated the people of England, and had produced his noblest poetry before he entered upon his German studies, should be supposed not to be truly and originally national is most unreasonable. In later life he was in the habit of connecting, even in a forced and unnatural manner, the language of Hooker and our early English divines with that of Kant—as if he felt almost reluctant to receive any thing from abroad which he could not associate with some national feeling or recollection. He did, however, most earnestly assert the truth of Kant's positions, believing that men had been withdrawn from their faith in those transcendent truths which their creeds and institutions embody by the loss (for such he deemed it) of the conviction that they had a faculty within them to which these truths could appeal. But on the one hand, no person was ever more zealous to maintain that the Practical Reason which demands these truths is not the inheritance of philosophers, but of every poor man in the land; on the other, no one ever took more pains to show that these truths must be proclaimed to both rich and poor by an organized body, whose understandings were to be diligently cultivated and exercised, and whose teachings were most profitable when they were not captiously disputed, but humbly received. He may possibly have dreamed, like other great thinkers, of constructing a philosophical system. If he had accomplished such a task, we should probably have found the same reason to complain of it as of any other which has passed under our review in this article. But in the writings which he has left, he has been happily preserved from this infirmity of noble minds. All the principles which he has developed refer to some practical subject, either connected with the regulation of our lives or with the political circumstances of the country. In his books he has thrown out many thoughts with which taken singly we entirely disagree. But we are convinced that he did not mean them to be taken singly. He had confidence that his readers, if they thought it worth while to study any part of his utterances, would study the whole of them, and would gather his meaning not from detached sentences, but from the impression which after a careful and affectionate study remained upon their minds. He felt, perhaps, more strongly than any philosopher since Plato the worth of the principle, that every deep truth comes forth in two imperfect and even apparently contradictory forms; in neither of which singly, but in the result of both, we have that wherein we can permanently rest. We should say, that in the development and application of this truth consisted his great and peculiar

Moral and
Metaphy-
sical Phi-
losophy.

merit as a philosopher. He derived it not from the schools of Germany, but of Greece. But he owed the clear sense of it to his English education. He had received from that our English habit of balance and compromise, which leads in low and feeble minds to the sacrifice of every thing positive and true; but which becomes in our practical Jurists an exquisite tact in deciding the worth of contradictory evidence; which leads our wiser Statesmen to seek after a constitutional equilibrium not in theory but in fact; which disposes our working Divines to be discontented till they can find some point of practical reconciliation between two opposite opinions; which has enabled our Philosophers and Theologians to recognise the Platonic idea as not the more visionary, but the more real, because it cannot reduce itself under the conditions of human logic, and to believe that the deepest principles of Christianity would want one sure test of their derivation from God, if they did not possess the same characteristic.

We have made these remarks, not for the sake of exalting our own countryman, but in order to show that each of the three nations has had its own appropriate task to fulfil, and that its philosophers cannot avail themselves of the labours of foreigners, unless they are content to assist in fulfilling it. If Coleridge understood the Germans better than Bentham, it was because he was a more truly English philosopher, because he possessed that national spirit in which the other was wholly wanting. The Englishman who derives his wisdom from France or Germany, the Frenchman or German who merely sits at the feet of English teachers, will, we are sure, be utterly unable to sympathize with those whom he copies and affects to admire. At the same time it is quite evident that the courses of thought which have been followed in Germany, France, and England, though starting from such different points, cannot be kept separate from each other, and that there is a common characteristic belonging to them all. It would be difficult to say in which country the historical spirit has had most opposition to encounter—whether in Germany, from the subjective tendencies of its philosophy; in France, from the worship of the present age; or in England, from the selfishness by which we have corrupted our institutions, and thus made the doctrine plausible that they were the work of barbarians. It would be difficult to say in which of these countries this spirit has put itself forth most strongly in defiance of all these contradictions, and often in spite of the very habits and tempers of those who have assisted in diffusing it. The mind of Schelling is probably much disposed to look upon truths as derived from the reason rather than as recognised by it. The mind of Guizot must be naturally much inclined to honour the time in which he lives, and to disparage the past. Coleridge's mind was essentially poetic and creative; to him of course the acknowledgment of truths, as given to us not as originated in us, must have been most difficult. Yet each of these men was by different processes led to feel that the highest vocation of the philosopher is to perceive that which is, to discover a meaning in that which has happened, to receive that which has been bestowed. To cultivate this spirit in the students of

Moral and Metaphysical Philosophy should, we conceive, be the great aim of the teachers in these nations. For the most plausible sophisms of our day are upheld by vague notions respecting the history of human progress, and can only be confuted by a patient meditation upon it. We are told in dim prophetic language that certain great metaphysical Ideas have ever clothed themselves in religious forms and notions, which after a while became antiquated, and are exchanged for others; that the period is come when the creed which our fathers have believed must be discarded like the Paganism of former days; that a free and glorious Pantheism will, in the minds of the wise few at least, supersede all the mythical, superstitious, and partial worships which have existed among men. It may be that the phrases of this rhetoric have made an impression upon the European mind which nothing but experience can efface. It may be that we are destined to learn by a discipline perhaps the most tremendous that the nations have ever undergone, what Pantheism is worth, what it can do for mankind, and what huge, shapeless, monstrous superstitions must emerge out of it, before the Light which we have scorned shall burst forth to restore Distinction and Order, and form the Universe anew. But the utterers of this language appeal to History, and by History they should be encountered. Let the records of Greek speculation be calmly studied that it may be seen what they could, and what they could not do; how much truth there was in them; how utterly false they must have been if there be no such divine and transcendent Facts as Christianity consists of. Let it be seen whether Jewish History be really a collection of startling prodigies, or whether it be not the most orderly coherent narration which the world possesses, revealing the deepest principles through events which make those that are happening to us intelligible. Let it be seen from the History of Modern Europe, whether the human spirit does not owe all its emancipation and all its progress to the belief that it is connected by an actual bond with its Creator; whether it did not fall again into bondage and confusion when it lost sight of the Facts by which that Union is declared; whether any attempt to establish Human Society and Human Life upon any other basis than these facts has not ended in the degradation of both. These things if the student diligently ponder in the only temper which can make History or Philosophy profitable, he will find the difference between receiving the truths which any age was appointed to impart, and making that age an object of idolatry; between an affectionate sympathy with those who proclaimed the wants and hopes of man, and the fancy that the announcement of these wants is a substitute for the satisfaction of them; between the belief that the philosopher has faculties given him which he is not to hide, but to use for the good of mankind and the glory of his Creator, and the notion that he can dispense with one of those helps in attaining a knowledge of himself and of God, which are necessary to the meanest peasant. Above all he will see these words inscribed upon the Halls of Philosophy not less legibly than upon the Temple of Religion, "WHOSO HUMBLETH HIMSELF SHALL BE EXALTED, WHOSO EXALTETH HIMSELF SHALL BE ABASED."

Moral and
Metaphy-
sical Phi-
losophy.

L A W.

Law.

Jurisprudence distinguished from Ethics.

PRELIMINARY OBSERVATIONS.

1. BETWEEN the sciences of Ethics and JURISPRUDENCE, or LAW, there is a connection so close that it is not very easy to determine accurately the respective limits of each. But in a general way, the circumstance which occasions the distinction between them is *compulsion by public authority*. The science of ethics acquaints us with our duty, and the reasons of it. Jurisprudence proceeds to the consideration of the means of enforcing that duty, by the artificial institution of a public and regular system of compulsion, and it inquires into the rules ordained for this purpose. A distinction of another kind is sometimes made. Morality, it is said, relates to principles of action; law, to precise and definite rules of conduct. There is some foundation for this distinction, arising from the peculiar object which each science has in view, though the fundamental principles, and, in a great measure, the practical rules also, are the same in both. In morals, we seek for a just method of self-government; our aim is the private regulation of our own conduct; and we look, not so much to the outward form of our actions, as to the spirit and intention by which we are actuated. Here then we may refer our conduct immediately to the most general standard of right, whereas compulsory enforcement and prevention, which form the proper and immediate objects of law, must necessarily be exerted in single and definite acts, are applicable only to outward conduct, and therefore require precise rules for their application. And, accordingly, we find that Hooker represents law, in its essential character, to be both coercive and precise: "That which doth assign unto each thing the kind, that which doth moderate the force and power, that which doth appoint the form and measure of working, the same we term a law."*

Jurisprudence universal and particular.

2. Jurisprudence is either universal or particular. The former relates to the science of law in general: it investigates the principles which are common to all positive systems of law, apart from the local, partial, and accidental circumstances and peculiarities by which those systems respectively are distinguished from one another. Particular jurisprudence treats of the laws of particular states; which laws are, or at least profess to be, the rules and principles of universal jurisprudence itself, specifically developed and applied.

Conflicting opinions of the limits of jurisprudence.

3. According to some eminent continental writers of the present day, the science of jurisprudence properly admits of being treated in four different ways, philosophically, historically, didactically or dogmatically, and exegetically. The philosophy of law embraces those moral considerations and reasons upon which all laws are grounded: it investigates the origin and principles of natural justice or right, and the sources and object or purpose of positive laws. The history of law traces its actual rise and progress. The didactic branch is concerned with the formal rules and doctrines of law, without directly regarding its spirit and object. And lastly,

* *Ecclesiastical Polity*, book i. sec. 2.

it is the office of the exegetic department to interpret and explain these rules and doctrines.* Universal or general jurisprudence is sometimes called the science of *legislation*. Legislation indeed is occasionally considered as the *art* of the law-maker, whilst jurisprudence is treated as the science. An able living writer, however, of our own country, whose work upon the present subject has attracted a considerable share of notice both here and on the continent, contends that jurisprudence and legislation are entirely distinct sciences. "General jurisprudence, or the philosophy of positive law," says Mr. Austin, "is not concerned directly with the science of legislation. It is concerned directly with principles and distinctions which are common to various systems of particular and positive law; and which each of those various systems inevitably involves, let it be worthy of praise or blame, or let it accord or not with an assumed measure or test. Or (changing the phrase) general jurisprudence, or the philosophy of positive law, is concerned with law as it necessarily *is*, rather than with law as it *ought* to be; with law as it must be, *be it good or bad*, rather than with law as it must be, if it be good."† The learned author, it will be readily seen, dissents from the writers above referred to, and we believe from most others also, in his use and application of the term *philosophy of law*. Legislation, he says, is a branch of morals, and as such distinct from the science of jurisprudence. It is not very clear what limits he would assign to the latter, but apparently he would make it little more than a science of terminology. At the same time his book abounds with many accurate and useful definitions and distinctions; but, like Bentham, for whom he professes to entertain a profound admiration, he betrays an excessive anxiety to arrive at a degree of accuracy, probably unattainable in such a science as law. Attempts of this kind tend rather to perplex than to assist the student. Some of the most instructive works relating to or connected with the science of jurisprudence, are such as it would be difficult to class under any single recognised head or division of the subject; nor, in our opinion, would any great advantage follow, were it possible to satisfy the most restless inquirer upon this point.‡

Law.

* *Introduction Générale à l'Histoire du Droit*; par M. E. Lermier, ch. iii. Warnkœnig, *Doctrina Juris Philosophica*, cap. i. sec. 1.

† *The Province of Jurisprudence determined. Preliminary Observations*, p. iii.

‡ "Of what stamp," asks Mr. Bentham, "are the works of Grotius, Puffendorf, and Burlamaqui? Are they political or ethical, historical or juridical, expository or censorial? Sometimes one thing, sometimes another: they seem hardly to have settled the matter with themselves. A defect this to which all books must almost inevitably be liable which take for their subject the pretended *law of nature*; an obscure phantom, which, in the imagination of those who go in chase of it, points sometimes to *manners*, sometimes to *laws*; sometimes to what law *is*, sometimes to what it *ought* to be. Montesquieu sets out upon the censorial plan: but long before the conclusion, as if he had forgot his first design, he throws off the censor, and puts on the antiquarian. The Marquis Beccaria's book, the first of any account that is uniformly censorial, concludes as it sets out with penal jurisprudence." *Morals and Legislation*, ch. xvii. vol. ii. p. 264, n. He elsewhere

Law.
Lord
Bacon.

4. Lord Bacon has afforded us one of the clearest and most satisfactory descriptions with which we are acquainted of the science now under consideration, whether we designate it by the name of legislation, or of universal jurisprudence; and he enumerates several of the principal problems of which it is the object of that science to offer a solution. "All those," he observes, "which have written of laws, have written either as philosophers or as lawyers, and not as statesmen. As for the philosophers, they make imaginary laws for imaginary commonwealths; and their discourses are as the stars, which give little light, because they are so high. For the lawyers, they write according to the states where they live, what is received law, not what ought to be law: for the wisdom of a law-maker is one, and of a lawyer is another. For there are in nature certain fountains of justice, whence all civil laws are derived but as streams; and like as waters do take tinctures and tastes from the soils through which they run, so do civil laws vary according to the regions and governments where they are planted, though they proceed from the same fountains. Again, the wisdom of a law-maker consisteth not only in a platform of justice, but in the application thereof; taking into consideration by what means laws may be made certain, and what are the causes of the doubtfulness and uncertainty of law; by what means laws may be made apt and easy to be executed, and what are the impediments and remedies in the execution of laws; what influence laws touching private right of meum and tuum have into the public state, and how they may be made apt and agreeable; how laws are to be penned and delivered, whether in texts or in acts, brief or large, with preambles or without; how they are to be pruned and reformed from time to time, and what is the best means to keep them from being too vast in volumes, or too full of multiplicity and crossness; how they are to be expounded, when upon causes emergent and judicially discussed, and when upon responses and conferences touching general points and questions; how they are to be pressed, rigorously or tenderly; how they are to be mitigated by equity and good conscience, and whether discretion and strict law are to be mingled in the same courts, or kept apart in several courts; again, how the practice, profession, and erudition of law is to be censured and governed; and many other points touching the administration, and, as I may term it, animation of laws."*

His
maxims.

5. He attempted to supply in part the deficiency here complained of, by his short but highly valuable treatise entitled *Exemplum tractatus de justitia universali sive de fontibus juris*, appended to the eighth book of his great work *De augmentis scientiarum*. This treatise consists of ninety-seven aphorisms upon the heads enumerated by him as above. The consummate ability, learning, and skill of its author, added to his extensive experience as a practical statesman and lawyer, eminently qualified him for advancing the science of legislation; and these aphorisms are deservedly held in high estimation. By many their appearance is regarded as an important epoch in the history of legal science. Some even

expresses himself dissatisfied with Blackstone, on similar grounds. See Bentham, *Fragment on Government*, Pref. p. xii. &c.

Some sensible remarks, bearing upon the present discussion, may be found in Lieber's *Political Ethics*, part i. ch. iv. sec. 31, p. 67. London, 1839.

* *Advancement of Learning*, book ii.

go the length of asserting that Lord Bacon was the first, in modern times, who conceived the idea of an universal jurisprudence; that it was he who first sowed the seed which has only of late sprung up and produced fruit, in the shape of a more exact theory of law, and improved methods and principles of legislation, though as yet the science is far from being complete. It must indeed be admitted with Mr. Hallam, that the maxims in question are an admirable illustration of the principles which should regulate the enactment and expression of laws, as well as of much that should guide, in a general manner, the decisions of courts of justice, and that they furnish principles of general application, inasmuch as they are not limited to the circumstances of particular societies.* But others again bestow less unqualified praise on the services of our great philosopher, in relation to the advancement of the science at large. Thus M. Lermnier remarks the cautious timidity of Bacon with regard to political philosophy. "Concerning government," says Bacon, "it is a part of knowledge secret and retired, in both these respects, in which things are deemed secret, for some things are secret because they are hard to know, and some because they are not fit to utter;" and, accordingly, the only part which he handles of this branch of the subject, regards the means of extending the bounds of an empire. As to the science of law, considered apart from what belongs more properly to politics, M. Lermnier thinks it may fairly be questioned, whether Lord Bacon has, in reality, given us a treatise on universal justice; for he does not attempt to examine the rational foundation of society, or the basis and groundwork of law. He contemplates laws merely in the outward form they present, and the only principles with which he furnishes us relate exclusively to their practical regulation and administration. So far he is admirable, it is conceded; but with respect to the theory and philosophy of law, his maxims cannot, as this ingenious writer contends, be allowed any rank or importance in the history of the science.†

6. For the philosophy of law we may go back to the ages of antiquity, to those philosophers of whom Lord Bacon says, *proponunt multa, dictu pulchra, sed ab usu remota*; who have explored the fountains of justice without pursuing its streams. We do not now refer to works exclusively or principally of a moral as opposed to a political cast, however such works may assist the jurist, by the light which they afford him in his search after the common foundation of morals and law. At present, we are concerned only with such writers as have directly contributed to promote the science of jurisprudence. Plato is generally ranked amongst the earliest and most remarkable of these writers. His views are comprehensive, noble, and just; though perhaps his diffuse and unsystematic style, and the form of the dialogue in which he has chosen to clothe his ideas, may be looked upon as defects by the rigid lovers of system. Two of his dialogues, the *Commonwealth* and the *Laws*, are very commonly, though erroneously, supposed to be connected with one another, as if in the latter he intended to follow up and complete the plan which he

Law.

The phi-
losophers

Plato.

* See Hallam's *Introduction to the Literature of Europe in the XIVth, XVth, and XVIth Centuries*, vol. ii. p. 444, &c. See also the remarks of Mr. Basil Montagu, the learned editor of Lord Bacon's Works, vol. ii. p. lxvii., and note U.

† Lermnier, *Introd. Gén. à l'Hist. du Droit*, ch. vii.

Law. commenced in the former, by furnishing his imaginary community with imaginary laws. The two works in question, however, are wholly unconnected. Plato had no intention of representing any actual or probable condition of society, in his *Commonwealth*, a work which has reference to morals, rather than to our present subject. The supposition, therefore, occasionally entertained, that he seriously countenanced doctrines much resembling those which are attributed to the Saint-Simonians and certain other modern sects on the subjects of property and marriage, is equally absurd and groundless. In his *Laws*, he appears more properly in the character of a politician and a statesman; and if we consider the age in which he lived, and the lights which he possessed, he stands unrivalled as a *political philosopher*. We are to look to him chiefly for general views, respecting the true object of government, the duties of a legislator, and the fundamental principles of legislation; at the same time he is highly valuable on several matters of detail and of a practical bearing. In some respects, he seems to have anticipated the doctrines of modern theorists, as, for example, where he treats of the nature, object, and proportion of legal punishments, and the circumstances which he thinks ought to accompany the promulgation of laws. The concluding books of this work contain a system of municipal and international law.

Aristotle. The depth and precision of Aristotle, the anatomy of justice, which he gives in his *Nicomachean Ethics*, and the unrivalled sagacity, diligence, and exactness, with which he has traced and explained the structure of human governments, place him, even now, in the very first rank of political writers; but he far surpasses them in general, in one most important respect, namely, in the regard he pays to experience, and in his close observation of human nature. His theory of justice is encumbered with a whimsical attempt to establish a sort of moral arithmetic, justice being made to depend upon a geometrical proportion; but when he extricates himself from this, his good sense is apparent. His political doctrines are to be found in his *Politics*. The first book contains the theory of slavery and sociability. His doctrines with regard to the former are amongst those which have been most frequently and severely censured: whole races of human beings, he asserts, are decreed by nature to be slaves; and on this unsatisfactory principle he justifies the practice of reducing our fellow creatures to the mere condition of *things*. In the second book, he criticises the theories of Plato and of other philosophers, and reviews the constitutions of Sparta, Crete, and Carthage. In several succeeding books, he establishes the principles on which civil society is instituted, and traces the nature and forms of the various sorts of government, simple and compound, the course of revolutions, and the causes of the decline of states. That he did not theorize without regard to facts and experience, appears from the circumstance, that, as a preparatory step, he examined and compared one hundred and fifty-eight different constitutions of Greece and Italy. Afterwards, he proceeds to investigate the conditions of private justice and of social happiness, and many points of practical legislation and government. The eighth and last book is occupied with the subject of education.

Cicero.

7. The next philosopher to be noticed is Cicero. But he is distinguished, not so much for having advanced the principles of the science, by any original discoveries of his own, as for the conspicuous place which

he occupies in Roman jurisprudence, which first began, in his day, to be enriched and improved by its alliance with Grecian philosophy then recently imported into Rome. Before the time of Cicero, law was merely practised there as an art. He entertained the idea of reducing it to a regular science,* a task which was partly executed by his friend and contemporary, Servius Sulpicius. Cicero was not a professed jurisconsult, but he was intimately acquainted with the Roman laws and constitution, and to him we are indebted for some of the most valuable information we possess respecting them. He may justly be regarded as pre-eminently the philosophical jurist of Rome. Two treatises of his, on political and legal philosophy, have been transmitted to us, but in a very mutilated condition, about one half of each being lost. One of these, his treatise *De Legibus*, has long been known. Of the other, bearing the title *De Republica*, we possessed only a few fragments, chiefly quoted by Lactantius, until, in the present century, the work, as we now have it, was brought to light by the learned Cardinal Mai. In the titles of these works, in his adoption of the form of a dialogue, and in his general philosophy, Cicero has imitated Plato. His political science, relative to the structure of governments, is chiefly borrowed from Aristotle. In his *Commonwealth*, he discusses the excellences and defects of the different forms of government. In remarking upon the excellence of the Roman constitution, he attributes its superiority over those of the states of Greece to the fact, that whilst they were originally framed or largely reformed by the single efforts of distinguished individuals, the Roman constitution was slowly and gradually developed in the course of many ages, and by the successive labours of many statesmen.† Such, he maintains, must always be the process, by which a constitution is brought to maturity and rendered really and substantially good. Hence he proposes, as the most proper and rational method of prosecuting his inquiry into the best form of government, to trace the rise and progress of the Roman constitution from its earliest infancy; a method, in his opinion, far preferable to that adopted by Plato, who feigned an imaginary commonwealth. He then proceeds, with consummate ability, to review the principal epochs and changes in the constitution of his country; but, unfortunately, we are prevented from following him to the end, by the loss of the latter portion of this work. In his *Laws*, Cicero eloquently maintains the divine origin and immutable character of natural justice or right, independent of all human conventions and opinions. He shows the connection between law and religion, and strenuously asserts the obligation of states to support a national faith and established religious institutions. He then passes on to the appointment and

* "Si enim aut mihi facere licuerit, quod jamdiu cogito, aut alius quispiam, aut, me impedito, occuparit, aut mortuo effecerit, ut primum omne jus civile in genera digerat, quæ perpauca sunt; deinde eorum generum quasi quædam membra dispertiat; tum propriam cujusque vim definitione declaret; perfectam artem juris civilis habeatis, magis magnam, atque uberem, quam difficilem atque obscuram." *De Oratore*, i. 42.

† "Is (scilicet Cato) dicere solebat, ob hanc causam præstare nostræ civitatis statum cæteris civitatibus, quod in illis singulis fuissent fere, qui suam quisque rempublicam constituissent legibus atque institutis suis, ut Cretum Minos, Lacedæmoniorum Lycurgus, Atheniensium, quæ persæpe commutata esset, tum Theseus, tum Draco, tum Solon, tum Clisthenes, tum multi alii; postremo exsanguem jam jacentem doctus vir Phalereus sustentasset Demetrius. Nostra autem respublica non unius esset ingenio, sed multorum, nec unius hominis viâ, sed aliquot esset constituta sæculis et ætatibus." *De Rep.* lib. ii. 1.

Law.

functions of magistrates, the constitution of senates and popular assemblies, and the other topics of constitutional law and policy. The imaginary republic, represented in this work, bears a close resemblance to the Roman state, for which it has frequently been mistaken.

8. Of the ancients, those whom we have noticed are commonly regarded as the chief promoters of philosophical jurisprudence. After them, we do not, until we come to modern times, meet with the names of any writers in this particular department, of sufficient importance to render their appearance an epoch in the history of the science. We shall now proceed to mark out the principal eras, and notice a few of the chief events connected with the gradual development and progress of practical jurisprudence.

The Roman law.

9. The science of positive or artificial law in general may be considered to have but recently emerged from a state of infancy, in the time of Cicero, when, as we have already intimated, the Roman law first attained the rank and dignity of a science. "From that time," to use the language of Lord Bacon,* "there was such a race of wit and authority, between the commentaries and decisions of the lawyers, and the edicts of the emperors, as both law and lawyers were out of breath." Without pausing to enumerate the particular circumstances which led to such a result, it is sufficient at present to notice the fact, that the Roman law gradually lost much of the narrow character of a purely national system, intended for the regulation of a single people, and accommodated itself to the widely differing manners, customs, and ideas, of the various nations composing the empire. The progress of the system towards maturity, and its ultimate establishment on a broad philosophical basis, was materially promoted by the labours of a series of profound and accomplished lawyers, who followed each other in uninterrupted succession down to the age of Alexander Severus; in and about whose reign the illustrious names of Papinian, Paul, Ulpian, and Modestinus occur. So far as principles were concerned, the Roman law had now attained its full maturity; for the lawyers of the next three centuries possessed no originality, but contented themselves with a tame use of the rich materials left to them by their predecessors, without attempting to develop any new principles. Their highest ambition did not rise above several attempts,—some of them tolerably successful,—to reduce and abridge into a more convenient size and form, the contents of the unwieldy mass of their law books. Gregory and Hermogenes, two private lawyers of the age of Constantine, collected and arranged, in the Gregorian and Hermogenian Codes, the imperial edicts down to their times. Nearly a century after, a similar collection, in continuation of the others, was made by order of the Emperor Theodosius the younger. This collection was called the Theodosian Code, and it was the first work of the kind undertaken by public authority. But the most remarkable achievement in this way, indeed we may perhaps say, the most remarkable event in the annals of jurisprudence, was the remoulding of the whole body of Roman law, in the early part of the VIth Century of our era, by command of the Emperor Justinian. Tribonian, an eminent lawyer of the day, was the chief instrument employed in this task. The body of law, as digested and arranged by him assisted by other lawyers, comprehended origin-

The classical juris-consults.

Codes.

The civil law.

ally three distinct works: 1. the Code in twelve books, consisting of the imperial constitutions methodically arranged, from the earliest times of their promulgation; 2. the Digest or Pandects, in fifty books, containing the opinions and writings of the most eminent lawyers, digested in a systematical manner; 3. the Institutes, being an elementary treatise on the Roman law, in four books. To these were afterwards added the Novels, or imperial constitutions of posterior date, which formed a supplement to the Code; for, notwithstanding Justinian's determination that the first three works should form the only standard of reference in all civil matters, for his own and all future times, it was found impossible to dispense with subsequent legislation. These four compilations together compose what is technically called the *Civil Law*, in contradistinction to the Roman law, which is a more general term, and includes the ante-Justinianean jurisprudence as well, though frequently the two names are used indifferently to denote the same thing. With the compilation of the *Corpus Juris Civilis*, the history of the Roman law, properly speaking, ends.

10. During the dark ages, which succeeded the final The dark overthrow of the Roman empire, legal and all other ages. science was overwhelmed in confusion and ignorance.

The descendants of the ancient Romans, however, were still permitted, by their barbarian conquerors, the use of their own laws; and, though jurisprudence was no longer cultivated as a liberal science, the books of the civil law, as Savigny has clearly proved,* were in the hands of the lawyers of those days, and, however imperfectly understood, formed one source from which a knowledge of Roman law was derived. For a long time,—indeed the general impression so long cherished is yet far from being entirely removed,—the favourite theory, so successfully overturned by Savigny, was, that the Theodosian Code and other inferior compilations were the only known repositories of Roman law, until accident led to the discovery of a copy of the Pandects at Amalfi, in the beginning of the XIIth Century. While the civil law,—so far as its liberal cultivation, as an object of science, was concerned,—was thus in abeyance, two other great systems were partially brought to light. These were the Feudal and the Canon laws. The former The feudal law. was that system of polity which was introduced by the northern nations who overran the Roman empire, and whose customs were, for the most part, nearly similar. It is generally supposed to have acquired something like a systematic form about the time of Charlemagne, who was desirous of effecting, as far as possible, an uniformity of law throughout his extensive dominions; but the principal collections of this law were not compiled until after the dark period we are now considering. The account which is given in our HISTORICAL DIVISION of the Rise, Progress, and principal Features of the Feudal system, will supersede the necessity of saying much upon it in the present article. The Canon law was the rule of the Christian church, and, properly speaking, it The canon law. was coeval with the general adoption of Christianity in Europe. But before the Papal dominion attained its full height in the latter part of the XIth Century, this system was described simply as a *rule*; it then was dignified with the name of a *law*; and thenceforward it

* An offer to King James of a Digest to be made of the Laws of England. *Works*, vol. v. p. 357

* *Geschichte des Römischen Rechts im Mittelalter*. Heidelberg, 1815–1831. It has been translated into French by M. Guenoux, Paris, 1839; and into English by Dr. Cathcart, Edinb. 1829.

Law. obtained great influence in most countries of Europe. Several collections of canons were made in the dark ages, but the chief collections, forming what is called the *Corpus Juris Canonici*, were not compiled till a later period.

Jurisprudence of the middle ages. 11. The subsequent history of jurisprudence is separable into two grand periods, that of the middle ages, and that of modern times. The former begins about the year 1100, and occupies a space of about four centuries; and it is remarkable principally for the zealous cultivation of the civil law in Italy, and, in a subordinate degree, for the ulterior developement of the feudal and canon laws. The Italian civilians of the middle ages are commonly divided into three schools, those of Irnerius, Accursius, and Bartolus. Irnerius was a native of Bologna, where he professed and taught the civil law, with such remarkable success, that it soon became the favourite study, and continued to hold the chief place among the sciences throughout the whole period now under consideration. Students from every quarter of Europe flocked to the Italian universities, for the purpose of gaining an intimate acquaintance with this law; and such was the estimation in which it was held, that in a short time it was adopted by almost all the European nations as the basis of their legislation.

The Italian civilians. Irnerius. The Glossators. Irnerius was the founder of what is called the school of the glossators, from the glosses or marginal notes which the professors of the civil law were in the habit of writing upon the text. After some time, these glosses usurped, in a great measure, the place and authority of the text, and at length they multiplied to such an extent, that a general digest of them was thought desirable. The project of composing such a digest was conceived and executed by Accursius, who compiled his *Glossa Ordinaria* about the middle of the XIIIth Century. This compilation had immense and rapid success, for it soon obtained the force of law in the public tribunals, and it put the old glosses so completely out of fashion, that they were soon forgotten, and in many instances destroyed as useless lumber. Savigny speaks very slightly of it, and he says that its indifferent execution, and the extraordinary favour it met with, attest the decline of the science since the times of the earlier Irnerians.

Accursius. Bartolus. Bartolus, the chief of the third school, flourished about a century later than Accursius. He was chiefly celebrated as a commentator, and it was for his great superiority over his predecessors in this department, more than for any thing original in the character of his writings, that he was regarded as the head of a school. Zeal and industry constituted a large portion of the merits of the Italian civilians, for they had neither philosophy, history, nor polite learning, to aid them. Of all the collateral departments of knowledge, logic alone, at a somewhat advanced stage of this period, was blended with jurisprudence, but that in a way rather to encumber than to assist the latter science.

The feudal and canon laws. 12. Whilst the feudal law regulated the tenure of property, and the canon law the ecclesiastical relations of the states of Europe, each of those systems, in a scientific point of view, occupied, during the middle ages, a very inferior position to the civil law, which was regarded as the great repository of legal science, embodying all the essential principles of universal jurisprudence; and it continued to be viewed in this light till after the Reformation. The feudal and the canon laws borrowed largely from it. Codes and digests of feudal and canonical learning were from time to time compiled

on the model of the books of the *Corpus Juris Civilis*. Like the latter too, these compilations became the subject of numerous glosses and commentaries. Bulgarus an eminent civilian of the XIIth Century, supposed to have been a pupil of Irnerius, is said to have been the first who wrote a gloss on the Book of Feuds, the principal collection of feudal law. His example was followed by many others, who were, for the most part, civilians likewise. With regard to the canon law, when it obtained the influence we have above alluded to under Pope Gregory VII., in the XIth Century, a knowledge of it became a common road to the highest honours, and hence it had many followers. For some time the glossators and the canonists formed two entirely distinct classes; occasionally too, some degree of rivalry and jealousy was observable between them. By degrees, however, the canonists regarded the civil law as an integral part of their studies; and, on the other hand, the civilians, in their works, made use of many principles derived from the canon law. The first instance of the same individual professing both laws at once, occurs about the beginning of the XIIIth Century, in the person of Lanfrancus, a professor at Bologna. Afterwards the practice became not uncommon.

13. The modern period begins in the XVIth Century, about the time of the Reformation, when the science of jurisprudence underwent a complete revolution. Towards the close of the preceding period, the defects of the existing modes of studying and cultivating the civil law began to be pretty generally felt and acknowledged, and some eminent scholars, such as Politian, brought history and polite learning to bear upon the civil law. These latter, however, studied and wrote upon the books of the civil and Roman law, as philologists, not as lawyers, and until the following century, no material improvement took place amongst the regular professors, who still persisted in the old barbarous and vicious system. France, Holland, and Germany, may be singled out as the three countries where the civil law has been most successfully cultivated in modern times, though other countries as well have produced many eminent civilians. The founder of the *Erudita Jurisprudentia* was Alciatus, an Italian by birth, who settled in France, originally at Avignon, from whence he was removed in 1529 by Francis I., who was ambitious of promoting the study of the civil law in his dominions, and, with that view, placed Alciatus as professor of it at Bourges. The attempt of Francis completely succeeded, for the French became the most illustrious of all the modern schools. Of the many most learned and accomplished lawyers whom it produced, we need only mention the names of Duaren, Govea, (by birth a Portuguese,) L'Hospital, Hotman, Cujacius, Donellus, Brisson, James Godefroi, Domat, D'Aguesseau, and Pothier. Of all these, the reputation of Cujacius, or Cujas, stood so high, eclipsing that of the founder Alciatus, that the French is commonly called the Cujacian school. He is supposed to have been the profoundest civilian who has appeared since the days of Justinian. His works are very numerous. Most of them are of an expository character; hence he was regarded as the head of the exegetic, whilst his contemporary and rival Donellus, who wrote treatises, was considered as the chief of the didactic school of their day. Cujas died in the year 1590.

14. France took decidedly the lead for nearly two Dutch centuries, until it found a rival in Holland. The Dutch school.

Law.

school may be considered to have commenced with the founding of the university of Leyden in 1575, to which place Donellus was invited about that time, he being obliged to leave France on account of having embraced the reformed religion. It was he who gave the first great impulse to the study of the civil law in Holland. Other universities were subsequently founded, each of which had its law professors. Grotius and Vinnius flourished about the middle of the XVIIth Century; Van Leeuwen and Huber towards its close. In the following century we have the names of Voet, Westenberg, Schulting, Noodt, and Bynkershoek. These raised the reputation of the Dutch schools so high, that students in general repaired to Leyden and Utrecht, instead of to Bourges and Toulouse. In the political changes and revolutions which afterwards took place, the universities and law schools of Holland suffered; and though these afterwards partially recovered, Holland has not regained its former position. It has now a respectable school of law, but it cannot pretend to vie with the schools of Germany.*

German school.

15. Zasius was the founder of the German school. For a long time the voluminous works of the civilians of that country were extremely heavy and barbarous in their style, though exhibiting much industry and research. Haloander, who died in 1531, is one of the first names really deserving of notice. He was distinguished chiefly for his learned edition of the *Corpus Juris Civilis*, in which the Greek Novels, hitherto known through a Latin translation, were for the first time given in the original. During the latter part of the same century many professors of distinguished merit appeared; most of them, however, were foreigners, as Giphanius "the boast of Germany," Donellus,—already noticed as adorning the jurisprudence of France and Holland,—Balduinus, and Gothofredus the elder, whose edition of the *Corpus Juris Civilis* makes an epoch in jurisprudence, being the text-book generally received. In the XVIIth Century, this school degenerated; but towards the close of the same century, the historical study of the Roman law began to make some progress. The most conspicuous German civilian of the XVIIIth Century was Heineccius, whom Sir James Mackintosh designates as the best writer of elementary books with whom he was acquainted on any subject.† At present Germany is unrivalled in this branch of learning. A new school of historical jurisprudence, founded several years ago by professor Hugo, who is now living at an advanced age, has firmly established itself there, and is gradually extending its branches to other countries. A conspicuous ornament of the same school, Haubold, died a few years ago. The works of these two authors are extremely valuable. But its acknowledged chief is professor Savigny of Berlin, the learned historian of the Roman Law in the Middle Ages. The field of historical jurisprudence is comparatively new, for none of the French Cujacian school undertook a work strictly historical, unless Gruchy and a few others can be considered as exceptions; and although something was formerly done in Germany and Italy, much yet remains to be explored.‡

* *Introduction to the Study of the Civil Law.* By David Irving, LL.D. 4th edition, London, 1837.

† *Discourse on the Study of the Law of Nature and Nations*, p. 27.

‡ Irving's *Introduction to the Study of the Civil Law*, p. 165, &c.

16. The civil law has been cultivated with more or less of success in other countries. Spain and Portugal have produced some eminent civilians. So has Italy; though, as compared with other countries, it has never in modern times resumed the rank it formerly held. Gravina, however, a native of Calabria, and professor of civil law at Rome, somewhat more than a century ago, deserves to be singled out as the most elegant and classical of modern civilians. His *Origines Juris Civilis*, is a work of learning, and of great taste and judgment. In England the civil law has been much neglected, though it has contributed very largely,—much more largely than is commonly imagined,—to our national jurisprudence. Dr. Duck, the author of an elaborate and very useful treatise on the Use and Authority of the Roman Law in the Dominions of Christian Princes, is one of the few English civilians who enjoy any share of continental reputation. He lived in the earlier part of the XVIIth Century. For a fuller account of the principal civilians and their works, our readers may consult with advantage Dr. Irving's useful publication on the *Study of the Civil Law*, to which we are indebted to some extent in the preceding observations. Although the civil law has ceased to be regarded with the same unqualified deference which was paid to it in former times, although, too, modern changes have in several instances superseded its direct authority, and given it but a secondary place in countries where it formerly held the first, it still forms a most important object of study for all who would wish to become acquainted with the origin of a very large portion of the jurisprudence of the modern world. A strong testimony is borne to its philosophical excellence, by several modern theorists, who often silently borrow many of their principles and distinctions from it, whilst they affect to be wholly independent of any positive system.

17. Under the effects of increasing civilization, the rigour of the Feudal law was gradually modified and softened, and many of its incidents were at length superseded. Yet it has left so many and deep traces behind it, that a knowledge of it is not only interesting, but indispensable, in order to understand the principles which govern the modern laws of property. Feudal learning received a considerable share of attention in France from the contemporaries and successors of Cujacius, as well as from several writers of other countries, according to the various national forms in which that law was modified. In Germany, this, like every other branch of jurisprudence, has been assiduously cultivated. In England, we had our own Littleton, before the commencement of the modern period; and afterwards, Spelman, Wright, Gilbert, and many others. In Scotland, the feudal law, as there applied, has been illustrated with much learning and ability by Sir Thomas Craig, one of the greatest lawyers whom his country has produced.*

18. With regard to the Canon law, the different authoritative collections contained in the *Corpus Juris Canonici*, were completed before the commencement of the modern period. Though the authority of the Canon law was much impaired by the Reformation, it still forms the basis of the ecclesiastical law of every country where the Roman Catholic religion is professed, and even in Pro-

Law.

Cultivation of the civil law in other countries.

Feudal learning in modern times.

The canon law in the same period.

* Irving's *Introduction*, &c. p. 218.

Law.

testant countries, generally speaking, it has the force of law, where it is not repugnant to the law of the land.* The most distinguished canonist of the last century was Van Espen of Louvain. Spain has generally been distinguished for its canonists. In Germany, the Canon law has received much attention, even at the Protestant universities; we are told that when Boehmer was dean of the University of Halle, early in the last century, he acquired so great a reputation in Italy as a canonist, that difficult and complicated processes were frequently sent for the decision of the law faculty in that Protestant university. At the present day, the Canon law is by no means neglected in Germany. Though, in point of importance, it cannot be placed upon an equal footing with the feudal and civil laws, it is nevertheless a study of occasional importance even to lawyers.

19. Without going too much into detail, or anticipating what remains to be said in the body of this essay, the preceding observations may suffice to convey some general notion of the advance and progress of the three great systems which together constitute the triple basis of the national jurisprudence of Europe. Modern times have been tolerably fruitful in theories and speculations on the abstract principles of law; they have also developed another great positive system, regulating the intercourse of civilized states, and called the Law of Nations.

Modern
speculative
writers.

20. The spirit of free inquiry which accompanied the revival of learning and the Reformation, extended itself to politics and jurisprudence. Machiavel first reduced statecraft to system; but his works, though nearly related, scarcely belong to our present subject. Several theoretical writers on law and politics appeared in the latter half of the XVIth Century, of whom Bodin, an eminent French lawyer, held by far the most distinguished place. "No former writer on political philosophy," observes Mr. Hallam, "had been either so comprehensive in his scheme, or so copious in his knowledge; none perhaps more original, more independent and fearless in his inquiries. Two names alone—Aristotle and Machiavel—could be compared with his."† The work of Bodin which elicited this eulogium, is his *Republic*, written in Latin; a work which has often been compared with Montesquieu's *Spirit of Laws*, and which in many respects bears a close resemblance to it; and a comparison of the two together will go far to lessen Montesquieu's claims to originality, in some important respects. The *Republic* excited extraordinary interest in its author's own day, and it became and continued to be the text-book of political students until the time of Grotius. An erroneous idea has gained ground with some persons, that it was made the subject of public lectures at Cambridge in the lifetime of its author, an error probably arising from the favour the book met with, as an object of private study, at that university. At present, it is not much read or known. The first book commences with an inquiry into the end or object of political society. Man is considered successively in his private and in his public relations, as a member of a family, and a member of the state. Under the former relation, the subjects of marital and paternal authority, and domestic servitude, are discussed; under the latter, the origin of commonwealths, (which Bodin founds in an original compact,) the general rights and privileges of

Bodin.

citizens, and the nature of sovereign power. The second book is occupied with an examination of the different forms of government. The third book treats of senates and councils of state, of the powers and duties of subordinate magistrates, of corporations and other public bodies, and of the different orders of citizens,—freemen as well as slaves,—considered in their relation to the state. The topics discussed in the fourth book, which is disfigured by some strange astrological fancies of the author, are the rise and fall of states, and the causes of revolutions, the impolicy and danger of sudden changes, the extent and limits of the judicial authority of the sovereign, and toleration in matters of religion. In the fifth book, he examines at large the influence of climate upon the principles of government. Here he anticipates Montesquieu, this being one of the main topics for which the latter is celebrated, and on which his claim to originality is chiefly founded, however little value the theory itself may be thought to possess. Bodin next proceeds to consider the means of checking luxury, and obviating great inequalities in the possessions of individuals; how property confiscated for crime should be disposed of; and how the state should regulate its rewards and punishments; lastly, he considers the subjects of war, domestic and foreign, alliances, and the rights of ambassadors. The first part of the sixth book relates to the public census of property, public revenues, taxation, and coining. The author then proceeds to a comparison of several forms of government, in which he gives the preference to the monarchical form; and he concludes his treatise with a theory of justice.

21. Bacon, the next great name that follows in chronological order, directed his attention chiefly to the machinery of law; but it is unnecessary to add any thing further to our previous observations respecting this great philosopher. Selden, who succeeded him, and was one of the most learned men whom England has produced, enriched the literature and jurisprudence of his country by his researches into legal and constitutional antiquities. In his celebrated treatise entitled *Mare Clausum*, he claimed for England the exclusive dominion of the seas, in opposition to Grotius, who espoused the cause of his countrymen, the Dutch, in their claim to the right of fishing upon the British coast. The general principles involved in this controversy invest it with an interest beyond the mere matter in dispute. Selden is also considered to have made some advances towards a development of the law of nature, in his treatise *De jure naturali et gentium juxta disciplinam Hebræorum*; in which he separates what he considers to be of universal and essential obligation, from the more positive and purely national parts of the Jewish law.

22. But the man who has done the most, in modern times, for the science of jurisprudence, is Grotius. Dr. Adam Smith considers him to have been the first who attempted to give the world any thing like a system of those principles which ought to run through and be the foundation of the laws of all nations.* It is true, that before Grotius wrote, Bacon started the idea of an universal jurisprudence; yet, after all, the latter did little more than suggest the existence of such a science, and offer a few though valuable axioms, of general application. Grotius investigated the subject in all its bearings. He placed the science on a sure and sound footing, and may justly be regarded as the parent of what has since

Law.

* Butler's *Horæ Juridicæ Subsecivæ*, p. 183.

† *Introduction to the Literature of Europe*, &c. vol. ii. p. 229.

* *Theory of Moral Sentiments*, p. 610.

Law.
The law of
nations.

been called the Law of Nature and Nations. The same remarks will apply to him considered as the author of the positive Law of Nations, meaning thereby, that system of artificial rules and customs by which the intercourse of civilized independent nations is governed; for the labours of Albericus Gentilis, and other authors of inferior note, who preceded Grotius in this department, were not so successful as to entitle any of them to be accounted the parent of International Law. The two sciences, here spoken of as distinct, were blended and treated of together in the celebrated work of Grotius *De jure Belli et Pacis*; and they have since been frequently regarded as parts of the same science. In more recent times, a clearer and more marked distinction has been drawn between them; and it is now pretty generally admitted that there is no more necessary or intimate connection between the Law of Nature and the Law of Nations, properly so called, than there is between the former and the municipal law of any well regulated state. The Law of Nature contains in itself the seeds of all positive laws, national as well as international. It is true, indeed, that in discussing and deducing the rules of the Law of Nations, we are necessarily compelled, from the absence of any common and paramount authority over nations in general, to resort more frequently to reasons derived from principles of universal right; yet, in examining the municipal laws of any particular state, in discussing what the laws of such a state *ought to be*, a similar reference to universal justice is no less strictly necessary. Such was the opinion of Grotius himself, when he pronounced the Laws of Nature to be the *proavia juris civilis*. The phrase, Law of Nature and Nations, is open to some criticism. When it is intended to embrace international law, we should understand the phrase, *of nations*, in the same sense in which the Roman classical jurists understood their *jus gentium*, corresponding more nearly to our *law of nature* than to our modern international law. Perhaps it would have been desirable that this confusion had been avoided. The phrase here criticised seems, however, in a great measure, to be going out of fashion in the present day. As we shall have occasion hereafter to notice the work of Grotius, we shall at present abstain from offering any further remarks upon it.

Puffendorf. 23. Grotius was followed, at an interval of half a century, by Puffendorf, who, in his *Law of Nature and Nations*, further developed and methodized the system of his illustrious predecessor. In the year 1661, he was nominated by the Elector Palatine professor of the law which forms the title of his work, at Heidelberg; and this is the earliest instance of the foundation of such a professorship in any university. Similar professorships were afterwards instituted elsewhere, at Coimbra, for example. Puffendorf separates the law of nature from the law of nations properly so called, and though far inferior to Grotius in genius and learning, he effected an improvement by introducing a better order into the system.

Vattel. 24. A century later, Vattel's *Law of Nations* appeared, which, though it by no means supersedes the works of Grotius and Puffendorf, is more generally used as a text-book; and, as such, it is, in many respects, superior to them. In general, the author adopts the principles of the Saxon philosopher Wolff, and he is therefore regarded as belonging to a different school from Grotius and Puffendorf. Other writers of name have illustrated this science, but it is unnecessary to

swell these observations, by enlarging at present on this branch of the subject. We must, however, mention Law. Burlamaqui, a contemporary, though not the country-
man, of Vattel, and an author of sound principles and superior merit. His treatise on Natural Law is a favourite text-book on the general principles of morality and politics; but the law of nations is not indebted to him to any material extent.

25. In modern political philosophy, amongst a crowd Locke. of inferior names, we may select that of Locke, than whom few have acquired a higher celebrity, or exercised a greater influence on speculative opinions. His *Essay on Government* tended so powerfully to bring the doctrine of the Social Compact into fashion, that although that doctrine was by no means new, it is frequently ascribed to him.

26. Half a century later, we have Montesquieu, who Montes- belongs to a different class, but attained a still higher quieu. celebrity than Locke. His *Spirit of Laws* was published in the year 1748, and it placed its author nearly if not quite at the head of modern legal philosophers. At one time, few books were more read and studied, and though it has now lost much of the great popularity it once enjoyed, it still bears a high reputation. "What former age,"—says Sir James Mackintosh, with reference to the superior advantage in historical knowledge enjoyed by writers of the last over the preceding century,—“could have supplied facts for such a work as that of Montesquieu? He has been, perhaps justly, charged with abusing this advantage, by the undistinguishing adoption of the narratives of travellers of very different degrees of accuracy and veracity. But if we reluctantly confess the justness of this objection; if we are compelled to own that he exaggerates the influence of climate, that he ascribes too much to the foresight and forming skill of legislators, and far too little to time and circumstances, in the growth of political constitutions; that the substantial character and essential difference of governments are often lost and confounded in his technical language and arrangement; that he often bends the free and irregular outline of nature to the imposing but fallacious geometrical regularity of system; that he has chosen a style of affected abruptness, sententiousness, and vivacity, ill suited to the gravity of his subject: after all these concessions, (for his fame is large enough to spare many concessions,) the *Spirit of Laws* will still remain not only one of the most solid and durable monuments of the powers of the human mind, but a striking evidence of the inestimable advantages which political philosophy may receive from a wide survey of all the various conditions of human society.”*

27. Towards the end of the last century, two distinguished political writers appeared in Italy, Filangieri. Filangieri and Beccaria. The former of these, a Neapolitan, was a political speculator of a high order; and the work for which he is celebrated, on the science of legislation, (*Scienza della Legislazione*), published between 1780 and 1789, though it does not appear to have led to any remarkable practical result or movement, is yet deserving of notice, for the learning and knowledge it displays, the spirit in which it is written, and, on such a subject, the rare qualities of temper and moderation in its author. Shortly before this work was published, appeared Beccaria's *Essay on Crimes and Punishments*, ap- Beccaria.

* *Discourse on the Law of Nature and Nations*, p. 24.

Law. one of the earliest works in which attention was much called to criminal jurisprudence. More than two centuries before, Sir Thomas More had condemned, as wanton and unjust, the bloody severity of our own criminal code,* which, during a single reign, consigned to death no fewer than 72,000 thieves and rogues, exclusive of all other malefactors.† Beccaria called in question the necessity of inflicting the punishment of death in any case whatever. "This treatise,"—we quote the words of an anonymous writer,—“contains many acute and just general observations, but applies chiefly to the codes of criminal law which at the date of its appearance were in force throughout the states of Italy; and its principal value, in the present day, must be admitted, even by its greatest admirers, to consist not so much in what the author has himself done, as in what he has taught others to do. His well-founded aversion to the cruelty which marked the administration of criminal justice in his own country seems to have driven him to an opposite extreme, and he now and then suffers language to escape him, which cannot fail to bring the whole of his doctrines into suspicion with those who reverence the foundations on which the good order of society has hitherto been supposed to rest. Such as his doctrines were, however, they were approved and followed by some of the most popular and powerful princes then reigning in Europe. The Grand Duke Leopold of Tuscany took the lead in the career of reform, and from the success which that sovereign says had attended his previous mitigation of the penal code of his dominions, he was induced, on the 30th of November, 1786, to issue that celebrated edict from Pisa, by which he proclaimed the total abolition of capital punishment throughout the states of Florence.”‡ Subsequently, Austria, Russia, and France, followed the example thus set, and revised their criminal codes; and although the well-meant endeavours of these and other states have not in all cases been crowned with success; though they have been obliged, in many instances, to retrace their steps; yet, on the whole, an improvement has been effected by the introduction of a milder spirit into modern legislation; and some part, at least, of the merit belongs to Beccaria.

28. For the last three-quarters of a century, dating from the rise of Kant and his school, a number of philosophers and theorists, in addition to those we have noticed, have appeared in Germany and France. Our readers may find the chief of them mentioned, and distinguished into schools, in the works of two living authors, to whom we have already referred.§ One of these authors, M. Lermnier, seems to make it a subject of reproach to England, that it has contributed so little

* *Forte fortuna, quam die quodam in ejus mensâ essem, laicus quidam legum peritus aderat; is nescio unde nactus occasionem, cepit accurate laudare rigidam illam justitiam, quæ tum illic exercebatur in fures, quos passim narrabat nonnunquam suspendi viginti in una cruce; atque eo vehementius dicebat se mirari, cum tam pauci elaborarentur supplicio, quo malo fato fieret, uti tam multi ubique grassarentur. Tum ego, (ausus enim sum libere apud Cardinalem loqui,) nihil mireris, inquam: nam hæc punitio furum et supra justum est, et non ex usu publico. Est enim ad vindicanda furta nimis atrox, nec tamen ad refrænanda sufficiens. Quippe neque furtum simplex tam ingens facinus est, ut capite debeat plecti, neque ulla pœna est tanta, ut ab latrocinii cohibeat eos, qui nullam aliam artem quærendi victus habeant.—Utopia, lib. i.*

† Hume, *History of England*, vol. v. p. 533.

‡ *Quarterly Review*, vol. xxiv. p. 234.

§ Lermnier, *Introduction Générale à l'Histoire du Droit; Philosophie du Droit*. Warnkœnig, *Doctrina Juris Philosophica*.

to theoretical jurisprudence.* Until very lately, Bentham appeared almost alone to remove the reproach, and he is open to objection for his strong utilitarian views and principles, and for his disregard of and contempt for all past experience. For our own parts, we can scarcely consider this dearth of speculative writers in England as a subject of regret. In too many instances, the modern principles which have been broached, are either very unsound, or very fanciful; and where this is not the case, we question whether the science of jurisprudence, or legislation, has derived much benefit from the labours of the writers in question. The love of system and method, however, which has given birth to or been engendered by this philosophizing spirit, has been attended with one remarkable effect in recent times. We allude to the rage for codification, which took its rise in the latter part of the last century, and which has led to a general revision and systematic arrangement of their laws in codes, by several of the governments of Europe. The subject of codification engaged the attention of the principal lawyers of Germany, in a remarkable controversy which took place among them about the year 1815. The immediate subject of discussion concerned the propriety of making one general code for the whole of that country. MM. Thibaut and Savigny, professors of law at Heidelberg and Berlin respectively, were the leaders on either side, the former in favour of such a code, the latter against it. Savigny's views upon the subject are contained in a little work, which he entitled the *Vocation of the present Age for Legislation*. This, like every other work of the same profound writer, has met with a very extensive and favourable reception. A translation of it into English has been written and privately printed by a learned member of the English bar. On the question of codification at large, Savigny considers that, in a general revision and consolidation of the laws of a state, codifiers too often lose sight of, or are unable to trace, the true motives and reasons which led to each particular enactment, and that in endeavouring to arrive at an ideal perfection, they are too apt to neglect the peculiar wants, feelings, and customs of the people whose laws it is proposed to remodel and systematize. What though there be some confusion and difficulty, (affecting chiefly the regular professors of the art or science of law,) arising from the variety of customs prevailing in the different provinces of an empire, added to the multiplicity of the laws themselves. Such a state of things is not inconsistent with prosperity and order, with happiness and justice, among the people at large. But these blessings might be endangered by a rash attempt to introduce simplicity and uniformity, on the part of those who are presumptuous enough to imagine themselves competent to provide, at a single stroke, for the multifarious exigencies, which had singly demanded all the care, vigilance, and abilities of generations of bygone statesmen. If ever a measure, of such vast extent and importance, as a new general and uniform code for a large empire, is at all likely to be attended with happy results, it is when the science of jurisprudence has attained its very highest point of perfection, and when men are found fitted for the task of codification, by the greatest abilities and the most profound knowledge, historical, philosophical, and practical, of the

* *L'Angleterre se trouve en ce moment entre ses praticiens obstinés et l'école de Bentham; pour la science du droit proprement dite, elle y sommeille toujours. Introd. Gén. à l'Hist. &c. ch. xix.*

Law.
Modern
philosophers.

Modern
codification.

Law. whole subject submitted to them. These qualifications were not sufficiently possessed by the compilers of the three principal codes of modern times, those namely which were in force at the time when Savigny wrote the treatise above referred to,—the French, the Austrian, and the Prussian codes. Savigny considers them to have been signal failures, and the influence which they exerted on legislation, he thinks, was any thing but salutary. Of late, the rage for wholesale codification seems to have subsided in a great measure.

English
legislation.

29. In our own country, which has always been distinguished for its sober good sense and distrust of abstract theories, no universal revision of our laws has been attempted in the way of a general code. Bentham, indeed, who entertained any thing but respect for the jurisprudence of his country, vehemently urged the necessity of a totally new code for England, and proposed that the duty of compiling one should be intrusted to a foreigner! But very few, we imagine, could be found to second him in such a scheme. In the mean time, however, the task of consolidation and revision has been carried on in England, in a much more safe and wholesome manner; not by pulling our system to pieces under pretence of imparting to it an appearance of greater regularity, and dressing it up in what might be considered a more scientific form, but by taking up successively different branches of our law, expunging what is obsolete or useless, curtailing redundancies, supplying what is manifestly wanting under those heads, and arranging the rules under them in a better order. Whatever exception may be taken by self-styled philosophers to our laws and constitution, as an unsightly and unscientific mass of heterogeneous materials, not only have those laws and that constitution been productive of unexampled prosperity and happiness at home, but for a length of time they have been so largely the theme of admiration, and the object of imitation to foreigners, that English jurisprudence must form a prominent object of attention to the general lawyer. On this account, therefore, in addition to a just national predilection, English law, though a private national system, deserves a place in an Article like the present. Before, however, we proceed to notice any of the positive systems we have alluded to above, we propose to give a very brief consideration to the philosophy of law, chiefly with a view to explain what we conceive to be the true grounds upon which all human law and legislation rest; a subject, which, though frequently abused by sciolists, and therefore sometimes liable to suspicion and distrust, is nevertheless of deep interest and momentous importance. Without further preface, we shall now proceed with that part of the subject.

CHAPTER I.

THE PHILOSOPHY OF LAW.

Sect. 1. *Of the Nature of Laws in general.*

A law in its
most gene-
ral sense.

30. Various definitions have been given of a Law by different writers; as, a Rule of action;* a Necessary Relation arising from the nature of things;† &c. But whatever definition we may accept, a law, in its most extensive signification, involves two essential properties, generality and necessity. A law is general, inasmuch

as the mode of action or relation which it implies, is not confined to a few objects or a single occasion, but embraces a number of objects, and operates constantly and uniformly. By the second property, we understand that this general relation or systematic mode of action is not the effect of chance, nor spontaneously or capriciously adopted by those who observe it, but that it is produced by some external necessity or coercive power prescribing the relation or form of action, and compelling its observance. We may once more refer to Hooker's definition given at the commencement of this Article, (No. I.) as containing, perhaps, the most general, and, at the same time, the most exact description of a law with which we are acquainted.

Law.

31. Much discussion, and, as it appears to us, much verbal trifling, has been expended to little purpose, by some recent writers, in endeavouring to fix with precision the degree of generality necessary to constitute a law, in what they consider to be its only strict and proper sense. We shall, however, give the result of their ex-cogitations, so far as we think them worthy of remark. A law, in its only proper sense, say these writers, by which they mean a law imperative, or a mode of conduct prescribed by one intelligent agent to others, must be strictly general, in the two respects mentioned in the preceding paragraph. First, it must apply to a large though indefinite number of individuals; and although it may allow of numerous and extensive exceptions, although not only single individuals, but even large classes placed under the authority of the same lawgiver, may be privileged to be exempt from its observance, a law must be so general, in respect of its objects, as to form the rule, and not the exception. Secondly, it must prospectively have a perpetual or rather indefinite duration; it must not even be temporary, much less limited to a special occasion. When either of these requisites be wanting, call the prescribed rule what you will, it is not, they assert, a law. They profess to determine, in the following manner, the names properly applicable, where that of law is improper. When a mode of conduct is prescribed to all or the greater part of the members of a community, but on a special occasion only, it is an *Order*. When it is prescribed, though perpetually, to one or a few comparatively out of the whole number of members of the community, it is a *Privilege*, as the Roman jurists would have understood the term, for we have none exactly answering to it. Lastly, when not only the occasion is special, but the number of persons affected is limited, the order or privilege is narrowed into a simple *Command*.

32. So far with regard to the generality of a law. Various What we have above designated by the general name of necessity, is of various kinds; and laws may primarily be classified according to the particular species of necessity which occasions them. Thus there is a mathematical necessity, on which depend the general relations, or laws, of numbers, lines, spaces, &c. Again, there is a physical necessity having for its objects passive agents, and giving rise to the various physical or material laws of the universe. We however are concerned with a different species of necessity, that namely which is applicable to moral agents only; which directs and influences their conduct, without absolutely coercing their will or destroying their freedom; and which is commonly known by the name of *Obligation*. Upon this moral necessity depend all the laws, human and divine, by which man, considered as a rational and conscious

Obligation.

* Blackstone, *Commentaries*, vol. i. p. 38.

† Montesquieu, *Spirit of Laws*, book i. ch. i.

Law.

being, is governed. Burlamaqui has written so clearly, and so much to the purpose, upon this topic, though with some slight drawbacks, that what he has said may not upon the whole be unacceptable to our readers. "When we mention necessity," he says, "it is plain we do not mean a physical, but a moral necessity, consisting in the impression made on us by some particular motives, which determine us to act after a certain manner, and do not permit us to act rationally the opposite way."

33. "Finding ourselves in these circumstances, we say we are under an obligation of doing or omitting a certain thing; that is, we are determined to it by solid reasons, and engaged by cogent motives, which, like so many ties, draw our will to that side. It is in this sense a person says he is obliged. For whether we are determined by popular opinion, or whether we are directed by civilians and ethic writers, we find that the one and the other make obligation properly consist in a reason, which, being well understood and approved, determines us absolutely to act after a certain manner, preferable to another. From whence it follows, that the whole force of this obligation depends on the judgment by which we approve or condemn a particular manner of acting. For to approve, is acknowledging we ought to do a thing; and to condemn, is owning we ought not to do it. Now *ought* and *to be obliged* are synonymous terms."

34. "We have already hinted at the natural analogy between the proper and literal sense of the word *obliged*, and the figurative signification of the same term. Obligation properly denotes a tie; a man *obliged* is therefore a person who is *tied*. And as a man bound with cords or chains cannot move or act with liberty, so it is very near the same case with a person who is *obliged*: with this difference, that in the first case, it is an external and physical impediment which prevents the effect of one's natural strength; but in the second, it is only a moral tie, that is, the subjection of liberty is produced by reason, which, being the primitive rule of man and his faculties, directs and necessarily modifies his operations in a manner suitable to the end it proposed."

35. "We may therefore define obligation, considered in general, and in its first origin, a restriction of natural liberty, produced by reason; inasmuch as the counsels which reason gives us are so many motives that determine man to act after a certain manner preferable to another."*

Reason and conscience. 36. The only fault we find with this clear statement is, that here, as elsewhere in the same work, Burlamaqui seems to give a somewhat undue preference to human reason over conscience. The *sense* of obligation, which forms a distinct subject of consideration from the *nature* and *origin* of obligation viewed objectively, may be presented to the mind in either of two ways—by reason or by conscience—and in either way may produce its due effect. By reason, we not only conclude that a certain line of conduct is our duty, but we arrive at that conclusion by a rational process; and in consequence of having first satisfied ourselves of the existence of some adequate cause to make that line of conduct obligatory upon us, we distinctly see that the observance of such conduct tends to our good, and therefore we conclude that we ought to observe it. Conscience, on the other hand, simply judges a thing to be right, without con-

Law. sidering or inquiring how or why it becomes a part of our duty. Reason and conscience have each had their advocates among moralists; of whom some have chosen to name reason, and others conscience, as the sole or chief guide in moral conduct. Neither reason nor conscience, however, should be exclusively favoured: each of them in its proper place has its distinct function to perform. At the same time, we cannot for a moment hesitate which of those classes of moralists ought, in our estimation, to be preferred. In all the more important questions relating to our moral conduct, conscience is the safer and the better guide: it is less liable to be deceived, and exercises a more salutary influence than reason.

37. The question, however, with which we are now Their pro- concerned, is not what should be the practical test and per office, criterion of duty or right, in matters of difficulty and doubt; nor whether, or in what respects, we ought, as a general rule, to obey the dictates of either reason or conscience, or both. At present we have to consider, philosophically, the ground and cause of that obligation, which, whatever be the sentiments of those whom it affects, imparts to law its real force and efficacy. Let it then be remembered,—in order to separate two distinct questions, frequently confounded,—that reason and conscience are simply informants, and nothing more; that, as such, they are not infallible; but that each of them is liable occasionally to be impaired, abused, or deceived. When we say that our conscience or our reason deceives us, what do we mean? We mean that it makes a false representation; either that it suggests the existence of an obligation to do a particular act, where no obligation really exists; or that it represents an act as improper or indifferent which we are under an obligation to perform. Our conscience and our reason, therefore,—whatever may be their absolute or relative value as guides and informants,—can never, properly speaking, *create* an obligation; they can never make that to be intrinsically right, which is essentially wrong. Obligation and right, it may not be improper to remind our readers, are here considered objectively.

38. Keeping the above distinction in view, we must look for the philosophical ground and reason of obligation beyond and independently of any persuasion or feeling entertained by the person obliged respecting his duty. The true reason of obligation, then, is implicitly contained in the fundamental axiom in moral science, that every rational being is bound to consult his own true welfare and happiness. This proposition, properly understood, is an ultimate truth which does not admit of discussion or proof. The only question can be respecting the application of so strong a term as *obligation* to the expediency of acting in accordance with one's interest. But surely no term can be too strong, which is intended to express the reasonableness, the propriety, of choosing happiness, or the gross folly of wilfully abandoning it, when attainable by us. As such an abandonment, on the part of a free agent, would imply a forfeiture of his claim to be regarded as a reasonable being, so long as he sustains that character, he may, without much impropriety, be considered to be *tied down* to the choice of happiness. An obligation therefore supposes, first an action proposed to a free agent, for his voluntary observance; secondly, some consequence of good or evil, connected with that action; either a good consequence, as the reward of his compliance with the proposed condition, or an evil consequence, to punish

* *Principles of Natural Law*, part i. ch. vi. sec. 9.
VOL. II.

Law. him in case of non-compliance. The consequence, good or bad, must be such as the agent cannot of himself otherwise command or avoid; and it is called the **Sanction of law.** *sanction*. It may, we see, be either a reward or a punishment. Some jurists, indeed, contend that the latter only can properly constitute the sanction of a law; but as the attainment of a positive good affords as rational a ground, and, in many cases, as efficacious a motive, for directing the agent in the choice of his conduct, as the avoiding of a positive evil, we feel fully warranted in classing rewards amongst legal sanctions. Whether, under particular circumstances, punishments may not be preferable to rewards, with a view to effect the main purpose for which positive laws are instituted, is quite a distinct question, and one into which we decline entering at present. In another aspect, the attainment or the loss of a good may be regarded, respectively, as the avoiding or the incurring of an evil; and in this way, sanctions of both species may be reduced to a single class.

Errors of utilitarianism. 39. The connection, however, between interest and duty, though rarely questioned, is apparently regarded by some moralists, of a high order too, as accidental rather than essential. Their disinclination to adopt, fully and explicitly, the doctrine of obligation as above stated, may be ascribed, in a great measure, to the prevalence of two dangerous errors in connection with that doctrine, and to their wish to discountenance them. The first error assumes that no obligation, however solemn, ought to be recognized as such until the sanction be perceived, and the force of the obligation actually proved. Not content with establishing interest as the true and only basis of obligation, the holders of this opinion seem to require that motives of interest should always be present and uppermost in the mind. Any faculty, which enables us to apprehend moral truth, without going through the preliminary steps requisite to satisfy our purely intellectual inquiries, is apparently treated by them as fallacious. So that they not only would destroy one of our most useful faculties, conscience, but would necessarily confine all duties within the narrow limits of each individual's own weak and fallible intellect. This is one of the worst errors into which Paley and others of his school have fallen. Or if he is not justly chargeable with this error in its full extent, he has at least spoken of conscience in terms so disparaging as to convey the impression that it ought to be disregarded as a worthless prejudice.* We have already endeavoured to show the distinction between the ultimate reason and the practical apprehension of obligation; a confusion between these two has in no small degree given birth to the error here pointed out.

Motives of expediency. 40. The other error consists in partial and inadequate conceptions of true expediency, which is confined, for the most part, to advantages of an inferior order. Paley endeavours to obviate any objection on this head, by defining obligation to be a *violent* motive or inducement. This however does not afford a complete or accurate explanation of the matter, and his use of the word *motive* is calculated to mislead. On the one hand, obligation,—unless we would affix a purely arbitrary meaning to the word, as expressive merely of a certain though indefinite amount of expediency,—is susceptible of every degree of strength, from the lowest to the highest we can imagine. It would be no less unphilosophical than

untrue, to deny the reasonableness, and therefore the obligation, of endeavouring to procure even a slight good, where the pursuit of it does not interfere with the attainment of a higher good. All that can be said in such a case is, that the obligation is less stringent indeed, but not on that account the less reasonable. On the other hand, a particular inducement, viewed separately and apart from ulterior considerations, may be extremely violent, without affording any true and rational motive for action; as where there are conflicting inducements of a still more powerful nature, suggesting a contrary line of conduct. In such a case, reason would suggest the propriety, and therefore the obligation, of pursuing that course which is sanctioned by the most powerful inducements. An obligation, therefore, may be distinguished from a simple motive, however violent: strictly speaking, we should estimate the former from a consideration and comparison of all the separate consequences, good and bad, present and future, attending the action which it involves. The obligation consists in the resulting motive of all such consequences. In the case of our most important obligations, indeed, there are consequences so momentous and overwhelming that, in comparison with them, all inferior inducements may safely be disregarded. To conclude, any theory of obligation must necessarily be false which adopts a low and incomplete standard of expediency.

41. We have now sufficiently explained the nature of obligation, or that moral binding power which is derived from the existence of a sanction. A sanction may be specially annexed to a single prescribed action, and so create a special obligation. It is only where it is generally and permanently annexed that we have a law.

42. But in order to render a law operative, a third condition, besides generality and obligation, is required, viz., that the law be promulgated or made known to those who are to be bound by it: first, with regard to the rule, as distinguished from the obligation. This being the standard and measure by which they are to regulate their conduct, an accurate knowledge of it is indispensably necessary; otherwise they would be left in the dark how they ought to act. But with regard to the sanction, a far less precise knowledge will often suffice. A simple persuasion of duty, without any attempt to investigate its cause and nature; a knowledge of the existence of some superior agent, with the will and power to reward or punish the observance or breach of the law; or even a general and indistinct apprehension of some impending but undefined good or evil, connecting itself with the rule, may be, and often is, sufficient to influence the agent, and dispose him to compliance. The modes and instruments used for imparting a knowledge of laws and of their obligations are various; but the examination of these belongs rather to the discussion of the different species of laws than to the abstract description of law in general.

43. A law is commonly distinguished from some other things, which bear a near relation to it: 1. From *counsel* or *advice*; which is the communication of a binding precept by one who has no share in imposing it; 2. From *covenants* and *contracts*; which are made between equals, and do not suppose any superiority to reside in either of the parties over the other; 3. From a *liberty*, or simple *permission*; which, however, is often confounded with a *right*. These distinctions relate chiefly to positive laws. Some jurists have

* *Moral and Political Philosophy*, book i. ch. v. &c.

Law.

taken the trouble of discussing whether a simple permission be an effect, or a mere negation, of law. Those who hold the former opinion, sometimes make a distinction between Preceptive and Concessive law. The preceptive part is the law properly so called: it includes the whole of the commands and prohibitions of the lawgiver. The concessive law marks the extent of the subject's liberty. A liberty implies, simply, indifference on the lawgiver's part: it does not necessarily include protection for its exercise. When protection by the lawgiver is added to either a liberty, or a duty, the latter becomes a right. It is a liberty or a duty, in relation to the lawgiver. It is a right, as against those who are prohibited from interrupting its exercise or performance. As they are prohibited by means of legal obligations, right and obligation are correlative.

A liberty distinguished from a right.

Sect. 2. Of Necessary Laws.

Laws either necessary or positive.

44. Having explained the chief properties of all laws in general, we next distinguish their several kinds. And first, laws are either Necessary or Positive. Necessary laws are those inevitable relations of right which must exist in the very nature of things, and which cannot be ascribed to the institution of any lawgiver. Some writers, indeed, contend, that the name of laws cannot properly be applied to such: however, we have Hooker's authority for our application of the term. "They apply the name of law unto that only rule of working which superior authority imposeth; whereas we, somewhat more enlarging the sense thereof, term any kind of rule or canon, whereby actions are framed, a law."* Positive laws are artificially instituted by a lawgiver; and they are occasionally designated by such names as *arbitrary*, or *voluntary*, implying that they were established at the will, as well as through the power, of the lawgiver.

Discordant opinions respecting the division and nomenclature of laws.

45. No small difficulty and confusion have been occasioned by the disagreement of jurists, not only respecting fundamental principles, but also with regard to the nomenclature and division of laws. The most common primary division is into Natural and Positive. The term *natural law*, however, is ambiguous, the same writers frequently using it in different senses. The meaning of it which most nearly combines the various significations generally attributed to it, is that law which, antecedently to all human institutions, is binding on the whole of mankind. So far most authors are agreed; but in other respects, where this law is in question, many points of difference are observable among them. Several writers—and Grotius is of the number—make the law of nature to be the supreme law of all, and regard it as purely necessary, or independent even of Divine institution.† The great jurist just named distinguishes natural law from a Divine positive law;‡ but then he mixes up the former with many principles peculiar to mankind; thereby proving that his natural law partakes of both the *Eternal* and the natural law of Hooker. Hooker's Law of Nature is subordinate and specific, being that particular modification of the Law Eternal (or supreme necessary law) which the Almighty has adapted to the state of man.§ The most common idea of the law of nature regards it as of Divine institution: consequently, when so regarded, it is in strict-

ness deemed to be positive, because instituted by God; and, therefore, when it is opposed to positive law, the latter must be understood to mean *human* positive law. But then, with some inconsistency, a third species of law is introduced, and opposed in turn to each of the others: viz. a Divine positive law, not natural. The Roman jurists understood the law of nature in a different sense from any of the above, for with them it included the brute creation.* Lastly, some modern writers, bewildered amongst all these conflicting meanings, cut the matter short by denying that there is any such thing at all as the law of nature. An illogical division of the subject matter has occasioned much of the confusion in the question. With a view to avoid it, we have adopted division stated at the commencement of this section. We shall class the law of nature, then, amongst positive laws, giving it a limited meaning, and, with Hooker, regarding it as a positive, or artificial modification of the necessary law, which is the supreme rule, not only of mankind, but of all moral beings. To this necessary law our attention must now be specially directed.

Law.

46. In one respect, every law is necessary; that is, In what respect a law is said to be necessary. as it affects those who are subject to it; for as they cannot, at their pleasure, command the reward, or evade the punishment, which constitutes the sanction of the law (No. 38), they are under an unavoidable or necessary obligation to observe it. In the case of positive laws, this necessity extends no further: the good or evil of the sanction has no necessary or inevitable connection with the law beyond the pleasure of the lawgiver, who may repeal the law by withdrawing the sanction. But the necessary laws, of which we now treat, cannot be so annulled. Their sanctions are, in relation to every moral being without exception, the inevitable consequences of certain actions, or rather, we should say, of certain active habits and dispositions, involving the unalterable and essential laws of morality, and constituting the fundamental rule of right.

47. The existence, however, of necessary laws has not always been asserted in sufficiently clear and distinct terms; occasionally, too, it has been called in question. We do not now allude to such writers as Hobbes, who holds that "we ourselves are the authors of the principles of justice, of right and wrong, by making those laws and compacts whence the measures of justice are taken: since before any such laws or compacts were instituted, there was no such thing as justice or injustice, public good or evil, among men, any more than among beasts." Assertions of this kind are scarcely deserving of serious refutation. Those whose opinions upon such matters are really entitled to respect, agree in recognizing some paramount and invariable law, or rule of justice, independent of and superior to all human institutions and opinions. The only question is respecting the source of this law. A practice has very extensively prevailed of ascribing it, in terms at least, to an arbitrary or positive institution of the Almighty, as if it were within the range of possibility for him to have approved of or sanctioned another and different rule. One slight observation will go far to place the matter in its true light. The very practice of styling God *just* and *good*, shows that we acknowledge some original and independent principles of

The existence of necessary laws asserted,

* Ecclesiastical Polity, book i. sec. 2.

† De Jure Belli et Pacis, Proleg. vi. &c.

‡ Ibid. lib. i. ch. i. x.

§ Ecclesiastical Polity, book i. secs. 1, 3.

* Jus naturale est, quod natura omnia animalia docuit. Nam jus istud non humani generis proprium; sed omnium animalium, quæ in terra, quæ in mari nascuntur, avium quoque commune est.—Dig. I. i. l. 3.

Law. justice and goodness. Leibnitz remarks, that justice would not be an essential attribute of the Deity, if he by Leibnitz, had established it by an arbitrary decree of his own. Justice, he adds, observes laws of equality and proportion as certain and unchangeable as those of arithmetic and geometry. Hooker speaks to much the same effect. In treating of the Law Eternal, as of that law which God himself observes, he remarks, "that counsel is used, reason is followed, a way observed, constant order and law is kept." "They err, therefore, who think that of the will of God to do this or that there is no reason besides his will. Many times no reason known to us; but that there is no reason thereof, I judge it most unreasonable to imagine." And he concludes the first book of the *Ecclesiastical Polity* with that noble, but now somewhat trite passage, wherein he says of law, that "all things in heaven and earth do her homage, the very least as feeling her care, and the greatest as not exempted from her power."* Expressions such as these, sufficiently indicate Hooker's recognition of one essential, necessary law; and we are left to infer that, had not God chosen and observed it, he would not be the reasonable, good, and perfect Being that he is.

by Grotius. 48. The emphatic language of Grotius should not be omitted here. No writer we are acquainted with has been equally explicit on this topic. After maintaining, against Carneades and all such objectors, the immutable character of right, and its nature as it applies to man, he adds, that "all he had said would take place, though we should even grant, what without the greatest wickedness cannot be granted, that there is no God, or that he takes no care of human affairs." We have already (No. 45) pointed out the sense in which Grotius understands the law of nature, which, we have also stated, he distinguishes from a voluntary Divine law. The latter, he says, "does not command or forbid such things as are in themselves, or in their own nature, obligatory or unlawful; but, by forbidding it, renders the one unlawful, and, by commanding, the other obligatory." But, "the law of nature," he shortly afterwards adds, "is so unalterable, that God himself cannot change it. For though the power of God is infinite, yet we may say that there are some things to which this infinite power does not extend, because they cannot be expressed by propositions that contain any sense, but manifestly imply a contradiction. For instance, then, as God himself cannot effect, that twice two should not be four, so neither can he, that what is intrinsically evil should not be evil."†

49. Those who, without fully comprehending the meaning of Grotius, and startled by his decided language, have objected to his view of the subject, when they come to reason upon it, cannot help arriving at the very same conclusion with him. Puffendorf does so, apparently,

* *Ecclesiastical Polity*, book i. secs. 2, 15.

† *De Jure Belli et Pacis*, ubi supra. It would be superfluous to multiply authorities to establish what perhaps is already sufficiently evident. But we cannot forbear noticing that the profound Bishop Butler hints at the existence of similar moral impossibilities. "Why the Author of nature does not give his creatures promiscuously such and such perceptions, without regard to their behaviour; why he does not make them happy, without the instrumentality of their own actions, and prevent their bringing any sufferings upon themselves—is another matter. Perhaps there may be some impossibilities in the nature of things which we are unacquainted with." *Analogy of Religion Natural and Revealed*, ch. ii. See also, Burlamaqui, *Natural Law*, part 2, ch. v. sec. 5.

without being aware of it.* He felt alarmed, it seems, because he imagined that Grotius favoured the idea, that without supposing a Divine Creator, there could have been room for the actual exercise of the necessary law. But Grotius contended for no such thing. He contended only for necessary *relations* of right and wrong. His meaning is identified with that of Montesquieu, who says, that "before [positive] laws were made, there were relations of possible justice. To say that there is nothing just or unjust, but what is commanded or forbidden by positive laws, is the same as saying that before the describing of a circle, all the radii were not equal."† In other words, he meant, that wherever Omnipotence interposes, and establishes the requisite conditions for the possible breach or observance of the necessary law under any particular form,—wherever God creates free agents, endows them with moral and intellectual faculties and perceptions, and places them in juxtaposition with objects of the same nature with themselves, towards whom it is in their power to exercise benevolence, justice, truth, &c., and without whose existence and presence it would be impossible for them to exercise those virtues,—there nothing further is required on the part of God to determine in general what shall be right or wrong; for that it would be impossible, in the very nature of things, for any exertion of power to annex moral happiness, and therefore an obligation (No. 38), to the habitual exercise of malevolence, injustice, and falsehood; or moral misery,—involving its inevitable consequence, wrong,—to benevolence, justice, and truth.

50. Of the virtues here named, and assumed as constituting the essential relations of right, no one will feel disposed to question the intrinsic excellence and paramount obligation. Yet we do not consider the preceding discussion by any means superfluous. By having a clear and definite knowledge of the foundation of our duty, we are enabled to form a better estimate of the stability and just proportions of the superstructure; many difficulties, otherwise wholly inexplicable, are accounted for; Divine wisdom and justice are vindicated by the removal of all suspicion that we are subjected to a law capriciously chosen; we become satisfied that there is, that there must be, one paramount rule, absolutely binding upon us and all moral beings; a rule which it is of the highest moment that we should observe, which no power can surmount or annul, and which therefore offers an end worthy of the greatest efforts of Divine and human wisdom, labouring to establish the supremacy of that law in and over us. Having fully satisfied ourselves of the reality and nature of that law, and taken it as the basis of all our obligations in particular, we can the more readily disembarass ourselves of the many groundless theories with which the science of jurisprudence has been encumbered. But first we should observe, if any such observation is now required, that the necessary moral law strictly fulfils all the conditions of a law enumerated in the preceding section. Its obligation we have already shown to be necessary and essential. That it is more than general—that it is universal—follows from its essential character. With regard to its promulgation, whatever occasional mistakes and misapprehensions may occur respecting its

* *Law of Nature and Nations*, book i. ch. ii. sec. 6; book ii. ch. iii. sec. 4.

† *Spirit of Laws*, book i. ch. i.

Law.

application in particular cases, there is no reasonable human being so devoid of moral sense, as not to feel its paramount obligation, without any formal announcement of it to him. It is on this latter account that it is frequently styled *natural*; though, as we have stated, we reserve that term for one particular and positive form of it.

Man necessarily a social being.

51. A particular consideration of the *social* character of the necessary obligations of benevolence, justice, and truth, will serve to remove the charge of selfishness, sometimes, though without much reflection or justice, brought against the account of obligation, which takes interest for its basis. (No. 38.) Recollecting how large and important a share of our duty is included in the obligations we have named, and how, in their very nature, they imply that our immediate regards should be fixed on others, and therefore withdrawn from self, all objection of the kind alluded to ceases. A like consideration enables us to expose the very imperfect, and therefore false, view, not unfrequently taken of the ultimate end and purpose of human societies and laws; a view, which regards the ends of government to be merely temporal, and intended only for protection against the material wants and dangers to which man is exposed in a state of unassisted nature. It is true, that the existence of such evils furnishes the immediate occasion for the institution of temporal government, and that one important end of the latter is protection against them. But this is not all. The social nature of man is not a mere accident of humanity; it belongs to him essentially, as he is a moral being; and sociability is enjoined as an indispensable part of the necessary law, because it is inseparable from the cultivation and exercise of benevolence, justice, and the other social virtues. In the more complete and correct view of the subject, human laws are subsidiary to the necessary moral law; and the material wants of man are only so many artificial bonds, for uniting our species in a society which is designed to be matured into one of a superior nature.

Sect. 3. Of Positive Law in general.

Standard of perfection.

52. The necessary, or, as we may now call it, the moral law, furnishes the only true ultimate standard of right. And that standard, though there are various degrees of approximation towards it, cannot itself stop short of absolute perfection. Now moral perfection implies a complete and spontaneous accordance with the high requisitions of the moral law: it is not enough that the outward actions be right: the principles and motives must be so too; the moral taste and disposition must be so entirely in harmony with the dictates of the highest morality, that no suggestion, either of reason or of conscience, much less any external compulsion, can be required, as an incentive to habitual active good.

Positive law a process of moral regulation.

53. So far with respect to the standard of right. We have next to regard man as a moral being, whose proper aim, therefore, is his perfection, and whose duty it is to make the nearest practicable approach towards it. The natural or ordinary processes, which may be put in action for promoting his moral advancement, comprehend the regulation of his spirit and inward principles, and the regulation of his outward actions. It is the former which it is proposed ultimately to affect; but though a direct appeal may, in some measure, be made to them; yet this cannot be done effectually without the concurrent regulation of his external conduct. The converse of this is also true; that is to say, even supposing there

Law.

were only some subordinate object, depending on man's outward actions, in view, such as his temporal welfare, this object could not be promoted in any tolerable degree, by simply attending to his conduct, without at the same time endeavouring to guide his principles. Thus there are two essential processes, distinct, though intimately related. They are distinct, not only in their immediate objects, but in the instruments they employ, and the methods they pursue, and they require distinct agents to put them in motion, though their ultimate aim be directed to one common point. In civil society, the regulation of the inward principles belongs to the Church; that of the outward actions,—in those respects, at least, where the standard of conduct is rendered compulsory,—to the State. *Government*, in a somewhat enlarged sense, may be considered to embrace both departments, as subordinate and necessary branches of the same whole, its ultimate object being to promote, chiefly and principally, the moral welfare of the community over which it is set.* Having premised so much, our attention must now be directed to outward actions, and to positive law, which comprehends the rules framed for regulating them.

54. Positive law is a rule of conduct, instituted by a lawgiver, and enforced by artificial sanctions.

55. I. Positive law is *instituted*. It therefore presupposes a lawgiver or sovereign; and this suggests a question respecting the right in general of imposing laws, or the right of sovereignty. As an abstract proposition, this right is commonly stated to depend upon the possession of the three qualities necessary for good government; namely, wisdom to discern the real interest of the governed, and the best mode and means of promoting that interest; goodness, to dispose the sovereign to pursue those means; and power to apply them effectually. The application of this principle to establish God's right to the government of mankind is easy enough; for granting that he possesses the three qualities here mentioned, it follows that his government cannot be otherwise than beneficial in the highest degree to man, who is therefore morally bound to submission, inasmuch as he is bound to pursue his highest interest. And obligation and right being correlative, (No. 43.) man's obligation to submit, involves God's right to govern. It would be absurd to reduce God's right of government to his supreme power alone, though such an opinion has occasionally been put forward; for could we suppose the case of a being of transcendent power imposing laws unnecessarily severe and unjust, he must of necessity be an evil and malignant, not a good being; and as he could not change the eternal laws of right, nor oblige beings of less power to any course morally injurious to themselves, his right could not extend to any unjust exertions of power, which are mere

Institution of positive law.
Right of sovereignty.

* Our greatest political philosopher, Burke, observes, "An alliance between Church and State in a Christian Commonwealth is, in my opinion, an idle and a fanciful speculation. An alliance is between two things, that are in their nature distinct and independent, such as between two sovereign States. But in a Christian Commonwealth, the Church and the State are one and the same thing, being different integral parts of the same whole." See the whole passage in his works, vol. x. p. 43. ed. 1818. The reader will find the same idea more fully expanded in Coleridge's essay *On the Constitution of the Church and State, according to the Idea of each*, and in Mr. Gladstone's Treatise, *The State in its Relations with the Church*. Cicero, it may be observed, dwells especially, and with more than his ordinary force and eloquence, on the moral purpose of law and civil government. *De Legibus, passim*.

Law.

tyranny. We can therefore admit no such modification of the principle which establishes wisdom, goodness, and power, as the three essential elements of the title to sovereignty.* How to apply this principle, however, to human potentates, is much more difficult; but at present we shall not enter into the question as applicable to them. Only we may observe that rightful human authority is a *moral* right, or, still more strictly, a *natural* right, because it precedes human law, and generally arises upon a state of circumstances, which happen in the ordinary course of human affairs, and without the device and contrivance of man.

Positive law concerns specific actions.

56. II. Positive law is a rule of *action*; it "appoints the form and measure of working." In this respect it is distinguished from moral law, which is not restricted to any specific mode of action, but inculcates, as a principle, the performance of right, and the abstinence from wrong, universally, and under all circumstances, but without particularizing any of such circumstances. We need not here repeat what has been said on this topic, in the first paragraph of this article. At present we must observe, that positive law, having to deal with the baser part of our nature, applies itself exclusively to the coarser and more mechanical department of our moral training, namely, the regulation of our *conduct*. Because it would be useless to lay the abstract truths and principles of morality before rude and untutored beings whom it is necessary to compel, it embodies those principles in some selected and definite form, and prescribes rules of conduct more or less specific. The natural order of progress in practical morality is from single actions to habits, from habits to principles, and from principles to a confirmed disposition. Positive law having guarded the outward and visible conduct of man, and having laid the foundation of right habits, in the most ordinary and obvious cases, can do no more. It is the business of other and higher appliances, and of self-regulation, to confirm, enlarge, and mature those habits into a settled principle of right.

Artificially enforced.

57. III. Positive law is enforced by artificial sanctions. The potency of a sanction, as theoretical writers inform us, depends on four circumstances, the intensity of the good or evil it involves, its duration, its certainty, and its proximity in point of time. With regard to the first three, wherever they can be known and measured, they afford a fair test for estimating the genuine degree of obligation, and therefore they ought, in reason, to possess an influence proportioned to their amount. It is otherwise with regard to the last. We can only allege the frailty and imperfection of man as the cause of those remarkable differences which are observable, in the present power of sanctions over him, proportioned to their relative proximity or remoteness. "Whatever withdraws him from the power of the senses, whatever makes the past, the distant, or the future, predominate over the present, advances him in the dignity of thinking beings." But this power of realizing what is distant, does not belong to him in his natural state, nor is it readily attainable. Hence the sanctions of the moral law, which are commonly supposed to be remote, lose much of their just influence; and, in a multitude of instances, it would be unable to act upon man, without the artificial institution of more immediate sanctions. The compa-

rative violence with which the latter act, gives them, in a measure, the character of compulsion; but this is a moral and not a physical compulsion: man is still left a free agent, and the feeling of restraint occasioned by the trammels of positive sanctions will gradually subside, in proportion as he rises to the influence of higher motives.

Law.

58. As to the subject-matter of positive laws, they may include any human actions, those which terminate in the person himself, as well as those which affect others. Subject-matter of positive laws.

It would be impossible, however, by any abstract rule, to assign the just limits of legal enforcement. "To distinguish," says Mr. Austin,* "the acts and forbearances that ought to be the objects of law, from those that ought to be abandoned to the exclusive cognizance of morality, is, perhaps, the hardest of the problems which the science of ethics presents. The only existing approach to a solution of the problem, may be found in the writings of Mr. Bentham. (See, in particular, his *Principles of Morals and Legislation*, ch. xvii.)" Mr. Bentham, however, has not solved the problem, and for the simple reason, that it does not admit of a general solution: the selection of topics for enforcement depends upon a number of circumstances constantly varying. There are some obligations and rights, indeed, so obvious and clear, and so necessary to be observed and respected, that there can scarcely ever be a question about the propriety of enforcing them. Jurists have distinguished such by the name of *perfect* obligations and rights.† There are others, of a more refined morality, not less certain it is true, at least in the apprehension of a sound moralist, but unsuitable to be extorted by force: these are called *imperfect*. But there is no well-defined line separating the two sorts of obligations and rights. The progress of society and civilization has a tendency gradually to advance the recognised boundaries of perfect obligations. We have a remarkable proof of this in the Law of Nations, a complicated system of acknowledged perfect obligations, but of modern growth. All laws, properly speaking, are of perfect obligation, since without sanctions they would not strictly be laws. However, the Roman jurists gave the name of *laws of imperfect obligation* to such imperfect rules as the legislator, accidentally or designedly, had omitted to provide with sanctions. Perfect and imperfect obligations.

59. A positive law must be sufficiently notified to those who are to obey it. But no general rule is laid down respecting the mode. The remaining conditions, commonly stated to belong to positive laws, are, 1. that they be *possible*, or such as the subject can fulfil; 2. that they be *just*, or conformable to the moral law, and the constitution of man; 3. that they be *useful*, or ordained, not simply for the sake of restraint, which would be unreasonable and tyrannical, but with a view, and in a manner, to secure some benefit or advantage to the subject on whom they are imposed.‡ Other conditions.

Sect. 4. The Law of Nature.

60. Having settled the general principles of law,—of moral or necessary law as the foundation of all other, and of positive law, as a means to an end,—our next business is to apply these principles to the Law of nature.

* Burlamaqui, *Principles of Natural Law*, part i. ch. ix.; Puffendorf, *Law of Nature and Nations*, l. 6. 10, 11.; 1 Blackstone's *Commentaries*, 40, 48.

† Puffendorf, *Law of Nature and Nations*, l. vii. 7. The expression, *sum jus*, used by Latin writers, denotes a perfect right.
‡ Burlamaqui, *Natural Law*, part i. ch. x. sec. 9.

Law. Nature in particular, or to that positive system which God has established for the government of his creature man, and upon which all human laws are built. The questions that we shall have to examine are, what the law of nature is; in what its obligation consists; how we become assured of its existence and reality, and acquainted with its rules; and how human laws connect themselves with it. But first in searching for a deter-

Meaning of the word natural. minate meaning of the word *natural*, amongst the various loose, and confused, and confusing senses in which different authors use that word, we cannot find a better explanation of its true meaning, than that given by Butler, who says that it signifies *stated, fixed, or settled*;* in which sense, it is opposed to what is commonly considered as supernatural or miraculous, because of rare or extraordinary occurrence. So that Natural Law, or the Law of Nature, will mean that stated, fixed law, which governs the human race, considered as one general community. It will appear, that, as a law, it possesses the two essential requisites of permanency and universality. (No. 31.) As the law of nature, its authority will be shown to be coextensive and coexistent with nature, or with the human race and the constitution of man in all ages. Again, it will be seen that, as a positive law, it possesses all the properties of such a law enumerated in the preceding section:— that it is instituted by God, with a view to perfect our nature; that it is most wisely and beneficently adapted for that purpose, first, by the definite form of its rules, as compared with the larger principles of the moral law, and, secondly, by the sensual character of its sanctions; in both these respects accommodating itself to human weakness. Lastly, it will appear to be sufficiently notified to all mankind. (No. 42.) Though this law be, in the highest degree, *possible, just, and useful*, (No. 59.) and though we are bound to receive it without questioning the wisdom, goodness, or power of Him who has ordained it, yet for any thing that we can know, or may observe in the law itself, God might have effected his purpose equally well, by a measure of a different form: the law of nature, like every other positive, or *arbitrary* (No. 44.) law, implies a certain choice of means on the part of the lawgiver, but without supposing that he could do violence to the end proposed by a wanton exercise of power. But though we must ascribe so much to God's power of choice, the indulgence in any speculations of what he might have done, can only lead to the most absurd results or perplexing confusion.† “Let us

then, instead of that idle and not very innocent employment of forming imaginary models of a world, and schemes of governing it, turn our thoughts to what we experience to be the conduct of nature with respect to intelligent creatures; which may be resolved into general laws or rules of administration, in the same way as many of the laws of nature respecting inanimate matter may be collected from experiment.*”

61. In speaking of natural law, we have hitherto adopted the definition given by Butler, but it is necessary to go a little more into detail, to explain the several senses in which ethical writers understand the term. Not only are different authors inconsistent with each other in the meaning which they attach to it, but scarcely one of them can be found who is throughout consistent with himself. Many of them appear to have had no very definite idea on the subject; and even those who set out by defining what they understand by the expression, “Natural Law,” yet afterwards employ it in such a manner as proves they have forgotten their own definition.

Law. Further explanation of the term natural.

62. A little attention will enable us to discover that the predominant idea which this term conveyed to the minds of the writers on this subject was one or other of the following:

63. 1st. The universality of the extension, and perpetuity of the continuance, of this law. Being a law imposed by the Deity upon the whole human race for their universal and perpetual observance, it is used in this sense in contradistinction to all those practical propositions of which the obligation is special or variable, and which might be ranked among natural laws, only that they want the qualities of universality and perpetuity. Thus the preservation of our health might be deemed a natural law; but not so the obligation to consult or follow the prescription of any particular physician, although there can be no doubt that the latter is morally binding.

2dly. The natural character of its sanctions; the rewards or punishments by which its observance is enforced; being such consequences as follow generally from human actions in the ordinary course of nature, without any miraculous interposition on the part of the Deity, or any systematic interference on the part of man. By systematic interference, we mean the interference of the human magistrate in execution of the sanctions imposed by a human legislator, in contradistinction to the casual interference of man, such as the pain which he inflicts on an offender who has provoked his resentment; this casual interference being ordinarily looked upon in the light of a natural sanction.

64. 3dly. The nature of its promulgation; where the law is such that it may be readily discovered by the ordinary exercise of reason and observation, in tracing the connection between human actions and their consequences. The word ordinary is here employed in contradistinction to any of the more subtle and far-fetched

* “Though we were to allow any confused undetermined sense, which people please to put upon the word *natural*, it would be a shortness of thought scarce creditable to imagine, that no system or course of things can be so, but only what we see at present. . . . But the only distinct meaning of that word is *stated, fixed, or settled*; since what is natural as much requires and presupposes an intelligent agent to render it so, i. e. to effect it continually, or at stated times, as what is supernatural or miraculous does to effect it for once.” *Analogy*, ch. i. sub fin.

† For example, such as the following. “Considering the Creator only as a being of infinite power, he was able unquestionably to have prescribed whatever laws he pleased to his creature, man, however unjust or severe. But as he is also a being of infinite wisdom, &c. 1 Blackstone's Commentaries, 40. “The Creator might certainly have formed all men at the same time, though separated from one another, by investing each of them with the proper and sufficient qualities for this kind of solitary life.” Burlamaqui, *Natural Law*, II. iv. 11. When we recollect the moral necessity of sociability, (No. 51.) and the provisions in nature for maintaining and strengthening it, even the momentary supposition of such a possibility as that hinted at by Burlamaqui, however subsequently explained, at once perplexes our reason, by the confusion

it makes between moral and human nature, and revolts our ideas of God's goodness and justice, by supposing he could do any thing so absurd and unjust. Burlamaqui, indeed, afterwards explains his meaning. But why suppose an absurdity for the sake of afterwards answering it? We quote the passage on account of the principle it involves. “These (the laws of nature) are not arbitrary laws, such as God might not have given, or have given others of a quite different nature. Supreme wisdom can no more than supreme power act any thing absurd and contradictory. It is the very nature of things that always serves for the rule of his determinations.” *Ibid.* II. v. 5.

* Butler, *Analogy*, Introd.

Law. deductions of reason. All men are not competent to carry out a long series of reasoning; many are not capable of understanding such a process when discovered and communicated to them by others. These more refined deductions of reason, therefore, because of their difficulty and consequent infrequency, are not considered to come within the meaning of the term natural, using that term in reference to the ordinary depth of men's understandings, and not to the abstract nature of things.

Uncertain limits of natural law.

65. It is scarcely necessary to observe, that when the signification of the term "natural law" is so unsteady, the limits of this law cannot be fixed, but will vary according to the particular sense which each individual jurist attaches to this term, and according to the particular notion he may entertain of the degree of generality, kind of consequence, or amount of intelligence and acuteness, which may, in his opinion, be requisite or sufficient to constitute *natural*.

66. Having premised thus much, in reference to the various significations of the term natural law, we will now proceed to consider the subject under the three following heads:—

1st. The proofs and sources of our knowledge of natural law.

2d. Its sanctions.

3d. Its rules and precepts.

Proof and sources of our knowledge of the law of nature.

67. I. The proofs and sources of our knowledge of natural law. On the source from whence the knowledge of natural law should be deduced, and the mode of proof by which it should be established, jurists have not agreed. Some rest their system on the foundation of revelation; some on the universal consent of mankind; whilst the generality of more modern jurists, since the time of Bishop Cumberland, reject these two foundations, and rest entirely upon that of reason, observation, and experience. We shall proceed to consider each of these three sources in order, premising that any system of morals which does not derive its supply from them all, must be incomplete.

Revelation.

68. 1st. With regard to revelation. A distinction has frequently been taken between the law of nature and revelation, to which we cannot assent. The law of nature does not prescribe one thing and revelation another. Revelation therefore is not a distinct code; it is only a peculiar expression of the natural or moral law, one or the other; in fact, it is not a substantive law in itself, but rather a distinct version of the moral law, impressed with the high authority of supernatural testimony. If we take the law of nature in its widest sense, as embracing the whole compass of morality, any system must be very incomplete which fails to include revelation among its sources. Whence but from revelation can we derive any knowledge of the great leading principles of morality? It is true the Deity has implanted in man a moral sense or faculty, and that moral sense suggests to some extent the precepts by which our conduct should be governed; but if we desire an acquaintance with the great and *perfect* moral scheme, we must look for it in the Scriptures, since there, and there alone, we find the great principles of eternal morality rising to their highest perfection, and the doctrine of universal love for the first time fully and clearly revealed as an established principle of action. It is true that among the speculations of the more enlightened heathens we find the love of mankind at large highly commended. Their mode of theorizing on the

subject is beautiful, but it never went beyond theory. In the Scripture it is, that the law of love is made the rule of life, and for the first time declared to be the only right and true law of human action.*

But although revelation be an indispensable foundation of a perfect moral system, yet it is not to be supposed that the moral code will be found there carried out into all its details, since, as has been well observed, "whoever expects to find in the Scriptures a specific direction for every moral doubt that arises, looks for more than he will find."† We must have recourse, in general, to other sources, for the details of duty.

69. 2dly. The consent of mankind. Grotius has used this mode of proof very extensively in establishing the law of nations. "I have likewise towards the proof of this law," he observes, "made use of the testimonies of philosophers, historians, poets, and, in the last place, orators; not as if they were to be implicitly believed; for it is usual with them to accommodate themselves to the prejudices of their sect, the nature of their subject, and the interest of their cause; but that when many men of different times and places unanimously affirm the same thing for truth, this ought to be ascribed to a general cause which, in the questions treated of by us, can be no other than either a just inference drawn from the principles of nature, or an universal consent."‡ Puffendorf rejects the consent of mankind as a proof of natural law;§ but it is evident that, in condemning Grotius, he has mistaken his meaning. The masterly defence of Grotius, by Sir J. Mackintosh, in answer to those who criticise his citation of authorities, is too good to be omitted:—"He was not of such a stupid and servile cast of mind as to quote the opinions of poets or orators, of historians and philosophers, as those of judges, from whose decision there was no appeal. He quotes them, as he tells us himself, as witnesses whose conspiring testimony, mightily strengthened and confirmed by their discordance on almost every other subject, is a conclusive proof of the unanimity of the whole human race on the great rules of duty and the fundamental principles of morals. On such matters, poets and orators are the most unexceptionable of all witnesses; for they address themselves to the general feelings and sympathies of mankind; they are neither warped by system, nor perverted by sophistry; they can attain none of their objects, they can neither please nor persuade, if they dwell on moral sentiments not in unison with those of their readers. No system of moral philosophy can surely disregard the general feelings of human nature and the according judgment of all ages and nations. But where are these feelings and that judgment recorded and preserved? In those very writings which Grotius is gravely blamed for having quoted. The usages and laws of nations, the events of history, the opinions of philosophers, the sentiments of orators and poets, as well as the observation of common life, are, in truth, the materials out of which the science of morality is formed; and those who neglect them are justly chargeable with a vain attempt to philosophize

* The mode in which the Mosaic law inculcates the universal law of love is beautifully exhibited by the late learned and pious H. J. Rose, in his *Notices of the Mosaic Law*, London, 1831.

† Paley, *Mor. Phil.* book i. ch. iv.

‡ Grotius, *De Jure Belli et Pacis*, Proleg. 41.

§ Puffendorf, *Law of Nature and Nations*. II. iii. 7.

Law.

Law. without regard to fact and experience, the sole foundation of all true philosophy.*

Observation, experience and reason.

70. 3dly. Observation, Experience, and Reason. As a source of our knowledge of natural law, revelation is not sufficient; its proper object, as we intimated above, is not the natural but the moral law. Neither is the consent of mankind alone sufficient, since this consent is not easily collected, and, even if it were, would still require the test of individual experience, to make our knowledge perfect. It is by examining nature, by observing human impulses and human actions, and their effects, in what general direction our instincts, prejudices, and passions point, and what are the external objects, in the attainment or avoidance of which the gratification of our physical appetites and preservation of our physical nature consist, that we perceive the primary indications of natural law. These general tendencies to action, when kept within due bounds by that sense of right which belongs to the moral nature of that complex being, man, constitute the system of natural laws. In tracing them out, it is not necessary to enter upon any profound metaphysical discussion; it is only requisite to observe the effects that follow from human actions, and to judge if those effects be universal. When we judge them to be so, it is impossible to attribute that universality to accident; we are irresistibly led to attribute that universal connection of our actions with their results, to an institution of Divine Providence, appointed for the purpose of indicating that law of our conduct to which they were calculated to lead us.

Sanctions of natural law.

71. II. We have next to consider the Sanctions of natural law. But in order to distinguish those sanctions from others to be presently noticed, to which they are nearly related, we must be careful not to confound man's moral with his human nature. As a moral being, man's perfection and complete happiness must ultimately depend upon his moral tastes and habits, and not upon his physical constitution. And it is essential to bear in mind that he was placed in the world for the purpose of exercising and improving, according to the opportunities afforded him, the several moral faculties with which he is endowed, but which as yet are imperfect and corrupted. Many writers, as, for example, Puffendorf and Burlamaqui, add considerably to the intrinsic difficulties of the subject, by regarding man, in the first instance, as a complex being, made up of soul and body, and then investigating the various provisions in nature, for satisfying his material concurrently with his moral wants; thereby unconsciously confounding together means and ends. The more proper, and, in our opinion, the only correct way of considering the subject, is the following. Granting man to be such an imperfect moral being as we have described above, how has God, by superadding the human nature which man now possesses, provided for his moral advancement? First, then, according to his human, and what we may regard as his purely adventitious constitution, man has many physical wants and requirements to be supplied and satisfied, and dangers to be avoided; and the supplying and satisfying of the former, and the avoiding of the latter, afford him an immediate object for active and strenuous exertion of a particular kind, and for the employment of all his faculties in a certain way. Thus a specific and definite form, or rule of action is marked

out for him, and not only marked out, but enforced through the instrumentality of those various wants and necessities to which his human nature is rendered subject. But in the mean time, while the outward man is thus acted upon, and by motives not of the highest kind, the moral man within is undergoing a salutary course of discipline, overcoming obstacles which stand in the way of his happiness, and gradually acquiring habits which will fit him for the purest enjoyments. The law of nature, then, regarded as a scheme of external compulsion, is subservient to moral discipline. Present or temporal good, which is promoted by observing the law of nature, is only incidental and subsidiary to the attainment of moral good, to the formation of habits, tastes, and dispositions, which are destined to have their full development and exercise in a future state of existence. Our attention must now be directed to the machinery by which the observance of the law of nature is enforced.

72. The sanctions of the law of nature consist in rewards as well as punishments; though, as already observed, (No. 38.) some writers have thought fit to make a gratuitous assumption that all sanctions consist exclusively of punishments. It does not appear under what category they would range rewards: that the latter certainly may and do form inducements, is apparent from the commonest observation of the ordinary course of nature, which not only allures to the performance of duty by the prospect of advantage, but deters from the commission of wrong by the dread of evil. The distinction after all is purely technical.*

73. Several of the most eminent and popular writers on the present subject, include among the sanctions of natural law, the moral happiness and misery which, considered as temporal consequences, result from our voluntary actions. They insist upon the intrinsic excellence of benevolence, justice, truth, and all the other moral virtues. They assert, what cannot indeed be disputed, that, even in the present life, the practice of virtue is attended with enjoyment of the highest and purest kind;† that putting aside all contingent and outward advantages, virtue is its own reward in the internal satisfaction, peace, and tranquillity of mind which it brings to its possessor; and this internal happiness they regard as the chief sanction of what they call the law of nature. As Cumberland states the proposition, all the virtues and that perfection of mind which accompanies them, are to be reckoned among the happy consequences or natural rewards of universal benevolence. Thus the practice leads to the growth of virtue; practical goodness becomes both means and end.‡ Vice is treated in a similar way. Its intrinsic evil is regarded as the chief discouragement to it.

74. Now this account of the matter is very beautiful, and in several respects perfectly true. And if the writers in question had confined their sanctions of natural law exclusively to moral good and evil, we could readily understand that by the law of nature they meant to express that which we have termed necessary law. But from the way in which they blend moral with physical

* See Puffendorf, *Law of Nature and Nations*, I. vi. 14. with Barbeyrac's notes; Burlam. *Nat. Law*, part i. ch. x. 11, 12; Cumberland, *Laws of Nature*, *Introd.* 14. ch. v. 40.

† Cumberland, *Laws of Nature*, v. 35. Burlam. *Nat. Law*, part ii. ch. xii. 5, 6. Puffendorf, *Law of Nature and Nations*, II. iii. 21.

‡ Cumberland, ch. v. 12.

* Mackintosh, *Law of Nature and Nations*, p. 16.
VOL. II.

Law.

or material happiness and misery, from the co-ordinate rank which they assign to these two distinct species of good and evil,* considered in the light of sanctions, and from the indistinct mode in which they speak of the origin of these sanctions respectively, and of the final purpose of natural law, we are often left in some doubt as to their precise views or meaning. It is not very easy to discover, for example, whether their law of nature be a scheme of internal moral rectitude, or a rule of outward action; a law subsisting of itself, *i. e.* necessarily arising from the very nature of things,—or a positive institution of the Almighty; whether it be designed for man's ultimate moral perfection, or intended solely or chiefly to secure his present temporal welfare, morally as well as physically.† We have endeavoured, by the mode in which we have treated the subject, to avoid these sources of doubt and confusion.

75. Moral good and evil, then, must be regarded as the inevitable consequences of virtue and vice respectively. (No. 46.) They are not made the consequences of our voluntary actions, by an exercise of Almighty Power, in which respect they differ essentially from every species of good and evil depending purely on the physical constitution imparted to us by our Creator. Looking upon the law of nature therefore as a positive system, ordained by the will of God, we cannot properly consider moral good and evil as its sanctions, that is to say, as parts of the artificial machinery, by which God seeks to enforce the practice of virtue. An analogy in this instance holds with the municipal laws of a state. We are not at liberty to reckon among the sanctions of these, any consequences but those which the legislator himself may impose for the purpose of enforcing his will; though unquestionably obedience to the law is followed and may be further recommended by many ulterior advantages, which indeed suggested the reason for its institution. It was for the promotion of our moral good that the positive system of natural law was instituted by the Almighty. Moral good, however intrinsically excellent, however well calculated to recommend itself, without any adventitious inducements, to a mind duly prepared to appreciate it, demands a certain degree of moral proficiency in the agent before it can effectually influence him to any considerable degree. But such high proficiency is unattainable by man in the present life. Hence, then, the divine institution of the law of nature.

76. In another view, it will appear improper to class moral good and evil among natural sanctions. We can assign no bounds to the degrees or kinds of goodness and virtue to which a sincere desire of moral good would carry us, (for we may omit the dread of moral evil as belonging only to the lower degrees of virtue;) whereas the determinate, specific character of what are ordinarily regarded as natural obligations, and the distinction tacitly drawn between them and more imperfect obligations, show that they belong to a law whose

sanctions are applied to the enforcement of specific and limited rules.

77. Hitherto we have spoken of moral good and evil as they affect our happiness in the present world. For similar reasons to those above stated, the rewards and punishments of a future life should not be included among natural sanctions. Without any vain endeavour to pry into their precise nature, it is generally admitted that they are of a moral character, and that they are the necessary results of habits and dispositions acquired here. The soundest views commonly entertained of future happiness or misery, represent it as a continuation of that which is felt here, only greatly heightened. In the next place, future happiness is an end, and not a mean. It was with a view to it that the whole scheme of God's moral government was instituted; and it would therefore be incorrect to state, that future happiness and misery are imposed, in order to regulate our conduct here, as a final end or purpose. A future state of happiness or misery is disclosed to us by the Almighty as a distinct species of inducement: this disclosure partakes more of the nature of counsel than of law, the essence of which is enforcement. Lastly, the consideration of the happiness to be enjoyed in a future state of existence, like the desire of present moral good, is calculated to carry us much further than the mere observance of natural duties. These are our reasons for excluding future rewards and punishments from the category of natural sanctions. The hesitation of several moralists of the school of Cumberland and Puffendorf to admit these as sanctions, proceeds on a different ground; it is wholly connected with their peculiar acceptance of the word *natural*, which, according to them, includes whatever we are able to collect by the light of reason only without the aid of revelation; their recognition therefore of the sanctions here in question depends upon their opinion of the extent and power of our natural reason in arriving at the knowledge of a future state. Some of them, however, incline to the opinion that the system of nature must appear imperfect, unless we suppose a future life. The distinction indeed between natural and revealed law is very generally recognized; still we cannot help thinking that the ordinary method of explaining the subject does not sufficiently show the full force of the word *law*, in the phrase, law of nature.

78. The moral obligation of obeying God's will is wholly independent of natural law, as we understand it; that is, independent of those additional sanctions to which we now come, and by which He urges us to the outward practice of virtue. Motives of the highest self-interest—the consideration that practical discipline systematically enforced must be submitted to, in order to the acquisition of habits essential to our final happiness—ought in reason to have sufficient weight to induce us to observe any system of practical rules proposed to us by Infinite Wisdom, even if the rules of practice marked out for us were simply declared, without being enforced. But God's natural, which is subservient to his moral, government of mankind goes further than simple declarations; it consists, as Bishop Butler observes, in God's annexing pleasure to some actions, and pain to others, and giving us an opportunity of knowing beforehand that such consequences will follow. In our mode of treating the subject, we are obliged to restrict those consequences within narrower limits than Butler seems to have contemplated; we

Law.

Nor are the rewards and punishments of future life.

* See Paley's *Moral Phil.* book i. c. 6.

† Leibnitz, in a little tract, entitled *Monita quædam ad Samuelis Puffendorfi Principia*, charges the author whom he criticises, with the commission of three fundamental errors. First, with confining the science of natural law and its ultimate aim, within the limits of the present life; as if it were possible to afford a complete and correct view of the whole subject, without embracing the consideration of man's final destiny. Secondly, with paying too exclusive a regard to outward actions, as opposed to inward principles. Thirdly, with making all duty, right, and obligation to depend on the arbitrary will of a lawgiver.

Law. confine them to physical or purely material advantages. These constitute the artificial, positive, or compulsory part of the Divine scheme of government; and they consist in those external consequences which follow our outward actions, in the common course of nature.

Sanctions of natural law are of an external and temporal nature. 79. Under this view, these various sanctions of the natural law may briefly be asserted to consist in all the external consequences which restrain or impel human conduct; any temporal advantages or disadvantages, which, operating upon human instincts, passions, prejudices, reason, or forethought, induce mankind in general either to acts or forbearances; all those benefits which may be summed up under the general name of utility. The great difficulty and abstruse nature of this comprehensive topic must forbid an enlargement upon it in a treatise of this kind: more especially as the complex constitution of man's nature has blended together motives essentially different in themselves, and has induced jurists and moralists to confound them in their statements and definitions.

Reason not a sufficient guide, from man's imperfection. 80. It is a trite observation that, in order to follow the law of nature, "right reason" is to be our guide. It is unquestionable that by reason is exhibited the standard of what is right: but it does not express the successive steps by which we are led to that exalted height. Were man a perfect being, reason, under which we would include all higher moral motives, would be sufficient, since he would require no inducements: but the very existence of moral and physical pain and pleasure are not only inducements in themselves, but evidences that he stands in need of them. In fact, there is a mutual connection and dependence between reason and those subordinate matters which form the sanctions of natural law. Reason will instruct the instinct, correct the passions, soften, without eradicating the prejudices, and direct our views to more distant objects: while, on the other hand, all these call our dormant energies into action, when the reason would be too slow of itself to incite us.

81. "We can easily imagine," observes Sir James Mackintosh, "a percipient and thinking being without a capacity of receiving pleasure or pain. Such a being might perceive what we do: if we could conceive him to reason, he might reason justly; and if he were to judge at all, there seems no reason why he should not judge truly. But what could induce such a being to *will* or to *act*? It seems evident that his existence could only be a state of passive contemplation. Reason, as reason, can never be a motive to action. It is only when we superadd to such a being sensibility, or the capacity of emotion or sentiment, (or what in corporeal cases is called sensation,) of desire or aversion, that we introduce him into the world of action. We then clearly discern, that when the conclusion of a process of reasoning presents to his mind an object of desire, or the means of obtaining it, a motive of action begins to operate; and reason may then, but not till then, have a powerful though indirect influence on conduct. . . . The general tendency of right conduct to permanent well being is indeed one of the most evident of all truths. But the success of persuasives or dissuasives addressed to it, must always be directly proportioned, not to the clearness with which the truth is discerned, but to the strength of the principle addressed, in the mind of the individual; and to the degree in which he is accustomed to keep an eye upon its dictates. . . . It is apparent that the influence of reason on the will

is indirect, and arises only from its being one of the channels by which the objects of desire or aversion are brought near to these springs of voluntary action."*

82. That state of perfection which we have already intimated (No. 52.) should form the ultimate aim of every reasonable being, is such a condition of moral taste and disposition, as pursues *spontaneously* what is good. It is the spontaneous quality of that disposition which we now wish to point out. Religion and morality both concur in representing the most excellent state to be, without a figure, a state of perfect freedom. But how is man to be brought to such a condition, that every principle of good may be in full activity, without any tendency to deflect towards evil, and all this without any external coercion? The gracious arrangement of Providence is this: first, to impel him, almost mechanically, by means of those sanctions which take their rise in his instinct and passions: then, to influence him less coercively by motives still of a temporal nature, but more distant, more the objects of forethought and reason, and therefore more consistent with free-agency; and, by gradually withdrawing the objects of sense, in proportion as the distant and the future predominate over the present, to train him for the contemplation of things still more remote and exalted. We may observe, that in the course of nature the greater is made the more distant good. It is evident that in this last condition, he is a more spontaneous agent than when under the influence of the immediate and urgent pressure of the lower kind of natural inducements.

83. The attainment of this spontaneous condition is further promoted by the occasional uncertainty in the event of distant natural consequences. Were the results of our exertions undeviatingly successful, we should become the mere slaves of sensible objects. Whereas, by the present constitution of nature, the general event of our voluntary actions is just sufficiently certain to direct us in the right course, but is sufficiently uncertain to prevent the object originally in view from being the ultimate end of our pursuit. Further, by this occasional disappointment and uncertainty, the limits between spontaneous and coerced action is left somewhat indefinite; a circumstance evidently favourable to a gradual transition of the agent from the trammels of natural law to a state of moral freedom.

84. It thus appears, that the sanctions of the law of nature must be regarded only as helps to the sanctions of the moral law, that is to say, as additional and inferior inducements to the observance of that inward practice, which the moral law requires as an inward principle.

85. The sanctions or natural consequences which we have been discussing are universal, and therefore it must be that the law to which they are referable must be universal in its operation, being coextensive with the human race, and with the system of this world: having been instituted at the creation by the Divine Legislator himself, and sustained ever since by his perpetual ordinance. We do not of course here mean some of those consequences, which though on other grounds they might be called natural, yet, wanting the property of universality, cannot properly be considered as parts of the natural Law. Natural consequences are also perpetual or constant in their operation. "We may say, perpetual,

* Sir James Mackintosh, *Dissert. on Progress of Ethical Philosophy*, sec. 5.—p. 152—154.

Law. if we will, that the laws of nature are eternal: though, to tell the truth, this expression is very incorrect in itself, and more adapted to throw obscurity than clearness upon our ideas. Those who first took notice of the eternity of the laws of nature did it, very probably, out of opposition to the novelty and frequent mutations of civil laws. They meant only, that the law of nature is antecedent, for example, to the laws of Moses, of Solon, or of any other legislator, in that it is coeval with mankind; and so far they were in the right. But to affirm, as a great many divines and moralists have done, that the law of nature is coeternal with God, is advancing a proposition, which, reduced to its just value, is not exactly true; by reason that the law of nature being made for man, its actual existence supposeth that of mankind. But if we are only to understand hereby, that God had the ideas thereof from all eternity, then we attribute nothing to the laws of nature but what is equally common to everything that exists." *

and
generally
known.

86. The last thing to be observed with regard to these sanctions of natural law or natural consequences, is, that they are generally known: they are recognized and assented to by our instincts, our senses, our reason, and our experience; in short, they are universally promulgated, and written upon the heart of all mankind by Him who devised and framed them.

Rules and
precepts of
natural
law.

87. III. We now come to consider the rules and precepts of natural law. Being a positive law it is specific in its character. Though doubtless containing principles, yet these are of a less abstract character than those which belong to the moral law. This conclusion must naturally follow from what has just been said with respect to its sanction. As the moral law is too distant and elevated in its nature to be sufficient for inciting man, in his present state of weakness, without the intervention of these more proximate inducements which the law of nature affords, so is it too wide and expanded to guide his narrow and shortsighted will and judgment, without the interposition of certain landmarks, which may confine his steps within determinate bounds. The specific nature of these rules may best be shown by a few examples. To select, then, one among the most obvious and important of our instincts, namely, love. If this primary affection of our nature were, on the one hand, to indulge in a too extensive range of objects, it might be dissipated away, or, on the other hand, were it without immediate objects, it might perish for want of exercise: whereas, by the present constitution of nature, it is capable of being gradually refined into the highest and noblest affection, and indefinitely extended without danger of attenuation. Hence it is that love, in order to its ultimate purer and more perfect exercise hereafter, is here, under the influence of the law of nature, determined by the specific rules of domestic affections. And here it is observable, that the strongest affections are those which, in their legitimate exercise, are confined within the narrowest limits: the filial, or paternal, or conjugal affection for instance, is stronger than that which regards the domestic circle in general; the domestic affections are more powerful than the love of country; which, again, is more powerful than universal philanthropy. And it is further to be remarked, that the more confined and powerful the affection is, in the

Their
specific
character
exempli-
fied.

Law. same degree it the more nearly partakes of the character of animal instinct. To use the words of Hume, "This partiality, then, and unequal affection, must not only have an influence on our behaviour and conduct in society, but even on our ideas of vice and virtue; so as to make us regard any remarkable transgression of such a degree of partiality, either by too great an enlargement or contraction of the affections, as vicious and immoral. This we may observe in our common judgments concerning actions, when we blame a person who either centres all his affections in his family, or is so regardless of them, as, in any opposition of interest, to give the preference to a stranger or mere chance acquaintance." * The same considerations will obviously apply to other instinctive predilections: but it is not necessary to specify them here in detail. It may be sufficient to observe, that it is only by attending to the ultimate purpose to which they contribute so effectually, and for which unquestionably they were designed, that we are enabled to reason about them at all, or to view them otherwise than as so many unmeaning weaknesses, unworthy of rational creatures, and which ought therefore to be got rid of as speedily as possible. This indeed can seldom be the case with the social predilections above noticed. Their temporal, if not their moral utility is so obvious, even to shallow observers, that their propriety is seldom brought into question. Though even with regard to them, the world has occasionally afforded specimens, happily rare, of *philosophers*, who have betrayed a ridiculous ignorance of human nature and its requirements and an incapacity of taking a combined view of ends and means, by seriously advocating an universal philanthropy of the most impartial kind, at the expense of our more confined and therefore warmer affections; utterly forgetful that a pure, abstract, perfect justice, which bestows its regards strictly in proportion to the merits of its objects, can only be approached, by such a being as man, gradually and through the medium of that wisely adjusted scale of natural affections, which, though symptomatic of imperfection, and implanted in us because of such imperfection, constitutes nevertheless an essential element in human or *natural justice*.† The order of nature prescribes not that our primary and deeper and more partial affections should be depressed to the lower level of our more distant and extended, though more impartial regards, but that the latter should be gradually elevated to the higher level of the former.‡ So far

* Hume, *Philosoph. Works*, vol. ii. p. 258.

† "Eademque ratio fecit hominem hominum appetentem, cumque his naturâ, et sermone, et usu congruentem; ut profectus a caritate domesticorum ac suorum, serpat longius, et se implicet, primum civium, deinde omnium mortalium societate; atque, ut ad Archytam scripsit Plato, non sibi se soli natum meminerit, sed patriæ, sed suis, ut per exigua pars ipsi relinquatur." Cic. *De fin.* ii. 14.

"In omni autem honesto, de quo loquimur, nihil est tam illustre, nec quod latius pateat, quam conjunctio inter homines, et quasi quædam societas et communicatio utilitatum, et ipsa caritas generis humani; quæ nata a primo satu, quo a procreatoribus nati diliguntur, et tota domus conjugio et stirpe conjungitur, serpit sensim foras, cognationibus primum, tum affinitatibus, deinde amicitis, post vicinibus; tum civibus, et iis qui publice socii atque amici sunt: deinde totius complexu gentis humanæ: quæ animi affectio, suum cuique tribuens, atque hanc, quam dico, societatem conjunctionis humanæ munificet et æque tuens, justitia dicitur: cui adjunctæ sunt pietas, bonitas, liberalitas, benignitas, comitas, quæque sunt generis ejusdem." *De fin.* v. 23.

‡ It may be remarked that such passages as *St. Matth.* vi. 46, 47; *xxii.* 39; &c., do not depreciate the lower standard of

Law. with regard to our social predilections: their value and obligation depend altogether, it is generally admitted, on their ultimate moral effects. The same principle must hold good with regard to all other instincts and prejudices of universal occurrence, however unmeaning they may appear at first sight: they also must be judged of, not abstractedly, but by reference to the ends they produce. And it will be found, that several of the most valuable human institutions, property and government, for example, respecting which so many theories have been set afloat, depend, for their stability, on some divinely appointed and useful prejudice.

88. To give one other example: it was necessary that the principle of activity should be developed and exercised in man. But in this respect, he is not simply furnished with the principle, and left to discover for himself modes of exercising it. The necessity of providing for his physical wants directs his energies into certain determinate channels, in which they are nursed and matured. It is the law of nature that gives them that particular direction for purposes of discipline; it remains for morality to point them to higher and better objects.

Division of the laws of Nature. 89. But it is time to consider, in their proper order, the rules themselves of the law of nature. They may be divided, with reference to their immediate objects, into our obligations towards God, towards ourselves, and towards other men. We shall consider them under these heads with the utmost possible brevity.

Duties towards God. 90. To begin with our natural duties towards God. They include both speculative belief and practical observances. First, as to speculative belief, which must necessarily precede all external acts and duties of religion. The notion of a Deity is one which has been generally entertained, at all times, and in all countries, with the exception of a few of the most barbarous. It has been held, says Grotius, as well by those whose understandings were too gross to impose on others, as by those who were too wise to be imposed on themselves. Grotius attributes, with much apparent probability, this general persuasion to an universal tradition derived from the early patriarchs; for though, he observes, God's existence may be satisfactorily demonstrated by reasons and arguments drawn from nature, it is not every one that is capable of understanding the proof. In general, men are prejudiced in favour of their religion before, if ever, they reason about it. The ideas entertained of a Divinity are, in many cases, very imperfect and distorted; but there are two points essential to every religion, true or false, upon which the great majority of mankind are agreed, namely, the existence and the moral government of God. The notion of a divine providence is one which finds something congenial to it in the human mind, and it derives great strength from the authority which the mere fact of its general reception is calculated to convey. For these reasons it is a truth upon which jurists affirm no difference of opinion can be tolerated. We mean the doctrine of a moral Governor, for a perfect knowledge is not contended for.

Practical. 91. The practical part of natural religion includes worship, consisting in acts of prayer, thanksgiving, and adoration; and reverence, consisting chiefly in forbear-

our more common or natural affections; they only exhibit a still higher one to be aimed at. The sentiment in *St. Matt. xii. 50*, could not, with propriety, have been uttered by an ordinary man.

Law. ance from open disrespect.* It is natural, because man's natural constitution, by an impulse or instinct, leads him to manifest sentiments of respect, towards whomever entertained, in outward expressions and significant actions. Accordingly, some form of worship or other has always obtained in every place where a Deity has been acknowledged. What applies to worship will apply still more strongly to the negative duties of reverence, which can scarcely be withheld consistently with a true belief in a Supreme Governor of the world.

92. The outward part of religion is connected with the inward, both as *cause* and *effect*. Worship springs from inward reverence, and also contributes to it. It is from considering outward worship in the latter aspect, that we derive its moral obligation (No. 38.); because, by increasing our reverence towards, and therefore our love of, God, it promotes our highest happiness.

93. We next come to our natural duties towards ourselves, which may be disposed of in a summary way. They consist, according to Puffendorf, in duties of self-perfection and self-preservation. But the duties of self-perfection treated of by him have properly no place in a treatise on jurisprudence. The natural duties of self-preservation include all that relates to the preservation and integrity of our life and body, and consequently enjoins all the acts necessary towards their sustentation. They forbid self-destruction, and self-abuse of every kind. The natural sanctions by which these become laws of nature, after what has been said respecting natural sanctions, must be too evident to require any observation.

94. Our social or relative obligations form the third and largest class of natural precepts. We shall consider the several relations men stand in towards each other, first, in their personal capacities; secondly, in relation to things. To these a third division is sometimes added by modern writers, especially those of Germany; namely, the natural law of facts. Some censure the addition of this third division, as unphilosophical.

95. Were we to examine at length the various duties and rights incidental to the general relation in which all men stand towards each other, *as men*, it would lead to too long a discussion, and would be inconsistent with the plan we have adopted. Certain rights are represented as naturally belonging to every man, who has not forfeited them by his misconduct. They are principally four: viz. 1. a right to his life and the safety of his person; 2. a right to acquire property; 3. a right to his reputation; 4. a right to liberty. These are rights which every one will be disposed to claim, and which every right feeling man conceives himself under an obligation of conceding to others. It is evident that natural inducements will not be wanting to assert them. But some of them, the fourth especially, are somewhat indefinite; and under this general head, it will not be desirable to attempt a discussion of their exact limit. We shall therefore proceed with the particular relations of persons. "Almost all the relative duties of human life," observes a writer frequently before quoted, "will be found more immediately, or more remotely, to arise out of the two great institutions of property and marriage. They constitute, preserve, and improve society. Upon their gradual improvement depends the progressive civilization of mankind; on them rests the whole order of civil life. * * * These two great insti-

* Paley, *Mor. and Pol. Phil.*, book v. ch. 1.

Law.

tutions convert the selfish as well as the social passions of our nature into the firmest bands of a peaceable and orderly intercourse; they change the sources of discord into principles of quiet; they discipline the most ungovernable, they refine the grossest, and they exalt the most sordid propensities; so that they become the perpetual fountains of all that strengthens, and preserves and adorns society; they sustain the individual, and they perpetuate the race. Around these institutions all our social duties will be found at various distances to range themselves; some more near, obviously essential to the good order of human life; others more remote, and of which the necessity is not at first view so apparent; and some so distant, that their importance has been sometimes doubted, though, upon more mature consideration, they will be found to be outposts and advanced guards of these fundamental principles; that man should securely enjoy the fruits of his labour, and that the society of the sexes should be so wisely ordered as to make it a school of the kind affections, and a fit nursery for the commonwealth."*

Relation of husband and wife.

96. The liberty which has been just described as one of the rights naturally incidental to every person, includes the right of contracting marriage; from whence spring the two *private* natural relations of husband and wife, and parent and child. And first, of husband and wife.

97. The sexual appetite implanted in all animals serves to perpetuate their species, but in man becomes the instrument of still higher purposes, through the natural institution of marriage. If we look merely to the propagation of the species, or the general good of society, we get a very imperfect view of the importance and obligation of marriage. To estimate these fully, we should consider its lasting moral effects on the parties themselves; how their natural passion of love, by being permanently confined within strict limits, is refined as well as deepened, and gradually deprived of its selfish character by its extension to new objects, viz.—the offspring of their union.

The marriage contract.

98. The mutual consent of both parties is essential to establish the relation of husband and wife. But on the subject of the *natural* mode of signifying this consent, jurists seem to have forgotten that here, as confessedly on other subjects, *natural* admits of comparative degrees; whereas they assert that the lowest, that is, the least formal manner of signifying consent, which is only just sufficient to bind the parties, is the *natural* mode. Undoubtedly, where no particular form of compact is established by law, or prevails by custom, any mode, which in the eyes of the parties themselves amounts to such a compact, will be naturally sufficient to bind them. But there is, on the part of mankind, a disposition, if not universal, at least very common, to invest with importance the entrance into the state of marriage, by accompanying it with ceremonies, often of a religious cast. Men are also naturally disposed to observe engagements contracted with circumstances of solemnity, and above all of religious solemnity, with more scrupulous fidelity than any others. We state facts without attempting to account for them. It might therefore be fairly argued, both on the ground of general disposition and of reasonableness, that a solemn religious mode of contracting marriage is the *most* natural. The disclaimer, on the part of many jurists, of such epithets as *holy* or *sacred*, as applied to marriage

in its civil relations, is absolutely puerile. By whom ever the *state* of marriage (not the mere mode of entrance into it) is viewed, the importance of availing ourselves of all means to maintain its obligations inviolate, can scarcely be over-estimated.

99. It will be evident that the *moral* purposes of marriage above slightly traced out, involve two essential conditions, exclusiveness and permanence, and are best answered by making those conditions as strict as possible. Hence it appears that monogamy, terminating only with the life of one of the parties, is the most perfect form of conjugal union. Monogamy has therefore been established in all Christian communities, as Grotius expresses it, in order that the wife, bestowing herself entirely on the husband, may receive the equal return of his whole heart and affection. The equality of the sexes, and the general practice of mankind, point this out as the *most* natural form. Puffendorf remarks that, where a man has several wives, he usually singles out one as a favourite.* Polygamy has however been practised in many nations both ancient and modern, without disapprobation by the wise and good; and all the ends of marriage may be answered, consistently with it, though less perfectly than by monogamy. For these reasons it has been held that the law of nature does not absolutely forbid it, though it ought not to be tolerated in a Christian community. Observations not very dissimilar will apply to the power of terminating the matrimonial relation before its regular conclusion, by divorce. The law of nature at all times prescribes, as its *most perfect rule*, the indissoluble maintenance of the matrimonial relation during the joint lives of the parties, except in the single case of adultery; a rule generally followed, though with exceptions. Here it coincides with the only rule permitted by Christianity; but the law of nature tolerates an inferior rule of morality; as we find in the comparative facility of divorce indulged to the Jews, for example, and not condemned, even followed, by confessedly good men.

100. There is but one more topic connected with marriage, to be touched upon here; namely, respecting the degrees of prohibition, according to the law of nature, on account of affinity and consanguinity. Three classes of the natural degrees of prohibition are noticed by jurists: first between parents and children, and more remote descendants; these have been justly held in universal and unqualified detestation at all times and in all countries. Secondly, between brothers and sisters; which, though strongly condemned in general, as contrary to the law of nature, have been so in a somewhat modified degree. Thirdly, between more remote collaterals; about which there has been not only some variety of practice in different ages and countries, but some variety of opinion among jurists. Writers have attempted in several ways to account for such prohibitions; but our space will not allow us to follow their theories. Grotius alleges the question to be most nice and difficult, and that whoever attempts to assign cer-

* *Law of N. and N.*, VI. i. 19. On the subject of marriage, Puffendorf, though learned and methodical, is anything but profound. He takes a low view of its purposes, and argues upon the subject too much like a lawyer. He not only mistakes in some instances the *invariable terms* of the conjugal union, but he carries the initial compact with him throughout the whole matrimonial status, instead of treating it merely as the barrier which gave entrance, but prevents egress.

* Mackintosh, *Disc. on the Study of the Law of N. and N.*, 35.

Law. tain and natural rules why marriages in such cases are unwarrantable, in the manner they are prohibited by the laws and customs of nations, will find by experience that the task is both difficult and impracticable.* May we hazard the conjecture that the God of nature implanted these natural aversions, along with the physical defects incidental to unions of the kind here supposed, in order to induce mankind to extend their alliances beyond their own families, and thereby cultivate a more comprehensive social affection?

Parent and child.

101. On the relation of parent and child, little need be said. The right of parental authority is commonly stated by jurists to be founded in generation;† but this only indicates the person, not the foundation of the right. How, indeed, can the mere fact of having been the instrument of a child's earthly existence, abstractedly confer any authority of the kind? Puffendorf builds the right partly on a supposed tacit compact,—he loves compacts,—balancing the right of parental government against the duty of support! But the only true ground of natural parental authority is natural parental affection, for it secures the probability of good parental government. (No. 55.) The want of it therefore will go far towards depriving the parent of that right, on the principle of law, that *cessante ratione, cessat etiam lex*; but it must be a very gross case indeed, which will, in point of natural justice, warrant a compulsory legal interference with parental rule. On principle, we must be careful how we follow the dangerous example, set by some writers, of proposing for rules of exception extreme cases of very rare occurrence. It is to be lamented that both Puffendorf and Grotius should have committed an unwise mistake of the kind, by formally propounding it as a general rule, that parents, if unable to support their children, may, by the law of nature, pawn or sell them.‡ Silence, upon such a distressing case of extreme necessity, would surely have been preferable. The question, to which of the parents the right of government over the children exclusively or principally belongs, has occasioned some discussion and difference of opinion among jurists: some give it to the father on account of the greater dignity of the male sex; some to both parents equally. The latter probably comes nearer the truth; but as the age and sex of the child, and the disposition of it and its parents respectively, will point out, in the best possible way, how, when, and by whom the authority on each occasion ought to be exercised, it is surely idle to speculate further upon the point. At the proper age, nature will indicate, by the feeling of and qualifications for independence on the child's part, that the strict exercise of parental authority ought to cease. But, of course, no abrupt line, in this, or perhaps in any instance, offers itself in nature.

Law of Things.

102. The order of our subject now brings us to the natural law of things. The first point to be established is the right of the human species, exclusive of other

beings, to the free use of and dominion over the inferior creation, both animate and inanimate. The attempts of philosophers hitherto to render a rational account of this right, are far from affording complete satisfaction; indeed their arguments are often extremely fanciful. It seems best, therefore, to rest it altogether upon authority; upon the general practice and sentiments of mankind, and still more upon the indisputable authority of Scripture, which expressly declares the Creator's will in this respect.* The texts, however, here referred to, establish nothing more than the right of the human species generally; they give no intimation of the mode in which it was intended that mankind should avail themselves of the divine permission to take and use the objects designed for their support; they are silent respecting the relative and conflicting claims of individuals; they do not, for example, determine whether, or how far, a particular person, in taking what he requires, is at liberty to consult, exclusively or chiefly, his own convenience, or is bound to pay regard to the wants of others. In the actual distribution of worldly substances which has, somehow or other, been effected, one of the most striking phenomena, and that which at first sight appears most repugnant to our preconceived notions of abstract justice, is the gross inequality that everywhere prevails. It might be argued, that all having equal need of support, and none being expressly excluded in the terms of the original gift to man, every one might fairly lay claim to an equal share. Yet not only is the contrary system rigorously maintained by law, but jurists are all agreed at least upon one point, namely, in their determination to support it by reason, though the diversity of their arguments shows that the several reasons alleged by them are neither so complete nor so satisfactory in all cases as might be desired.

Law. The general right of mankind to things.

103. Exclusiveness and permanence are the chief essential characteristics of a private dominion over things, or property. By exclusiveness it is implied that the proprietor has or claims the sole right of using or directing the use of some particular object; by permanence, that he has that sole right, not to serve a temporary occasion only, but for an indefinite period to come; or, as it is expressed, he has a right to the very *substance* of the thing to be used. The inequality above noticed is perhaps an accident of property; but it is an accident which, as we have intimated, universally accompanies its establishment, and it is the circumstance which has occasioned the chief difficulty attending the examination of the subject. Opposed to the system of permanent distribution implied by private property, stands what is called the "original community of things." The term *original* points to the real or supposed state of things at some early period of the world, not exactly defined. The term *community* is equivocal. Sometimes it implies that condition of unheeded neglect in which objects of possible utility continue, before any one appears to use or appropriate them. In this sense all things were at one time common, and some continue so to the present day. Writers who aim at a superior degree of exactness distinguish this species of community by the epithet *negative*. The other, or *positive community*, implies that the objects said to be common have indeed been permanently converted to the use of man, but not made the property of any one or more individuals exclusively of all others; but that every one, without dis-

The nature of private property.

Its inequality.

Original community of things, regarded as an historical fact.

* *De J. B. et P. II. v. 12.* The whole of what Grotius says on the subject of marriage in this chapter is well worthy of careful perusal. Upon this, as upon most topics, this great jurist shows himself vastly superior to Puffendorf.

† *Puff. Law of N. and N. VI. ii. 4. Grot. De J. B. et P. II. v. 1.*

‡ *Puff. VI. ii. 9. Grot. II. v. 5.* Grotius, in support of his position, refers to the Mosaic law. The passages he cites (*Exod. xxi. 7.* and *Deut. xv. 12.*) prove the previous existence of the practice; but the law of Moses at most but tolerates it, and interposes expressly for the purpose of securing good treatment to the objects of sale.

* *Genesis, i. 28, 29; ix. 2, 3.*

Law.

pute or contest, uses the objects as he has occasion, from time to time, leaving them, (if not of a perishable nature,) when the occasion ceases, to be used in like manner by any one else. There is an essential difference, it may be observed, between these two species of community. In the former, there is no *property* in any sense, no severance from the general stock. In the latter, there is a severance: the dominion of mankind is asserted over the objects positively common; the human species collectively is the proprietor, and the brute creatures are the excluded parties. But the property, or permanent and exclusive dominion, extends no further; it does not obtain between man and man. Now this latter state is the condition represented in the fables of poets concerning the golden age; and it is gravely asserted, without much thought, by some jurists, to have been the primitive state of things in the first ages of the world. Blackstone, for example, in his elegant and much admired sketch of the rise and progress of property,* (by the way, chiefly borrowed from Grotius,) treats it as a positive fact. But is this at all likely to have ever been the case? (we can scarcely suppose that the dreams of poets would have been adopted as a mere embellishment, wholly unsuited as they are to the severity becoming a philosophical inquiry after truth;) or rather, is the assumed fact consistent with the nature of man, which has been the same ever since the fall, for with man before his fall we have no concern? What are the circumstances universally attendant on human nature, which, by an inevitable consequence, lead to the establishment of private property? Externally, there are man's wants, and the indispensable necessity of labour for supplying them; internally, there are his natural selfishness and love of ease. Be it that, in the early ages of the world, before arts or luxury had made much progress,† his wants were few; still it required no inconsiderable share of labour, even under favourable circumstances of climate, &c., to supply those few wants; and, taking him in the average, his love of labour and of his species was never so strong as to induce him to forego the fruits of his labour, either from an indifference to the necessity of repeating it, or from a disinterested generosity towards others. In proportion to the trouble expended in acquiring possession of some useful object, or in preparing it for use, would be the tenacity with which he would hold, and the vigilance with which he would guard it, in other words, his assertion of a private and exclusive dominion over it. His natural selfishness and love of ease would be aided further by an instinctive persuasion, or prejudice, that he had a right to keep to himself whatever he found unoccupied, and chose to improve by his own labour and skill. Labour, then, taken in connection with the natural wants and feelings of man, is a chief and primitive cause of private property. A further

The natural rise of property.

* *Commentaries*, vol. ii. p. 2, &c.

† Jurists sometimes favour us with a sketch of the supposed order in which the principal arts came into operation, as they naturally tend to promote private property, by suggesting new uses of things, and new modes of labour. Agriculture is supposed to have been of comparatively late growth; but, it has been observed, the first-born man was a tiller of the ground. It is therefore highly probable that he asserted a private dominion in the soil, at least during the interval between preparing the ground and reaping the crop. We may add, that it would not be consistent with his character to suppose him capable of exercising that forbearance which a community of property must constantly demand.

cause, but one, for the most part, of later growth, is the scarcity of useful objects; human selfishness will naturally induce a man to grasp at what but few can obtain. This cause, we say, is, *for the most part*, of later growth; for though, with regard to most objects of utility, a scarcity is not experienced till the increase of population has increased the demand, whilst the supply remains stationary or even declines, still some things are by nature originally and absolutely scarce. We conclude then that private property was the order of the world from the commencement, though less regular and systematic, and embracing fewer objects, compared with the number of inhabitants, than in later times. Hitherto, however, we have spoken of separate property as a fact, apart from its equity or justice; for whether our view of the early state of things be correct, or whether, as those who depict mankind as originally holding all things in common would have us believe, the system of private property be a departure, gradually introduced, from the mode observed in primitive times, no historical delineation, however true, of the perpetual and universal triumph of a seemingly selfish, monopolizing spirit over a disinterested or at least equal regard for the wants and requirements of all, can amount to a complete proof of the rightfulness of the system which accords in perpetuity to individuals whatever they or their ancestors, finding without an owner, have succeeded in laying their hands upon. From the question of fact, then, we shall now turn our attention to the question of right.

104. Neither our limits, nor our plan, which is to exhibit principles rather than opinions, will permit us to follow the several schools of jurists through their more minute shades of difference.* Almost all the accounts given of the origin of the *right* of property will be found, under various modifications, to be connected with one or other view of the "original community of things," expressive, not, as heretofore, of an historical fact, but of a doctrine of primitive right. Jurists in general may, upon this point, be distributed into two great sects,—those who hold respectively the doctrine of a positive and of a negative community of things. The positive community asserts that each man has an original right to an equal distributive share of the surface and productions of the earth, of which, whatever distribution may have been made provisionally or by usurpation, he cannot be justly deprived without some act naturally (that is, according to acknowledged principles of universal justice) calculated to have that effect. We state the doctrine without exaggeration, though somewhat more explicitly than its favourers in general venture to do. How then do they endeavour to reconcile, or rather to force an agreement between their notions of primitive justice and the claims of existing proprietors? By the convenient fiction of a tacit compact,—the ready refuge of a certain class of jurists on all occasions of difficulty,—whereby every one is supposed, at some period or other, to have voluntarily surrendered whatever original claim he had to a distributive share of the material objects of property, and to have confirmed the defective titles of the actual holders. This theory, however, will not bear the test of examination. Senseless as a

Law.

The right of private property.

The original community of things, treated as a doctrine of right.

Doctrine of a positive community

Fiction of a tacit compact.

* The reader will find in Professor Warnkœnig's work on philosophical jurisprudence, referred to in our preliminary observations, a brief and lucid statement of the chief theories on the subject of property. See c. vii., sec. lxxxvii. &c.

Law.

fiction, groundless as a fact, even if we could attribute to this airy phantom some substance of reality, it labours under every species of incurable vice, and leaves unexplained difficulties at least as great as that which it is intended to solve.* It may indeed be said that the allegation of a tacit compact is only a mode of expressing the general approval of the institution of property in all ages and countries, evidenced by its general adoption. But this explanation only nullifies the intended solution. It substitutes authority for reason. It asserts, in substance, that mankind in general feel and act one way, whilst right and justice are in the opposite direction. If there be any truth in the assumption that all men have or had originally the right that is pretended, then the general acquiescence of mankind is an acquiescence in general injustice, until some substantial *bona fide* transaction, superseding that right, be *proved*, not feigned. Again, it is a misrepresentation and abuse of terms, to ascribe in any degree to such an acquiescence the character of a compact. If the matter acquiesced in be naturally right, the giving or the withholding of the consent of men does not constitute or impair its rightfulness. There is, however, some foundation of truth in the doctrine of equality; but its extent and the mode of its assertion are misunderstood by those who make use of the fiction of a tacit compact. We shall have to consider this point presently.†

Doctrine of
a negative
commu-
nity.

105. The doctrine of a negative community asserts, that things, originally, that is, prior to any actual use or occupation of them, instead of being the property of all equally, are the property of none. Things being common in this sense, it is held that whoever first reduces them into his own possession, in order to prepare them by labour for use, thereby makes them his own. Some consider that it is the fact of occupancy alone, others that it is the labour which he adds to them, that gives him his title: it will be unnecessary at present to discuss which is the sounder view, or whether there is any substantial difference between them. Locke is commonly regarded as one of the ablest and most successful supporters of the doctrine of a negative community, and of the consequent title by labour, and his views upon the subject of property do not substantially differ from those of most other supporters of the same doctrine.

Locke.

His theory
of labour.

We shall give the pith of his argument in his own words. "Though the earth and all inferior creatures be common to all men, yet every man has a property in his own person: this nobody has any right to but himself. The labour of his body, and the work of his hands, we may say, are properly his. Whatsoever then he removes out of the state that nature hath provided

* How, for instance, are we warranted in assuming, as a principle of original justice, that an individual can, by his own act of consent, bind his descendants of all future generations? Yet without some such assumption, we could not give to the supposed compact, if real, the complete, the binding validity commonly attributed to it, at whatever epoch we imagine it to have been entered into.

† Grotius—the great doctor, in jurisprudence, of the identical principle which the kindred science of theology acknowledges in the well-known rule, *quod ubique, quod semper, quod ab omnibus creditum est*—seems to have forgotten himself in the application of this principle to the subject of property. Deserting his favourite ground of authority, he resorts to a groundless and unsatisfactory fiction, herein more resembling Puffendorf than his ordinary self. The terms he employs plainly show that the consent alleged by him, in reference to the present subject, is not the concurring testimony of witnesses, but the consent of contracting parties. *De J. B. et P.*, lib. II., c. II., s. III. 5.

Law.

and left it in, he has mixed his labour with, and joined to it something that is his own, and thereby makes it his property. It being by him removed from the common state nature hath placed it in, it hath by this labour something annexed to it, that excludes the common right of other men; for this labour being the unquestionable property of the labourer, no man but he can have a right to what that is once joined to, *at least where there is enough, and as good left in common for others.** The concluding words, which we have marked by italics, deserve particular attention, both because they assert in substance the doctrine of a positive community, which Locke is ordinarily supposed to reject, and because they mark the limit which he virtually sets to the title by labour. His whole argument applies only to that state of circumstances in which the supply of natural objects, unappropriated or in common, exceeds the demands of the comparatively few claimants. So long as this state of things continues, no one can deny that it would be both wanton and unjust, and a dishonest encroachment upon the fair privileges of labour, for any person, under pretence of an equal right, however abstractedly just, to lay claim to any share of what another, without detriment to a single individual, has appropriated and improved. Here therefore we may safely assign some period for the continuance of the right of private property, arising from the intermixture of private labour, namely, whilst the season of superabundance lasts. But does the argument apply, and is it not a paralogism to extend it, to a different state of circumstances, namely, when people have multiplied, and new claimants appear upon the stage, who, finding every thing of possible utility already occupied, demand to know the reason why they—guilty of no neglect in so tardily asserting their claim, but only labouring under the accidental disadvantage of being born, without their own will, in a later age—why they, able and desirous of benefiting themselves and others by their industrious exertions, are debarred from receiving their equal share of things? To say that everything has already become the property of the ancient occupants by the just title of labour, is to assume the whole matter in dispute. Their labour, indeed, did originally give them a title to the things occupied, but (if we would give consistency to Locke's theory, which recognizes a positive community) not absolutely, entirely, and for ever: according to Locke, there must have been from the first a dormant claim for unborn claimants to an equitable share of the things in the mean time occupied, awaiting the coming in of the full tide of population, when, if ever, the claim for the first time would be ripe for assertion. If this claim had any existence, then the original occupants mixed their own property (their labour) with what, *pro tanto*, was the property of others; and according to a recognized principle of universal equity, they must bear the consequences of the voluntary confusion, though it may involve the loss of their labour. The obligation of a re-distribution of property once being introduced, the process must be incessantly repeated, since population is always shifting and increasing. In substance, then, there is an end of private property; and this is the inevitable conclusion at which we must arrive, if we adopt the notion of a positive community of things as a law of nature.

106. Locke saw the dilemma, and endeavoured to

* *Essay on Government*, ch. v.

Law.

evade it by a sophism. He alleges the superabundance, *in the whole world*, of good, but unreclaimed and unoccupied land; for land is the particular subject matter which he chiefly selects. But do not the distant (to most poor people, *inaccessible*) locality, and other disadvantages of those unappropriated lands, violate his own condition expressed in the words "as good"? At last, as if half conscious of the defects in his argument, he adopts the favourite and clumsy expedient of varnishing them over with a tacit compact, which, he says, "settled the property which labour and industry began." Rather let us call it the instrument whereby the disinterested victims of voluntary poverty signed away for ever, and without an equivalent, the right assumed for them to a comfortable subsistence.

Theory of aggregate social utility.

107. Such is the much applauded theory of labour, whereby it is attempted to justify the "rigour of the rights of a proprietor." Such is a specimen of the manner in which theorists burst through the difficulties that beset the subject. There is another class of reasoners, who may be considered to hold the doctrine of a negative community, though they do not either formally state their adherence to that doctrine, or, in general, embarrass themselves with the assertion of any abstract doctrine of justice. They avoid fictions, too; they look chiefly to the practical benefits attending the system of private property, in promoting the good of society *in the aggregate*. Paley enumerates, and admirably illustrates in a few short sentences, the chief social advantages of the institution of property: it increases the produce of the earth, and preserves it to maturity; it prevents contests; and it improves the conveniency of living.* But Paley offers them rather as an apology for the seemingly inequitable, because unequal, system, than as a conclusive argument to justify the assumed rights of individual proprietors. Indeed, for the latter purpose, it would be absurd to allege the preventing of contests, if it be admitted that there are such things as *perfect* rights, (or rights which may be asserted by force,) and that the natural right of property is at once such a right, and a fair subject of difference of opinion. To quiet the violent contest of two honest maintainers of contrary opinions, determined each to assert to the full his own view of his right to a certain object, by capriciously assigning the whole or the greater part of that object to one of the combatants, would indeed be a summary mode of settling the difference, but it would not adjust their claims in point of natural right. Again, no demonstration of the general or aggregate plenty caused by property can account for the justice of its gross inequality, which, let it be remembered, is the point at issue. Paley is not quite candid in asserting a fact, which, if strictly true, would go far towards proving, not indeed that the system of property is naturally just in itself, but that it is just to submit to it. He alleges that even the poorest and the worst provided, in countries where property and the consequences of property prevail, are in a better situation than *any* are where most things remain in common. Without pausing to inquire whether it be proper to compare the relative conditions of individuals under different systems, instead of under the same system, we meet the alleged fact with a direct negative: though generally, it is not universally true. Paley must have been aware that, in all countries, there are several not undeserving individuals, reduced to such

a state of destitution, that it can scarcely be doubted that they would be benefited by a general dissolution of the bonds of property; certainly their comforts would be promoted if they set the laws of property at defiance. When Paley comes to consider the topic of *right*, as distinguished from utility, and when he is driven to confess that the foundation of the right of property is the law of the land! he in truth gives up the question. The law of the land is but the public resolution of those who compose the governing body, and being the resolution of fallible mortals, it may be right or wrong, just or unjust. The tacit compact of society may very often amount to the same thing expressed in a different form. But as all the individuals assumed to be bound by the compact, or by the law of the land, are not in reality parties to the transaction which binds them, whatever the forgers of specious fictions may pretend; until we have some probable proof of the intrinsic justice of the matter agreed upon or imposed, we are at liberty, in a strictly philosophical inquiry, to treat the compact as public chicanery, and the law as systematic violence. Paley ends his speculations on the subject of property, as he ends them on most subjects handled in his shallow book, by referring it ultimately to the "will of God." A truly philosophical mode of proceeding for a professed philosopher! He exposes in glaring colours, without in the least explaining, the apparent injustice of the inequality of property, only, it would seem, to attribute it at last to an arbitrary and, in the state in which he leaves it, unreasonable institution of the All-just!

108. So far as yet appears, it must be confessed, that the modes of arriving at the conclusion are far from satisfactory. In order, if possible, to place the line of argument on a better footing, we shall notice briefly three leading and prevalent defects to which may be attributed, as we conceive, most of the imperfections of the favourite theories.

109. In the first place, those writers who propose the doctrine of equality as a law of nature, rarely understand it fully. Whilst they deny the natural right of any individual to more than his strictly equal share of things, they assert the most absolute, exclusive, and selfish right of every individual to the whole of his own labour, without reference to the relative abilities of himself and others. No reason can be alleged why the principle of equality, if tenable as a law of nature, should be applied to natural material things, and not to labour. Both are essential for the support and comfort of man. Again, the wants of all men are tolerably equal, but their abilities vary. Why then should a person be under a natural obligation to manifest a just consideration for the equal wants of all, passively, by abstaining from an usurpation of more than his just share of things, and not actively, by contributing his just quota of labour? It will be observed that we do not contend for an absolute, but a relative equality; that is to say, an individual who can, with equal ease, effect twice as much as a man of average strength and ability, ought to contribute to the common stock a double portion of the species of labour for which he is best suited; an individual, whose ability falls below the average, should be excused in proportion; each participating equally in the common benefit. This is the true principle of equality. Sir Thomas More perceived it, and has amusingly carried it out in fiction. In practice, it would be neither desirable nor possible to carry it into effect *as a law of*

Law.

* *Mor. and Pol. Phil.* book iii., part i., ch. ii.

Law. *nature*; nor do *we* maintain it as such; but they who do so are rarely consistent with themselves.

The natural and moral laws of property distinguished.

110. In the second place, a confusion is frequently made between natural and moral right, especially with regard to the doctrine just explained. The law of equality is a law of morality, and not of nature. By a law of nature, the reader will recollect, is meant a rule of conduct which is not simply agreeable to the dictates of abstract justice, but suited to the imperfect condition of man; which is provided with an artificial, though a divinely instituted sanction; and which involves a *perfect obligation*, or an obligation which may be asserted by force. The natural law of property, which possesses all these characteristics, is a law of inequality. But it is not inconsistent with the co-existence of a moral law of equality; for whilst the unequal possession of property, for purposes of preservation, improvement, and economy, may be rigidly maintained by force, the ultimate and spontaneous distribution of it, measured by an equitable standard of moral equality, (one element of which is the relative merit of the participators,) may be inculcated by means not violent,—by conscience, or by persuasion, for example. Although, therefore, it be true that there is an equal law of property, they who assert it as a *law of nature*, unconsciously confound the distinct offices of law and religion; for they thereby tacitly assert that public or private force, instead of persuasion or private conscience, is the proper mode of providing for its observance.

The ultimate or moral purpose of property

111. In the third place, the divine purpose of the institution of private property is, in general, very inadequately represented. We speak not of the immediate application and use of material objects, but of the mode in which, and the purpose for which, the possession of those objects is moulded. The objects of property, in themselves, are destined for the physical support of man. The particular manner in which the tenure of them is arranged, (so different where mankind, and the brute creation, respectively, are in question,) in separate and permanent properties, contributes, indeed, more effectually and fully to the temporal prosperity of most men; but chiefly it promotes the moral advancement and discipline of all. That “difficulty is good for man” is a proposition which none can question. But the encountering of difficulties, though a most essential part of moral discipline, is most irksome, and would never, in the absence of some exciting cause, be spontaneously undertaken. The perfection of his moral character demanded that his noblest faculties of thought and action should be aroused and called forth into unceasing and boundless activity; that his energy and power of endurance should be strengthened, and his foresight quickened and enlarged; attainments which, it would seem, are impossible, except as the result of habit and perseverance. His natural imperfection and indolence required the systematic application of natural means. Accordingly, his path, at every turn, has been strewn with multiform difficulties; a necessity for encountering those difficulties has been created by the necessity for labour imposed upon him, even in order to the attainment of every end and purpose connected with his comfort and existence; and a motive to encounter those difficulties has been supplied, in according to him the enjoyment of the objects of his desire, hope, and labour, in the indulgence of the natural feeling with which he embraces his *τὸ ἴδιον*. We have before shown how the concentration of the narrow domestic affections for a time, in the *τὸ ἀγαπῆδον*, con-

tributes to the development of an enlarged love. But it is no less certain that the direct inculcation of a spurious universal philanthropy would terminate in indifference, than that the direct enforcement of a system of vapid equality in respect of property would end in utter indolence. It is for the moralist to enlarge upon, the jurist can only hint, in a general and imperfect way, at the several powers and moral qualities elicited in the different classes of society, through the instrumentality of private property. Let it suffice to observe, that although a well-sustained and honourable course of successful exertion, crowned externally with wealth, is frequently a most healthful exercise to the party who engages in it, the possessors of property are often those who least benefit, morally, by the institution which renders them prosperous. The strenuous exertions called into action in the honest endeavours to escape from poverty, the patience with which it is learnt to be endured, when it cannot be avoided, and the gratitude and sympathy felt and exercised in the receiving and giving of private charity, would in a great measure become dormant, were that system altered which sentences some to poverty.* Upon no other grounds than what are here intimated, can we justify the otherwise inequitable differences of condition which are incidental to the natural laws of property. It is not that philosophical jurists, from the days of Aristotle to the present day, have abstained from even frequently, though partially, pointing out the moral effects produced by property; but they generally treat those effects rather as collateral accidents than as the main cause of the system. They seldom assign them that conspicuously prominent place, which they should hold, in order to become beacons to direct our steps through the tangled thickets of fictions and theories.

112. But it will be said, that property, whilst it encourages exertion, fosters at the same time covetousness and other evil passions. To this we reply that the machinery destined for the best purposes may be abused as well as used; and whilst law keeps it in action, the business of morality and religion is to provide against its abuse.

113. Having endeavoured to establish the justice of property on moral grounds, we shall now show the means by which it becomes a law of nature. In preceding parts of this chapter, we have not been solicitous to avoid the appearance of countenancing prejudice, recollecting that “prejudice engages the mind in a steady course of wisdom and virtue, and does not leave the man hesitating in the moment of decision, sceptical, puzzled, and unresolved;” that “prejudice renders a man’s virtue his habit, and not a series of unconnected acts;” that “through just prejudice, his duty becomes a part of his nature.” Considering that man is naturally more disposed to follow the impulses of feeling than

* Our limits forbid us from entering upon the subject. But here would be the natural place for considering, philosophically, the nice and delicate question of a poor-law, which is a partial attempt to supersede the natural law of unequal property, and introduce in its room a compulsory rule of equality. It would be an interesting, though by no means an easy task, to endeavour to adjudicate between the conflicting claims of morality to encourage individual exertion, and of physical destitution to receive public relief, in lieu of precarious private charity; to establish a just medium between the natural heat of individual beneficence, and the lukewarmness universally attendant on the corporate exercise of every moral virtue, though coupled with the advantage of public example.

Law. to obey the dictates of reason, and that therefore it is of more immediate consequence to influence his will than to inform his intellect, prejudice seems the more proper way of urging him to the adoption of this law of nature. Accordingly, the instinctive sense of property is implanted in him: he feels that he has a right, he knows not why, to keep whatever he is the first to possess. This feeling is born with him, and it accompanies him, with undiminished force, through every period of his life, and through every vicissitude of intellectual and moral attainment. This feeling is common to all men; and therefore, whilst it creates, it protects private property: those who are accustomed to measure their sense of justice by their natural feelings will feel under a natural obligation to respect in others the right which the same feelings prompt them to assert for themselves. Thus *occupancy* becomes the natural foundation of the right of property; and this, whilst the ultimate purpose of its institution is concealed from those who maintain it. And this sense of property will seldom betray them into an unmeaning and unprofitable usurpation of things: in general, they will not appropriate what cannot be used; and they can seldom use what they appropriate, until they have exerted more or less of salutary labour, either in reducing it into possession, or in fitting it for use. Thus the fact of occupancy or possession, being either an evidence of past, or an earnest of future exertion, is rightly made, in accordance with our natural feelings, the commencement of a permanent and exclusive dominion. The gratification of the natural desire of possessing, taken in connection with the innate sense of justice that accompanies the practice and assertion of the right, constitutes a natural sanction of property; it constitutes that original sanction which prompts and enables individuals to become proprietors. A further, more artificial, but still natural sanction, is the good of society—we mean the temporal good,—which is composed of the good of *most* of the individuals constituting the society, not literally of all. This social utility is ordinarily recommended as a proper motive, acknowledged as a sufficient reason, and suggested as a good practical measure, for the adoption, application, and enforcement of the natural principles of property in civil societies. And unquestionably, temporal social utility is a safe every-day standard; but until the view is extended to the higher moral purposes of property, the importance, the wisdom, and the justice to *all*, of this natural institution, cannot be fully appreciated.

Instinctive sense of property.

Occupancy.

Social utility.

National differences in the laws of property.

114. Above we described the chief characteristics of property to be, inequality in the relative amount held by different individuals, exclusiveness of possession, and permanence of enjoyment. The circumstantial varieties, in these several respects, observable in the laws and customs of different countries, have led some writers to imagine that property is purely the creature of civil law. This want of minute uniformity, however, is consistent with a general resemblance in the leading features; and we are not to conclude that, because the laws of two countries may have sanctioned different rules in some particulars relating to property, therefore one or other of them must be so far a departure from or an addition to the law of nature. In England, a local act of parliament is regarded as a part of the public law of the land; philosophically, it comes under the technical definition of a *privilege*. (No. 31.) So, the particular law of property established in a certain country, if

it be the best suited to the habits and feelings of the people, is the local law of nature for that country. It will be seen that the question is a question of terms; it turns upon the applicability of the phrase *law*, which in strictness imports a generality altogether, or nearly co-extensive with the whole community, to a rule of limited application. The community subject to the law of nature is the whole world; in the case supposed, the particular law is not universal, but it has every other characteristic of *natural*. Confining our attention to that which is most general, we shall discover what the law of nature ordinarily lays down, in the several particulars mentioned at the beginning of this paragraph, by considering what rules of property are at once most in accordance with the natural feelings of man, and most conducive to healthful exertion.

115. First, as to the extent of a man's just acquisitions; let us inquire whether any limit in general is set to the amount, by the law of nature. Locke tells us that "Nature has well set the measure of property by the extent of men's labour, and the conveniences of life;"* and Puffendorf speaks very nearly to the same effect.† If an individual assumes to lay exclusive claim to a thing which it would be absurd to appropriate, such as a part of the sea, the water of a river, or an irreclaimable and extensive desert, his claim would be justly treated with contempt. Such things in their own nature always remain in common. A nation, indeed, may appropriate them as against other nations; but there, an useful purpose—such as public security—justifies the public appropriation; which, after all, is not an appropriation in the ordinary sense for purposes of positive use, but rather an exclusion of other nations from doing certain acts within certain limits. Again, an individual cannot reasonably, or agreeably to the law of nature, appropriate such an exorbitant quantity of land or anything else, that there can be no rational probability of his ever being able to turn it in any way to account. This can seldom take place, except in a new country; there, if an individual, who is not restrained by the law of the land from so doing, made the attempt, his claim would not be respected to its full and unreasonable extent. It has indeed sometimes happened that prodigal and useless grants of land were made, (in our colonies, for example,) which are held sacred; but in those cases measure was taken of the *probability* of improvement within some reasonable time, regard being had to the extensive operations of modern civilization, though in the result public expectation was disappointed. To the same principle of unreasonable excess may be reduced the limit that Locke lays down, where he says, that if, before the appropriation of land, a person gathered more fruit, or killed more venison, than he and those whom he allowed to participate with him could consume, so that the fruit rotted, or the venison putrefied, and was wasted, he offended against the law of nature, and might be punished, for he thereby invaded

* *Essay on Government*, ch. v., sec. 36.

† One man is then adjudged to be the occupant of land, when he tills and manures it, or when he encloseth it with settled boundaries and limits: yet still with this proviso, that he grasp at no more than what, upon a fair account, seems tenable by one family, however enlarged and multiplied. Should one man, for instance, be, with his wife, cast upon a desert island, sufficient to maintain myriads of people; he could not, without intolerable arrogance, challenge the whole island to himself, upon the right of occupancy, and endeavour to repulse those who should land on a different part of the shore. *Law of N. and N.*, book iv., c. vi. 3.

Law. his neighbour's share. Waste of this kind will rarely occur to any great extent; but if we can imagine a case where it has become very common, palpable, and excessive, there, probably, direct and compulsory prevention would be neither improper nor unnatural. In general, however, the wrong is a *moral* wrong, or one which ought not to be controlled by law; because considerable latitude must be allowed for the possibility of abuse, on the very same ground that property itself is tolerated, namely, in order that private enterprise may not be checked. By no very violent extension of the principle of interference in case of waste, a person who does not cultivate his estate so advantageously as might be thought possible, might be deprived of all or part of it; this evidently would violate every feeling and every good purpose connected with the right of property. Subject to the limitations we have stated, the bounds of which cannot be nicely defined,—and perhaps, also, under very peculiar circumstances, subject to limitation on grounds of public policy,—the amount of private acquisition ought not to be restricted. If a limit were set, when a person's property had attained its maximum, enterprise, so far as his acts or influence extended, would come to a stand. The unlimited power of acquisition, generally permitted, has a tendency to produce inequalities in the relative extent of private possessions.

Exclusive-
ness of
property.

116. The second point is the exclusiveness of property. Property may be more or less exclusive, according as it belongs to one or to several partners, few or many. Sole property, which is the most exclusive, is also the most natural; both because the selfish feelings of mankind naturally tend more to severance than to union of possession, (we do not speak now of the very artificial unions of modern and highly civilized times;) and because, in general, the interest felt by a sole proprietor is stronger, and consequently the incitement to exertion in improving and preserving his property is greater, than would be the case were there several proprietors of the same thing. In proportion as the number of owners is increased, the interest of each individually is diminished; and as it is not certain that the united interests of all will be made to bear with their full natural force in the same direction, except under very good arrangement, the probability is that the total amount of interest felt will not be so great, as if concentrated in one person; but it is not our purpose here to discuss the artificial arrangements by which this disadvantage may be overcome. However numerous the joint owners of property may be, their right to exclude all others from its use and enjoyment is as natural as the similar right of a sole owner with respect to his property.

Perpetua-
tion of pro-
perty by
descent or
otherwise.
Life owner-
ship.

117. Thirdly, we have to inquire what is the natural extent of the right to property, in point of duration or continuance. Jurists for the most part are agreed that the right to property, which involves the rights of using, selling, or giving it away at the pleasure of the proprietor, subsists during his life at least. On his death, does it once more become common; does it devolve upon his children; or does it go to any successor whom he may have appointed? Neither of these modes ought to be gratuitously assumed, as exclusively or peculiarly natural: they should all be subjected to the test of reason and experience, in order to discover which is the most ordinary, probable, and reasonable. The supposition, entertained by some, that a man's property

VOL. II.

naturally returns to a negative state of community after his death, rests on no better foundation, that we can discover, than the purely gratuitous assumption of those, who, in their zeal to simplify the system of nature, make it incomparably more artificial than those more ordinary conditions of things which they peremptorily pronounce, with as little reality of truth as of reason, to be departures from what is natural. Nothing, for example, could be more artificial than the equality of possessions attempted to be enforced by the Spartan law, and by the custom of Tanistry under the ancient Brehon law of Ireland, if we may judge by the necessity that was experienced for occasional state interference, to rectify the inequalities inevitably introduced by time. Or, to take an example more nearly bearing on the matter in hand, the rule of the early feudal law, whereby the feudal proprietor's interest legally ceased with his life, was purely artificial; for, in spite of this positive arrangement, custom, which was only nature re-asserting herself, first established as an indulgence, and at length gradually matured into a right, the practice of inheritance. The same motives and reasons which recommend property itself, recommend its perpetuity; but as to which is the more natural way of perpetuating the proprietor's ownership beyond his life,—by inheritance, or by testamentary disposition,—and even whether the latter be in strict accordance with nature, have been subjects of difference with jurists. The majority decide that inheritance by children is the natural way, and that the power of bequeathing to a stranger derogates from their natural right, and is purely an indulgence of municipal law. If so, its permission is indefensible. In strictness of reason, however, children have no right to anything more than what suffices for a maintenance during the helpless period of infancy, and to put them in a fair way of earning a livelihood at a proper age. This reasonable claim satisfied, the single paramount principle, by which the question must be governed, is THE PROMOTION OF INDUSTRY. In order to give this principle its due effect, we ought to regard not the expectant but the immediate possessor of property. The certainty of succeeding to property, without any efforts of his own, far from exciting the industry of the expectant, tends to relax it. The natural right of the children to succeed to their father's property; then, is inheritance. It is his, because the interest he naturally takes in them is second only—if second—to the interest he takes in himself; and because the feeling that thus prejudices him usefully in their favour—whilst it is a passion which, under due regulation, may be innocently indulged—gives point, energy, and an object to his industrious efforts. By a just extension of the same principle, the utmost possible effect that can be safely accorded ought to be given to the greatest possible number of the natural passions which ordinarily actuate the human breast, excite man to undertake an energetic course of industry, and sustain him throughout it. Accordingly, free indulgence should be afforded to his love of having, upon which depends the right of getting and holding property for himself; to his love of power, whereby he acquires the right of disposing at will of his property, not only during his life, but to any one he pleases after his death; to his love of his children, and, in subordinate degrees, of his remoter relatives, (in general sufficiently just and strong to prevent a capricious abuse of the disposing power,) which gives birth to the rights of inheritance, first of

Law.

Natural
right of
children.

Right of
inheritance.

Right of
property.
General
disposing
power.
Testa-
mentary
power.

Inherit-
ance.

Law. lineal descendants, and, failing them, of collaterals, but subject to the disposing power above mentioned; and lastly, to his love of family distinction, which originates the right of primogeniture. The order in which these several rights of nature are here named, expresses the order of their relative strength. They may all be reduced to the single right of absolute disposition: if the proprietor names his successor in estate, this is an express disposition; if he suffers his property to devolve on his heir, this is a tacit disposition, since, without any groundless fiction, it may fairly be presumed, from the ordinary course of human affections, that the proprietor's wishes pointed in that direction. Any form or mode, which can fairly be relied upon, adopted by the proprietor to express his intention to dispose of his property, either in his lifetime or after his death, is naturally sufficient: civil laws establish particular set forms, in order to guard against misconceptions and false pretences; and as the real danger of such, or the judgment of legislators respecting the danger of them, varies under different circumstances, nations differ in their established forms of transferring and transmitting property. If to our mode of argument above it be objected, that although, indeed, inheritance by children benefits the parent morally, by adding to the exciting causes of his industry, it at the same time tends to damp the exertions of the children, by encouraging relaxing expectations on their part; we reply, first, that in many cases the inheritance is not more than a fair provision for them; secondly, that where it is more, the absolute power of disposition in the parent partially obviates the disadvantage of certainty of expectation; thirdly, that there would be no alternative but the parent's property becoming common on his death; which, if general, must create too great a facility of acquiring, by reason of the constant supply of property becoming once more common, thereby occasioning a general relaxation of industry, which would extend to the children themselves. The last point here to be considered is, the proprietor's right to forestal the disposing power of his successors by creating entails, or directing some other strict mode of devolution. By allowing this right, on the one hand, the possessor's disposing power is enlarged, and consequently the interest he feels is increased; on the other hand, the power and the interest of his successor will be proportionably diminished; probably with corresponding effects upon the zeal and exertion of each. The mode of adjusting the right, which would give neither of them any advantage over the other, would evidently be to allow only absolute or unfettered dispositions. This mode is the most simple and the most equal, and may therefore perhaps be considered as the most natural. At the same time, we do not mean to imply that family consequence, which is a natural feeling that has its advantages, ought not to be supported by entails. But we cannot enlarge on this topic.

Prescription.

118. We shall close our observations upon property, by explaining how the owner of anything may, by lapse of time, lose his right to it, and how another person may succeed him, as rightful proprietor, by the title of Prescription. We have already (No. 113.) mentioned occupancy as the rightful commencement of property; but the extent of the occupant's ownership must depend upon his intention. If he intends to appropriate the thing occupied only for the brief period of actual occupation, upon his relinquishing it, it becomes once more

common. If he designs to make it permanently his own, as he cannot in all cases carry it about with him, or be present with it, his *animus possidendi*, or virtual and intentional occupation, ought to be made apparent, in order that others may not, in his absence, be induced to appropriate the same thing, under the idea that it is common. To take two extreme cases: if there were no obligation upon proprietors to mark their private property as such, or to assert their right with promptitude, a general carelessness on their part might be the consequence; it might become difficult to distinguish what is common from what is private; and then, as none could know with certainty what they might safely appropriate, few would be willing to apply their labour to that which might be claimed the next moment. The result would be, on the one hand, indolence and laxity; on the other, hesitation and doubt, detrimental to enterprise and industry. If we take the other extreme, the result would not be very different: were the obligation of asserting the right of property so strict as to preclude the possibility of doubt in any case, many would be unwilling to incur the trouble of constantly maintaining their right; and the fear of losing possession, by some accident or oversight of ordinary occurrence, and not always to be provided against—to take advantage of which there would always be abundance of designing persons on the watch—would introduce a sense of insecurity, highly prejudicial to that confidence which is essential to industry. To define exactly the just medium between over-strictness and over-laxity is evidently impossible; indeed the line between them is not invariable, for it will change with the subject matter, and with the prevalent habits and ideas of different ages and countries. We can only lay it down generally, that an appropriator ought to take a fair and reasonable degree of care to indicate the existence of a right of property. Appropriation, however, rarely takes place, without the thing appropriated bearing permanent marks of private ownership. The same argument which applies to things supposed to be strictly in common, nearly applies to things evidently belonging to some one, though the title is either in doubt between two or more, or not asserted by the rightful owner. We will suppose, in such a case, that a person who has no pretence of title, or not the best right to the subject of dispute, no visible owner appearing, takes possession of it, and is suffered to continue for a length of time in undisturbed occupation: we have to consider what effect ought to be attributed to lapse of time in such a case. Putting out of view the temporal interests of the parties immediately concerned, it is of no importance, socially or morally, how property commences, or who is the proprietor. But it is of vital importance, both socially and morally, that a right of property once commenced in any individual should not easily be disturbed; for the permanent security of property is only another name for the encouragement of permanent and progressive industry. Therefore, on general principles, it is desirable to allow the largest space of time that can be in reason desired, or with safety granted, for the lawful proprietor to recover his property. But this time must have a limit; for so long as the lawful owner neglects to assert his claim, the unlawful possessor continues in a state of suspense, between hope that the other may never claim his own, and fear that he may; and this state of suspense discourages exertion, and ought not to be continued for ever. As time advances, the hope will

Law

Law.

gradually become stronger, and the fear weaker; and this growing confidence will exhibit itself in an increasing disposition, on the part of the occupant, to treat the property as his own, and to give it all the improvement he can, which it is capable of receiving. To disappoint him, by robbing him at last of the fruits of his exertions, would be to check similar exertions in parallel cases, the moral and social mischiefs of which need not be enlarged upon. Turning now our attention to the relative claims of the parties, regarded as individuals, to a just share of consideration; the occupancy and the right of the lawful owner (as we may still call him for the sake of distinction) originated perhaps in the accident of finding the property lying in common; the occupancy of the other originated in the wilful neglect or forgetfulness of the lawful owner: the moral reason for the lawful owner acquiring a permanent right of property at all, consisted in the probability that his right of property would act as a spur to exertion, a probability unfulfilled in the event; the moral reason for the intruder becoming in time the rightful owner is the certainty that he occupies the place which the other ought to have asserted, of a vigilant and industrious proprietor, in will at least, if not in act. Lastly, the enjoyment of the property, if considerable, becomes, in process of time and by habit, essential to the comfort of the intruder, and a part of his nature; and it ceases to be so with the other. From these several personal considerations, in addition to the public and moral grounds above stated, we conclude that the title by prescription or lapse of time ought at length to prevail over an originally better title. If it be asked what is the natural period of prescription? we answer, that it is not uniform, nor ever can be, until there is an uniformity in the prospective and retrospective habits of all mankind; until the time for superinducing forgetfulness and begetting confidence is the same throughout the world; and until the arts of life are equally known, and the time for bringing them to maturity is the same everywhere.* We have now stated, in general terms, the philosophical foundation of the prescriptive right of property. The original owner may claim his property at any time within the period of prescription. Particular circumstances may have a natural tendency, and have been occasionally allowed to operate, either to prolong or to abridge the ordinary period. The infancy of the rightful owner, whose infancy excuses his neglect, is an instance of the former: the fact of the lawful owner, fully aware of his title, standing by in silence, whilst improvements are carried on by the intruder, is not an unreasonable ground for the latter, or rather for presuming his consent to the assumed ownership of the other.† There is one material consideration, which must not be omitted,

*The periods of prescription have varied excessively in different countries and ages. We may give a few examples. By the early Roman law, a period of one year for moveables, and two years for immoveables, was fixed. In our own law, the periods of prescription for different purposes have frequently been varied; at present the period of prescription, in regard to lands, is, in general, twenty years. By the Hindoo law, an uninterrupted possession for three generations confers an absolute title to lands; an occupancy by three successive owners, for sixty years in the aggregate, counting as an occupancy of three generations. The period for moveables is commonly shorter than that for lands; because the former are apt to pass through a greater number of hands in a given time than the latter.

† This is the ground taken by Grotius, who founds the right of prescription in a tacit abandonment. *De J. B. et P.*, lib. ii., c. iv.

viz., the honesty or dishonesty of the intruder, when he takes possession: there is a wide difference whether he comes in under an honest though erroneous impression respecting his right; or with a full knowledge, or strong suspicion, that another is the rightful owner. Puffendorf, in common with several other writers, considers honesty to be so essential, that the intruder himself can, by no lapse of time, purge away the original infirmity of his possessory title, but that he must be succeeded by some second holder, (an heir or alienee,) who was not aware of the dishonesty; from the commencement of whose occupation the period of prescription is to date.* Puffendorf, we are inclined to think, lays down the rule too absolutely. Doubtless, the dishonesty of the intruder is an element to be considered, along with the other grounds above described; but we are not sure that, even in spite of dishonesty, time ought not, on those grounds, often to prevail in "mellowing into legal right" an usurpation of property "that was violent in its commencement." The shades of moral and natural dishonesty, between a full consciousness of another's title, and an utter ignorance of it, are infinite; and there may be great moral culpability in the negligence of the former owner. Lastly, it is to be observed that the right by prescription is sanctioned by a strong instinctive feeling or prejudice of our nature; and, in point of authority, the universal practice of mankind, in every age, has been to respect it.

119. Those who, following the arrangement of the Natural Roman law, adopt the natural law of facts as one of law of facts. their chief divisions, investigate, under that name, the several facts upon which legal obligations and liabilities are founded. Under this head, they examine most of the principles of civil and criminal justice, and treat the copious subject of contracts. Our plan does not exactly coincide with theirs; nor does our allotted space allow of a minute discussion of the several topics they examine, especially that of contracts, which has been so ably and so fully handled in the Roman law. We may here briefly remark that in the endless conflicts and transactions of human life, there is a daily necessity, as well as opportunity, for the exercise of man in the practice of the two eternal principles of JUSTICE, and GOOD FAITH, or TRUTH. The natural principles and measure of human justice must be collected partly from the preceding discussion of relative obligations, and partly from the few remarks about to follow in the next section.

120. Good faith, or truth, relates either to speech or Truth. to action; it prescribes the avoiding of all deceit, and the faithful fulfilment of every engagement. Writers Its moral obligation. universally concur in treating the obligation of observing it as a postulate in morals; but the mode in which it is developed in the law of nature is worthy of observation. What we wish to point out is the universal recognition of degrees of obligation respecting it; for although no limit can be set to the moral duty of truth, yet, in human estimation, some kinds of truth are held in more solemn respect than others. Additional circumstances of outward solemnity,—amongst which we Its natural degrees. may particularly distinguish a direct appeal to the Supreme Being,—are held to impart an additional degree of sanctity and efficacy to truth itself. Yet are not the bare intention to deceive, and the breach of any engagement, highly culpable violations of this great moral

* *Law of N. and N.*, book iv., ch. xii., 4.

Law.

principle? Men, however, in all ages, have made distinctions; some even, who have no regard whatever for their bare word, exhibit no small respect for their oath; and all acknowledge that the latter is *more* binding. Again, with respect to contracts: some are considered to have a more peculiarly stringent obligation; those, for example, which are of a more equal reciprocity, with reference to the relative degree of advantage thereby secured to each party; and this wholly independent of the certainty and freedom of the engagement and the expectations thereby raised.

Natural sanctions of truth.

121. If it be asked, why this is so? we may probably (for it would be unbecoming to speak with dogmatical certainty on such a subject) here apply the same principles and reasons, previously offered in explanation of the duties already disposed of. Man requires to be practically disciplined in the observance, in order to the love, of truth. As his attainments in this, as in other instances, must be gradual, truth has been accommodated by the law of nature to his imperfect condition. His mind, accordingly, has been naturally impressed, through his instincts, with an artificial esteem for certain kinds and modes of truth above others; and the Author of Nature has made the observance of them especially important to the temporal welfare of human society. There is then a certain kind and degree of truth, respecting which the law of nature leaves us but little choice; which all mankind, but the very lowest, venerate; and which, from interested and worldly, but not therefore improper motives, is vindicated by human laws. But higher and more complete degrees of truth are left, as they must be left, both in nature and in law, entirely free and unimpelled.

Use of oaths, &c.

122. There are some natural and legal obligations, in themselves abundantly important, to which a further sanction is commonly added by the imposition of an oath or other solemn engagement. We may instance the duties arising from the relations of husband and wife, and sovereign and subject. In such cases, the independent or original causes of duty ought alone to be sufficient to impress the parties with a practical sense of obligation; but as those causes often are either unknown, or not readily borne in mind, advantage is taken of the plain and obvious obligation of truth, to effect the practical object in view by an additional and different species of obligation. Thus truth aids the observance of other duties; and other duties, where the independent reasons of them are practically recognized, aid the observance of truth: for the mind will not always discriminate which of the two concurrent obligations predominates. But caution is necessary in thus uniting truth with other obligations; for there is a tendency in the human mind to measure by each other two obligations united in the way supposed: if, on the one hand, it sometimes happens that a duty of but secondary moment is elevated to excessive importance by the addition of a solemn oath, there is a danger, on the other, lest the estimation in which the oath is held should be depressed to the comparatively low level of the duty it is intended to strengthen. It follows that the interests of truth forbid the imposition of oaths on trivial occasions; and the same principle ought to be extended to all solemn engagements.

Practice of alleging tacit compacts.

123. The preceding remarks suggest one concluding reflection. Scarcely anything can be less philosophical than the practice of alleging tacit compacts as the sole or chief foundation of certain social and political obli-

gations. No one will deny that compacts must be observed; but from philosophers we are entitled to know what is the original and independent obligation which thus calls for and justifies ratification by a solemn compact. We are not to be deluded by the trick of substituting the obligation of good faith (which, in many of the cases supposed, has no existence, and if it has, can be no substitute) for the *fundamental* obligation, which such philosophers do not condescend to investigate. Possibly, the practice may have partly arisen from an undue preference given to truth above the *co-ordinate* and *co-equal* principles of love and justice.

Sect. 5. *Of the enforcement of the Law of Nature by mankind, in civil society.*

124. In the preceding section we have attempted to give a rough sketch of the principal obligations of the law of nature. That law, we have seen, is a particular measure of conduct, adapted to the present condition of man, and calculated to promote his moral advancement; and its sanctions, or the natural means by which it is pressed upon him, include instincts and motives of self-interest; gradually ascending, however, and mingling themselves with care for the public welfare, and with reason and conscience; thus both rendering him less selfish, and tending to that state of spontaneous action, which it was observed (Nos. 52. 53.) is essential to perfection. So far, this law is the law of nature. We have now to consider it as the subject-matter of human law, or as receiving an additional sanction and protection at the hands of men by means of human punishments.

125. Though the force and power of its natural sanctions will never permit the law of nature to be wholly disregarded, yet the freedom which is indispensable to man's moral character, and to his progressive advancement as a moral being, requires some degree of laxity in those sanctions, and consequently some liability to error on his part. Had the mechanism of instincts been carried much farther, greater regularity of conduct, indeed, might have been produced; but it would have been the mechanical regularity of brutes, not the orderly virtue of free agents. Again, a more extensive employment of superior motives,—those, namely, which act upon man's sense of aggrandizement, which operate through the noble faculty of reason, and call forth much intelligence, knowledge, and foresight,—would, in all probability, only have made him a more highly cultivated intellectual machine, and confirmed him in selfishness. These several moving powers have probably been employed as far as they could be with safety or advantage, room being still left for the free play and final supremacy of the moral sense. But the liberty so far accorded to man, whilst it is the cause of his greatness, is also the source of much disorder. A necessity still subsists for further direction and control, but of a different kind, the direction and control, namely, of an intelligent agent; and these are provided in the power which is administered by man himself.

126. In treating of human compulsion, we will consider, first, the species and principles of interference; and, secondly, the organization of the authority which directs it.

127. I. The duties of men towards each other, or the rights answering to those duties, form, if not in strictness (as some contend) the only, at all events the most common and indisputable subjects of human enforcement. But in order to be enforced, they must

Law.

Laxity of the law of nature.

Enforcement of relative obligations.

Law. be certain and definite, with respect both to their extent and objects; and they may either be so naturally, (that is, in a manner too plain and obvious to all to require any further definition,) or be rendered so artificially, by the private contract of the parties interested, or by the laws of states. Accordingly Nature, Contract, and Law are said to be the three sources of perfect rights.

Modes of exerting force. 128. Next, as to their enforcement. Human force is exerted to obviate the injuries resulting from a violation of them in three ways: 1. in self-defence, to repel *present* injuries; 2. to compel reparation for *past* injuries; and, 3. to prevent *future* injuries, by the punishment of the wilful violators of such rights.

Self-defence. 129. The right of self-defence is more comprehensive in a state of nature than in civil society. In the former, every individual is the protector of his own rights, with respect to the future as well as to the present. But in civil society, private individuals are permitted, and that from unavoidable necessity, to use violence only in the immediate defence of their persons or property; all steps for their future security being reserved for the public authorities. With respect to *present* self-defence, which we have now to consider by itself, its just limits, we conceive, whatever some writers may assert to the contrary, are precisely the same in civil society and in a natural state. This right, as Grotius observes,* arises directly and immediately from the care of our own preservation, which nature recommends to every one, and not from the injustice or crime of the aggressor who may threaten our safety. So that if a person, involuntarily, or by mistake or accident, assaults me, or in any way places my life in imminent danger, I am not to be debarred, by the consideration of his innocence, from consulting my own safety, even at the expense of his life. The law of charity does not require me to have a *greater* regard for him than for myself. But, on the other hand, the same law requires that I should not have a *less*. If, therefore, my own defence will probably do him a greater injury than he will do me, I should submit to be the sufferer. This is the broad and intelligible rule. Jurists sometimes introduce other collateral or subordinate considerations, which we cannot here examine, to modify the right of self-defence under particular circumstances. Cases of this kind, where, without any blame attaching to either party, the sacrifice of one or other of them becomes unavoidable, are of rare occurrence, and fall more properly within the province of casuistry than of jurisprudence. The right of self-defence, as jurists commonly treat it, relates principally to the repelling of wilful and malicious assaults. Puffendorf,† and others after him, contend, but assuredly with very inadequate notions of the obligations of humanity, that the unjust intention of the assailant, however slight an injury he may attempt, deprives him of *all* title to consideration; and that, therefore, although it may sometimes be more *prudent* for the party assailed to submit to a trivial wrong, than to avert it by the death, even, of the assailant, there can be no *obligation* upon him to that effect. Where then should the line be drawn? The limit of rightful self-defence is, we conceive, identified with the limit of just punishment (of which presently, No. 134); that is to say, whatever would be the utmost amount of

punishment that could properly be inflicted for the crime apparently intended, supposing it to have been completed, to that length of harm the right of self-defence may be justly carried; beyond it, the obligation of endurance begins. Such appears to be the principle recognized in the laws of most well-regulated communities, which rarely suffer with impunity any crime to be prevented by death, unless the same, if committed, would also be punished by death.* And thus the defence of life, limb, or chastity has, in all or most countries, been held a sufficient justification for the taking of life. But it must be understood that the party assailed is in no case privileged to proceed even to the length we have stated, unless it be absolutely necessary for his own immediate safety. It may be objected that the limit drawn is merely theoretical, because in the hurry, danger, and excitement of the moment, and moreover where self is in question, it is rarely possible to weigh the matter very nicely. However we have only attempted to describe generally what is strictly and rigidly *justifiable*. Some excess must always be *excused*, provided it does not surpass the fair allowance claimable on the score of ordinary human infirmity, and become an act of cruelty; in which case the gross excess would be a just ground of punishment. It only remains to be observed that, as a general rule, whatever violence an individual may lawfully exert in his own defence, he may equally exert in defence of another person who stands in a near domestic relation to him.

130. So far as to the repelling of an injury before it can be carried into effect. When an injury has actually been inflicted, it may be viewed in a two-fold aspect—Distinction between civil injuries and crimes. as it may be repaired, and as it may be punished. In the former aspect it is called, in the language of jurisprudence, a *civil injury*; and the object of reparation is to place the injured party as nearly as possible *in statu quo*, with reference to the past. In the latter aspect the same wrong is called a *crime*, and is made the subject of punishment with a view to prevent the perpetration of similar injuries in future.

131. With respect to reparation; the civil injury, Reparation. upon which a right to it is based, is not, as in the case of self-defence, necessarily of a violent character; nor is it always attended with criminality. A right withheld, whether in ignorance or otherwise, or a wrong attributable in any degree, however slight, to neglect or carelessness, is such an injury as will give the sufferer a right to reparation from the person who has thus caused it. The claimant, however, must not himself be in fault; and he must prefer his claim within a reasonable time after the injury has happened, otherwise a hardship might be inflicted upon the other party, by subjecting him to an unexpected and stale demand. The period for preferring the claim must depend a good deal on the subject matter. As to the mode of reparation: where the injury consists in the withholding of some specific right or thing, restitution in specie, if practicable, is obviously the proper remedy. Where this mode of redress is impossible or inapplicable, compensation in value ought to be rendered. Lastly, where redress of the proper kind and amount is denied, it may be extorted by force; but, in civil communities, private individuals are not allowed to exert this force for themselves, but must obtain redress through the civil tribunals constituted for

* *De J. B. et P.*, II., i. 3.

† *Law of N. and N.*, book ii. ch. 5.

* Blackstone, *Comm.* iv. 182.

Law.

the purpose. The statement of these leading principles must suffice on this head; for our limits will not suffer us to go into the details, which are complicated and difficult.

Crimes and punishments.

132. We have now to consider injuries as they become crimes. The circumstance which makes an injury a crime, and calls for the punishment of the criminal, is his guilty knowledge, or the wilfulness of his offence. This affords a just ground to apprehend from him and from others the perpetration of further injuries; from him, whose evil disposition has been already sufficiently manifested; from others, who may be instigated by his bad example. Punishment, therefore, is inflicted upon him with a double view; as a check upon him, and a check upon them. So far as it is intended as a check upon him, the punishment may be of two kinds; either such as incapacitates him physically from repeating his offence, by depriving him of the power or opportunity of doing hurt, as, for example, perpetual imprisonment, or death, or banishment to a distant part of the world; or such as tends to reform his character and make him a better member of society, by the wholesome fear it produces in him of incurring a like punishment again. The former kind is proper for heinous offenders, whose crimes afford proof of an incorrigibly bad disposition, incapable of being acted upon by any moral restraint; the latter for less atrocious offenders. The same punishment, of either kind, operates upon others by the force of example; and in order to give it effect in this way, it should be made as public as possible. It is also frequently necessary to add to its severity, as compared with what would suffice to restrain the individual culprit merely. Thus, though perpetual imprisonment will always be an effectual protection against the designs of any given offender, however dangerous, nothing less than death may be sufficient to strike terror into some whose evil disposition has not yet broken out into open crime. On account, then, of public example, punishments are usually adjusted less according to the nice measure of each individual culprit's depravity than according to the general experience of their efficacy. The extent of punishments, however, with this or any other view, must be limited by considerations of justice and mercy towards the offender, of which presently. We have only now to add that the efficacy of every punishment depends much upon the speed and certainty with which it follows the offence.

In what sense punishments are inflicted for the benefit of the public.

133. Punishments are said to be inflicted for the benefit of "society," "the public," or "the community." Each of these terms is ambiguous: it may designate either the individuals composing "the public," &c., considered as individuals, or the same persons collectively "in their social aggregate capacity,"—in other words, the state. Those who, like Blackstone,* endeavour to make all crimes turn upon a supposed injury to the state, misrepresent the matter. His distinction between a civil injury and a crime consists, it would seem, in the former being a (past) injury to an individual, and the latter being a (past) injury to the state. Is an individual murdered?—the state has lost a citizen, and fears that it may lose more, and therefore it punishes the murderer. But what is the case in theft? Here the state suffers no loss, no diminution of its power or resources, though

some of its wealth passes, dishonestly it is true, from the hands of one citizen to another; and yet it punishes the thief. But, in truth, the primary distinction between civil injuries and crimes, as we have already mentioned, consists in their relation respectively to the past and the future. The distinction of private and public, which also belongs to them, is secondary, and in some measure accidental. In far the greater number of cases,* a private individual is the only immediate sufferer by a crime; therefore he alone is entitled to civil satisfaction; but when he has obtained it, he has obtained only part of that to which, as an individual, he is entitled, namely, reparation for the past. He has still a right to protection for the future; but here he no longer stands alone; for all other individuals of the community, as individuals, and not in their collective capacity, participate in the same danger, arising from the very same cause, and have therefore an equal right to protection in the same way, namely by the punishment of the criminal. As all then are equally, though separately, interested in his punishment, there is a peculiar propriety in the action for that purpose being instituted in the name of the sovereign, as the *parens patriæ*. The fiction according to which the wrong is said to be done to the sovereign (who is often confounded with the state) is explained by considering the interest he has in his subjects' welfare to be paternal rather than beneficial; consequently his interference is not on selfish grounds.

134. We shall now say a few words on the principles which ought to regulate the proportion of punishments. It does not follow, because prevention is the end, and the only end,† of punishment, that therefore it may be pursued at any cost. It is now pretty generally admitted that neither the difficulty of preventing a particular crime, nor its frequency, nor the number of persons interested in its suppression, will, either separately or jointly, justify the use of preventives disproportionately severe. In such cases it is the duty of society rather to tolerate the partial inconvenience arising from an imperfect check. There are two distinct measures of just punishment: the depravity of the offender, and the magnitude of the evil that would result from his unpunished offence. As to the first, it is unnecessary here to exhibit in detail the various circumstances which are ordinarily taken as proofs of an offender's depravity of heart: they will readily suggest themselves to most readers. In proportion to his depravity, as evidenced by his crimes, an offender's title to consideration diminishes, and, as it would seem, wholly ceases when his crime is of a nature that evinces a state of utter and hopeless depravity. It then becomes lawful to put him to death for the sake of public

Law.

* Not in all: for some criminal acts, such as treason and sedition, may be directly levelled at the state, and these are punished for the future safety of the state as such. Everything relating to punishments on this latter ground, properly falls within what is called public law (public being here used in the second sense, explained in the beginning of the paragraph). Punishments for the protection of private rights, which are the only ones contemplated in the text, belong to private law. The distinction between public and private law is thus expressed in the *Digest*, l. i. 1, 2:—"Publicum jus est, quod ad statum rei Romanæ spectat: Privatum, quod ad singulorum utilitatem."

† Some, without much attention to verbal accuracy, contend for retribution as an end of punishment. A palpable mistake; since retribution is but another expression for punishment itself. They probably mean to refer to the measure.

Law.

example. Grotius, in his admirable chapter on punishments,* mentions the opinion of several philosophers, that possibly death may sometimes be expedient even for the sake of incorrigible offenders themselves, since, apparently, a prolongation of life can only serve to harden them still more. This opinion, however, he advances with becoming diffidence, and at most as a conjecture. Some, rather too hastily, contend that depravity ought to be the sole measure of punishment. However, we ought not to overlook the other measure above mentioned, namely, the magnitude of the evil contained in the offence, which will frequently compensate for a total or partial absence of depravity. Of course it is not meant that punishment should ever fall on one who is not a voluntary agent. To warrant punishment, there must at least be an antecedent and known responsibility, with a power in the agent of commanding his own actions. Under these circumstances, it is still very possible to occasion the most deplorable mischief, without any depravity of heart. We may instance, as an example, a sentinel sleeping at his post, overcome by *almost* involuntary drowsiness, whereby the safety of an army is compromised. For such a crime death may be justly denounced, supposing that no lighter punishment is found effectual to secure vigilance by timely exertion; because here the question rests between the lives of *many* innocent persons, and the life of one or a very few offenders in this way, who know beforehand the magnitude of the interests entrusted to them, and the consequences of their conduct. This single instance will suffice to illustrate the principle. It is not allowable, however, to impose artificial obligations of this kind without a strong necessity for so doing.

Offences
against
self.

135. Hitherto our attention has been exclusively devoted to offences against the rights of others. The offences which a man may commit against himself are in general, as such, unfit subjects for prevention by punishment; not because every man has an abstract right to do what he pleases with himself, (than which nothing can be more untrue,) but because an inquisitorial visitation of the private morals of individuals would be detrimental to their independence of character, which is connected with all that is truly great and noble in their nature. But immoralities which would be passed over with impunity, if committed in private, are properly made subjects of punishment when done in public, on account of their evil example. There are, however, some crimes of the present class too gross and detestable to be tolerated under any circumstances: these form exceptions to the general rule. This observation applies to suicide, the greatest offence a man can commit against himself. It is curious to observe how some writers, fettered by a false or overstrained notion, that with the conduct of individuals, as such, society has no concern whatever, go out of their way to find reasons or excuses for preventing this crime, when the true and only tenable ground is so plain and obvious, viz., the tremendous and irremediable injury the self-destroyer commits *against himself*. The loss of a citizen to the state; the grief, distress, or loss he occasions to his family; such are the frivolous grounds often put prominently forward—reasons which, it may safely be affirmed, never would, by themselves, have induced society to interfere. In fact, upon this,

as upon other points, the instinctive feelings of men, which are generally condemnatory of suicide, appear to have informed them much more truly than their reason.

Law.

136. Observations nearly similar to the preceding will apply to offences against God and religion. It is the duty of society to provide the means of inculcating upon its members the performance of their obligations in this respect; but, in general, compulsion is an improper instrument for the purpose. Punishments are wholly improper for enforcing duties of *performance*, and they must be sparingly and cautiously used as a preventive of what is positively injurious. As the temporal welfare of men in society is but an auxiliary, and ought not, therefore, to be made the substitute of their moral and spiritual welfare, we can regard the political and social benefit of religion only as a natural inducement, and not as the ultimate reason, for society's interference for the defence of the outworks of religion. But whether we take the religious, or merely the temporal ground, the limit of interference will be the same. The rule is toleration, and the exception interference. Grotius, with his usual good sense, draws the line where this toleration ought to cease—of course it cannot be done with exact precision. In the cases where he considers interference to be justifiable, not only are the external interests of society vitally implicated in the suppression of the offences proscribed, but there is no undue interference with the most extensive freedom which can be rationally claimed. Those cases may be thus enumerated: the public propagation of downright atheism (see No. 90); the open worship of confessedly evil spirits, under the notion that they are such; human sacrifices; open and wilful profanity.*

Offences
against
God and
religion.

137. We may not improperly conclude this division of the subject with two observations. The first relates to a topic alluded to in a former page (No. 38.), respecting the supposed superiority of punishments over rewards, as inducements to right conduct, and properly reserved for this place, because we were not then treating particularly of *artificial* inducements. The means possessed by civil governments for the ordinary control of their subjects' conduct consist exclusively of punishments (for we ought not to reckon the honours and rewards bestowed for rare and distinguished merit). Some, therefore, have chosen to assert that rewards are, in their own nature, less calculated to influence *the conduct*. But this does not seem to be the case. The objection, in a moral point of view, to an excessive use of artificial rewards seems to be this: that while they may equally influence *the conduct*, they are calculated to give permanence to artificial *motives*, and so to impede progress towards that *spontaneous* course of action in which moral perfection consists. When punishments are the inducements made use of, in proportion as the conduct becomes more and more correct, the necessity for their application becomes more rare, and the sense of their presence more faint; until at length they silently and of themselves practically cease to operate, though retaining their dormant power unimpaired. But rewards, when employed for the same purpose, must be constantly accumulated upon the agent, in proportion as his conduct becomes habitually conformable to the rule proposed for his observance; the natural consequence of which must be to enslave him to them. Now the great object of man's life being the cultivation of

Punish-
ments and
rewards
contrasted.* *De J. B. et P.*, lib. ii., c. xx., sec. 7.* *De J. B. et P.*, lib. ii., ch. xx., sec. 44—51.

Law. both his conduct and motives, with a view to his ultimate perfection, which perfection imports freedom from all necessity for outward constraints or inducements, we may, upon the grounds here intimated, partly account for the Deity having placed punishments rather than rewards at man's disposal, as the ordinary means of guiding the conduct of his species.

Probable ground of man's right to punish.

138. The other observation relates to the probable reason for conferring the administration of these punishments and of the other modes of lawful violence above explained, upon *man*; who, it may perhaps be thought, is but imperfectly qualified to administer them with just discrimination, and in danger of contracting a hardness of heart from the constant habit and necessity of putting them in force himself. But what might at first seem objections to, are perhaps the very reasons for, the delegated authority thus conferred upon him. Without the power to discriminate between good and evil, and without—we do not say the *hardness*, but—the *firmness* of heart to exercise severity upon just and necessary occasions, his moral character must be essentially imperfect. Probably it was to rectify his errors, and to supply his weaknesses and defects, in both these particulars,—in a word, to teach him *justice*, not as a mere sentiment, nor as an expedient for his present regulation, but, practically, as an energetic and eternal principle of action,—that the task has been imposed upon him of himself discovering and defining the limits of his own rights, and of inflicting evil upon evil-doers, as the only means of preserving those rights.

Civil government.

139. II. The second division of the subject which we proposed to discuss (No. 126.), relates to the organization of the compulsory power possessed by men for exacting a due observance of their perfect obligations. The only way in which this power can be effectually exerted is, in civil society, under the permanent and systematic direction and control of one or a few individuals exercising exclusive and supreme authority over all the other members of the community. Though, in conformity with established usage, the expression *state of nature* is employed to denote a state of things opposed to civil society, yet, as it has been frequently observed, the latter is, in truth, the natural state of men, because it is their ordinary condition in all ages, the state in which Providence designed that they should live, and one which they are as surely and unerringly led by their natural wants and instincts to adopt, as property, marriage, or any other institution confessedly natural. The circumstance that some voluntary act on their part is necessary at first to establish the relation of sovereign and subject, does not in the least detract from its natural character; unless we choose to say, that property and marriage are not natural, but adventitious, because they originate respectively in the voluntary acts of occupancy and consent. (Nos. 98, 113.) Indeed it would be difficult to name any natural right, which has not, in some respect or other, a voluntary commencement.

Ends of the institution of civil government.

How then does the right of sovereignty begin? Before attempting to answer this question, it will be desirable to have a clear and correct notion of the *ends* to be answered by the institution of civil government. Now a mistake is very commonly committed in supposing the furtherance of justice to be ultimately the sole end; as if (to borrow an expression of Hume*) obedience to civil government were a *new* and *fictitious* duty, invented to

support the *primitive* and *natural* duty of justice. Law. The cultivation of the habit of obedience or submission is itself an independent and co-ordinate object, essential to the perfection of man; who, as a created and finite being, must for ever be dependent on the will of his Creator. Now right submission does not imply an abject and idolatrous deference to mere *power*: to be genuine or praiseworthy, it must be based upon an intelligent and right apprehension of the *wisdom* and *goodness* of Him to whom it is due; uniting the two opposite, and seemingly contrary, qualities of independence and deference;—of independence, inasmuch as such submission is freely and willingly, and rationally yielded; and of deference, because, though not blind, it is profound and implicit. Next, with respect to the mode of cultivating and gradually developing this manly principle; a necessity of habitually bending to the will of some visible ruler materially tends to render men submissive. But were this ruler greatly their superior in wisdom and goodness, his superiority would probably awe them into an undiscerning and childish subjection, with but little prospect of advancement or improvement, or union with manly freedom. If again, without any such superiority in wisdom or goodness, the person of their ruler, and the precise commencement, termination, and extent of his authority, had been so exactly and manifestly defined by the Deity, as to leave no room for doubt or mistake, it is not difficult to divine the probable result:—men, ever prone to run into one extreme or other, would permanently become, in heart and disposition, either abject slaves or downright rebels: for unqualified submission could scarcely be rendered to such a ruler as last supposed without a wilful blindness to all his imperfections, or rather, perhaps, without a total surrender of all freedom of thought and action; and yet, in the face of an authority so unquestionable, to venture upon any the slightest or fairest criticism of his acts of government would be to stand clearly self-convicted of undutiful disaffection. It was probably with a view to avoid these extreme consequences, but at the same time to afford men proper and fitting objects and opportunities for the cultivation of the virtue of submission, that they have been subjected to the rule of individuals of their own species undistinguished from the mass by any extraordinary natural endowments, and that a certain indeterminateness, not apparent, but real, hangs over every thing connected with the natural origin, extent, and limits of human sovereignty.

140. If the preceding argument be well founded, it is idle to refer the origin of sovereign right exclusively to any one definite mode of commencement. Without either affirming or denying absolutely any of the favourite and conflicting theories on the subject, it will suffice to observe that there is much of truth in each of them, in the most slavish, as well as in the most licentious. When carried to their extreme lengths, or adopted exclusively, they either become objectionable, or afford but a very lame and imperfect account of the matter they profess to clear up. The doctrine of a Divine Right affords no explanation whatever of the legitimate mode of acquiring a title to sovereignty, or of the just limits of sovereign power: when properly understood, it implies nothing more than what every believer in a superintending Providence must hold: as for the stolid and extravagant lengths to which it was formerly carried, nothing will be expected from us here. Some derive civil from parental authority. This specious but fanci-
Origin of sovereign authority.
Divine right.
Parental authority.

* Essay on the Origin of Government.

Law. ful theory (so far as it applies to the title of modern dynasties) has been completely disposed of by Locke; nevertheless many still pertinaciously adhere to it. Let us see what they profess to tell us. That originally the parent, himself independent, ruled over his own children; that by degrees his family increased into a tribe; and at length, from a tribe, into a nation; and that the power, which he at first wielded as a father, naturally expanded itself, in him or his heirs, into the authority, first of a chief, and then of a king. So far, well. But why stop there? Why not carry the history on to its natural conclusion? In process of time, this kingdom becomes too large and unwieldy to be governed by one man, and falls to pieces by its own weight; or sections of the people separate from the parent stock, and set up for themselves; or else the people become dissatisfied with their sovereign, and, justly or unjustly, depose him and his immediate family. In all these cases, some new expedient must be hit upon, to procure a legitimate sovereign. But it never will be pretended that none but the next heir in succession of the original progenitor is capable of lawfully succeeding, for this would be to brand with illegitimacy, in all probability, every sovereign of the present day: and yet such must be the conclusion, or else the theory, as an universal theory, means nothing. Another favourite mode of making out the right of supreme dominion commences by asserting the original independence of all men; a state which is put an end to by their own voluntary choice and consent, each individual transferring the natural power he possesses to one person, whom all agree to treat as their common sovereign. It is curious to observe how this pretended natural independence of mankind, which is made the foundation of a right by delegation, might just as readily be made the foundation of a right by assumption. This will appear evident, when we consider that complete independence implies two things, an unlimited right of self-protection, and an unfettered right of private judgment. Now the rights of no individual can receive full protection, without the regular and systematic punishment of all offences by whomsoever and against whomsoever committed (No. 133.); and this implies the exercise of sovereign functions. So that any naturally independent individual has a perfect right, if he only has the ability, to assume supreme and exclusive authority over all. In exercising sovereign functions, he does only what is necessary for the full protection of his own rights; and in preventing the application of any other form, rule, or measure of enforcement, except his own,—in a word, by assuming and maintaining exclusive dominion, he merely asserts his right of private judgment. Thus, according to the doctrine of natural independence, might becomes right.

Parallel between government and property. 141. A strong analogy holds between the dominion over persons, and the dominion over things; only the comparison must not be pressed too rigorously. It is with government, as with property, of very little moment how it commences, or in whom it becomes vested; but it is of the last importance, that, once completely vested in a particular individual, it shall not easily be disturbed. Occupancy, we have seen (No. 113.), is the origin of the right of property, the natural title to which at once becomes full and perfect, the moment possession is taken, provided there be no adverse right. But it is otherwise with respect to government. If the throne, in any country, happens to be vacant, either in consequence of a revolution, or in any other way, the

first occupant, who succeeds in becoming sovereign *de facto*, does not thereby acquire a full title *de jure*, though he has a nascent right. It is impossible for any one to be so far successful without the support of a considerable number of adherents possessed of weight and power; and his ability to conciliate them is in itself a proof of a superior fitness for command. It has frequently been maintained, both in ancient and modern times, that superior natures have a natural right to govern: such persons, it is said, attaining supreme command, "are not so much like men usurping power, as asserting their natural place in society." But at the same time, it must depend upon a variety of considerations that cannot be reduced to rule,—upon the necessity of the case, the individual's fitness for his office, the probability of a successful issue without a disproportioned cost, and in some measure, on his motives,—whether he has, or has not, a right voluntarily to assume the reins of government at all: under such circumstances, the question of right, with respect to him, is not for lawyers, but for moralists and casuists; and it is for individuals themselves to determine, on their own moral responsibility, whether and how far it is their duty to yield obedience, or incur risks and penalties if they withhold it.

142. Passing on from the first origin to the more advanced stages of supreme power, whatever doubts may be entertained respecting its commencement, every day's continuance of it unbroken has a natural tendency to mature it into a full and perfect right. This will be apparent, if we consider the natural effect of time, which is, to improve the relation that subsists between a *de facto* sovereign and his subjects, by obliterating the remembrance of any real or supposed injustice attending his first rise to power, and by begetting and encouraging habits of submission, generous confidence, and loyal attachment towards him on the part of his subjects, sentiments which will enable him to pursue a milder policy than perhaps the object of maintaining his power, in the teeth of distrust and jealousy, permitted at first. The most satisfactory title, then, of sovereign dominion, is that by Prescription. It will be borne in mind how prescription was applied to the right of property, in the last section. Almost every argument and conclusion there offered might be here nearly repeated, with reference to the present subject, by making a few simple alterations, which the reader will be able to do himself. (Nos. 117, 118.) The like may be said with regard to the right of inheritance (No. 117.); as to which, some not very profound thinkers are perplexed to know, why the fact of an individual reigning now, should give his son a title to reign after him. Now the benefit of the public, for which kings reign, is the principle upon which we would wish to rest this as well as every other circumstance connected with sovereign right; and, accordingly, we say, that it is *the subjects'* privilege that the son should succeed to the father; first, because it encourages the latter to govern his kingdom well, in order to transmit it in a flourishing condition to his posterity, and, secondly, because men naturally feel a stronger attachment and respect towards the descendant of an illustrious line of kings, though of but ordinary capacity, than towards an upstart: and this attachment often affords a stronger guarantee of good government on the part of him towards whom it is felt, than even the possession of superior qualities, without it. It is unnecessary, after what has been said in a former page, to defend this feeling from the charge of being a prejudice.

Law.

Progress of the right of government.

Prescriptive right.

Hereditary title.

Law.
Cessation
of the right
of sove-
reignty.

143. As to the mode in which the right of government ceases, reasons similar to those which govern the question between contending proprietors (No. 118.) must decide the just but indeterminate period of time for extinguishing the title of a sovereign *de jure*, unjustly dispossessed, and giving validity to that of an usurper, reigning *de facto*. It would be here superfluous to give those reasons in detail. One important difference, however, must be noticed. The exact period of prescription sufficient for this purpose may be precisely defined by a positive law, in the case of property, but cannot well in the case of sovereign right; because the sovereign himself is the law maker, or at least forms an essential part of the legislature, and it is not likely that he would by anticipation limit the period of his own right as a sovereign; nor indeed would it be safe to contemplate beforehand such a dreadful occurrence as a revolution, which must precede the practical determination of the question. The period of prescription, applicable to such cases, must, therefore, remain for ever in doubt, and be determined by the particular circumstances of each case as it arises. A like doubtfulness must for ever hang over the solution of that "tremendous problem"—What degree of tyranny will justify subjects in rising against their sovereign? The strict maintenance of the relation of sovereign and subject is of the highest importance to man's best interests. Allegiance, therefore, is one of his most sacred natural duties: it must bear up against much provocation from the injustice of a bad ruler. But as human nature is not infallible, as the most exalted station and sacred functions cannot absolutely exempt a man from the possibility of committing the greatest excess of wickedness, there is not strictly, in human nature, any such thing as an absolutely indefeasible right. Sovereign right itself furnishes no exception to this general principle. "There is a limit, at which forbearance ceases to be a virtue." But it is utterly impossible to define the limit. Those who have made the attempt, all come to some vague conclusion, expressive of a case of the most dire necessity, when the very existence of the commonwealth is in jeopardy, when the sovereign has become the *enemy* of his people. The question itself should never be approached or discussed, except with the utmost caution, and in the most guarded terms. The grounds of resistance can never, with propriety or safety, be made an expressed topic in public law; for the aim of every constitution must be to prevent the occurrence of an event requiring such an extreme remedy, rather than to pave the way for it, by laying down rules of procedure, in anticipation of its occurrence.*

Resistance
in extreme
cases of
tyranny.

Functions
of govern-
ment.

144. Having now disposed of the difficult question of the *right* of government, we have next to explain, very briefly, its several functions or powers; which are threefold: the legislative, the judicial, and the executive. These three powers are, in all well-ordered states, kept

distinct, by being vested in the hands of distinct bodies or persons.

145. The functions of the legislature consist in ascertaining and defining, by general rules, the rights and obligations of all the members of the state. The laws or acts of the legislature, made for this purpose, may be regarded as so many standing interpretations of the laws of nature, applicable to the particular circumstances of the community in general, and absolutely imperative upon every member of it.

146. It belongs to the judicial power to interpret and apply the acts of the legislature to every particular case as it arises. It is generally considered to be a matter of importance to narrow the discretion of judges within strict limits. But as legislation is conducted on a *general* survey of what is good for the subjects of the state, and as many minute points must necessarily escape observation before they arise in practice, it follows that a number of subordinate questions, capable of being resolved in a variety of ways, are left for judicial determination; and thus, it is not uncommon for judges to become, in effect, legislators, to a certain extent; that is, where the practice prevails of succeeding judges adopting the decisions of their predecessors, as a general rule, and so giving a permanent authority to such decisions. Bentham is loud and vehement in his vituperation of *judge-made law*; without much reason, we conceive: for though judicial discretion should be restrained within bounds, it neither can nor ought to be altogether prevented; and it is desirable in general that well considered decisions should be afterwards adhered to in like cases. There is, however, one important species of laws, neither judge-made, nor expressly enacted by the legislature, which form a considerable part of the jurisprudence of every country, and become regularly the subjects of judicial cognizance: we mean *customary laws*. A general and long established adherence to an ancient custom is considered as proof of its propriety and binding obligation, for otherwise it would be difficult to account for its extensive and permanent adoption. As it would be superfluous for the legislature to express, by any formal act, the obligation of customs already well known and commonly received, they are properly allowed the same force as express enactments.

147. The executive functions of government may be briefly summed up in the actual administration of force, for whatever purpose it may be required, whether for the execution of the laws, or for the internal or external defence of the realm. Under this head, also, it is usual to include all acts of administration, federal or otherwise, that immediately concern the state in its external relations with other states.

148. In preceding parts of this section, sovereignty was treated generally, as importing a right to direct and control permanently the actions of a multitude of persons forming a community or state. To be now more specific, its essence consists in the possession of supreme *executive* power; from which it follows that a sovereign must also have at least a substantive voice in the legislature, so as to make his consent necessary to every legislative enactment; for otherwise he is but the first subordinate executive officer in the state, under the supreme power of an independent legislative body, which would then in reality be the true sovereign. It is not essential to sovereignty to exercise judicial functions, though it is usual for the sovereign to name the

* Speaking of the English revolution, Sir W. Scott says: "The foreigner who peruses our constitution for the forms of procedure competent in such an event as the Revolution, might as well look in a turnpike Act for directions how to proceed in a case resembling that of Phaeton." *Life of Napoleon*, vol. viii. p. 204. On the general question of resistance, the reader may consult with advantage, Grotius, *De J. B. et P.* lib. i. ch. iv.; Puff., *L. of N. and N.* book vii. ch. viii.; Sir J. Mackintosh, *Hist. of the Rev. in Eng.* in 1688, ch. x.; Burke, *Appeal from the New to the Old Whigs*, Works, vol. vi.

Law.
Forms of
govern-
ment.

judges. The sovereign is the only person in the state legally irresponsible, in all respects.

149. The *form* of government depends chiefly on the form of the legislature. There are three simple forms; viz. a monarchy, an aristocracy, and a republic. In a monarchy, the sole legislative, and consequently, the supreme executive authority, is lodged in the hands of a single individual; in an aristocracy, it is lodged in the hands of a select council; and in a republic, in the hands of a number of persons connected with the mass of the people, commonly by some species of representation more or less extensive. From various combinations and modifications of these simple forms, several kinds of mixed governments may be constructed. It is not our purpose to enter into an examination of their several structures, or a discussion of their relative merits, a discussion, upon which political writers have given free scope to their fancy. Only this is to be observed, that a government, like ours, properly compounded of

all the three simple kinds, is admitted, on all hands, to be the best.

150. We must now conclude this Chapter, which has been extended far beyond its fair limits. In it, we have attended more to the fundamental principles which lie at the root of all legal obligations, and connect law with morals, than to the forms and modes of arranging and giving expression to those obligations. A knowledge of the superstructure of law in general is to be gained from an examination of the principal systems of particular jurisprudence, and especially of the great systems of positive law which may be considered to belong not to any nation in particular, but to the civilized world in general. THE LAW OF NATIONS, though not the first in point of time, considered as a formed and regular system, stands first in the extent and universality of its obligation; and therefore it shall form the subject of the next Chapter.

Law.
Conclusion.

LAW.

CHAPTER II.

THE LAW OF NATIONS.

Law.
Definition
of the law
of nations.

151. THE Law of Nations* is that law by which the relative rights and duties of nations, whether belligerent or neutral, at war or peace, are defined and enforced.

It is not to be confounded with the *Jus Gentium* of the Roman civilians, who by that term intended what has been usually understood and discussed as the law of nature. *Quod naturalis ratio inter omnes homines constituit, id apud omnes peræque custoditur, vocaturque jus gentium.* (D. i. 1, 9.) We have already remarked (45.) that the law of nature, according to the civilians, embraced the whole animal kingdom in its operation; with us the law of nature is, like the *jus gentium*, considered as operating upon mankind alone.

In proceeding to examine the system, we shall discuss (1) its origin, (2) its history, (3) its sources, (4) its authority, (5) its general rules, (6) its sanctions.

Sect. 1. Origin of the Law of Nations.

Opinions
of Hobbes
and Puf-
fendorf;

152. On this subject considerable difference of opinion subsists amongst jurists: Hobbes and Puffendorf† deny that the distinction between the law of nations and the law of nature is otherwise than verbal, and affirm that "what, speaking of the duty of particular men, we call the law of nature, the same we term the law of nations when we apply it to whole states, nations, and people."

of Grotius; Grotius, on the other hand, treats the distinction as real and substantial: "When," he says, "many men of different times and places unanimously affirm the same thing for truth, this ought to be ascribed to a general cause, which in the question whereof we are treating can be no other than a just inference drawn from the principles of nature or a universal consent. The former shows the law of nature,‡ and the latter the law of nations." "That," he adds, "which cannot be deduced from certain principles by just consequence, and yet appears everywhere observed, must owe its origin to a free and arbitrary will." The authority and origin of the law of nations he ascribes to "the will of all, or, at least, many nations;" its proofs, he affirms, to be "the same as those of the unwritten civil law, viz., continual use and the testimony of men skilled in the law."§ Vattel,

and of
Vattel.

* Dr. Zouch, an eminent English civilian, has suggested the title *Jus inter Gentes*, or, as we translate it, international law, as more appropriate; and the suggestion has been adopted by the Chancellor D'Aguesseau, Mr. Bentham, and Dr. Wheaton, the author of the latest, and in some respects the most useful work on the subject. Doubting, however, with Sir James Mackintosh, "whether innovations in the terms of science always repay us by their superior precision for the uncertainty and confusion which the change occasions," we have thought it preferable to retain the name by which this system is best known.

† Hobbes, *De Cive*, ch. xiv.; Puff. *Law of Nature and Nations*, ii. iii. xxiii.

‡ *De J. B. et P.* prel. xli.

§ *Omni in re consensio omnium gentium lex naturæ putanda est.* Cicero, *Tusc. Quæst.* lib. i. cap. xiii.

perhaps the most popular writer on the subject, seeks to reconcile these discordant opinions. He observes* that "the application of a rule cannot be reasonable or just unless it is made in a manner suitable to the subject. We are not to believe that the law of nations is precisely and in every case the same as the law of nature, the subjects of them only excepted, so that we have only to substitute nations for individuals. A civil society or state is a subject very different from an individual of the human race, whence, in many cases, there follows, in virtue of the law of nature itself, very different obligations; for the same general rule applied to two subjects cannot produce exactly the same decision when the subjects are different, since a particular rule, which is very just with respect to one subject, may not be applicable to another. There are many cases, then, in which the law of nature does not determine between state and state, as it would between man and man. We must therefore know how to accommodate the application of it to different subjects; and it is the art of applying it with a justice founded on right reason that renders the law of nations a distinct science."

153. In proceeding to examine the position so broadly laid down by Hobbes, we find that he bases it on an assumption which is altogether at variance with truth, and which he states thus: nations, when once instituted, become endued with the personal properties of individuals, and are therefore, like individuals, subject to the law of nature. This fashion of considering states as beings possessed of the qualities and capacities of individuals, and, consequently, subject to the same general law, is gravely vindicated by Sir James Mackintosh,† who observes that, so to consider them, is no "fiction of law," but "a bold metaphor," and pregnant with the useful moral, that states in their dealings with each other should respect those great principles of justice which avowedly ought to regulate the intercourse of individuals. The metaphor is bold, and may carry a moral, but most assuredly metaphors form no safe foundation for reasonings, and, if we may credit history, have in every age served only to perplex and obscure the researches of the philosophical inquirer.

Plainly to speak, we may safely affirm that states differ from individuals in every quality in virtue of which individuals are subject to the natural law. A state is a metaphysical entity, a mere abstraction, a general term which convenience has dictated to express the combined action of a number of individuals. Thus where a war is declared by the sovereign of one state against the sovereign of another state, it implies that the whole nation

What a
state is.

* *Droit des Gens*, prel.

† *Discourse*, p. 7. *Nescio quo modo nihil tam absurde dici potest, quod non dicatur ab aliquo philosophorum.* Cicero, *De Divin.* lib. ii.

Law.

declares war. "Every man," says Chancellor Kent, "is in judgment of law a party to the acts of his own government; and a war between the governments of two nations is a war between all the individuals of the one and all the individuals of which the other nation is composed." 1 *Comm.* 55. A state has no conscience, no capacity of suffering, no mortal attributes whatever.

The members of a state (that is, those whose union constitutes the state) are, in their capacity of individuals, of course, subject to the natural law. From the operation of this law their quality of citizenship, which subjects them also to the authority of a municipal law, exempts them in no degree; and this natural law regulates their conduct in all possible conditions and relations of life, as sovereigns and as subjects, and in their transactions with the sovereign and with the subjects of their own or any foreign state. The authority of the law of nature is in fact co-extensive with mankind.

The law of nations distinct from the law of nature.

154. The law of nations is, however, very distinct in its character; its rules are not the same, its sanctions are wholly unlike, its obligation is limited. As is every system of law, it is of course based on the principles of the law of nature; but these, if it defines, it modifies and enforces with its own peculiar sanctions. Indeed, writing not of ethics but of law, of *what is* and not *what ought to be*, we must acknowledge, that so far from the law of nations being identical with the law of nature, the former confessedly permits, regulates, sanctions, and approves of transactions wholly repugnant to the sovereign and fundamental principles of the latter. An instance of this will at once suggest itself to the mind of the reader in any degree familiar with the history of modern diplomacy. In an elaborate and most masterly judgment of the supreme court of the United States of America, the slave trade is admitted to be "contrary to the law of nature." Still in this trade for a long series of years the principal and most enlightened nations of Europe have been engaged; the trade then was sanctioned by their usage, and, with such a sanction, could not be pronounced contrary to international law. The fact that some of these nations have lately forbidden the pursuit of this traffic to their subjects, and have entered into treaties with one another for the abolition of the commerce, would not operate so as to render it unlawful as far as the subjects of other states were concerned. The municipal regulations of no state can operate upon foreigners beyond its frontiers, nor can the special arrangements of any two or more states limit the independent rights of the others.*

The language of Sir William Scott, in reference to the same subject, was similar.† This great judge has well expounded the relations which subsist between the two systems, the law of nations and the law of nature. "A great part of the law of nations," he says, "stands on the usage and practice of nations, and on no other foundation: it is *introduced*, indeed, by general principles, but it travels with these principles only to a certain extent, and, if it stop there, you are not at liberty to go further, and to say that more general speculations would bear you out in a further progress; thus, for instance, on mere general principles, it is lawful to destroy your enemy, and mere general principles would make no great difference as to the manner in which this was effected, but the conventional law of mankind, which is evidenced in

their practice, does make a distinction, and allows some and prohibits other modes of destruction, and a belligerent is bound to confine himself to those modes which the common practice of mankind has employed."*

155. The law of nations originates in the will of nations; its authority is their consent, and its evidence is their practice and conventions. Practice evidences what is called the *customary* law of nations, and conventions the *conventional* law of nations.

Law.

Origin and evidence of the law of nations. Its divisions.

(1.) The larger proportion of the law of nations owes its origin to the practice and usages of nations. Custom is a prolific source of municipal jurisprudence, and Lord Erskine used to affirm, somewhat rashly it is true, that "there is no other law in England than custom."† The obvious conveniences of certain rules has obtained for them, in the intercourse of civilized communities, an acceptance and consequent authority inferior in no degree to that which attaches to the results of express compact and solemn convention. In so far as this acceptance extends, so far extends the operation of the rules thus recognized. Some of these rules are partial in their operation, because the usages which have originated and evidence them are only local in their nature, such, for instance, as is the case with the customary law relating to the whale fishery.‡

It is competent for any nation, that so pleases it, to renounce any of her customs, and so to exempt herself from the jurisdiction of such portions of the law of nations as those customs warrant. A timely notice of her intention so to do has usually been esteemed proper on the part of the state.§

(2.) The conventional law of nations is recorded in the treaties to which at different times the various independent powers have become parties. These are considered to have an operation beyond the contracting parties. The circumstance that they are for the most part constructed in reference to the same principles and by diplomatists educated in the same school of public law, and referring as authorities to the same writers, has given them an authority in the resolution of vexed questions, and in the exposition of what the law actually is upon any point which may have been doubted.||

Having thus explained the origin of the law of nations, we proceed to trace its history.

Sect. 2. History of the Law of Nations.

156. The Law of Nations is the natural result of a state of comparative civilization, in which the benefits of commerce and international intercourse are known and appreciated. By the Greeks every foreigner was esteemed a barbarian, and barbarian and enemy were synonymous terms; without an express compact, men in their belief owed no duties to each other, and we may reasonably refer the indignation expressed by the Ithacans against Eupheithes for having joined the plundering Taphians against the Thesprotians, to the fact of a

Amongst the Greeks.

* The "Flad Oyen," 1 Robinson, *Rep.* 140.

† J. J. Park, *Contre-projet to the Humphreysian Code*, p. 81.

‡ Fennings v. Lord Grenville, 1 Taunton, *Rep.* 248.

§ *Ibid.* Martens, *Précis du Droit des Gens Moderne de l'Europe*, liv. ix. sec. 5.

|| 1 Wheaton, *Elements of International Law*, 60, 61; Martens, *Précis*, Introd. sec. 3, liv. ii. ch. ii. sec. 5; Klüber, *Droit des Gens Moderne de l'Europe*, tit. prel. ch. i. 84. For an instance of treaties being taken as declaratory and expository, see Emerigon, *Traité des Assurances*, ch. xii. sec. 35.

* The "Antelope," 10 Wheaton, *Rep.* 120.

† The "Le Louis," 2 Dodson, *Rep.* 238.

Law. league subsisting between themselves and the latter nation. There is much force in the expression—

*οἱ δ' ἡμῶν ἄρθροισι ἦσαν.**

Greek public law.

Subsequently the mutual relations of Grecian states were subjected in some degree to regulations which, being religious in their character, were enforced by the *βουλή* or council of the Amphyctionic league, itself a religious institution. The public law to which we refer may, according to M. Sainte Croix, be reduced to the following principles:—(1) those who perished in battle were to be buried; (2) no permanent trophy was to be raised in commemoration of a victory; (3) the lives of such as take refuge in the temples of a captured city were to be spared; (4) burial was to be denied to the sacrilegious criminal; (5) no molestation was to be offered to the Greek resorting to the public games and temples and there sacrificing.† The Amphyctionic league itself was, however, a party to the most cruel and oppressive acts when the interests of its favourite Delphians were in question. Witness its conduct in the first Sacred War, when at its instigation the Athenians, by an artifice of Solon, succeeded in poisoning the Crisseans, and levelling their city with the ground.‡

Amongst the Romans.

157. The Romans appear to have made but little advances towards recognizing an international law, although of such a system Cicero conceived the necessity, and advocated the adoption, whilst his great rival, Sallust, did not hesitate to denounce a Numidian massacre, by Marius, as *contra jus belli*. These facts, with the circumstance that the Romans established a college of heralds, and instituted a fetial law, have led writers to infer that amongst this people subsisted just notions of the relative duties of states; but an examination of their history will show the supposition to be erroneous. Some formalities§ were certainly adopted in the declaration of hostilities, unknown as it would appear to the Greeks; the relation of allies was recognized in principle, however neglected in practice, but the moderation towards the vanquished, certain fruits of international law, ascribed by the patriotism of Seneca and Tacitus to the founder of Rome, was, in fact, no more than the policy of an infant state anxious to increase its strength by the incorporation of conquered enemies, and was found no ways incompatible with the Roman victor dragging at the chariot wheels of his triumph the kings and senates whose captivity he had purchased with his sword. With the fetial law we are only imperfectly acquainted, but it seems to have been simply a law peculiar to the Romans, and regulating their conduct towards foreigners, and not properly an international law common to other nations, and determining on general principles their respective rights and duties.

Roman Fetial law.

International law of the Middle Ages.

158. If the middle ages were distinguished by a larger appreciation of the rights of nations, and an approach to the true principles of a law of nations, it is owing chiefly to the diffusion of Christianity through Europe. According to Barrington,|| merchant strangers were by the laws of the Wisigoths not only protected, but permitted to enjoy the benefit of their own laws. In Sicily the plunder of shipwrecked goods was made a capital offence; whilst the Bavarians assured safety and immunity to all foreigners entering their territories. Our

Magna Charta also (cap. xxx.) extended protection and offered encouragement to merchant strangers. The extension of commerce* and the rise of a spirit of community in Europe, to which the Crusades and the system of chivalry gave birth, originated by degrees the rudiments of an European public law and treaties, formal declarations of war, and a more general recognition of the rights of ambassadors were all so many evidences of the march of civilization. This public law, or law of nations, to the development of which contributed, in a great degree, the study of the Roman jurisprudence, was considered as operating only upon Christian nations; and the Pagan and his possessions were still considered the lawful prey of the Christian conqueror. The Roman church, to whom this perversion of Christian doctrine is fairly attributable, was, however, an important agent in the establishment of a law of nations in Europe; and this not simply by the impetus which it communicated to European civilization, but also by constituting a species of European social system of which it became the head. It is difficult to overrate the importance of such a central power which, being independent and antiphysical, and owing its existence wholly to moral influences, was able and disposed to become the arbiter of national differences.

Law.

The Roman church an agent in establishing a law of nations.

159. The perfection of the law of nations has been owing chiefly to two causes: First, to the formation of the political system of modern Europe; and, secondly, to the writings of Grotius and other eminent publicists.

Causes of the perfection of the law of nations.

160. (I.) The political system of modern Europe originating in the relations that have arisen amongst European states is referred by Heeren† generally to the progress of civilization, which necessarily multiplies the points of contact between neighbouring states, and specially to four causes: (1) the Italian wars; (2) the affairs of religion after the Reformation; (3) the necessity of opposing the Turks; (4) the commerce of the colonies, and the commercial interests resulting therefrom. To which he properly adds (5) the facility of communication which printing and the establishment of posts afford. Another bond of union amongst European nations was the similarity of their laws and habits

The political system of modern Europe.

* Some notice must be taken of the maritime codes which contributed greatly to develop the elements of international law in Europe. The Rhodians—a people to the extent of whose commerce and power Cicero, who studied amongst them, bears testimony (*Orat. pro Leg. Manil. xviii.*)—possessed a code of this kind, which was recognized at Athens, in all the islands of the Ægean, and throughout the coasts of the Mediterranean. The Roman emperors professed to embody this code in their jurisprudence. It was, perhaps, the foundation of the famous *II Consolato del Mare*; the merit of compiling which has been contested by Pisa, Venice, and Barcelona, and which was first published in the Catalan dialect in the latter place in 1494. Although the jurists of the latter city may have conferred on this code the form in which it is known to us, it is probable to have emanated in the first instance from Pisa. Its antiquity is probably not much earlier than the middle of the thirteenth century. By it were defined the mutual rights of neutral and belligerent vessels; and so far is international law indebted to its framers. Its authority was formally recognized by the King of France and the Count of Provence, and its operation extended to all the Mediterranean states. Regulations borrowed from this source, and which were compiled in France under the reign of Louis IX., were recognized in this country. Known as the laws of Oleron, it has been supposed that we owe them to the wisdom of Richard I. In addition to these, we may mention the ordinances of Wisburg, a town in the island of Gothland, chiefly compiled from those of Oleron, just before 1400: by these the Baltic traders were governed. Hallam, *Middle Ages*, ch. ix. part ii. vol. iii. pp. 396-8.

† *Manual of the History of the Political System of Europe*, vol. i. p. 7.

* *Odys.* lib. xvi. 427.

† *Gouvernemens Fédératifs.*

‡ *Æschines* con. Ctesiphon, 125; *Pausanias*, x. 37, 84.

§ *Cicero, De Re publicâ*, lib. ii. 17.

|| *Observations on Statutes, chiefly the more ancient*, p. 22; Dublin edit.

Law arising from their common origin. It may finally be observed that, as the power of the Roman church declined, the Germanic empire became the centre of the European system, to the solidity and compactness of which it greatly contributed. This end was secured, moreover, by the circumstance that monarchy was the prevailing form of government in Europe, and the direction of public affairs being entrusted to the hands of a few, greater steadiness of international policy was secured than was compatible with the character of democratic institutions.

161. (II.) The second cause of the perfection of the law of nations is owing to the writers on the subject. Some of the principal of these we propose here to enumerate.

Francisco a Victoria. 162. Francisco a Victoria, a Salamanca professor, who commenced teaching at Valladolid in 1525 (reputed the restorer of theological learning in Spain), discusses, in his *Relectiones Theologicæ* (Lyons, 1557), the general right of war: the difference between public war and reprisals; the just and unjust causes of war; its proper ends, and the right of subjects to examine its grounds. He bases the rights of the king of Spain over the American Indians on the ground (*inter alia*) that the Indians had denied permission to trade in their country, which he esteems a just cause of war, but denies that the war was just simply because the Indians were Pagans.

Dominic Soto. 163. The next writer on this branch of jurisprudence claiming a passing remark is a pupil of Victoria, Dominic Soto, a Spanish Dominican, distinguished for the active part he took in the Council of Trent against both the Papal and Scotist factions. He was consulted by the Emperor Charles V., to whom he was confessor, on occasion of a conference held before him at Valladolid, in 1542, at which Sepulveda appeared as the champion for the Spanish colonists and Las Casas as an advocate for the oppressed American Indians. The opinion of Soto was conformable to that of his great master,—“*Neque discrepantia (ut reor) est inter Christianos et infideles, quoniam jus gentium cunctis gentibus æquale est.*” His celebrated work, *De Justitia et Jure*, from which this passage is taken, was published in 1568. Soto condemned in strenuous language the disgraceful atrocities of the slave trade.

Balthazar Ayala. 164. Probably the earliest work in which the practice of nations in time of war is to be found fully discussed is one by Balthazar Ayala, judge advocate (if we may so use the term) to the Spanish army in the Netherlands, under the command of the Prince of Parma. It is entitled *De Jure et Officiis Bellicis et Disciplinâ Militari*, lib. iii.; was published about 1582; and was highly commended by Grotius. Like Victoria and Soto, the author denies the lawfulness of levying war against infidels, even by the authority of the Pope, on account of their religion, for their infidelity does not deprive them of the right of dominion, inasmuch as the sovereignty of the earth was given to every reasonable creature. “*Et hæc sententia*,” he adds, “*plerisque probatur, ut ostendit Covarruvias.*”

Albericus Gentilis. 165. Albericus Gentilis, the next international jurist of note, was an Italian protestant, who, through the Earl of Leicester, obtained the professorship of civil law at Oxford. His work *De Legationibus* was first published in 1583, and another by him, *De Jure Belli* (Lyons, 1589), was imitated in its plan by Grotius in the first and third books of his great work.

We may also mention the name of the renowned jesuit, Francisco Suarez, the casuist, who flourished in

the latter part of the sixteenth century, and who by his treatise, *De Legibus ac Deo Legislatore*, proves that he was the first to see, according to Sir James Mackintosh, “that international law was composed not only of the simple principles of justice applied to the simple intercourse between states, but of those usages long observed in that intercourse by the European race which have since been more exactly distinguished as the consuetudinary law acknowledged by the Christian nations of Europe and America.”*

Law.
Francisco
Suarez.

166. Of Grotius, and some of his more distinguished successors, we have already (Ch. I.) spoken. We cannot, however, pass over the name of Samuel Puffendorf, who (born in 1681) was chosen by Charles Lewis, the Elector Palatine, to fill the chair of Law of Nature and Nations at Heidelberg, the first chair of the kind instituted in Europe. His *De Jure Naturæ et Gentium* first appeared in 1672, and is a highly valuable work. The author was styled by Leibnitz *Vir parum jurisconsultus et minime philosophus*.

Samuel
Puffendorf.

167. Subsequently to this time have flourished the translator and editor of Grotius, Puffendorf, and Bynkershoek, Jean Barbeyrac, who translated Grotius into French, with notes, in the year 1724, Puffendorf (the best edition of which is that of London, 1740), and Bynkershoek. He added valuable notes to all his translations. Christian de Wolf (born 1679) was professor at Halle, 1707, and at Marbourg, 1723 to 1754. His *Jus Gentium* was published at Halle in 1749. Bynkershoek, a jurist of Holland, is a useful and learned writer of authority. His *De Dominio Maris* was published in 1702; his *De Foro Legatorum* in 1721. His principal work, *Quæstiones Juris Publici*, has been translated into French by Barbeyrac (Hague, 1724), and into English by that learned jurist, Dr. Du Ponceau, provost of the law academy of Philadelphia. Both of these editors have added valuable notes. Professor Von Martens is a distinguished jurist; and we may fairly add the name of a living writer, Dr. Henry Wheaton, long a reporter of the Supreme Court of the United States of America, a Scandinavian antiquary of no mean reputation, and who has filled, with credit to himself and with advantage to his government, the post of resident minister at the court of Berlin. His *Elements of International Law* was published in London in 1836. We ought not to omit all mention of Sir James Mackintosh, whose devotion to public life alone prevented him from acquiring high reputation as a philosophical and highly accomplished jurist.

Jean Bar-
beyrac.

Sect. 3. Sources of the Law of Nations.

167. The Law of Nations, according to Sir William Scott, is “fixed and evidenced by general, ancient, and admitted practice, by treaties, and by the general tenor of the laws, ordinances, and formal transactions of civilized states.”† It does not enjoy the advantage possessed by every system of municipal jurisprudence, of being expounded and enforced by an independent and impartial judiciary; and if in some points it should be deficient in that precision and certainty which, taught by one of his great oracles, the English lawyer is prone to esteem the highest perfection of every jurisprudential system,‡

Sources.

* *Dissertation on the Progress of Ethical Philosophy*, Whewell's edit., p. 110.

† The “Le Louis,” 2 Dodson, Rep. 249. See also Lord Mansfield's observations in *Triquet v. Bath*, W. Bl. Rep. 471.

‡ “Certainty,” says Lord Hardwicke, “is the mother of repose, and therefore the law aims at certainty.” *Walter v. Tryon*, 1 Dickens, 245.

Law.

1. Text-writers.

it is chiefly owing to this fact. The sources of the law of nations may be enumerated as follows:—

169. (1.) Text-writers of authority, an authority which they obtain whenever they record the usages and practice of nations, and whenever their speculations are conceived in a spirit of impartiality. This is a character which especially belongs to the great work of Grotius, and is consistent with the disposition of one whose candour and love of justice so far subdued his personal feelings as to enable him rightly to appreciate the character of his inveterate persecutor.* The writings of Puffendorf and the other publicists already mentioned have always been considered as forming a part of the *corpus* of international law. The degree of authority to which the writings of certain publicists is entitled will always be a subject of controversy. Such, for instance, as that of Selden, who in his work, *Mare Clausum* (1635), following in the steps of Alberic Gentili (*Advocatio Hispanica*, 1613), asserts the right of England to the sovereignty of the British seas. But generally it is to the works of jurists that the nations of Europe appeal as authorities in the determination of their mutual differences. Our own country has been distinguished by the deference she has on all occasions paid to these venerable and enlightened expositors of international jurisprudence.†

2. Decisions of international tribunals and courts of prize.

170. (2.) The adjudications of international tribunals as boards of arbitration and courts of prizes. The degree in which the authority of these adjudications is admitted depends of course in a great measure on the constitution of the tribunals from which they emanate. "Greater weight," says Mr. Wheaton, "is justly attributable to the judgments of the mixed tribunals appointed by the joint consent of the two nations between whom they are to decide, than to those of Admiralty courts established by and dependent on the instructions of one nation alone."‡

3. Ordinances of states.

171. (3.) Ordinances of particular states prescribing rules for the conduct of their commissioned cruisers and prize tribunals.§

4. Histories.

172. (4.) The history of all diplomatic transactions.

5. Treaties.

173. (5.) Treaties, whether of peace, alliance, or commerce.||

Sect. 4. Authority of the Law of Nations.

Subjects of the law of nations.

174. The subjects of the law of nations are what are usually designated *sovereign* states; that is, states which govern themselves independently of foreign powers. It is not necessary in this place to enter into any lengthened description of the signs and tokens in which this quality of sovereignty manifests itself. The right of negotiation is, as far as the law of nations is concerned, the

most important of these; and this is a right which may be modified by compact without the state forfeiting its title to be considered as a sovereign state. Such is the case with the states which form the Germanic Confederation, who, by the terms of their union, are precluded from negotiating a separate treaty, or even concluding an armistice, without the consent of the Confederation, with any power against whom the Confederation have declared war.* On the other hand, in the Swiss Confederation, while the Diet possesses the exclusive right of declaring war and effecting treaties, each canton is authorized to conclude for itself military capitulations and treaties relating to economical matters and matters of police, so long as they do nothing inconsistent with the federal part and with the rights of the other cantons. By the constitution of the United States of America, no state is permitted "to enter into any treaty, alliance, or confederation by itself, keep troops or ships of war in time of peace, enter into any agreement or compact with a foreign power, or engage in war, unless actually invaded or in such imminent danger as will admit of no delay."† The president has power, by the consent of the senate, provided two-thirds of the senators present concur, to levy war, make peace, and enter into treaties.‡ The right of negotiation, apart from compact, depends upon the internal organization of the state, and of this, to use an expression familiar to municipal jurisprudence, every other state is bound to take notice.

175. The authority of the law of nations, owing its existence to the will of nations, is necessarily limited to those nations that have evinced their willingness to be bound by it. To learn what nations have done so we must resort to history for information; but speaking generally, we may say that, as a system, it has been adopted by, and therefore binds, all the nations of Europe§ and their emancipated colonies in the other hemisphere. Formerly it would seem to have been doubted how far the Turkish empire could have been considered within its range;|| but modern events teach us to consider that power as a member, and no unimportant member, of the European commonwealth, and as clearly subject to the law of nations. As to those nations whose policy forbids our considering them in the same light, we may conclude that they are entitled to the protection of international law no further than what an enlightened appreciation of the natural rights of states will obtain for them. To treat these nations with cruelty—to violate their independence—to oppress their citizens—to despoil their dominions—these are acts so plainly inconsistent with the character of civilized nations—so wholly repugnant to the obvious dictates of natural justice, that the state who indulged in them would draw down on itself the just indignation of Europe; but at the

Law.

Right of negotiation.

Limits of the authority of the law of nations. European nations and their emancipated colonies. Turkey.

* Prince Maurice of Nassau. Gustavus Adolphus, during the war he undertook to secure the liberties of Protestant Europe, always slept with the Treatise of Grotius under his pillow. It was the study of Lord Bacon's writings, we are told by Barbeyrac, that first suggested to Grotius the idea of reducing the law of nations to a science. *Notes to Puffendorf*, sec. 2.

† See the report made to the king in 1753, in answer to the Prussian memorial, prepared by Sir George Lee, Dr. Paul, Sir Dudley Ryder, and Mr. Murray (Lord Mansfield). Smollett's *Hist. Eng.* ii. 301. Amongst the writers quoted in our courts are Barbeyrac, Vattel, Wicquefort (*Viveash v. Becker*, 3 Maule & S. 284), Bynkershoek, Grotius, and Puffendorf (*Wolf v. Oxholm*, 6 Maule & S. 192).

‡ 1 *Elem.* 57, 58.

§ 1 Kent, *Comm.* 70, 71.

|| *Ibid.* 59. Martens gives a list of the most useful works for a student of this science. Of course in modern times it may be advantageously enlarged.

* 1 Wheaton, *Elem.* 74.

† *Constitution of the United States of America*, art. i. sec. 10.

‡ *Ibid.* art. ii. sec. 282.

§ The law of nations is adopted in Great Britain in its full and most liberal extent by the common law, and is held to be part of the law of the land; and all statutes relating to foreign affairs should be framed with reference to that rule. 4 *Comm.* 67; *Ricord v. Bettenham*, W. Bl. Rep. 564; *Triquet v. Bath*, W. Bl. Rep. 471; *Hopkins v. De Robeck*, 3 Durnf. & East, 79; *Viveash v. Becker*, 3 Maule & S. 284.

|| Martens, *Précis*, *Introd.* sec. 4. It observed, however, according to this writer, none of the forms of diplomatic etiquette, liv. vii. c. vi. sec. 1. See also 1 Wheaton, 221, and Co. *Inst.* 155; Klüber, *Droit des Gens Mod. de l'Europe* 2. 1, ch. ii. sec. 5.

Law. same time these uncivilized states are not entitled to those formal courtesies or privileges which the comity of civilized nations consider as naturally due. Instances of the kind alluded to are such as respect the ransom of prisoners of war, the rights of ambassadors, &c.* They cannot expect to receive advantages which they deny to others.

Thus a formal sentence of condemnation by a court of admiralty was not considered requisite to transfer the property in a vessel captured by the Algerines, and subsequently sold *bonâ fide* to a Christian purchaser. It was held sufficient if the confiscation had taken place by a public act of the competent authority, according to the custom established in that part of the world.† This was the distinction between them and more civilized states. The African states were not, however, deemed as exempt from an observance of the law of blockade, and this for good and sufficient reasons. "On a point like this," said Sir William Scott, "the breach of a blockade—one of the most universal and simple operations of war in all ages and countries, except such as are merely savage—no indulgence can be shown. It must not be considered by them that if an European army or fleet is blockading a town or port that they are at liberty to trade with that port. If that could be maintained, it would render the obligation of a blockade perfectly nugatory. They, in common with all other nations, must be subject to this first and elementary principle of blockade,—that persons are not to carry into the port supplies of any kind. It is not a new operation of war; it is almost as old and general as war itself. The subjects of the Barbary States could not be ignorant of the general rules applying to a blockaded port so far as concerns the interests and duties of neutrals."‡

Indian states and Morocco. 176. The native princes of India appear also to have been considered as within the operation of the general principles of the law of nations;§ and, as it would seem by a case decided in the reign of Queen Elizabeth, the Emperor of Morocco also.||

177. In the earlier periods of European history, the rights of every Christian people to subject savage nations to their dominion, and to dispossess them of their territories, were not only vindicated in theory but asserted in practice. The popes, who considered themselves as lords paramount of the princes, were accustomed to secure the support and reward the fidelity of their royal subjects by ample donations of territory "*in partibus infidelium*." Thus did Alexander VI., in 1493, bestow on Ferdinand and Isabella of Spain the lands not previously occupied by a Christian nation, discovered and to be discovered beyond a line drawn from pole to pole one hundred miles west of the Azores; and thus did Nicholas V. confer the sovereignty of Guinea, and the

right of subduing its inhabitants, on Alphonso of Portugal and the Infant Henry.* The sovereigns thus favoured did not, it would seem, consider it altogether prudent to rest their title to their new acquisitions on these grants. The rights of discovery and of priority of occupation were frequently alleged by them. The lawfulness of spoiling the idolator is assumed in the patent granted by Henry VII. of England to the celebrated John Cabot (Giovanni Gavottæ) and his sons, who are thereby empowered "to seek out and discover all islands, regions, and provinces whatsoever that may belong to heathens and infidels," and "to subdue, occupy, and possess these countries as his vassals and lieutenants." Sir Humphrey Gilbert was authorized by Queen Elizabeth "to discover such remote heathen and barbarous lands, countries, and territories not actually possessed of a Christian prince or people, and to hold, occupy, and enjoy the same, with all their commodities, jurisdictions, and royalties."

Sect. 5. General Rules of the Law of Nations.

178. Before proceeding to state those principles to which the Law of Nations appears most readily reducible, it is desirable that some consideration should be given to two circumstances which jurists do not appear to have sufficiently attended to, and their neglect of which has, in many instances, deprived their speculations of that practical value which otherwise would have belonged to them. This neglect appears to have resulted from an imperfect appreciation of the true character and scope of the law of nations.

179. In the first place, it may be remarked that this system of jurisprudence in so many points approximates to the science of general politics, that it is difficult to ascertain its appropriate boundaries. By the science of general politics—a term tolerably intelligible, though assuredly not very precise or determinate—we would be understood to mean the science of civil prudence, or, as Lord Bacon styles it, civil knowledge; and which belongs to the province of ethics rather than of law. This science, being "conversant about a subject which of all others is most immersed in matter and hardliest reduced to axiom,"† is beset with so many difficulties of its own, that to confound it with the law of nations is only to perplex and impede the resolution of questions properly distinct in their character, and in themselves sufficiently complicated and embarrassing. States, it must be remembered, may not always enforce their just claims‡—privileges may be waived—positive injustice may be endured; at times things lawful may not be thought convenient; and on all occasions we cannot safely argue from the fact to the right, and conclude from what is done, what ought to have been done. With examples illustrative of this, the diplomatic history of Europe is

* 1 Wheaton, *Elem.* 52.

† The "*Helena*," 4 Robinson, *Rep.* 3. Formerly it was considered that there was no change of property in case of a recapture so as to bar an original owner in favour of a recaptor or his vendee until there had been a sentence of condemnation. (*Goss v. Withers*, 2 Burr. *Rep.* 696; *Lindo v. Rodney* and another, 2 Douglas, *Rep.* 616; the "*Flad Oyen*," 1 Robinson, *Rep.* 139.) See also 13 Geo. II. c. 4, and 29 Geo. II. c. 34.

‡ The "*Hurtige Hane*," 3 Robinson, *Rep.* 324.

§ *Elphinstone* and another *v.* *Bedreechund* and others, Knapp, *Rep.* 316. In the case of the Advocate-general of Bombay *v.* *Amerchund*, the authority of both Puffendorf and Bynkershoek were relied on in argument, Knapp, 329. See also Martens, *Précis*, liv. iv. c. ii. sec. 7, Note.

|| 4 Co. *Inst.* 152-4.

* 1 Wheaton, *Elem.* 209; Vattel, liv. i. ch. xviii. sec. 208.

† *Advancement of Learning*, p. 272; edit. 1633.

‡ Martens, *Précis*, liv. ii. ch. iii. sec. 1. As an instance of this we may refer to the reply of the British government (25 Sept. 1807) to the Russian declaration of war in 1807. This latter power insisted on the proceedings of the British fleet in entering the Sound and attacking the Danish capital, in disregard of the inviolability of the Baltic. In answer, the principles on which the inviolability of that sea had been rested were denied, although it was admitted that our government had at particular periods, for special reasons, forborne to act in contradiction to them. *Papers laid before Parliament*, January, 1808. On such doubtful questions, see some observations by Mr. Wheaton. 1 *Elem.* 96-7.

Law. rife; and there is nothing that the jurist ought more sedulously to avoid than ascribing a scientific value to what is the result simply of prudent policy—a policy which has regard less to positive rights than to times and occasions.

Distinction between international and public law. 180. In the second place, jurists have not sufficiently apprehended the true province of the law of nations which is conversant solely with the *mutual* relations of nations, and which is distinct from that public law* by which the internal organization of states and the rights and duties consequent thereon are defined and enforced. When Vattel wrote “of nations considered in reference to themselves,”† described the constitutions of various communities, and eloquently disserted on “the objects of a good government,”‡ he occupied himself with a subject distinct from “the Law of Nations.” A similar confusion of ideas is visible in the *Manual* of Mr. Wheaton. A charge so serious against a writer of such high reputation must not be hazarded without proof. Every state possesses, by the very terms of its existence, certain rights. These are incidental to its very character as a state. They arose antecedently to the law of nations, and subsist independently of it. They are recognized by the law of nations, which, indeed, for their very security, subjects them to certain modifications in practice. Such are the undoubted rights of every state to its security—its independence—its equality, and its property. In terming them the “absolute international rights of states,”§ Mr. Wheaton employs an improper term, for though these rights are guaranteed by international law, they cannot be called international rights, for their existence was prior to international law.

181. They are the rights of states *as states*; they result from the law of nature, and not from the law of nations; and to characterize them, as has been done by Mr. Wheaton, can only serve to perpetuate the confusion apparent in the writings of so many jurists who have been unable to distinguish the frontiers of natural and international law.

We proceed to show how the law of nations deals with these rights, and to exhibit the modifications to which it subjects them.

Right of security. 182. (I.) *The Right of Security*.—It is the undoubted duty of every state to protect its subjects from foreign aggression in the lawful exercise of their industry and the enjoyment of their property. Bound by this duty, every state, on general principles, is entitled to raise armies, build fortresses, and levy taxes which may furnish a revenue efficient for these purposes.

183. By convention, this right may be, and often is, limited. Thus, by a treaty with France, in 1685, the Genoese republic agreed to disarm a portion of its navy.|| By the treaty of Aix-la-Chapelle (1748), and of Paris (1786), the French agreed to destroy the fortifications of Dunkirk, so long the terror of our Channel trade; while, by the treaty of 1815, they covenanted to demolish the fortifications of Huningen, which for many years had menaced the city of Basle, and stipulated to raise none others within three leagues of that city. These may be thought the hard terms which the power of the conqueror enabled him to impose in the insolence

Law. of victory, but they were justified on the ground that they were requisite to the preservation of that very principle—the right of security—which they appeared at first sight to contravene.

184. It was long a moot question amongst jurists how far an interference is justified when a state already powerful is increasing her power to such an extent as to become an object of terror to her neighbours.* It is the unquestionable right of every state to multiply its resources, as well by internal improvement as by external aggrandizement, provided it does not violate the rights of other states. “Nevertheless,” says Professor Martens, “it may so happen that the aggrandizement of a state already powerful, and the preponderance arising from it, may sooner or later endanger the safety and liberty of the neighbouring states. In such case there arises a collision of rights, which authorizes the latter to oppose by alliances, and even by force of arms, so dangerous an aggrandizement, without the least regard to its lawfulness.”† Grotius, on the other hand, denies that “the dread of our neighbour’s increasing strength is a warrantable ground for our taking up arms against him;”‡ and with him Vattel§ concurs. The wars undertaken for the preservation of that famous system, known, from its operation, as *the balance of power*, naturally suggested this question. It is one on which no doubt can reasonably be entertained at this day. We have no right to interfere with a neighbour who is enlarging his dominions by colonization, or strengthening his frontier with fortifications, unless we have good reason to apprehend that he is meditating aggressions on us. If we have reason to suspect that his intentions are hostile, we shall naturally place ourselves in a posture of defence; but assuredly the naked fact that he is increasing his power, and by means in themselves perfectly legitimate, will give no title for our interference.

185. (II.) *The Right of Independence*.—This is a right which belongs to every sovereign state as such, and one especially tendered and protected by the law of nations. No nation, unless authorized thereto by compact, is entitled to interfere in the internal concerns of another, whether those concerns relate to government, legislation, or the administration of justice. Compact, indeed, in some cases modifies this right. In pursuance of treaties to that effect, the kings of Denmark were empowered to arbitrate in any differences that might arise between the kings of Sweden and their senates; and the kings of Sweden had similar authority in cases of disputes in the Danish government. The princes and states of West Friesland in like manner agreed to submit to the decision of the republic of the United Provinces matters on which they were divided; and many other cases of a similar kind might be mentioned.¶ A title of interference may also ensue treaties of mediation and guarantee. Thus, France and Sweden, at the peace of Westphalia, in 1648, guaranteed the Germanic constitution on the basis on which it was then settled; and the present constitution of the Helvetic Confederation was adjusted by the mediation of the allies, in 1813. So also may the constitution of any of the states composing the Germanic Confederation be guaranteed by the Diet, on the application of the state: the Diet then acquires a right

* Klüber, *Droit des Gens Mod. de l'Europe*, ch. i. p. 4.

† Liv. i. *passim*.

§ 1 Wheaton, *Elem.* 107.

|| Martens, *Précis*, liv. iv. ch. i. sec. 2, Note.

† Liv. i. ch. vi. xi. xiv

* 1 Wheaton, *Elem.* 108.

† See the authorities quoted in a note to Martens, *Précis*, at *sup. cit.*

‡ Martens, *Précis*, liv. iv. ch. i. sec. 3.

§ *De Jure Belli et Pacis*, lib. ii. ch. xxii. sec. 6.

|| Liv. iii. ch. iii. sec. 42. ¶ Vattel, liv. ii. ch. vii. sec. 52.

Law.

to determine the construction and enforce the maintenance of the constitution so guaranteed.* It is usual when intestine divisions vex a state, for those states with whom it is in amity to proffer their mediation of any intestine divisions which may distract any member of the great European commonwealth; and the acceptance of such an offer by both parties gives to the state offering the right to interfere.

The Holy Alliance.

186. There are other circumstances under which this right originates; and modern history has recorded the establishment of a league (at Vienna, in 1815), well known as the Holy Alliance, whose object was to check the progress of revolutionary principles and to sustain the menaced monarchical institutions of Europe. The justification of such an association is to be found in the peculiar circumstances of the times, which rendered it a measure of absolute necessity on the part of states not prepared to surrender their freedom of action. The considerations which govern the policy of the British government in such matters may be learnt from a declaration of the court of St. James's, issued in consequence of the steps taken by Austria, Russia, and Prussia, in consequence of the Neapolitan revolution of 1820.

Policy of Great Britain.

These were steps which the government of Great Britain esteemed inconsistent with the undoubted right of every nation, as far as its neighbours were concerned, to establish what constitution was agreeable to its wishes. Great Britain, on that occasion, admitted the right of interference so far as a regard to the independence of other states made an interference necessary, but still it regarded the assumption of such a right as only to be justified by the strongest necessity, and to be limited and regulated thereby. The exercise of such a right it considered as an exception to general principles of the greatest value and importance, and as one that properly grows out of the special circumstances of the case, but at the same time considered that exceptions of this description can never, without the utmost danger, be so far reduced to rule as to be incorporated into the ordinary diplomacy of states or into the institutes of the law of nations.†

French invasion of Spain in 1822.

187. The intervention of France with the affairs of Spain in 1822, which led to the overthrow of the Spanish constitution, was also protested against on similar grounds by the British government. It declared the original alliance between Great Britain and the other powers to have been "an union for the reconquest and liberation of a great proportion of the European continent from the military dominion of France, and having subdued the conqueror, it took the state of possession as established by the peace under the protection of the alliance. It was never intended as an union for the government of the world or for the superintendence of the internal affairs of other states."‡ "No proof was produced to the British government of any design on the part of Spain to invade the territory of France; of any attempt to introduce disaffection amongst her soldiery; of any project to undermine her political institutions; and so long as the struggles and disturbances of Spain should be confined within the circle of her own territory, they could not be admitted by the British government to afford

any plea for foreign interference. If the end of the last and the beginning of the present century saw all Europe combined against France, it was not on account of the internal changes which France thought necessary for her own political and civil reformation, but because she attempted to propagate first her principles and afterwards her dominion by the sword."*

Law

188. The interference of the Christian powers of Europe in behalf of the Greeks, when groaning under the oppression of Ottoman misrule, was without doubt prompted by a desire to rescue a gallant and Christian people, to whom Europe was so deeply indebted, from the fate to which the ambition and cupidity of their governors had subjected them. Instances might have been cited from the history of international law which would have sanctioned an interference on a religious ground. From the time of the Crusades to the sixteenth century, when Protestant states confederated together to secure the religious freedom to the Protestant citizens of Roman Catholic communities, examples of this kind have been gathered; but it was thought more prudent in the treaty of alliance (London, 6th July, 1827), to state the *casus fœderis* in other terms. In the preamble of the treaty it is set forth, that the contracting parties were "penetrated with the necessity of putting an end to the sanguinary contest which, by delivering up the Greek provinces and the isles of the Archipelago to all the disorders of anarchy, produces fresh impediments to the commerce of the European states, and gives occasion to piracies, which not only expose the subjects of the high contracting parties to considerable losses, but besides render necessary burdensome measures of protection and repression."

Recognition of Greek independence.

189. As a consequence of the independence of a nation, its laws affect and bind directly all property within its territory, and all residents, native and foreign, and all contracts made and acts done within it.† They may regulate the enjoyment and transfer of property within its territory, fix the civil rights, capacities, and states of those who become its subjects by birth, domicile, or even temporary residence; they may determine the validity of contracts and other acts done within it, and the legal import which those contracts shall bear; and they may establish the forms and proceedings by which the judicial tribunals may entertain, investigate, and adjudicate on claims, and permit the execution within its own territory of foreign judgments. They do not, *proprio vigore*, operate beyond the territory of their state.‡ but the comity of nations (*comitas gentium*, or, as it is sometimes styled, *la nécessité ou bien public et général des nations*) recognizes them as operating universally, so far as they do not prejudice the independence of foreign states or the native rights of their citizens.§ The bounds and nature of this operation it will be our business briefly to consider. This involves the discussion of the *jus gentium privatum* and the *collisio legum*, or *conflict of laws*, to which the attention of jurists has of late been frequently directed.

Authority of the laws of a nation.

190. The subject is one of considerable difficulty, but

* Mr. Secretary Canning to Sir Charles Stuart, 31 March, 1823. *Papers*, April, 1823, p. 57.

† Story, *Comm. on the Conflict of Laws*, ch. ii. sec. 1.

‡ 1 Burge, *Comm. Extra territorium jus dicenti impune non paretur*. D. 2, i. i. 20.

§ Instances of the adoption of this rule in England are to be found in *Robinson v. Bland*, 2 Burr. Rep.; *Holman v. Johnson*, Cowp. 341.

* 1 Wheaton, *Elem.* 132.

† Lord Castlereagh's Circular Despatch. *Papers laid before Parliament*, session 1821.

‡ Confidential Minute of Lord Castlereagh on the Affairs of Spain, May, 1820. *Additional Papers laid before Parliament*, April, 1823

Law.
Real pro-
perty.

the following summary is sufficient for the purposes of the present essay:—

(1.) As to real property. The title to real property, its tenure, and the forms by which it is conveyed, are governed by the law of the place in which it is situate. If any question respecting these is raised before a foreign tribunal, such tribunal will adjudicate according to that law.*

(2.) As to personal property. Foreign states recognize and give effect to those laws which affect personal property, which, as its situs is that of the domicile, are such as prevail in the place of domicile.

(3.) As to the state and capacity of persons. This is generally governed by the law of domicile as to acts done, rights acquired, and contracts made in the place of domicile. If they are valid there, they are valid everywhere else.† But as to acts done, rights acquired, and contracts made in other countries, the law of the country in which they are done, acquired, or made will generally govern in respect of the capacity, state, and condition of persons. Personal disqualifications arising from the customary or positive law of a country—such as the disqualifications resulting from heresy, Popish recusance, &c.—are not recognized by foreign states or their tribunals. In questions of legitimacy, the *lex loci* of the marriage will generally govern as to the issue subsequently born. Acts or contracts done or made in a foreign country in evasion of the laws of the native country will not necessarily be recognized by that country. The rule cannot be laid down more positively, as it is obviously a matter of discretion.

(4.) As to contracts; their form, interpretation, obligation. A contract, if valid in the place where it was made, is usually considered as valid elsewhere.

These are general principles, but they cannot be taken as invariably true; the exceptions in some cases being so numerous as to invite the doubt how far they are to be considered principles at all. Mr. Burge, in his *Commentaries on Foreign and Colonial Law*, and Mr. Justice Story, the American jurist, have exhausted the learning on this subject, and to their works the reader is referred for further information.

191. Sovereigns and their diplomatic representatives while in a foreign country are considered not amenable to its laws; so also is a foreign fleet or army while within the territorial jurisdiction of a friendly state. The vessels, public and private, of every nation on the high seas are considered as subject to the jurisdiction of their country.

With the exceptions above mentioned, the judicial power of every independent state extends,—

192. (1.) To the punishment of offences against the laws of the state committed by foreigners as well as subjects within its territory or on board its vessels on the open seas or in foreign ports, and to the punishment of piracy and other offences against the law of nations.

193. (2.) To all civil proceedings *in rem* relating to personal or real property within its territory; and

194. (3.) To all controversies respecting personal rights and contracts, or injuries to the person or property, when the party resides within the territory, wherever the cause of action may have originated.‡

* 1 Burge, *Comm.* 25-9. *Locus regit actum* is the maxim of law: see *Trotter v. Trotter*, 1 Wilson & Shaw, *Appeal C.* 407; and *Curling v. Thornton*, 1 Addams, *Rep.* 21.

† Story, *Comm.* ch. iv. sec. 101.

‡ 1 Wheaton, *Elem.*

195. (III.) *The Right of Equality.*—One of the fundamental principles of public law generally recognized is, according to Sir William Scott, “the perfect equality and independence of all distinct states. Relative magnitude creates no distinction of right; relative imbecility, whether permanent or casual, gives no additional right to the more powerful neighbour, and any advantage seized on that ground is mere usurpation. This is the great foundation of public law, which it mainly concerns the peace of mankind, both in their politic and private capacities, to preserve inviolate.”* This great principle is, however, somewhat modified in its application, both by the usage of nations and by compacts, whereby the forms of international etiquette are defined and established. These may be considered as they relate (a) to the relative rank and precedence of states, (b) to their titles, (c) to the language employed in their negotiations, and (d) to their maritime ceremonials.

196. (a) Certain honours, denominated royal, are enjoyed by the empires and kingdoms of Europe, by the Pope, by the grand duchies of Germany, by the Germanic and Swiss Confederations, and, as we presume, the United States of America.† The right to send an embassy of the first order, the right of preceding all other states, together with several distinctive ceremonies, make part of the *royal honours*.‡

By the Roman Catholic states precedence is yielded to the Pope, who by the Protestant states is considered simply as the prince-bishop of Rome. The text-writers appear disposed to consider the great republics inferior in point of precedence to crowned heads; but these states have never acknowledged their inferiority, and the question has been by tacit consent of all parties suffered to fall into obscurity. Indeed the whole question of relative rank, as far as concerns states enjoying royal honours, was felt to be so doubtful as well as so delicate that, on the discussion, in the Congress of 1815, of the report of a committee appointed specially to consider the subject, it was resolved to leave it in its original state of uncertainty. The usage at present, as it appears, may probably be stated thus:—

197. Monarchs enjoying royal honours, without being crowned heads, give precedence on all occasions to crowned emperors and kings.

Monarchs not enjoying royal honours yield precedence to such as do.

Below this last class rank demi-sovereigns or dependent states.§

198. This matter of precedence never having been formally settled, various expedients are resorted to in order to avoid an inconvenient contest at the period of a negotiation, without compromising the rights of any party. Of these the best known is the usage of the *alternat*, by which the rank and places of various powers is changed from time to time in a certain regular order, or one determined by lot.|| For instance, in treaties between two powers two copies are prepared, and each power, in the copy it keeps, is named first. If there are several parties to the treaty, the same number of copies is made, and the same usage observed. The right to this alternation has sometimes been denied to certain states.

* The “*Le Louis*,” 2 Dodson, *Rep.* 243.

† This state never accredits ministers of the first class to any foreign power.

‡ Martens, *Précis*, liv. iv. ch. ii. sec. 3. Vattel, liv. ii. ch. iii. sec. 37.

§ 1 Wheaton, *Elem.* 196.

|| *Ibid.* 197.

Law.
III. Rights
of equality.

Law.

The king of Great Britain refused to recognize the title of the King of Prussia to this privilege in 1742, and Hungary and Sardinia experienced great difficulties in obtaining admission to the alternation at the peace of Aix-la-Chapelle.* Sometimes it is arranged that the signatures to a treaty shall range according to the order assigned by the French alphabet to the respective powers parties thereto.†

Titles of states.

199. (b) While it is competent to any state to confer what title it pleases on its prince or chief magistrate,‡ it is not incumbent on foreign states to recognize that title, and sometimes when they do recognize it, a condition is annexed to the recognition. France and Spain recognized Russia as an empire, stipulating that the title of emperor should not affect the sovereign's precedence; but the Empress Catherine II., on her accession in 1762, refused to renew this stipulation, declaring, at the same time, that the imperial title should not alter the ceremonial between the two courts. France, in turn, renewed her recognition, with a declaration that if any such alteration were made, she would cease to give the imperial title to Russia. Neither the royal title for Prussia nor the imperial title for Russia were generally acknowledged until the powers of Europe had successively consented to them.§

Diplomatic language.

200. (c) With respect to the language employed in negotiation, in the early periods of diplomatic history, the Latin, as the European language, was that generally used; and this, towards the end of the XVth Century, was, in consequence of the political influence of Spain, superseded by the Castilian tongue, which, in its turn, was supplanted by the French, now the most usual medium of diplomatic intercourse. Some nations, as Spain and the German-Italian states, employ their own language; but it is customary for them, when treating with a power whose language is different, to transmit a translation of the treaty in the language of that power, if it is understood that the courtesy will be reciprocated. When both contracting parties have a common language, their treaties are worded in that language.||

Maritime ceremonials.

201. (d) With respect to maritime ceremonials, it may be observed that they have frequently formed subjects of contest between powers. The honours they involve are paid either by a salute with cannon, a salute with the flag, or with the pendant, by furling it up, lowering it, or hauling it quite down; or by a salute with sails by lowering or hauling down the foretopsail.¶

As far as its own maritime jurisdiction extends, every state may impose what regulations it pleases in respect of these ceremonials, but they are usually made the subject of express compact. On the open seas it is usual for an admiral to be saluted by a ship carrying only a pendant, if the ship belongs to a friendly power; and detached ships generally salute fleets.**

Right of property.

202. (IV.) *The Right of Property*.—"The dominion of a nation," says Vattel, "extends to everything which she

possesses by a just title. It comprehends the ancient and original possessions and all acquisitions made by means which are just in themselves, or admitted as such by nations—concessions, purchases, conquests made in regular war, &c."*

Law.

203. How far prescription may be considered as operating upon nations, jurists do not appear to have agreed; but the uniform practice of nations shows that they recognize the long and uninterrupted possession of a territory as excluding the claims of all other nations, and that this principle, whose exposition fills so large a head in municipal jurisprudence, is equally recognized, as reason dictates that it should in international law.

First occupancy.

204. As to national possessions, which being of comparatively recent acquisition, other sources of title are acknowledged: when the Russian government claimed the sovereignty of the north-west coast of America, from Behring's Straits to the 51st degree of northern latitude, they rested their claim, as they said, "upon the three bases required by the law of nations, that is, upon the title of first discovery; upon the title of first occupancy; and, in the last place, upon that which results from a peaceable and uncontested possession of more than half a century;† a space of time longer than that during which the United States had enjoyed a national existence. As to the two first sources of title named by the Russian envoy, they are unquestionably good as against every nation except that nation whose liberties their assertion may prejudice. The right of a state in quality of her superior power—the power lent her by her civilization—to subject to her dominion a territory inhabited by another people, be they ever so savage, may be questioned on the principles of the law of nations, however consistent with the customs and usages of nations. Vattel discusses the question, which is that of almost every European colony, "whether it be lawful to possess a part of a country inhabited only by a few wandering tribes?" and he justifies his reluctant assent to the affirmative on the ground that these tribes "cannot exclusively appropriate to themselves more land than they have occasion for," and that "their unsettled habitation in these immense regions cannot be accounted a true and legal possession."‡ An argument framed with a view to a conclusion does not deserve much mercy, and it may fairly be asked, (1) Who is to be the judge of the necessities of these tribes? Their usual occupations, hunting and fishing, notoriously require a large range of territory to enable them to support subsistence. (2) How, on Vattel's principles, can the integrity of the Russian empire, with its 150 inhabitants to every square mile of territory, be secured? Australia contains perhaps 5,000,000 of square miles, and a population truly insignificant. Surely this jurist's principle would impugn the inviolability of our dominion in that vast continent.

205. Still, as a fact, it is not to be denied that in savage countries the rights of the natives have in every instance been treated as subservient to those of the first Christian or civilized settler. Vattel's principle was carried further on one occasion by the British government, who, when Spain, on the ground of prior discovery and long occupation, confirmed by the treaty of Utrecht (in 1790), claimed the sovereignty of the

* Martens, *Précis*, liv. iv. ch. ii. sec. 6.

† 1 Wheaton, *Elem.* 198.

‡ Anciently the Pope and the Emperor of Germany claimed the right of conferring the royal dignity on any house they pleased. In this way the Emperor Henry IV., in 1086, made the Duke of Wratisslaus King of Bohemia, and Pope Eugene made Alphonso King of Portugal. The Pope for a long time refused to acknowledge Frederick as King of Prussia.

§ Martens, *Précis*, liv. iv. ch. ii. sec. 2.

|| Wheaton, *Elem.* 198-9.

¶ Martens, *Précis*, liv. iv. ch. iv. sec. 15.

** Martens, *ib.* sec. 17.

* Liv. ii. ch. vii. sec. 80.

† Le Chevalier Poletica to Mr. John Quincy Adams (American Secretary of State), *Ann. Reg.* vol. lvi. p. 579.

‡ Liv. i. ch. xviii. sec. 209.

Law. north-west coast of America, as far north as Prince William's Sound (lat. 61°), asserted "that the earth is the common property of mankind, and of which each individual and each nation has a right to appropriate a share by occupancy and cultivation."* This is almost the language of the German chieftain of Nero's time: *Sicut diis cælum, ita terras generi mortalium datas, quæque vacuæ, respublicas esse.*

Property
in seas.

206. As to the extent to which the right of property operates; and first, as to the sea:—So much of the sea as is bounded by the territories of a state (such as St. George's Channel, which runs between England and Ireland) is considered as belonging to that state. And thus all harbours, ports, bays, and *embouchures* of rivers are considered as belonging to the state whose land forms their boundaries. The usage of nations has, for obvious reasons, extended this possession to as much of the open sea as lies within cannon range (that is, about three miles) of the shore,† in obedience to the maxim, *Terræ dominium finitur, ubi finitur armorum vis.* The term *shore* includes not only what is generally so denominated, but all islands lying off the coast, although not sufficiently firm for habitation;‡ but it does not include shoals. By the 9th Geo. II., c. 5, a jurisdiction of four leagues from the shore is assumed by the British government for revenue purposes, so that foreign goods transhipped within those limits are subject to the payment of duties.

207. Being "a claim of private and exclusive property over a subject where a general, or, at least, a common use is to be presumed," says Sir William Scott, "the general presumption certainly bears strongly against such exclusive rights, and the title is to be established on the part of those claiming under it in the same manner as all other legal demands are to be substantiated—by clear and competent evidence."§ Still the property in whole seas have at various times been claimed by various nations. Thus the possession of the Indian seas was claimed by Portugal, while the Venetian republic asserted similar pretensions to that of the Adriatic, the Ottoman empire to the Black Sea (subsequently renounced by the treaty of Adrianople in 1829), and Denmark to the Baltic.

Close and
open seas.

208. The question of *mare clausum* and *mare apertum* was formerly one of considerable interest, and occupies a large place in the writings of jurists. It may generally be stated, that while the sea (subject to the exceptions first mentioned) is for the most part open and common to all nations, as to certain parts the general right may be modified by compact and usage. Even Grotius, the stout advocate of the general right, is forced to admit that this is authorized by numerous passages in ancient writers, to whose authority on other subjects he is, in Mr. Bentham's opinion, always too ready to defer.|| Father Paul Sarpi, the well known

historian of the council of Trent, vindicated the claim of Venice to the supremacy of the Adriatic; and Bynkershoek (no mean authority) acknowledges that certain portions of the open sea may become the property of a state, grounding his denial of the claims of England to the Four Seas simply on the fact of a deficient length of possession.* Vattel, too, asserts that tacit agreement evidenced by a non-usage of the general right, may confer property or dominion in a sea.†

209. Secondly, as to rivers and lakes. These, when wholly included within the territory of a state, are its exclusive property, and every state is considered as possessed of so much of a river as flows through its territories. If the river divides two states, the mid-channel is considered as the boundary line, unless prior occupation has given to the one or the other the right of possession to the whole. In the case of rivers flowing through several states, all the nations inhabiting its banks possess the right of navigating it for commercial purposes. This is what is called an *innocent use*, and is considered to be subject to the convenience and safety of any state which its exercise may affect. This innocent use appears to involve the right of doing whatever is necessary to its enjoyment, such as mooring vessels to the banks, landing cargoes, &c.; but usually these, as well as the general right itself, are settled and determined by convention.

210. The right of navigation to which we allude may be renounced by treaty, as was the case in the treaty of Westphalia, whereby the navigation of the Scheldt was closed in favour of the Dutch provinces. The treaty of Vienna, in 1815, declared that the commercial navigation of the great rivers of Germany and ancient Poland should be open, provided that the regulations of the police were observed. These, it further declared, should be uniform and as liberal as possible to the commerce of all nations.‡

211. Having discussed those rights of nations which are intrinsic and result from their character as nations, we have now to consider such rights as are properly international, and originate in their mutual relations. These relations being either *pacific* or *hostile*, we may consider these rights as they have reference to a state of war or of peace.

212. (I.) The international rights of nations in their pacific relations are, the rights of legation and the rights of negotiation and treaties. The rights of legation are so fully treated of in another part of this work, that it is unnecessary we should consider them further in this place.§

213. As to the right of negotiation, certain public conventions are effected by subordinate powers, authorized thereto by the terms of their commissions and the nature of their duties. Thus the governor of a town and the general who besieges it have a power to settle the terms of the capitulation, and the authority to effect such an agreement being annexed in their respective offices, the capitulation is obligatory on the states whose servants and representatives they are.

214. Admirals and generals have usually authority, within the spheres of their review, to grant special licences to trade, cartels for the exchange of prisoners, capitulations, truces, &c.

* 1 Wheaton, *Elem.* 211. A singular admission of the rights resulting from discovery appears in the conduct of the Turkey Company in 1581, who being desirous of engaging in the Indian trade, and considering the passage round the Cape, discovered by the Portuguese, to be their exclusive property, and not to be interfered with, chose the overland route, where they were exposed to the powerful competition of the Venetians.—Macpherson, *Hist. of European Commerce with India*, p. 75.

† Martens, *Précis*, liv. iv. ch. iv. sec. 4. 1 Wheaton, *Elem.* 215.

‡ The "Union," 5 Robinson, *Rep.* 385.

§ The "Twee Gebroeders," 5 Robinson, *Rep.* 339.

|| Bentham, *Introd. to Morals and Legislation*. See Sir James Mackintosh's eloquent vindication of the Dutch Jurist, *Prel. Disc.* p. 18.

* *Quest. Jur. Pub.*

† Liv. i. ch. xxiii. sec. 286.

‡ 1 Wheaton, *Elem.*

§ Div. iv. vol. v. Art. DIPLOMACY.

Law. and truces for the suspension of hostilities; and these are conventions to the validity of which the ratification of their principals is not required, unless by express stipulation.*

Sponsio. 215. "By the Latin term *sponsio*, we express an agreement made by a public person who exceeds the bounds of his commission, and acts without the orders or command of his sovereign."† Conventions of this kind are not valid until they are ratified, which may be done either formally or by implication, as if one of the states whose agents effected the convention should cause troops to pass through the territories of the other on the footing of friends; this would be considered as a tacit ratification of the convention. Mere silence would not be so construed, although good faith requires that if one state does not purpose to recognize the acts of its servant, it should notify the fact to the other. If on the belief that the agent was duly authorized, the convention has been wholly or partially acted on by one party, it would seem it has a claim to be indemnified or replaced in its former position.‡

Ratification. 216. A minister, properly to conclude a public treaty, ought to be fully empowered by his state to do so; but before the treaty can be considered binding on the contracting parties, it is usual to require that it should be ratified by them. This power of ratification is generally reserved in a treaty, and no state should refuse to exercise it without a powerful reason, such as that the minister has not followed his instructions.§

Obligation of treaties. 217. It is for the internal constitution of every state to determine in whom the right of negotiating and effecting treaties resides; but treaties so far lawfully made are obligatory on the state. When the treaty requires the payment of money to carry it into effect, the legislature are morally bound to pass the law, because to the performance of the treaty the public faith has been pledged by competent authority. In such a case, however, it is usual for the British government to stipulate in the treaty that the crown will recommend to parliament to vote the necessary monies. If the treaty involves an alienation of the public domain, whether it be public or private property, it would seem to be obligatory on the party so contracting, if they are clothed by their respective states with the whole treaty-making power. And thus in a case decided by the Supreme Court of the United States, it was held as a clear principle of national law that private rights might be sacrificed by treaty to secure the public safety; although it was admitted—and Grotius is an authority on the point—that the government would be bound to compensate the individuals whose rights might be surrendered.|| The fundamental laws of a state may deny to the treaty-making power the right of thus alienating the public domain; in such a case the whole treaty-making power is not intrusted to one department of the state, and no treaty involving such an alienation would be obligatory, unless it were ratified by the legislature. Commercial treaties often require legislative sanction; and the commercial treaty of Utrecht, between France and England, was never carried into effect, parliament having rejected the bill necessary for that purpose.

218. Compacts between nations are either transitory

covenants (*pacta transitoria*) or treaties (*fœdera*), properly so called. When a transitory covenant has been fulfilled, and has continued without being renewed, or its future duration has been defined by the contracting parties, it still continues in force. Change in the person of the sovereign, the form of the government, or the sovereignty of the state, do not impair its validity, if any one of the parties do not violate it. A war only suspends a convention of this kind, and the return of peace restores its operation. Such are treaties of a first boundary or exchange of country, &c.*

219. Treaties cease to be obligatory when the sovereign power with whom they were made ceases to exist—when one of the states contracting changes her internal constitution so as to render the treaty inapplicable to her condition—and when a war breaks out between the parties. These two last rules are subject to exceptions. As to the first:—

220. Jurists distinguish between *real* and personal treaties. Personal treaties depend for their continuance on the person of the sovereign, or ruler, or his family. The former bind the state. The death of the ruler, extinction of his family, or the severance of his or their political connection with the state, dissolves the treaty. Political revolutions do not affect a treaty which is real.† The reader is referred to the Article on Diplomacy, already mentioned, for further information connected with this subject; the ceremonial etiquette connected with legations is therein fully detailed and explained.

221. War is defined by Vattel, as "that state by which we prosecute our right by force." It is with public war alone (that is, war carried on between independent nations) that the law of nations concerns itself. It may be *perfect* or *imperfect*, *civil*, or *national*, *offensive* or *defensive*.

222. A *perfect* war is, where a whole nation is at war with a whole nation, and all the members of one are empowered to exercise hostilities against all the members of the other. A perfect war.

223. An *imperfect* war is limited, as to persons, places, and things, as was the case with the hostilities authorized by the United States, against France in 1798.‡ An imperfect war.

224. A *civil* war is a war between members of the same state, and, according to Grotius, is a *public* war, as far as the government are concerned, and private on the part of the insurgents. A civil war.

225. A *national* war is a war between nation and nation, undertaken and carried on by the authorities, according to the political organization of the nation, constitutionally competent so to do.§ A national war.

226. A war is *offensive* on the part of the sovereign who commits the first act of violence against the other. Offensive war.

227. It is *defensive* on him who receives the first blow. Defensive war.

228. The sovereign power is alone possessed of authority to make war; a civil war of course does not fall within this observation.|| Authority of making war.

229. As far as the law of nations is concerned, every war commenced and prosecuted in form, and consistently with its principles, is just. The right of determining Just war.

* Vattel, liv. ii. ch. xiv. sec. 207. 1 Wheaton, *Elem.* 290-1.

† Vattel, liv. ii. ch. xiv. sec. 209. ‡ 1 Wheaton, 291.

§ *Id.* 192. Martens, *Précis*, liv. ii. ch. i. sec. 3.

|| Ware v. Hylton, 3 Dallas, *Rep.* 199, 245. 1 Kent, *Comm.* Grotius de Jure Belli et Pacis, lib. iii. ch. xx. sec. 7.

* Martens, *Précis*, liv. ii. ch. i. sec. 7. Vattel, liv. ii. ch. xii. sec. 192. 1 Wheaton, *Elem.* 296.

† Martens, *Précis*, *ib.* sec. 5.

‡ 2 Wheaton, 10.

§ Martens, *Précis*, liv. viii. ch. ii. sec. 1.

|| Vattel, liv. iii. ch. i. sec. 4.

Law.

under what circumstances it shall take up arms is the natural prerogative of every state, and is incidental to its right of independence. The law of nations simply regulates this right, and prescribes the mode in which it shall be exercised.

Declara-
tion of
war.

230. The custom of making a declaration of war to the enemy, previous to the commencement of hostilities, is of great antiquity, and was practised even by the Romans.* Louis IX. would not attack the Sultan of Egypt until he had sent him a herald to announce his intention of doing so. The earlier jurists, with the exception of Bynkershoek, generally consider a war, undertaken without this previous declaration, to be contrary to the law of nations,† and Grotius bases its necessity, not on the ground that an enemy may be put on his guard, but that it may be clear that the war is undertaken, not by private persons, but by the authority of the community.‡ Since, however, the peace of Versailles, in 1763, such declarations have been discontinued, and the present usage is, for the state with whom the war commences to publish a manifesto within its own territories, communicating the existence of hostilities, and the reasons for their commencement.§ The publication of this manifesto was looked on as so essential, that nations have demanded a restitution of everything taken before such publication; || but although this publication is usual, as being necessary for the direction of the subjects of the belligerent state,¶ it may be questioned whether its omission would have such an operation.** When war has once been declared, whether by manifesto or by acts equivalent thereto, it is a war not simply between governments in their politic capacities, but binding on their subjects.††

Confisca-
tion of
enemy's
property.

231. A nation in a state of war is considered authorized, on general principles, to seize the persons and confiscate the property of the enemy's subjects.‡‡ This is a principle which modern usage has practically modified to a considerable extent. Grotius himself considers that, as far as respects debts due to private persons, the right of demanding is only suspended by the war. Vattel, while admitting the general principle, qualifies it by the exception of immovable property (*les immeubles*), held by the enemy's subjects within the belligerent state, and which, having been acquired by the consent of the sovereign, cannot be sequestered without a breach of faith. Debts, and other things in action, he holds liable to seizure. He at the same time admits that "at present the advantage and safety of commerce have induced all the sovereigns of Europe to relax from this rigour. §§ The state does not touch even the sums which it owes to the enemy; everywhere, in case of war, the funds confided to the public are exempt from seizure and confiscation." In the absence of express convention,

by which this matter is sometimes regulated, the modern rule, according to Mr. Wheaton, is, that neither the property of the enemy within the belligerent state, nor the debts due to his subjects, are confiscated on the breaking out of war.* In England it has been usual, in maritime wars, for the government to seize and condemn, as *droits of Admiralty*, the property of the enemy found in her ports at the breaking out of hostilities; but as to the debts due to his subjects, we consider them as only suspended by the war, and as restored by the peace.† Our ancient law was, however, much more liberal than this,‡ and, in comparatively modern times, the subjects of an enemy "residing here and demeaning themselves dutifully, and not corresponding with the enemy," have been, with their effects, taken under the special protection of the crown.§

232. It would appear that persons *domiciled* in the enemy's country are subject to reprisals as well as the natives.|| The nature of the residence which constitutes the domicile is determined in several cases decided by our courts.¶ If a person during hostilities enters a house of trade in the enemy's country, or continues the connection during war, his residence in a neutral country will afford him no protection.** There is no principle of international law more undoubted than that which prohibits all trade between belligerent nations, unless authorized by their governments,†† as it renders void all commercial contracts between subjects of the same. A remittance of supplies to a colony during its temporary subjection to an enemy is equally prohibited, although permission had been given to export the produce of that colony.‡‡ Numerous expedients have been resorted to by English merchants to evade the operation of this rule; but they have been effectually defeated, by the determination of our municipal courts to enforce it with the utmost strictness.§§

233. The rule extends equally to allies: "Between allies," says Sir William Scott, "it must be taken as an implied, if not express contract, that one state shall not do anything to defeat the general object. If one state admits its subjects to carry on an uninterrupted trade with the enemy, the consequence may be, that it will supply that aid and comfort to the enemy which may be very injurious to the protection of the common cause, and the interest of its ally."||| This prohibition operates of course no further than the necessity which justifies it, and has no existence when the trade is of such a nature as can in no manner interfere with the common operations, or has the allowance of the confederate state.

234. The modern law of nations prohibits those barbarous customs which distinguished the warfare of early times. Considering war simply as a means to protect nations in the enjoyment of their just rights and lawful

* Cicero, *De Off.* lib. i. ch. ii. Dig. 49, 15, 24.

† Grotius, *De Jure Belli et Pacis*, lib. i. ch. iii. sec. 4. Puffendorf, lib. viii. ch. vi. sec. 9. Emerigon, *Traité des Assurances*, tome i. p. 563. Vattel, lib. iii. ch. 4, sec. 81. Bynkershoek, *Quest. Jur. Pub.* lib. i. ch. 2. See Sir William Scott's *Judgment in the "Nayade,"* 4 Robinson, *Rep.* 252.

‡ Vattel, lib. ch. iv. sec. 60. *De Jure Belli et Pacis*, lib. iii. ch. iii. sec. 11.

§ 1 Kent, *Comm.* 54.

|| Martens, *Précis*, liv. viii. ch. ii. sec. 4.

¶ 2 Wheaton, *Elem.* 11.

** 2 Kent, *Comm.* 54. See 3 Campbell, *Rep.* 66. The "Herstelder," 1 Robinson, *Rep.* 114.

†† Vattel, lib. iii. ch. v. sec. 70.

‡‡ Martens, *Précis*, liv. viii. ch. ii. sec. 5, quoting Grotius, Puffendorf, and Wolf. Finch, *Law*, 28.

§§ Sec. 76-7.

* 1 Wheaton, *Elem.* 18.

† The "Hoop," 1 Robinson, *Rep.* 196. *Ex parte Bousmaker*, 13 Ves. Jun. 71. *Furtado v Rogers*, 3 Bosanquet and Puller, 191. The "Nuestra Señora de los Dolores," Edwards, *Rep.* 60.

‡ See *Magna Charta*, ch. xxx. and 27 Edw. III. stat. ii. cap. xvii. 1 Hale, *Pleas of the Crown*, 93. Bro. tit. *Property*, pl. 38.

§ Foster, *Crown Law*, 183.

|| Grotius, *De Jure Belli et Pacis*, lib. iii. ch. ii. sec. 7.

¶ The "Harmony," 2 Robinson, 324. The "Moran Chief," 3 Robinson, 12. "La Virginie," 5 Robinson, 99.

** 2 Wheaton, *Elem.* 70-1. The "Citto," 3 Robinson, *Rep.* 38. The "Portland," 86, 41.

†† The "Hoop," *ut cit. ante*. The "Jonge Pieter," 4 Robinson, *Rep.* 83, Potts v Bell, and others, 8 Term *Rep.* 548.

‡‡ 1 Kent, *Comm.* 61. The "Bella Giudita," 1 Robinson, *Rep.* 207.

§§ Chitty, *Law of Nations*, p. 13 15.

||| The "Neptunus," 6 Robinson, *Rep.* 405.

Law.

possessions, it condemns all cruelty not absolutely necessary to secure that end. If to anything further than the progress of civilization, and the more general appreciation of the dictates of natural justice, consequent on the diffusion of a purer and sincere spirit of religion, the benevolent aspect in which war is thus regarded may be ascribed with some propriety to the influence which the writings of Grotius have exercised on international law. Even against the language of some of those authorities, in whom his confidence has been esteemed too implicit, this great and eminently wise publicist has protested, in eloquent terms, against those practices which the customs of the times had sanctioned, and which regarded war as incompatible with moderation, and a regard to the maxims of ordinary humanity. Wolf and Bynkershoek, subsequent to his age, considered that no measures which could injure an enemy were improper; but these writers, however highly esteemed on other points, have not in this prevailed; and a disposition has been constantly manifested, by the enlightened nations of Europe, to mitigate the horrors of hostilities, as far as is consistent with the occasion which produces and justifies them. The laws of war will now occupy our attention.

Laws of war.

235. It is now universally agreed, that hostilities can be undertaken by none who are not lawfully authorized thereto by their sovereign, or the supreme power of the state to which they belong. This does not of course prohibit the subjects of a state, when attacked, from defending themselves;* but it has been contended, though on what principle of justice it does not appear, that even such would be treated by the enemy with more rigour than those acting under the express orders of their superior.† Captures, however, made by a private armed vessel, without a commission, are not considered as piratical;‡ but the property seized does not pass to the captors, but is condemned to the crown,§ as prize of war, or, as it is styled, droit of Admiralty. The same result follows when vessels commissioned against one power seize the property of another, with whom war afterwards breaks out. This probably arises from the recognized distinction, maritime and land warfare;|| but it does not appear to have been approved by Sir Matthew Hale.¶ Modern writers** have deprecated the employment of privateers; that is, private cruizers commissioned by the state.†† The question of the liability

of the owners and officers of privateers in damages for illegal acts, beyond the security given,* is a question of municipal and not international law, and therefore cannot properly be discussed here; so likewise is their interest in the captures made.† It seems to have been settled, Privateers that a cruizer commissioned by two powers is to be treated as a pirate, even although the two powers are allies,‡ and many states have prohibited their subjects from arming in any way the fitting out of private vessels, intended to cruize against the subjects of friendly powers. The French Marine Ordinance of 1681 considered such an act as piratical.§

236. The law of nations prohibits as unlawful the use of Unlawful certain means of offence, such as poisoning,|| assassina- arms, &c. tion, and, according to Martens, the loading of cannon with nails, pieces of iron, &c.¶ The same writer considers as properly exempt from the extremities of war, children, women, old men, and others incapable of bearing arms, and such retainers of the army as are not employed in actual warfare;*** and also soldiers and others actually so employed, who have submitted, and entreated quarter. As to these last, he contends that their treatment will be subject to three considerations:—

237. (1) Whether sparing their lives will be consistent with the safety of the conqueror? (2) Whether he has a right to subject to the *lex talionis*?†† (3) Whether they have become his captives through their commission of a crime worthy of death, or whether they are spies, &c.?

238. The distinction to which we have before alluded is Distinction founded on the circumstance, that the presumed object between maritime and land warfare. of maritime warfare is “the destruction of the enemy’s commerce and navigation—the sources and sinews of his naval power, which object can only be attained by the capture and confiscation of private property;” while the object of wars by land is treated as being “conquest, or the acquisition of territory, to be exchanged as an equivalent for other territory lost.” In this latter, “the regard of the victor for those who are to be, or have become, his subjects, naturally restrains him from the exercise of his extreme rights in this particular.”††

239. A prisoner of war is entitled to protection and Prisoners good usage, but may be strictly confined if he attempt of war. to escape. Officers are frequently liberated on their parole, or word of honour, that they will not serve against the power by which they are released during the war, or during a stipulated time. The exchange of prisoners

* Vattel, liv. iii. ch. xv. sec. 223. Dig. 47, 22, 4. Vattel (sec. 225) considers this to be a rule relating rather to public law in general than to the law of nations properly so called; but unquestionably it is a rule adopted by the law of nations, by which it is enforced, as is apparent from the difference of treatment to which unauthorized belligerents are exposed from that which the regular combatants are subjected.

† Martens, *Précis*, liv. viii. ch. iii. sec. 2.

‡ The American Commissioners to the court of France, to M. De Sestine, 1 Sparks, *Dipl. Amer.* 443.

§ Viner. *Ab. Prerog.* N. 3. pl. 22.

|| Chitty, *Law of Nations*.

¶ Hargrave, *Law Tracts*, 246.

** Mr. Wheaton takes credit to the United States for having by treaty with Prussia, in 1785, agreed in any future war with that power to employ no privateers. It appears, however, that the privateering system was carried further by America than any other power, for during the war with Great Britain the legislature of New York passed an Act which constituted every association of five or more persons desirous of embarking in the trade of privateering, should it comply with certain formalities, a body politic and corporate, and conferred on it the ordinary corporate powers, 1 Kent, *Comm.* 98. Note.

†† “The privateers in our wars are like the *Mathematici* of old Rome, a sort of people that will always be found fault with, but still made use of.” Sir Leoline Jenkin’s *Works*, vol. ii. p. 174.

* 1 Kent, *Comm.* 98-9.

† Vattel, liv. iii. ch. xv. sec. 229. The “Elsebe,” 5 Robinson, *Rep.* 173. At common law it would seem that the whole seizure went to the captors: “goods that belong to an alien enemy, anybody may seize to his own use,” Finch, *Law*, 17 and per Wright J. *Murrough v. Conyns*; 1 Wilson, *Rep.* 213; but see *Home v. Lord Camden*; 1 H. Blackstone, *Rep.* 476 and 2; *ib.* 533. It is usual to require of the owners of privateers severally, that they will conduct their cruizers according to the laws and usages of war and the instructions of the government. 1 Kent, *Comm.* 97.

‡ Sir Leoline Jenkin’s *Works*, vol. ii. p. 174.

§ 1 Kent, *Comm.* 100; see also 2 Vernon, 592.

|| *Armis bella non venenis, geri debere*, *Vol. Max*, lib. vi. cap. v. sec. 1; and see Vattel, whose observations are more than usually indistinct, liv. iii. ch. viii. sec. 155.

¶ Martens, *Précis*, liv. viii. ch. iii. sec. 3.

** *Ibid.* sec. 4; Vattel, *ut cit. sup.* sec. 145; *Edinburgh Review*, No. xv. p. 13.

†† When King John, in 1215, took the castle of Rochester, which had resisted his assaults for a long period, he ordered the garrison to be hanged, but Sauvery de Mauleon reminded him of the probability that such treatment would at a future time be retaliated on his own officers. Lingard, *Hist. Eng.* vol. iii. p. 1, (new edit.). Compare Rutherford, *Hist. Nat. Law*, with Martens, *Précis*, *ut sup. cit.* sec. 5.

‡‡ 2 Wheaton, *Elem.* 84-5.

Law. during the continuance of hostilities is a practice common to all civilized states.*

Exchange of prisoners. 240. The persons of artisans, labourers, merchants, and persons whose occupations are peaceful, it is customary to respect.

Enemy's property. 241. As to the enemy's property:—In the rigour of international law, to capture or destroy this is lawful; but this rigour has been mollified by the humane usages of nations, which have acquired the force and obligation of laws. A distinction must, however, be taken between at sea; hostile operations conducted on land or at sea. It is in land warfare that the progress of civilization has most decisively manifested itself.

on land. 242. The religious edifices, works of art, repositories of sciences, and public buildings of a decidedly civil character, belonging to an enemy, are considered as sacred from spoliation and destruction by the customs of all enlightened nations.† Private property on land is also respected, subject to occasional certain exceptions: (1) property taken from the enemy in the field; (2) property in a town taken by storm, after having repelled all overtures for a capitulation; and (3) contributions levied by a belligerent, for the support of his army, and towards defraying the expenses of the war.‡ In a case of extreme necessity, it is lawful to devastate and lay waste an enemy's territory, and to destroy all buildings, &c., therein as far as is requisite for the success of military operations, but the lawfulness of such proceedings is limited by that necessity in the view of Retaliation all communities.§ A departure from these rules will be justified, it is thought, by the *lex talionis*, which is considered to exercise a vast influence in modifying the humane usages of modern warfare.

Captures. 243. When the capture has been effected, the title to the property so captured is considered, as between belligerents, to pass from the original owner to the captors. The law of rule is, that the transfer is effected by occupation. *Occupatione dominium prædæ hostibus acquiritur.*|| A possession for twenty-four hours is, according to Grotius¶ and others, essential to this transfer; and although Bynkershoek** does not concur, this appears to be sanctioned by modern authority.††

244. In the case of ships and goods taken at sea, the title does not pass until the validity of the capture has been affirmed by a competent prize court of the captor's government,‡‡ sitting in its own country. §§ By the practice of Great Britain and the United States, their prize courts may try captures, which have been carried into neutral ports. |||| When the capture has been effected within, or by vessels fitted out within the territorial limits of a neutral state, the tribunals of that state have juris-

diction to try the capture, and some, as the price of the permission they afford belligerents to bring their captures into their ports, have, by their municipal laws, reserved a right of adjudicating on such captures, when the original owners of the captured property have been their subjects, and of restoring the property to them.* The right of condemning prizes is one that no neutral state can concede to a consular tribunal sitting within her territories.† The sentence of a competent prize court renders the title of the captor conclusive, as far as personal property is concerned.‡ The distribution of this property is a matter regulated by the internal laws of every state. Whatever is captured is, in intendment of law, captured by the state; *bello parta cedant reipublicæ*,§ although the common law of England considers it otherwise.||

245. The law of Postliminy is one of the few portions of the Roman *Jus Feciale* which has descended to us. Jus Postlimini. By this law, according as it is at present understood, if a vessel, even although it has been two or even four years in possession of the captors, be recaptured *before* condemnation by a ship belonging to the country of the original owners, they may claim its restoration, on paying a proper salvage to those by whom the recapture has been effected.¶ The operation of this law, as far as concerns cases arising between her own subjects, or between her own subjects and those of such of her allies as evince a disposition to a reciprocal liberality, has been extended by Great Britain to any recaptures effected during the war, and without regard to any sentence of condemnation having passed.** The right of postliminy takes place only within territories of the captor's nation, or its allies, and does not include neutral countries.†† It is, however, in reference to real property that an allusion to this law is chiefly necessary; and it is this law which avoids, on the return of peace, all alienations of such property, by a belligerent state, in occupation of the enemy's country. To impart stability to them, they must be confirmed by the treaty of peace.

246. The observance of good faith to an enemy is one of those duties on the obligation of which all jurists unite, and which it is the obvious interest of all belligerents to practise.†† Good faith towards an enemy.

247. We have, in concluding this rapid review of the laws of war, to consider those relaxations of their stern rigour, which are familiar to modern, and in some degree even to ancient practice. (1) A truce is a suspension of hostilities, either for a long or indefinite period, or sometimes only as to a portion of the military operations. The authority to effect the former is not always implied in the authority of the commanders, but usually so as respects the latter. Whenever effected, a truce is obligatory on all the subjects of the belligerent states, after it has been duly promulgated. (2) The right to effect capitulations for the surrender of fortresses, &c., is involved in the authority committed to every commander

* Vattel, sec. 150; Martens, *ut cit.*

† Vattel, liv. iii. ch. ix. sec. 168. In the case of the Marquis de Somermeles, (Stewart's *Vice Admiralty Rep.* 482,) the Vice-Admiralty Court of Halifax restored to the Academy of Arts in Philadelphia a case of Italian paintings and prints, captured on their passage to the United States by a British vessel in the war of 1812, "in conformity to the law of nations, as practised by all civilized countries," and because "the arts and sciences are admitted to form an exception to the severe rights of warfare," 1 Kent, *Comm.* 93.

‡ Martens, liv. viii. ch. iii. sec. 9. Vattel, *ut cit. sup.* sec. 165, 2 Kent, *Comm.* 92; 2 Wheaton, *Elem.* 81.

§ Kent, *Comm. ut cit. sup.*

|| Voet. *ad Pandect.* lib. ix. tit. xv. sec. 3. Goss v Withers, 2 Burr. 683.

¶ *De Jure Belli et Pacis*, lib. iii. cap. vi.; and Vattel, liv. iii. ch. x. sec. 196. Martens, *Précis*, liv. viii. ch. iii. sec. 11.

** *Quest. Jur. Pub.* lib. i. ch. iv. v.

†† 2 Wheaton, *Elem.* 88-9.

‡‡ 2 Kent, *Comm.* 102.

§§ Wheaton, *Elem.* 89-90. The "Flad Oyen," 1 Robinson, *Rep.* 134.

|||| The "Henrick and Maria," 4 Robinson, *Rep.* 43.

* 2 Wheaton, *Elem.* 91-4.

† 2 *Ibid.* 94.

‡ The schooner "Sophia," 6 Robinson, *Rep.* 142. This is a case where a prize had been transferred to a neutral, and a peace was concluded, without a sentence of condemnation having been passed. The transfer was held valid.

§ Martens, *Précis*, liv. viii. ch. iii. sec. 10.

|| See ante. p. 14, Note †.

¶ The "Constant Mary," 3 Robinson, *Rep.* 97, Note. The "Huldata," *ibid.* 235.

** 13 Geo. II. cap. iv.; 17 Geo. II. cap. xxxiv.; 19 Geo. II. cap. xxxiv.; 43 Geo. III. cap. clx. The "Santa Cruz," 1 Robinson, 49. As to the salvage payable, Chitty, *Law of Nations*, 104-7.

For the law of the United States, see 1 Kent, *Comm.* 112.

†† 1 Kent, 169.

‡‡ Grotius, *De Jure Belli et Pacis*, lib. iii. ch. xix.

Law.
Passports,
&c.
Licences to
trade.

by the terms of his commission. (3) Passports, safe conduct, and licences are granted in war, for the protection of persons and property. Licences to trade are the most important of these. Grotius considers that the interpretation to be put on these permissions should be liberal rather than strict, *laxa quam stricta interpretatio admittenda est*.^{*} Sir William Scott adopted a different principle, and considering a licence as a high act of sovereignty, and consequently *stricti juris*, concluded that it "must not be carried further than the intention of the great authority which grants it may be disposed to extend." It is not "to be construed with pedantic accuracy," nor should "every small deviation be held to vitiate it. An excess in the quantity of goods permitted might not be considered as noxious to any extent: a variation in the quality or substance of the goods might be more significant, because a liberty assumed of importing one species of goods under a licence granted to import another might lead to very dangerous abuses."[†] The time mentioned in the licence must not be exceeded,[‡] and the port of shipment therein named is a material point.[§] A greater liberality of construction has been evinced by the courts, when the question has respected the parties for whose advantage the licence has been obtained.^{||} It has been decided that a general licence is to be construed so far strictly as not to extend to a protection of an enemy's property,[¶] but a licence specifying any flag protects even an enemy's property.^{**} The conditions contained in a licence must, to render it valid, be truly and fairly performed;^{††} and it is a permission which the war is considered to terminate.^{‡‡} In the first instance it must be granted by competent authority,^{§§} circumstances sometimes forbidding property captured at sea to be sent into port. The captor, according to the general law of nations, may either destroy it or permit the original owner to ransom it. The effect of a ransom is to give, on the authority of the state to which the captor belongs, a safe conduct to the vessel captured, which will protect it from all cruizers of that state. By the 22 Geo. III. c. 25, British subjects are prohibited from ransoming enemy's property.

Ransom.

Rights of
war as re-
spect
neutrals.
Definition
of neutra-
lity.

248. We have now to discuss the rights of war as concerns neutral nations. Properly "neutral nations are those who in time of war do not take any part in the contest, but remain common friends to both parties, without favouring the arms of the one to the prejudice of the other."^{|||} To be neutral, the nation should give no assistance when she is under no obligation to give it. In whatever does not relate to war, she is not on account of his present quarrel to deny to any of the parties what she grants to the other. Such conduct is the very essence of neutrality, and a nation forfeits her neutral character when she departs from it. It does not necessarily, however, preclude her, if so bound by treaty previous to the war, furnishing a belligerent party

Qualified
neutrality.

^{*} Grotius, *De Jure Belli et Pacis*, lib. iii. ch. xxi.
[†] The "Cosmopolite," 4 Robinson, *Rep.* 11. The "Oxienoxhap," *ibid.* 96.
[‡] The "Cosmopolite," *sup. cit.*
[§] The "Twee Gebroeders," 1 Edwards, 95.
^{||} *Defflis v. Parry*, 3 Bos and Puller; 3 Timson *v.* Merac, 9 East, 35; *Rawlinson v. Janson*, 12 East; 223; *sed contra*; the "Jonge Johannes," 4 Robinson, 263; the "Aurora," *ibid.* 218.
[¶] The "Josephine," 1 Acton, 313.
^{**} The "Hendrick," 1 Acton, 322.
^{††} "Vandyck *v.* Whitmore," 1 East, 486, see also 12 East, 302.
^{‡‡} The "Planters' Winsch," 5 Robinson, *Rep.* 22.
^{§§} The "Hope," 1 Dodson, 226.
^{|||} Vattel, liv. iii. ch. vii. sec. 103.

with a limited succour in money, troops, ships, or munition, or from opening her ports to receive his prizes. From such an obligation he is said to be released if his ally be the aggressor in the war.^{*} Hostilities cannot lawfully be exercised within the territories of a neutral state, the common friend of both parties. Nor can such neutral state permit, to one or certain of the belligerent parties, the passage of their armaments through her dominions, unless she is prepared to concede a like indulgence to the opponents. Such a preference would destroy her neutrality.

249. A neutral territory must not be violated for the purposes of war. No capture effected within its limits is lawful; and, when illegally made, the neutral state is bound to restore it to its original owners. Nor can such capture be lawful when made by a vessel hovering at the mouth of the river, or bays, or round the coast of a neutral. Bynkershoek[†] has excepted from this rule a vessel chased within a neutral jurisdiction, whither he thinks the belligerent may follow and capture her; but this doctrine, though it has not wanted supporters, is now generally disowned.[‡]

Neutrality
of a terri-
tory not to
be violated.

250. The restitution of property, illegally captured within neutral limits, is effected by an application to the captor's government, by the neutral state, as it is her rights which the law considers to have been violated, and the hostile claimant has no right to appear for the purpose of suggesting in the validity of the capture.[§]

Illegal
captures in
neutral
states.

251. A belligerent cruizer innocently passing through a neutral jurisdiction is not considered to have violated its rights, so far as to invalidate a subsequent capture; and a neutral is not compelled, in virtue of his neutrality, to deny such a passage, nor even to refuse to a belligerent vessel pursued refuge in its harbours; but it ought not to permit it to lie there, and wait a favourable opportunity of renewing a conflict. It need not deny to such vessel provisions and refreshments, which the law of nations universally tolerates; but no proximate acts of war are in any manner to be allowed to originate on neutral ground.^{||} For this reason, when belligerent vessels meet in a neutral port, or one pursues the other there, hostilities cannot be permitted between them during their tarrying; but should one sail, the other must not follow for twenty hours: such at least is the opinion of Professor Martens.[¶]

Conduct
of neutrals
towards
bellige-
rents.

252. A neutral state that permits the arming and equipment of ships or troops for the purposes of a war within its territory violates its neutrality;^{**} but it is no breach of neutrality to suffer a belligerent to bring in his prizes for the purposes of sale.^{††}

Permission
to arm in a
neutral
territory.

253. It may be here remarked that, if a neutral acquiesces in an outrage inflicted on him by one belligerent, the other has a right to retaliate; and that if a deed interdicting a neutral from trading with us, or visiting our ports, is executed upon him, it is an interdiction he has no right to submit to, because its execution will be our injury. If his submission is the result of favour to the belligerent, the neutral becomes constructively a party to the war, and his neutral character with its consequent immunities terminates forthwith. If, on the other hand, it

Acqui-
escence of
neutral in
an outrage

^{*} Bynkershoek, *Quest. Pub. Jur.* lib. i. cap. ix.
[†] Lib. i. cap. viii.
[‡] The "Vrouw Anna Catherine," 5 Robinson, *Rep.* 15.
[§] 2 Wheaton, *Elem.* || The "Anna," 5 Robinson, 373.
[¶] Martens, *Précis*, lib. viii. ch. vi. sec. 6.
^{**} This is forbidden in Great Britain, by the Foreign Enlistment Act, 59 Geo. III. cap. 69.
^{††} 2 Wheaton, *Elem.* 148-9.

Law.

originates in his weakness and inability to resist, we may insist, for our own protection, and without denying to him his neutral character, that what he has suffered from our enemy he may suffer from us, otherwise he would be keeping an open trade with the enemy to our disparagement, and becoming an instrument of its illegal pressure on our resources.*

Commerce of neutrals.

254. As to the commerce of neutrals, it has been decided,† that not only has a neutral a right to pursue his general commerce with the enemy, but even to act as the carrier of the enemy's goods, from the enemy's country to his own, without being subject to the confiscation of the ship, or of any *neutral* goods on board. This is a right which was formerly disputed, but which is now universally recognized; and the neutral owner, when the enemy's goods on board his ship have been seized, is considered entitled, provided his conduct has been fair, to his reasonable demurrage, and his claim for freight.‡

Enemy's goods on board a neutral vessel.

255. That a belligerent is entitled to seize an enemy's goods on board a neutral vessel is an undoubted principle;§ if, however, it should appear that the enemy's interest in the goods was only partial, or that they were the joint property of an enemy and a neutral, the share of the neutral will be saved harmless, and that of the enemy alone confiscated.|| It has been, however, usual in commercial treaties, to stipulate that free ships shall make free goods, and thus this principle, like many other principles of international law, as we have frequently had occasion to observe, has been modified by convention.

The effects of neutrals on board enemies' ships are, upon general principles, considered as exempt from confiscation.¶

Contraband of war;

256. The freedom of commerce to which neutral states are entitled does not extend to *contraband of war*, such as warlike stores and other articles directly auxiliary to warlike purposes.** "The catalogue of contrabands," says Sir William Scott, "has varied very much, and sometimes in such a manner as to make it very difficult to assign the reason of the variations, owing to particular circumstances, which have not accompanied the history of the decisions."†† Grotius distinguishes between those articles which are useful only for the purposes of war, those which are not so, and those of indiscriminate use in war and peace. With other writers, he agrees in prohibiting to neutrals the carrying of the first to the enemy; the second he permits; the third he sometimes permits, and sometimes forbids. Bynkershoek‡‡ considers that the third ought, under no circumstances, to be considered contraband, and the whole question is involved in much confusion. "In 1673," says Sir William Scott, "when many unwarrantable rules were laid down by public authority§§ respecting

contraband, it was expressly asserted by a person of great knowledge and experience in the English Admiralty that, by its practice, corn, wine, and oil were liable to be deemed contraband. In much later times, many sorts of provisions, such as butter, salted fish, and silk, have been condemned as contraband. The modern established rule was, that generally they are not contraband, but may become so under circumstances arising out of the peculiar situation of the war, or the condition of the parties engaged in it. Among the causes which tend to prevent provisions from being treated as contraband, one is that they are of the growth of the country which exports them.* Another circumstance to which some indulgence by the practice of nations is shown, is when the articles are in their native and unmanufactured state. Thus iron is treated with indulgence, though anchors and other instruments fabricated out of it are directly contraband. Hemp is more favourably considered than cordage. Wheat is not considered so noxious a commodity as any of the final preparations of it for human use. But the most important distinction is, whether the articles are destined for the ordinary use of life or for military use. If the port is a general commercial port, it shall be understood that the articles were going for civil use, although occasionally a frigate or other ships of war may be constructed at that port. On the contrary, if the great predominant character of a port be that of a port of naval equipment, it shall be intended that the articles were going for military use, although merchant ships resort to the same place.‡ It is the *usus bellici* which determine an article to be contraband."

The transportation in a neutral vessel of military persons or despatches in the service of the enemy exposes the vessel to confiscation.§

257. The penalty of confiscation for engaging in a contraband trade is not held generally to attach, if the goods are not taken *in delicto*, and in the actual prosecution of the voyage;|| but a different rule is held to apply to cases of contraband carried from Europe to India, with false papers and false destination, intended to conceal the real object of the expedition, where the return cargo, the proceeds of the outward cargo taken on the return voyage, was held liable to condemnation.¶¶

258. The *rule of the war of 1756*, as it has been called, has formed so frequently a subject of controversy amongst publicists that it cannot be passed over without remark. The superiority of Britain as a naval power was conclusively established by the war with France in 1756, when the communication of this latter country with her naval power was effectually interrupted by our fleets. In order to avert the disastrous consequences with which the French colonies were threatened, their government permitted a neutral power, the Dutch, to carry on the trade, the advantages of which had previously been enjoyed

* Chitty, *Law of Nations*, 151-2.

† Barker v. Blakes, 9 East, 283.

‡ Vattel, liv. iii. ch. vii. sec. 115; 1 Kent, *Comm.* 125; 2 Wheaton, 160-1; the "Twilling Riget," 5 Robinson, *Rep.* 82.

§ Grotius, *De Jure Belli et Pacis*, lib. iii. ch. vi. sec. 6.

|| The "Franklin," 6 Robinson, *Rep.* 127; The "Zulerna," 1 Acton, *Rep.* 14.

¶ 2 Wheaton, 162; 1 Kent, 138.

** Grotius, *De Jure Belli et Pacis*, lib. iii. ch. i. sec. 5.

†† The "Jonge Margaretha," 1 Robinson, *Rep.* 189.

‡‡ *Quest. Jur. Pub.* lib. i. cap. ix. sec. 10.

§§ By his prerogative the King of England is empowered to declare what articles are contraband (Chitty, *Law of Nations*, 119); but this declaration can have no operation beyond his own subjects.

* See the "Sarah Christina," 1 Robinson, 237.

† This doctrine is modified by the case of the "Charlotte," decided by the same judge, 5 Robinson, 305.

‡ The "Jonge Margaretha," *ut cit.* See also the "Maria," 1 Robinson.

§ The "Carolina," 4 Robinson, *Rep.* 256. The "Atalanta," *ib.* 440. The "Friendship," 6 Robinson, *Rep.* 420. The "Orozembo," 6 Robinson, *Rep.* 430.

|| The "Imina," 3 Robinson, *Rep.* 168.

¶ The "Rosalie and Betty," 2 Robinson, *Rep.* 343.

Law.

by the French marine exclusively. Some of these Dutch vessels, having been captured, were condemned on the principle stated by the Court of Appeal in the case of the "Wilhelmina."* "By the general law of nations, it is not competent to neutrals to assume in time of war a trade with the colony of the enemy which was not permitted in time of peace."† This principle applies equally to all species of trade, whether coasting or colonial trade. The extent to which it has been relaxed, and the arguments which have been urged against it as a principle, we have no space here to discuss.

259. "The law of blockade," says Bynkershoek,‡ "is founded on the principles of natural reason as well as on the usage of nations. In order that this law may apply, (1.) there must be an actual blockade in existence."§ The mere declaration of a blockade will not suffice. An adequate naval force must be stationed at the blockaded port; and properly "that denomination is given only where there is, by the power which attacks it by ships stationary or sufficiently near, an evident danger in entering."|| An accidental absence of the blockading squadron, provided the blockade is speedily renewed, forms an exception to the rule, and an attempt to take advantage of the absence to break the blockade is considered fraudulent.¶

(2.) The neutral must have had notice of the existence of the blockade. It is usually notified to all neutral governments, and "it is the duty of foreign [neutral] governments to communicate the information to their subjects, whose interests they are bound to protect. I shall hold, therefore, that a neutral master can never be heard to aver against a notification of blockade that he is ignorant of it."** This is a case where the blockade has been notified to the government; but if the individual is personally informed of it, the consequences are the same.

To enter or quit a blockaded port with a cargo laden after the commencement of the blockade is punished with confiscation of the ship and cargo, and the offence is not considered discharged until the end of the voyage.

260. The right of visiting and searching merchant ships on the high seas exists "for the purpose of ascertaining what the ships and their destination are :†† and the penalty for a violent contravention of this right is the confiscation of the property so withheld from visitation and search.‡‡ But the forcible resistance of an enemy master will not in general affect neutral property laden on board of an enemy's ship.§§

261. We may notice that the English Court of Admiralty held a neutral to have no right to charter and lade his goods on board a belligerent armed merchant ship, without forfeiting his neutrality;||| but this is a

doctrine which the American Courts have refused to sanction.*

262. How far neutral vessels, under an enemy's convoy, are subject to capture is still a moot point.

Sect. 6. Sanctions of the Law of Nations.

263. Having thus detailed the fundamental rights which either result from, or are secured by, the law of nations, we proceed to explain the sanctions by which such rights are enforced. These sanctions are of various kinds, and forcible in various degrees. They may, however, be reduced to two classes :—

264. (1.) Reprisals.† There are two kinds :—*negative*, Reprisal. when a state refuses to fulfil its obligations, or to permit another nation to enjoy rights which it claims. They are *positive*, when they consist in seizing persons and effects belonging to the other nation in order to compel them to give satisfaction.‡ Reprisals may also be either *general* or *special*. They are styled *general* when a state, General. which has, or supposes it has, received an injury from another, formally commissions its subjects to take the persons and property of the other state wherever they may be found. "I do not," said De Witt, "see any difference between general reprisals and open war."§ They are, in fact, according to modern usage, the first step taken at the commencement of a war, and considered equivalent to a declaration of hostilities, unless an immediate satisfaction is made by the other state.|| *Special* Special. reprisals are where letters of marque are granted in time of peace to individuals who have suffered an injury from the governments or subjects of another nation. These were common in very early times in England, and are specially authorized by the 4th Hen. V., cap. 7. They have been regulated by treaties; by those of Munster between Spain and Holland in 1648; by those between England and Holland in 1654 and 1667; by that of Rhyswick in 1697; and of Utrecht in 1713; by the French Ordinance of Marine in 1681; by the Articles of Confederation of the United States of America in 1781, and by the treaty between that republic and the republic of Columbia in 1825 (1 Kent, *Comm.* 61). This kind of reprisals in time of peace has, however, been condemned generally by the jurists, and has fallen into almost total and deserved disuse. The effect of the confiscation of the property of a foreign state antecedent to an open rupture is ably

* 2 Wheaton, 257.

† Many writers on the law of nations have discussed, in connection with this subject, the law of retaliation, *lex talionis*. This is not, however, a sanction of the law of nations, as it is not a punishment for the violation of any principle of that law. Retaliation ensues the breach of what are called *imperfect obligations*, and which do not justify a resort to forcible measures. Where a state, for instance, is guilty of the breach of a simple custom, or establishes some partial right or law prejudicial to foreigners, the state whose citizens are prejudiced retaliates by exposing the citizens of the offending state to similar disadvantages. This is amicable retaliation (*retrosion de droit*).—Martens, *Précis*, liv. viii. ch. i. sec. 2. Vattel, liv. ii. ch. xviii. sec. 339-41; 2 Wheaton, 4.

‡ 2 Wheaton, 5.

§ Vattel, liv. ii. ch. xviii. sec. 345, Note.

|| Per Lord Mansfield, *Lindo v. Rodney*, Douglas, *Rep.* 613. The Syracusans, in the time of Dionysius the Elder, voted a declaration of war, and immediately seized the Carthaginian property in their warehouses, and the Carthaginian ships in their ports, and then sent a herald to Carthage to negotiate. Mr. Mitford considered this a breach of the law of nations. *Hist. Greece*, vol. v. p. 402-4.

* 4 Robinson, *Rep. Appendix* 4.

† On this subject, see 2 Wheaton, *Elem.* 225-8. Chitty, *Law of Nations*, 153-83.

‡ *Quest. Jur. Pub. lib. i. ch. iv. sec. 11.*

§ The "Betsey," 1 Robinson, *Rep.* 92.

|| Convention of 1801 between Great Britain and Russia, art. iii. sec. 4.

¶ The "Columbia," 1 Robinson, *Rep.* 134. The "Hoffnung," 6 Robinson, *Rep.* 116.

** Per Sir William Scott, The "Neptunus," 2 Robinson, 112.

†† The "Maria," 1 Robinson, *Rep.* 340.

‡‡ Vattel, liv. iii. ch. vii. sec. 114.

§§ The "Catherine Elizabeth," 5 Robinson, *Rep.* 232.

||| The "Fanny," 1 Dodson, 443.

Law.

explained by our jurist, Sir William Scott, on occasion of an embargo laid on Dutch property, after the breach of the treaty of Amiens in 1803, under circumstances which Great Britain considered an hostile aggression on the part of Holland: "The seizure," he said, "was at the first equivocal, and if the matter had terminated in reconciliation, the seizure would have been converted into a mere civil embargo, so terminated. Such would have been the retroactive effect of that contrary course of circumstances. On the contrary, if the transaction end in hostility, the retroactive effect is exactly the other way." — (The "*Boedes Lust*," 5 Robinson, *Rep.* 246.)

265. (2.) War. On the subject of war, we have already spoken at length; and here we may observe, that the long discussions in which jurists have indulged themselves, respecting the justice and injustice of wars, pertain rather to morals than to law. Every nation, in right of its independence, is justified in commencing any war when it sees fit: no other state has a right to dictate to it in this—the exercise of its own sovereignty. That all wars prosecuted in obedience to the rules of the law of nations are by that law treated as just, is evinced by the fact, that in every such case the nations engaged are considered entitled to the benefits, and subject to the obligations, which the law of nature imposes.

Law.

LAW.

CHAPTER III.

ROMAN LAW.

Sect. 1. Introduction.

Law. 266. THE history of legal systems is a subject of great interest to philosophic minds, though it does not possess the popular attraction of ordinary history. The details of political strife;—of wars begun, continued, and ended;—of treaties made and broken;—such is the stuff of which ordinary history is made. We love to hear of battles,—to follow the fortunes of a nation in prosperous and in adverse times,—to watch the windings of diplomatic intrigue,—to sympathize with the fate of our adopted party. But our study of ordinary history may have higher aim than mere intellectual pleasure. We may call forth our mental powers and our moral feelings into wholesome exercise, while we rejoice with the success of virtuous heroism, of patriotic spirit, of honest simplicity,—or mourn over the triumphs of fraud, ambition, and violence, while we investigate the causes of national improvement or decline, and seek to deduce general principles of policy applicable to other times and circumstances.

In strictly *legal* history,—at least, in the history of private law,—there is not so much to excite popular interest. There is little of romance in the detail of its progress. It presents no Marathon,—no Cannæ,—to our view. Yet some of the greatest minds have felt its claims to deep attention. “What,” to use the language of Burke (*Abridgment of English History*, ch. ix.), “can be more instructive than to search out the first obscure and scanty fountains of that jurisprudence which now waters and enriches whole nations with so abundant and copious a flood;—to observe the first principles of right springing up, involved in superstition and polluted by violence, until by length of time and favourable circumstances it has worked itself into clearness;—to view the laws, sometimes lost and trodden down in the confusion of wars and tumults, and sometimes overruled by the hand of power,—then victorious over tyranny, growing stronger and more decisive by the violence they have suffered; enriched even by those foreign conquests which threatened their entire destruction; softened and mellowed by peace and religion; and improved and exalted by commerce, by social intercourse, and that great opener of the mind—ingenious science.”

In this celebrated passage, Burke dwells principally on those *pleasures* of scientific contemplation which legal history affords; but it has also its practical *uses*. We cannot enlarge our acquaintance with human nature, and extend our knowledge of the dealings of God with men, without acquiring powers which may be wielded for good or for evil. By a philosophical study of legal history we learn the means of producing certain results,

and,—inasmuch as our moral approbation of laws may often depend, in some measure, on our observation of their effects,—we are furnished with additional data for estimating the propriety or impropriety of any proposed enactment. As the natural philosopher, during the performance of an experiment, endeavours, with comprehensive eye, to observe all the concomitant circumstances, and watches the successive changes which take place; so will the jurist survey the workings of ancient legislation, and, relying upon a certain constancy in human nature,—after making allowance for causes in operation now and not formerly, or formerly and not now,—will expect some identity in the residuary phenomena. It is true, indeed, that the theory of legislation, conversant as it is with the behaviour of moral agents capable of originating force, is liable to be affected by many capricious disturbances. It is also true that the course of legislation is, to a considerable extent, determined by circumstances,—such as soil, climate, race, antecedent history,—to which the systematic calculations of science can oppose little or no effectual resistance. Great caution is requisite in any attempt to transfer the institutions of one country to a people of different temperament, manners, and habits. The law may be good for Rome, which would be intolerable in Constantinople or in London. We may elicit principles and discover tendencies; but in the application of our principles, a tolerable approximation is all that can in general be hoped for. Still, it is something that a scientific lawgiver may even approximately predict the consequences of a law: some scope is still left to human energy and wisdom, acting with the weapons which experience furnishes. When the conscientious statesman reflects that his own feeble oars, plied with vigour and skill, may stem the drifting of a current, and that the laws which he introduces or supports may impress a new direction on the course of his country's fortunes, he will feel a deeper responsibility and be encouraged to redoubled efforts. The practical advantages of legal history, notwithstanding the inevitable drawbacks we have adverted to, are great and manifold.

The remarks which we have made with respect to legal history are applicable, more or less, to the study of *all* systems. There is no system, however barbarous and remote, from which some lessons of experience may not be drawn, and applied to existing events. It is with comparative history as with comparative anatomy; the dissection even of monsters and of brutes will enable the scientific physician to understand with clearer insight the functions, structure, and diseases of an ordinary man. There is no system in which, abstract-

Law.

On the peculiar claims to attention of the history of Roman law.

Law.

ing from practical application, we may not gratify our natural curiosity by diving into the recesses of human nature, and gaining a more extended prospect of the dispensations of Providence.

Ancient
Eastern
lawgivers.

We may read, with pleasure and instruction, the earliest accounts of the incipient legislation of hunters, fishermen, pastoral and agricultural people;—may view, as in a picturesque panorama, the priests of Egypt and the Brahmins of India;—may regard, in the sluggish dynasties of the East, the birth and pilgrimages of those philosophic legislators whose dim figures appear through the mists of mythology with an undefined, but gigantic, outline; and may follow the footsteps of Orpheus, Musæus, or Pythagoras, as they travelled to enrich the versatile West with mutilated and corrupted, but still valuable, treasures of holiness and wisdom.

The Jew-
ish law.

The theocratic legislation of the Jewish people presents views of still higher, but of very different, interest. The importance of the Mosaic Law to the jurist is not small, even when we abstract those religious considerations which more properly belong to the province of the divine. But Christians have occasionally erred in attempting to apply to the circumstances of modern times rules and institutions adapted only to the unique position of a peculiar people. Selden, in his treatise *De Jure Naturali et Gentium juxta Disciplinam Ebræorum*, endeavours to separate the laws which exclusively bind the Hebrews from those which are of universal obligation. In such researches there is obviously much danger of incorrect interpretation and of incorrect reasoning from analogy. In supernatural revelations of the Divine will we should rather look for general rules applicable to the *hearts* of men than expect to discover in detail political maxims and legislative principles intended for the compulsory regulation of their external conduct. The historical records of the *Old Testament* are, without doubt, intended to inculcate by example some lessons of general importance to the temporal interests of nations;—to show that righteousness exalts, —to show that sin is a reproach; and in this sense we may yield our assent to the noble lines of our own Milton (*Paradise Lost*, iv. 357), when he speaks of the Jewish prophets—

“As men divinely taught, and better teaching
The solid rules of civil government
In their majestic, unaffected style,
Than all the oratory of Greece and Rome.
In them is plainest taught, and easiest learnt
What makes a nation happy and keeps it so,
What ruins kingdoms, and lays cities flat.”

If, however, we read the polemical pamphlets and sermons of the time of the civil wars in the reign of our first Charles, we shall often meet with passages where the special institutions which seem to be expressly intended (among other purposes) to distinguish and separate the Jews from the Gentiles are quoted as universally binding; and where the special commissions given to the children of Israel to subjugate the land of Canaan, and to execute the judgments of Jehovah upon idolaters, are treated of as precedents of the *jus belli* applicable to existing events.

The work of Michaelis (Joh. Dav.), *Gründliche Erklärung des Mosaischen Rechts* (8vo.; Frankfort, 1776-1780), is one of the best commentaries upon the Law of Moses. It was translated into English by Alexander Smith, D.D., Minister of the chapel of Garrioch, Aberdeenshire (London, 1814; 4 vols. 8vo.) In giving it this character we must by no means be con-

sidered as sanctioning many of the opinions and views found in it, and still less as approving the prurient spirit in which many parts of it are written. There are some able remarks on the Mosaic Law in the *Notices of the Mosaic Law*, written by the late Hugh James Rose, in which the works of M. B. Constant and M. Salvandy on this subject are animadverted upon. There are also many useful reflections in the admirable *Lectures* of Dean Graves upon the *Pentateuch*; but the questions to which this writer directs attention are rather theological than legal.

The early legislation of the Grecian states is involved in uncertain tradition. The greater part of their legal systems was probably indigenous and the result of gradual development, although the Grecian authors, in their fondness for historical theories, usually attributed to foreign derivation any law or custom to which they discovered a resemblance in foreign countries. We cannot, however, reject the tradition that many of their most ancient lawgivers borrowed from Oriental sources, and introduced. *per saltum*, novel rules into preceding institutions; but we regard with scepticism the details of such introduction. As we English are prone to attribute to Alfred a great number of reforms in which he had no share, so the Grecian historians have a tendency to the mythical idealization of such lawgivers as Solon or Lycurgus. Each is a legal Hercules—a collective person—carrying off the praise of exploits belonging to other men and other times.

Even in the early and dark ages of Grecian legislation, we may learn to trace events as they emanate partly from human will, partly from overruling causes, local or general;—may remark the changes effected in law by intercourse with strangers, increase of wealth, inventions and discoveries in arts and sciences, division of labour, the propagation of new opinions maintained by leading spirits, and the shocks of external attack or internal revolution.

The later laws of the Grecian states,—especially the laws of Athens,—have strong claims upon the attention of the scholar; but it cannot be denied that they are too imperfectly known in some cases, and that in others the bustling action is too rapid and of too local an interest, to render them proper subjects of detailed research in an Article necessarily so limited in extent as the present.

That a people so many-sided, so æthereal, so full of native fire, so abounding in the inspiration of genius as the Greeks—a people never equalled in painting and sculpture—wonderfully proficient in music, history, oratory, poetry—profound as well as acute in metaphysical research, and passionately fond of litigation—should not have displayed, in the cultivation of jurisprudence, remarkable ingenuity and refinement could not have been anticipated, and is, in fact, contradicted by extant specimens of Grecian legislation. Yet, gifted as they were with talents the most diversified, keenly alive to every allurement of sense, easily impressed with every form of beauty,—nay, often enthusiastic in their admiration of virtuous sentiments and actions,—the Greeks were not distinguished by a simple, diligent, and permanent worship of justice. They were distracted by too many exciting pursuits to develop calmly and patiently the principles of private law. Faction was their element; and the legislation of victorious faction was tinged with the bitterness of party spirit. Ambition, fear, revenge—the consequences

Law.

Law. of party spirit—were excited to frenzy by the rhetorician and the demagogue, and often unduly agitated lawgivers, jurors, and magistrates.

Of the private law of the Athenians, our knowledge is principally derived from the remains of the orators; from the explanations of lexicographers, as Harpocration and Pollux; and from the incidental notices of ancient scholiasts. From the time that Greece had attained its highest civilization, it did not remain long enough independent to foster a distinct class of lawyers, devoting all their powers to the cultivation of their profession, and seeking, like the Roman juriconsults, through a succession of ages, to give symmetry to the body of law, to establish sound principles, to solve doubts, to reconcile cases of apparent conflict, and to follow legal rules into their remotest deductions. Even with the laws of Athens our acquaintance is fragmentary; not a single systematic or institutional work upon Athenian jurisprudence having come down to us.

Those who desire to study the existing fragments of Athenian law, in a collected form, may consult the following works:—

Desiderii Heraldi Quæstionum Quotidianarum Tractatus; ejusdem Observationes ad Jus Atticum et Romanum cum Salmasii Defensionibus (Paris, 1650).

Jo. Meursii, *Themis Attica* (Traj. ad Rhen. 1685; 4to.).

S. Petiti, *Leges Atticæ*, with the observations of Wesseling and others (Lug. Bat. 1741; folio); being the third volume of the collection entitled *Jurisprudentia Romana et Attica*. Sam. Petit is often inaccurate.

Potter's *Antiquities of Greece*,—a work well known and esteemed in this country,—is learned and useful.

Platner (Ed.), *Der Prozess und die Klagen bei den Attikern* (2 Bände 8vo.; Darmstadt, 1824-5.)

Meier (M. H. E.) and Schömann (G. F.), *Der Attische Prozess* (8vo.; Halle, 1824.) This is the most valuable treatise which has yet been written on the subject of Attic Procedure.

The English reader who desires to investigate the public law and political institutions of the Grecian States, as treated of in modern works, will consult the *Histories* of Gillies, Mitford, and Thirlwall; Clinton's *Fasti Hellenici*; and the treatises, which have been translated into English of Heeren, Müller, Boeckh, C. F. Hermann, and Wachsmuth.

The Articles upon the Laws of Ancient Greece signed with the initials C. R. K. (Charles Rann Kennedy), in *The Dictionary of Greek and Roman Antiquities*, edited by Dr. Smith, and lately published by Taylor and Walton, are exceedingly accurate, and contain the fruit of much original research. They have the further advantage of directing the reader to the best modern sources of information.

Sect. 2. *Advantages of the Study of Roman Law.*

267. The institutions of the ancient Grecian States bear a fraternal resemblance to each other, and to the early law of ancient Rome. But the history of Roman law possesses an interest far less temporary and local. From the length of time during which the empire of Rome continued, from the intrinsic merits of the Roman system, from its subsisting effects upon the institutions of the modern world, from its connexion even with our own jurisprudence, and its utility as an object of comparison and contrast, the Civil Law (as the Imperial Law of Rome has been called by way of

eminence) deserves to be studied with peculiar diligence; but in order fully to understand the Civil Law or the Imperial Law of Rome, it is necessary to trace back to its origin the earlier history of Roman jurisprudence.

In this Chapter,—which is intended to recommend the study of Roman law to English readers,—we shall examine it as affording a type of that blending of popular and technical elements—that progress from the religious, the poetical, and the symbolical to the artificial and practical,—and of that decline of living energy when the period of active growth is ended, which, according to Savigny,* mark the course of all systems of law; we shall then view it in its historic bearing on our own law as administered in some of our various courts of justice; we shall point out its importance in the illustration of Roman history and philology, and we shall conclude by attempting briefly to discriminate some of its characteristic merits and defects.

Experience seems to shew that, in the growth of Roman law legal systems, the combined result of necessary causes and of the arbitrary exertions of the human will is manifested in a deviation from uniformity which is not unlimited; and Rome endured as a living empire for so many ages that, in following the origin, progress, and vicissitudes of her legal institutions, we are presented, on a great scale, with a pattern (if any where such a pattern can be found) of average development. We see law in its infancy, youth, manhood, and decline. We are not allowed indeed to witness its very birth. That is shrouded in the darkness of antiquity.

There can be no doubt that the greater part of the early Roman law, not relating to legal procedure, but regulating the primary rights subsisting between man and man, was first observed as custom.

Slight and almost imperceptible is the boundary between mere custom and law. The origin of custom, as distinguished from law, is not traceable to an individual or to a body of men. It seems to arise spontaneously in the minds of multitudes. Mere custom lives on without taking notice of its own existence. It has no future aim. It exercises no active powers. In a system regulated by law there is consciousness of volition: a conception of a governing power,—a rational standard to which it is felt that human conduct ought to conform in time to come. Custom passes into law. When a custom begins to be associated with the feeling of right, when the omission of its observance is felt as an unjust disappointment of well-founded expectations, and when those who, either by usurpation or by consent, exercise authority in a community, begin to employ their force in punishing the violation of such custom,—the custom is already converted into a law. The execution of laws, thus originating out of custom, and sanctioned by general feeling, adds to the authority of the magistrate, and gives him the power of imposing as legal commands the dictates of his own will, though in part morally unjust or socially inconvenient. The legality of the better is imparted to the worse. However much the notions of legality and justice may become separated in the progress of events, they are originally closely connected. The arbitrary commands of a despot, not respecting to a considerable extent the customary usages of a people, and not coinciding with their feelings of justice, might be obeyed from fear, but would never be regarded as legal.

* See his celebrated *Essay on the Vocation of our Age for Jurisprudence*, translated by Mr. Hayward.

Law.

Perhaps in the earliest period of Roman jurisprudence there were several branches of law which derived their existence immediately from the creation of legislators without any previous existence in the form of custom. In early law, there is often much that is palpably arbitrary, and there is much which evinces by its unity and system the invention of some single scientific head. There are several gradations from that order which is created at once,—springing like Minerva from the brain of Jove,—to the order which is the result of the slow victory of the more reasonable among conflicting customs. To distinguish by internal evidence the creature of direct legislation from the creature of custom, and to determine the departments in which one or the other predominates, is an interesting and instructive speculation, for which the early history of Roman law affords ample scope.

Religious
character
of early
Roman
law.

In the early Roman polity, so far as it appears to have been derived from the Etruscans, (*see Festus v. Rituales*), may be observed that close connection of law with *religious* solemnity which almost invariably characterizes the legislation of ancient people. Among the Romans, as among almost all ancient people, the dark doubts which, in the absence of supernatural revelation, are felt as to the future and the past; the belief of superhuman powers, and the irrepressible longing for the unknown and the infinite, were connected with worldly affairs, and made the instruments of action on the hopes and the fears of men.

Symbolical
character
of Roman
law.

Ceremonies of different kinds, varying with the genius of the people, are always found in early laws. Of these, some are obviously symbolical; some, from the commencement, are more abstruse, and live on in practice when their inner meaning is forgotten, or known only to antiquarians and lawyers. Rome presented no exception to this rule. Among the early Roman juriconsults, one of the greatest mysteries of the legal craft was the preservation and due application of the more ancient symbolical forms. These were employed partly to excite the attention and impress the memory; but there was a necessity for their existence in the same mental constitution, which, without thought of utility, fills the vocabulary of primitive people with metaphor, and their style with parable. Most legal forms, however they may differ in felicity of imagery, are originally a natural drama. In later times, artificial science and the prose of busy worldliness may alter their character and deaden their spirit, but they often still retain a remnant of their original nature. They have two of the three great elements of poetry. They are "simple and sensuous."

Introduc-
tion of the
technical
element.

In early Rome, though the customs or laws which regulated the private rights of individuals, anterior to and independent of the rights connected with legal procedure, were known to the people at large; yet there always existed a *technical* branch of law, unknown to the people at large, regulating legal procedure and those secondary rights which are connected with it. Barbarous and oppressive in many particulars as the early Roman Law of Procedure undoubtedly was, it had the effect of bringing into existence a body of professional lawyers, and this consequence in the sequel proved highly conducive to the improvement of private law. When the affairs of life became complicated: when commerce created new wants, and reflection gave birth to new arts, the common people were no longer able to master the increasing store of legal knowledge relating

even to their own private rights, and were obliged to resort for the resolution of doubts and difficulties to the technical juriconsult. The opinions of a class of men, qualified by education and experience to judge correctly on the questions proposed to them, had, as might have been expected, considerable influence on the development of private law. The decisions of the prætor felt this influence. It was manifested in his edict. It was shewn in the systematic and logical form which distinguished the institutional treatises written by lawyers. And it was not alone in form and symmetry that the law was improved. The Roman mind seems to have been always sensible to the claims of justice; and the rigid doctrines of the Stoic philosophy which was professed by many of the most eminent jurists contributed to give an air of virtuous dignity to many of the rules admitted into practice. The learned leisure of the professional lawyers, and their forward-looking character, combined with the rarity of interference by the legislature in the regulation of private rights, were the means of saving the Roman system from that patchwork appearance which is usually presented by a system formed from time to time under the pressure of the occasion. To the great abilities, accurate reasoning, sound judgment, and correct principles of the juriconsults of the golden age of Roman law (from Cicero to Alexander Severus), we owe many of its most valuable and enduring passages. They seem to have aimed to reconcile the dictates of reason with the rigid rules of law. They taught the magistrates how to employ legal fictions, when ancient technicalities would otherwise have been too strong in a direct contest with the rising demand for reformation. Powerful dialecticians, and great masters of a clear and concise style, their genius was yet so eminently practical that they rarely descended into scholastic divisions. Leibnitz compares their writings for close logic to those of the ancient geometers. "*Ego—auctorum labores,*" says he, "*ex quibus Digestorum opus exceptum est, admiror, nec quidquam vidi, sive rationum acumen, sive dicendi nervos spectes, quod magis accedat ad mathematicorum laudem.*"* They calculated with words, and the steady use of legal terms in a fixed sense, and the studious rejection of redundancy, of ornament, occasioned a close resemblance between the diction of most of them. Their workmanship was often more valuable than the materials upon which it was employed. It was happy indeed for Roman law that it was principally in legal procedure and in relations peculiarly affecting the Roman citizen that the system was hampered by barbarisms, which the plentiful introduction of fictions could not altogether remove. In the branches of law which were sometimes designated by the terms *jus gentium*, the rules that reason and justice would dictate were usually stated with much perspicuity, and reduced into highly scientific order. It was the authority of the juriconsults which conferred upon Roman law the impress of high morality and distinguished intellect. It was the juriconsults who made it immortal. If it were only for the models of legal style and reasoning which Roman law presents, its study would be as highly desirable to the modern lawyer as is the contemplation of the master-pieces of ancient art to modern painters and sculptors.

A time came when the law was oppressed by its own weight, when the complexity of practice, the long series

Law.

* Leibnitz, *Œper*. tom. iv. pars iii. p. 254. Edit. Genev.

Law. of authoritative writings, the unwieldy bulk of express enactments, and the multitude of voluminous commentators, were sufficient to bewilder the most resolute jurist. Then came the collections of law of which the most celebrated was made under the auspices of Justinian, and has survived entire to the present day.

Phases in the history of Roman law since the time of Justinian. From the origin of the city to the legal consolidations of Justinian, the Roman law extended over a period of about thirteen centuries. The fate of the Imperial Law from the death of Justinian,—its long slumber, or rather drowsy existence, during the ascendancy of Teutonic jurisprudence; its rapidly increased vitality in the XIIth Century of the Christian era; its introduction into the legislation of modern Europe under the authority of adoption by the sovereign powers of several states, and the recent diminution of its influence as a living system on account of the codes which regulate the administration of justice in some of the continental nations; these are events of no mean importance in the history of mankind,—unparalleled in importance by any other system of law, and exhibiting phases which illustrate all other systems. Some of the reasons which make it expedient for a philological student to select a *dead language* as the subject of his investigation, and some of the motives which should guide his choice, apply, *mutatis mutandis*, to fix the attention of a scientific jurist upon the Roman law. The analogies, indeed, are many which subsist between the studies of law and language. As a man of one language can scarcely understand that language well, so a man who has studied but one legal system will be deprived of many of those suggestions and associations which comparison and contrast furnish, and, for want of their aid, will be unable to connect those disjointed fragments which otherwise would be viewed in their proper places as parts of a continuous whole. Legal systems, like plants, may differ greatly from each other, but the botanist who has watched a single noble plant through all its processes from its germination to its withering, will have acquired a species of knowledge which will usefully direct observation and suggest experiments as to the properties of other plants.

Utility of the study of Roman law to the English lawyer. We have dwelt upon the importance of carrying these views into the study of Roman law, because the English lawyer is too often contented to confine himself to the investigation of his own system apart from all others. Among the evils which arise from this exclusiveness and isolation is a want of capacity,—prevalent among English lawyers,—to comprehend the reasonings or even to understand the nomenclature which pervade continental legislation and the writings of foreign jurists. Crippled and constrained by the contemplation of a single object, the mind loses the power of free attitude and unbiassed view. This evil has not been unfelt in diplomatic controversies on disputed questions of international law; for misunderstanding and want of sympathy are occasioned by diversities in the use of language and the association of thought.

Connection between Roman and English law. The *Corpus Juris Civilis* is to the majority of English lawyers an unknown book, and yet if we were to investigate the beginnings of many doctrines which now hold undisputed place in our law, we should be obliged to own that Rome was their true birthplace. In making this statement, we would, at the same time, guard the reader against the common error of supposing that historical derivation must exist wherever there is internal resemblance. Some principles of right are

understood by all men; and if two nations prohibit theft and murder, no inference can thence be drawn as to the originality of the one and the imitation of the other. Similar circumstances among different people will produce similar laws, and here and there a resemblance may be due to accidental coincidence. Legal antiquarians, like etymologists, have often committed gross errors of criticism, by deciding upon the derivation of legal systems according to fanciful and superficial theories. We are far from asserting that any of our common law has descended to us from the time when the Romans occupied this island—when, according to the conjecture of Selden, based upon a passage of Dion Cassius (lib. lxxii. in *excerpt. Xiphilin.*), Papinian might have been heard pronouncing judgment in a British court. Beyond, perhaps, a few popular traditions, some branches of royal prerogative, and some uneffaced imitations of Roman institutions apparent in our municipal corporations, the Saxon dynasty seems to have obliterated the traces,—never probably very widely nor very deeply impressed,—of Roman jurisprudence. In the barbarous specimens of legislation due to the era of Saxon and Danish rule, the few texts of Roman law which occur appear to us to be traceable through the papal canons. How faint is the impression which even the Anglo-Saxon laws have left upon our system! We have still the local court and the local officer; and some of the rude democratical elements of judicial procedure and constitutional law have been matured into institutions of real, civilized liberty; but, happily for us, the harsh and partial regulations, savouring of original Teutonic savageness, which awarded the various penalties of crime, have passed away, and the ancient absence of all express regulation in many most important points has been supplied by the legislation of more enlightened times and more cultivated men. Of the refined yet exceedingly oppressive system of Norman feudalism,—with its *maritagium*, its aids and wardships,—which was introduced throughout England after the Conquest, as rapidly as circumstances would permit, the antecedent history is exceedingly obscure, and notwithstanding many striking analogies, it is impossible to make out that the Norman writs and pleadings had any historical connection with the Roman *formulae*, or the process which was despatched in the name of the emperor.

The history of the Roman law in this country has been treated of by Selden in his *Dissertation on Fleta*, by Dr. Duck, by Hurd in the first of his *Dialogues on the Constitution of the English Government*, and by Von Savigny in various parts of his *Geschichte des Römischen Rechts im Mittelalter*. Soon after the establishment of the school of Irnerius at Bologna, the Pandects were brought to this country by Vacarius in the reign of Stephen. All that is known of Vacarius, who was the first professor of the Civil Law at Oxford, has been collected by the late Dr. Wenck of Leipzig. The early text-writers of English law have borrowed, more or less, from Roman sources, principally through derivative channels. Scarcely anything in Glanville (who wrote in the reign of Henry II.), except the formal introduction, savours of Justinian; but large portions of the first three books of Bracton's *Treatise De Legibus Angliæ* (which appeared in the reign of Henry III.) are taken from the Institutes of Justinian, not directly, but through the medium of Azo's *Commentaries*. Whoever (as we have done) compares with Bracton the *Summa Azonis* will see that the Roman portions of the former are, for the most part,

Law.

Roman dominion in Britain.

Anglo-Saxon period.

Introduction of Norman feudalism.

Roman law taught in England.

Early text-writers. Bracton.

Law.

Reasons
why the
Roman law
made no
progress in
the courts
of Common
Law.

copies or abbreviations of the latter, with occasional accommodations to existing English institutions. The English legislature, though not remarkably attached to originality, as may be seen by comparing (see Houard in his introduction to his Treatise on the *Coûtumes Anglo-Normandes*) some of our ancient laws with the earlier capitularies of the Frankish kings,—rarely, in early times, adopted the language or the principles of the Roman law. Every one has heard of the repudiation by the parliament of Merton of the doctrine of legitimation *per subsequens matrimonium*, when, to repeat an often-quoted sentence, “*Omnes comites et barones unâ voce responderunt, nolumus leges Angliæ mutari quæ hucusque usurpatæ sunt atque approbatæ.*” In this decision, it is probable that the jealousy of ecclesiastical ascendancy had much weight. The canon and civil laws were associated in idea, and the barons dreaded the introduction of a system of jurisprudence, which might have impaired the vigour of the feudal tenures, with all their lucrative incidents. Higher motives of patriotism may have mingled with prejudice and dislike. Among the best known portions of the imperial law—the only portions still known to many—were certain sentences concerning the authority of the emperor, which occupied a prominent place in the van of the Institutes and the Pandects. “*Quod principi placuit legis habet vigorem, cum, Lege Regiâ, quæ de Imperio ejus lata est, populus ei et in eum omne suum imperium et potestatem conferat.*”^{*} Bracton, by a kind of pious fraud, misinterprets, and paraphrases in an extraordinary manner, the words “*cum Lege Regiâ quæ de Imperio ejus lata est.*” “*Id est,*” says he, “*non quicquid de voluntate regis temere præsumptum est, sed quod consilio magistratuum suorum, rege auctoritatem præstante, et habitâ super hoc deliberatione et tractatu, rite definitum est.*”[†] In other words, the king’s pleasure is law; but that is the king’s pleasure which is indicated by a royal law passed in the imperial council. Other passages, equally at variance with the spirit of our constitution, which, even in early times when the limits of the prerogative were not well defined, was always adverse to autocratic monarchy,—other passages, equally requiring a decent veil or a fallacious gloss, are to be found straggling in several parts of the Imperial Law. Upon the main body of Roman Law, formed in republican times, they were mere excrescences. They did not pervade the system, but probably if the system had been partly introduced into England, occasion might have been taken gradually to broach doctrines adverse to liberty. The feeling prevalent among common lawyers, of the dangerous and slavish nature of a study with which their acquaintance was small, was fostered by the most celebrated of our early constitutional writers, Fortescue; and, for the combined reasons to which we have alluded, there was little systematic borrowing from Roman stores on the part of the parliament and the common law judges. A maxim or a rule was, ever and anon, stolen from the Institutes and the Pandects, but, as usually happens when plagiarism is committed by men whose knowledge is superficial and hastily acquired, was almost always distorted or misunderstood. The most learned civilians among our judges, as Holt, were mere amateur players with the Roman Law. Even that which was tolerably unadulterated upon its introduc-

tion was soon modified by English notions; and thus it happens that, though many phrases and expressions are shared by our common law and the civil law, there are few which are used in the same sense in both.

There are other English tribunals—of which the Admiralty and Ecclesiastical Courts are the principal—wherein the law that is administered and the manner of administering law, bear in many particulars plain marks of immediate Roman derivation. To the advocates who practise in those courts, the study of the Roman law is advantageous for reasons which do not so strongly apply to the student of common law. The advocate, in the imperial collections and the commentaries of civilians may often find the very doctrines which he is in need of for practical application: doctrines, which, from the mere circumstance of their origin, would be addressed with a shew of authority to the ears of the judge, though unaided by very particular precedents and very clear analogies, and independent of their intrinsic reasonableness. In our Common Law Courts, the civil law has no such *primâ facie* welcome, and closer proof of antecedent adoption in circumstances similar to the case in question seems to be required. In both kinds of courts, any reception of the civil law depends, according to orthodox legal theory, upon customary adoption in particular cases; but, in fact, the adoption is more specific and not so wholesale in the courts of Common Law. In the Courts of Common Law, occasions do not often arise of illustrating a doubtful question by a quotation from the Institutes, the Code, the Pandects, or the Novells. Some such questions indeed may occur. In putting a construction upon the Statutes of Distribution, for example, reference might be made, without any imputation of misplaced learning, to the 118th Novel of Justinian. Cases where the jurisdiction of the Common Law Courts and the Courts Christian is concurrent may afford opportunities for such quotation; and there are branches of mercantile law, which have been developed by Lord Mansfield and other great judges, not without attention, either immediately to Roman jurisprudence, or to foreign usage based upon rules of Roman law. Upon such subjects, a citation from the *Corpus Juris Civilis* might afford so apposite an illustration, that by a barrister of wary discretion and delicate tact it might be hazarded even in a Common Law Court, without exciting much ridicule or provoking a remonstrance from the bench.*

In an intermediate position, with respect to the civil law, between the courts of common law and the courts to which we have last alluded, is the Court of Chancery. Many of the leading doctrines of English equity, though they have been much modified in our tribunals, are manifestly taken from the collections of Justinian.

* “One day, *in banco*, a learned gentleman, who had lectured on the law, and was too much addicted to oratory, came to argue a special demurrer. ‘My client’s opponent,’ said the figurative advocate, ‘worked like a mole under ground, *clam et secretè*. His figures and law Latin only elicited an indignant grunt from the chief justice. ‘It is asserted in Aristotle’s Rhetoric’—‘I don’t want to hear what is asserted in Aristotle’s Rhetoric,’ interposed Lord Tenterden. The advocate shifted his ground, and took up, as he thought, a safe position. ‘It is laid down in the Pandects of Justinian’—‘Where are you got now?’ ‘It is a principle of the Civil Law’—‘Oh, sir!’ exclaimed the judge, with a tone and voice which, to a punster’s ear, would have abundantly justified his assertion, ‘we have nothing to do with the Civil Law in this court.’”—26 *Law Magazine*, p. 73.

* *De Constitutionibus Principum*, l. 1, pr. D. (i. 4.)

† Bracton, lib. i., c. xvii. s. 7.

Law.

Progress of
Roman law
in other
English tri-
bunals.

Connec-
tion of Ro-
man law
with the
Court of
Chancery.

Law.

Our ancient chancellors, up to the time of Wolsey, were mostly ecclesiastics, and had not that horror of a foreign jurisprudence which distinguished some of our common lawyers,—which made Lord Coke exult in the fancied *autochthonous* character of English legislation, and praise his countrymen as legally deserving to be styled "*Penitus toto divisos orbe Britannos*." The notions of "*Forum Romanum*" and of "*Jus Prætorium*" have been generally present to the minds even of more modern chancellors; some of whom, as Lord Nottingham and Lord Hardwicke, afford indications of having studied the civil law with considerable attention. To the equity barrister, not merely for general discipline, but for the special illustration of his own department of law, we recommend an original knowledge of the *Institutes* of Justinian, and of those parts of the *Digest* which, in legal treatises on equitable subjects, are often quoted at second-hand and explained incorrectly.

Dulcius ex ipso fonte bibuntur aquæ.

Light shed
by Roman
law on Ro-
man his-
tory and
philology.

We have endeavoured to point out, without exaggeration, the connection which subsists between English and Roman law, and the extent of the benefit which an English lawyer may acquire by becoming a learned civilian. We proceed to advert shortly to the light shed by Roman law upon Roman history and philology.

Many historians have been ignorant of law; and yet there are few nations whose history can be properly written without a careful study of their jurisprudence, not only as it affects the constitution of the governing power, but as it regulates the private rights of individuals. Perhaps, of all histories, the Roman would be the most incomplete, if it did not include an account of the progress of the legal institutions of Rome. The Roman people, even more than the people of our own country, were imbued with the spirit of their laws. They were accustomed to legal solemnities, and often regarded with greater jealousy an infringement of established forms than an inroad upon their civil liberty made under cover of constitutional observance. This is particularly observable in the commencement of the reign of Augustus. The Roman child was catechized in the Twelve Tables or the Prætor's Edict; and the Roman man delighted to witness the litigious contentions of the Forum. If we examine the Roman character for the sake of discovering its most conspicuous features, we shall be led to designate it, as, in the first place, warlike, and, in the next, *legal*.

*Tu regere imperio populos, Romane, memento,
— pacisque imponere morem.*

Philosophy, fine arts, and polite literature, the Romans borrowed from the Greeks; but in the cultivation of military science, and in the development of an elaborate system of jurisprudence, they manifested something like original genius.

That legal genius which was so essential an ingredient in their lives, and those peculiar institutions which coloured all their nationality, must be duly appreciated by all who would enter into the spirit of their history. The historians and other classical writers of Rome are full of allusions to legal notions. In the Latin books which are ordinarily read in schools and colleges, many passages occur which to ordinary readers, ignorant of legal history, are either incomprehensible or wholly misinterpreted. Take, for instance, the orations of Cicero *pro Quintio* and *pro Cæcina*, and see

VOL. II.

how slightly the majority of English scholars can penetrate their sense and spirit.

Law.

Character-
istic me-
rits and de-
merits of
Roman
law.

It has been our object, in what we have said in recommendation of the study of Roman law, to give a correct and critical estimate of its real advantages without indulging in the hyperbolic panegyric which is common among those who have either devoted much time to the investigation of its niceties, or are willing to earn a cheap reputation by praising a species of learning in which there are not many able to detect their deficiency. The enduring and wide-spread authority of the Roman law, surviving the power of the Roman arms and the destruction of the Roman empire, is a proof of its great merit, and is a memorable example of the ascendancy of reason over brute force. It had also great defects. The marks of its narrow and barbarous origin were never entirely effaced; and the indirect attempts to efface them were the cause of excessive technicality and over-subtle refinement. In the style of the later Imperial constitutions, there is an Asiatic pomp and tumour which contrasts unfavourably with the vigorous conciseness of an earlier age. Even the real improvements of later times disturbed the symmetry of the ancient system, which, with elaborate logical consistency, harmonized and connected the evil and the good.

No enlightened legislator in modern times would resort to the tortuous fictions,—the exclusive distinctions in favour of citizens,—the exorbitant powers of the father, husband, and master,—the want of protection against the abuse of authority in the government of dependencies,—the rudeness in the administration of criminal justice, which characterized even the brightest periods of Roman jurisprudence. The jurist of modern times, on the other hand, would be rewarded by an investigation of the Roman doctrines of successions, the estimation of damages, the determination of the *onus probandi*, the principles of interpretation, the analysis of obligations, the nature of the events by which rights begin, are transferred, and ended. Even where the subject is worthless, the manner in which it is handled by a Gaius or an Ulpian affords a lesson rich in practical use.

Sect. 3. *On the Study of the Origin and Changes of the Legal System of Ancient Rome.*

268. The most improving mode of studying Roman law is to regard it not merely in one of its several phases, but to investigate its commencement and trace its changes. To English scholars and jurists in particular, an historical study of Roman law, rather than a technical study of the compilations of Justinian, is to be recommended. We proceed to point out some of the authorities which may be consulted with advantage, and to give some account of the literature of Roman law.

Leibnitz seems to have been the first who divided the history of law into *internal* and *external*. Though these two parts are not strictly separable, yet most modern writers have adopted the distinction, using the words in a sense rather different from that in which they were originally employed by Leibnitz.

"Historical jurisprudence," says Leibnitz, "is either internal or external. The former enters the very substance of jurisprudence. The latter is only an auxiliary and support. The internal history of law gives an account of the laws themselves of various states. In

Division of
the history
of law into
internal &
external.

5 c

Law.

my opinion a more special enumeration is still wanted. In Roman law, for example, we want a statement of the changes made by each tribune through *plebiscita*, by each consul through *senatus consulta*, by each prætor through his edicts, by each emperor through his constitutions, until the system grew into its present form.* To explain the meaning now usually attributed to the distinction:—External history treats of the modes by which laws come into existence, and of the manner in which the science of law has been from time to time cultivated. It explains the political events which have had an influence upon the progress of legislation, and gives a chronological account of laws, lawgivers, and legal works. It is sometimes divided into two parts: (a) history of the sources; (b) history of the literature.

No history can do all that external legal history is required to do, without adverting to the contents of the laws themselves; but a history, the *principal object* of which is to trace the variations of doctrines of law—to consider, for instance, the different rules prevailing at different times with respect to property, judicial organization, procedure, penal law, the relations of personal *status*, and so forth—is called an *internal history*.

It is obvious that a history, in order to be complete, ought not to confine its views either to the *sources* of law—that is to say, the modes by which law comes into existence,—or to the *contents* of law—the variations which the propositions and conceptions of law from time to time undergo. A complete history of Roman law must be both external and internal. It will direct attention not only to the constitutional changes in the legislature and the magistracy, and to the dry names of laws and lawgivers and legal works, but to the progressive development of the social state of the Romans, and to the general spirit of those institutions, civil and political, which affected the character of their public and private life. How inseparable, in the reality of things, are the subjects of external and internal history, may be rendered plain by considering that as the habits and thoughts of people give rise to actions and political events which produce changes in legislation, so, alternately, do the contents of new laws give rise to new thoughts and habits. External and internal history resemble those geometrical figures *which circumscribe each other*.

Necessary
imperfec-
tion of the
History of
Roman
law.

In a history of law like the Roman, we must expect to find many absolute gaps, and many doubtful questions. The most diligent research has left many things obscure and unelucidated. We may meet with a statement, for instance, that a law has existed, without being told when, or how long. This happens frequently when we have recourse to writers not professionally legal. Sometimes, intentional alterations and interpolations have been made, without notice, by subsequent compilers.

Different writers, in treating of the history of law, have

* "*Jurisprudentia historica est vel interna vel externa: illa ipsam jurisprudentiæ substantiam ingreditur: hæc adminiculum tantum et requisitum. Historia juris interna est, quæ variarum rerum publicarum jura recenset; desidero tamen specialem recensionem, quid a quolibet tribuno per plebiscita, aut a quolibet consule per senatus consulta, prætorum per edicta, et imperatore per constitutiones ordine innovatum sit, donec in hanc formam jus Romanum crevit.*" Leibnitz, *Nova Methodus discendi docendæque Jurisprudentiæ*, pars ii. sec. 29.

adopted different methods. Some have systematically classed the subject of law, and have treated separately the history of each subject. Thus, in one book or section, may be found the history of the law of persons; in another, that of property; in a third, that of successions; in a fourth, that of obligations; in a fifth, that of actions.

Law.

Syn-
chro-
nistic
and
successive
systems of
legal his-
tory.

Other writers divide the history of Roman law into periods, and give a complete view of every department of law in one period before they proceed to the next. This mode of writing the history of law has been called *synchronismus*, and there has been much dispute among jurists whether it is to be preferred to the former mode which treats successively of different subjects from the beginning to the end of the history of each. Both plans have their respective advantages and disadvantages. If the synchronistic plan gives a more commanding *coup d'œil* of a given period, and traces more accurately the relations of different parts, and the mutual influence of contemporaneous changes; the chronological or successive presents more distinct views of separate departments, does not distract the attention by bounding from one topic to another, and is exempt from the objection of grouping together subjects whose relative velocities and *crises* of development are so different, that the natural halting-places in the history of one are unsuitable to the history of another.

Gibbon, in the 44th chapter of his *Decline and Fall*, divided, as others had done before him, the history of the Roman law, up to the reign of Justinian, into periods, and his division presents such marked differences in the successive periods that, with slight alteration, it has been followed by Hugo. Hugo, *ö pärv*, translated into German, and annotated this celebrated chapter, and Warnkœnig published at Liege, in 1821, Guizot's French translation under the title, "*Précis de l'Histoire du Droit Romain, par E. Gibbon*," adding his own and Guizot's notes to a selection from Hugo's. In place of the words *moveable* and *immoveable*, Gibbon uses *real* and *personal*, in the sense familiar to English readers. He falls into some positive errors, but his well-arranged, luminous, and comprehensive sketch shows the master's hand.

Notices of
some of the
Histories
of Roman
law.

We shall now mention some of the principal historians of Roman law both before and after the publication of the *Decline and Fall*. The fragment of the *Enchiridion* of Pomponius (*De Origine Juris*), which has been preserved in the second title of the first book of the *Digest* or *Pandects* of Justinian, is the principal ancient authority professedly treating of the history of Roman law. It has been commented on by Bynkershoek, (*Oper.* vol. 4, Lug. Bat. 1743, p. 1) and Vandermaelen; and the valuable preface of Heineccius (*Opera*, tom. iii. sect. 2, p. 66), to J. L. Uhlii *Opuscula ad Historiam Juris pertinentia* (Hall, 1735), may be consulted with advantage. One of the last editions is that of Pernice, Göttingen, 1822. This fragment, which Gibbon supposes to have been abridged, and probably corrupted, by Tribonian, is, in many parts, difficult and obscure.

The works of Sigonius, *De Antiquo Jure Civium Romanorum*, *de Antiquo Jure Provinciarum*, and *De Judiciis*, are among the most valuable early historical essays on Roman law. Notwithstanding the more accurate information upon some points furnished by the recent researches of Niebuhr, Savigny, and Madwig, the materials collected by Sigonius will be always of great use to the scholar.

Law.

Jac. Gothofredi *Manuale Juris* contains one of the earliest systematic histories of Roman law. It has great merit, and is still popular in France. There is an edition by Berthelot, 8vo. Paris, 1806.

J. V. Gravina, *Origines Juris Civilis* (with the notes of Mascovius, Leipzig, 1737; and of Sergius, Naples, 1756—58), is celebrated for the excellence of its rather diffuse and declamatory latinity. It contains a great variety of valuable information, and, in spite of some inaccuracies, will be consulted with pleasure.

J. Gottl. Heineccii, *Historia Juris Romani et Germanici*. The later editions of this well-known work are enriched by the Notes of Ritter and Silberrad (and Kugler?). It treats chiefly the external history of Roman law. In the excellent *Antiquitatum Romanarum Syntagma* of the same author (of which the last edition is that of Mühlenbruch, Frankfurt, 1841, incorporating the notes of Haubold), the internal contents of Roman law are arranged according to the order of Justinian's *Institutes*, and each subject in succession is treated historically. Heineccius was deemed by Sir James Mackintosh (*See his Discourse on the Law of Nature and Nations*) to have been the best writer of elementary works that ever lived,—not even excepting Euclid. Heineccius has the art of writing clearly, concisely, and intelligibly. He begins at the beginning, and unfolds his subject regularly, citing the most material authorities at sufficient length in his text, instead of taking for granted much subsidiary knowledge in his reader, and merely referring to many books which it might be troublesome and inconvenient to consult. It is amusing to read the contemptuous criticisms of some modern authors upon the works of Heineccius, and to view the small holes they attempt to pick. Let, for example, Hugo have espoused a different side from Heineccius on some doubtful question, and straightway they set down Heineccius as guilty of error. They sometimes blame him for being ignorant of facts only brought to light by recent discoveries. The writer of a critique on Adam's *Roman Antiquities* in the 4th volume of the *Thémis*, p. 365, says of Adam, "*Il suit pas à pas Heineccius, que toute l'Ecosse prend encore pour guide, lorsque l'Allemagne et la France l'ont délaissé.*" One of the principal faults of Heineccius, in common with the civilians of his day, is the use of technical expressions, of barbarous latinity, though sometimes clear and expressive, in place of the classical phraseology of the ancient Roman juriconsults. Thus, he speaks of *obligationes, reales, verbales, literales, consensuales*, instead of "*obligationes, quæ re, verbo, literâ et consensu contrahuntur.*" He was imperfectly versed in the remains of Ante-Justinian law. The notes of Haubold on his *Antiquitatum Rom. Syntagma* are very good but short. The notes of Mühlenbruch and the portions of new text written by him are learned, but remarkable for the inelegance of style* which characterizes the works of this author. The title, *De Actionibus*, (lib. iv. tit. 6) is left in a very confused state from the mixture of heterogeneous materials. A skilful hand is required neatly to restore a valuable old work.

The *Historia Jurisprudentiæ Romanæ* of Bach is a convenient and satisfactory book of reference. It

* In lib. iv. tit. 6, s. 24, we observe the following specimen of Latin grammar: "*Neque verò propter damnum infectum illâ etiam ætate usu venit legis actio, si quidem commodiore et pleniore jure uti poterant cives Romani, ex quo stipulatio damni infecti nomine præstanda præponerat prætor.*"

has been pronounced by Haubold,—a most competent judge,—to be "*liber aureus, tironum manibus nunquam excutiendus.*" The last edition of this work appeared, Leipzig, 1807, with the notes of Stockmann. Dr. Irving, misled possibly by the circumstantial announcement in Ersch's *Literatur der Jurisprudenz*, No. 234, asserts in his *Introduction to the Study of the Civil Law*, 4th edit. 1837, p. 189, n. 4, that "there are other editions of a more recent date." Wenck, Steinacker, and Schilling have been successively mentioned as engaged in preparing a new edition, which is much wanted.

Hugo's *History of Roman Law* (of which the 11th edition appeared in 1832) is the result of profound learning, combined with vigorous and original intellect. The style of the book is harsh and crabbed. It abounds in long and involved sentences. Facts and theories are often briefly and indirectly alluded to as known, and errors are often corrected, or important information conveyed, in a parenthesis. Schilling's *Kritik* (Leipzig, 1829) upon the 10th edition of this history is a useful accompaniment. Jourdan (not the late eminent doctor of law, but a doctor of medicine) translated, in French, the 7th edition of Hugo's work, but has not successfully performed his difficult task. He rarely assigns their proper force to the thickly-scattered particles, and not seldom misses or distorts the true sense. To use the words of Hugo himself, "he understood neither the law nor the language." There is an unfinished Latin translation by Warnkœnig.

Schweppe's clear and well-written history often affords satisfactory information not easy to be met with elsewhere. As an example of this, we may mention his explanation (p. 900 in sect. 522) of the "*senes ad coemptiones faciendas, interimendorum sacrorum causâ,*" alluded to by Cicero *pro Murænâ*, 12. The third edition, with notes by Gründler, was published at Göttingen in 1832. Hence it appears that Dr. John Reddie, in his *Historical Notices of the Roman Law*, (Edinburgh, 1826,) was mistaken in supposing that "Schweppe's attempt at a history of Roman law had fallen still-born from the press." Schweppe is often in opposition to Hugo, who is over-fond of novelty and minute criticism.

Zimmern's *History of Roman Law* is unfortunately left incomplete on account of the author's death. The first and third volumes only have been published, containing "*Private Law,*" and "*Civil Process.*" It is stated by Tigerström (*Innere Röm. Rechts-Geschichte*, p. 30,) that Huschke has undertaken to finish the work by writing a second volume. From the acuteness and originality which it displays, as well as its vast erudition, it is the most important contribution to the legal history of Rome since Hugo's book. The author's sentences are often carelessly composed, and his style has some of Hugo's defects.

The history finished by Professor Walter of Bonn, in 1840—the first part having appeared in 1834—is written in easy and perspicuous language, and, from the comprehensiveness of its view, may be recommended as exceedingly well calculated for the "*homo unius libri.*" Walter does not enter at length into the discussion of controverted points, but is judicious in his choice of results. A good translation of his book would be a valuable acquisition to English literature.

The number of professed histories of Roman law is so great that it would occupy too much space to mention them all. Tigerström's *Internal History* is a tole-

Law.

Law.

nable compilation, though it is not free from unsound views and inaccurate statements. The account it gives of the old law of procedure is carefully composed. In sect. 128, we observe that the author, from the error of supposing *lanx* to mean a lance, falls into some absurd misunderstandings of the authorities with respect to *furtum per lancem et licium conceptum*. The companion volume on *External History* appeared in 1841. It abounds in misprints and cannot be commended. Macieowski's *Historia Juris Romani* (Warsaw, 1825,) and Holtius, *Historiæ Juris Romani Lineamenta*, are short, and adapted to the use of commencing students. They chiefly follow Hugo. The brief outlines of Klenze (1835) and Rudorff (1841) are very useful from their well-chosen selection of authorities. The reader may consult with advantage the section on Roman law in Falck's *Juristische Encyclopädie*, of which the fourth edition has been translated into French by Pellat (Paris, 1841).

French
histories of
Roman
law.

In French, there is a short history by C. J. Ferrière, known in this country by an English translation. There are also slight sketches by A. M. J. J. Dupin, (*Précis Historique du Droit Romain*, Paris, 1825,) whose inaccurate little work has been translated into German and Spanish; and by Ortolan, *Histoire de la Législation Romaine*, (2nd edit. Paris, 1842). The agreeable essay of Warnkœnig, *Histoire Externe du Droit Romain* (Bruxelles, 1836), may be read by the student who will be content with a very rapid survey, but in several parts it requires further elucidation. The most ambitious works in French on the history of Roman law are those of Terrasson and Berriat St. Prix. The former is designated by Gibbon "pompous but feeble," and his work is stated to be "of more promise than performance." It derives some value from the "*veteris jurisprudentiæ Romanæ monumenta*" appended to it; but the text of these monuments is exhibited in a very corrupt state, and will be found more purely given in the collections of Spangenberg and Haubold, of which we shall presently speak. The work of Berriat St. Prix (Paris, 1821) contains the fruit of some original research, and furnishes much amusing anecdote and illustration. The author idolizes Cujas, whose life is subjoined to the history. In this respect (and in most others) he differs widely from Hugo, who frequently cites Cujas for the sake of finding fault. M. St. Prix is very severe in his treatment of the glossators, and a little blind to their merits. The English reader will (not unjustly) form an unfavourable opinion of his accuracy from the following curious passage, although, when M. St. Prix wrote, there was more foundation than now there is for his oddly mistaken representation of English law.

"Ici la surprise des modernes redouble. Sont-ce donc là, s'écrieront-ils, ces législateurs dont nous suivons encore, après vingt-cinq siècles, des décisions que les juristes vantent sans cesse? Mais les anciens pourraient récriminer dans cette occasion comme dans la précédente. Chez eux les actions des lois avaient été imaginées par la politique. Les Anglais, qui, choisissant en quelque sorte dans le droit Romain ce qu'il y a de plus défectueux, n'en ont retenu que les actions, pourraient-ils donner une semblable excuse? Comment justifieraient-ils entr'autres cette caution ridicule exigée pour l'obtention de leurs Writs peremptoires, qui sont proprement des actions des lois?—caution, dont ils ont tellement senti l'inutilité, que, malgré leur respect servile

pour leurs coutumes, ils la réduisent dans toutes les affaires à la simple prononciation de deux noms, John Doë et Richard Roë, qui ne designent que des individus imaginaires."—pp. 57, 58.

To this information are added the following reference and note. *Voy. Baert, Tableau de la Grande Bretagne*, (an 8 (1800), t. 2, p. 354, et suiv.)

"Lorsqu'il n'y a pas de Writ pour une action d'une espèce imprévue, (et tel était aussi l'inconvénient du système des Romains), il faut pour créer ce nouveau Writ, un Acte du Parlement." *Voy. ibid.* p. 353. "Ainsi, l'on met en mouvement toute l'autorité législative pour une formule inutile!"

In our own language, there is no separate work of much importance on the history of ante-Justinian law. Dr. Bever's *History of the Legal Polity of the Roman State* is a heavy book. It has been translated into German, and Schomberg's *Historical and Chronological View of the Roman Law*, into French. Mr. Charles Butler has given outlines of the history in his *Horæ Juridicæ Subsecivæ*. R. P. Burke's *Historical Essay, &c.* (2nd edit. 8vo., Cambridge, 1830) is the production of a man of cultivated mind and good sense. There is much historical information in that interesting repertory of miscellaneous learning, Dr. Taylor's *Elements of the Civil Law*.

Spence's *Inquiry into the Origin of the Laws and Political Institutions of Modern Europe*—an able and scholar-like performance—enters upon the legal history of the Roman provinces, and treads on ground which Savigny has since more minutely explored.

We subjoin the names of some of the works in various languages on the history of Roman law of which our limits do not permit us to give a detailed account.

With respect to the history of the history of Roman law, it is to be observed, that for a long time legal history was little attended to, but when, in the middle of the 16th Century, the study of it began to be diligently pursued, legal professors were generally contented to introduce historical remarks, as it were parenthetically, into their dogmatical lectures. In the 18th Century, separate lectures upon Roman history first became general. Works upon external history, which were the most common, were usually published under the title *Historia Juris Romani*; and works upon the internal history under the title *Antiquitates Juris Romani*. After Hugo, it became not unusual to blend external and internal history throughout; and, at the present day, systematic works under the title *History and Institutes of Roman Law*, making the knowledge of external history introductory to dogmatical information, are coming more and more into vogue.

Onuphrii Panvinii, *Reipublicæ Romanæ Commentariorum libri tres* (Paris, 1588).

Aymari Rivalii, *Civilis Historiæ Juris libri V.* (Mogunt. 1533).

Valentini Forsteri, *Historiæ Juris Civilis libri tres* (Basil, 1565): inserted also in the first volume of the great collection of Ziletus, usually known by the name of *Tractatus Tractatum*.

M. Ant. Ferratii, *Epistolarum (a) libri tres* (Patav. 1699); (b) *libri sex* (Venet. 1738).

G. Schubart, *De Fatis Jurisprudentiæ Romanæ, curâ Tillingii* (Lips. 1797).

Ch. G. Hoffmanni, *Historia Juris Romani* (tom. ii. Lips. 1718—26). A valuable work.

Law.

Law.

Jo. Sal. Brunquelli, *Historia Juris Romano-Germanici* (Amst. 1751).

Rud. Frid. Telgmann, *Einleitung zu der Geschichte der Römischen Rechts-Gelehrsamkeit, von Scheidemantel* (2 Theile. Leipz. 1780).

Jo. Aug. Hellfeld, *Historia Juris Romani* (Jen. and Lips. 1740).

Udalr. Oetti, *Historia Juris Civilis* (Styr. 1769).

Jo. Ant. Riegger, *Historia Juris Romani Privati potissimum* (Frib. Brig. 1770).

Reitemeier, *Encyclopädie und Geschichte der Rechte in Deutschland* (Gött. 1785).

Hufeland, *Lehrbuch der Geschichte und Encyclopädie aller in Deutschland geltende Rechte* (Jena, 1798).

The first part of this unfinished work contains an introduction to the history of Roman law.

We may here remark that the word *Encyclopädie* conveys very different notions to German and to English ears. To the German, *Juristische Encyclopädie* means an introduction to the study of law, containing general outlines and *præcognoscenda*, and is either formal, confining itself to method and nomenclature, or material, giving a cursory view of the contents of law, or both formal and material.

Jos. Tosc. Mandatoritii, *Juris Publici Romani Arcana* (tom. v. Neap. 1767—80).

J. H. C. de Selchow, *Elementa Juris Romani Ant Justiniani* (Gött. 1778).

C. A. Güntheri, *Historia Juris Romani* (Helmst. 1798).

A. Hummel, *Handbuch der Rechts-Geschichte* (3 Bände, Giessen, 1805).

Th. Max. Zachariæ, *Versuch einer Geschichte des Röm. Rechts* (Breslau, 1814). The author of this work is not to be confounded with C. S. Zachariæ, who has written a curious tract *De Originibus Juris Romani ex Jure Germanico repetundis* (Heidelberg, 1817).

C. A. Gründler, *Handbuch der Rechts-Geschichte* (1er Band, Bamb. 1821. Unfinished, 1842).

W. Rein, *Das Römische Privat-Recht, &c.* (Leipz. 1836).

Ch. Ch. Dabelow, *Römische Staats- und Rechts-Geschichte im Grundrisse* (Halle, 1817).

Lud. Pernice, *Geschichte, &c. des Röm. Rechts im Grundrisse* (Halle, 1823).

H. R. Stockhardt, *Tafeln der Geschichte des Röm. Rechts* (Leipz. 1828).

Christiansen, *Die Wissenschaft der Röm. Rechts-Geschichte im Grundrisse* (1er Band, Altona, 1838. Unfinished).

The author is a disciple of the philosophy of Hegel.

Danz, *Lehrbuch der Geschichte des Röm. Rechts* (1er Theil. Leipz. 1840. Unfinished).

Ch. Giraud, *Introduction Historique aux Eléments de Droit Romain de Heineccius* (Paris et Aix, 1835).

Guérard, *Essai sur l'Histoire du Droit Privé des Romains* (Paris, 1841).

Burchardi, *Staats- und Rechts-Geschichte der Römer* (Stuttgart, 1841).

This book contains much in a small compass.

Geib, *Geschichte des Römischen Criminal-Processes* (Leipz. 1842).

This interesting and elaborate treatise, though we often see cause to differ from its results, merits the praise of originality, and supplies an important desideratum in the history of Roman law.

An account of some of the works above mentioned

may be found in Dr. Irving's *Introduction, &c.* A complete list has not been attempted. Some of the omissions of modern names may be supplied by referring to Engelmann's edition of the *Bibliotheca Juridica* (Leipz. 1840). See the head *Römisches Recht* in the index, p. 581. Where there are several editions of the same work, the latest has usually been mentioned, and the titles have been abridged. There are excellent chronological tables of Roman law, forming the greater part of what is called the second volume of Haubold's *Institutionum Juris Romani Privati Historico-dogmaticarum Lineamenta*, as edited after the author's death, by D. C. E. Otto (Lips. 1826).

In none either of the French or English works which we have named can the reader expect to find any very detailed and satisfactory account of the views which have been disclosed by the recent discovery of additional sources, or by diligent research into the old authorities. In the successful cultivation of Roman law the Germans at this day take precedence of every other nation. Some of the modern speculations are important in themselves; some have been raised to importance by temporary fashion, to the comparative neglect of investigations which in the preceding generation were in vogue. In some of the modern German manuals, a student would be disposed to remark a certain dogmatism or pedantry in the mode of statement, and a want of subordination and symmetry in the composition. Minute points and doubtful conclusions are dwelt upon at undue length, as if they were certain, and not, after all, insignificant.

The reader who desires to make himself master of the progress of modern speculation will find great assistance from some of the continental journals and reviews of jurisprudence, which have been made the organs of literary party, and have received contributions from the ablest civilians of the day. The most celebrated of these for their illustration of the history of Roman law are,—

Hugo's *Civilistisches Magazin* (7 Bde, Berlin). This magazine was commenced in 1791, and some of the early volumes have been several times reprinted.

Zeitschrift für geschichtlichen Rechts-Wissenschaft, von T. C. Von Savigny, &c. (Berlin). This journal is now edited by A. A. F. Rudorff, and has reached the 11th volume.

K. Von Grolman's *Magazin für Rechts-Wissenschaft*, continued by Von Löhr (4 Bde, Giessen, 1800—25).

Rheinisches Museum für Jurisprudenz (7 Bde, 1827—1834). Up to 1830 inclusive, this journal was published at Bonn. It subsequently appeared at Göttingen.

Thémis, ou Bibliothèque du Jurisconsulte (10 tom. Paris, 1819—29). Complete copies of this valuable review are now scarce.

The *Kritische Zeitschrift, &c.* published at Heidelberg under the editorship of Mittermaier and Zachariæ, and Fœlix's *Revue de Législation Française et étrangère*, occasionally advert to modern works on Roman law.

The historical cultivation of the Roman law received its first great impulse in modern times from Hugo, and this impulse was most ably enforced by Haubold and Savigny. The sound learning and critical judgment of A. W. Cramer gave him high authority with the most distinguished scholars of his day, though his published works on historical subjects were not voluminous. In the history of Græco-Roman law, Biener, the two brothers Heimbach, and the younger Zachariæ, occupy the

Law.

Superiority of the modern Germans in the cultivation of Roman law.

Journals and reviews.

Names of modern writers eminent for their historical researches in Roman law.

Law.

first rank amongst the moderns; in the study of the most ancient documents of Roman law, Dirksen. The new sources recently discovered have employed a host of professional *savans*, and stimulated the zeal of historical research. Among the names which, in this pursuit, have ornamented the present century may be enumerated Schrader, Von Löhr, Erb, Unterholzner, Goeschen, Clossius, Wenck, Beck, Otto, Bethmann-Hollweg, Böcking, Hasse, Blume, Puggäus, Witte, Buchholz, Barkow, Klenze, Heffter, Francke, Schilling, Keller, Haenel, Huschke, Sell. To these we might add many a "*fortem Gyam, fortemque Cloanthum*."

In exciting such remarkable activity, the spirit of party had no small influence. The spirit of party, arising from the controversy between Savigny and Thibaut on the expediency of introducing a general code into the states of Germany, soon after the year 1814, divided the continental jurists into the philosophical* school, followers of the latter, and the historical school, disciples of the former. The collision of opinions kindled excessive ardour on both sides, and for some time there seemed to be danger of a divorce between philosophy and history. Thibaut thought that sound principle was better than precedent, science than erudition, good laws than ancient customs. The philosophic Thibaut, honoured and lamented, has been carried to the grave, and no one now is readier than the illustrious Savigny to admit that history and philosophy ought to go hand in hand:

*Alterius sic
Altera poscit opem res, et conjurat amice.*

Notice of
elementary
works on
Roman
law

The number of elementary works on the imperial law of Rome is even greater than the number of historical treatises. The ancient course of instruction was confined rigidly to the elucidation of the original text of Justinian's collections by notes or glosses. This plan was gradually abandoned in the 17th Century; but still the best academic manuals, if they did not assume the shape of continuous commentaries or lectures on portions of the *Corpus Juris Civilis* (as was sometimes the case), ordinarily followed the arrangement of the *Institutes* or *Pandects*.

A course of instruction upon the *Institutes* usually presented a general survey of the science, while a second course upon the *Pandects* necessarily descended more into practical details. We have favourable examples of these later modes of teaching in the *Prælectiones* of Huberus, the excellent *Recitationes* of Heineccius, the two *Elementa Juris* of the same author (one *secundum ordinem Institutionum*, the other *secundum ordinem Pandectarum*), and the two *Principia Juris* of Westenberg.

Towards the end of the last century, elementary writers began very commonly, in explaining the law of the *Institutes* or *Pandects*, to adopt arrangements of their own; and this mode of teaching was called the *systematic*, to distinguish it from the former, which was called the *legal*.

Recourse to the systematic plan was rendered expedient from the old method not being in itself sufficiently scientific. Though the arrangement of the *Institutes* was more scientific than that of the *Pandects*, neither arrangement was well adapted to a course of

* The principal periodical publication, which served as an organ of the philosophical party, is the *Archiv für die Civilistische Praxis*, published at Heidelberg. It has reached (1842) 22 volumes.

instruction in those developments of the civil law which were most important in modern practice. The advantages of the change which admitted the adoption of arbitrary system were found to counterbalance the facility of reference, the power of easily comparing and estimating different treatises, and the authoritative catholicity, which were the recommendations of the older method.

Many elementary works combine the *history* of doctrines, with exegetical and dogmatical information; and thus some good works,—as, for example, those of Rein and Pernice,—which we have classed among the historical treatises, might as well have been reserved for mention here. On the other hand, many elementary works confine themselves to the dogmatical illustration of such parts of the civil law as are of practical use at the present day.

It is not unusual for a German professor to publish *Outlines* (*Grundrisse*), with references to authorities, for the use of students attending his lectures, and, after some years, to transform the lectures themselves into a detailed elementary treatise. From the general plan of teaching by lectures in the German universities,—and of teaching not only those branches of knowledge which serve to discipline the mind, but those which are actually employed in the business and professions of mature life,—there arises a closer connection than with us between academic education and the general supply of literary productions. This remark is necessary to enable Englishmen to appreciate the character of many German law-books.

To begin with German elementary works: there are valuable outlines by Heise, Burchardi, Gans, Blume, and Goeschen. German elementary works.

The most eminent scholars have not disdained to compose manuals for beginners. Haubold's *Doctrinae Pandectarum Lineamenta* has not been reprinted since it first appeared in 1820. Hugo's *Lehrbuch des heutigen Römischen Rechts*, which forms the fourth volume of his course of civil law, reached a seventh edition in 1826.

Of the highest order of merit is the condensed and profound *System des Pandekten-Rechts* (8te Ausg. Jena, 1834) of the late celebrated Thibaut. The second volume of the legal remains of the same author has been recently (1842) published at Berlin under the title *Lehrbuch der Geschichte und Institutionen des Römischen Rechts: Hermeneutik und Kritik des Römischen Rechts*. It is a concise and masterly production. More discursive and more fully unfolded in its various parts is the admirable *System des heutigen Römischen Rechts*, which presents the matured production of a long life of study, guided by the sound judgment and illumined by the penetrating intellect of Savigny.

In the choice of riches it is difficult to know whom to mention, and in what order.

The practical elementary treatise which has had the largest circulation is that of Mackeldey, *Lehrbuch des heutigen Römischen Rechts* (11te Ausg. Giessen, 1838). The eleventh edition, published after the author's death, has the advantage of K. F. Rosshirt's remarks and additions. The tenth has been badly translated into French by Jules Beving, of Brussels; and there are other translations into most of the modern European languages.

Almost equally popular is Mühlenbruch's manual, published in Latin under the name *Doctrina Pandectarum* and in German under the name *Lehrbuch des*

Law.

Law.

Pandekten-Rechts. The last genuine editions bear date Halle, 1838. There are Brussels reprints; and a separate collection has been made, by way of supplement, containing at length all the authorities referred to in the work.

Warnkœnig's *Commentarii Juris Romani Privati* (3 vols.; Liege, 1825), and his shorter *Institutiones Juris Romani Privati* (3tia editio; Bonn, 1834) unite the clear and pleasing writing of the French school with a fair share of that erudition which distinguishes the Germans.

Wening-Ingenheim's *Lehrbuch des gemeinen Civil-Rechtes* is a work of long-established reputation. It follows the outline of Heise; and the fifth edition, in three volumes (München, 1838), was revised by J. A. Fritz.

Schweppé's *Römisches Privat-Recht in seiner heutigen Anwendung* (5 Bände; 4te Ausgabe; Göttingen, 1828-1833.) The last three volumes were prepared for publication after the death of the author by W. Meier.

An improved edition (the second) of Marezoll's succinct *Lehrbuch der Institutionen des Römischen Rechts* came out at Leipzig in 1841. A manual for students by Boecking, of Bonn, has recently been published (1842.) Schilling's *Lehrbuch der Institutionen und Geschichte des Römischen Privat-Rechts* (unfinished, 1842) combines institutional with historical information. It enters elaborately into useful details, and exhibits rare accuracy and industry.

Lang, J. J., *Lehrbuch des Justinianisch-Römischen Rechts* (2te Ausgabe; Stuttgart, 1837.)

The posthumous lectures, published at Göttingen, of Goeschen on the civil law, do not belie the reputation of their late distinguished author. Von Vangerow's *Leitfaden für Pandekten-Vorlesungen* (Marburg, 1839, &c.) are said to be worthy of Thibaut's successor; and Kierulff's *Theorie des gemeinen Civil-Rechts* (Altona, 1839) has acquired a high character.

Haensel's *Lehrbuch der Institutionen des Rechts* (Leipz. 1842) promises well. The learned Puchta, whose excellent *Lehrbuch der Pandekten* (Leipz. 1838) is only one of his many important contributions to Roman jurisprudence, has given, in his *Cursus der Institutionen* (Leipz. 1842), a systematic course, historical and dogmatical, which, like Savigny's, traverses all the principal departments of the civil law.

We have mentioned no work which does not possess considerable merit. Probably several good books have been omitted from sheer ignorance; and certainly not a few inferior productions have been designedly passed over. The dates given in the preceding list will show how great has been the recent activity of publication.

Few works, if any, of civil law are better known to English readers than Domat's valuable arrangement, translated by Dr. Strahan. Since Pothier's labours on the *Pandects*, several publications on separate heads of Roman law have appeared in France. In particular, the law of possession has received culture, in consequence of the impulse given to the study of it throughout Europe by Savigny's finished treatise on the *Recht des Besizes*. In this department, M. Pellat's recent work demands honourable notice. The elucidation of the sources has not been neglected; and if the active spirit of research which immediately followed the discovery of Gaius has been somewhat diverted to feudal jurisprudence and the development of the existing legal system, the pursuit of the civil law still obtains a reasonable amount of patronage from the countrymen of

Cujas. Of modern French elementary writers we shall now name but two.

The *Chrestomathie* of Professor Blondeau (Paris, 1830) contains an excellent choice of texts for an elementary course, and there is much good sense in the author's remarks. Difficulties which would be slurred by a less logical and clear thinker are not disguised, and, even in cases where more complete solutions might have been given, the attention of the student is well directed to interesting objects of inquiry.

To the first volume of Ortolan's *Explication Historique des Instituts* (2me edition; Paris, 1840) is prefixed an introduction of about 100 pages, entitled *Généralisation du Droit Romain*. In these pages is manifested all the sparkling vivacity of which many French authors are so fond. The pointed enunciation of abstract propositions,—the bold induction of general principles,—the wide views lucidly presented within the compass of a few pages,—the happy command of metaphysical language—give to this introduction greater attractions than are due to the profoundness of its learning or the minute correctness of all its statements. Readers who look with suspicion on the dashing generalizations,—the sacrifice of laborious research and detailed fact to (so called) philosophical ideas,—the abandonment of sober criticism in search of epigrammatic and antithetical turns of expression—which are common in Lermnier, Michelet, and others of the school of writers to which Ortolan belongs, will scarcely be disposed to give him credit for that modicum of learning and correctness which are really displayed in his clever and brilliant performance.

There are few English elementary works on the civil law. This is principally owing to the practical pursuits which engross professional men after their release from the liberal studies of youth, and to the severance of our own from the imperial jurisprudence. In old times, the learned apprentices of the English common law, who had heard of the arbitrary maxims of the *Institutes*,—who were told that the civil law justified torture,—who saw, in an adoption of the Justinianean compilations, an acknowledgment of some extrinsic authority beyond the authority of reason—a stepping-stone to the domination of the Emperor or the Pope,—were loth to spend upon the illustration of a rival system those precious hours which were demanded by the exclusive jealousy of their own. The works of our civilians are not very important. Wood's *Institutes* or Ayliffe's *Pandect of the Civil Law* are now seldom referred to. Dr. Browne's *Compendious View of the Civil Law* (2nd edition, London, 1802) would not satisfy a modern inquirer. The well-arranged *Analysis* of Dr. Halifax (Cambridge, 1836) requires some editor who would not, like Dr. Geldart, scruple even to mention the name of Gaius.

A complete library of all English books, whether elementary or not, which treat of Roman law, would not be a load for many camels. Who, now-a-days, would consult Fulbeck's *Parallel*, look at Ridley's *View*, or displace from its dusty shelf Wiseman's *Law of Laws*? Some of our writers on ecclesiastical law, as Swinburne and Godolphin,—better casuists and canonists than civilians,—though they often refer to the *Corpus Juris*, seem generally to quote at second-hand. The *Jurisprudentia Philologica* of Eden, and the translation of Justinian's *Institutes* by Harris, are more frequently met with than read. Yet England has pro-

Law.

French
modern
elementary
works.

English
works on
the Civil
Law.

Law.

duced a few names not unregarded nor unvalued by foreign civilians. Of these the most eminent are Selden, Zouch, Duck, and Taylor.

The *Dictionary of Greek and Roman Antiquities* (London, 1842), edited by Dr. W. Smith, contains many very able disquisitions on separate departments of the law and constitution of Rome. It seldom, like the vulgar herd of commentators, dilates unnecessarily on things that are trite and futile; and it generally is honest enough rather to acknowledge than to shirk a felt difficulty. Most of its defects are inseparable from a work published periodically, and having many independent contributors. Hence the discordance of opinion which may sometimes be observed in the treatment of the same subject under different heads. We have occasionally, but more rarely, perceived that a single writer, puzzled between conflicting views, has been content, in different parts of the same article, to make two statements, each probably supported by high authorities, but irreconcilable one with the other. In matters connected with Roman law, the contributions of Mr. George Long will be found safe and consistent. To the Roman lawyer the study of Roman antiquities is essential. The profound antiquary will refer to the ample treasures amassed by Grævius and others, and will acknowledge the surpassing excellence of the classical-minded Italians of the 16th century—men like Manutius, Fulvius Ursinus, and Sigonius, “native there, and to the manner born;”—but the tiro in legal antiquities, in addition to his Heineccius and Adam, Fuss, or Creuzer, will do well to procure Dr. Smith’s *Dictionary*, for, upon the whole, there is no work in the English language from which he would be able to collect so much sound information on Roman jurisprudence anterior to Justinian.

Scotch
modern
writers on
Civil Law.

Scotland—a country more Roman in the Roman, more feudal in the feudal parts of its jurisprudence than England—has given some signs of attention to the Civil Law. Blair’s *Inquiry into the State of Slavery among the Romans* is in some respects a meritorious compilation, but has little pretension to rank high as a legal work; for, though it enters upon legal questions, it altogether overlooks some of the most important technical sources.

Reddie’s *Historical Notices of the Roman Law* (Edinburgh, 1826) is a book that deserves praise for having led the way in furnishing information as to the recent progress of the study in Germany. It does ample justice to Hugo and his followers. It was succeeded by Dr. Irving’s *Introduction*, a work which we have already frequently cited, and from which the student will derive much useful knowledge on bibliographical matters, and will learn some interesting—as well as some trifling—particulars with respect to the private history of modern continental civilians. In some instances, real scholars and men of genius, as Gans, are omitted, or, as Cramer, rapidly dismissed, while minor luminaries, as Maciejowski, occupy a disproportionate space. The author is minutely accurate in names and dates, but sometimes descends to frivolous gossip, and lugs in the result of his literary researches unseasonably. There is a smallness in his original criticisms, and “the legend of Amalfi” is made far too much of as a criterion of scholarship. The account of the materials of Roman law, upon which modern commentators work, is given in the commencement, which is much the best written part of the book.

In the United States of America, the Civil Law is perhaps more generally studied than in this country, but our knowledge of American literature does not enable us to name any original elementary works. Mr. Justice Story is glad to adorn his learned and popular treatises with illustrations drawn from the *Corpus Juris Civilis*. The polite and various reading of Story furnishes a copious repast, which is sweetened by the complacency of the host, and his lavish generosity of praise; but the English lawyer does not expect to discover in his pompous and conventional diction marks of that sharp-edged intellect which touches the true distinction and probes the hollow plausibility;—of that logical precision which tracks the vague or ambiguous phrase wherein error lingers or conscious hesitancy lurks;—of that sound and steady judgment which knows where to reconcile apparently conflicting authorities, and where to choose between them, guiding its choice not by number but by weight;—of that critical acumen which can appreciate intuitive sagacity, and assign to it the superiority over undigested erudition;—of that pure taste which will not, with the most unfastidious latitudinarianism, blend the good and the bad—white wine and red—Mitford, Beames, and Cooper. It is necessary to caution such of the numerous readers of Mr. Justice Story as have not deeply studied the Roman law, against the danger of so far trusting him as to accept his application of its rules and his construction of its language with undoubting confidence. In justification of this caution, we proceed to make two quotations from his celebrated *Commentaries on Equity Jurisprudence*.

Law.
American
works on
Roman
Law.

“The term Interdict was used in the Roman law in three distinct but cognate senses. It was, in the first place, often used to signify the edicts made by the prætor declaratory of his intention to give the remedy in certain cases, chiefly to preserve or to restore possession. And hence such an interdict was called edictal; *edictale, quod prætoris edictis proponitur, ut sciant omnes eâ formâ posse implorari*. Again, it was used to signify his order or decree, applying the remedy in the given case before him; and then it was called decretal; *decretale, quod prætor pro re natâ implorantibus decrevit*. And in the last place, it was used to signify the very remedy sought in the suit commenced under the Prætor’s Edict; and thus it became the denomination of the action itself.” (Vol. II. p. 143, sect. 865; 2nd edition, London reprint. 1839. Citing Livingstone *On the Batture Case*, 5 *Amer. Law Journ.* 271, 272; Brisson, *De Verb. Sig. Interdictum*; Vicat, *Vocab. Interdictum*; Heineccii, *Elem. Pand.* ps. 6, sects. 285, 286; but neither Brisson, Vicat, nor Heineccius gives any sanction to the text.)

To say, as Story in substance says, that the Interdict has *three* (not *four*) kindred meanings;—that it means sometimes, 1st. the Edict by which the Prætor promises to give the remedy; sometimes, 2nd. the order or decree applying the remedy; sometimes, 3rd. the remedy sought; and sometimes, 4th. the action itself,—is to make a statement which, to those who understand Roman procedure, is absurd, and to those who understand it not must appear so involved and incomprehensible—so pure a specimen of that “true no-meaning” which “puzzles more than wit,”—that one is surprised to find a writer of Mr. Justice Story’s eminence palming such an explanation upon himself and his readers for sense. We may be allowed to con-

Law.

jecture that the crass obscurity of the paragraph we have quoted arises from a grievous misunderstanding of a division which Oldendorp* supposed to exist, of interdicts into two kinds, analagous to the division of *bonorum possessio* into *edictalis* and *decretalis*, according to the different modes in which such possession was obtained. The forms of certain interdicts applicable to given circumstances were contained in the Prætor's Edict. According to Oldendorp, the interdicts obtained under circumstances to which the forms in the edict were applicable, were, *interdicta edictalia*. Those interdicts he calls *decretalia* of which there were no precedents in the edict, but which were issued by the prætor upon the exigency of a particular case. An English lawyer may illustrate to himself the difference by thinking of the difference between writs of which there were precedents in the *Registrum Brevium*, and writs *in consimili casu*. We have not been able to find any authority for Oldendorp's division of interdicts. It is probable that no interdict could properly have been granted in a case which did not come within the meaning or spirit of those general terms in which the prætor in his edict promised to give such a remedy to suitors. A decree (*decretum*) was an order formally made in court (l. 9, s. 1, *D. de Officio Proconsulis*, (i. 16)); and, if the analogy of the division of *bonorum possessio* into *decretalis* and *edictalis* were followed† we should be led, according to our interpretation of that division, to assign the name *interdicta decretalia* to those interdicts which could only be obtained by formal decree *pro tribunali*. Such may have been the case with all interdicts positively commanding a party to exhibit or to restore something; for these, as distinguished from prohibitory interdicts, were sometimes called *decreta*. It is not impossible that in some, at least, of the cases particularly defined in the prætor's edict, an interdict might have been obtained as of course, out of court, *de plano, per libelli subscriptionem, sine causæ cognitione*, and that such an interdict might have been called *edictale*.

In thus diverging, for the purpose of explaining the origin of Story's confusion on the meaning of the word interdict, we have taken the opportunity of making some statements which will be found of use hereafter, when we come to examine the nature of the prætor's jurisdiction.

The following example of incorrectness is scarcely less glaring.

Mr. Justice Story,—after asserting that an English Court of Equity, when the true owner of an estate can recover it at law, will give no relief, unless there was some fraud on the part of the owner, to a *bonâ fide* possessor on account of the money he may have expended in repairs and improvements,—goes on thus:—

"The Civil Law appears to have proceeded on a far broader principle of natural justice. For, by that law, any *bonâ fide* possessor, as for instance a creditor,

* Oldendorp, cited in *Calvini Lexicon Juridicum*, tit. *Interdictum*, was, we believe, the author of the Latin words inserted in Story's text,—words which, from their strange inapplicability, must have astonished the surrounding English.

† See l. 3, § 8, *D. de Bon. Poss.* (xxxvii. 1); l. 2, § 1, *D. Quis Ordo in Poss. servetur* (xxxviii. 15); l. 3, § 4, *D. de Carbon. Edict.* (xxxvii. 10); l. 1, § 4, *D. Si Tab. Testam. null.* (xxxviii. 6); l. 14, § 1, *D. de B. P. cont. Tab.* (xxxvii. 4); l. 1, § 7, *D. de Successorio Edicto.* (xxxviii. 9).

Law.

who had laid out money in preserving, repairing, or substantially improving an estate, was allowed a privilege or lien for such meliorations. *Creditor, qui ob restitutionem ædificiorum crediderit; in pecuniam quam crediderit privilegium exigendi habebit* (Dig. lib. xii. tit. 1, l. 25; 1 Domat, b. iii. tit. 1, sec. 5, art. 6, 7). *Pignus insulæ, creditori datum, qui pecuniam ob restitutionem ædificii extruendi mutuam dedit, ad eum quoque pertinebit, qui redemptori, domino mandante, nummos ministravit*, (Dig. lib. xx. tit. 2, l. 1; 1 Domat, b. iii. tit. 1, sec. 5, art. 5, 6, 7.)" (2 Story, *Eq. Jur.* p. 454, sec. 1239.)

Now, the cases adverted to in these citations, made through Domat from the *Pandects*, are quite inapposite to the proposition for the proof of which they are quoted. That a man who has lent money expressly for the purpose of rebuilding a house should have a priority of charge over other creditors upon that house, when rebuilt with his money, to the amount of the sum so lent and so expended, does not prove that a *bonâ fide* possessor, expending money upon his own account in building upon ground which, without his knowledge, belongs to another, is entitled to reimbursement as against the rightful owner to the amount either of the actual sum expended, or of the increased value imparted to the land. The creditor in Dig. lib. xii. tit. 1, l. 5, is not a *bonâ fide* possessor,—is not, in fact, a possessor at all. The second passage, cited from Dig. lib. xx. tit. 2, l. 1, is equally inapplicable. It means that a man who pays the builder by the direction of the owner for rebuilding a house, is as much entitled to a tacit charge upon that house as if he had lent the money in the first instance to the owner in order to rebuild it. Here is nothing like a lien for the benefit of a *bonâ fide* possessor making improvements without any legal compulsion on the one hand, and without the sanction or default of the legal owner on the other. Here, again, the creditor is not in the situation of a possessor at all. The following texts would have been more serviceable to the learned commentator:—§ 30, *J. de Rerum Divisione* (ii. 1); l. 33, *D. de Condict. Indeb.* (xii. 6); l. 14, *D. de Doli mali et Metûs Exc.* (xlv. 4). They prove that a *bonâ fide* possessor had the means of making his claim for *impensæ* available against the true owner by refusing to give up possession until he was indemnified; but that he lost all remedy whatever if, before his expenses were paid, the true owner obtained just possession.

English, as well as American, lawyers are so much indebted to the excellent tendencies of this accomplished judge, who has contributed greatly to liberalize his profession, by choosing worthy subjects of investigation, and treating them in no narrow spirit;—so much deference is paid to his authority in parts of the law not frequently traversed by the mere *leguleius*—the class of men to whom "*omne ignotum pro magnifico est*" is so numerous,—and there is so much danger of the baser ore gaining currency along with the genuine metal,—that it is the more necessary, in acknowledging the high merits of Story, to give due warning of his deficiencies and errors.

It would be impossible, without occupying a disproportionate space, to enumerate the best writers on the most important heads of Roman law. The following modern works are considered classical:—

Savigny, *Das Recht des Besitzes* (6th edition; Gies-sen, 1837).

Writers on the single heads of the Roman Law.

Law.

Hasse, *Die Culpa des Römischen Rechts* (2nd edition; Bonn, 1838).

Mühlenbruch, *Die Lehre von der Cession der Forderungs-Rechte* (3rd edition; Griefswalde, 1839).

Hasse, on *Dos* (unfinished); Puchta, on *Customary Law* (*Gewohnheitsrecht*); Keller, on *Litis Contestatio*; Unterholzner, on *Prescription* (*Verjährungslehre*); Marezoll, on *Civil Reputation* (*Bürgerliche Ehre*); and M. S. Mayer, on *Successions* (*Erbrecht*), rank high among the modern works illustrative of Roman law.

Some of the leading commentators on the sources will be mentioned in their proper place. As a specimen of select bibliography and biography, Haubold's *Institutiones Juris Romani Literariæ* (Leipz. 1809), so far as it extends, displays the soundest judgment and learning. The same author's *Institutionum Juris Romani Privati Historico-dogmaticarum Lineamenta*, edited by D. C. E. Otto (Leipz. 1826), deserves the highest commendation for its comprehensive and methodical distribution of subjects, accompanied by ample references to the ablest commentaries and disquisitions. Hugo's *Civilistische Bucherkentniss* (2 Bände; Berlin, 1828-9) is also a classical production. *Struvii Bibliotheca Juris Selecta*, edited by Buder (Jena, 1756), and Camus's *Bibliothèque Choisie des Livres de Droit*, appended to the *Profession d'Avocat*, and edited by Dupin, *aîné*, are useful works, and common in English libraries.

A fuller list of modern books is given in Ersch's *Literatur der Jurisprudenz, &c. seit 1750* (Leipz. 1823); but the most complete legal bibliography is contained in the *Bibliotheca Realis Juridica* of Lipenius, with its modern supplements.

For a detailed history of the cultivation of the Roman law in the middle ages, the famous work of Savigny (*Geschichte der Römischen Rechts im Mittelalter*; * 6 Bände; Heidelberg, 1826-1831) may almost be said to supersede all others. The work is mutilated in the feeble French version of Guénoux, but the first volume has been well translated into English by Cathcart.

In giving a general account of the progress of law at the commencement of this Article, we have shown how different countries in Europe have successively acquired pre-eminence in the cultivation of Roman Law, and have mentioned the names of some of its most distinguished votaries. English civilians have principally adhered to authors of the Dutch school, partly because of the former intimacy of our commercial intercourse with the Low Countries, partly because the Dutch commentators usually followed the laudable practice (almost deserted by their German successors) of writing in Latin, and partly because of the intrinsic and enduring excellence of their practical works.

It is almost superfluous to state, that even a moderate acquaintance with Roman law cannot be acquired without the study of general Roman history and Latin philology. Greek must be mastered by the deeper proficient; and, of modern languages, German affords the greatest help.

Works relating to the general history of Rome.

The English student of Roman law will peruse the historical works of Adam Ferguson, Malden (*Library of Useful Knowledge—History of Rome*), and Arnold, some of whose valuable contributions adorn this Encyclopædia. The whole of Niebuhr's *History* is now added to English literature. We also possess the brief

Manual of Heeren. The inquiries of Wachsmuth into the most ancient times of Rome (*Die ältere Geschichte des Römischen Staates*, Halle, 1818) deserve translation, proceeding as they do from the ablest of Niebuhr's opponents.

Among the recent German contributions to the constitutional history of Rome are the following works:—

K. D. Hüllmann, *Römische Grundverfassung* (Bonn, 1832).

Rubino, *Untersuchungen über die Römische Verfassung und Geschichte* (1er Band; Cassel, 1839). These researches are highly interesting and important.

Huschke, *Die Verfassung des Königs Servius Tullius* (Heidelberg, 1838).

K. W. Goettling, *Geschichte der Römischen Staats-Verfassung bis zu C. Cæsar's Tod.* (Halle, 1840). Very little reliance is to be placed upon Goettling's critical judgment or logical faculties.

Peter (Carl, Dr.), *Die Epochen der Verfassungsgeschichte des Römischen Republik* (Leipz. 1842).

For the interpretation of the technical terms of Roman law, the student could scarcely require a better guide than the admirable *Latin Dictionary* of Facciolati, himself no mean civilian. Some additional help, however, may be obtained from the work of B. Brissonius, *De Verborum Significatione*, edited by Heineccius and Boehmer (Halle, 1743); to which may be joined the *Addimenta* of J. Wunderlich (Hamburg, 1778), and the *Supplementum* of the deeply-learned A. G. Cramer (Kiel, 1815). Few books can be named more useful than the treatise *De Formulæ et Solemnibus Populi Romani Verbis* of the same Brissonius, edited by Bach (Leipz. 1754). The *Vocabularium Utriusque Juris* of Vicat, though it contains little that is original, is highly esteemed in this country. The *Clavis Ciceroniana* of Ernesti is in every one's hands. It often usefully explains laws and legal phrases, but the author occasionally betrays a want of familiarity with technical language. Thus,—to give an example pointed out by Hugo (*Röm. Rechts-Gesch.* 195),—he interprets the expression “*Χρήσει μὲν tuus, κτήσει δὲ Attici nostri*” (*Cic. ad Fam. vii. 29*), by the words “*usu quidem tuus, possessione autem Attici*” (*Ern. Clav. verb. Χρήσει*). A lawyer would have remembered the line, “*Vita mancipio nulli datur, omnibus usu,*” and would never have thought of translating κτήσει by *possessione*. The *Promptuarium* of Elvers (Göttingen, 1824) may be referred to in reading Gaius and Ulpian. The *Manuale Latinitatis Fontium Juris Civilis Romanorum, in usum tironum*, by H. E. Dirksen (Berlin, 1837), though it is the production of a well-read scholar and sound lawyer, does not supersede the older work of Brissonius. It may sometimes be necessary to refer to the invaluable *Glossarium mediæ et infimæ Latinitatis* of Ducange, of which there is an octavo abridgment by Adelung.

For the interpretation of the terms occurring in Græco-Roman law, may be consulted Ducange's *Glossarium mediæ et infimæ Græcitalis*, the Glossaries of Jac. Godefroi to the *Codex Theodosianus* and to *Harmenopulus* (the latter, with additions by Reitz, has been inserted in the Supplement to Meerman's *Thesaurus Juris Civilis*); Reitz's Glossary to Theophilus; and the *Glossæ Nomicæ*, edited by Labbe.

The collection of Latin Inscriptions by Orelli, and the numismatic works of Spanheim, Eckhel, Mionnet, and Rasch, will occasionally be found of assistance to the historical investigator of Roman Law.

Law.

* A second edition of the first three volumes appeared in 1834.

Law.

Sect. 4. *On the Original Authorities for the Materials of Roman Law.*Original
authorities
for the ma-
terials of
Roman
law.

From the preceding imperfect indication of some of those modern writers who have treated of the law of Rome, we pass to a statement of the authorities from which their materials are taken.

These authorities are of two kinds, untechnical and technical.

The untechnical authorities consist of those notices which unprofessional writers, historians, orators, philosophers, grammarians, poets, &c., have left to us in relation to the laws of Rome. With respect to the untechnical authorities, much reliance is not to be placed upon their accuracy in the incidental mention of legal subjects. It is not safe, for example, to draw a positive inference from the legal phraseology sometimes put, by way of joke, into the mouths of his characters by Plautus. A lawyer's eye may now and then detect legal inaccuracies in the classics. Thus Horace, who is vulgarly believed to have been a cordial hater of lawyers, speaks of the prætor taking away from a spendthrift the disposal of his property, and committing the *tutela* to his sane relations—(*Sat. lib. ii, sat. 3, l. 218*)—

Interdicto huic omne adimat jus
Prætor, et ad sanos abeat tutela propinquos.

The more technical word here would have been *cura*, though *tutela*, even in legal writers, is sometimes used as a generic word, comprising the *cura* as well as the *tutela* strictly so called. The inaccuracy is, therefore, scarcely so great as would be that of an Englishman who should speak of the *guardian*, in place of the *committee*, of a lunatic. Again, Livy several times (*e. g. xxxix. 9*) uses the word *manus*, which technically denotes the peculiar authority of the husband over the wife, in a much more general acceptance. In estimating the constitutional and legal information given by historians, it is necessary to trace the causes likely to lead them into a biased narrative,—to examine whether they are credulous,—whether they desire to make their accounts interesting by anecdotes, however apocryphal,—whether they accommodate the institutions of Rome to those of their own country,—whether they espouse a party, or have a theory to support. In reading the speech of an orator in defence of a client, it must be recollected that considerable liberty was taken in mis-stating the law for the purpose of gaining the cause, or winning the applause of the audience. Not seldom, to use the words of Cicero,—(*2 de Fin. c. 22, extr.*) “*à iudicibus oratio avertitur: vox in coronam, turbamque effunditur.*” Cicero acknowledges this of himself. (*4 de Fin. 27*).

Untechnical
authorities.

There are scarcely any of the Roman classical writers from whom the industry of civilians has not derived some inference or elicited some illustration.

Of all the Roman writers not strictly professional, or not writing professionally, Cicero, the advocate, the augur, the prætor—we will not add,* the juriconsult—furnishes the largest fund of information upon Roman law and antiquities. His *Topica*, addressed to the jurist Trebatius Testa, is a treatise abounding in

legal explication. The treatises *de Officiis*, *de Legibus*, *de Oratore*, and *de Republica*, are important to the civilian. His other writings on philosophy and rhetoric, and his private letters, are rich in legal allusions. His speeches to a *judex*, to the senate, or to the *populus*, throw light upon many obscure points of practice and procedure. The fragments of speeches recently discovered by Mai, Niebuhr, and Peyron, were edited separately by Engelbronner in 1830.

Next to Cicero, perhaps Aulus Gellius is the unprofessional writer most valuable to the student of Roman law. In his amusing miscellany, the *Noctes Atticæ*, are preserved legal forms, antiquarian information connected with legal history, and fragments of ancient juriconsults. The rhetoricians, M. Annæus Seneca, Julius Victor, and especially Quinctilian, contain many passages relating to Roman law. The *Scriptores Linguae Latinæ*,—especially *Ter. Varro, Verrius Flaccus, Festus, Nonius Marcellus, Fulgentius Plantiades, and Isidore,—to whom may be added Priscian,—abound in explanations of juridical terms. Festus, in particular, preserves many curious notices of legal history. Along with these may be mentioned the Greek grammarian Dositheus Magister, and the lexicographer Suidas. Of the poets, Virgil and Ovid were manifestly well versed in legal antiquities, especially in matters connected with religion. The scholiasts, Asconius and Boethius, in their commentaries on Cicero, Donatus on Terence, Servius on Virgil, Acron and Porphyrio on Horace, are often serviceable to the jurist. The *Scriptores Rei Rusticæ*, especially M. Porc. Cato, Ter. Varro (before mentioned as a grammarian), and Columella, throw occasional light upon the forms of rustic contracts and other portions of the law. More important are the *Agrimensores*, or *Scriptores Rei Agrariæ*—a class very distinct from the preceding. They often illustrate the law of boundaries, the rights of neighbouring proprietors, and the nature of *possession*, legally and historically considered. The study of them led Niebuhr to those researches which produced his Roman history. Under this class of writers are comprehended Frontinus and his commentator Aggenus Urbicus, Siculus Flaccus, and Hygenus.† Frontinus, who lived under Domitian, and wrote, among other treatises, a treatise upon the stratagems of war, possessed very accurate legal knowledge, and is not to be confounded (as has sometimes been done) with that Fronto, a contemporary of Antoninus, whose letter and orations, recently brought to light by Mai, are but of little use in illustrating the political institutions of his day. The constitution of the Roman army—a difficult and perplexing subject—is closely connected with the civil constitution of early Rome; and we venture, without examination, to suggest a doubt whether the external historians of Roman law have sufficiently pressed into their service the *Scriptores Rei Militaris*. Some of the fathers of the Church, especially Tertullian, Lactantius, Justin Martyr, and Augustine, have yielded to lawyers good store of materials. Lucius Annæus Seneca, the philosopher,—the two Plinies, especially the naturalist,—Aurelius

Law.

* There has been much dispute whether Cicero was a juriconsult, and high modern authorities decide in the affirmative. (See Zimmern, 1 *Röm. R. G.* 289, n. 16.) We hold with Bynkershoek, that he did not consider himself to be one, though he knew that he could be, if he chose. (See *pro Muræna*, 13, and 1 *de Leg. c. 13*.)

* The designations *Scriptores Linguae Latinæ*; *Historiæ Augustæ Scriptores*, &c., are, to some extent, ambiguous. The groups of authors comprised under them by different editors have not been invariable.

† This name, usually written Hyginus, is invariably written as above in the best ancient manuscripts.

Law.

Symmachus (*præfectus urbi* under Theodosius II.),—Macrobius,—Valerius Maximus (a contemporary of Tiberius),—Balbus and L. Volusius Mæcianus, who wrote on weights and measures, the former under Augustus, the latter under the Antonines,—M. Valerius Probus (not the contemporary of Nero, but a later grammarian), who wrote, among other things, on legal abbreviations (*notæ juris*)—the chronologer Censorinus,—the architect Vitruvius, have all been subjected to examination and contribution.

Historians
who have
written in
Latin.

To this miscellaneous assemblage we have postponed the professed historians of Rome, not because they are unimportant, but because they form a considerable class of themselves. "*Il faut éclairer,*" says Montesquieu, "*les lois par l'histoire, et l'histoire par les lois.*" The external historian of law especially must have frequent recourse to general history. Among the authors who have written in Latin, he must occasionally consult Julius Cæsar, Cor. Nepos, Sallust, Velleius Paterculus, and Florus: he must not overlook Aurelius Victor, Eutropius, Sextus Rufus, Ammianus Marcellinus; but his principal attention will be bestowed on Livy, Tacitus, and Suetonius. For the times of the later emperors, he will turn to the *Historiæ Augustæ Scriptores*, under which name may be enumerated Lampridius, Spartianus (who is not improbably the same person with Lampridius), Gallicanus, Trebellius Pollio, Vopiscus, Julius Capitolinus, Cassiodorus. These writers are peculiarly valuable for their elucidation of the political and social institutions of the Empire.

We proceed to the historians who have written in Greek.

Historians
who have
written in
Greek.

Of the early legal history of Rome, Dionysius of Halicarnassus, a careful and learned antiquary, but of no poetic taste or feeling, is a most important source. Polybius, the friend of the Scipios, is very accurate, and abounds in maxims of political wisdom. Notwithstanding the fragments discovered by Mai, his *General History* unfortunately remains very imperfect. The loss of the greater part of the sixth book is severely felt. Some of his observations on the Roman constitution are translated by Cicero in his treatise *de Republica*, Plutarch's most interesting *Lives*, and his *One Hundred and Thirteen Roman Questions*, have much corn which a critical fan may separate from the chaff. He is more at home, however, in the affairs of Greece than in those of Rome. Dion Cassius (called, by the precisely accurate Hugo, Cassius Dio), Diodorus Siculus, and Appian, all begin their histories from the commencement of the Roman state. Dion is continued by Xiphilinus and Zonaras.—Herodian and Zosimus treat exclusively of the time of the emperors. The Byzantine historians contribute largely to legal history. Among these, Procopius of Cæsarea (continued by Agathias and Menander) is especially important for the time of Justinian, though his veracity and fidelity have been called in question from the hostility to his master which is exhibited in his later secret history (*Ἀνέκδοτα*), in striking contrast with the spirit of his former work on Justinian's wars.

The Jewish antiquary, Flavius Josephus, ought not to be omitted; and some of the ecclesiastical historians of the Christian Church, as Eusebius, Theodoretus, Socrates Scholasticus, and Evagrius, illustrate the bearing of the imperial constitutions upon Christianity, and the incipient relationship of the Civil and Canon Laws. Falck, in his *Juristische Encyclopädie*, remarks

(see Pellat's French translation, Paris, 1841, p. 413 n.) that Zirardini, in his *Commentary on the Post-Theodosian Novells* (Faenza, 1776), has availed himself of the Christian martyrologists, for the purpose of elucidating some of the obscurities which still overhang the judicial organization and procedure of the Romans.

A few works remain to be mentioned before we quit the unprofessional authorities.

Joannes Laurentius ("*Laurentii filius?*" Hugo) Lydus, a man without judgment or sound learning, and, like Procopius, a contemporary of Justinian, wrote a work *Concerning Magistrates*, of some consequence to the Roman lawyer, inasmuch as the subject was peculiarly within the province of the author, who was employed in the chancery of the *præfectus prætorio*. The manuscript of this work was first discovered by Villoison, in 1785, in the neighbourhood of Constantinople, and the first printed edition appeared at Paris, in 1812, by Fuss, with a preface by Hase. This, which was long the *only* edition, requires to be completed by the letter of Fuss to Hase, (*Fussii ad Hasium Epistola, in quo Lydi textus et versio emendantur*. Leodii, 1820.) A new edition of the extant works of the Lydian Joannes, including the treatise *Concerning Magistrates*, revised by Bekker, appeared at Bonn in 1837.

In 1829, Huschke published at Breslau (under the title *Incerti auctoris magistratuum et sacerdotiorum populi Romani expositiones*) a dry little work, discovered in manuscript at Paris, and by some modern scholars referred to the end of the fourth century; by others to the beginning of the seventh, or to a still later date.

With these works may be consulted the more full and satisfactory Latin treatise on the same subject, called *Notitia Dignitatum*, which, from internal evidence, appears to have been composed after the death of Honorius (A.D. 425), and before the destruction by Attila (A.D. 452) of *Concordia* and *Aquileia*. It may be found, with the notes of two of its former editors, Pancirolli and Labbe, in *Grævii Thesaurus*, tom. vii., p. 1309, and it has been lately re-edited at Bonn by Boecking. It must not be confounded with the similar work of borrowed title inserted in the last volume of Jac. Godefroi's edition of the *Codex Theodosianus*.

We must also name the *Notitia Provinciarum*, which was first published by Maffei, in the appendix to his *Istoria Teologica*, from a Veronese manuscript; and the *Libellus Provinciarum Romanarum*, which may be found in the *Varia Geographica*, edited by Gronovius. (Lug. Bat. 1739.)

We now proceed to the technical authorities for the materials of Roman law. Technical authorities.

The technical authorities may be divided into two classes:—1st. Private legal documents.—2nd. Legislative enactments and legal works.

Here it is necessary to remark, that the distinction between technical and untechnical is not logically precise. An unprofessional writer, for instance, as Aulus Gellius, is classed as an untechnical authority, while the legislative enactments and other legal documents preserved in his works are, when considered by themselves, ranked among the technical authorities. Moreover, some of the above-named works might as well be classed under a distinct head, as half technical.

The greater part of the private legal documents which have come down to us are contained or referred to in the very interesting collection of Spangenberg, Private legal Documents.

Law.

Law.

entitled *Juris Romani Tabulæ Negotiorum Solemnium, modo in ære, modo in marmore, modo in charta superstites* (Lipsiæ, 1822). Spangenberg's own dissertations, prefixed to the collection, abound in valuable information.—Every lawyer knows how conducive to a right understanding of the law is the study of formal precedents.—The comparison of the modes in which the same objects are effected in different systems of jurisprudence has the advantage of correcting the prejudice into which narrow-minded practitioners not seldom fall, of supposing that the way of transacting business to which they have been accustomed is the only possible or the only rational one. At the same time, the resemblances between wills, conveyances, and other legal proceedings, of different ages and countries, afford curious proof of the force of tradition in such matters, and the extent of servile copyism.—The scribe or conveyancer, looking at an old form, frequently adopted words applicable to a state of law which was become obsolete, or had been altered. In reading the description of the abutments, and the verbose transfer of all the right and interest of the vendor, and of all ways, watercourses, &c., with the appurtenances, in a Roman *venditio*, the lawyer will often be reminded of the style of an English purchase-deed. Few, perhaps, are aware of the antiquity of the form still so common in the commencement of wills, "In the name of God, Amen;" or of the still greater antiquity of the declaration, borrowed probably from the Athenian law, that the testator, though weak in body, is of sound mind and memory. The ancient records of forensic proceedings, and the honourable discharges, usually conferring certain franchises, given to soldiers (*tabulæ honestæ missionis*), are among the most valuable legal monuments we possess. The authorities which we are now treating of are for the most part either contained in Spangenberg's work, or reference is there made to more copious collections, such as Marini's *Papiri Diplomatici*, where they are given at large. Few, we believe, of much importance have been since discovered.

In the year 1820, there was found in the vineyard of St. Amendola, on a part of the Via Appia, which bore evident marks of having been an ancient Roman burying-ground, a marble slab broken into two fragments. On these was an inscription wanting the beginning and end, and presenting, probably, the oldest specimen in existence of a Roman will. The person first named as heir is required, as a condition, to take the name of the testator, which appears to be Dasumius. Among the legatees are mentioned Plinius Secundus and Cornelius Tacitus. The inscription was first published by the advocate Carlo Fea, in 1820, in the *Diario di Roma*, and afterwards in the *Varietà di Notizie*. It was subsequently brought before German scholars by Puggé, in the 1st volume of the *Rheinisches Museum für Jurisprudenz*, p. 249 (Bonn, 1827). A further fragment of the same inscription was discovered at Rome in 1830, and published in the 2nd Part of the *Annali del Istituto di Corrispondenza Archeologica*, 1831.

In 1838, Huschke published for the first time, with a commentary, as an offering on the occasion of Hugo's jubilee from the faculty of law at Breslau, an instrument of donation of one T. Flavius Syntrophus—*T. Flavii Syntrophii instrumentum donationis ineditum edidit et illustravit, &c.* Ph. Ed. Huschke (4to. maj. Vratislaviæ, 1838). This instrument, which was dis-

covered by Ritschel at Rome in 1837, in a manuscript collection of inscriptions, contains a gift, by Syntrophus, of a piece of land to one of his freedmen in trust for himself and the other freedmen, present or to come, of the donor. The trust is enforced by a *pænæ stipulatio*. Then follow the *mancipatio*, with the assistance of an *antestatus*, and the *traditio*. An English conveyancer would compare the donation to a deed of feoffment with livery of seisin endorsed. In a *Mémoire* by Warnkœnig, on the progress of the Science of Law in Germany since 1815, in the 8th Volume of Fœlix's *Revue de Législation* (January, 1841), the will of Dasumius, first published by Fea, is erroneously identified with the *instrumentum donationis* of T. Flavius Syntrophus, first published by Huschke. Warnkœnig (p. 30) enumerates among the recent German contributions to Roman Law "*l'édition des fragmens du testament récemment trouvé de T. Flavius Syntropus* (sic), dans lequel Tacite et Pline sont nommés légataires, donnée par M. Huschke à Breslau, et enrichie d'un Commentaire (1838)."

In the year 1841, J. F. Massmann published at Leipzig an account (*Libellus Aurarius*) of two sets of wax-covered tablets containing inscriptions. The first set was made of three fir boards connected book-wise, so as to afford four internal pages of wax (*triptycha*), and was found in 1790 in a gold mine in the neighbourhood of the village of Abrudbánya, in Transsylvania. The second set, made similarly of beechen boards, and in less perfect condition than the first, was found in 1807 in another gold mine situate within the village itself. In the fir tablets, the wax, hard and blackened by time, is inscribed with a legal document, written in Latin from right to left, and commencing at what we should call the fourth page. It attests, in due form, certain matters connected with a corporation or guild, probably of mining adventurers or workers in gold (*collegium aurariorum*). It is not only interesting in a legal view, but remarkable as containing the oldest specimen in existence of Roman running-hand, strangely preserved. Fortunately the same form of words was written twice, and the difficulty of decyphering was thus considerably diminished. Means of comparing the shape of letters were presented, and the effaced or doubtful writing in one copy of the testation was supplied or explained by the counterpart. The names of the consuls are affixed (*Imp. L. Aur. Ver. III. et Quadrato Cs.*), and thence we are able to assign to the document the date A.D. 168. The contents of the beechen tablets are in an unknown character, supposed to be Dacian, with the exception of the old oracular response (not inappropriate to the place) scribbled in Greek:—

Ἀργυρεαὶς λογχαῖσι μάχου, καὶ πάντα κρατῆσεις.

A detailed description of the fir tablets and their contents may be found in the 28th Volume of the *Foreign Quarterly Review*.

Of the technical authorities, not consisting of legal documents relating to private transactions, but classed by us under the head of legislative enactments and legal works, many are known merely from monumental inscriptions. Of such inscriptions, the most valuable are copied in the posthumous work of Haubold, edited by Spangenberg, and entitled *Antiquitatis Romanæ Monumenta Legalia* (Berolini, 1830). This collection gives the history of the monuments, and also indicates the particular passages of ancient authors which contain legal fragments: it is indispensable to

Law.

Law.

the student of Roman law. A monument of early date (A.U.C. 568), valuable philologically as well as legally, is the *Senatus-consultum de Bacchanalibus*, known to English readers from the commentary upon it appended to Taylor's *Civil Law*. Among the most interesting of the monumental inscriptions is the law concerning Cisalpine Gaul (A.U.C. 705, B.C. 49). The name of this law was detected in a singular manner. It contains a formula which it directs to be used in certain legal proceedings; and this formula, as Puchta was the first to observe, refers to the law ordaining its use by the name *Lex Rubria*. The *Lex Servilia* and the *Lex Thoria*, with a not unusual economy, are engraved on opposite sides of the same bronze tablet. There has been much controversy as to the respective dates of these laws, which may probably be referred to the middle of the seventh century of the city. The recent commentaries of Klenze and Dirksen on the former, and of Rudorff on the latter, have met with high commendation from eminent scholars. The tablet found at Bantia has on one side the *Lex Acilia Repetundarum* (enacted probably between the years of the city 654 and 665), and on the other side a municipal statute in the Oscan language. Our knowledge of the mode of administering justice in Italy and the Provinces, and of the law relating to the *ager publicus*, receives many accessions from these sources. The *Obligatio Prædiorum*, commonly, but not so properly, called *Tabula Alimentaria Trajani*, is a charitable foundation of Trajan (or sanctioned by Trajan), for the support of orphan children. A similar document of Hadrian is stated by Blume (in the 4th volume of the *Rheinisches Museum für Jurisprudenz*) to have been lately discovered at Naples. The *Tabula Heracleensis*, one fragment of which was brought to England in 1735, but in 1760 restored to Naples, contains fragments of several interesting laws supposed to have been enacted between the years of the city 644 and 680, and relating in part to Rome, in part to municipal towns and colonies.* On the opposite side is engraven a Greek psephisma. Two Greek Edicts of Cn. Virgilius Capito and Tib. Julius Alexander, Præfects under Claudius and Galba, were found by F. Cail'aud in 1818, engraven on the portico of a temple in the Great Oasis of Egypt, and were again copied in succession by Mr. Hyde and Sir Archibald Edmonstone, or, as Haubold (*Mon. Leg.* p. 199), by a natural but ludicrous mistake of the contracted title for the name, tells us, "*a duobus Anglis, Hyde et Bart.*" They are given in *Edmonstone's Journey* (London, 1822), with the English translation and notes of Dr. Young, and have since been amply commented upon by Rudorff in the 2nd Volume of the *Rheinisches Museum für Philologie*. They illustrate the public state of Egypt. The *Edictum* of Diocletian (A.D. 303), fixing a maximum price for labour and provisions throughout the Roman Empire ("our world"), is more curious in an economical than a legal view. It was first found and copied by Sherard, the British Consul at Smyrna, in 1709, on a stone at Stratonice (Eski Hassâr). This stone was lately re-discovered by Mr. William Bankes, who made a fac-simile, which is deposited

* The Latin inscription on the *Tabula Heracleensis* is identified by Savigny with the *Lex Julia Municipalis* mentioned by Orelli, *Inscr.* No. 3676, and is referred by him (*9 Zeitschrift*) to A.U.C. 709.

in the British Museum. Another stone, containing (remarkably enough) the beginning of the same edict, with some various readings, was brought from Egypt in 1807, and is now in a private museum at Aix. Mr. Martin Leake's *Essay* upon the Edict of Diocletian may be found in the 1st Volume of the *Transactions of the Royal Society of Literature*. We must not omit the *Lex de Imperio Vespasiani*, which is famous for its bearing on the controversy with respect to the reality of the *Lex Regia*, mentioned in the commencement of Justinian's *Institutes* (§ 6, *I. de Jur. Nat.* (I. 2.)). In concluding this subject, it is proper to remark that great caution is necessary in discriminating between genuine and forged inscriptions. Some very learned antiquaries (*e. g.* Ursinus) have not scrupled to disgrace themselves by literary fabrications, and some of the most valuable of the early collectors (*e. g.* Gruterus) were more industrious than critical.

The legislative enactments and legal works, known, not from monumental inscriptions, but from manuscripts and printed books, may conveniently be classed, according to order of time, into three groups: 1, the Ante-Justinianean; 2, the Justinianean; 3, the Post-Justinianean authorities.

The earliest Ante-Justinianean authorities consist of such remains as we possess of the Laws of the Kings.

In various authors mention is made of particular laws, which are attributed by name to one or other of the kings of Rome, and there have been many attempts to collect all these scattered notices, and to restore the laws to their ancient form and order. Niebuhr thinks that the laws of the kings, as we now have them, are not earlier than the restoration of the city after the invasion of the Gauls. In support of this opinion, two arguments are urged, derived from internal evidence: first, the traces which may be discovered in them of reference to times subsequent to the kings; and, secondly, the Latinity of the fragments cited by Festus and other ancient writers. These, unlike a genuine monument of the earliest times, the hymn of the *fratres Arvales*, are perfectly intelligible, and their language is not more ancient in its form than that of the fifth century of Rome.

These arguments tend to prove, not that the so-called laws of the kings were a subsequent invention, but that their contents had been partially modernized in style, and in some instances corrupted by adaptation to recent events. Similar things have happened in our own legal literature. Our old law-book, called the *Mirror of Justice*, is said by Bradshaw, Attorney-General, in *Reniger v. Fogossa*, Plowden, 8, to have been made before the Conquest. The original stamina of that work belonged probably to the age to which Bradshaw attributes it, but, as we now have it, with later alterations and additions, it cannot be considered as older than the reign of Edw. II.

We may also remark that, without supposing any considerable change to be made in the language of an old law, it might be expected to be more easily understood than the contemporary song of an agricultural brotherhood. The general currency of the metal would prevent its rusting. If Shakspeare had never written, how much more of Shakspeare's language would now be obsolete! A text of Luther's translation of the Bible would appear to a modern German less antiquated than some of the ballads which were sung by rustics in Luther's days.

Law.

Law.

The laws of the kings, as they are to be met in ordinary collections, are not in the state in which they are cited by ancient authors. Collectors and antiquaries, as Ursinus, Joseph Scaliger, and Terrasson, have endeavoured to re-invest them in their ancient weeds. They have rummaged the oldest monuments, the Salian verses, the rostral column, the inscription in honour of Scipio, son of Barbatus, and have set to work to produce an imitation of what Terrasson erroneously calls *Oscan* orthography. To give an example cited in the 5th Volume of the *Thémis*, p. 258, in a review of M. Daunou's *Lecture on the Jus Papirianum*,—Pliny speaking of a law of Numa, says, "*Eadem lege ex imputata vite libari vina Diis nefas statuit, ratione exco-gitata ut putare cogere alios aratores (i. e. those who would otherwise confine themselves to ploughing), et pigri circa pericula arbusi.*" Out of this, Fulvius Ursinus, in his *Notes on Augustinus, de Legibus et Senatus-consultis*, has invented the text, SARPTA. VINIA. NEI. SIET. EX. EAD. VINOM. DIS. LEIBARIER. NEFAS. ESTOD.

Historical objections have been made to the hypothesis that the Royal Laws were preserved, from the defect of evidence with reference to early times and the destruction of monuments consequent upon the Gaulish invasion (compare Liv. vi. 1). We see no reason, however, why they might not have been transmitted either by unwritten tradition, or in the written commentaries of the priests, or the records of private families. We believe with Wachsmuth that the art of writing was known in the earliest days of Rome, and that it was not confined to engraving on wood or bronze, but comprehended the use of a style and waxed tablets, and the painting of coloured letters on boards covered with chalk. A collector of the laws of the kings, named Papirius, is mentioned by Dionysius of Halicarnassus (iii. 36), and Pomponius (l. 2, § 2 *D. de Origine Juris* (i. 2); § 36, *ibid.*). The former speaks of *Caius* Papirius; the latter, with apparent inconsistency, in one section, uses the prænomen Sextus, and in a following section speaks of Publius Papirius. Dionysius says that Papirius collected the *sacred* laws after the expulsion of the kings; Pomponius says that Papirius collected *all* the laws of the kings, and that he was a contemporary of Superbus, who, in the fragment *de Origine Juris*, is erroneously styled "*Demarati Corinthii filius.*" A commentary was written upon the *Jus Papirianum* in the time of Julius Cæsar, by Granius Flaccus (l. 144, *D. de Verb. Sign.* (L. 16)), who entitled his book, or one of his books, *De Indigitamentis*, i. e. concerning invocations or sacred rites, the religious laws of the kings having doubtless remained the longest in use. Indeed, it is probable that Papirius, who was himself a pontiff, directed his attention principally to religious ceremonies, which may be well supposed at the period when he lived to have constituted a large portion of the technical law, and to have been connected with the principal transactions of life.

Servius on *Æneid.* xii. 836, and Macrobius, *Sat.* iii. 17, cite passages from the *Jus Papirianum*, preserved, it may be, in the work of Granius Flaccus; and some of the moderns give the same name to the collections, such as they are, that we now possess of royal laws.

* The word *indigitare*, according to the etymologists, is derived from *indu*, the old form for *in*, and *citare*, signifying to invoke. Modern Latinists usually employ the word in the sense of *pointing out*, as if it was derived from *digitus*.

Law.

Cicero (*de Republica*, ii. 14) speaks of laws of Numa, *quas in monumentis habemus*; and again (V. 2), in reference to Numa, he says, *quæ legum etiam scriptor (not lator) fuisset, quas scitis extare.*

There is a strange story reported by Pliny (xii. 13 or 27). Cassius Hemina, he says,* a most ancient author of annals, states in his fourth book that Cn. Terentius, the scribe, in turning up the soil of his field in the Janiculum, came upon a coffin which had once contained the body of Numa. In the same coffin were Numa's books, of paper (*è chartâ*). Then follow circumstantial details, showing the precautions which had been taken in packing the books, to preserve them from decay.

This took place 535 years from the beginning of Numa's reign, in the consulate of P. Cornelius L. f. Cethegus, and M. Bæbius Q. f. Pamphilus (A.U.C. 573). The books, according to Hemina, contained the philosophical writings of Pythagoras, and, because they were philosophical writings, were burnt by Q. Pætilius, the prætor. L. Piso Censorius agrees with Hemina as to the burning, but reports that seven of the books related to pontifical law, and that there was an equal number of Pythagorean books. Tuditanus says that the books contained the decrees of Numa; Varro and Antius, that there were two pontifical books in Latin, and two philosophical books in Greek. The same thing is told, with several material variations of detail, by Livy (xl. 29), Valerius Maximus, Plutarch, and other authors.

Joannes Lydus, in his work *De Ostentis*, first published by Hase at Paris, in 1823 (a work which contains some curious traditions respecting the book of Tages and the early history of Roman augury), quotes (p. 63), on the authority of Fulvius Nobilior, a remarkable passage from the writings of Numa, to the effect that "the science of the stars (astrology), instead of turning us from religion, leads us to acknowledge the Providence of the Omniscient and Ineffable Father of all things, as manifested in His works, and to observe with admiration that the mind of man, under the governance of God, is able to discourse concerning the things of Heaven."

The remains which we possess of the *Leges Regiæ* are for the most part exactly of the kind which might be expected from the manners and circumstances of the age to which they are referred. Let us take the following examples.

Sei parentem puer verberit, ast oloë plorassit "Parentes," puer diveis parentum sacer esto.

Festus (v. *plorare*) informs us that *plorassit* here means "should call upon."

From Cassius Hemina (referred to in Pliny, xxxii. iii.) and Festus (v. *Pollucere*), Scaliger has just put together the following law of Numa.

Pisceis qui squamosei non sunt, nei polluceto.

Squamosos omneis præter scarum, polluceto.

The object of this law, according to Pliny, was to prevent expensive purchases of fish for the purpose of votive feasts, and to put a stop to the increase of price that would thence be occasioned to persons purchasing for other purposes; for fishes without scales were dearer than most kinds of fish with scales. The scar,

* According to Censorinus, c. 17, Cassius Hemina was alive at the date of the fourth secular games, A.U.C. 607, in the consulate of C. Corn. Lentulus and L. Mummius Achaicus.

Law. it is supposed, was excepted, as being a rare fish (See *Hor. Epod. Ode ii. v. 50*). This reasoning, though ingenious, is not entirely satisfactory. There is an analogous provision in the Mosaic Code.

Marliani pretended to give the laws of Romulus from a tablet found in the Capitol, but the forgery of the *Tabula Marliana* is now generally admitted.

Franc. Balduinus, Charondas, Merula, Lipsius, and others have made collections of the *Leges Regiæ*. One of the most common books in which the fragments may be found, with a commentary, is Gravina's *Origines Juris*.

C. F. Glück wrote an elaborate work on the *Jus Papirianum* (*Liber singularis de Jure Civili Papiriano*, Hal. 1780). Nearly all that is really known of the laws of the kings is contained in Dirksen's *Review* of preceding essays on the criticism and restoration of their text. This review is contained in Dirksen's *Versuche zur Kritik und Auslegung der Quellen des Römischen Rechts* (Leipz. 1823).

Law of the Twelve Tables. We have now to treat of the Laws of the Twelve Tables, and shall first say a few words concerning their origin and history.

The expulsion of the kings was ratified by a *Lex Curiata de Imperio Consulari*, passed on the proposal of Junius Brutus, and called by Pomponius a *tribunian law*, because Junius Brutus at the time was *Tribunus Celerum*. The consuls now succeeded to that jurisdiction which had formerly been exercised by the kings. It is likely that some of the royal laws and regulations went speedily out of use; but that this was the case with all of them, as Pomponius asserts, is at variance with probability, and contrary to facts attested on the better authority of Dionysius, from whom we learn that the collection of Papirius was made after the expulsion of the last Tarquin, and that some of the royal laws were incorporated in the Twelve Tables. If their force as written law ceased, many of them must at least have retained influence as *jus moribus receptum*.

Causes which led to their enactment. There were several co-operating causes which, soon after the establishment of the consular constitution, rendered the *plebs* anxious to obtain a body of revised and written laws. The existing customary laws were scanty, and silent as to many events which required some legal provision. They were partial, and established differences of rights between the patricians and plebeians. The interpretation and application of unwritten laws, labouring under these imperfections, were intrusted to the favoured and dominant caste, who studiously kept them from the knowledge of the many. Under such circumstances, a large discretion was necessarily exerted in the administration of justice; and scope was afforded for its exertion in an arbitrary, unfairly biassed, and oppressive manner. When, in the institution of tribunes of the people, the plebeians found an organ of their will, their disappointment at the ill success of the consular government soon assumed definite form and expression. The perpetual strife between the tribunes and the consuls ended in the proposal of a plan for a review of the laws and constitution, in order to deprive both of their abuses.

Of the causes we have assigned for this result, some have been mentioned by one ancient historian, some by another. The most recent legal writers sometimes adopt one or more of those causes, to the controversial exclusion of the rest. The open or tacit refutation of the errors of contemporaries or predecessors, and the

desire of advancing something new, prevail in the majority of modern continental works in place of that freer movement of spirit which led the older writers to the topics most striking or most important.

Law. In the year of the city 292, the tribune C. Terentillus Arsa proposed a law for the appointment of five plebeians to restrain the consular power within proper limits. The proposition was successfully opposed by the senate, but was renewed by the tribunes in the following year (Dion. Hal. x. 3). The plan seems to have made no outward progress till the year 300, when the tribunes manifested a disposition to concession. In the words of Livy (iii. 31), "*tribuni lenius agere cum patribus: Finem tandem certaminum facerent. Si plebeia leges displicerent, at illi communiter legum latores, et ex plebe et ex patribus, qui utrisque utilia ferrent quæque æquandæ libertatis essent, sinerent creari.*" The patricians required that the legislation should proceed wholly from themselves; but though no decision was come to as to the persons to whom the delegation of legislative power should be made, the first step towards an adjustment was taken. Three deputies, Sp. Postumius Albus, A. Manlius, and P. Sulpicius Camerinus, were sent to Athens with directions to copy the renowned laws of Solon, and to make themselves acquainted with the legal institutions, written and unwritten, of the other states of Greece. This occurrence took place about fifteen years before the Peloponnesian war, in the most brilliant period of the age of Pericles. It was a proceeding perfectly in accordance with the spirit of ancient times, and the practice of most of the celebrated early lawgivers.

Upon the return of the deputies from their mission, in the year 302, the contending parties came to an agreement that the revision of the laws and the constitution should be intrusted to ten patricians, with the title *decemviri legibus scribendis*. These decemvirs were invested with extraordinary powers. The tribunate, the consulship, and all other magistracies, which could in any way impede or obstruct their operations, were suspended, and from their judicial sentence there was to be no appeal. At their head was Appius Claudius, as having been designated consul for the year, and the three deputies formed part of their number.

In those days there dwelt at Rome a philosopher named Hermodorus, a native of Ephesus. Being of good family, and a benefactor to his state, he had obtained the appellation *οὐρίστος*, and had excited by his eminence the jealousy of his fellow-citizens, who, without attempting to deny his merits, determined that he should carry them elsewhere, and banished him, saying, "*ἡμέων μηδὲ εἰς οὐρίστος ἔστω*" (Dion. Laert. in Vita Heraclit.).

Hermodorus probably accompanied the deputies in their mission (though of this there is no record), and was afterwards of so much use to the decemvirs in the accomplishment of their task that a statue to his honour was erected in the forum. Pliny, the naturalist, who mentions the statue, speaks of it in the past tense, as if it were not in existence in his day. With respect to the exact part which Hermodorus took, the accounts transmitted to us and the opinions of the moderns vary. According to some of the moderns, he acted merely as an interpreter, while others suppose him to have been in reality the leading mover in the work of compilation, if not the actual composer of the decemviral laws. The dissidence of modern opinion has

Law. been increased by misconstruction of the classical authorities on the subject. The words of Pliny are, "*legum quas decemviri scribebant interpretes*" (H. N. xxxiv. 5); of Pomponius, "*legum ferendarum auctor*" (l. 2, § 4, D. de O. I. (i. 2)); of Strabo, "*δοκεῖ νόμους τινὰς Ῥωμαίους συγγράψαι*" (xiv. i. 25). It is likely that Hermodorus directed and facilitated research by his superior knowledge of the Greek language and literature, and that, in deference to his abilities and peculiar acquirements, his advice had some influence in the selection of laws that was made.

At the end of the same year (302) ten tables of laws were ready. They were first set up in a conspicuous place, in order that they might be read and examined by the people, who were invited to suggest such alterations as they might judge requisite. Afterwards, the revised tables, having received the previous sanction of the senate, were approved as law in the *comitia centuriata* and the *curiæ*. They were then openly erected in the forum, that all men might know and observe them—erected *pro Rostris*, says Pomponius, by anachronism or by anticipation, for the Rostra were not set up in the forum until more than a hundred years afterwards (Liv. viii. 14).

The Ten Tables were probably intended to be a complete work, and by their fairness and impartiality they gave satisfaction to the people. Through the contrivance of Appius Claudius, who had succeeded in obtaining popular favour, and wished to make the new constitution perpetual, a rumour became rife that further laws were wanted, and a new election of decemvirs was brought about in the year 303 for the purpose of making some additions to the former legislation. Appius was placed at the head of the newly-elected decemvirs, who were his creatures. Among them, Dionysius says (x. 58) there were three plebeians, and Niebuhr will have it that there were five plebeians, though Livy (iv. 3, *fin.*) speaks only of patricians. Two new tables, "*quorum*," says Cicero (*De Rep.* ii. 36), "*non similiter fides nec justitia laudata*," were composed in the year 304, but were first published in 305, after the abolition of the decemvirate, along with the other ten tables by the consuls L. Valerius and M. Horatius (Liv. iii. 57). This publication seems to have been a final promulgation, and not to have resembled the preliminary publication of the first ten tables, when they were set up for public revision. It is not likely that the efficacy of the last two tables was allowed to depend on the legislative power previously delegated to their compilers. There is indeed nowhere any express statement that they were ratified by the centuries, but the fact may be inferred from the name *lex* given to the whole collection. The law of the Twelve Tables was early known by various names: *Lex XII. Tabularum*, *Lex decemviralis*, *Leges XII.*, or simply *Leges*, or *Lex*, or *XII.* The epithet *legitimum* is often peculiarly applied to legal transactions regulated by the Twelve Tables.

In the preceding account, we see one of the ordinary lessons of history—long and steady efforts, stimulated by real wants, at length successful, and great questions involving the interests of contending factions settled by mutual concessions. The memory of the classical reader will revert to those pages of surpassing beauty wherein Livy relates the circumstances that led to the restoration of the old constitution—the tragic tale of lust veiling its dark purposes under the forms of justice—the poem of Appius and Virginia.

VOL. II.

The collection consisted of public as well as private laws; so that, even in Livy's time, it might be called "*Corpus omnis juris Romani; fons publici privatique juris*" (Liv. iii. 34). Tacitus, with that sententious sarcasm upon his own age to which he was much addicted, gives to it the appellation of the last impartial law that existed in Rome, "*finis æqui juris*" (Ann. iii. 27). Cicero, in his boyhood, as other boys were then wont, learned it by heart, "*ut carmen necessarium*," (*De Leg.* ii. 23), although when he was a man the prætor's edict had begun to take its place. The term *carmen*, which is applicable to any solemn form of words, perhaps gave origin to the notion that the whole collection was written in verse. Vico, from the fragments and imitations preserved in Cicero *de Legibus*, imagined that it was composed in Adonic metre. The praise which Cicero bestows upon it by the mouth of Crassus (*de Or.* i. 44) is high and exaggerated. "*Fremant omnes licet, dicam quod sentio: bibliothecas mehercule omnium philosophorum, unus mihi videtur XII tabularum libellus, si quis legum fontes et capita viderit, et auctoritatis pondere, et utilitatis ubertate superare.*" It never was repealed, and up to the age of Justinian its maxims and rules were quoted as authority. From Sidonius Apollinaris, a writer in the fifth century of our era, we learn that it was publicly taught in his day (xxiii, v. 448—451):—

*Sive ad doctiloqui Leonis ædes,
Quo bis sex tabulas docente juris,
Ultero Appius Claudius taceret,
Claro obscurior in decemviratu.*

The frequent occasional legislation on public affairs in times subsequent to its enactment must have materially derogated from its provisions with respect to the constitution and administration of government; and this is probably the reason why so few of the fragments that have come down to us relate to public law. It prescribed certain forms of proceeding called *legis actiones*, and there is reason to believe that one of the earliest considerable reforms which made a large part of it obsolete, and diminished its importance in private law, was the enactment of the *Lex Æbutia* (of uncertain date, but probably to be referred to the sixth century of the city), which abolished the old *legis actiones* except in causes tried before the *centumviri* (compare Gell. xiv. 10 with Gai. iv. 30). L. Charondas (L. Caron) brought forward (*ad Udalr. Zasii Catalogum legum*) the following inscription stated to have been found on an ancient tablet, but there is no doubt of its spuriousness. "*L. Æbutius Tr. Pl. vir popularis, legem tulit ad populum, ut XII tab. capita quæ inutilia essent reip. tollerentur, quæ lex multis contradicentibus tandem rogata est.*" Though many other laws besides the *Lex Æbutia* might be cited as derogating from the Twelve Tables, it was not so much by express and direct enactment as by the *jus honorarium* gradually introduced by the magistrates, particularly the prætors, and by the interpretation of the juriconsults, that they were, slowly and cautiously, modified or supplanted. One of the evils of superstitious adherence to a code was visible in the barbarism which sheltered in this ancient sanctuary, was never entirely banished from the Roman law.

What was the material of the tables upon which the laws were engraven? This has been a subject of much controversy. Some say oak, some brass or bronze (*æs*), and others, with Pomponius (who pro-

Law.
Subsequent
fate of the
decemviral
laws.

Fate of the
tables
themselves.

Law.

bably judged from the manners of his own time), ivory. Some give one material to the first ten tables, and another to the last two, or one material at first and another afterwards. Those who wish profoundly to study the grounds of the divers opinions touching these matters (on which "the same skill and the same degree of skill" have been expended as on greater things) may consult Zimmern (R. R. G. i. § 31).

Upon the burning of the city by the Gauls, the Twelve Tables perished, but search was made for the laws they contained, which were found and engraven on new tables that were set up in the forum near the Curia Hostilia (Liv. vi. 1, and Val. Max. ix. 5). Cyprian (who died a martyr, A.D. 258) has the following remarkable passage (*De Gratia Dei*. lib. ii. ep. 4, *ad Donatum*): "*Forum fortasse videatur immune—plura illic quæ detesteris, invenies. Incisæ sint licet leges duodecim tabulis, et publice ære præfixo jura præscripta sint, inter leges ipsas delinquitur, inter jura peccatur.*" From this it has been inferred that the Twelve Tables were in existence at Rome in the beginning of the third century of the Christian era; but it is not clear that Cyprian was not speaking of Carthage, or of more recent tables of criminal laws. One of the old glossators, Odofredus, who died 1265, assigns to certain fragments of two tables still longer duration by asserting that they actually existed in the Lateran at Rome in his own time. He was a man of little historical knowledge or critical power, but we think it worth while to insert his words (Odofred. in *Dig. Vet. l. Jus Civile* (6) *de Justitia et Jure* (i. 1)): "*Et de istis duabus tabulis aliquid est apud Lateranum Rome: et male sunt scripte: quia non est ibi punctus, nec (§) in litera (i. e. in the text) et nisi revolveritis literas, non possetis aliquid intelligere.*" This cannot be understood as a jocular invention, like those in which the grave glossators sometimes indulge, as in the account of the origin of the Twelve Tables to be found in the gloss on l. 2, § 4, *D. de O. J.* (ii. 1). There we read how the Athenians sent a wise man to Rome to see whether the Romans were worthy to receive laws from Greece; how a fool was selected to dispute with the wise man, in order that, if the fool should be defeated, *tantum derisio esset*; how the wise man held up one finger, the fool three; the wise man his open palm, the fool his clenched fist; and how the wise man was satisfied that the Romans were worthy by the deep meaning which he assigned to the gestures of the fool. In the law-schools of the middle ages, the dunces, with simple credulity, would swallow all this; the smarter freshmen, tittering, would humour the joke. Modern dunces are acute enough to discover the absurdity of the story, and to wonder at the ignorance of the old glossators.

Controversy as to the mission to Athens and the adoption of Athenian laws.

The materials of which the collection was composed consisted in part of the laws and customs of the Romans themselves. So far it resembled many of the old Teutonic collections of laws, which rather established uniformity by legislative sanction on the foundation of pre-existing institutions, than introduced enactments completely new. Its language was more vigorous and concise than that of the French *coutumiers*, to which it may in some respects be compared. It was not, however, a mere consolidation and amendment of rules already in force, but it borrowed novel improvements from foreign jurisprudence (Dion. H. x. 66). In many of the marked characters of the political and social constitution of early Rome,—the

senate—legislative assemblies of the citizens—the system of slavery—of clientship,—we discern the lineaments which were common among the Grecian races, and which must have been previously familiar to the founders of the city. The central and southern parts of Italy afforded specimens of Grecian colonies, and the ancient Romans were not so proud as to reject what was useful because it was foreign, nor so unenlightened as to be wholly ignorant of the arts and fame of their more eastern Athenian brethren. In sending deputies to Athens, they cannot be justly charged with seeking to engraft upon their own system the misplaced produce of an uncongenial stock;—their plan was rather "*antiquam exquirere matrem.*" How far the decemvirs extended their researches may well admit of doubt. Tacitus (*Ann.* iii. 27), with relation to them, employs the wide expression, "*accitis quæ usquar egregia.*" Similar are the words of Gellius (xx. 1, *init.*), "*inquisitis exploratisque multarum urbium legibus.*" Servius says (*ad Virg. Æn.* vii. 695), that part of the Twelve Tables, comprising the Fecial Law, was taken from the *Æqui Falisci*; but Livy (i. 32) reports that this introduction of the Fecial Law was the work of Ancus Martius. Dionysius mentions not only Athens but the Greek cities of Italy. Several authors, in reference to the regulations against luxury, speak of the Lacedæmonian polity (which was based on unwritten laws) as having been the model of the Roman (See Plin. *Ep.* viii. 24; Amm. Marc. xvi. 6; Sym. *Epp.* iii. 11; *Athen. Deipn.* vi. 106).

Whatever conclusion may be arrived at with respect to other sources, there is good authority for maintaining that much was borrowed from the Athenian laws attributed to Solon. By some distinguished writers,—among whom Vico and Gibbon are pre-eminent,—the mission to Greece, and even the existence of the Grecian laws, in the decemviral legislation, have been doubted. The controversy which has arisen upon this subject has employed many modern pens. Among others, Bonamy, Damiano, Gratama, Levesque, Stramigioli, Lelièvre, Valeriani, Ciampi, Cosman, and Maciejowski, have all entered the field, some on one side, some on the other. For an indifferent account of the controversy, by Berriat St. Prix, the reader may be referred to two articles in the *Thémis*, tom. iv. 304, and tom. vi. 269.

We hold with those who see no improbability in the mission, and who consider the insertion of Athenian laws in the Twelve Tables to be established beyond doubt. The Grecian contemporary historians certainly pass over the mission without notice; but then, how insignificant in Grecian eyes would have been the advent of a few barbarians, whose visit produced no visible effects on the state of political parties! Cicero certainly, in a rhetorical passage (*de Or.* i. 44), extols the Roman *jus civile* above the meagre and uncultivated legislation of Lycurgus, Draco, and Solon; but then, no one asserts that the Twelve Tables were all in origin Greek, and the *jus civile* in Cicero's time was more than the Twelve Tables. It comprised all the jurisprudence developed by technical skill. Besides, there is nothing either strange or inconsistent in looking down on those who have helped us to attain our present elevation. Cicero certainly nowhere expressly mentions the embassy,—not even when relating the history of the decemviral legislation (*de Rep.* ii. 36);—but then nothing can be more express than the testimony of Cicero himself to a point of much greater importance, namely, the adoption of parti-

Law.

Law. cular laws of Solon by the decemvirs. In his treatise, *de Legibus* (ii. 23), he says, "*Jam cætera in XII. minuendi [sunt] sumptus lamentationisque funeris translata de Solonis fere legibus.*" Again, with reference to the same matter, he adds, "*quam legem (sc. Solonis) eisdem prope verbis nostri decemviri in decimam tabulam conjecerunt. Nam de tribus riciniis et pleraque illa Solonis sunt. De lamentis vero expressa verba sunt.*" Against the authority of these passages it is useless to object the undeniable fact that many of the laws given by Cicero, in his treatise *de Legibus*, are mere imitations of the style of the Twelve Tables.

Readers who may wish to see how generally the tradition that the decemvirs copied from the laws of Athens was received by ancient authors, may refer to the following passages, in addition to those which we have already quoted:—Sallust, *Cat.* 51; Flor. i. 24; Pomponius, in l. 2, § iv. *D. de O. J.* (i. 2); Aurel. Victor, 21; Oros. xi. 13; Zonaras, vii. 18; Amm. Marc. xxii. c. ult.; August. *de Civ. Dei*, ii. 16; Isid. v. l. The only author whose words have been cited in opposition to this tradition is Polybius, who (ii. 12) speaks of the embassy sent to the Ætolians and Achæans (A. U. C. 526), in order to explain the Roman proceedings in Illyricum, as the first ἐπιτοκῇ μετὰ πρεσβείας (i. e. the first instance of diplomatic intercourse) between Greece and Rome. Niebuhr asserts that in the law relating to personal *status*, and to the course of procedure, there is a total divergence between the systems of Greece and Rome in all that is substantial and characteristic. The minor resemblances which exist he would attribute to natural circumstances apart from imitation, or to the wide diffusion of certain institutions, such as the division into *gentes*. We cannot, however (*pace tanti viri*), admit so great a divergence as is asserted by Niebuhr in all the leading principles of the law relating to *status*, though it be true that the father and the husband had greater power at Rome than at Athens. In both states the privileges of citizenship consisted of franchises, which were sometimes partially, sometimes altogether, conceded to strangers. In both, adoption was common, and the perpetual guardianship of females prevailed. Between the Athenian and Roman principles of procedure, we see, instead of divergence, a very remarkable correspondence. The narrowing of the question to be tried, by means of artificial pleading calculated to separate the law from the fact, existed probably even in the earliest period of Roman law before *formulae* were substituted for the ruder proceeding of the *legis actiones*. The distinction, which is prominent in our own common law, between the functions of the judge and the jurymen, was at the foundation of the ancient Roman as well as Athenian procedure.*

None of these coincidences may be derivable from the Twelve Tables: some may be accidental, some natural, some traceable to the Pelasgian origin (to use a somewhat indefinite but convenient term) of the Latin race. This cannot be said of the laws to prevent expense in funerals, of which Cicero speaks in the passages already cited. To the identities which they present others may be added, which are equally incapable of being rationally accounted for on any supposition but that of direct borrowing. We know that in early Rome,

as at Athens, there were corporations (*collegia*). Gaius, in his commentary on the Twelve Tables (l. 4, *D. de Collegiis* (xlvii. 22)), cites a decemviral law to the effect that *collegia* might make bye-laws binding on their members, provided that no public laws were thereby infringed. He then quotes the words of an exactly similar Athenian law, attributed to Solon. Again (l. 13, *D. finium regundorum* (x. 1)), he quotes at length a law attributed to Solon (compare Plutarch, in *Solone*, 23, extr.) as being the source of certain Roman rules relating to boundaries.

The provision in the Twelve Tables (see l. 14, § ult. *D. de præscriptis verbis* (xix. 5)) concerning damage done by cattle—*si quadrupes pauperiem fecerit*—is very like a translation of the law of Solon mentioned by Plutarch (Solon, 25)—βλάβης τετραπόδων νόμος. Compare with Plutarch the passage in Paulus (*pr. Si quadrupes*, Paulus, *Rec. Sent.* i. 15), where Cujas, instead of the corrupt reading, *Lege Pesulania*, would introduce *Lege Solonia*.

The law of *furtum* in the Twelve Tables presented many remarkable resemblances to the Athenian law of Φῶρα. "*Sei nox furtum factum esset, sei im occisit, jure caisus esto.*" A law of Solon, preserved in Demosthenes (*adv. Timocrat.*), confers, in similar words, impunity on the slayer of the nocturnal thief. It may be urged that such a distinction between *burglary* and diurnal theft is natural. It is made by Moses (*Exod.* xxii. 2, 3), and it appears in some of the Teutonic codes (See Grotius, *de J. B.* lib. ii. c. 1, sec. 12, n.).

Be it so. What shall be said of *furtum per lancem et licium conceptum*? A man who suspected that stolen goods belonging to him were concealed in another's house was allowed by the Twelve Tables to search for his property, but he was obliged to go into the suspected house with no more clothing than a linen clout (*licium*) around his loins, holding a platter (*lancem*) over his head. Thus, naked, and with both hands engaged, the searcher had no opportunity of carrying anything about him and putting it into his neighbour's house for the purpose of making a false charge. The platter was wanted to occupy the hands, not, as some have supposed, to carry away the stolen property, if found (Gai. iii. sec. 192, 3). Festus says (*v. Lance*) that the searcher "*lancem ante oculos tenebat, propter matrum familias aut virginum præsentiam.*" Some one naturally asks, if the platter be used to prevent the searcher from seeing the ladies, how can he look for his goods? Heineccius (*Ant. Synt.* iv. 1, 19) answers,—Ridiculous question! *Lanx* was not a platter but a mask, and was used not to prevent the searcher from seeing the ladies, but to prevent *them* from recognizing his features. From Petr. Arb. *Sat.* c. 97 (cited by Heinecc. *Ant. Synt.* lib. iv. tit. 1, sec. 21), we may collect what the practice was at a comparatively advanced age. There the search for a runaway or kidnapped slave is made by public officers, while the owner, who accompanies them, is dressed in a *versicoloria vestis* instead of the *licium*, and makes use of a silver *lanx* to contain the money offered as a reward for the discovery (*in lance argentea indicium et fidem (?) præferebat*). The constant adherence of the Romans to their ancient customs is proved by their clinging to this ceremony when its meaning was forgotten, and the proper manner of going through its forms had become a matter of dispute among lawyers. That a custom nearly in all respects similar to this was in force among the

* This was long ago observed by Dr. Pettingal, in his *Inquiry into the Use and Practice of Juries among the Greeks and Romans*, London, 1769.

Law.

Athenians, appears from the *Clouds* of Aristophanes (v. 496, *et seq.*), where the scholiast explains the nature and reasons of the proceeding. It is not unlikely that the very terms of the Athenian law correspond with the fictitious law given by Plato (*de Legibus*, lib. xii). Neither Plato, however, nor Aristophanes, nor the scholiast, mentions anything in the Grecian ceremony answering to the Roman *lanx*.

Perhaps we have said enough to establish the high probability of the common belief that the laws of the Twelve Tables derived some rules from Athens. Their fragments have been compared with the fragments of Solon's laws in several elaborate works. Petrus, Præteius (in Otto's *Thes. Jur. Rom.* tom. iv.), and Rittershusius, have laboured in this field; but none have equalled in celebrity Salmasius and his generally victorious opponent, the more critical and sagacious Heraldus. The bitter *Animadversiones* of the latter is truly an extraordinary monument of immense erudition. Haubold (*Inst. Hist. Dog.* i. 89) refers to various modern writers whose works illustrate the memorable agreement between Roman and Attic law in the various kinds of actions, the different kinds of bail (*vadimonia*), the use of *sponsiones* and *sacramenta*, the *clepsydra* in judicial proceedings, the mode of attesting legal documents, the employment of advocates, the examination by torture of slaves, the distinction between *jus* and *judicium*, and the practice of appointing *judices* by lot. Some Greek technical terms, as *chirographum* and *syngrapha*, appear to have been early introduced; others, as *hypotheca*, *antichresis*, *parapherna*, and *emphyteusis*, were probably later importations from the *jus gentium*. How early the maritime law of the Rhodians may have been adopted by Rome it is impossible to collect with certainty. It seems that, in the time of the free republic, Servius Sulpicius and Ofilius (l. 2, *pr.* and § 3, *D. de Leg. Rhod.* (xiv. 2)) gave opinions on its construction. This we mention, in connexion with the subject of the Twelve Tables, as an example of later recourse to Grecian jurisprudence.

Spirit and character of the decemviral laws.

When we come to examine the remaining contents of the Twelve Tables, we do not find any examples of the abolition of pre-existing distinctions in the legal rights and constitutional privileges of the patricians and plebeians; and we know that many political disqualifications of the latter existed long afterwards. On the other hand, we find no provisions recognizing such distinctions as already in force. The law which prohibited *connubium* between the patricians and plebeians is mentioned by Cicero (*de Rep.* ii. 37) as one of the partial enactments of the last two tables, and is treated by him as a novel hardship,* though Niebuhr looks upon it as only a confirmation of a very early rule. The disability was abolished by the *lex Canuleia* in the year 311. The partial distinctions which we find in the Twelve Tables relate to citizen and foreigner, freeman and slave,—not, except in the case of *connubium*, to patrician and plebeian. For example, in the case of a citizen, by usucaption of immoveable things (*fundus*) property was acquired in two years; of all other things, in one year: but in the case of a stranger, no prescription could defeat the Roman owner's claim—"Adversus hostem æterna auctoritas." The penalty for breaking or crushing the bone of a freeman was 300 *asses* (pounds

of brass); of a slave, 150 *asses* (Gai. iii. 223). We see here nothing like the inequality which pervaded our own Anglo-Saxon laws, in which each man's *were* was adjusted with great nicety according to his rank, ecclesiastic or civil.

Perhaps the most ancient rules in the Teutonic codes are those which, with careful minuteness, fix the pecuniary compensation to be paid for every bodily injury according to its position, severity, and extent. The Twelve Tables went upon the same principle, but their scale was not sufficiently sliding. A practical commentary on the roughness,—the want of philosophic refinement,—of their legislation was afforded by a waggish madcap (*egregie homo improbus atque immani vecordia*), one Lucius Veratius, whose great delight it was to go about slapping the faces of free citizens with the palm of his hand. A slave used to walk behind him, laden with a heavy purse (for money in those days was a ponderous commodity); and Veratius, as soon as he had boxed a cheek, ordered his slave to weigh out the legitimate 25 *asses* to the insulted citizen, in pursuance of the rule, "*Si injuriam faxit alteri, xxv. æris pænæ sunt*" (Gell. xx. 1).

The Twelve Tables, without expressly making different laws for different orders, did yet, by the severe regulations with respect to debtors, which we proceed to explain, make in effect one law for the rich, another for the poor. Their words may possibly have been "*Forti sanatique, nexo solutoque, idem jus esto*," but such was not the spirit of their legislation.

Debtors, proved to be such by confession *in jure* or by trial, were allowed 30 days to seek for the means of discharging their debts. If a debtor, after the 30 days, was unable to pay, he might be summoned before the magistrate (*in jus vocatio*), and, if necessary, arrested (*manus injectio*) by the creditor. In speaking of the magistrate, we must be understood to mean, before the institution of the prætorship, one of the consuls; afterwards, the *prætor urbanus*. If no one now offered to be his security, his sentence was severe. Taken to the creditor's house, he was there kept for 60 days, unless he could pay or make terms, bound with a chain not exceeding 15 pounds in weight, and, unless he chose to live at his own cost (which might well be, as his property could not be touched by his creditor), fed with a pound of corn daily. In the latter part of this period, he was brought before the magistrate in the forum on three successive market-days, and proclamation made of the adjudged debt, in order to see if any one would yet come forward on his behalf. On the third market-day (the 60 days being expired), if no friend appeared, he was put to death (*capite pœnas dabat**) or sold as a slave into a foreign land beyond the Tiber, the execution of the sentence being probably left to the creditor. If there were several creditors—we will give the very words of the law,—

"*Tertiis nundinis partis secanto, si plus minusve secuerunt, se fraude esto.*"

Many and ardent have been the controversies of the learned upon the meaning of these words (See Zimmern, *R. R. G.* 1, sec. 46, n. 17). Some, of whom Bynkershoek is the great champion, and among whom our

* *Duabus tabulis iniquarum legum additis, quibus etiam quæ dijunctis populis tribui solent, connubia, hæc illi ut ne plebei cum patribus essent inhumanissimâ lege sanxerunt.*

* *Capite pœnas dabat* is here undoubtedly used to denote what we ordinarily understand by capital punishment. In a general sense, any punishment was *capitalis* which involved the loss of life, freedom, or citizenship (l. 2, *pr. D. de Pœnis*, xlviii. 19).

Law.

countryman, Dr. Taylor, may be mentioned, have taken them figuratively, understanding by the *sectio* the sharing of the debtor's labour, or the division and sale of his property. We agree with the conclusion of those who, in accordance with every allusion to the subject extant in the ancient classics, take the literal sense. Dr. Arnold (*Roman History*, i. 136, &c.) well remarks,—

“Some modern writers have imagined that the words ‘*partis secanto*’ were to be understood of a division of the debtor's property, and not of his person. But Niebuhr well observes, that the following provision alone refutes such a notion—a provision giving to the creditor that very security in the infliction of his cruelty which Shylock had in his bond omitted to insert: ‘*Si plus minusve secuerunt, se fraude esto.*’ Besides, the last penalty reserved for him who continued obstinate was likely to be atrocious in its severity. What do we think of the *peine forte et dure* denounced by the English law against a prisoner who refused to plead? a penalty not repealed till the middle of the last century, and quite as cruel as that of the law of the Twelve Tables, and not less unjust.”

The concluding remarks of Dr. Arnold upon this subject appear to be in the main so just and important that we cannot refrain from adding them to the preceding quotation:

“Aulus Gellius, who wrote in the age of the Antonines, declares that he had never heard or read of a single instance in which this concluding provision had been acted upon. But who was there to record the particular cruelties of the Roman burghers (patricians) in the third century of Rome? and when we are told generally that they enforced the law against their debtors with merciless severity, can we doubt that there were individual monsters, like the Shylock and Front de Bœuf of fiction, or the Earl of Cassilis of real history, who would gratify their malice against an obnoxious or obstinate debtor, even to the extremest letter of the law? It is more important to observe that this horrible law was *continued* in the Twelve Tables, for we cannot suppose it to have been introduced there for the first time; that is to say, that it made part of a code sanctioned by the commons (plebeians) when they were triumphant over their adversaries. This shews that the extremest cruelty against an insolvent debtor was not repugnant to the general feeling of the commons themselves, and confirms the remark of Gellius, that the Romans had the greatest abhorrence of breach of faith, or a failure in performing engagements, whether in private matters or in public. It explains also the long patience of the commons under their distress; and, when at last it became too grievous to endure, their extraordinary moderation in remedying it. Severity against a careless or fraudulent debtor seemed to them perfectly just: they only desired protection in case of unavoidable misfortune or wanton cruelty, and this object appeared to be fulfilled by the institution of the tribuneship,—for the tribune's power of protection enabled him to interpose in defence of the unfortunate, while he suffered the law to take its course against the obstinate and the dishonest.”

Perhaps, in the passage last cited, may be traced an excess of favour to the plebeian party. It is not improbable that those plebeians who were beginning to acquire political ascendancy when the law of the Twelve Tables was sanctioned, could number at their head

many wealthy citizens—the foremost classes in the centuries—who already, by the influence of money, were able to present an opposing front to the influence of birth and rank. Merchandise and commerce were greatly extended even in the first period of Roman history; and it has often happened in the annals of the world that a rising monied interest has been a powerful instrument in democratic hands. If the plebeian knights and wealthy plebeian money-lenders were at the head of their party, it is easy to see why their zeal in the reformation of abuses should have stopped short where an alteration of the old law would have diminished their own power.

One of the most profound philologers and antiquaries that ever lived, Jacob Grimm, in his excellent *Deutsche Rechts-Alterthümer* (p. 613, *et seq.*), gives some examples in Teutonic law of similar severities to debtors, which are evidently borrowed from the law of the Twelve Tables. He, too, adverts to the horrid tales founded upon the horrid facts, and refers to the *Pecorone* of Giovanni the Florentine (Giorn. 4, Nov. 1), which was written in 1378, as the probable original of Shakspeare's Shylock.

The provision for selling a Roman beyond the Tiber deserves notice. By the direction that the sale should take place in a foreign country, the conversion of a freeman into a slave was less anomalous and less repugnant to received notions. A free Roman could not become a slave by his own private convention (l. 37, *D. de liberali Causâ* (xl. 12)). To effect so great a change by peaceful means a judicial sentence was required. The law, however, admitted that strangers might, by right of conquest, make slaves of Roman citizens. Whether debtors might become *nexi* (who, though not slaves, were *servorum loco*) without judicial sentence is a disputed question, which we cannot enter upon now.

It is not unlikely that the extreme cruelty of the law against debtors arose from a cause similar to that which gave occasion to the *peine forte et dure* in our own jurisprudence (See Barrington's *Observations on the Ancient Statutes*, p. 86). With us, the felon who stood mute, and was pressed to death, might thus save his broad lands. The forfeiture was not incurred which would have followed conviction and sentence. In like manner, the early Roman law was severe because it was imperfect. It gave no execution against property, which could only be reached indirectly by the constraint of execution against the person. Private property was not transferred by the law, and could not pass except by acts emanating from the will of the owner. This defect was gradually remedied, partly by direct legislation, and partly by the *jus honorarium*. The *peine forte et dure* is not the sole example in our English system of powers short in one direction and unduly stretched in another. The Court of Chancery could give effect to its decrees respecting property only by operating against the person. If a defendant chose to endure close imprisonment, and to brave all the terrors of process for contempt, the plaintiff had no redress except the gratification of revenge. Cases might still be pointed out in our law in which courts, from the want of appropriate powers, are compelled to work upon the *will* in such a manner, that judges appear to be waging an unseemly contest with an individual, while their sentence is liable to be baffled by a conscientious, an obstinate, or a wrong-headed man.

Law.

Law. Thus the law may have all the odium of inhuman and disproportionate severity without any of the merit of efficaciousness.

The creditor's right to take to his house (*ducere*) the sentenced debtor resembles the Athenian practice concerning ἀγῶγμοι.

The statement we have given of the law of debtor and creditor as it existed in the Twelve Tables, is extracted principally from the animated discussion between the jurisconsult Sextus Cæcilius and the philosopher Favorinus, reported by Gellius (*N. A.* xx. 1). There is something almost burlesque in the cold pedantry with which Cæcilius attempts to vindicate the barbarity of the "*partis secanto*," by reference to the legend of Metius Fufetius Albanus, and the harsh remark of Virgil,—

At tu dictis, Albane, maneres.

Ancient commentaries on the Twelve Tables.

The Twelve Tables, from the great conciseness of their enactments, stood in need, like our ancient Acts of Parliament, of a very liberal interpretation and extension by analogy. They required also that the particular modes of proceeding to be used in taking advantage of their provisions should be settled by precedent and practice. The earliest commentator on the Twelve Tables, whose name is known, was S. Ælius Catus, who was consul A.U.C. 556. His work was entitled *Tripertita*, from containing, first, the text; secondly, the interpretation (sometimes called specially *jus civile*); and thirdly, the forms used in procedure. We also know that Antistius Labeo and Gaius wrote extended commentaries, and the same may be said with much probability of L. Acilius (or Atilius), of L. Ælius (or Lælius), of Valerius Messala, and of S. Sulpicius Rufus.

Modern literary history of the laws of the Twelve Tables.

We possess but few fragments of the Twelve Tables literally. More often the purport of their enactments has been given by an author in his own words. The principal writers from whom our knowledge of their contents is derived are Cicero, Gaius, the elder Pliny, Festus, Gellius, Macrobius and Servius. From the commentary of Gaius on the Twelve Tables, twenty extracts are contained in the *Pandects*, and one in Joannes Lydus. The treatise *De Republica* of Cicero, the *Institutes* of Gaius, and Mai's Vatican fragments, have furnished the present century with some additional information unknown to former ages.

In attempting to restore the text, the moderns have taken the same liberties with the Twelve Tables as with the laws of the kings, and the controversy on their Latinity between Funccius and Branchu may still be consulted with interest by philologists.

The modern collectors and interpreters have been very numerous. Of these, J. Lipsius, Fr. Balduinus, Jac. Rævardus, Theod. Marcilius, Ant. Augustinus, and Conr. Rittershusius are among the foremost. As an Englishman, Ricardus Vitus Basinstochius (*i. e.* Richard White of Basingstoke) deserves notice. He is said by Bach to have been professor of the *Pandects in Academia Duacensâ* (Douai). His work, which is extremely rare, contains notes on the laws of the Twelve Tables according to the order of the *Institutes* and the first part of the *Digest*. The strange and confused title of it is (according to Bach, *Hist. Jur. Rom.* ed. 1807, p. 35) *Ricardi Viti, Basinstochii, ad Leges Decemviorum in duodecim Tabulis, Institutiones Juris Civilis in quatuor libris, primam partem Digestorum in quatuor*

libris notæ juris interpretati (8vo. Atrebatii, (Arras), 1597). But in collecting, arranging, and interpreting, no one of his predecessors approached the merits of Jac. Godefroi, whose labours have been the groundwork of nearly all that has been subsequently effected. Dirksen, who, since the time of Godefroi, has done more than any other person for the Twelve Tables, acknowledges that any attempt to restore the original order of the text must be founded on the principles which the ingenuity of Godefroi suggested. The Twelve Tables, as they appear in all ordinary books, are for the most part borrowed from Godefroi's restoration, which may be found in his *Fontes quatuor Juris Civilis*. It is inserted by Otto with a valuable preface in the third volume of his *Thesaurus*. The second edition of M. A. Bouchaud's voluminous *Commentaire sur la Loi des Douze Tables* appeared at Paris in 1803 (2 tomes, 4to). It is one of the works most frequently referred to, on account of the mass of materials it contains. Dirksen is celebrated for the sound and cautious criticism with which he has conducted his investigations, and for the ability with which he has reviewed all preceding researches. His work (which is a *chef-d'œuvre*) is entitled *Uebersicht der bisherigen Versuche zur Kritik und Herstellung des Textes der Zwölf-Tafel-Fragmente* (Leipz. 1824).

Our knowledge of the method observed in the Twelve Tables, so far as it depends upon positive authority, is exceedingly small. We know from Cicero (*De Leg.* ii. 4) that they began with the law of procedure. A rule quoted by Festus (*v. Reus*) as the second law of the second table relates to reasons for postponing the trial of a cause. Dionysius (ii. 27) states that a law of Romulus concerning the power of the father was inserted in the fourth table. Cicero places the regulations as to funerals in the tenth; and we are told by Cicero and Dionysius (x. 60) that the law prohibiting *connubium* between patricians and plebeians was in one of the last two. The Twelve Tables treated of succession by will before they proceeded to succession *ab intestato* (l. 1, pr. *D. Si Tab. Test. null.* (xxxviii. 6)). After directing that the *agnati* should be called to the *hereditas* of an *ingenuus* in default of *sui heredes*, the Twelve Tables, before they proceeded to make provision with respect to the *hereditas* of a *libertinus*, directed that the *tutela* of one who required guardianship should belong to the *agnati* (*see Theoph. ad. § 1, I. de Legit. Patr. Tut.* i. 17). This is all we know. For further arrangement we have only conjecture. Where distinct passages are cited in succession, the order of their citation affords strong evidence of their order in the tables. There is a passage in Ausonius (*Idyl.* xi. 61—64) from which it has been supposed by some, without sufficient reason, that there was a triple division into *jus sacrum*, *privatum*, and *publicum*, to each of which four tables were allotted. The words of Ausonius are,—

*Jus triplex, tabulæ quod ter sanxere quaternæ :
Sacrum, privatum, et populi commune quod usquam est.
Interdictorum trinum genus ; UNDE repulsus
VI fuero, aut UTRUBI fuerit, QUORUM ve BONORUM.
Triplex libertas, capitisque minutio triplex.*

He merely means to state that law, which admits of a threefold division, is the subject of the Twelve Tables; and it suits both his metre and his search for triplets to observe that twelve is a multiple of three.

Law.

Order of subjects in the Twelve Tables.

Law. There is no intention to assert an internal correspondence between the triple division of law and that of the Twelve Tables.

It has generally been taken for granted that each table was complete in itself, and the manner of quoting by the number of the table and the law (of which an instance, already referred to, occurs in Festus), would seem to show that, apart from the division into tables, there was no marked division of subjects or titles which would have given rise to a more natural mode of citation. Puchta, however, has observed that the decemviral laws might have been written consecutively, like the leaves of a book, without any attempt to make the conclusion of a table coincident with a termination in the sense. Illustration may be sought from the analogy of similar tables. Were Solon's tables engraven with such unscientific disregard to system? We think not. It seems that some of the Athenian tables were called *ἄζονες*, others *κύρβεις*, and that the sacred and public laws were engraven on the latter, the private on the former (*Plut. Solon* 25). The scholiast in Aristophanes refers the military laws to the *κύρβεις*. Demosthenes usually quotes the Athenian laws by the number of their tables. The expression *ὁ κάτωθεν νόμος*, occurring in the oration of Demosthenes against Aristocrates, is rightly explained by Taylor, and has nothing to do with the position of the law in one of the tables. Plutarch refers in one passage (*Solon* 19) to the eighth law of the thirteenth table, *ὁ δὲ τρισκαίδεκατος ἄζων τῶν ὀγδοῶν ἔχει [τον] νόμον οὕτως αὐτοῖς ὀνόμασι γεγραμμένον*. The reading of the Bodleian MS. *των νομων* gives some countenance to the opinion that one table was not always large enough to contain one entire law, and that the eighth law extended into the thirteenth table.

The conjectural grounds on which Jac. Godefroi endeavoured to establish the order of the Twelve Tables are ingenious and probable. All the six books of Gaius on the Twelve Tables are quoted in the *Pandects*; and, each book being supposed to relate to two tables, some indication of the subjects of the table is given by the twenty quotations. The tables themselves are supposed to have formed six pairs, two tables of each pair being connected in subject. Rather inconsistently with the last supposition, the eleventh table is supposed to have been supplementary to the first five, the twelfth to the following five, thus severing the third pair. The order of the Prætor's Edict is supposed to have been the same as that of the Twelve Tables, where both treat of the same matters. Some inferences also are deduced from later compilations,—the *Codex Theodosianus*, the *Codex* of Justinian, and the *Pandects*—which are known to have followed (more or less) the order of the Edict. Upon these grounds Jac. Godefroi arrived at the following arrangement: Tab. I. *In jus vocatio*; II. *Judicium et furtum*; III. *Res creditæ, depositum, fœnus*; IV. *Patria potestas et connubium*; V. *Hereditas et tutela*; VI. *Dominium et possessio*; VII. *Delicta*; VIII. *Jura prædiorum*; IX. *Jus publicum*; X. *Jus privatum*; XI. *Supplementa ad Tabb. I.-V.*; XII. *Supplementa ad Tabb. VI.-X.*

Dirksen departs from Jac. Godefroi's arrangement principally in the following particulars. He transfers *connubium* from the fourth of Godefroi to the fifth table. He gives the law of landed property to the sixth table. He makes the seventh contain *obligationes*

ex contractu, including *depositum* and *fœnus*, which are transferred from the third of Godefroi; and he makes the eighth contain *obligationes ex delicto*, including *furtum*, which is transferred from the second.

The Prætor's Edict and the Edict of the Curule Prætor Edict. *Ædiles* can scarcely be treated here as a separate branch of ante-Justinian law, inasmuch as nearly the whole of the remains we possess of them are embodied in the collections of Justinian.

These remains are a portion of the *jus honorarium*, the nature of which we shall very briefly explain.

The Roman magistrates, on their entrance into office, were accustomed to expose publicly (*in albo*) the rules according to which they intended to proceed during their magistracy. The rules thus published as a guide for the whole year of office were called *edicta perpetua*, in contradistinction to the *edicta repentina*, which were promulgated on particular occasions.

Thus, where the law was defective it was supplied, and in many instances its harshness was corrected; but in modifying the *jus civile*, the magistrates, who were sworn to obey the laws, proceeded gradually, and with the sanction of public opinion. Their right of acting with such an instrument of gradual reform was recognized by the constitution, and was exercised more freely in early times, when the wants of mankind extended widely beyond the narrow limits of the *jus civile*, than at a subsequent period, when those wants had been supplied by precedent and direct legislation.

The edicts in general were founded on principles of reason and equity, and the magistrates in composing them often made use of the advice of judicious friends acquainted with the civil law. As the whole set of rules published by a magistrate was called his edict, so each separate rule was often called an edict, as in the Jewish law each section of Mishna is itself called a Mishna. A magistrate usually inserted in his edict those rules of his predecessor which had been approved and found useful. To such *edicta tralatitia* he sometimes added *edicta nova* of his own. The former gradually came much to exceed the latter in number; and, in Cicero's time, the law of the entire prætor's edict had been almost settled by custom (*Cicero de Invent.* ii. 22). The power of making *edicta repentina*, suited to the emergency of particular cases, was sometimes abused; and to remedy such abuse on the part of prætors a *lex Cornelia* (A.U.C. 687) forbade them to depart from their *edicta perpetua*.

Priests and generals, as well as civil magistrates, had their *jus edicendi*. The only magistrates who appear not to have been invested with this power were the *ædiles plebis*, who were bound by the edict of the curule ædiles (*Theoph. lib. i. tit. ii. sec. vii.*). Of all the edicts, those of the prætors were the most important; but mention is frequently made in ancient writers of the edicts of the censors, tribunes, and governors of provinces. The name *jus honorarium* given to the department of law thus introduced by magistrates arose from the circumstance that the magistrates possessed *honores*. Some of the ancients were in favour of a different and less probable origin. They thought that *jus honorarium* was so called because it was observed *in honorem magistratum* (§ 7, *I. de Jur. Nat.* (i. 2); l. 7, § 1, *D. de J. & J.* (i. 1)).

Many of the Roman jurists occupied themselves in commenting upon and arranging the prætors' edicts when they began to assume a permanent shape.

Law.

The first who distinguished himself in this pursuit was Ofilius, a friend of Cæsar. Servius Sulpicius, the master of Ofilius and well-known friend of Cicero, had previously composed a short compendium of prætorian law. The edict of the prætor peregrinus, and those of provincial governors, probably agreed in most respects with those of the prætor urbanus, and differed only where the peculiarities of their jurisdiction and local circumstances required distinct provisions.

A memorable epoch in the history of prætorian law occurred in the reign of Hadrian, when Salvius Julianus, an eminent lawyer, revised and arranged, under the auspices of the emperor, the edicts of the prætors and curule ædiles. The eclectic edict, composed out of these materials, was sent to the urban and provincial magistrates as a permanent form. It was called in an eminent sense *edictum perpetuum*, the word *perpetuum* having now acquired a different meaning from that in which it was first employed. The reformed edict of Julian or Hadrian was the perpetual edict, inasmuch as it was intended to remain unchanged for an indefinite time; whereas a perpetual edict was anciently one which was intended to regulate the conduct of the magistrate *continuously* during his period of office.

The magistrates, however, even after the composition of Salvius Julianus, retained the *jus edicendi* upon points not determined by the *edictum perpetuum* or by other positive legislation (Cassiodor. *Varia*, vi. 18; l. 2, *C. de Off. P. P.* (i. 26)). An example of this may be seen in three edicts of the *præfectus prætorio*, which, in ordinary editions of the *Corpus Juris*, are printed as *Novells* of Justinian, and numbered 166-168. The exact extent and effect of the labours of Salvius Julianus are much, and, from the paucity and obscurity of historical record concerning them, perhaps hopelessly, controverted. Some difficulties will be surmounted, if we suppose that the main body of his work consisted of a fusion of edicts, urban and provincial, which might be adapted, by means of variable formal parts and minor alterations, to the use of different magistrates. It is not unlikely that the change was connected with the new judicial organization of Italy, which was effected under Hadrian.

In the fragments of juriconsults collected in the *Pandects*, certain parts of the perpetual edict are quoted much more than others. We have thus many more fragments relating to interdicts than to *possessio bonorum*. When a commentator on the prætor's edict is cited in the *Pandects*, the number of the book of the commentary is stated; and thus means are afforded of arriving at the order of arrangement observed in that edition of the Prætor's Edict, namely, the *Edictum Perpetuum Hadriani*, to which all the commentaries cited in the *Pandects* refer. The arrangement was probably more historical than scientific, and it is likely that sometimes subjects were brought together, not because of any natural connexion, but because they were associated by means of a technical term which had different meanings. Thus, under the head *de edendo*, it is likely that the accident of similar name brought together clauses concerning the *editio* of actions, and the *editio* of the accounts of an *argentarius*.

It is generally supposed to have been divided into seven parts, like the *Pandects*, which Justinian states to have been digested in imitation of the perpetual edict (§ 5, const. *Deo Auctore*). The *partes* were divided

into *tituli* with rubrics, and these again into *capita*.

Law.

Single sentences are sometimes called *clausulæ*.

The division of parts usually adopted by the moderns is as follows:—I. *De jurisdictione et iis quæ fiunt in jure*.

II. *De judiciis*. III. *De rebus creditis*. IV. *De nup-*

tiis, dotibus et tutelis. V. *De furtis, bonorum posses-*

sionibus, testamentis, novi operis nunciatione, damno

infecto, et aquâ. VI. *De manumissionibus, publicanis,*

vi, injuriis, missione in bona et privilegiis creditorum.

VII. *De interdictis, exceptionibus et stipulationibus*.

G. Giphanius, one of the early modern arrangers, proposed a division into ten parts. The collected remains may be seen in many common books: e. g. Van Leeuwen's edition of the *Corpus Juris Civilis*, Jac. Gothofredi *Fontes quatuor Juris Civilis*, Pothier's *Arrangement of the Pandects*, Haubold's *Instit. Hist. Dog. Lin.* tom. ii. On the provisions of the edict Noodt is a most able commentator; and the work of Heineccius, "*Historia Edictorum et Edicti Perpetui*," is exceedingly learned and elaborate. Bouchaud has written several papers on the edicts of the Roman magistrates, in the *Mémoires* of the French Institut. The latest important work on this subject is that of De Weyhe, *Lib. iii. Edicti, sive libri de origine fatisque jurisprudentiæ Romanæ, præsertim Edictorum Prætoris, ac de formâ Edicti Perpetui* (4to. Cellis, 1821). The principal fault of some of the writers we have mentioned is their affecting to be wise beyond what is written in the original sources.

In looking over the remains of the prætor's edict, the English lawyer will observe that from it was derived our writ *de ventre inspiciendo*, which was in established use before the time of Bracton (See l. 1, § 10, *D. de Inspic. Vent.* (xxv. 4)).

The remains of the edict of the Curule Ædiles are contained in the first title of the 25th book of the *Digest*. Aulus Gellius and Festus refer to a more ancient edition than that which is there cited. Thibaut's *Essay on the Ædilitian Edict* is contained in his excellent *Civilistische Abhandlungen*.

The remaining parts of the law of Rome, older than Justinian, and not incorporated in his collections, may be classed under two heads.

A, private technical works, as well

[a,] those which have come down to us separately, in their original purity, as

[b,] those which have been preserved to us in private technical compilations.

B, technical compilations made by the public authority, including as well

[a,] single works (which are, for the most part, mutilated, interpolated, altered, and abridged), as

[b,] collections or codes.

A a. The works and fragments of works of juriconsults which have come down to us in their original purity, without being incorporated in the collections of Justinian, or in earlier compilations, public or private, are of the greatest value. Unfortunately, the number of them is small.

A b. It is remarkable that nothing approaching to a complete collection has yet been formed of the passages in the ancient classics which specially illustrate Roman Jurisprudence, and particularly of those passages which contain extracts from the writings of jurists. There is an exceedingly valuable list of some such passages in the "*Notitia historico-literaria*" at the commencement of Haubold's *Monumenta Legalia*. In 1815, Dirk-

Edict of
Curule
Ædiles.
Other re-
mains of
Ante-Jus-
tinian law.

Law.

sen published at Königsberg *Bruchstücke aus der Schriften der Römischen Juristen*. A complete collection of such fragments in continuation of Dirksen's work is promised in the prospectus of the Bonn edition of the *Corpus Juris Ante-Justiniani*.

Pure private technical works and fragments of works.

To the heads A [a] and A [b] belong, in addition to some works already mentioned, the following remnants of antiquity. (1), a fragment of Sextus Pomponius. (2), a fragment of Herennius Modestinus. (3), the work entitled *Tituli ex corpore Ulpiani*. (4), Fragments from the *Institutiones* of Ulpian. (5), *Fragmentum Regularum, ex veteri jureconsulto*. (6), *D. Hadriani sententiæ et epistolæ*. (7), *Consultatio veteris jureconsulti*. (8), *Mosaicarum et Romanarum Legum Collatio*. (9), *Gaii Institutiones*. (10), *Fragmentum de Jure Fisci*. (11), *Fragmenta Juris Romani Vaticana*. (12), *Fragmentum Græcum de obligationum causis*. Of these we shall now treat in order.

Fragment of Pomponius.

(1.) Arnoldus Ferronius, in his *Consuetudinum Burdigalensium Commentarii* (4to., Lugd. 1536), published a minute fragment (18 words) of Sextus Pomponius, taken from a most ancient worm-eaten MS. which had formerly belonged to the library of Petrus Crinitus of Florence, and had been given to Ferronius by J. C. Scaliger. It was copied by Cujas from Ferronius, and was lately brought again to light by A. G. Cramer in Hugo's *Civilist. Mag.* vi. 1-33.

Fragment of Modestinus.

(2.) A fragment of Herennius Modestinus, still shorter than the fragment of Pomponius, was first published (4to. Lutet. 1573) by P. Pithæus, from a manuscript of his father, along with the *Mosaicarum et Romanarum Legum Collatio*. Another fragment of Modestinus, preserved by Isidore, concerning the difference between relegation and deportation is mentioned by Böcking, *Ulpiani fragmenta*, p. 109 (Bonn, 1836).

Tituli ex corpore Ulpiani.

(3.) From these unimportant fragments we pass to a source of considerable value, namely, the fragments of Ulpian, first published by Tilius (Du Tillet) (8vo. Paris, 1549) under the name *Tituli ex corpore Ulpiani*. The text of Tilius was taken from a Vatican manuscript (codex 1128), from which all other existing manuscripts of the work are copied (See Zimmern, *R. R. G.* i. 21, n. 25). This position has been lately controverted by G. E. Heimbach (*über Ulpian's Fragmente*, Leipz. 1834), who has been successfully answered by Savigny (9 *Zeitschrift*, p. 157). The 29 titles of Ulpian thus preserved are now generally thought to have been part of Ulpian's *Liber Singularis Regularum*. G. E. Heimbach, indeed, has supposed, without sufficient ground, that they are a collection from the works of various jurists. The Vatican manuscript of these fragments of Ulpian is at the end of a manuscript of the Westgothic compilation, called the *Breviarium Aniani*, of which we shall hereafter speak, but the fragments are pure, and form no part of that compilation. Ulpian was never popular in the Western Empire. The present work has no trace of interpolation or Westgothic interpretation, and it deals too much in antiquated law (of which it was the most valuable source before the discovery of the genuine Gaius), to suit the purpose of the Westgothic compilers. Its style, indeed, is more concise than is usual with Ulpian. It is manifestly imperfect at the beginning and the end. It treats of persons, property, successions, but not of obligations nor of actions. It has been illustrated by the notes of Cujas, Schulting, Meerman, and Cannegieter. The first really accurate edition of the text was given by Hugo,

Berlin, 1822. Among the most valuable observations on the *Tituli ex corpore Ulpiani* are those of Schilling, *Animadversionis Criticæ ad Ulpiani fragmenta Specim.* i. ii. (Vratislay, 1830), and those of Savigny in the 9th volume of the *Zeitschrift*. The last and best edition is that of Böcking, contained in his *Ulpiani Fragmenta* (Bonn, 1836).

(4.) The recent discovery made by Stephen Endlicher of some short but not unimportant fragments of the Institutes of Ulpian deserves to be mentioned on account of its singularity. In the Royal Library at Vienna was a papyrus manuscript, in its original binding. Strips of parchment had been employed by the book-binder to fasten the leaves together. S. Endlicher perceived that five of these strips were cut out of a legal manuscript, and that in one of them the column of writing was headed *Ulp. Inst.* Of the decyphered characters, a part relates to contracts, and the rest furnishes some new and interesting information concerning the interdict *Quem Fundum*. A fac-simile may be seen in Endlicher's *Catalogus codd. philolog. lat. bibl. palat.* (Vindob. 1836, tab. iii.) The text is given by Savigny, with valuable remarks, in the 9th volume of the *Zeitschrift*; by Pellat, in the 4th volume of the Brussels *Revue de Législation et de Jurisprudence*; and by Böcking in the work already cited, *Ulpiani Fragmenta* (Bonn, 1836).

Law.

Endlicher's fragments of Ulpian's Institutes.

(5.) The *Fragmentum Regularum ex veteri jureconsulto* has been published with various names. We follow that which is given by Schulting in his *Jurisp. Ante-Just.* In the *Corp. Jur. Ante-J.* edited by the Bonn Professors, it is called by Böcking, *Disputatio forensis de Manumissionibus*.

Fragmentum Regularum ex veteri jureconsulto.

The history of this fragment is curious. In the beginning of the third century of our era (A. D. 207, *Maximo et Apro consulibus*), Dositheus Magister, a schoolmaster who taught Greek and Latin, composed a school-book under the title *ἐρμηνεύματα* or *interpretamenta*, in three books. Of these, the first two contained a grammar and glossaries; the third, a collection of subjects for translation from Latin into Greek, and *vice versa* (1 Puchta, *Inst. R. R.* 495). These subjects consisted for the most part of extracts from other writers. Among them is the fragment we are now treating of, extracted from a legal writer, and containing some remarks on the division of *jus* into *civile, naturale*, and *gentium*, the division of persons into freeborn and freedmen, and the law of manumissions. The manuscripts of Dositheus contain a Greek text, and a miserable, often unintelligible, Latin one. It cannot be doubted that the Greek is translated from a Latin original; and the Latin text that has come down to us has been conjectured by Lachmann (*Versuch über Dositheus*, Berlin, 1837) to have been formed in the following manner. The pupils of Dositheus, or other boys who made use of the Greek as a theme for re-translation, were accustomed as an exercise to place the words of the original text out of their natural order under the corresponding words of the Greek translation. The manuscripts of Dositheus were in fact specimens of a kind of ancient converse of modern Hamiltonian school-books. Proceeding on this idea, Lachmann has attempted, and on the whole with success, out of the disjointed Latin, to restore the original. The author of the fragment is unknown. Schilling, against the probable inference to be derived from internal evidence, supposes it to have been a compilation from several jurists, and

Law. in this opinion is followed by Zimmern. It has, without ground, been attributed to Ulpian; and as Dositheus himself calls the work *Regulæ*, it is supposed by Lachmann to have been an extract from *Pauli Regularum Lib. vii.* The Latin text was first published at Paris in 1573 by Pithæus, at the end of his edition of the *Collatio Legg. Mos. et Rom.*; the Greek and Latin together were first published by Röver (Lug. Bat. 1739). In the Berlin *Jus Civ. Ante-Just.*, the text, which in many books is stated to have been revised by F. A. Biener, was really edited by Beck. There are able observations on this fragment by Cujas (*Obs.* xiii. 31), by Valkenaer (*Miscell. Obs.* tom. x. p. 108), and by Schilling (*Diss. crit. de fragmento juris Rom. Dositheano*, Lips. 1819).

Divi Hadri- (6.) The same third book of Dositheus contains the
ani Senten- *Divi Hadriani sententiæ et epistolæ*, a collection of
tiæ et Epis- legal anecdotes and opinions of little value. It is the
tolæ. first extract in the *ἐπιτηνέματα*; and between it and the fragment concerning manumissions is inserted the genealogy of Hyginus. A complete edition of the third book of Dositheus was given by Böcking (Bonn, 1832).

Consultatio (7.) The *Consultatio veteris jureconsulti* is the
veteris work of an unknown lawyer, who probably lived
jurecon- shortly before the time when Justinian's compilations
sulti. were made, as he does not cite them, though he cites the Gregorian, Hermogenian and Theodosian codes. It really consists of four consultations, but the first consultation branching into three points concerning pacts, has occasioned the whole to be sometimes called *Consultatio veteris jureconsulti de pactis*. It was first edited by Cujas, in 1577, at the commencement of his own *Consultationes*. It appears in the *Jurisp. Ante-J.* of Schulting. In the Berlin *Jus Civ. Ante-J.* it is edited by Biener, and in the Bonn *Corp. Jur. Ante-J.* by Puggé.

Collatio Le- (8.) The *Collatio legum Mosaicarum et Romana-*
gum Mosai- *rum* is, like the *Consultatio*, written by an unknown
curum et author, who probably lived at the end of the fifth or
Romana- the beginning of the sixth century. It has been at-
rum. tributed without foundation to Licinius Rufinus, but internal evidence proves that the author was posterior to that Licinius Rufinus, at least, who was a jurist contemporary with Paulus. It is valuable for its literal citations from Gaius, Papinian, Ulpian, Paulus and Modestinus, as well as from the three codes which preceded the compilations of Justinian. It is sometimes called *Lex Dei*, from the words which commence a sentence at the beginning of the manuscripts of the work. The first edition is that of Pithæus (Paris, 1573). Of the many subsequent editions, the best beyond comparison is that of Blume (8vo. Bonn, 1833), which is the basis of the edition given in the Bonn *Corp. Jur. Ante-J.* The author was either a Jew or a Christian,—more probably the latter. His object in comparing the Mosaic and Roman systems of law appears to have been to furnish Christians and Jews with an apology for the Roman system, and to recommend the Mosaic system to the Pagans.

The Insti- (9.) Of all the discoveries recently made, none has
tutes of been so valuable, and none has excited so much at-
Gaius. tention, as that of the *Institutes* of Gaius. A corrupt and mutilated abridgment of these *Institutes* had been preserved in the *Breviarium Aniani*, but it was reserved for modern times to decypher a manuscript of the original work.

It was a common practice in the middle ages, to

Law. wash out the relics of antiquity, in order to economize the parchment on which they were written. When ing alone would not expunge the writing,—as often happened in the case of manuscripts written on the once hairy side of the parchment,—the characters were further scratched out with a knife. A father of the church sometimes covered the pages which had before contained the works of some profane dramatist. Not unfrequently the parchment was a second time submitted to the same treatment. The father, who had supplanted the dramatist, was himself washed and rubbed out, in order to give place to some scholastic doctor

In the Library of the Chapter at Verona is a *codex* formerly numbered xv., but now xiii., containing a manuscript of the letters of St. Jerome (*Hieronymus*), written over an older manuscript. Nearly one-fourth part of the *codex* was *bis rescriptus*, and, where this was the case, it seems that St. Jerome had also been the second occupant. It has been stated, we know not on what authority, that the letters of St. Jerome had been edited from this very manuscript, without any notice having been taken of the original purpose to which the parchment had been applied. The manuscript first written on the parchment consisted of 251 pages, and each page of 24 lines. It is a remarkable fact, that one leaf, or two pages, 235 and 236, concerning prescriptions and interdicts, had been detached from the rest of the manuscript, and escaped being overlaid by St. Jerome. These two pages had been found in the Library of Verona, before the year 1732, by the celebrated Scipio Maffei. He alludes to them in his *Verona Illustrata*, Parte terza, c. 7, p. 464 (Verona, 1732). In his *Istoria Teologica* (Trento, fol. 1742), p. 90, *et seq.*, the greater part of the text which they contain is printed. He gave a fac-simile of part of the writing in plate iii. cut x., subjoined to his *Istoria Teologica*, and part of this fac-simile was republished, not very accurately, in the *Nouveau Traité de Diplomatie* (Paris, 1757, see tom. iii. p. 208, tab. 46). Maffei had observed the correspondence between the fragment he published and the 15th title of the 4th book of Justinian's *Institutes*; but, instead of recognizing Gaius, whose text was the ground-work of Justinian's work, he supposed that the leaf he had found was part of an interpretation or compendium of Justinian's *Institutes* made by some later jurisconsult.—To Maffei, however, belongs the credit of having first given to the world two pages of the genuine Gaius.

It had not escaped the notice of Maffei that the manuscript of the letters of St. Jerome was a *codex rescriptus*.—This appears by his remarks in the catalogue of the Library, but he did not know what the subject of the obliterated writing was, and was not aware of the connexion between that manuscript and the detached leaf which had drawn his attention.

The fragment concerning Interdicts, published by Maffei, had not been unobserved by the learned Haubold, "*cujus ingentem eruditionem nihil fere solebat fugere.*" He determined to recall it to the memory of German jurists, and prepared an essay for that purpose, which was published at Leipsic in 1816, under the title *Notitia fragmenti Veronensis de Interdictis*, and is to be found in his collected *Opuscula*, tom. ii., p. 327-46.

By chance, while the *Essay* of Haubold was in

Law.

preparation, but not yet published, in the year 1816, Niebuhr was despatched to Rome by the King of Prussia, as Minister to the Apostolic See. On his way, he spent the greater part of two days in examining the Cathedral Library of Verona.—Whether he was induced to make this examination by previously excited curiosity concerning Maffei's fragment, we have not been able to ascertain; but certain it is that he made wonderfully good use of his limited time. He copied, fully and accurately, the fragment concerning interdicts and prescriptions, which he did not hesitate to ascribe to its real author, Gaius.—He also copied the manuscript of a fragment *de jure fisci* (of which we shall presently speak), and by means of the infusion of nut-galls, he was able to decypher the 97th leaf of the obliterated writing of *codex xiii.*, which he at once recognized as an important work of a most ancient jurisconsult, whom at first he supposed to be Ulpian. The fruit of his researches he communicated by letter to Savigny, by whom they were printed in the 3rd volume of the *Zeitschrift*. Savigny added a learned and acute commentary of his own, and put forward the felicitous conjecture, amply verified in the sequel, that the ancient text of *codex xiii.* contained the genuine *Institutes* of Gaius, and that the fragment concerning prescriptions and interdicts had formerly been a part of that *codex*.—The fame of this discovery soon spread among the jurists of the continent. Hugo, in particular, exerted himself in turning the attention of the learned to the subject, and pledged himself that the manuscript would amply repay the labour of decyphering: accordingly, in May, 1817, the Royal Academy of Berlin despatched to Verona two distinguished scholars, Goeschen and Bekker, charged with the task of transcribing the manuscript. Bekker was prevented by other occupations from accomplishing the task, and departed for Milan before the end of July; but his place was supplied by Bethmann-Hollweg, who, from the deep interest he felt in the subject, volunteered his valuable services, and greatly assisted Goeschen. The difficulty of transcription was excessive. The original manuscript, according to the opinion of the celebrated palæographer, Kopp, as stated in the 4th volume of Savigny's *Journal*, was anterior to Justinian's legal reforms. The scribe, like the majority of legal writers in our own country, at the present day, employed a great variety of contractions, and words were often expressed by initial letters. The old order of the leaves was much deranged.—There were very few pages where the parchment had not been written over: in 60 pages, it was *bis re-scriptus*, and the new writing was in general directly over the old. In order to prepare the parchment, it had been washed, apparently bleached in the sun, and in some parts scraped with a knife. Various chemical re-agents were tried in order to render the ancient characters again legible. Infusion of nut-galls succeeded the best. Notwithstanding these difficulties, by far the greater portion of the *Institutes* of Gaius has been preserved to us. Probably not one-tenth of the whole work is wanting. It is true that certain parts of the extant leaves resisted all attempts at decyphering, and that three leaves are missing, *viz.*:—1st. The leaf following p. 80, of which the argument may be collected from the Gothic epitome. 2nd. The leaf following p. 126, and commencing the third book. The whole contents of this leaf have been luckily pre-

served in the ancient extract, made by the author of the *Collatio*, before described. 3rd. The leaf following p. 194. The loss of this leaf is very tantalizing, for it doubtless contained some particulars relative to the old *legis actiones*, which we are left without any means of supplying. A few of the gaps which are occasioned by the impossibility of decyphering are also very lamentable, for they occur in the most obscure parts of the work,—in passages where the curiosity of the antiquary is raised highest,—and all the ingenuity of conjecture, possessed by the ablest critics, has been unable satisfactorily to fill them up.

With scrupulous accuracy and consummate ability did Goeschen and Hollweg finish their irksome and difficult labour. In 1819, the first printed sheet of the volume appeared; but not until 1821 was the first complete edition of the work brought out by Goeschen. Its appearance excited an unusual sensation among the jurists of the continent. Every copy was soon eagerly bought. It was considered to form an era in the study of Roman law. It was found to elucidate doubts and clear up difficulties before regarded as hopeless. By the true explanations it afforded, many an ingeniously constructed theory was demolished. It was felt that there was something unique and most remarkable in this resuscitation. Modern jurists were thus suddenly placed upon a vantage ground, from which they looked down upon their less fortunate predecessors. They blessed their stars, and thought it luxury to be admitted to a pure spring of knowledge, for many previous ages untasted and unknown.

A new edition was loudly called for, and in 1824, the learned Bluhme (as his name is generally written), or Blume (as he writes himself), made a fresh collation of *codex xiii.*, and the result of his renewed examination was given to the world by Goeschen, in the celebrated edition of 1824. An improved reprint of this edition, by Lachmann, was published in 1842, the editor having completed a critical revision, which had been interrupted by the death of Goeschen.

The civilians of the continent have, from the first publication of Gaius, laboured assiduously in interpreting the text, in composing dissertations on the doctrines contained in it, and in conjectural supply of the *lacunæ*; but no edition of the whole work with a good commentary has yet appeared. Among the most valuable productions to which Gaius has given birth, we would mention Gans' *Scolien zum Gaius* (Berlin, 1821), Dirksen's remarks in his *Versuche zur Kritik*, &c. (Leipz. 1823, pp. 104–136), Heffter's edition of the 4th book, with commentary, 4to. (Berlin, 1827), and Huschke's observations on Gaius in the first volume of his *Studien des Römischen Rechts* (Breslau, 1830). Klenze and Böcking's *Gaii et Justiniani Institutiones*, 4to. (Berlin, 1829), is a useful book. The texts of the two elementary works are there placed side by side, but Gaius is made to yield to the order adopted by Justinian. Among the latest editions of Gaius are that of Böcking (Bonn, 1841), and that of Lachmann, 8vo. and 4to. (Bonn, 1841). Lachmann's edition, in 4to. form, is now the first part of the Bonn *Corpus Juris Ante-J.* This, however, is distinct from the *new edition of the edition of 1824*, which we before mentioned. All the copies of an edition of the entire work by Heffter, which was originally intended to form the first part of the Bonn *Corp. Jur. Ante-J.*, have been long since exhausted.

Law.

Law

This account of the discovery of Gaius has been principally compiled from the well known prefaces prefixed to Goeschen's first and second editions. There are a few errors in the article relating to the same subject in the 48th volume of the *Edinburgh Review*, p. 381. &c. The bantering doubts which are thrown out by the critic (p. 385) as to the authenticity of the discovery were successful in provoking the ire of some matter-of-fact Germans, who looked upon them as serious.—(See Hugo's *R. R. G.*, 11th edition, p. 22, n. 23.) The authenticity of the *Institutes* of Gaius is beyond dispute. This is clear from internal evidence, which would prove a forger to have possessed miraculous knowledge and sagacity. The work agrees with the *Institutes* of Justinian (which were derived from it), is the manifest source of the Gothic epitome, and contains all the passages cited from Gaius in the *Pandects*, in the *Collatio*, by Boethius and by Priscian.

Gaius wrote under the Antonines, in the most flourishing period of Roman jurisprudence; and his *Institutes*, as they were a popular manual at Rome, are perhaps to the modern student the best initiation into the civil law, especially if they are read along with the *Institutes* of Justinian and the *Paraphrase* of Theophilus. The method of Gaius is based on the celebrated threefold division of law, "*Omne jus quo utimur, vel ad personas pertinet, vel ad res, vel ad actiones.*"—It is necessary to observe, that he treats rather of the *dynamics* than of the *statics* of law—rather of those events or *forces* by which classes of rights begin, are modified or terminate, than of those rights and duties which accompany a given *stationary* legal relation.—Thus, in treating of the *jus quod ad personas pertinet*, when he comes to the *patria potestas*, it is not his object to explain the mutual rights and duties of parents and children, but to point out the cases and events in which those rights and duties arise or cease.

Fragmentum de Jure Fisci.

(10.) The "*fragmentum de Jure Fisci*" is derived from a manuscript of two leaves in the Library at Verona. The characters, though not written over, were faded and difficult to decypher. The two leaves were observed by Maffei, who gave some extracts from them in his *Istoria Teologica*, in conjunction with the extracts from the detached leaf of Gaius; but, as we have before stated, a complete copy was first taken by Niebuhr. The best edition of the text is that of Böcking, with critical notes, in *Ulpiani Fragmenta* (Bonn, 1836). It is generally supposed to be a fragment, not of Ulpian, but of Paulus, who wrote two books "*de jure fisci.*" A passage of the fragment (§ 19) is repeated almost verbatim, but in a different connexion, in another work of Paulus, *Liber Quintus Sententiarum*, as appears by an extract in the *Pandects* from the latter book (L. 45, § 3, D. de Jure Fisci, (xlix. 14)).

Fragmenta Juris Romani Vaticana.

(11.) The Vatican fragments were taken by Mai from a manuscript of the *Collationes* of Cassianus in the Vatican Library. Twenty-eight leaves of this manuscript were observed to be *folia rescripta*, and upon examination it was found that the characters first written and covered by the work of Cassianus belonged to a collection of extracts from constitutions and juridical writings. Of this collection seven titles are imperfectly preserved, and were first published by Mai, at Rome, in 1823. Of subsequent editions, the best are those of Buchholz (Regiomontani, 1828, with explanatory notes, and of Bethmann-Hollweg (Bonn,

1833), with critical notes. The author of the collection, which was probably made a short time before the compilation of the Theodosian code, appears to have had a practical end in view, viz. the reduction into a convenient size of sources commonly quoted before the courts. He gives extracts from Papinian, Ulpian, and Paulus; and in those extracts Celsus, Julianus and Pomponius are frequently cited. On several points, especially the doctrines of *dos*, *ususfructus* and *donationes*, the Vatican fragments throw important light. How much they contribute to our knowledge of Roman law is well shown by G. Bruns, in a little work under the title *Quid conferant Vaticana Fragmenta ad melius cognoscendum Jus Romanum* (Tubingæ, 1842).

(12.) A Greek fragment, *de Obligationum Causis et Solutione*, first discovered by Mai, was reprinted with notes by Haubold (Leipz. 1817), and may be found in the second volume of Haubold's collected *Opuscula*, p. 394. It elucidates the *stipulatio Aquiliana*, the form of which is given and well explained in *Brissonius de Formulis*, lib. vi., c. 194.

Before we proceed to enumerate some less pure sources, we cannot avoid expressing a wish that some of the many palimpsests or *codices rescripti* which exist in the British Museum were examined by competent scholars. The continental jurists who have been deputed to explore Great Britain in search of legal treasures seem not to have been acquainted with the existence of such palimpsests, nor do they seem to have investigated the manuscripts belonging to our Inns of Court.

B a. To the head *B a*, consisting of the separate works and fragments of works of single jurists, preserved in collections made by public authority, may be referred (1.) the epitome of Gaius; (2.) the fragment of *Pauli Sententia*; (3.) a fragment of Papinian. All these are preserved in the *Breviarium Aniani*.

B b. To the class *B b* belong (4.) the *Codex Gregorianus*; (5.) the *Codex Hermogenianus*; (6.) the *Codex Theodosianus*; (7.) the *Ante-Justinianean Novells*; (8.) *Edictum Theodorici*; (9.) *Edictum Athalarici*; (10.) *Lex Romana Burgundiorum*; (11.) *Institutio Gregoriana*.

To the head *B b* might be referred the *Breviarium Aniani*, as it was a law-book made by public authority; but it is itself a collection not only of private legal writings sanctioned by public authority, but of abstracts of former collections. The head *B b* might hence be sub-divided into two classes: the first consisting of those codes and collections, or parts of codes and collections, which have come down to us separately and independently; the second, of those which are only known from their preservation in the *Breviarium Aniani*.

Of this compilation, we shall now give some account before we proceed to the description of the several sources enumerated under the heads *B a* and *B b*.

When the Western Empire submitted to Germanic rule, the Roman law was still allowed to retain its force by the side of a newly introduced Germanic jurisprudence. The *Lex Romana*, as it was barbarously called, remained the law of the subjugated Romans, while the *barbari*, as the Germans were proud to be styled, continued to live under their own Teutonic institutions. Hence arose various difficulties. In the case of contracts between Germans and Romans, or crimes committed by a man of one race against a man of the other, which system was to prevail? These

Law.

Fragmentum de Obligationum Causis.

Private writings preserved in Compilations made by public authority. Collections or Codes.

Law. questions were differently decided in the different kingdoms of the Western Empire, and the reader may be referred to Savigny's *History of the Roman Law in the Middle Ages* for a full elucidation of the anomalous system of *personal laws*.

In the Westgothic kingdom, which was established in Spain and a part of Gaul, it was found necessary, during the reign of Alaric II. (484--507), to make a collection of such sources of Roman law as were to be recognized for the future, in regulating the rights and duties of the subjugated Roman provincials. The collection was made, and, in the year 506, copies of it were sent to the higher magistrates, with a *com-mo-nitorium*, in which it is stated that the duty of publication was intrusted to Goiaric, the *comes palatii*; while the duty of authenticating true copies with his signature was left to Anianus.—Many have supposed, without sufficient reason, that Anianus was the author of the whole compendium, and hence it has received, in comparatively modern times, the name *Breviarium Aniani*, a name which first appears in the edition of Tilius (Paris, 1550). It has also sometimes been called *Breviarium Alaricianum*, or simply *Breviarium*, or *Lex Romana*, or (from the *Codex Theodosianus*, which is its first and chief ingredient) *Lex Theodosii*. By Anianus himself, in his subscription, it is termed "*Codex de Theodosiani legibus, atque sententiis juris vel diversis libris electus*." There are considerable variances in the manuscripts of the *Breviarium*. The oldest manuscript is that which is now at Munich. One of the best belongs to Sir T. Phillipps, bart., the owner of the celebrated *Bibliotheca Phillippica*, of Middlehill.—The first complete edition of the *Breviarium* (with an Appendix of other works relating to Roman law) was given by Sichert (Basil, 1528). Of subsequent editors, Cujas deserves especial notice. This Westgothic collection consists of an abstract of the Theodosian Code, in 16 books, followed by a collection of later constitutions called *Novells*. This part of the collection has been called *leges*, while the name *jus* has been given to the part which follows, consisting of the epitome of Gaius, the extract from the *Sententiæ* of Paulus, the extract from the Gregorian and Hermogenian Codes, and the fragment of Papinian. A barbarous interpretation (*interpretatio* or *scintilla*) is subjoined to the text of each work, with the exception of the epitome of Gaius, whose *Institutes* are so remodelled in barbarous fashion, and so mutilated, as not to want an *interpretatio*. The text of Gaius, the oldest author employed, was so full of antiquated law, that, in its original state, it would have been unsuitable to the character of the times. The *interpretatio* was made by the original compilers. It sometimes paraphrases, sometimes modifies the law contained in the text, and sometimes refers to passages in other parts of the compilation. The texts themselves are occasionally garbled and corrupted.

The *Breviarium*, with its corrupt *interpretatio*, has been itself the theme of a still more corrupt *interpretatio* of the second order, in base Italian Latin. Those who are anxious to see to what extent an ancient monument may be defaced and deformed, may consult the *Lex Romana Utinensis*, at the end of the 4th volume of Canciani's *Leges Barbarorum*.*

Law. The *Westgothic Breviarium* must not be confounded with the *Lex Wisigothorum*, which is a code, not of Roman but Germanic Law.

We return to the sources which we have separately enumerated.

(1.) The Westgothic epitome of Gaius, imperfect and disfigured as it is, is now of little use since the discovery of the genuine *Institutes*, except for the purpose of understanding ancient quotations, in which it is referred to, and of assisting in the restoration of the valuable original. It has been ably commented upon by Schulting (*Jurisp. Ante-J.* p. 1—186), and by Meerman (*Thesaurus*, tom. vii. p. 669—686). It is edited by Haubold in the *Berlin Jus. Civ. Ante-J.* and by Böcking in the *Bonn Corp. Jur. Ante-J.* It consists, according to the ordinary division, of two books, and contains no abstract of the 4th book of the genuine Gaius, concerning actions.

(2.) The famous jurisconsult, Julius Paulus, wrote, in the reign of Caracalla, a treatise entitled *Sententiæ Pauli ad filium Libri V.*, in which he followed the order observed by the commentators upon the edict, while, in his usual succinct style, he gave a collection of practical legal propositions established by custom. Great part of this work was adopted, with some curtailments affecting the sense, and perhaps some alteration of the text, by the Westgothic compilers. The work, as it has been preserved to us in the *Breviarium*, is usually called *Pauli Sententiæ Receptæ*. It has been published, sometimes in its Westgothic shape, sometimes with the insertion of additional passages from the *Sententiæ* of Paulus, which are to be found in the *Pandects* and other sources. The first edition is that of Paris, 1525. The text and *interpretatio* are given in Schulting's *Jur. Ante-J.* with notes by the editor, and a selection from those of Rittershusius and Cujas, and in Meerman's *Thesaurus*, tom. vii., with the commentary of Petrus Faber. The text, edited by Biener, without the *interpretatio*, appears in the *Berlin Jus. Civ. Ante-J.* The last edition is that of Arndts, with a mass of various readings by Hänel, in the *Bonn Corp. Jur. Ante-J.*

(3.) The only fragment of Papinian preserved in the *Breviarium* is an unimportant passage "*e libro primo Responsorum, sub titulo, De Pactis inter virum et uxorem*." (Schult. *Jur. Ante-J.* p. 810.)

(4.) The *Codex Gregorianus*, or *Corpus Gregoriani* (as it has been called from its compiler Gregorianus), was a collection containing the constitutions of Roman Emperors, arranged in books, and titles with rubrics, according to the order of subjects adopted by writers on the edict. Nothing is known of the compiler. Whether his name was Gregorianus or Gregorius has been doubted, though the better opinion is in favour of the longer name. The number of books which this code contained is unknown. The 13th book is cited in the Vatican fragments, and the 14th (if Blume's reading be correct) in the *Collatio*, tit. i. c. 8. The oldest constitution which can be referred to it with certainty, is one of Septimius Severus, of the year 196 p.c.; the latest, one of Diocletian and Maximinian, of the year 295. It is not unlikely that the constitutions contained in this code have been for the most part

* The following may be taken as a favourable specimen. "Incipit Liber Gaii I. Interpr. Ingenuorum statum unum est.

Nam liberorum vero trea genera sunt. Ingenui vero sunt, qui de ingenuos parentes nascuntur. Liberti sunt, sicut jam diximus, trea genera: hoc est, cive Romanum, et Latini et Divicicii."

Law.

embodied in other later collections of constitutions, viz.:—those of Theodosius and Justinian, and that in this way we really possess a majority of them; but the Westgothic *Lex Romana* preserves only 13 titles of the *Codex Gregorianus*, containing about 22 constitutions, and in some manuscripts a few more by way of Appendix. From citations in other sources, we are able to pronounce without doubt of some other constitutions that they were included in this collection. Upon the whole, we are not able to refer to the *Codex Gregorianus* more than 70 constitutions, though it originally contained some thousands. The best critical edition is that of Hänel, in the *Bonn Corp. Jur. Ante-J.*

Codex Hermogenianus.

(5.) Of the author of the *Codex Hermogenianus* nothing is known beyond the name Hermogenianus. By some, he has been identified with the Hermogenianus who was the author of *Libri Epitomarum*, from which extracts are made in the *Pandects*. The *Breviarium* preserves but two titles of this collection, each containing one constitution. It does not seem to have been divided into books, for it is always quoted by titles only, and it is generally believed to have been a supplement to the *Codex Gregorianus*, though we find, from an assertion in *Coll. Leg. Mos. et Rom.* vi. 5, that one constitution of Diocletian, inserted in the *Codex Hermogenianus* with the date 291, existed in the *Codex Gregorianus*, with the date 290. In the *Consultatio*, c. 9, seven constitutions of Valens and Valentinian, about the date 364 and 365, are ascribed to this collection; but critics have generally believed that the *Codex Hermogenianus* was of earlier date—that, in fact, it was not carried down later than the reign of Constantine—and that there must be some error in the reading of the text in the *Consultatio*. Hänel, the latest editor, has not inserted these seven constitutions in his text. This and the preceding code are usually mentioned together. Both were probably the work of private individuals; but we are told in the *interpretatio* appended to tit. 4 of the 1st book of the *Theodosian Code*, that their authenticity was confirmed by imperial sanction. The constitutions of the emperors anterior to Constantine, inasmuch as their contents were often either originally derived from, or subsequently embodied in, the writings of jurisconsults, came afterwards to be classed not as *leges* but as *jus*; a distinction which would be paralleled in our English system if our oldest Acts of Parliament were classed not as statute, but as common law. This accounts for the fact already mentioned, that the *Codex Gregorianus* and *Codex Hermogenianus* are included in the *Breviarium* under the head of *Jus*.

Codex Theodosianus.

(6.) After the reign of Constantine, the imperial constitutions began greatly to multiply, and the writings of jurisconsults had already extended to a formidable bulk. To remedy this inconvenience, Theodosius the younger, emperor of the East, formed the design of making a code like the Gregorian and Hermogenian, and supplementary to them. The code was to contain public and general constitutions from Constantine onward. He also entertained a project of compiling a work on the Civil Law of Rome from the writings of the jurisconsults and the constitutions anterior to Constantine. The latter project was dropped, but the former design was carried into execution, and very recent discoveries have elicited some hitherto unknown historical particulars respecting its completion. A commission of eight or nine members was appointed

Law.

in 429 to compile a code of constitutions according to certain instructions, among which was a provision that conflicting clauses were not to be altered or omitted, but the dates were in all cases to be carefully appended, in order that, in case of collision, it might be known what law was to prevail. This commission did not accomplish the work assigned to it; and another, consisting of 16 members, was appointed in 435, with the ampler power of removing what was superfluous, adding what was necessary, changing what was doubtful, and correcting what was inconsistent. The code was completed by the second commission in 438, was first promulgated as law in the Eastern Empire, and in the course of the same year was transmitted to Valentinian III. and confirmed as law for the Western Empire. The *Gesta in Senatu*, an account of the proceedings in the Roman senate when the code was formally submitted to them for approval, is a remarkable and in some parts enigmatic document lately brought to light among the discoveries of Clossius.

The Theodosian code consists of 16 books, which are divided into titles, and the titles subdivided into sections or laws. It is usually referred to thus: *L. 5, C. Th. de Scenicis* (xv. 7). This signifies the fifth law of the title of the Theodosian code, with the rubric *De Scenicis*; and in order to prevent the necessity of consulting an alphabetical index of rubrics of titles, the figures xv. 7 are intended to denote that the title with the rubric *de Scenicis* is the 7th of the 15th book. The reference would indeed be sufficient thus: *L. 5, C. Th. de Scenicis*; or thus: *L. 5, C. Th. xv. 7*; but it is better to give the name as well as the number of the title; as an English lawyer, in quoting a case, gives the names of the parties as well as the volume and page of the reporter. A more simple, but less usual mode of reference is, *C. Th. lib. xv. tit. 7, l. 5*; or *C. Th. xv., 7, 5*.

In classing the constitutions under titles, the compilers, in pursuance of their powers and instructions, disposed chronologically under one title those constitutions or parts of constitutions which were connected in subject. The first V. Books are arranged according to the order of the commentators on the edict, and contain the greater part of the constitutions which relate to private law. Books VI.—VIII. are principally occupied with constitutional and administrative regulations. The IXth contains criminal law; the Xth and XIth financial law, to which are appended some matters connected with legal procedure; XII.—XV. treat of the constitution and administration of towns and other corporations; and the XVIth Book is devoted to ecclesiastical rules. The whole is a vast collection of the highest importance and interest to the historian and antiquary, as well as the lawyer.

The 16 books of the Theodosian Code were at first known to the moderns only through the imperfect abstract and *interpretatio* of the *Breviarium*. In this shape it appeared in the editions of Ægidius (Antwerp, 1517), and of Sichard (Basil, 1528). But there were incomplete manuscripts of the genuine Theodosian Code in existence, and many of the manuscripts of the *Breviarium* were interpolated from the original source. Availing himself of such aids, Du Tillet gave, in 1550, a mixed edition in two parts, of which the first contained eight books, according to the Gothic epitome; the second, the remaining eight books—the 16th in a very imperfect condition—according to the genuine code.

Modern literary history of the *Codex Theodosianus*.

Law.

In the editions of Cujas (the first of which appeared at Lyons in 1566), three more books (vi.-viii.) of the genuine code were added from divers sources, and the 16th book completed; but the first five were still drawn from the *Breviarium*, and such was substantially the state in which, until very recently, the text of the Theodosian Code remained.

After Cujas, the labours of that most learned and judicious critic and scholar, Jac. Godefroi, supplied an invaluable commentary, which was first published after the death of the author by Marville, in six volumes, folio, (Lugd. 1665). A second and improved edition of the *Codex Theodosianus*, with the commentary of Godefroi, was given by Ritter (Lipsiæ, 1736-45); and there have been reprints of Ritter's edition at Mantua and Venice. One of the editions of the Theodosian Code, with Godefroi's commentary, is an indispensable adjunct to the library of every Roman lawyer. The text was not materially altered by Godefroi, who added but three genuine constitutions.

In the present century, further discoveries have been made. A few constitutions have been brought to light by Hänel and Mai. In 1823 Clossius gave to the world a list of the titles in the *Breviarium*, the *Gesta Senatus*, and a constitution addressed to the *constitutionarii* (i. e. the officers appointed to authenticate the copies). In addition to these, which serve to illustrate the code, but form no part of its contents, he published no fewer than 79 new genuine constitutions, from a manuscript of the *Breviarium* which he found in the Ambrosian Library at Milan, and which fortunately was interpolated from the original source. In the same year Peyron published 85 new constitutions, decyphered from some palimpsest leaves, contained in a manuscript volume* in the Library of the University at Turin. The first writing on the leaves was part of the genuine code, over which had been written a narrative of the exploits of Alexander the Great. 16 leaves out of the 30 decyphered by Peyron only furnished new readings for constitutions which were already known.

It seems that there were 14 other palimpsest leaves which Peyron had laid by in a book in his own library, and had forgotten to take out again. These were found in 1835 or 1836 by Baudi de Vesme. They give 22 new constitutions, besides various readings. The various readings were communicated by De Vesme to Hänel, who, with enormous labour, has completed a new and very valuable edition of the text of the Theodosian Code for the *Bonn Corp. Jur. Ante-J.* The publication of the 22 new constitutions is said to be reserved for an edition to be brought out by De Vesme himself.

Notwithstanding these acquisitions, which have re-

* Peyron maintains that this volume had formed a part of the library of the monastery at Bobbio, which was founded by St. Columbanus, a native of Ireland, about the beginning of the seventh century. Dr. Irving, in his *Introduction*, p. 49, quotes a statement of Peyron which we think it worth while to transcribe, in the hope of drawing the attention of British and Irish philologists to the Latin books in Saxon characters, and with Saxon glosses, originally belonging to the library at Bobbio, and seen by Peyron in the libraries at Milan and Turin. *Ad primam hanc ætatem pertinet primus bibliothecæ fundus, veneranda illa scilicet codicum supellex a D. Columbano ejusque discipulis Bobium comportata. Fidi in Ambrosiana, atque Taurinensis habet, libros Latinos Saxonis litteris descriptos, glossisque Saxonis illustratos: hos pro varia eorum antiquitate vel a D. Columbano ex Hibernia, vel a Caniano ex Scotia, aliisque Anglicis Monachis allatos autumo.*

stored to us a considerable portion of the first five genuine books, the Theodosian Code is still far from perfect. Hänel calculates that the first five books, which are the only books still incomplete, contained 1000 or 1200 constitutions; that of these between 600 and 700 are wanting; that of those which are wanting, about 350 are wholly lost, while the rest are to be found in the Code of Justinian. The existing constitutions in the 16 books amount to about 2800.

An excellent edition of the first five books, including all the modern discoveries, except those of De Vesme, and enriched with explanatory as well as critical notes, was published by Wenck, 8vo. (Lips. 1825.)

7. The formation of a code places but a slight check upon further legislation. New constitutions (*Novellæ constitutiones*, or *Novellæ*) were promulgated in the Eastern and in the Western Empire. The necessity of this was foreseen by Theodosius the Younger and Valentinian; and, in order to preserve a legal unity, it was arranged that the *Novellæ*, applicable to the circumstances of both empires, which should be published in the one, should, as soon as conveniently might be, be sent to, and adopted in, the other. In pursuance of this arrangement, *Novellæ* of Theodosius the Younger, and of his successor Marcian, were sent to and adopted by Valentinian. After the death of Valentinian in 455, the connexion between the two empires became more loose, but some *Novellæ* of Leo I. were sent to the West in the reign of Anthemius, whom Leo had helped to the throne. The *Novellæ* of the Eastern Emperors Theodosius, Marcianus, and Leo, were not only sent to the West, but adopted there practically, and were inserted in the Western collections.

On the other hand, Valentinian sent constitutions from the West, but it has been doubted (1 Puchta, *Inst.* 655) whether they were ever practically adopted in the East, and whether Majorianus, Severus, and Anthemius went so far as even to send their *Novellæ* into the Eastern Empire, although still, as before, they bore, for form's sake, the names of the two contemporary Cæsars at their head. Certain it is that the compilers of Justinian's Code made no use of the *Novellæ* either of Valentinian III., or of any other succeeding emperor of the West.

Six books of Ante-Justinianean *Novellæ*, divided into titles with rubrics, are appended to Jac. Godefroi's edition of the Theodosian Code; but the 6th book, containing the *Novellæ* of Anthemius, is deficient in most of the manuscripts of the *Breviarium*. Five constitutions of Theodosius and Valentinian, not inserted in previous printed editions, were found in the Ottobonian manuscript, which now belongs to the Vatican Library, and were published by Zirardini (Faventia, 1766), and by Amaduzzi (Romæ, 1767). The last edition is that of Beck in the *Berlin Jus. Civ. Ante-J.* Hänel has undertaken to review the text for the *Bonn Corp. Jur. Ante-J.*

(8.) Odoacer, after having overturned the Empire of the West, and conquered Italy in 476, yielded in 493 to the arms of the Ostrogoths, who became masters of Italy.

It was principally to regulate the mixed legal relations between Romans and Ostrogoths that Theodoric the Great, in 500, published a set of rules based on the Roman, not the Gothic, law. It seems that, where express legislation was silent, the Roman law,

Law.

The Ante-Justinianean Novellæ.

Edictum Theodoric.

Law.

with some modifications introduced by practice, was suffered to prevail over the whole population in the kingdom of the Ostrogoths, although the jurisdiction, unless both parties were Romans, was committed to Gothic magistrates. The edict of Theodoric, though compiled principally out of imperial constitutions and the *Sententiæ* of Paulus, has not hitherto been inserted in collections of Roman law. It is to be found in the collections of Germanic law by Lindenbrog, Canciani, Georgisch, and Walter. The best edition is that of Rhon, *Commentatio ad Edictum Theodorici* (Halæ, 1816).

Edictum
Athalarici.

(9.) The short edict of Athalaric contains some further provisions directed against offences, and borrowed in part from Roman law. It is given in the 1st volume of Canciani, *Leges Barbarorum*, and in Manso, *Geschichte des Ostgothischen Reichs in Italien* (Breslau, 1824). See also Gretschel *ad Edict. Athalarici* (Lips., 1828).

Lex Roma-
na Burgun-
diorum.

(10.) Towards the middle of the 5th century the kingdom of the Burgundians was formed on the banks of the Rhone.

The *Lex Romana Burgundiorum* was compiled after the year 517, for the use of the Burgundian Romans, the Burgundian conquerors having already a similar code of their own called *Gundobada*. It is composed from the same sources as the Westgothic *Breviarium*, but follows the order of the *Gundobada*. It is a meagre compilation, consisting of only 46 titles.

In many manuscripts, this collection follows, without interval and without new title, the Westgothic *Breviarium* which ends with a fragment from *Papiniani lib. i. Responsorum*. To this circumstance is usually attributed the name *Papian*, by which the Burgundian *Lex Romana* is commonly known. In the manuscripts, *Papiani* is the almost invariable corruption of *Papiniani*; and Cujas is supposed to have mistaken the last portion of the *Breviarium* for the commencement of the ensuing treatise. We are not quite satisfied with this explanation, and doubt whether the name had not an earlier origin than the first edition of the *Papian* brought out by Cujas. In the edition of Cujas printed at Paris, 1586, the book is described as *Burgundionis J. C., qui Papiani Responsorum titulum præfert liber*. We think it likely that the Roman, as well as the Burgundian, law-book of the Burgundians had a distinct appellation. (See Savigny's *Zeitschrift*, vol. ix, p. 237, &c.)

The best, and the only separate, edition of the *Papian* is that of Barkow, *Lex Romana Burgundiorum*, (Gryphisw, 1826.)

Institutio
Gregoriana.

(11.) In the 9th volume of Savigny's *Zeitschrift*, Klenze published for the first time, from a manuscript of the *Breviarium* at Berlin, a work entitled *Institutio Gregoriana*. Its author and purpose are unknown. It contains not only extracts from the *Codex Gregorianus* (from which by some it has been supposed to derive its name), but from the *Theodosian Code*, from *Pauli Sententiæ*, and from a work of Papinian. It is later in date than the *Breviary* of Anianus, and it is doubtful whether it is an authorized, independent compilation.

Having now enumerated the principal sources of Ante-Justinian law, considered apart from the compilations of Justinian, it is proper to state that the three best known collections of those sources are, (1) Schulting's *Jurisprudentia Ante-Justiniana*;

(2) the Berlin *Jus Civile Ante-Justinianum* (to which Hugo promised a preface, which has never appeared); and (3) the still unfinished *Corpus Juris Ante-Justiniani*, now publishing at Bonn.

Of these, the first alone has a commentary, and the notes are so excellent that it is not likely to be ever superseded. It omits the *Theodosian Code* as being already pre-occupied by Jac. Godefroi.

The second contains the *Theodosian Code*, with the *Appendix of Novells*, but it omits throughout the Westgothic interpretations. It is still the best and most complete collection of texts, but it will be superseded in these respects by the third, which, when completed, will be a highly valuable work. The best separate editions of the sources, however, will always be found more agreeable to consult, and often, on account of their exegetical notes, more useful.

We have now to describe the separate contents of the Roman law, as it appeared in the compilations made under the direction of Justinian. The com-
pilations of
Justinian.

The idea of forming a collection of positive laws has been attributed to Pompey, to Cicero, and to Julius Cæsar. We have seen how far Theodosius the Younger carried his plan into effect, and we have considered the nature of the Roman legislation of three Germanic kingdoms, founded on the ruins of the Western empire. Justinian was more vast than all who had preceded him in his conception, and more successful in the complete execution of his plan. It seems to have been his intention to establish a perfect system of written legislation for all his dominions; and for this purpose to make two great collections, one of the imperial constitutions, the other of all that was valuable in the works of jurisconsults, who were now quoted at second-hand.

The history of Justinian's legislation has often been told, and our limits warn us to be brief in treating a subject where accurate information may be obtained in easily accessible channels.

The first work attempted by Justinian, as the most practical and the most pressing, was the collection of imperial constitutions. This he commenced in 528, the second year of his reign. The task was intrusted to 10 lawyers, among whom was the distinguished professor Theophilus. At their head was Tribonian, a man who was endowed with rare qualifications for such an appointment. He was himself deeply learned in law, and possessed in his library a matchless collection of legal sources. He had passed through many gradations of rank, knew mankind well, and was remarkable for energy and perseverance. "His genius," says Gibbon, "like that of Bacon, embraced as its own all the business and knowledge of the age." In compiling all that was useful in preceding constitutions, the commission was authorized to correct, to retrench, to consolidate, but not to make material alterations. The commissioners executed their task with excessive haste. The new code received the imperial sanction in April 529, but it was soon found to be incomplete, and was in the sequel entirely suppressed.

Soon after the completion of the first code, Tribonian was authorized to choose fellow-labourers to assist him in the other division of the undertaking,—a part of Justinian's plan, which the emperor justly regarded as the most difficult, but also the most important and the most glorious. In December, 530, sixteen commissioners were chosen to lay under contribution the works The Digest
or Pandects.

Law.

Law. of those jurists who had received from former emperors authoritative permission *jura condere*. They were to select all that was best, with permission to alter and interpolate, in order to accommodate their extracts to the circumstances of the time. Rapid was their progress. In little more than three years the immense compilation was complete, and on the 30th December, 529, was published with the imperial sanction. It comprehends upwards of 9000 extracts, in the selection of which the compilers made use of above 2000 different treatises. This extraordinary work has been blamed by men of divers views on divers accounts. Tribonian and his associates, regarding more practical utility than the curiosity of searchers into ancient law, did not scruple at times so to adulterate the extracts they made, that a theorizer in legal history might easily be misled if he trusted implicitly to their accuracy. Hence the *emblemata Triboniani* have been to many critics a fertile topic of reprehension. On the other hand, the complaints of Thibaut (*Inst. und Herm. des R. R.* § 88) are levelled against scientific rather than historical delinquencies. He saw that unity and system could result only from a single complete code of remodelled laws, and not from the lazy plan of two separate collections, made out of independent pre-existing writings; and though, from the circumstances of the time, Justinian was forced to adopt the latter alternative, it was unphilosophical to commence with the constitutions instead of the jurisconsults. Those principles which lie at the foundation of jurisprudence pervade the writings of the Roman lawyers, and their works are in reality more full of practical law than the constitutions to which occasional exigency gave birth. Then the arrangement of the *Pandects* sins against science. They are divided into seven parts or 50 books; the parts begin respectively with the 1st, 5th, 12th, 20th, 28th, 37th and 45th books. Each book is divided into titles with rubrics. The separate extracts under each title were not originally numbered, but the name of the author and a reference to his work were prefixed. In the general order of the books, and in the order of titles, the order of the edict, which was itself based upon the order of the twelve tables, was followed. But this arrangement was historical, not systematic. There is no *pars generalis*, no connected statement of first principles, no regular development of consequences. Leading maxims are introduced incidentally; and matters of the greatest moment, as the law of procedure, are scattered under divers heads,—here a bit and there a bit. The internal arrangement of the separate fragments of jurists under each title would appear at first sight to be completely fortuitous. It is neither chronological nor alphabetical, nor does it follow any rational train of thought depending on the subject treated of. Blume has elaborately expounded a theory which, though still rejected by Tigerström and others, seems to us to rest upon the foundation of facts, and must at least be something like the truth. No one can form any opinion of the merits of Blume's theory, without a careful examination of a great number of titles in the *Digest*. We can now do no more than state its general nature. He finds that the extracts under each title resolve themselves into three series,—that one series of adjoining extracts is taken from Commentaries on Sabinus; a second from Commentaries on the Edict,—and a third from Commentaries on Papinian. Hence he supposes that the

commission was divided into three sections, and that to each section was given a certain set of works to analyze and break up into extracts. Blume's celebrated Essay on the *Ordnung der Fragmenta in den Pandectentiteln*, is to be found in the 4th volume of Savigny's *Zeitschrift*. From the works of 35 jurists there are direct extracts in the *Pandects*. The jurists from whom extracts are made often cite other jurists, and there are four jurists from whom the *Pandects* contain pure or literal extracts at second-hand. The 39 jurists from whose works the *Pandects* contain literal extracts, whether directly or at second-hand, are often called the *classical jurists*, a name sometimes extended to all those jurists who lived not later than Justinian, and sometimes confined to Papinian, Paulus, Ulpian, Gaius, and Modestinus, upon whose works a peculiar authority in the courts was conferred by a constitution of Valentinian III. (*C. Th. lib. i. tit. 4, 5*). Extracts from Ulpian constitute about one-third of the *Pandects*; from Paulus, about one-sixth; from Papinian, about one-twelfth. In Hommel's *Palingenesia Pandectarum* the fragments of each jurist are collected and printed separately; an attempt is made to reanimate the man—to restore his individuality—by bringing together his dispersed limbs and scattered bones.

Justinian was desirous that the body of law compiled under his directions should be all in all, not only for practice, but for academical instruction. He forbade all commentary on his collections, and prohibited the citation of older writings. It is said that Napoleon exclaimed, when he saw the first commentary on the *Code Civil*, "*Mon Code est perdu!*" and Justinian seems to have been animated with the same spirit. He allowed no explanation save the comparison of parallel passages and the interpretation of single words. As the *Pandects* and the *Code* led far into detail, which could not be understood by beginners, it became necessary to compose an elementary work for students.

This task was assigned to Tribonian, in conjunction with Theophilus and Dorotheus, who were respectively professors in the two great schools of law at Constantinople and Berytus. The *Institutes* of Gaius were taken as the basis of the work, which was to be brought into harmony with the *Code* and the *Digest*, but other treatises were made use of in its formation. Hence there is an occasional incongruity in the compilation, from the employment of heterogeneous materials. For example, the discordant notions of Gaius and Ulpian on the *jus naturale* and the *jus gentium* are brought together, but refuse to blend in consistent union. It is divided into four books, arranged according to the principle adopted by Gaius.—Law treats of rights.—Differences of rights result from permanent differences in those who possess rights—the subjects of right—*persons*; and also from differences in that over which rights are exercised—the objects of right—*things*. Besides the modifications of right attributable to permanent differences in persons, and natural or conventional differences in things, there are transitory rights which arise from voluntary acts. Further, in order to redress any violation of those primary rights which alone would have to be considered, if men acted in obedience to law, the law establishes secondary rights—remedies for violation of right, and rights of action. In the *Institutes*, first is treated the *jus personarum*,—then follows the doctrine of *things*, so far as they are important in law. The law of obligations,

Law.

The *Institutes*.

Law. connected as they are with *incorporeal things*, follows; and at the end there is a very short notice of the law of actions, and of criminal law.

Codex Re-
petitæ
Prælec-
tionis.

L. Deci-
siones.

The *Institutes* received the imperial sanction, at the close of 533, on the same day with the *Pandects*.

“Man, and for ever!” exclaims Gibbon. Had it been possible to make law for ever fixed, and had the emperor’s workmen successfully accomplished the unchangeable petrification, the desire of Justinian’s heart would now have been fulfilled. There were many questions upon which the ancient jurists were divided. Under the early emperors these differences of opinion gave rise to permanent sects, nor were they afterwards entirely extinguished when party-spirit had yielded to independent eclecticism. The compilers of the *Pandects* had tacitly, by their selection of extracts, manifested their choice; but a Catholic doctrine, the great object of Justinian’s wishes, was not thus to be accomplished. In order to settle the orthodox legal creed, it became necessary for him to promulgate fifty decisions (*L. Decisiones*) on the most controverted points. Moreover, the imperfection and ambiguity of the existing law required to be remedied by further constitutions, of which a considerable number was issued during the three years occupied in the composition of the *Pandects*. Accordingly, immediately after the publication of the *Pandects*, orders were given for a revision (*repetita prælectio*) of the first code, which was to be purged from errors, and reinforced by the admission of the subsequent constitutions, including the fifty decisions. On the 15th November, 534, the new edition of the code came out; and to this new edition, in contradistinction to the former (which was now superseded and carefully suppressed), has been usually given the name *Codex repetitæ prælectionis*. It is now ordinarily called simply the *Code* of Justinian, though Hugo, considering that the other collections of Justinian are also entitled to the name of codes, would term *this* in accordance with the *Turin Gloss*, *Constitutionum Codex*. The code is divided into 12 books, and the books into titles. The general arrangement corresponds on the whole with that of the *Pandects*, so far as the two works treat of the same subject, but there are some variations which cannot be accounted for. For instance, the law of pledges and the law of the father’s power occupy very different relative positions in the *Pandects* and in the *Code*. Under each title, the constitutions are arranged chronologically. They do not go further back than the reign of Hadrian, the matter of older constitutions having been fully developed in the works of jurists.

The No-
vells.

Justinian, though fond of legal unity, was also fond of law-making. If he had lived long enough, there might probably have been a third edition of the *Code*, and perhaps a second edition of the *Pandects*. When the new *Code* was published, he anticipated the formation of a separate collection of *Novells*, but there is no decisive proof that the *Novells* were collectively published by authority in his lifetime, though between 534 and his death in 564 or 565, upwards of 150 new constitutions (*νεαπαὶ διατάξεις*) were separately promulgated, which made many reforms of great consequence, and seriously affected the law as laid down in the *Pandects*, *Institutes*, and *Code*. There is a marked diminution in the number of *Novells* after the death of Tribonian in 545.

Such were Justinian’s legislative works—works of no

mean merit—nay, with all their faults, considering the circumstances of the time, worthy of extraordinary praise. They preserve precious remains (remains precious as materials for thinking, and as models of legal reasoning), which would else have been wholly lost. They have long exercised, and, pervading modern systems, continue to exercise, enormous influence over the thoughts and actions of men. If adherence to their contents cramped, on the one side, the spontaneity of indigenous development, it opposed barriers, on the other, to the progress of feudal barbarism. In spite of a certain enslavement to elements originally base, they exhibit a striving in chains towards the natural, the rational, and the good.

We must now say a few words concerning the editions, separate and collective, of Justinian’s compilations. The editions up to the end of the first third of the 16th century, called *incunabula*, are scarce, for, from the inconvenience of their form, the variety of contractions they employ, and the want of numbers to the fragments or paragraphs, they have been subjected to the same fate as the early manuscripts; but, like the early manuscripts, they are often of use in correcting the text.

The first printed edition of the *Institutes* is that of Petrus Schöffler (Mainz. 1468, fol.). The last edition of importance is that of Schrader (Berlin, 1832). This is intended to form the first part of a projected *Berlin Corpus Juris Civilis*, which is still promised, but has hitherto made no further visible progress.—Among the commentators, Vinnius, Otto, and Costa are in the first rank. There are useful modern French commentaries and translations by Blondeau, Ducaurroy, and Ortolan. We regard the Greek paraphrase of Theophilus as the most useful of all commentaries, but the work is so clear as seldom to require explanation; and not without reason was an Essay, as long ago as the first year of the 18th century, composed by Homberg, professor of law at Helmstadt, *De multitudine nimia Commentatorum in Institutiones Juris*. The most valuable critical editions, anterior to Schrader’s, are those of Haloander (Nuremb. 1529), Contius (Paris, 1567), Cujas (Paris, 1585), or re-edited by Köhler (Göttingen, 1773), Biener (Berlin, 1812), and Bucher (Erlangen, 1826).

The literary history of the *Pandects* has been a subject of hot and still unextinguished controversy. The most celebrated existing manuscript of this work is that called the Florentine, written by Greek scribes, probably not later than the end of the 6th or the beginning of the 7th century. Many have supposed it to be one of the original copies transmitted to Italy in the lifetime of Justinian; but, though it is in general free from contractions and abbreviations, which were strictly forbidden by the emperor, letters and parts of letters are sometimes made to do double duty, as *necesse* for *necesse esset* (*geminationes*) and *AB* for *A B* (*monogrammata*). The Florentine MS. was for a long time at Pisa, and hence the glossators refer to its text as *litera Pisana* (*P.* or *Pi.*) in contradistinction to the common text, *litera vulgata*. Its history before it arrived at Pisa is doubtful. According to the testimony of Odo-fredus, who wrote in the 13th century, it was brought to Pisa from Constantinople; and Bartolus, in the 14th century, relates that it was always at Pisa. We are strongly inclined to put faith in the constant tradition, supported by Brenemann (*Historia Pandectarum*) with

Law.

Literature
of the Jus
tinianean
Compila-
tions.

Editions
of the
Institutes.

Editions
of the
Pandects.

Law.

much weight of evidence, that it was given to the Pisans by Lothario the Second after the capture of Amalfi in 1133 (?) as a memorial of his gratitude to them for their aid against Roger the Norman. The truth or falsehood of this tradition would be a matter of little importance if it were not usually added, among other more apocryphal embellishments, that the finding of the MS. and its removal to Pisa attracted the attention of the neighbouring school of Bologna to the study of Roman law. It is certain that, soon after the capture of Amalfi, the Roman law, which had long been comparatively neglected, was brought into remarkable repute by the teaching of Irnerius; but this resurrection is attributed by Savigny to the growing illumination of men's minds and that felt want of legal science which the progress of commerce and civilization naturally produces. He thinks that curiosity, excited by these causes, not by any sudden discovery, had only to put forth its arm and seize the sources of Roman law, which were previously obvious and ready for its grasp.

Pisa was conquered by the Florentine Caponus in 1406, and the MS. was brought to Florence in 1411, ever since which time it has been kept there as a valuable treasure, and regarded with the utmost reverence. Where the Florentine MS. may have been before the siege of Amalfi is of little consequence; but it is of great consequence to decide another much disputed question, namely, whether the Florentine MS. be or be not the sole authentic source whence the text of all other existing MSS., and of all the printed editions, is derived. In favour of the affirmative opinion there are several facts which have not, we think, been very satisfactorily accounted for. The leaves of the Florentine MS. are written on both sides, and the last leaf but one, in binding the volume, has been so placed as to reverse the order of the pages. This fault is copied in all the existing manuscripts. The order of the 8th and 9th titles, in the 37th book, is reversed in the Florentine MS., but the error is corrected by the scribe by a Greek note in the margin. There are fragments similarly reversed in lib. xxxv. tit. 2, and lib. xl. tit. 9, and similarly corrected. In the other existing old MSS., written by men who did not understand Greek, the error is re-produced, but not the correction; whereas omissions in the Florentine text, which are supplied in Latin in the margin, are to be found supplied in the other MSS. The words "*pro soluto*" are thus supplied at the end of the 3rd title of the 41st book, and the words "*non ad mortis tempus referuntur*," in l. 1, *D. Unde Liberi* (xxxviii. 7). In the 20th and 22nd titles of the 48th book, the Florentine MS. omits several fragments, which were first restored by Cujas from the *Basilica*. The omissions exist in all the other manuscripts. In general, where the text of the Florentine MS. presents insuperable difficulties, no assistance is to be derived from the other manuscripts; but they all retain the errors of the Florentine. Moreover, they appear to be all later than the beginning of the 12th century.

In opposition to these facts, the supporters of the conflicting theory, of whom Savigny, being one of the latest and not the least able, is the best known, adduce many passages of the ordinary text in which the omissions and faults of the Florentine MS. are corrected and supplied. Some of the variations are not improvements, some may be ascribed to critical sagacity and happy conjecture, and some may have been drawn

Law.

from the *Basilica* or other Eastern sources; yet, in the list which Savigny has given, a few variations remain which can scarcely be accounted for in any of these ways. Such being the state of the question, we are disposed to adopt the hypothesis, that the early glossators may have been acquainted with one manuscript or more, containing the whole or part of the *Pandects*, and not copied from the Florentine, but copied from the same original—not exempt from faults—from which the Florentine was immediately copied. The Florentine, and the issue descended from the parent of the Florentine, exhibit many of the defects of their common ancestor, with some independent beauties and deformities of their own (*See Thibaut, Inst. und Herm. des R. R. p. 409*).

Four palimpsest leaves of a MS. of the *Pandects*, probably nearly as old as the Florentine, were discovered at Naples by Gaupp, and an account of them was published by him at Breslau in 1823. They belong to the 10th book, but are nearly illegible.

In most of the manuscripts and early editions the *Pandects* consist of three nearly equal volumes. The first, comprehending lib. i.—xxiv. tit. 2, is called *Digestum vetus*; the second, comprehending lib. xxiv. tit. 3. lib. xxxviii., is called *Infortiatum*; the third, comprehending lib. xxxix.—lib. l., is called *Digestum novum*. The *Vetus* and *Novum* are each again divided into two parts. The second part of the *Vetus* begins with the 12th book; that of the *Novum* with the 45th. The *Infortiatum* is divided into three parts; of which the second begins with the 30th book, and the third (*mirabile dictu!*) with the words *tres partes*, occurring in the middle of a sentence in l. 82, *D. ad Leg. Falc.* (xxxv. 2). This third part is hence called *Tres Partes*. The glossators often use the name *Infortiatum* exclusively for the first two parts of the second volume, e. g., *Infortiatum cum Tribus Partibus*, and sometimes the *Tres Partes* are attached to the *Novum*. In order to explain these peculiarities many conjectures have been hazarded. It is most probable that the division owes its origin partly to accident,—that the *Vetus* first came to the knowledge of the earliest glossators; that they were next furnished with the *Novum*; then with the *Tres Partes*, which they added to the *Novum*; and that then they got the *Infortiatum*, so called, perhaps, from its being forced in between the others; and that finally, in order to equalize the size of the volumes, they attached the *Tres Partes* to the *Infortiatum*. The common opinion is that the *Infortiatum* derived its name from having been reinforced by the *Tres Partes*.

The editions of the *Pandects*, with reference to the character of their text, may be divided into three classes—the Florentine, the Vulgate, and the mixed. Of the Florentine editions (so called from their representing the text of the Florentine MS.), the most beautiful, the most celebrated, and the most pure, was that published at Florence in 1533 by Lælius Taurellius, who, out of paternal affection, allowed his son Francis to name himself as the editor. The Vulgate editions have no existing standard text to refer to. The ideal standard is the text formed by the glossators, as revised by Accursius. Their number is immense. The first *Digestum Vetus* appeared at Venice, 1477; *Infortiatum*, Rome, 1475; *Novum*, Rome, 1476. In the early vulgate editions, the Greek passages of the original are given for the most part in an old Latin translation, and the inscriptions prefixed to the extracts, and referring

Law.

to the work and the author, are either imperfect or wanting. Of the mixed editions, the most distinguished is that of Haloander (Nuremberg, 1529), a daring and adventurous critic.

Among the commentators on the *Pandects*, the best known in this country, and among the best, are Noodt, Voet, and Pothier. The *Meditationes in Pandectas* of Leyser, and the *Notæ* of A. Schulting, with observations by Smallenberg, are very voluminous,—the latter exceedingly good. Still more voluminous are the German *Erläuterungen* of Glück, extending, with the continuations of Mühlenbruch and Reichardt, to above 40 parts. They are interesting as shewing the construction put upon the law of the *Pandects* in cases that occur in modern practice. Some of the ablest modern civilians admire the older commentators—Duarenus, Cujas, Faber.

Editions of the Code.

In the early editions of the *Code*, the first nine books are often separated from the latter three, which, relating principally to the public law of the Roman empire, were often inapplicable in practice under a different government. The inscriptions and subscriptions of the constitutions are almost always wanting, and the Greek constitutions are omitted. Haloander (1530) considerably improved the text, but Contius (*Le Conte*) was the first to restore to their proper place the omitted Greek constitutions (*leges restitutæ*). In modern times, Witte and Biener have directed their attention specially to the critical examination of the *Code*. In Goeschen's preface to the first edition of *Gaius* is mentioned the discovery at Vienna of some *folia rescripta* of a very ancient manuscript, which contains some new Greek constitutions. A copy of the decyphered text, made by Blume, has been placed at the disposal of the editors of the unfinished Leipzig edition of the *Corpus Juris Civilis*, first brought out under the auspices of the brothers Kriegel.

Cujas and Wissenbach are among the best commentators on the *Code*.

Editions of the Novells.

Justinian published most of his *Novells* in Greek; some in Latin only (e. g., the *Novells* now numbered 9, 11, 23, 33, 62, 65), from their relating exclusively to the affairs of the West; some in Greek and Latin (e. g., *Novells* 32 and 111, and also *Novells* 17 and 18, though the original Latin of the latter two no longer exists). We possess the *Novells* in three ancient forms: the Latin *Epitome* of Julian, containing 125 *Novells*; an ancient Latin translation, containing 134 *Novells*; and a Greek collection, numbering 168 *Novells*. We shall speak of these in order.

The *Epitome Novellarum* was made by Julian, a professor of law at Constantinople, and a contemporary of Justinian. The last constitution it contains bears date 556, and it is probable that the *Epitome* was made soon after that date, especially as from its language it appears intended for the Italians, who had been reduced under the dominion of Justinian by the conquest of the Ostrogoths in 554. The earliest manuscript is that of Ranconetus, of the 8th century, now in Paris. The earliest printed edition is that of Boerius, in 1537, from a Mantuan manuscript; the best known, that of Pithou (fol. Basil, 1576). Manuscripts of the work have been found by Blume at Vercelli, and by Lancizolli at Vienna. An extract from the latter is given by Biener in the 5th volume of the *Zeitschrift*. The first 39 constitutions have no marked chronological arrangement; constitutions 40 to 117 inclusive, are

arranged chronologically; and the remaining eight constitutions appear to be an appendix without order. The work is often an improvement on the original *Novells*, leaving out the *proæmia* and *epilogi*. The constitutions in the *Epitome* are invariably quoted by the glossators under the name *Novellæ*.

The ancient Latin translation is ascribed by the glossators to the time of Justinian, and, notwithstanding the badness of its Latinity, it is not improbable that it was made, perhaps by different authors in succession, in the lifetime, or soon after the death, of Justinian, for the use of the Italians. It contains none of those marks of Byzantine interpolation which appear in the Greek collection; and, unlike the Greek collection, it contains no constitutions later than the time of Justinian. Schweppe (*R. R. G.* § 126 c.) conjectures that it was a translation from a Greek collection made in Illyricum. Of the 134 *Novells* which it numbers, the first 129 are in chronological order; the last five are an appendix, arranged with some regard to dates. To this translation the glossators gave the name *Authenticum*, to distinguish it from the *Epitome* of Julian, and as a recognition that it contained the legal text—a proposition which had once been doubted by Irnerius. A single *Novell* in the *Authenticum* was called an *authentica*. The Greek collection was wholly unknown to the glossators. After the discovery of the Greek originals, the *Authenticum* was distinguished from subsequent Latin translations by the name *versio vulgata*. Of the 134 *Novells* contained in the *versio vulgata*, the glossators recognized only 97 as practically useful, and these were the only *Novells* to which they appended a gloss. As the *Institutions*, *Pandects*, and *Code* were divided into books and titles, the glossators pedantically divided the 97 glossed *Novells* (which they arranged chronologically) into nine books, intended to correspond with the first nine books of the *Code*. These books were called *collationes*. Under each collation were placed a certain number of constitutions, and each constitution formed a separate title, except the eighth, which was divided into two titles. There were thus 98 titles. The rubrics of the constitutions, and the division into chapters and paragraphs, though not due to Justinian, were probably older than the glossators, and to be attributed to the original collectors or translators. The 97 glossed *Novells*, thus divided, constituted the *liber ordinarius*; the remaining *Novells* of the *Authenticum* (which were called *extravagantes* or *authenticae extraordinariae*) were divided into three *collationes*, to correspond with the last three books of the *Code*; but, as they were not used in forensic practice, they soon ceased to be copied in the manuscripts. Among the later manuscripts, however, there are a few in which, if Puchta's statement be correct, *extravagantes* (sometimes more, sometimes fewer) may be found introduced among the *novem collationes* (1 Puchta, *Inst. und Gesch. R. R.* 708). The oldest manuscript of the *Authenticum* is one of Vienna of the 13th century, containing 133 *Novells* out of the 134. The ancient Latin version of the 62nd and 75th *Novells*, before known only from the *Epitome* of Julian and the Greek *summa*, was first published from this manuscript by Savigny (*Zeitschrift*, ii. 100), and an original Latin constitution of Justinian concerning debtors in Italy and Sicily, before wholly unknown, was published from the same manuscript by Biener (*Zeitschr.* v. 352). The oldest printed edition of the *versio vulgata* is that of Vit.

Law.

Law. Pücher, containing the 97 *Novells*, with the gloss, followed by the three last books of the *Code* (Rome, 1476).

A Greek collection of the *Novells* of Justinian was made for the use of the Oriental lawyers, probably under Tiberius II., who reigned 578-82. It is cited by numbers in the *Basilica*, and it is numbered up to 168, but it does not really contain 168 *Novells*, for the Latin constitutions, which were not of practical use in the East, are not translated, but are given only in Greek *summæ* or abstracts. Some of the *Novells* it contains are numbered twice. This is the case with the *Novells* numbered, in the present editions of the *Corpus Juris*, 32 and 34, 41* and 50, 75 and 104, 143 and 150. This repetition may be accounted for, in the case of the first and last pairs, by the fact that the same *Novell* was published in Greek and Latin, and, in those different forms, sent into different parts of the empire. The Greek collection was not confined to constitutions of Justinian. There are four of Justin II., viz.—*Nov.* 140, 144, 148, 149; three of Tiberius II., viz.—*Nov.* 161, 163, 164; and four edicts (*eparchica, formæ*) of the *præfectus urbi* and *præfectus prætorio*, viz.—165-168. These edicts, possibly, were introduced as having obtained the imperial sanction. That the genuine Greek collection consisted of neither more nor less than 168 numbers, appears from a Greek list of the 168 titles, which was found in the Queen's Library at Paris, and thence called *Index Regineæ*. A Latin translation of this list was published by Cujas in his *Expositio Novellarum* (1570), and the original Greek text is given in the second volume of Heimbach's *Anecdota* (1840), p. 237. The first 120 *Novells* of this collection are in chronological order; the remainder are intermixed.

We have already stated that the Greek *Novells* were wholly unknown to the glossators. Haloander was the first who published a collection of them at Nuremberg, in 1531, from a Florentine manuscript. Scrimger, a Scotchman and professor of civil law at Geneva, afterwards published a much more complete collection of the Greek *Novells* from a Venetian manuscript. The collection of Scrimger was printed by H. Stephanus at Geneva, in 1558. It is stated by Schweppe (*R. R. G.* § 126 b) that the Florentine MS. was of the 15th century, and contained 168 *Novells*; and we are further told by Tigerström (*Aeussere R. R. G.* p. 295, n. 20), who follows Schweppe in these statements, that the edition of Haloander of 1531 contained 165 *Novells*, and that the number was completed in the edition of the Haloandrine *Novells* by Hervagius and others (*in officina Hervagiana*, Basil, 1541). Schweppe also states that the Venetian MS. was of the 13th century, and contained 162 *Novells*, exclusive of an appendix. The appendix contained (a) 13 constitutions of Justinian, improperly called *edicts*; (b) five *Novells* of Justin; (c) a *Novell* of Tiberius; (d) the four edicts of the *præfects*, which appear at the end of the ordinary collection. The 1st, 5th, and 6th (so called) edicts of Justinian are inserted in the body of the ordinary collection as *Novells* 8, 111, and 122 respectively. The 9th edict of Justinian is substantially the same as *Novell* 136. Of the *Novells* of Justin, in Scrimger's appendix,

four appear in the body of the ordinary collection as *Novells* 140, 144, 148, 149.

If the statements of Schweppe as to the number of *Novells* in the Florentine and Venetian MSS. be correct, it would seem that the editions of Haloander and Scrimger cannot be faithful transcripts of them. By reference to Osenbrüggen's edition of the *Novells*, it appears that many of the 168 are wanting in Haloander—a fact which seems to be inconsistent with Tigerström's statement. Again, if, in the body of Scrimger's collection, as in the Venetian MS., there were 162 *Novells*, it would seem that Scrimger must have *Novells* not included in the ordinary collections; for we find from Osenbrüggen that the *Novells* ordinarily numbered 122, 140, 144, 148, 149, 165, 166, 167, 168, being contained in Scrimger's appendix, are not inserted in the body of his collection.

These difficulties, which must occur to an investigating reader, are not explained in the professed histories of Roman law. The fact is, that neither the original edition of Haloander,—nor the subsequent Haloandrine edition of Basil, 1541, which gives at the end a few supplementary *Novells* in Greek, furnished by A. Alciatus and other friends of Hervagius,—nor the Haloandrine edition of Paris, 1542, in which the supplementary *Novells* of Hervagius are introduced in their proper places,—nor, lastly, the edition of Scrimger,—contains anything like the number of *Novells* ordinarily assigned to it. In some cases the mere title of a *Novell*, which was not found in the manuscripts, was inserted; in some cases not even the title of an omitted *Novell* is inserted, but its number is given, and the *lacuna* is marked by asterisks. It might as well be said that the body of Scrimger's collection contains 168 *Novells* as that it contains 162. The last *Novell*, indeed, which it contains at length is numbered (ρξβ) 162, but the remaining six are thus accounted for in a marginal note,—

“Here the old manuscript contains the following memorandum:—*Novell* 163 is not copied here, as being the 5th of Tiberius; in like manner *Novell* 164 is omitted, as being the 2nd of Tiberius; *Novell* 165 could not be found; *Novell* 166 is the 1st *forma* of the *eparchica*; *Novell* 167 is the 24th *forma* of the *eparchica*; *Novell* 168 is the 2nd *forma* of the *eparchica*.”*

There are similar marginal notes in the middle of Scrimger's collection, accounting for omitted *Novells*.

It seems from his preface that Scrimger was acquainted with a manuscript collection of *eparchica*. An *Index* to a collection of 39 *eparchica* has been recently published for the first time by C. E. Zachariæ in his *Juris Græco-Romani Delineatio* (Heidelberg, 1839).

We proceed now to give an account of the modern Latin translations, and of the origin of the modern mixed editions of the *Novells*.

Haloander gave a Latin version not only of such of the Greek *Novells* in his own collection as were not included in the *Authenticum*, but of those which had already appeared in the semi-barbarous and obscure *versio vulgata*. Scrimger's Greek collection was published without a translation; but the *Novells*, which are contained in Scrimger and not in Haloander, were translated by Agylæus (*Supplementum Novellarum*,

* In place of *Novell* 41, the Vulgate editions, following Contius, have inserted Julian's epitome of a constitution referred to in the preface of *Novell* 41.

* Ενταῦθα τὸ παλαιὸν ἀντίγραφον γεγραμμένον ἔχει ταύτην τὴν ὑπόμνησιν. Ἡ ρξγ οὐκ ἐγράφη ὡς, ὡς οὐσα εἰ Τιβερίου, ἀσάφως δὲ ρξδ, ὡς οὐσα β Τιβερίου, ἢ δὲ ρξε οὐκ εὐρίσθη, ἢ δὲ ρξς ἐστὶ τύπος α τῶν ἐπάρχων (sic), ἢ δὲ ρξξ ἐστὶ τύπος α τῶν ἐπάρχων, ἢ δὲ ρξη ἐστὶ τύπος β τῶν ἐπάρχων.

Law. Coloniae, 1560), who is often erroneously spoken of as if he had made a complete translation of the whole.

Such were the principal events in the literary history of the *Novells* before the editions of Contius, whose labours have been the foundation of all subsequent editions. For the basis of his Greek text, Contius took, as the best and fullest, the edition of Scrimger, and resorted to Haloander where Haloander's collection contained matter not in Scrimger. In this manner he made up, between both, 164 Greek (so called) *Novells*; but of these some are mere abstracts,—some nothing but titles of *Novells*. For his Latin text, he took as the basis the *versio vulgata*, so far as he was acquainted with it. He made use of 12 of the *authenticæ extravagantes* in addition to the ordinary 97. For the Greek text supplied by the collections of Haloander and Scrimger, so far as it was not translated in the *versio vulgata*, he formed a translation, partly his own and partly compiled from Haloander. To this he subjoined the matter contained in Julian's *Epitome*, so far as it was not contained in the *Authenticum* or the Greek *Novells*. In this manner he made up the 168 Latin *Novells*, which compose the stock of *Novells* in ordinary editions of the *Corpus Juris Civilis*.

Such is the account given by Thibaut, an accurate and careful writer, who attributes to Contius a pre-determination to make up the number 168, and charges him with using a little trickery (*"schelmerei"*) in order to effect this object; as if to Contius belonged the idea of counting the doubled *Novells* twice,—of inserting constitutions of Justin and Tiberius, and edicts of præfects (Thib. *Herm. und Kritik des R. R.* 427). This mode of counting seems to be more properly chargeable upon the persons who originally formed the Greek collection.

Contius published many editions of the *Novells*, differing among themselves in a way which it is of importance to remark. They appeared in the years 1559, 1562, 1565, 1566, 1571, 1576, 1579, 1581. The early editions contained only the Latin text of 168 *Novells*; the later contained also the 164 Greek *Novells*. Some of the editions contained the gloss, and in these the 97 glossed *Novells* were arranged, as usual, in the old nine *collationes*, while all the remaining *Novells* were subjoined as a tenth *collatio*. A momentous and memorable change, however, took place in the unglossed edition of 1571. In this, Contius classed the 168 *Novells* with reference to their dates, though there are some exceptions to the chronological order. Thus *Nov.* 113 is later than *Nov.* 125. Moreover, for ancient custom's sake, he distributed the 168 *Novells*, so arranged, into nine *collationes*, and subdivided the *collationes* into titles. After this came the appendix of Scrimger, and some supplementary *Novells* of Justinian, Justin II., and Tiberius II., known from the *Epitome* of Julian. The same order was reproduced in the edition of 1581, and has been followed ever since in all but the glossed editions.

We have taken considerable pains to present the reader with the result of all the authorities within our reach on an exceedingly perplexing subject. With respect to the literary history of the *Novells*, the information to be met with in ordinary books is unsatisfactory, and often erroneous. From the account which we have given, it will easily be imagined that great confusion has been occasioned in references by the varieties of arrangement in different editions of the *Novells*. For example, the 131st *Novell*, which, accord-

ing to the arrangement of Contius, forms the 14th title of the 9th *collatio*, was the sixth title of the 9th *collatio* of the old glossators. It may be useful to state, that the 97 glossed *Novells* are the following:—*Novv.* 1–10, 12, 14–20, 22, 23, 33, 34, 39, 44, 46–49, 51–58, 60, 61, 63, 66, 67, 69–74, 76–86, 88–100, 105–120, 123–125, 127, 128, 131, 132, 134, 143, 159. The following were the *extravagantes*, which, though known to the glossators, were not glossed; *Novv.* 11, 13, 21, 24–29, 30, 31, 35–38, 40, 42, 43, 45, 50, 59, 62, 64, 65, 68, 87, 101–103, 129, 130, 133, 144–147. Of the remaining *Novells* the glossators were ignorant. For the purpose of thoroughly understanding the references to the *Novells* made by the glossators and more recent civilians, it would be desirable to have several parallel lists. These should be, for instance, (1), a list showing how the glossed *Novells* were arranged in the original *collationes*, and designating by numbers the different places they occupy in the old and the new *collationes*; (2), a list of the *Novells* not known to the glossators, showing the places of their insertion in the modern *collationes*; (3), a list of the *Novells* of the *Epitome* of Julian, showing the correspondence of their numbers with the numbers of the modern *Novells*.

Much labour and learning have been recently expended in unravelling the intricacies of this part of literary history, and in correcting the errors of former writers on the *Novells*. Biener's *Geschichte der Novellen Justinians* (Berlin, 1824) is a work of very high reputation, containing the most elaborate and accurate information on the subject.

Of modern editions since the time of Contius, it is unnecessary to say much. Under the title *Novellæ Constitutiones Justiniani, a Græco in Latinum versæ operâ Hombergk zu Vach* (Marburg, 1717, 4to), more is performed than is promised. The author presents to us not only a very good new Latin translation, but the Greek text, and a series of Latin *Novells* from the *Authenticum*, of which the original Greek has not been preserved, and good critical notes. The translation of Hombergk zu Vach is the basis of that of Osenbrüggen, the latest editor. A more correct transcript of the 87th *Novell* than had previously appeared was given by C. J. A. Kriegel (Lips. 1832). G. E. Heimbach, in the first volume of his *Anecdota* (Lips. 1838), has announced his intention of publishing a new edition of the *Novells*, and it is to be hoped that he will fulfil his intention, for he comes prepared for the work with unusually important aids, derived from the collation of manuscripts and the abridgments of ancient scholiasts. The oriental ecclesiastical writers of the Græco-Roman Empire have preserved some fragments of *Novells* of Justinian connected with the church. Ten such fragments exist in a collection of *Johannes Antiochenus Scholasticus*, published in the second volume of G. E. Heimbach's *Anecdota*, 1840, and there is an account of another such collection in Richter's *Krit. Jahrb.*, 1838.

We have stated that the *Novells* were designated by the name *Authenticæ*, but that word has also another signification which it is necessary to explain in order to prevent the mistakes which have sometimes occurred in consequence of this verbal ambiguity. In their lectures on the *Institutes* and the first nine books of the *Code* (which are the books most practically useful), the earliest glossators were accustomed to insert in the margin of their copies abbreviated extracts from such

Law.

Law. parts of the *Novells* as made alterations in the law contained in the text. In reading the *Pandects*, they referred to the notes contained in the margin of the *Code*.—At a later period, these abstracts were discontinued in the *Institutes*.—In the *Code*, they were taken from the margin and placed under the text, where they still appear, distinguished by italic type in most of the modern editions. They are called *Authenticæ*, not, as some assert, from their representing the latest authentic state of the law, but from the name of the source whence they were taken, and which in practice they nearly superseded. Certain capitularies of Frederic I. and II., emperors of Germany, about the end of the 12th century, were treated by the glossators as *Novells*, and thirteen extracts taken from them are inserted in the *Code* with the inscription *Nova Constitutio Frederici*. They are known by the name *Authenticæ Fredericianæ*.

Corpus
Juris Civilis. The collections of Justinian, together with some later appendages, formed into one great work, are commonly known by the name *Corpus Juris Civilis*.

Later ap-
pendages
to the col-
lections of
Justinian. The later appendages are really arbitrary and misplaced additions, having no proper connexion with the Justinianean law, and they vary in different editions.

We have already spoken of Scrimger's edition of the *Novells*, which contained, in a kind of appendix, some constitutions of Justin II. and Tiberius II. The same volume contained also a collection of constitutions of the Eastern Emperor Leo the Philosopher, anterior to 893. These were appended by Contius to the Justinianean *Novells* in his edition of 1571, and his example has been followed by subsequent editors.

Some constitutions of Byzantine Emperors, from the 7th to the 14th century, were published for the first time, with a Latin translation, by Bonifidius (*de Bonnefoi*), in 1573, in his *Jus Orientale*, were admitted by Charondas into the *Corpus Juris* in 1575, and have been since generally retained.

Haloander added to his edition of the *Novells* the *Canones sanctorum Apostolorum*, since they were recognized in the sixth *Novell* as a source of existing ecclesiastical law. They are added to most editions of the *Corpus Juris Civilis*.

The *Feudorum Consuetudines*,—a Lombard compilation of feudal law, formed about the middle of the 12th century from the notes and treatises of several authors, especially the letters of Obertus de Orto,—followed by a few constitutions of German and French monarchs, and the *Liber de Pace Constantiæ* (1181), are, in many editions, the last appendages to the *Corpus Juris Civilis*. The *Libri* or *Consuetudines feudorum* are believed to have been appended to the *Novells* as a tenth *collatio* by Hugolinus de Presbyteris about the beginning of the 13th century.

The expression *Corpus Juris* was employed by Justinian himself (l. 1, *pr. C. de Rei Uxorice act.* (v. 13)); but the earliest editions of the whole of his collections have no single title. Russardus first chose the title *Jus Civile*. The modern name, *Corpus Juris Civilis*, appears first in D. Godefroi's edition of 1583 (*qu.* 1604), though the phrase had been employed by others before him. The old glossed editions consist of five volumes, folio, usually bound in five different colours, *viz.*,—I. *Digestum Vetus*; II. *Infortiatum*; III. *Digestum Novum*; IV. The first nine books of the *Code*; V. *Volumen*. The *Volumen* contained, first the 97 glossed *Novells*; then the last three books

of the *Code*, called the *tres libri*; then the *Libri Feudorum*; then a collection of constitutions of German and French monarchs; and, last of all, the *Institutes*. The *Institutes* are often spoken of as distinct from the *Volumen*, and the name *Volumen* is then confined to the preceding contents of the 5th volume. Originally there was no common title-page to the five volumes. The editions of the *Corpus Juris* may be divided into those which have the gloss, and those which have it not. The gloss is an annotation gradually formed in the school of Bologna, and finally settled by Accursius. It is of great practical importance, since, in the countries which adopted the Civil Law, the portions without the gloss did not possess legal authority in the courts. *Quod non recipit glossa, id non recipit curia*, was the general maxim. All the editions up to 1518 have the gloss. The latest edition with the gloss is that of J. Fehius, (Lyons, 1627). This celebrated edition has on the title-page of every volume (in allusion to the place, Lyons), the representation of a living lion surrounded by bees, with the motto *Ex forti dulcedo*. Hence it is known by the name *Edition du lion moucheté*. The very valuable index of Daoyz ought to be appended to it as a 6th volume. Of the editions without the gloss, some have notes and some have none. Of the unglossed editions with notes, the two most celebrated and useful are that of D. Godefroi and Van Leeuwen (Amsterdam, 1663, 2 tom. fol.), and that of Gebauer and Spangenberg (Göttingen, 2 tom. 4to.) Of the editions without notes, the most beautiful and convenient is the well-known Elzevir of 1664, distinguished as the "*Pars Secundus*" edition, from an error in p. 150. Two editions by Beck, one in 4to, one in 8vo, have lately been completed. That commenced by the brothers Kriegel, in 1833, and continued by Osenbrüggen and Hermann, is not yet finished. The edition undertaken by Schrader and other eminent scholars, will, if completed as it has been begun, supersede for some purposes all that have gone before it. The old editions of Contius, Russardus, Charondas, and Pacius are sought for by critics.

In the *Institutes* the titles are divided into paragraphs; in the *Pandects*, into *leges*, otherwise called *fragmenta* (separate extracts); in the *code*, into *leges*, otherwise called *constitutiones*; in the *Authenticæ* or *Novells*, into *capita*. Again, in the *Pandects*, *Code*, and *Authenticæ*, the *leges* or *capita* are in some cases subdivided into paragraphs or sections; but in the time of the glossators these divisions and subdivisions were not numbered, and hence initial words were usually used in citation.* The letter *D.* denotes the *Digest*; and it has been proved that from this letter, not from the

Law.

Mode of
citing the
collections
of Justi-
nian.

* Some have thought that the use of initial words in citing was introduced on account of the uncertainty occasioned by the variation of the numbering in different editions. The ancient mode of citation, before the time of Irnerius, was by numbers. This appears from Theophilus, the *Basilica*, and other Greek sources, from the *Turin gloss*, from Ivo of Chartres, &c. In the fragments of the MS. of the *Pandects* discovered by Gaupp, the *leges* are numbered. Even now, the editions in common use are not uniform in their numbering. For example: in some editions of the *Institutes*, the 6th title of the 3rd book is improperly divided into two titles. A 7th title, *de Servili Cognatione*, is made to begin with *tit. 6, § 10*, thus increasing by a unit the number of every subsequent title in the 3rd book. The mode of citing by numbers has been objected to on account of its easiness and simplicity, which would tend to diminish the industry of students, and derogate from the maxim that a civilian ought to have *ferreum caput, plumbeas nates, auream crumenam*.

Law.

Greek π , came the symbol *ff*, which also denotes the *Digest*. *ff* is, in fact, the corrupted form of an old *D.*, which, in writing, is left open at top, and has a stroke across it to denote abbreviation. *J.* denotes the *Institutes*; *C.* the *Code*; *Auth.* or *Nov.* the *Novells*. The following examples will illustrate some of the various modes of citation.

§ *Fratriis vero*. *J. de Nuptiis* refers to the section beginning with the words *fratriis vero*, of that title in the *Institutes* which has the rubric *de Nuptiis*. Upon referring to the letter *N* in an alphabetical list of titles, we find it to be the 10th of the first book. The old civilians from practice knew well the order of the titles without consulting any index. On looking through the sections of the 10th title of the first book of the *Institutes*, we find that the third section is the passage we are seeking for, since it begins with the words *fratriis vero*. It is unusual now to give the initial words of the paragraph, though the rubrics of the titles are still commonly given. The present ordinary mode of citing the passage above referred to would be § 3. *I. de Nuptiis* (i. 10); or, in the more simple mode approved by Gibbon, *I. i. 10, 3*, beginning at the larger and ending at the smaller division.

The sixth section of the fifth law of the third title of the twenty-third book of the *Pandects* would now be ordinarily referred to thus,—*L. 5. § 6, D. de Jure Dotium* (xxiii. 3); or thus,—*D. 23, 3, 5, 6*; or thus,—*L. 5, § 6, D. de Jure Dotium*. The latter is the usual mode of referring to titles frequently quoted, and therefore so well known as not to require indication by number. Such, for example, are the titles *de R. I.* (i. e. *de Regulis Juris*), and *de V. S.* (i. e. *de Verborum Significatione*). Various modes of citation were employed at different periods. *D. 23, 3, 5, 6*, would, by the ancient glossators, have been cited thus,—*D. de Jure Dotium, L. Profectitia, § Si pater*; or, later, thus,—*L. Profectitia, § Si pater, D. de Jure Dotium*; or, still later, *L. Profectitia 5, § Si pater 6, D. de Jure Dotium*. This last-mentioned mode is objectionable from its ambiguity; for when more than one *lex* in the same title begins with the same word, a similar notation is used; thus,—*L. Procurator 3 D. de Procur.* denotes the 3rd of the four *leges*, which, in the title *de Procuratoribus*, commence with the same word *Procurator*. *fr.* is sometimes used instead of *L.*, and *ff.* instead of *D.* The initial section of an extract or *lex* divided into sections is not numbered, and is referred to by the letters *pr.* For instance, *fr. 5, pr. ff. de Jure Dotium* (xxiii. 3). That which is really the second section is called the first, and numbered § 1. It is to be observed that the 30th, 31st, and 32nd books of the *Pandects* all treat *de Legatis et Fideicommissis*, and are not divided into titles. They are designated respectively as *D. de Legatis I.*, *D. de Legatis II.*, and *D. de Legatis III.* Thus, *L. 108, § 3, D. de Legatis I.* might be otherwise represented *D. 30, 108, 3*.

The *Code* is cited like the *Digest*,—now ordinarily thus,—*L. 22, C. Mandati vel contra* (x. 35). *Const.* is sometimes used instead of *L.* To prevent a multiplication of marks, the brackets enclosing the numbers of the book and title may be omitted. This remark is not confined to the *Code*.

The Prefaces to the *Pandects* and to the *Code* are still cited by initial words,—thus, *Const. Tanta*.

The ancient and modern modes of citing the *Novells* differ completely from each other. In the old mode,

the arbitrary division of the glossators into collations and titles was referred to, and the rubric of the title and the initial words of the chapter were employed. When a chapter is subdivided into paragraphs, the initial words of the paragraph alone are occasionally given; for memory, or an index able to guide at once to the smaller division, would render unnecessary the more vague indication. This remark is not confined to the *Novells*. The modern mode of citation is by the number of the *Novell* and chapter. Thus we now cite by the symbols, *Nov. 118, c. 1*, the same passage which would at one time have been referred to thus,—*Auth. de Hered. ab Intestat. § Si quis. col. 9, tit. 1*. Here we may recall attention to the confusion which has been produced by the introduction of newly discovered *Novells*, and their different arrangement at different times. A knowledge of the literary history of the *Novells* is necessary to those who would make successful use of the references which are occasionally to be met with.

The *Authenticæ of the Code*, of which we have given an account, are cited by reference to the title of the *Code* in which they are interpolated; e. g., *Auth. si qua mulier. C. ad S. C. Velleianum*.

It is necessary that we should preface our explanation of the mode of citing the *Consuetudines Feudorum* with an account of the extraordinary manner in which that work is commonly divided in modern editions of the *Corpus Juris Civilis*. We cannot enter into further details of its obscure and perplexing literary history, but desire to warn the reader that the statements usually made with respect to it are partly deficient in important points and partly erroneous. In its authentic and legal shape—the shape in which it is recognized in the courts of those countries where it is received as law—it consists of two books, the first containing 28, and the second 58 titles. In its most ancient shape, indeed, it formed a single book. In editions of the *Corpus Juris Civilis* since that of Pacius in 1580, supplements to the two books have been introduced which have received the name *capitula extraordinaria*, to distinguish them from the *capitula ordinaria* of the two authentic books. Cujas formed a new collection of feudal law in five books. The first three books of Cujas, and the first 72 titles of his fourth book, comprised the same matter as the two ordinary books, but the subdivision into titles by Cujas was different from that which was previously received, and more minute. The remainder of his fourth book was composed of some *capitula extraordinaria* derived from compilations formerly made by Jac. de Ardizzone and Jac. Alvarotus. The fifth book consisted of new materials added by Cujas himself; viz. some constitutions of Germanic emperors, two abstracts of *Novells* of Greek emperors, and the Golden Bull of Charles IV. In most of the modern editions of the *Corpus Juris Civilis* the whole work of Cujas is inserted with the exception of the Golden Bull, but from the mixture of the old and new divisions a strange confusion has resulted. The modern editors have reproduced the authentic text according to the ancient division into two books, whereas, for the unauthentic additions, they have followed the division of Cujas. After the first two books, we read *Liber tertius vacat*. Then follows the heading *Libri Quarti Fragmenta*, and the titles of the fourth book commence with the number 73. No indication is given that the third book and the greater part of the fourth, far from being lost, have been

Law.

Law.

already presented to the reader, inasmuch as the third book of Cujas consisted of what is now given as book 2, titles 23 and 24, and the first 72 titles of the fourth book of Cujas consisted of what is now given as book 2, titles 25-58.

The books of the *Consuetudines Feudorum* were divided from the first into numbered titles and sections, and therefore the work has been usually cited by numbers, not by initial words; thus,—I. F. 14, § 1. This denotes the first *Liber Feudorum*, 14th title, 1st paragraph after the *principium*. For the literary history of the *Libri Feudorum* consult Paetz, *De verâ Librorum Juris Longobardici origine* (Goett. 1805); Dieck, *Literär-geschichte des Longobardischen Lehnrechts*, (Halle, 1828); Laspeyres, *Ueber der Entstehung und älteste Bearbeitung der Libri Feudorum* (Berlin, 1830).

Those who desire to ascertain the particular dates and persons by whom changes in modes of citation were introduced, must refer to the authorities we shall presently name. We must warn the reader that we have not studied accuracy in stating the distinctions (critically and chronologically not unimportant) between capitals and small print, and Roman and Italic type. It may be useful to add that the first part of the *Pandects*, according to the division of Justinian, is called *Prota*; the 4th part, the *Umbilicus Pandectarum*; the 47th and 48th books, which treat of penal law, are called *libri terribiles*. Practice in reading the older writers will soon make a student familiar with the meaning of such contractions as *s.* (supra); *i.* (infra); *l. un.* (lex unica); *t. l.* (totus titulus); *h.* (hic); *e.* (eodem); *arg.* (arguendo); *v.* (videtur); *d. l.* (dicta lex).

For further particulars respecting modes of citation and their history, the following authorities may be consulted. Thibaut's *Civilistische Abhandlungen*, No. 10. Hugo, *Mag. t. iv.* p. 212; B. St. Prix, *Histoire du Droit Romain*, p. 218. Above all, Schilling, *Inst. und Gesch. des Röm. Privat-Rechts*, Einleitung, §§ 38-42.

Ignorance of the modes of citation has sometimes produced absurd mistakes. The citations of the imperial laws by our Bracton are cruelly mangled in the printed editions. Of this, Selden, in the third chapter of his *Dissertation on Fleta*, has advanced one or two examples. The author of *Fleta* was too prudent thus to refer to the sources. He introduces passages of Roman law without saying whence they come.

No subject connected with Roman law occupies at this day more earnest attention, or seems likely to be productive of more precious fruit, than the investigation of its history in the Eastern Empire, and the study of the numerous works which prove the zeal and industry of the oriental jurists. Many of these works have been long published; some have recently been printed for the first time, and several still remain extant in manuscript. An imperfect collection of oriental sources may be found in the *Jus Græco-Romanum* of Leunclavius and Freherus, and a short but masterly sketch of the history of Græco-Roman law by C. E. Zachariæ appeared in 1839.* We must content ourselves with briefly mentioning some of its most important remains.

* From the preface of this work (p. v.) we extract the following passage for the consideration of British statesmen.—“*Præterea si de lege ferenda quærat, neminem vestrum fugit, quam sit necessarium, aliorum populorum mores nosse atque instituta, ut intelligatur, numqua ejusmodi lex apud eos obtinuerit, et utrum comprobata fuerit usu, nec ne. Sic et imperatorum Byzantinorum leges examinasse juvat. Exempli gratia, juxjurandi usus restringendus ne sit nec ne, nuper magnopere disceptatum est. Jam vero extat lex*

Though Justinian disapproved of commentaries, he made the *Corpus Juris* a text-book for law students, who were required to expend five years in wading through its contents. The professors gave lectures and composed extensive paraphrases, which, even in Justinian's life-time, obtained considerable circulation. The laws, however, that had originated in Rome were found in many points not to be well suited to the meridian of Constantinople. While the lawyers were busy in making epitomes and explanations of a system of law which, being for the most part written in Latin, was not understood by the mass of the people, the Byzantine emperors sent forth an abundance of reforming constitutions. At length the necessity of a Roman-Greek code began to be felt, and, after a long term of parturition, the *Basilica* was brought into the world. This collection in turn gave birth to a multitude of *scholia* and commentaries. Nor was the legislative energy of the Byzantine Cæsars less manifest than the industry of the Byzantine lawyers. New constitutions kept coming into the world until the conquest of Constantinople by the Turks in 1453. We proceed to retrace some of the particulars of these events.

The paraphrase of Theophilus upon the *Institutes* (a treatise in which he himself had a share) is an excellent specimen of clear exposition. The best edition is that of Reitz (Hag. Com. 1751, 2 vols. 4to). The German translation by Wüsteman is remarkable for its accuracy and the goodness of its notes. A few extracts from Stephanus and Thalelæus, who belong to the same period, and commented respectively on the *Digest* and the *Code*, have been preserved in the *Basilica* and the *scholia* upon it. The compendiums of the *Novells* by Athanasius and Theodorus are still extant. The latter, with the exception of some fragments, is still in manuscript, but the former was published for the first time by G. E. Heimbach in his *Anecdota*, in 1838. The work *De Temporum Intervallis*, usually attributed to Eustathius, has been often reprinted. The best critical edition of it is that of C. E. Zachariæ (Heidelberg, 1836). The authors above mentioned precede the *Basilica*.

Concerning the successive constitutions of the Byzantine emperors of this period, some valuable notices by Witte and Biener appear in the 8th volume of Savigny's *Zeitschrift*.

Leo, the Isaurian (who reigned A.D. 717-41), perceiving the want of a compact code, compiled an *ἐκλόγη τῶν νόμων*, in 18 titles, about the year 740. The work is extant in manuscript, but its importance is diminished by the consideration that, on account of the arbitrary alterations it introduced, it never enjoyed complete validity, and one part of it after another was set aside by succeeding emperors. A *Manual*, which had more enduring influence, than the *Eclogæ* of Leo, was the *Πρόχειρος νόμος*, in 40 titles, published by the emperor Basilus Macedo about the year 878. It is extracted from the collections of Justinian, the *Eclogæ* of Leo, and *Constitutions* of Basilus himself. It had been long known to exist in manuscript, and it was lately edited for the first time by

imperatricis Irenæ, quæ ab omni juramento abstinendum esse præcipit. Ex cujus historia quam quæstio illa egregie illustrari possit, cur non idem, qui, quæ in Parlamento Anglico de eadem re nuper dicta atque acta sunt, studiose legamus et examinemus, ad Irenæ constitutionem animum advertimus? See also Leunclavius, *Jus Græco-Romanum*, vol. ii. pp. 135-138.

Law.

Law.

C. E. Zachariæ, (Heid. 1837), with very learned literary and historical prolegomena. A revision of this *Manual* was published before the death of the emperor, with the name *ἐπαναγωγή τοῦ νόμου*, and has come to us in an imperfect state. In addition to these compendiums, Basilus formed the plan of reducing into a systematic form all the existing law; and a work entitled *Reformation* (*ἀνακαθάρσις*) of the *Ancient Laws* is mentioned in both his *Manuals*, but it has not survived, and it is probable that it was never published nor introduced into the courts.

The Basilica.

The credit of completing the task and promulgating the projected code was reserved for his son Leo, the Philosopher (who reigned 886–911). This imperial code, commonly known by the name *τα βασιλικά*, consists of 60 books, divided into titles, chapters, and paragraphs. The name was probably chosen from its capacity for a double meaning—expressing not only the royal law-book, but alluding to the name of the preceding emperor.* It was intended to present a complete consolidation of the law, in which the *Institutes*, *Pandects*, *Code*, and *Novells* were to be blended together, and seasoned by the admixture of Byzantine constitutions and legal text-books. This most comprehensive and important work has reached us in an imperfect state. The edition of Fabrotus, in 7 volumes (Paris, 1647), is now not easy to be purchased. Of the 60 books which it contains, there are 27 which are not entirely genuine, but have been fabricated wholly, or in part, out of subsequent commentaries and citations. The Latin translation of Fabrotus was made in great haste and is very imperfect. Four genuine books were subsequently discovered, and may be found in Meermann's *Thesaurus*. A new and much improved edition, commenced by C. W. E. Heimbach in 1833, is now verging towards completion. The *Basilica* and the *scholia* upon it are of the greatest advantage in the criticism of the text of Justinian's collections. They have served to fill up gaps, correct corrupt readings, and elucidate the sense of difficult passages.

Græco-Roman law after the date of the Basilica.

Soon after the publication of the *Basilica* appeared the first *Synopsis Basilicorum*, published originally in alphabetical order. Leunclavius, who edited the *Synopsis* (Basil, 1575), thought fit to arrange the text in the order of the *Basilica*. Synopses of the *Basilica* abound. As a literary curiosity may be mentioned the *Synopsis* of Michael Psellus, composed for the Emperor Michael Ducas in 1070, and written in verses (*πολιτικοὶ στίχοι*), in which accent is supposed to supply the place of quantity. A sample of them may be seen in Irving's *Introduction*, &c., p. 74. "The work," says Thibaut, "betrays an extent of ignorance scarcely excusable even in a poet."

We pass over a crowd of Greek lawyers to come to Harmenopulus,—a jurist who had probably some acquaintance with the productions of the Italian schools. His *Πρόχειρον νόμων* (also called *Hexabiblus* and *Promptuarium*) is a short system of Græco-Roman law, composed about the year 1345, after the order of the *Pandects* of Justinian. The best edition is that of Reitz, in the 8th (supplementary) volume of Meermann's *Thesaurus*. This work long continued to possess authority in the East.

* The name of the collection, *Basilica*, cannot be analogically derived from the name of the Emperor Basilus. The opinions of Zachariæ (which we adopt), as to the authorship and publication of the *Eclogæ* and the *Basilica*, differ from those which had been previously the most generally received.

Until the time of the destruction of the Greek empire by the Turks, the Roman law appears to have been actively cultivated in the East. After that great event, the Greeks were still permitted to retain their own judges and their own laws; and up to modern times, the *Basilica* and the *Promptuarium* of Harmenopulus retained high authority in Greece. In 1830, the President, Capo d'Istria, appointed a commission to revise the *Basilica* and the constitutions of the Byzantine emperors, from which it was his intention to compile a code for Greece. Since the commencement of the new Greek monarchy many legal reforms have been made, founded chiefly on the basis of the modern French codes. The first article of the proclamation of Otho, dated Feb. 23, 1835 (O. S.), declares,—"The civil laws of the Byzantine emperors, contained in the *Manual* of Harmenopulus, shall preserve their force until the promulgation of the Civil Code which we have directed to be compiled." (See Fælix, *Revue de Législation*, tom. vii. p. 288).

If now we turn our eyes to view the fate of the Imperial Law in the Western Empire after the death of Justinian, how great is the contrast! What remaining legal works can we produce as specimens of its vitality, before the time when the school of Bologna started into sudden energy? In the domain of legal literature we travel over a waste, in which the research of Savigny* can scarcely cull a flower.—

"O the dreary, dreary moorland! O the barren, barren shore."

We have the *Turin gloss* on the *Institutes*, possibly contemporary with Justinian, or composed soon after his death during the Greek dominion that succeeded the kingdom of the Ostrogoths in Italy (3 Sav. *Röm. R.* p. 665). Between this and *Petri Exceptiones Legum Romanorum*—a compilation which seems to have been made in Valence—there is no separate legal treatise worth mentioning, and Petrus was certainly not very long antecedent to Irnerius. The Lombard *Brachylogus* was probably later than *Petri Exceptiones*, and the treatise entitled *Ulpianus de Edendo* (which seems to have been a favourite in England) was, in our opinion, not earlier than the first glossators. That the Justinianean law throughout the whole interval maintained a certain traditional existence in practice,—that it survived in part in the organization of the municipal towns in the kingdom of the Lombards,—that, textually, it was not wholly unknown to certain ecclesiastics and canonists, e. g., St. Lanfranc, archbishop of Canterbury,—that traces of it may be found in the collections of *formule*, such as those of Marculfus: these are all established facts. That, in academical instruction, especially in the schools of Ravenna and Rome, it formed, like the study of other branches of ancient literature, a part of general education, we regard as probable. On the other hand, we think it clear that, even in Italy and the kingdom of the Franks, it soon began to fall into disuse without being expressly abolished; and that, in other parts of the West, if we except the occasional and singular learning of some travelled scholar or friend of foreign scholars, whatever was known of Roman law was known not from the collections of Justinian but from other sources.

* Of the long list of quotations from the Justinianean collections which is appended to the second volume of Savigny's history, it is to be observed that the works containing the quotations from the *Corpus Juris* are not, for the most part, treatises upon Roman law.

Law.

Græco-Roman law after the destruction of Constantinople, 1453.

Fate of the Justinianean law in the Western Empire.

L A W.

CHAPTER IV.

C A N O N L A W.

Law. 269. HAVING, in the Miscellaneous and Lexicographical Division of this *Encyclopædia*, mentioned, under the title *Canon Law*, some of the most important particulars connected with the subject, we shall confine ourselves at present to the statement of a few facts which it was not necessary to insert in that article, and to an explanation of the ordinary mode of citing the *Corpus Juris Canonici*.

Uses of the canon law. The general canon law, so far as its principles are concerned, appeals much more to the religious and moral feelings than do most of the original doctrines of the Roman civil law; but it wants that purity of style and scientific method which constitute the great value of the juridical works of ancient Rome. The study of it is not calculated to form those habits of severe logic and precise expression which are characteristic excellences in legal authorship; but yet it has much attraction, and holds out much reward to attentive study. Mixed with bigotry and superstition, it will be found to contain many pure elements which deserve to be retained in every system of ecclesiastical law. It is not without its practical uses to the English lawyer, since there are parts of it which have been adopted by usage in our ecclesiastical courts. But it is in an historical aspect that it especially claims regard. It elucidates many important controversies, and unfolds interesting views of the human mind. No one can study the great problems of the constitution of the early Church—of the mutual rights and liabilities of the various ecclesiastical orders, from the highest to the lowest, and of the relations which subsist between the temporal and spiritual power—without deriving aid from the considerations presented by the rules and doctrines of the ancient canon law.

Origin of canon law. Canon law is not synonymous with ecclesiastical law. The latter term relates to the *matter*, the former to the *source*. The latter depends on a scientific division of law; the former refers to a certain historical origin. The English Parliament may make laws concerning bishops and advowsons, but those laws form no part of canon law. On the other hand, the ancient canon law contains many rules not touching upon ecclesiastical matters, and more suited to temporal than to spiritual cognizance. We shall therefore briefly examine the origin of canon law.

After the death of Christ, the early Christians were formed into separate bodies, which still communicated with each other in various ways, and retained a certain unity. As each separate body was a church (*Acts* xiii. 1), so the aggregate association, considered as subject to one Lord, constituted the Church of Christ (*Eph.* i. 22, 23; *v.* 23; *Coloss.* i. 18). It was natural that an association which, like the Christian Church in its

infancy, was not placed under the protection of the civil power, should avoid to bring before the magistrates appointed by the state complaints arising from the infraction of ecclesiastical discipline, or disputes upon points of doctrine and internal order. Hence, in such cases, care was taken that the infliction of punishments and the decision of disputes should be arranged within the bosom of the Church itself. The same tribunals which adjusted these matters took cognizance of temporal causes in which all the parties were Christians. The advice which the Apostle Paul gave to the Corinthians was generally followed (*1 Cor.* vi. 4—6). By means of private arbitration, the Church avoided the unseemly spectacle of submitting ecclesiastical controversies to courts of unbelievers, as well as the scandal which would have resulted from an exposure of the differences of the faithful.

A system of ecclesiastical polity was early developed in the Church. In each limited community, the appointed teachers of Christianity, by virtue of their high and sacred functions, as well as by the natural influence of superior information and intelligence, claimed and obtained a greater degree of consideration, in reference to the settlement of internal disputes, than their fellow-believers. It was by the intervention of those teachers that the mutual intercourse of the several communities was conducted, and power was thus increased by union.

A hierarchical subordination soon grew up among the separate communities which were thus connected. A pre-eminence of honour and respect was voluntarily conceded by the later and the less numerous societies to those which were more considerable and more ancient, and the rank of the bishop rose simultaneously with that of the community of which he was the spiritual head. The bishop, whose seat was a capital city, of large size and great political influence, or whose church could claim the honour of immediate apostolical foundation, was naturally listened to with more than ordinary attention. These distinctions of rank, having thus spontaneously originated among dignitaries of the same order, gradually led to claims of compulsory authority.

There were sometimes assemblies of the bishops of a single province, and sometimes more general assemblies of all the bishops of the Christian Church. The metropolitan or patriarchal bishop, by whom such general councils were convened, possessed an extent of real power which he was not slow to exercise on other fitting occasions. The intervention which, in process of time, conducted to ambitious pretension and undisputed sway, was usually manifested at the beginning under the form of advice or amicable exhortation.

Law.

Law.

There are very early traces of the decisive pre-eminence asserted by the Church of Rome over all the other Christian churches. She was considered as the rule and type of orthodoxy. Eusebius, in his *Ecclesiastical History* (v. 6), makes mention of a letter written in the first century from the Church of Rome to the Church of Corinth, to maintain the peace of the latter; and the same author (v. 24) relates circumstantially the manner in which the bishop of Rome conducted himself, towards the end of the second century, in a discussion on the solemnization of Easter (See Ranke, *Geschichte der Päpste*, i. p. 12, cited by Falck, *Juristische Encyclopädie*).

The idea of the separate unities of Church and State gained ground. Each was to subsist apart, by the side of the other, with independent but analogous systems of government. Even when the temporal and spiritual authorities were united in the same hands, they were regarded as distinct in conception. To use the language addressed by Justinian to Epiphanius, archbishop and patriarch of Constantinople, *Maxima quidem in hominibus sunt dona Dei à superna collata clementia, SACERDOTIUM et IMPERIUM: et illud quidem divinis ministrans, hoc autem humanis præsidens, ac diligentiam exhibens, ex uno eodemque principio utraque procedentia, humanam exornant vitam* (Nov. vi. pr.).

If such were the doctrines recognized in the Eastern empire, the same notions were practically influential in the West, where the spiritual power maintained an orderly existence amid the anarchy and revolutions of the temporal government.

Sources of canon law.

Based upon the notion of the distinct existence and governing power of the Church, a system of rules (*canones*) grew up, derived partly from custom, partly from a kind of direct legislation, and partly from professional opinion. Ecclesiastical observance was founded sometimes merely on reason or convenience; it rested in part on the authority of holy writ, or on early tradition; and sometimes it referred to the writings of the fathers, to the opinions of learned doctors, to the express resolutions of councils and synods, to the determinations of the highest authorities in the Church, to the decretal letters of popes, deciding questions, and establishing laws for the future. The system of rules thus formed has obtained the collective name of the Canon Law.

We have seen that the earliest canon law was administered between parties belonging voluntarily to the Christian Church, and not bound to submit to ecclesiastical jurisdiction. It was not sanctioned by civil penalties; it could not set in motion the temporal sword. It relied for its efficacy on the deprivation of spiritual advantages, and the censures of the Church. But the jurisdiction which was voluntarily submitted to when the civil power was wielded by pagan emperors, was not resigned when, through the conversion of Constantine, Christianity became the dominant religion of the Roman empire.

At first, the jurisdiction of the ecclesiastical authorities was limited to matters concerning religion. Afterwards, effect was given to those judgments touching temporal matters which were pronounced in the case of ecclesiastical persons who bound themselves to submit to the adjudication of the ecclesiastical authorities. Even laics, by consent, *præunte vinculo compromissi*, might oblige themselves to obey the sentence

of the bishop (See Nov. Valent. III. lib. ii. tit. 35, A.D. 452). Justinian altered this state of the law by giving to the bishop, in the first instance, without the consent of the litigating parties, the power of trying temporal causes in which the defendant was an ecclesiastical person. There was still, except in a few cases, an appeal to the ordinary tribunals (See Nov. 123, c. 21-23). At a later period of ecclesiastical history the spiritual tribunals in some countries enlarged their pretensions, and assumed the cognizance of all causes touching either an ecclesiastical matter or an ecclesiastical person. Thus it was that the canon law was extended far beyond its original sphere, and its domain became nearly concurrent with that of the civil law, from which it borrowed many important provisions.

The most general branch of the canon law is that which has been admitted by the Church of Rome, and prescribed for general use. Of portions of this Roman canon law certain authorized collections have been made, which form part of the volume entitled *Corpus Juris Canonici*. These authorized collections, with some modifications, have been adopted by the Roman-catholic states of Europe, and even, to a considerable extent, by countries in which reformed churches are established. But there are parts of the Christian Church which never admitted the supremacy of the Roman see. The Greek Church has canons of its own, and the churches of separate countries in communion with the papacy have also a special canon law, enacted with reference to their peculiar wants and circumstances.

Some of those who profess to give an historical account of the canon law divide it into three periods, the ancient, the middle, and the modern. The ancient begins with the first and ends before the middle of the ninth century, when the collection of canons attributed to Isidore Mercator, or Peccator, made its appearance. The middle begins after that event and ends with the Council of Pisa in 1409. The modern begins after that council and ends with the present time (See Butler, *Horæ Juridicæ Subsecivæ*, p. 106).

To the most ancient period of canon law are to be referred the Apostolic canons, a collection of ecclesiastical rules, which were probably for the most part in force before the Council of Nice in 325, though their claim to Apostolic authority is unwarranted. Of these canons the first fifty only have been admitted by the Latin Church, while the Greek Church recognizes as authentic thirty-five more.

In the most ancient period all the considerable synods were held in the East, and their resolutions were set forth in Greek; but the authority of the Greek canons was admitted in the West, and they were early translated into Latin. Until after the synod "in Trullo," held at Constantinople in 691, the Greek canons, in consequence of the bond of union which subsisted between the Greek and Latin churches, were the chief constituents of the ecclesiastical law of the West. One of the earliest Western collections was that which was made by Dionysius Exiguus, about the year 525. It was divided into two parts, the first consisting of a translation of Greek canons, the second of a collection of decretal letters by the bishops of Rome. The work of Dionysius was soon raised to the rank of a code for the Western church; but it received alterations from time to time, especially in the second part. The printed editions present this collection in the state in which it existed at the end of the eighth century, when

Law.

Canon law, general and particular.

Historical periods of canon law.

Ancient canon law.

Canons of Greek Church.

Collection of Dionysius Exiguus.

Law.

Pope Adrian gave the Emperor Charlemagne a copy of this *Codex Canonum* (*Codex Hadrianeus*). It was published by Pithou (Paris, 1687, fol.), under the title *Codex Canonum vetus Ecclesiæ Romanæ*.

The early canons of the Greek Church, of a date subsequent as well as anterior to its rupture with the Church of Rome, are contained in the valuable collection of Bishop Beveridge, published at Oxford, in 1672, in two folio volumes. The literary history of the early Greek collections has been carefully illustrated by Biener, in his work *De Collectionibus Canonum Ecclesiæ Græcæ* (Berlin, 1827).

Canons
of the
Spanish
Church.

After the collection of Dionysius the Little, other private compilations of canons were formed and circulated in various parts of the Christian world. Of these, the most important was a collection designed for the use of the Church of Spain, and generally attributed to the learned Isidore, who was bishop of Seville from 595 to 636. It was considered by Andreas Burriel, a Jesuit, who prepared an edition for publication, to have been the most pure, the most copious, and the best arranged which ever existed in any of the churches of the East or West; and, according to the same authority, the Church of Spain was invariably governed by it, until the close of the 15th century.

Middle
period of
canon law.
The
pseudo-
Isidorian
collection.

About the year 811, this collection was brought from Spain into France by Riculphus, bishop of Mayence; and it was not long before copies of a work now called the collection of Isidore Peccator, or Mercator, were diffused throughout Europe. Mercator is probably a corruption of Peccator, for bishops, with conventional humility, were accustomed to subscribe themselves "sinners," and the collection we are speaking of was probably at first passed off as the genuine collection of the bishop of Seville. It professed to contain the decretals of several contemporaries of the Apostles, and letters of almost all the popes from St. Clement to Sylvester. It abounded in tokens of forgery; but, in the age in which it was produced, the science of criticism was too little advanced to enable ordinary men of letters to detect the imposition. The maxims which it inculcated were favourable to the high pretensions of the Roman see. In the false decretals it was maintained that the pope alone had the right of convening councils; that he alone was judge in the last resort, where bishops were concerned; that he alone could translate bishops, and alone erect new bishoprics; that, in a word, it was within the scope of his power to alter any sentence pronounced by any tribunal, ecclesiastical or civil.

Some of the popes, cognizant of the forgery, or suspecting its existence, may have given countenance to the imposition, in order to extend their temporal power; others, finding the false decretals established by a formidable weight of authority, may have been reluctant *quieta movere*, or obliged in conscience to sustain the principles which were supported by so much evidence, and were so positively asserted to have emanated from the golden age of Christianity. For seven centuries after the first appearance of the collection, neither its authenticity nor its authority appears to have been questioned. It was first attacked by Marcillus of Padua, then by Cardinal Nicolas of Cusa, during the council of Basil, and afterwards by Erasmus. Writers of all religious persuasions were forced to acknowledge that it contained much which was spurious. It was given up by the Centuriators of Magdeburgh, by Bellarmine,

Law.

Van Espen, and Fleury. Even the author of the celebrated treatise, *Quis est Petrus?* (2nd edit. Ratisbon, 1791), though he be one of the ablest and most strenuous supporters of the papal prerogatives against the doctrines of the Sorbonne, is obliged to admit that the decretals of Isidore Peccator have been so far proved to be falsified that when they stand alone they are of no authority. By certain protestant writers the "crafty merchant" has been indiscriminately decried, with his thirty decretal epistles bearing the name of primitive bishops of Rome, and brought in, like "so many knights of the post," in a row, "to bear witness of that great authority which the Church of Rome enjoyed before the Nicene fathers were assembled" (See Usher, *Answer to a Challenge made by a Jesuit in Ireland*, London, 1631, p. 12). One of the most famous opponents of Isidore Peccator was David Blondel, in his work entitled *Pseudo-Isidorus et Turrianus Vapulantes* (4to, Genève, 1628).

It seems now to be ascertained that the collection of Isidore Peccator is in reality the Spanish collection probably made by Isidore of Seville, with some interpolations and the addition of some genuine and some spurious decretals, tending for the most part to relax the dependence of bishops upon their provincial superiors, and to extend the authority of the apostolic see of Rome. Whether the forgeries were the work of Riculphus is doubtful. Walter, in his *Lehrbuch des Kirchenrechts* (9th edition, Bonn, 1842), enters at large into the question, and comes to the conclusion that the spurious collection is due to Benedictus Levita of Mayence.

The pseudo-Isidorian collection is published in the first part of Merlin's *Decreta et Concilia* (Paris, 1523). The greater part of it is incorporated in the *Decretum* of Gratian, which forms the first book of the *Corpus Juris Canonici*.

M. De la Serna Santander, keeper of the Royal Library at Brussels, in the catalogue of his own library, printed in 1799, tom. i. p. 72, makes mention of a manuscript work which he possessed, in four volumes folio, containing the genuine Spanish collection revised by Andreas Burriel, *et ad fidem MSS. Cod. venerandæ antiquitatis exacta et castigata*. On the margin of this MS. are marked the various readings of several old manuscripts on vellum, of the 9th, 10th, and 11th centuries, preserved in the archives of the churches of Toledo, Girone, and Urgel, and in the royal libraries of Madrid and the Escorial. The Spanish manuscripts agree with each other, and are quite free from those errors and anachronisms which may be detected in the subsequent interpolations.

M. De la Serna Santander formed the intention of publishing the true collection of Isidore, as edited by the Jesuit Burriel, and for that purpose prepared a very interesting preface, in 114 pages 8vo, which was printed in 1803 in a supplement to his catalogue; but he was prevented by circumstances from completing his original design; and in 1809 his library, comprising the four MS. volumes of Burriel, was dispersed by sale. The genuine canons, however, have been since published at Madrid, in 1808 and 1821, in two volumes folio, with a preface by Gonzalez.*

* *Collectio Canonum Ecclesiæ Hispanæ, ex probatissimis et pervetustis codicibus nunc primum in lucem edita à publica Matritensi bibliotheca. (Matriti, ex typographia Regia, 1808). Epistolæ Decretales ac Rescripta Romanorum Pontificum. (Matriti, ex typographia hæredum D. Joachimi de Ibarra, 1821).*

Law.

The first volume of a work professing to contain the genuine Spanish collection had been previously printed at Rome, in 1739, under the editorship of Cajetano Cenni; but it seems that Cenni mistook a mere abridgment of that collection for the collection itself; and his work remains uncompleted. In the 6th volume of *Notices des Manuscrits de la Bibliothèque Nationale*, Paris, An. 9 (1801), is a paper by Camus on the pseudo-Isidorian canons; and in the 7th volume of the same work is an essay by Koch on a code of canons written by the direction of the bishop Rachion of Strasburg, in 787.

Compila-
tions pre-
vious to
the *Decre-
tum Gra-
tiani*.

Between the end of the 9th and the middle of the 12th century, several compilations of canon law were published, not, like the former collections of the Western churches, consisting of unmutated original documents, arranged in chronological order, but of ecclesiastical rules, extracted from the authoritative text, and arranged according to the order of subjects. This method had been followed in the Eastern empire, in ecclesiastical as well as civil law, ever since the age of Justinian, and was acceptable, especially to superficial readers, from its superior brevity and convenience. Among the most important collections of this class are those of Regino (915), Burchardi (1025), and Ivo of Chartres (1115). How much, in this country, the civil lawgivers of Anglo-Saxon times, and even of times subsequent to the Norman conquest, were indebted to the collections of Isidore and other ecclesiastical compilers antecedent to Gratian, appears to us not to have been fully understood by some of the modern editors of our ancient laws. We believe that the scraps of Roman jurisprudence, and the mutilated extracts from Cicero and other classical authors, which occur in the ancient English laws, were introduced at second hand through a canonical medium.

Formation
of the *Cor-
pus Juris
Canonici*.
The *Decre-
tum* of
Gratian.

All the compilations of ecclesiastical law of which we have hitherto spoken were eclipsed and superseded by the work of Gratian, a monk of Bologna. "He was a native," says Dr. Irving (*Introd.* p. 231), "of Clusium or Chiusi, near Florence; and, according to a very improbable account, was the brother of Petrus Lombardus and Petrus Comestor. One of these three individuals was a native of Tuscany; another, as his name denotes, of Lombardy; and the third, of France; but, in order to remove the difficulty which arises from this variety in the places of their nativity, some writers do their mother the honour of supposing that her three distinguished sons were the fruit of her unlawful intercourse with three different fathers, at different periods and in different places." His compilation was undertaken about the year 1140, probably at the instigation of St. Bernard, in order that the canon law might become, as the Roman law had lately become, an object of fashionable study and interpretation in the school of Bologna. In this object the author was successful. His work, originally called the *Concordantia Discordantium Canonum*, and afterwards *Corpus Decretorum*, or *Decretum Gratiani*, or simply *Decretum*, became a text for academical instruction, and was loaded, like the Imperial Law, with a copious gloss. In this celebrated collection there are many errors. Towards the middle of the 16th century Antonius Demochares and Antonius Contius published a corrected edition of it. A still more correct edition we owe to the Council of Trent. By a decree of that council it was ordered that accurate editions of missals, breviaries, and other

*Recensio
Romana*.

books relating to ecclesiastical matters, should be published. In consequence of this decree, Pope Pius IV. engaged several learned men in the revision of Gratian's compilation. The revision was continued through the pontificate of Pius V. Gregory XIII., the immediate successor of Pius V., when a cardinal, had been employed in the task. Under his auspices, a revised edition was finally published in the year 1580. Several faulty passages still remain in the work. Many of them have been pointed out by Antonius Augustinus, archbishop of Tarragon, in his learned and entertaining dialogues on the emendation of Gratian (*See Butler, Horæ. Jur. Subsec.* p. 114). The compilation of Gratian, without receiving any formal sanction, soon obtained universal authority in the Western Church. The first formal recognition of it was given by the bull of July 1, 1580, a memorable era in canon law, when the whole *Corpus Juris Canonici*, according to the *Recensio Romana*, completed by the 35 *correctores Romani*, in the time of Gregory XIII., was declared authentic.

Law.

The materials of Gratian's collection are drawn, for the most part, not directly from the sources but from preceding collections. The extracts, however, are preceded by inscriptions referring to the original sources whence they are severally taken. In some respects, the work may be compared to the *Pandects* of Justinian; but it differs in this respect, that, besides extracts from preceding authorities, it contains the personal observations of the author. Indeed, the *dicta Gratiani*, in the parts where they occur, are the texts, and the extracts which follow are the *pièces justificatives*. Many of the extracts or canons are entitled *Paleæ*, a name which has given rise to much doubt and conjecture. It is now ascertained that they were *addenda*, introduced into the *Decretum*, very soon after its publication, by one Paucapalea. The spirit of the *Decretum* is favourable to the pretensions of the papacy. "To the compilations of Isidore and Gratian," says the Catholic Butler, "one of the greatest misfortunes of the Church, the claim of the popes to temporal power by divine right, may in some measure be attributed. That a claim so unfounded and so impious, so detrimental to religion and so hostile to the peace of the world, should have been made, is strange; stranger still is the success it met with."

Soon after the collection of Gratian, a want was felt of supplementary compilations, of which many were published; but there were five of these which gained especial celebrity, from being honoured with a gloss, and taught in the school of Bologna. They were made respectively by Bernardus Papiensis (of Pavia), Johannes Galensis, Petrus Beneventanus, Innocent III., and Honorius III. The multitude of collections brought on the necessity of a new consolidated work, and accordingly Pope Gregory IX., in 1234, commissioned his chancellor, St. Raymond of Pennafort, to undertake the task. The five collections were embodied in this. It contains not only the decretal letters issued since the work of Gratian, but many of more ancient date, together with extracts from the canons of councils, and the writings of the fathers. St. Raymond was permitted to take the same liberties with ancient canons that Tribonian had taken with ancient constitutions of emperors. The compilation was executed greatly to the satisfaction of his

The five
collections.

The *De-
cretals* of
Gregory
the Ninth.

Law. Holiness, was published with formal sanction, and other similar compilations were forbidden. It became the commencement of the second portion of the *Corpus Juris Canonici*, and, like the *Decretum*, ere long received a *glossa ordinaria*. The full title is *Libri Quinque Decretalium Gregorii Noni*.

Liber Sextus Decretalium. The school of Bologna soon wanted a further collection. The spirit of research was awakened, and litigation was not inactive. To supply the demand, Boniface the VIII., in 1298, caused to be published, as a supplement to the five books of the *Decretals* of Gregory, a work which followed the same arrangement, and handled the subjects treated of much in the same manner. This work was called *Liber Sextus Decretalium*, or simply *Sextus*.

Clementine Constitutions. The following pope, Clement V., caused another collection to be made, which is known under the name *Clementine Constitutions*, or simply *Clementine*. It is sometimes called *Liber Septimus Decretalium*, and must not be confounded with another work under that title, compiled by Peter Matthæus, in 1590, and inserted in some editions of the *Corpus Juris Canonici*. The *Clementines* were sent by John XXII., in 1317, to the University of Bologna, where, like the *Sextus*, they received a gloss, and were accepted as an official text. The *Decretals* of Gregory the IX., the *Sextus*, and the *Clementines*, form the three parts of the second grand division of the *Corpus Juris Canonici*: they are usually known by the collective name of the *Decretals*,—a name which must not be confounded with the *Decretum* already described. The *Decretals* may be considered analogous to the *Code* of Justinian. In the revision of the *Corpus Juris Canonici* under Gregory XIII., the *Decretals* underwent a critical review, but were not altered and improved like the *Decretum*. For practical use, they are now the most valuable part of the general canon law.

Reception of the *Corpus Juris Canonici* beyond the dominions of the pope. Here ends the authoritative *Corpus Juris Canonici*, generally recognized by the European states, who received their law from the school of Bologna. The additions, now called *Extravagantes*, were made subsequently, and were declared to be authentic by Gregory XIII. in 1580; but the canon law had then ceased, according to the theories of continental jurists, to be susceptible of sudden increase; since its legal force, beyond the pope's own dominions, depended upon gradual customary adoption, or upon an ancient reception *en masse*, of which no repetition was possible.

The *Extravagantes Joannis* and *Extravagantes Communes*. At first, every collection of canon law, except the *Decree* of Gratian, was ranked among the *Extravagantes*. The name finally remained with the *Extravagantes* of John XXII. and the collection called *Extravagantes Communes*. The former contains decretal letters of John, distributed into fourteen titles by Jo. Chappuis. Twenty decretal letters of John XXII. had been previously published, with a gloss by Zenzelinus de Casanis, in 1325. The latter is a collection of subsequent *Decretals*, of which the most modern is one of Sextus IV., of the year 1483. It was made by Jo. Chappuis, and published soon after 1483. A subsequent edition, with some additions, appeared in 1503. The *Extravagantes* may be compared to the *Novells* of the *Corpus Juris Civilis*.

The *Institutes* of Lancellotti. To most editions of the *Corpus Juris Canonici* is appended the *Institutiones Juris Canonici* of Lancellotti. This work, first published in 1563, a few months before the dissolution of the Council of Trent, has never re-

ceived official sanction, although it was undertaken with the approbation of Paul IV., and, by the permission of Pius V., obtained a place in the *Corpus*. From its resemblance to the *Institutes* of Justinian it completes the analogy between the *Corpus Juris Civilis* and the *Corpus Juris Canonici*. This is an interesting and well-written little treatise, and its value is increased by the excellent notes of Doujat (Paris, 1685, and Venice, 1740).

The most prized edition of the *Corpus Juris Canonici*, with the gloss, is that of Lyons, 1671. The other editions of principal note are—(1) that of the brothers Pithou, of which the best impression is that of Paris, 1687; (2) that of Boehmer (Halle, 1747); (3) that of Richter (Leipzig, 1833-8).

We proceed to explain the manner of citing the different parts of the *Corpus Juris Canonici*, which, at first view, is rather more complicated and enigmatical than that which is in use for the *Corpus Juris Civilis*.

The *Decretum* is divided into three parts. The first contains 101 sections, called *Distinctiones*, and treats of the origin and different kinds of law, and particularly of the sources of ecclesiastical law; of persons in holy orders, and the hierarchy. The second treats of 36 particular cases (*causæ*), out of which questions of law (*quæstiones*) arise. The solution of these questions is given in the extracts. *Causa XXXIII.*, *Quæstio III.*, of the second book, constitutes a special treatise concerning penance (*de pœnitentia*), and is subdivided into seven *distinctiones*. The third book is entitled *De Consecratione*, and contains canons relating to the consecration of churches, the sacraments, and the celebration of divine service. It is divided into five *distinctiones*. It is remarkable that the *dicta Gratiani* are entirely wanting in the third part. Each separate extract is marked by the letter C., which stands for chapter (*caput*, *capitulum*), not, as it is ordinarily read, for canon. The whole contains about 3000 canons or chapters. In the early MSS. and editions of the *Decretum*, the larger divisions are numbered, but not so the chapters; hence the latter were necessarily cited by initial words. The first part was thus anciently cited: 1 *d. lex*, meaning the chapter beginning with the word *lex* in the first distinction. This is now numbered c. 3, and the fuller modern mode of citing would be, 1 *dist. c. 3, Lex*. The second part is thus cited: 3. *qu. 9, caveant*; or more fully, 3. *qu. 9, c. 2, caveant*; meaning, cause the third, question the ninth, canon or chapter the second. The 33rd cause and third question is cited in a peculiar manner, thus—*De Pœnit. d. 2, Radicata*. The same passage might be otherwise referred to, thus—33. *qu. 3, dist. 2, c. 2*. The third part is cited like the first, with the addition of the words *de consecratione*, thus: *de consec. d. 2, quia corpus*. This means the 35th chapter or canon (which begins with the words *quia corpus*) of the second distinction of the treatise *de consecratione*, which is the third part of the *Decretum Gratiani*. As the *Decretum* long stood alone, it is understood to be cited when there is no distinctive mark to denote a reference to other portions of the *Corpus Juris Canonici*. It is upon the same principle that the *distinctiones* of the first part of the *Decretum* are denoted simply by the letter *d*, while to the *distinctiones* in the second and third parts are respectively prefixed the specific symbols *de pœnit.* and *de consec.* We have seen that there is usually no

Law.

Editions of the *Corpus Juris Canonici*.Mode of citing the *Corpus Juris Canonici*.

Law.

mark to denote *causa*,* and that a mere number followed by *qu.* was considered a sufficient reference to a *causa* of the second part of the *Decretum*. Such is the explanation of a wretched technical system of citation, which seems designed to puzzle the uninitiated.

The *Decretals*, we have seen, consist of three parts—Gregory's, the *Sext*, and the *Clementines*. Gregory's *Decretals* (which are sometimes exclusively designated when the *Decretals* simply are spoken of) are divided (like the collection of Bernardus Papiensis, which they superseded) into five books, the contents of which are denoted by the verse—

Judex, Judicium, Clerus, Sponsalia, Crimen.

Each book is divided into titles with rubrics, and the titles are subdivided into chapters with *inscriptiones*. The *Decretals* of Gregory, having been the first collection that wandered beyond the *Decretum*, were originally denominated *extravagantes*, and they are still denoted in citation by the word *extra*, or the letter *X*, though the name *extravagantes* has been transferred to later compilations. Gregory's *Decretals* are usually cited by the chapter and title, without any reference to the number of the books. The varieties in the modes of citation are similar to those which we have explained in treating of the references to the *Pandects*. Thus *c. cum contingat X. de Offic. et Pot. Jud. Del.* is the 36th chapter, beginning with *cum contingat*, of the title in Gregory's *Decretals*, which is inscribed *de officio et potestate judicis delegati*; and which, by consulting the index, we find to be the 29th title of the first book.

The canonists seem to have been fond of the number five. The *Sext*, or sixth book of the *Decretals*, is itself divided into five books of the second order. The *Clementines* also consist of five books, the fourth book consisting of only three or four lines. The subdivisions of the *Sext* and *Clementines* resemble those of the *Decretals* of Gregory. The mode of citation, too, is similar, and admits of similar varieties; only, instead of *extra*, or *X*, there is subjoined in *Sexto*, or in 6, in references to the *Sext*, and *Clem.* or in *Clem.*, in references to the *Clementines*. Thus *c. Si gratiose 5, de Rescript. in 6*, is the fifth chapter, beginning with *si gratiose* of the title *De Rescriptis* in the *Liber Sextus Decretalium*,—the title with that rubric being the third of the first book. Again, *Clem. 1 de Sent. et R. J.*, or *ut calumniis, 1, de Sent. et R. J. in Clem.*, is the first chapter of the *Clementine Constitutions*, under the title *De Sententiâ et Re Judicata*,—which chapter begins with *ut calumniis*, and belongs to the eleventh title of the second book.

The *Extravagantes* of John XXII. are contained in one book, divided into fourteen titles. They are cited thus: *Extravag. ad conditorem, Joh. 22 de V. S.* This means the chapter beginning with *ad conditorem* of the *Extravagantes* of John XXII.,—title *De Verborum Significatione*, which is found to be the 14th title.

The *Extravagantes Communes*, in imitation of Gregory's *Decretals*, are divided into five books, though the fourth book is a blank! *Quartus Liber vacat!*—The mode of citation may be thus exemplified: *Extravag. Commun. c. Salvator de Præbend.* This means the chapter beginning with *Salvator*, in the title *De Præbendis*, among the *Extravagantes Communes*. There

* *Causa*, by the moderns, is often denoted by a capital *C*; the small *c* being still reserved to denote *caput*.

is no constancy in the position of the references to the larger or smaller divisions. Sometimes the rubric of the title precedes, sometimes follows, the initial words of the chapter. We have given, with some additions and explanations, the examples of citation which are contained in a note to the first chapter of Halifax's *Analysis of the Civil Law*, b. i.

In the modern period of canonical law, less alteration and advance have been made than in the ancient or the middle. While the ancient is chiefly signalized by the canons of the Greek Church, and the middle by the formation of the *Corpus Juris Canonici*, the modern, commencing after the Council of Pisa, in 1409, is remarkable chiefly for a few councils, among which the councils of Basil and Trent are the most important for some bulls and briefs of the popes, the transactions and concordats between sovereigns and the see of Rome, the sentences and ordinances of the various congregations of the cardinals at Rome, and the decisions of the *Rota*, the supreme tribunal of justice in Rome both for spiritual and temporal concerns.

All these are merely continuations of sources of canon law which existed in more ancient times; and no one can presume to call himself an accomplished canonist who rests contented with the extracts of the *Corpus Juris Canonici*, instead of resorting to the authorities whence they are taken. Of these, the most important are the transactions of councils. The earliest collection of councils is that of Merlin (Paris, 1523); the collection best known in this country is that of Labbe and Cossart, with the apparatus of Jacobatius (18 vols. fol. Paris, 1672). More complete is the collection of Coleti, with the Supplement of Mansi (31 vols. fol. Venice and Lucca, 1728–52). The most copious collection of all is that of Mansi, printed by Zatta (31 vols. fol. Florence and Venice, 1759–98). The 31st volume of this collection stops short some time before the middle of the 15th century. Chronological and alphabetical lists of councils, abridged from the excellent *L'Art de vérifier les Dates*, are given in the useful little book of Sir Harris Nicolas, entitled *Chronology of History*. Among the councils held since the Council of Pisa, in 1409, are that of Constance, 1414, and that of Basle, 1431. Of much greater consequence than these is the famous Council of Trent, 1545. Its history has been often written; it will be sufficient here to name the celebrated and opposing histories of Fra Paolo Sarpi and Sforzia Pallavicini. The translation into French of the former by Le Courayer is more valued than the original. The best edition is that of Paris, 1751, 3 vols. 4to. "The classical purity," says Charles Butler, "and severe simplicity of the style in which the decrees of the council are expressed are universally admired, and are greatly superior to the language of any part of Justinian's law." The best edition of the *Canones et Decreta Concilii Tridentini* is that of Rome, 1834. It has been reprinted (Leipz. 1837) as a supplement to the *Corpus Juris Canonici* of Richter. A preceding edition, of Augsburg, 1781, contains, in addition, the declarations of the *Congregatio Cardinalium, Concilii Tridentini interpretum*, and the decisions of the *Rota Romana*. There have been several collections of the bulls of popes: the best known is the *Bullarium Magnum*, published at Luxembourg, 1727–58, in 19 vols. fol. We know of no separate and complete collection of concordats; but for the names of treatises relating to some of the modern concordats, we would refer the reader to the

Law.

Law.

Bibliothèque de Droit of Camus, which is particularly copious in its information with respect to French works on ecclesiastical and canon law.

Into the detailed examination of the particular canon law of the several nations of Christendom we are unable here to enter. Some of the ecclesiastical laws of Rome are not intended for general adoption, but are designed only for the dominions of the papacy; others, intended to be general, have been by some countries rejected, or only partially received. The struggles which have been made in various Roman Catholic countries for the liberties and independence of their respective national churches fill some interesting chapters in ecclesiastical history. Canon law verges not only upon ecclesiastical history but upon dogmatic theology. It prescribes articles of faith as laws; it looks upon offences rather as sins than as crimes; and it holds forth punishments *pro salute animæ*. Of those branches of the Roman canon law which are intended to be general, some parts relate to matters of faith, some merely to matters of practice and discipline. The former are generally received by Roman Catholic countries, whereas the reception of the latter is variable and limited. If to give an account of the reception throughout Europe of the Roman canon law exceed our limits, still more is it inconsistent with the plan of so slight a sketch as the present to delineate the internal legislation of each national church. We must confine ourselves to a few words respecting the state of our own canonical law.

The canon
law of
England.

The canon law of England may be classed under two heads, foreign and domestic.

The foreign canon law of England consists of so much of the general canon law of Rome as has become incorporated into the English system by usage and by its reception in particular cases. Its efficacy rests upon the same foundation with the efficacy of those portions of the imperial civil law which have been adopted by usage in the Court of Admiralty. For example, by such usage and recognition, there are canons of the fourth Council of Lateran which have become part of the canon law of the land (*See Alston v. Atlay*, 7 Ad. & El. 289).

The domestic canon law may be divided into the canons anterior to the statute 25 Henry VIII., cap. 19, and the canons subsequent to that statute. We omit, as antiquated, the Anglo-Saxon canons.

The canons anterior to the statute 25 Henry VIII., cap. 19, consist of legatine and provincial constitutions.

The legatine constitutions were made in national councils held within this realm, in the time of Otho, legate of Gregory IX., in 1220, and of Othobon (afterwards Pope Adrian V.), who was legate here under Clement IV. in 1268. They were edited, with a gloss, by John de Athona, canon of Lincoln, about 1290. They extend to both provinces, Canterbury and York.

The provincial constitutions were made in convocations of the clergy of the province of Canterbury, in the times of various archbishops, commencing with Stephen Langton, under Henry III., and ending with Henry Chichele, under Henry V. These were collected, and adorned with the learned gloss of William Lyndwood, official of the Court of Canterbury, and afterwards bishop of St. David's, in the reign of Henry V. Although made only for the province of Canterbury, they were received also by the province of York, in convocation, in the year 1463.

VOL. II.

Law.

There were other constitutions, of divers prelates, both before and after; but these which have been mentioned, having been introduced to public notice by the two learned canonists above mentioned, have been principally regarded (*See Burn's Ecclesiastical Law*, Preface).

The printed editions of Lyndwood's *Provinciale* contain also the collection of John de Athona. The latest edition is that of Oxford, 1679.

By the statute 25 Henry VIII., cap. 19, after the following recital—

“Where divers constitutions, ordinances, and canons provincial or synodal, which heretofore have been enacted, be thought to be not only much prejudicial to the king's prerogative royal, and repugnant to the laws and statutes of this realm, but also over much onerous to his Highness and his subjects,”—

It was enacted, that the said canons, constitutions, and ordinances, provincial and synodal, should be revised by a commission of thirty-two persons, to be appointed by the king; and that such of them as should be disapproved by the majority of the commissioners, upon such review, should from thenceforth be void and of none effect;

“Provided that such canons, constitutions, ordinances, and synodals provincial, being already made, which will not be contrariant or repugnant to the laws, statutes, and customs of this realm, nor to the damage or hurt of the king's prerogative royal, shall now still be used and executed as they were afore the making of this Act, till such time as they be viewed, searched, or otherwise ordered and determined by the said two-and-thirty persons, or the more part of them, according to the tenor, form, and effect of this present Act.”

Upon this, Sir W. Blackstone (1 *Com.* p. 83) observes, “As no such review has yet been perfected, upon this statute now depends the authority of the canon law in England.”

The words of Blackstone, in this passage, are not expressly limited to the *domestic* canon law of England, although in a former page (p. 89) he expressly rested all the strength of the *papal* laws in England (so far as they had not parliamentary sanction) “on immemorial usage and custom in some particular cases and some particular courts.” This is probably an example of that studied ambiguity under which Blackstone not unfrequently shelters himself in cases of difficulty; so that he may be quoted on both sides. “*Magno se judice quisque tuetur*.” To us it appears that the Act, according to legal principles of interpretation, extends only to domestic canon law; and we much doubt that the domestic canons used and executed afore the making of the Act, and not repugnant to law or prerogative,* derived any authority from the Act which they would not have possessed from reception and usage, if the Act had never been made.

It is to be observed, that traces still subsist of that ancient jealousy which severed the barristers of Westminster Hall and the advocates of Doctors' Commons. For example, Bishop Gibson's legal notions of ecclesiastical jurisdiction are very different from those of Blackstone, who has been accused by Mr. Bowyer, in his recent work on the English constitution, of an Erastian spirit. But Westminster Hall is armed with

* In the early history of our constitution, prerogative is often contrasted with law, not as being *illegal*, but as being beyond or above law.

Law.

the power of prohibition, and its doctrines must therefore be received as the sounder when they clash with those which are in vogue near St. Paul's.

Under a similar Act, since repealed (3 & 4 Edw. VI., cap. 11), the well-known *Reformatio Legum Ecclesiasticarum* was compiled, under the direction of Edward VI., but was not confirmed before his death and the accession of Queen Mary, and so fell to the ground.

The statute 25 Hen. VIII., cap. 19, was repealed by 1 & 2 P. & M., cap. 8, but was revived by 1 Eliz., cap. 1, and is now in force.

With the exception of certain rubrics,* the Thirty-nine Articles, and some regulations relating to the Common Prayer, which, though made in convocation, have been incorporated into the statute law by the several Acts of Uniformity, the only canons subsequent to the 25th year of Henry VIII. affecting generally the law of the ecclesiastical courts, and the rights, order, and discipline of the Church, were those which were passed in the convocation of the province of Canterbury, under James I., in the year 1603. These canons have never received parliamentary authority, but they were ratified by the king for himself, his heirs, and successors, and about the year 1605 were received and passed in the province of York.

With respect to the canons of 1603, it was held by Lord Hardwicke, in the case of *Middleton v. Croft* (Strange, 1056), that they do not, *proprio vigore*, bind the laity even in ecclesiastical matters. How far they are binding upon ecclesiastical persons, as the clergy, and *quasi* ecclesiastical persons, as churchwardens, appears to be not completely settled. Here, again, Sir W. Blackstone speaks with studied ambiguity: "It has been solemnly adjudged," says he (1 *Com.* p. 83), "upon the principles of the law and the constitution, that where they are not merely declaratory of the ancient common law, but are introductory of new regulations, they do not bind the laity, *whatever regard the clergy may think proper to pay them.*"

We believe the fact to be, that the ecclesiastical courts look to the canons of 1603—canons far more suited to the circumstances of the present times than the older canons which were made before the Reformation—as their rule in cases to which they are applicable, and that they are constantly cited in those courts. What length of usage will sanction the practice of a court—how far the constant adoption of a rule not opposed to any definite authority would be valid, though its origin might be traced to an unparliamentary canon of no very ancient date—these are questions which, though not unworthy of discussion, we shall not now stop to discuss. Many of the doctrines and practices which now prevail in our courts of common law and equity might easily be shown to be more recent than 1603, and to have no Act of Parliament for their warrant.

Among the more important repertories and treatises of English ecclesiastical law, we would mention (in addition to Lyndwood) Burn's *Ecclesiastical Law* (of which a valuable edition, by Dr. Phillimore, appeared London, 1842); Gibson's *Codex* (2nd edition, Oxford, 1761); and Wilkins' *Concilia Magnæ Britanniae et Hiberniae* (Lond. 1737). The last-named work has recently become expensive, and is not always easily to be met with.

* See Cardwell's *Synodalia*.

Canon law is not now generally studied upon the continent with the learning and diligence which formerly distinguished the canonists. The investigation of its principles has not, however, been neglected in modern times by Roman and Spanish writers; and a new stimulus was given to the pursuit in Prussia, towards the end of 1837, by the controversies connected with the suspension of the Bishop of Cologne. To those who may be content with a succinct account of its history and contents, we recommend the perusal of Doujat's *Histoire du Droit Canonique* (Paris, 1677); G. L. Boehmer's *Principia Juris Canonici* (8th edition, Göttingen, 1802); and Fleury's *Institution du Droit Ecclésiastique* (Paris, 1771). The excellent work of Fleury, however, is not confined to general canon law, but treats of the ecclesiastical law of France. It was placed in the Index by the Congregation of the Holy Office at Rome. Those who desire to penetrate more deeply into the subject are referred, in addition to the original sources, to Van Espen's *Jus Ecclesiasticum Universum* (Lovanii, 6 vols. fol., 1753). This work, for learning and historical accuracy, as well as for its practical utility and detailed exposition, has acquired the greatest celebrity among Roman Catholics. C. Butler characterizes it as "a work which, for depth and extent of research, clearness of method, and perspicuity of style, equals any work of jurisprudence which has issued from the press; but which, in some places where the author's dreary Jansenism prevails, must be read with disgust." That a Protestant should write upon canon and ecclesiastical law might seem an interference with a property rightfully belonging to others—

"O touch not Hector! Hector is my due"—

yet has J. H. Boehmer, in his *Jus Ecclesiasticum Protestantium* (5 vols. 4to. Halæ, 1756–1789), commented upon the *Decretals* with so much sagacity and clearness, that Camus criticises his work in the following manner: "*Il traite certains articles suivant les préjugés de sa communion; mais, par rapport aux objets sur lesquels les Protestans sont d'accord avec les Catholiques, il y a beaucoup à profiter de son jugement, de ses lumières, et de son érudition.*" Some authors have attempted to exhibit a kind of *natural* ecclesiastical law, by applying the light of reason to deduce propositions from the hypothesis of a church government, and from the elements contained in the notion of a church. On this subject may be consulted the treatise of Droste-Hülshoff, *Ueber das Naturrecht als eine Quelle des Kirchenrechts* (Bonn, 1822). Many of the most recent Roman-catholic writers on the canon law in Germany have comprehended in their works the ecclesiastical law of the Reformed Churches. As may well be imagined, modern canonists are still divided on the question of the relations which ought to subsist between Church and State; some being favourable to the rights of the Church within the State; others, to the dominion of the State over the Church. The two most popular German manuals of the present day belong to these opposite parties. Walter's *Lehrbuch des Kirchenrechts* (9th edition, Bonn, 1842) has been accused of Ultramontanism; while Eichhorn's *Grundsätze des Kirchenrechts* (2 Bände, 8vo; Göttingen, 1831, 1833) inclines to the principles which have been espoused by the Gallican Church.

Law.

Modern study of canon law.

LAW.

CHAPTER V.

THE LAW OF ENGLAND.

Law.
Preliminary observations.

270. IN considering a subject so vast in its extent as the law of England, it will be obviously impossible to do more, in a work of this kind, than indicate its general character, larger divisions, and more remarkable doctrines.

However scientific in its structure, the law of England is not a science; nor is it easy to lay down with any confidence its doctrinal principles, seeing that those principles are in so many cases restrained and limited in their operation. Sometimes this is effected by positive legislation, which necessarily disturbs the harmony and impairs the symmetry of every jurisprudential system; and sometimes it is the effect of a desire on the part of the courts to avert that injustice to individuals, which a rigorous and invariable enforcement of general principles could not fail to ensure. Another circumstance may be noticed as enhancing the difficulty of discussing our jurisprudence, in the brief and dogmatic fashion imposed by our limits. Unlike the Roman law and the modern codes of continental Europe, the law of England does not profess to be comprised within one volume, in which the solution of all legal questions is properly to be sought. For the most part, it is an aggregate of customs, maxims and doctrines, accidentally preserved by tradition, originating in a dim antiquity—of decisions of courts of justice and of opinions of learned lawyers—all challenging an authority inferior in no degree to the most recent and explicit declarations of legislative will.

So vast in its integrity, so complex in its parts, the law of England is not easily reducible to system, nor comprehensible within the space allotted to this chapter. The lawyer, therefore, will readily pardon omissions dictated by necessity; and all the more when he remembers that the very purpose of an essay, addressed to the general mind, forbids the use of those pregnant *formulae* and that significant nomenclature which secure in the scientific treatise both brevity and precision.

The subject may be distributed* under, (1) Public Law (*Jus publicum*); and, (2) Private Law (*Jus privatum*).

The first, Public Law, defines the position and operation of the sovereign power, the rights and duties of the subject, and the sanctions by which the enjoyment of those rights and the performance of those duties are secured and enforced.

The second, Private Law, determines the rights and duties of individuals in their capacities as such, whether as respects their mutual personal relations, or the acquisition, enjoyment and transfer of their property;

* A distribution authorized by Ulpian. *Hujus studii duæ sunt positiones, publicum et privatum. Publicum jus est quod ad statum rei Romanæ spectat; privatum, quod ad singulorum utilitatem* (Dig. 1. 1, 1, 2).

and constitutes sanctions vindicating the authority of its commands.

Law.

PART I.

The Public Law of England.

271. "The sovereignty of the British constitution," says Blackstone, "is lodged in the three branches of the legislature" (1 *Comm.* 51). By this term, sovereignty, is intended that supreme, absolute and irresponsible power in the state, to whose authority the obedience of every one is due, and the location of which determines the character and name of the government. In the view of English public law, this power, the source of all law, public and private, resides in the Parliament—in *hoc orbis terræ sanctissimo gravissimoque concilio*—a body consisting of three members, King, Lords, and Commons.

The presumed object which this power consults in its operation, is the increase of the physical and moral happiness of its subjects.* Such object every command of the supreme authority is supposed to intend. *Finis enim et scopus, quem leges intueri, atque ad quem jussiones et sanctiones suas dirigere debent, non alius est, quam ut cives feliciter degant* (*De Augm. Scient. lib. viii. cap. ii. aph. 5*). To obtain such an object, the sovereign power will protect life and property from violence, external and internal—will cherish domestic industry, encourage foreign trade, and whatever may tend to the increase of national prosperity—will, to the extent of its power, promote the social comfort and physical well-being of the people; and, acknowledging the claims and vindicating the authority of the church of Christ, will secure the means of enjoying religious instruction and consolation to all its subjects.

The operations, then, of the sovereign power may be presumed to relate to the following six subjects:—

I. Justice; II. Defence; III. Social Economy; IV. Foreign Relations; V. Finance; and, VI. Religion.

272. I. JUSTICE.—By justice, we understand the performance of that sacred duty, recognised alike by ethicist and jurist, *suum cuique tribuere* (*Inst. I. 1, 1*). To induce its discharge, the sovereign power, pursu-

* The ablest casuist of the Anglican church takes a similar view:—*Potestas autem publicæ jurisdictionis, ordinatur primariò in bonum publicum ipsius communitatis, in bonum vero personæ tali potestati præditæ, id est, ipsius magistratûs, non nisi secundario et consequenter, quatenus nimirum utile est principi, ut respublica floreat* (Sanderson, *De Oblig. Conscien. Prel. 7, s. 4*). And so says the church herself: "The government of a prince is a blessing of God, given for the commonwealth, especially of the good and godly; for the comfort and cherishing of whom, God giveth and setteth up princes, and on the contrary part, to the fear and punishment of the wicked" (*Homily 21, against wilful Rebellion and Disobedience. Homilies, p. 509. Oxf. 1822*).

Law. ing its own mission, creates and defines rights,* institutes appropriate sanctions, and provides arrangements by which its definitions are interpreted and applied, and its sanctions enforced. Its operations in this respect are — (I.) *legislative*, which involves the making of laws; and, (II.) *administrative*, which relates to the application and execution of such laws.

Legislation. (I.) Legislation† is the appropriate business of the Parliament, in quality of its sovereign authority. By this it can make, confirm, abrogate, repeal,‡ restrain, enlarge, revise, and expound all manner of laws whatever, and can bind the king and the people,§ and all

* That is, *legal* rights; giving, as we must presume, to moral rights the efficacy and force of legal obligation.

—As Nature's ties decay,
As duty, love, and honour fail to sway,
Factitious bonds—the bonds of wealth and law—
Still gather strength and force unwilling awe.

I have ventured to substitute *factitious* for *fictitious*, as more consistent with Goldsmith's meaning.

† By *legislation*, the French lawyers intend the articular or codified law, as distinguished from the law contained in judicial decisions. This last they style *jurisprudence*. (Dupin, *De la Jurisprudence des Arrêts*. Paris, 1822).

‡ No law is exempt from the liability to abrogation, even should it contain a provision to the contrary. The 42 Edw. III. cap. 2, enacted that all statutes contrary to Magna Charta should be void; yet portions of Magna Charta have been repealed, and the repealing statutes have been constantly held in force (Jenk. Cent. 2).

§ The doctrine that a parliamentary law, or act of parliament, as it is often called, is of supreme authority, dates its origin from the middle or transition age of our constitution. It is now firmly established that Parliament can do no wrong, although "it may do several things that look pretty odd," as, for example, making Malta in Europe, and a woman a mayor or a justice of the peace (*per* Holt C. J., 2 Jon. 12).

Language, to say the least of it, highly indiscreet, has been used by learned judges in derogation of the authority of Parliament, whilst in exercise of its chiefest function, legislation. Hobart C. J. declares that a statute against natural equity, making, for instance, a man judge in his own cause, was void (*Day v. Savage*, Hob. 87); whilst Lord Coke considered that statutes were controlled by the common or traditional law, which adjudged them void when against common right and reason (*Bonham's case*, 8 Rep. 234.) Holt C. J. expressed his belief that this saying of Lord Coke was not extravagant, but was a very reasonable and true saying (*City of London v. Wood*, 12 Mod. 687). It is further asserted, in a book of authority (8 Bing. 491), *The Doctor and Student*, that an act of parliament to prohibit alms-giving is void—a doctrine which, if true, would invalidate the Statute of Labourers (23 Edw. III.), which contains (cap. 7) a provision to that effect. Lord Chancellor Ellesmere styles Coke's position "a paradox, which derogateth much from the wisdom and power of parliaments, that when the three estates—King, Lords, and Commons—have spent their labour in making a law, three judges on the bench shall destroy and prostrate all their pains, advancing the reason of a particular court above the judgment of all the realm" (*Observations on Lord Coke's Reports*, p. 11). Sir William Herle, who was Chief Justice in the reign of Edward III., observed, that "some acts of parliament are made against law and right, which they that made them perceiving, would not put them into execution; for it is *magis congruum* that acts of parliament should be corrected by the same pen that drew them, than be dashed to pieces by the opinion of a few judges" (8 Edward III., cited 8 Rep. 234). When the assertion of Lord Hobart was once quoted in court, the judges observed, that they would not treat a statute as void unless it were clearly against natural equity, and that they "would strain hard" rather than treat it as void (10 Mod. 115). An act was passed in the reign of Charles II. "concerning the size of carts and wagons, with many penalties upon the transgressors; and yet, when it appeared that the model prescribed by the act was not practicable, the judges (in their circuits) gave directions not to execute the Act" (Burnet, *Hist. own Times*, 428. Lond. 1838). The counsel for the crown, in Sir Edward Hales' case, relied on this in their arguments for the dispensing power. It is not difficult to apprehend the conclusions to which these doctrines point; and

the inhabitants of the channel islands, Isle of Man, and all colonies and plantations to the realm belonging, being subjects of the same realm. By laws, we would be understood to mean such commands of the sovereign as are *rules* of action, excluding from the signification of the term, those commands which are called, in common parlance, laws, only because they resemble them in their formal parts—in short, in their mode of expression, such as money acts, acts of attainder, and others.

In point of *fact*, the corpus of English jurisprudence does not consist wholly of parliamentary laws; and this for several reasons. In the first place, legislation was a function the supreme power exercised in early times rarely and imperfectly; and thus customs, unauthorized in their origin, supplying the deficiencies of governmental legislation, came in time to acquire the authority and obligation of laws—*vetustas pro lege semper habetur*. Secondly, as far as we know, Parliament had not always the supreme power, or the right of legislation, which latter formerly resided in the prince, though probably under limitations and restrictions which we cannot now perfectly apprehend. Thirdly, the interpretations of the laws have been themselves accepted as laws—sources whence legal principles may be drawn, and authorities by which legal controversies may be determined.

Thus we have still subsisting the ancient traditional law, which was the jurisprudence of our early age, when, like the Romans before the Twelve Tables were composed, we were governed, *incerto magis jure et consuetudine aliquâ, quam per latam legem* (Dig. I. 2, 2, 1). We have also our ancient charters, and the volumes of our judicial decisions; and all these, as well as parliamentary laws, constitute what we designate the law of England. Still, although not *created*, these laws are *permitted* by the power, now supreme, and may be taken as deriving their *present* authority from its sanction, expressed or implied: *Qui non vetat, si potest, jubet*.*

The law of England† may be considered as a system

although, as we shall presently observe, our judges have often covertly, and to the advantage of the subject, defeated the intentions of the legislature, the establishment of the principles so broadly laid down by Coke, Hobart, and Holt would be destructive of the very foundations of our constitutional polity. These principles suggest two reflections.

(1.) Their source we can readily understand to have been the notion, which our early constitutional history warrants, and which long lurked in the minds of our lawyers, that a statute might supply the defects, but could not repeal the provisions of the common law.

(2.) The duty of a judge called on to administer a law he believes contrary to the law of God (taking an extreme case) is tolerably clear; and our history has not been wanting in examples of judicial firmness under such circumstances. To resign an office he cannot conscientiously serve is the plain dictate of conscience. As to the citizen, of whom it has been sarcastically said, that he has nothing to do with the laws but to obey them, his obedience is clearly measured by his conscience; but this measure must not be stretched beyond its legitimate extension. He conceives the command of his sovereign to conflict with the commands of his God. The command, then, of the sovereign he disobeys, but cheerfully endures the penalty annexed to such disobedience.

* This maxim is borrowed from that of Aristotle, who considers the law as directing that which it does not prohibit, *ἡ μὴ κελεύει, ἀπαγορεύει* (*Ethic. lib. 1. cap. 11*).

† The enlightened patriots of the Revolution knew that true liberty could consist only with wise laws, and, in declaring "the true ancient and indubitable rights and liberties of the people of this realm" (1 Wm. & M. stat. 2, cap. 2), were careful to assert them by a new law; "the laws of England" being, as they said on another occasion, "the birthright of the people thereof" (12 & 13 Will. III. cap. 2). Two philosophers—*magis pares quam*

Law. framed in reference to certain principles. Its operation, by its very nature, is to afford those principles protection. In their relation to the law, we style them *principles*; in their relation to those who owe allegiance to the law, we style them *rights*. They are fitly styled *principles*, scientifically speaking, because they are the ultimate facts into which all the phenomena of the law are resolvable. With equal fitness are they called *rights*, practically speaking, because they result from obligations imposed by competent, that is, sovereign, authority, and are guaranteed by adequate sanctions.

Protection. (1.) First, then, the law of England assures its *protection* to every English subject who has not, by his misconduct, forfeited his claim to such protection. Arbitrary power is here unknown. Here, the empire of the laws is a phrase of significant import. Except in conformity with law, no man can be outlawed, that is, "barred to have the benefit of the law;" nor exiled; nor can he, against his will, be sent, even as an ambassador out of the land (*Magna Charta*, cap. xxix.; 2 *Inst.* 47), nor, in any office, to any of the colonies, which might, in reality, be no more than an honourable banishment. In order the better to preserve to the people the benefits of the law, no subject, inhabitant here, can be sent prisoner to the channel islands or to any of our transmarine possessions (31 Car. II. cap. 2, s. 12), and thus deprived of legal protection.*

Personal security. (2.) Secondly, the law of England assures to every man the enjoyment of *personal security*, and suffers him to commit even homicide in defence of his life or limbs. No freeman shall be destroyed, except according to law (*Magna Charta*, cap. xxix.); none prejudged of life or limb contrary to law (5 Edw. III. cap. 3); nor put to death without being brought to answer by due process of law (28 Edw. III. cap. 3). So dearly is life tendered by our polity, that it protects it, by "strict statutes and most biting laws," not only from the hand of violence, but even from the effects of misfortune, in establishing a provision for the support of those unable to labour, or unable, in return for their labour, to obtain the means of subsistence.†

similes (Locke, *On Gov.* part ii. s. 57; Mackintosh, *Disc. Law Nat.* p. 59)—have established the wisdom of their proceedings, and exposed, at the same time, the folly of those pretenders to philosophy, who

"Led their wild desires to woods and caves,
And thought that all but savages were slaves."

* It was one of the articles of impeachment against Lord Clarendon, that he had advised and procured divers of his Majesty's subjects to be imprisoned, against law, in remote islands and garisons, to prevent their having the benefit of the law (Hume, *Hist. Eng.* App., note to chap. xiv.). To avoid the consequences of an *Habeas Corpus*, Harrington, that dreamy and amiable enthusiast, was, assuredly illegally, hurried off from the Tower to St. Nicholas' Island, opposite Plymouth (see his *Life*, by Toland; Harrington's *Works*, pp. 36 et seq. Lond. 1700). Evasions of the numerous laws passed to protect personal liberty were common in those times. In vain were the laws multiplied in every shape, to meet every possible case; the evil they were intended to repress continued in full vigour; just as bribery, in ancient Rome, survived all the legislative efforts to subdue it: *De nullo crimine tam multæ apud Romanos latæ leges, nec ullæ minus observatæ.*

† Louis XIV., in his advice to his son, observes: *Si Dieu me fait la grace d'exécuter tout ce que j'ai dans l'esprit, je tâcherai de porter la félicité de mon règne, jusqu'à faire en sorte, non pas à la vérité qu'il n'y ait plus ni pauvre ni riche, car la fortune, l'industrie, et l'esprit laisseront éternellement cette distinction entre les hommes, mais au moins qu'on ne voie plus dans tout le royaume, ni indigence, ni mendicité; je veux dire, personne, quelque misérable qu'elle puisse être, qui ne soit assurée de sa subsistence, ou par son travail, ou par un secours ordinaire et réglé.*

(3.) The *personal liberty* of its subjects is also regarded by the law of England. Except by law, no man shall be taken, imprisoned, exiled, or in any manner destroyed (*Magna Charta*, cap. xxix.); a provision of our Great Charter repeated in a variety of subsequent statutes, which in effect declare that no man shall be taken or imprisoned except by due process of law (5 Edw. III. cap. 8; 25 Edw. III. st. 5, cap. 4; 29 Edw. III. cap. 3). The Petition of Rights enacts that no freeman shall be imprisoned or detained without cause shown (3 Car. I. cap. 1), an enactment rendered of inestimable value by the *Habeas Corpus* Acts (16 Car. I. cap. 10; 31 Car. II. cap. 2; 56 Geo. III. cap. 100), the effect of which is to enable the individual detained or imprisoned to bring under the notice of an independent and enlightened judicatory the cause or alleged ground of his imprisonment or detainer, and, constituting the judicatory judges of such cause, to enable them, as they think fit, to remand him to his former custody, to discharge him, or to admit him to bail.* A gaoler is not bound to detain a prisoner unless the warrant of commitment express the cause thereof; "and this doth agree with that which is said in the holy history (*Acts*, xxv. 27), 'It seemeth to me unreasonable to send a prisoner, and not withal to signify the crimes laid against him'" (2 *Inst.* 53).

(4.) The law of England protects every individual in the *enjoyment* of his *property*. No freeman shall be disseised or divested of his freehold, deprived of his liberties, free customs, or franchises, nor disinherited, nor shall his lands or goods be seized into the king's hands, which is against the Great Charter and the law of the land, unless he be duly brought to answer, and be prejudged by course of law (*Magna Charta*, cap. xxix.; 5 Edw. III. cap. 9; 25 Edw. III. st. 5, cap. 4; 28 Edw. III. cap. 3). Sacred as the rights of property

"Political society," says an able American statesman, "owes perfect protection to all its members, in person, reputation, and property; and it also owes necessary subsistence to all who cannot procure it for themselves. Penal laws to suppress offences are the consequences of the first obligation; those for the relief of pauperism, the second: these two are closely connected; and when poverty is relieved and idleness punished—whenever it assumes the garb of necessity, and presses on the fund that is destined for its relief, the property and persons of the more fortunate classes will be found to have acquired a security that in the present state of things cannot exist" (Livingston, *System of Penal Law for Louisiana*, Introd. Pref. p. 318. Philad. 1833). The establishment of a system of poor-laws, mischievous as its mal-administration has been, has, without doubt, greatly contributed to secure the general tranquillity of this country. The northern rebels in the reign of Henry VIII., who styled themselves "the Pilgrimage of Grace," told the Duke of Norfolk that "Poverty was their captain, the which, with his cousin Necessity, had brought them to that doing."

The object of every good system of poor-laws is that stated by the Presbyterian Reformers of Scotland, in their *First Book of Discipline*. "We are no patrons," they say "to sturdy and idle beggars, who, running from place to place, make a craft of begging, for these we think must be compelled to work, or else be punished by the civil magistrates; but the poor widows, the fatherless, the impotent maimed persons, the aged, and every one that may not work, or such persons as are accidentally fallen into decay, ought to be provided for" (Spotswood, *Hist. Ch. Scotl.* book iii. p. 158. Lond. 1677).

* See Greening's *Leading Statutes*, p. 63. Burnet tells a singular story respecting the manner in which the 31 Car. II. was passed. According to his account, the Lords would have rejected the bill, but Lord Grey, who was one of the tellers, relying on the inattention of Lord Norris, who was associated with him in his duties, counted one fat Lord for ten, and thus was the majority obtained (Burnet, 321). The measure was greatly opposed in the Lords.

Law.

are held, there are certain cases in which the supreme power, exercising its extreme authority, and preferring the general good to that of an individual, disregards private rights, in consulting the public advantage. Thus, it levies taxes, compelling the citizen, out of his personal possessions, to contribute to the necessities of the state; and thus it will carry a public road through the private property of individuals, contrary to their wishes. This it can do, because it is the supreme power, and knows no limit to its authority. "By the common law, every man may come upon my land for the defence of the realm. And in such case, upon such extremity, they may dig for gravel for the making of bulwarks, for this is for the public, and every one hath benefit by it. And in such case the rule is true, *Princeps et respublica ex justa causa possunt rem meam auferre*" (12 Rep. 13).

Security of
these prin-
ciples en-
sured by

perma-
nence of
laws re-
sulting
from

political
rights.

The fran-
chise.

Having enumerated the principal rights resulting from the law of England, we have now to ascertain in what way their enjoyment is ensured. This security of enjoyment is consistent only with, first, the permanence of the laws from which the rights result, and, secondly, the establishment and due enforcement of adequate sanctions.

As to the permanence of the laws, it may be observed, that the right to constitute and abrogate what laws it pleases, is involved in the very idea of sovereign power (Boullenois, *Traité de la Pers. &c. des Lois*, i. 2-4); and it is therefore competent to the sovereign power, at its discretion, and without regard to the interests of its subjects, by abrogating laws, to deprive them of any right they may possess. Still, it is presumed, that it will not, from mere wantonness, so prejudice its subjects; and that, if ever it deprives them of the rights it has conferred, it does so either for their advantage or through its own ignorance and inadvertence. Such is the reverent supposition of English law. It therefore assures to the subject certain powers of instructing the sovereign power in respect of the operation and consequences of its acts, and even of remonstrating with it on the subject of its conduct. By law, also, the subject has, under certain restrictions and to a certain extent, the right of nominating the body by whom sovereign power is exercised. The rights to which we refer may, from their public character, be designated *political rights*, and their exercise yields powerful protection to those fundamental rights we have already discussed.

(1.) Of political rights, we may rank as the most important, the right of designating the persons who constitute the Commons House of Parliament—a member of that honourable body possessing the supreme power in our state.* A right, in the exercise of which

* The possession of this right by a large number of the people has given birth to the absurd notion of the *sovereignty of the people*. The absurdity of the phrase will be abundantly manifest when we consider, first, that the right of election is possessed, not by the whole, but only by a portion of the people; secondly, that the House of Commons does not exclusively exercise sovereign power, but only shares in its exercise; thirdly, that there is a wide distinction between a right to do a thing, and a right to specify by whom the thing shall be done. The Earl of Warwick, as it has been well remarked, was called *king-maker*; was he therefore to be called *king*? The disposition of property through powers of appointment, to be exercised by those who never enjoyed the property, is a familiar illustration of the position. In jurisprudence, the maxim, *nemo dat quod non habet*, must be taken with some qualification.

In the United States of America the matter is far otherwise. There, "the people is the *only* source of power." * * * The body

the community is so largely interested, and in itself so readily made subservient to the evil purposes of self-seeking politicians, should be confided to none but trustworthy hands.* But the capacity of fitly exercising this important right does not manifest itself so palpably as to render our arrangements in this matter as perfect as could be wished. All that can be done is to confine the right, as far as may be, to those whose property and age may be presumed to exempt them from the temptations of extreme poverty and the indiscretion and inexperience of youth. This, according to the *actual* theory of the constitution, is the chief motive of the restrictions on the suffrage. There are other motives, which have reference to considerations foreign to our subject. The principal motive is that which we have stated. It must not, however, be supposed that such a motive is in accordance with the *historical* theory of the constitution, which has relation to the past and its opinions—a past, ignorant of natural rights, or indifferent to their assertion, and which, in limiting the franchise to the wealthy, limited it to the class whose interests it was alone accustomed to respect. A freehold estate of 40s. a-year was, in the reign of Henry V., when the freehold qualification was established, a considerable property.

(2.) The right of petitioning the king and the parliament is also a valuable privilege of Englishmen, as enabling them to make known their grievances to those capable of redressing them. The right of the subject to petition the king is *asserted* in the Bill of Rights, which further declares illegal all commitments and prosecutions for such petitioning (1 Wm. & Mary, stat. 2, c. 2). Whether this provision repealed the Act 13 Car. II. st. 1, c. 5, against tumultuous petitioning, and which threw great impediments in the way of petitioners, is doubtful (Rex v. Lord George Gordon, Douglas, 572; and Speech of Mr. Dunning, in the House of Commons, *Ann. Reg.* for 1781, part. ii. pp. 143-4). The Act has, however, fallen into desuetude.

(3.) The right of the subjects to have arms for their defence, suitable to their condition, and as allowed by law, is also asserted by the Bill of Rights, in which,

of electors, *after* having designated its magistrates, direct them in everything that exceeds the simplest ordinary business of the state" (De Tocqueville, *On Democracy*, by Reeve, vol. i. p. 75). Similar is the description of the German democracy by Tacitus (*De Mor. Germ.* xi). The nation was formed by the union of freemen, from whom all power and every legislative enactment proceeded (Savigny, *Geschichte des Römischen Rechts im Mittelalter*, b. 1. kap. iv. s. 157. Heidelb. 1815; see also Eichorn, *Deutsche Staats und Rechtsgeschichte*, b. 1, s. 50. Gött. 1821)—the people actually exercising the functions of government, and not simply designating those by whom they should be exercised. When the representative system was introduced among the Germans, and the Scabini or Echevins became the judges (Savigny, *ut sup. cit.*) and delegates of the people (Palgrave, *R. and P. Eng. Comm.* i. 89), unless, indeed, which was certainly the case for a time, the people retained their authority in more important matters, or, like the American demos, continued in such matters to *direct* the magistrate in the performance of his office, the sovereignty of the people vanished from the constitution of Teutonic polity.

* There is a singular exposition of the duties of a popular representative contained in a curious alliterative poem on the deposition of Richard II., edited by Mr. Wright, for the Camden Society (p. 28), 1838. The popular party in the 21st Richard II. are supposed to argue after this fashion: "We be servants and take salary, and [are] sent from the shires to show what them grieveth, and to parle [speak] for their profit and pass no further, and to grant of their gold to the great wattis [the meaning of which I do not know] by no manner [of] wrong way, but if war were." I have translated the passage.

Law.

Sove-
reignty of
the people.

Right of
petition-
ing.

Right to
arms.

Law. however, it is limited to the case of Protestants—a limitation now practically abolished. In such a state of advanced civilization as that in which our lot is cast, where the authority of the laws is so effectually established, and, by an improved police, so speedily enforced, such a provision as this has but little value. But it was otherwise of old; and its preservation in our statute-book is worth something, as a record of that love of liberty, and resolute assertion of lawful rights, which has given so much vigour and dignity to our national character—itself the source of our national greatness.

*Sic fortis Etruria crevit,
Scilicet et rerum facta est pulcherrima Roma.**

Enforce-
ment of
sanctions. It is, however, in the enforcement of proper sanctions that the chief security of our legal rights consists. This is accomplished either by the act of the parties possessing such rights, or by operation of the law itself.

By act of
parties. If, in developing the *rationale* of our jurisprudence, we were permitted to indulge our imagination, we should probably represent the defence of an invaded right as devolving on its possessor in the first instance, and consider the interference of the law as occurring only when this possessor was incompetent to the attempted defence. But the *facts* of our law point to principles of a precisely opposite character. Redress by the individual is permitted only when the law is defective: *when the law fails, then the individual acts*. The law (if we may so idealize sensible agencies) is tardy in its operations—scrupulous—deliberative—fearful lest its precipitation should defeat its own pur-

* It would be improper to dismiss this subject without exposing the absurd opinion that the constitution justifies a resistance to government under certain circumstances. Such a justification is wholly alien to the very nature of government, which is founded on the notion of obedience to authority. To speak of a sovereign power as legally responsible to any body or person—as being legally subject to duties, the infraction of which involves a forfeiture of its authority, is plainly absurd; for whatever is responsible must be responsible to some one, who must, therefore, be superior, and, *pro tanto*, sovereign. Again, legal duties imply legal means of enforcing them; and how shall we style that which is subject to anybody, or any one, a sovereign? "The governor," says Archbishop Whately, "is bound to make a good use of his power, no less than his subjects are to obey him; and he is accountable to God for so doing—but not to them; for if this merely conditional right to obedience be once admitted, it must destroy all government whatever" (*Bampton Lectures for 1822*, p. 292). The philosophical historian states the case forcibly: *Si, ubi jubeantur quærere singulis liceat, pereunte obsequio, imperium etiam intercidit* (Tacitus, *Ann.* lib. i. c. 83). The morality of resisting a government is a question of *policy*, and not of *law*; and on such a ground is it put by one who had some practical experience in the matter. Alluding to this nice question of resistance, Ludlow observes, that "we ought to be very careful and circumspect in that particular, and at least be assured of very probable grounds to believe the power under which we engage to be sufficiently able to protect us in our undertaking; otherwise I should account myself not only guilty of my own blood, but also, in some measure, of the ruin and destruction of all those that I should induce to engage with me, though the cause were never so just" (*Memoirs*, p. 235). The prudence of attempts against the authority of a government appears to have been, in Bishop Burnet's estimation, one element necessary to their morality (*Hist. own Times*, p. 404). On this subject Mr. Fox has made some sensible remarks (*Life of James II.* pp. 175-6). "The true interest of this kingdom," says Lord Clarendon's editor, "is supported *non tam famâ quam sua vi*; its own weight still keeps it steady against all the storms that can be brought to beat upon it, either from the ignorance of strangers to our constitution, or the violence of any that project to themselves wild notions of appealing to the people out of parliament, as it were to a fourth estate of the realm" (Preface to *Hist. Great Rebell.* vol. i. p. xiii. Oxford, 1807).

poses—*cunctando restituit rem*. But if the law suffered redress to be sought nowhere but at the feet of its tribunals, grievous wrongs, not to be compensated, might be perpetrated with impunity. In certain cases, therefore, it is permitted to the individual to do that justice to himself which the law cannot do for him; in short, borrowing an expression from vulgar speech, to take the law into his own hands.*

In defence of his life, limbs, or property, the individual may repel force by force, being careful to act on the defensive, and to use no violence not necessary for the purposes of defence. The defence of such as stand in the relations of husband and wife, parent and child, master and servant, are equally licensed by law, subject to similar restriction (Gregory and another *v.* Hill, 7 *T. R.* 299; Barfoot *v.* Reynolds, Str. 953).

It is not, however, by an attack on his person alone, that a man's life or health may be endangered; and, with respect to that large class of personal injuries, embraced by the comprehensive term "nuisance," the law empowers the sufferer to right himself without having recourse to legal remedies. But the term "nuisance" imports a great deal more than injuries threatening the *lives* of individuals; it is significant also of many things which damage individuals in person or property, or diminish their comforts. In all these cases, without awaiting the dilatory process of legal vengeance, it is competent to the sufferer himself to *abate* (*abattre*, to pull down,) the nuisance; and "the reason why the law allows this private and summary method of doing one's self justice, is because injuries of this kind, which obstruct or annoy such things as are of daily convenience and use, require an immediate remedy, and cannot wait for the slow progress of the ordinary forms of justice" (3 *Comm.* 5). It is also competent to every person deprived by another of goods or chattels, or of his wife, child or servant, to claim and retake them, wherever he happens to find them, "since it may frequently happen that the owner may have this only opportunity of doing himself justice: his goods may be afterwards conveyed away or destroyed," &c. (3 *Comm.* 4.) A similar remedy is possessed by parties of whose lands and tenements another party has taken possession. The power of distress for the non-payment of rent or otherwise, which is sufficiently explained in another part of this work, is another species of redress exercised by the individual.

As a principle, however, invasions of rights established are properly cognizable and to be redressed by law. In short, "every subject, for injury done to him, *in bonis, in terris, vel personâ*, by any other subject, be he ecclesiastical or temporal, free or bond, man or woman, old or young, or be he outlawed, excommunicated, or any other, without exception, may take his remedy, and have justice and right for the injury done to him, freely without sale, fully without any denial, and speedily without delay" (2 *Inst.* 55-6).

We have now to consider how the law deals with invasions of the rights to which it has given birth. It treats these invasions as divisible into two classes; one

* The ancient law was founded on a different principle (Palgrave, *R. and P. Eng. Comm.* p. 225, *et seq.*; Glanville, lib. ix. c. 8; Robertson's *Charles V.*, *View*, note xx. vol. i. p. 249. Lond. 1827). Still in criminal matters the Anglo-Saxon polity recognised the principles in my text, for it was a general rule that no one could take vengeance into his own hands, until he had demanded justice in vain (*In. 9*; *Anc. Laws and Instit. of England*, p. 47, folio; see also *Ælf.* 42; *ib.* p. 40; and some observations of Mr. Allen, *Rise and Growth of Royal Prerog.* p. 118, *et seq.*).

Law.

Self de-
fence.Abatement
of nui-
sances.Sanctions
enforced
by opera-
tion of law.Crimes and
wrongs.

Law.

ow
treated by
law.Punish-
ment of
crime.Satisfac-
tion of
wrongs.Adminis-
tration of
law judi-
cially.Ordinary
judica-
tories.

The judges

class comprehending those which it considers to be injuries affecting the state; the other class includes what it considers to be injuries affecting individuals only. The first class we would call *crimes*; the second, simply *wrongs*. In dealing with *crimes* the law *punishes* the offender, its sole motive being the repression of crimes. In dealing with *wrongs*, the law compels the offender to redintegrate the party injured, its motive being chiefly the satisfaction of the sufferer. To enforce moral justice is its aim in the latter case; in the former, to protect the community—*ne quid respublica detrimenti capiat*. Offences may belong to both classes; the distinction in point of fact, though not in theory of law, relates less to the actual nature of the act done, than to the sanction attaching on its performance. Thus, if one meet another in the street, and beat him, this is a breach of the public peace, and a crime—an offence against the state itself. It is therefore *punishable* by fine and imprisonment. But the offender also does a wrong to an individual, who shall have his action of trespass against him, and shall recover *satisfaction* for his injury in the shape of damages.

When a crime is committed, the law will reach it if it can, and punish the criminal. An injury to the state cannot be compensated; it is itself a crime to check the progress of public justice by compounding a felony or an information under a penal statute. Not so with a wrong: here it is permitted for him injured to arrange with his injurer the terms of a satisfaction of the injury done, and this arrangement will be enforced by law. The arrangement may be of two kinds: (1) the party injured may agree to accept a certain sum or other thing as a compensation—an arrangement technically styled *accord and satisfaction*; or, (2) the parties may agree to refer their dispute to an *arbitrator*, by whom, sitting as a domestic forum, the questions of injury and compensation may be determined. We have now to consider the enforcement of legal sanctions by the operation of the law itself, that is by its administration.

(II.) The administration of the law may be exercised judicially or ministerially. The judicial administration of the law is the appropriate duty of the judicatory, or that body or person whom the supreme power has clothed with judicial authority. This judicial authority extends to the determination of all controversies which have for their subject an alleged infraction of legal rights. To decide whether this infraction has taken place in the manner alleged, is the duty of the judicatory; and that this duty should arise, it is necessary that the infraction of legal rights should be controverted as a fact (Vinnius, in *Inst.* IV. xvii.). Unless it be so controverted, the judicatory have no jurisdiction.

In England there are ordinary and extraordinary judicatories. We will consider the ordinary judicatory in the first place.

A controversy may be single or complex in its character; it may involve questions of law, or questions of fact, or both. The parties may agree on the law and differ on the facts, or may agree on the facts and differ on the law, or may differ on both facts and law. Whenever the question raised is of a legal character simply, and does not require the investigation of facts, then is it to be decided by that branch of the judicatory styled, emphatically, *the judges*. Such questions as those which relate to the observance or non-observance by one or other party

of the prescribed forms of stating his complaint or defence, or to the competency of the particular tribunal whose aid has been evoked, these and the like are questions purely of law, and involving no disputed facts. So also is the question whether, assuming the facts stated by the complainant (or defendant) to be true, those facts, in the view of the law, establish his complaint (or defence) or not. Here there are no disputed facts; this a question purely of law, and therefore the judges are to try it. But when the question is of facts, wholly or partially, then have they no proper jurisdiction (Abbot of Strata Marcella's case, 9 *Rep.* 25). Such questions are referrible to another branch of the judicatory, known to us as *the jury*, who, in deciding them, may adopt either of two courses. They may, after such evidence has been adduced to them as the parties desire, subject to certain restrictions imposed by law, forming in their own minds a judgment as to the facts of the case, and determining for themselves what is the legal character of those facts, that is, what is the law applicable to them, decide generally that the complainant (or his opponent) has made out his case, and thus decide the controversy in his favour; or they may find specially the facts as proved before them, and refer to the learned judges the task of drawing from those facts the conclusion warranted by law, thus relinquishing to them the task of terminating the controversy. As the jury, however, are not presumed to be acquainted with law,* they are presided over by a

* Coke says, that "as the question between the parties is twofold, so is the trial thereof; for either it is *questio juris* (and that shall be tried by the judges upon a demurrer, special verdict, or exception, for *cuiuslibet in sua arte perito est credendum et quod quisque norit in hoc se exerceat*; and it is commonly and truly said, *ad questionem juris non respondent iuratores*), or it is *questio facti*. And the trial of the fact is in divers ways: of these a trial by twelve men is most frequent and common" (*Co. Litt.* 125 a). A very slight consideration will show that this statement is not precisely accurate; for although the jury never decide questions purely of law, yet, in returning a general verdict, they assuredly decide questions not simply of facts; they decide the facts, and they decide the law arising out of the facts—*ex facto oritur jus* (*Co. Litt.* 226). And to this purpose speaks our great master: "Where the enquest (jury) may give their verdict at large, if they will take upon them the knowledge of the law upon the matter, they may give their verdict generally." (*Litt. lib. iii. s. 368*). But formerly it was "dangerous for them so to do; for if they do mistake the law, they run into the danger of an attain; and, therefore, to find the special matter is the safest way, where the case may be doubtful" (*Co. Litt.* 228 a). Still they may, if they please, decide the law as well as the facts, and this not *incidentally*, as Mr. Hargrave contends (*Note, Co. Litt.* 155 b), but *directly*.

The true theory of the trial by jury, if the theory is to be gathered from the facts, is very different from that stated by Coke. That theory, which distinguishes so rigidly between the capacity to decide facts and the capacity to decide law, would consider the special verdict, finding facts only, as the rule, and the general verdict as an exception, in which the jury deviate from their orbit, and their eccentricity is excused by law. But this is not so, as will appear from a consideration of the Statute of Westminster II. (13 Edw. I.) cap. xxx., which permits special verdicts to be taken in certain real actions. "Also it is ordained that the justices ordained to take assizes may not *compel* the jurors to say precisely, if it be disseisin or not, should they *wish* to find the truth of the *fact*, and to seek the assistance of the justices; but if, of their own accord, they will say that it is disseisin or not, their verdict shall be taken at their peril." At first it was doubted in what cases special verdicts could be given; but it was at last held, "that in all actions, and upon all issues, the jury *might* find the special matter of fact pertinent and tending only to the issue joined, and thereupon pray the discretion of the court for the law" (2 *Inst.* 425). And, in the case in which this was decided, it was said that the law will not *compel* the jurors, who have not the knowledge in the law, to take upon them knowledge of points in the law (Downman's case, 9 *Rep.* 13):

Law.

and the

General
verdict.Special
verdict.

Law. judge, who informs them in all matters of law connected with the case; but they are at liberty to accept or reject his exposition as an element of their decision. In criminal controversies, if the person charged with the offence is, through their verdict, declared innocent thereof by the jury, he is ever quit and discharged of the accusation, nor can he be again tried upon it. If the jury find him guilty, he is said to be convicted, and must suffer the penalty which the law has annexed to the offence of which he has been convicted, unless the judgment against him be reversed, which is sometimes done on a writ of error, which is a form of proceeding by which he appeals from the inferior judicatory to a superior tribunal. Thus, if he be tried before the itinerant justices, hereafter mentioned, he may appeal to the Court of King's Bench; and if he be tried before that tribunal, he may appeal to the House of Peers, the court of last resort. These courts can reverse the judgment, if there have been any notorious defect *in regard of law* in the proceedings. A royal pardon will also relieve him from his punishment, or the crown may mitigate his sentence. In civil controversies, the decision of the judicatory is subject to reviewal by these

Writ of error.

so that, in fact, the liberty of finding a special verdict was given "for the relief of the jurors, who, upon their oath, shall not be compelled to find, at their peril, things doubtful to them in law" (*Ibid.*); an error in their verdict subjecting them to attain. This method of punishing jurors has fallen into desuetude, and, perhaps, in many cases, never pretended to be a remedy (per Lord Mansfield, *Bright v. Eynon*, 1 Burr. 390). It was, however, frequently resorted to in ancient times; and the granting, by the king's courts, of a writ of attain against a jury, for giving a verdict contrary to their oaths, is commanded by the Statute of Westminster I. (3 Edw. I.) cap. xxxviii. Sir Thomas Smith ascribes the disuse of attain to the difficulty of obtaining a conviction (*De Rep. Ang. lib. iii. cap. 2*; and see Hudson's *Treatise on the Star Chamber*, I. s. 4.). It must be recollected after all, and in forms we see preserved the original theory, from which modern practice often deviates, that when the jury do confine themselves to finding facts, and leave the points of law to the judgment of the judges, still the decision, both on law and facts, is *in form* that of the jury. The form is shortly this: in finding a special verdict, they state the naked facts as they find them to be proved, and pray the advice of the court thereon, concluding conditionally, that if, upon the whole matter, the court shall be of opinion that the plaintiff had cause of action, they then find for the plaintiff; and if otherwise, then for the defendant. Sometimes, and now most usually, the jury find generally for the plaintiff, subject to the opinion of the court upon a special case, stated by both parties or their representatives, with regard to a matter of law. Which ever way is adopted, it is a *general verdict of the jury* which terminates the controversy.

In Scotland, although the general verdicts in criminal cases were authorized by ancient practice, a custom crept in—probably in the reign of Charles II., when Sir George Mackenzie, the Lord Advocate (*Law and Cust. of Scotland in Crim. Matters*, tit. xxiii., *Works*, vol. ii. p. 242; 1722), contended for the abolition of juries—to consider the jury as triers of facts only; nor was this notion expelled the judicature until (1728) the trial of Carnegie for the murder of the Earl of Strathmore. In this case the judges found the facts in the indictment "relevant to infer the pains of law," and as the facts of the indictment were clear, the life of the prisoner would have been sacrificed, had not the jury, at the instance of the prisoner's counsel, Mr. Dundas, afterwards Lord Arniston, and Lord President of the Court of Session, asserted their right of returning a general verdict, and acquitted him (Lord Woodhouselee's *Life of Lord Kames*, vol. i. p. 35). Upon this subject consult Foster, *Cr. Law*, 279-80. On the other hand it may be observed, that, if it cannot with propriety be stated that the jury are judges only of facts, their decisions appear in very early times to have been subject to judicial revision. *Sed cum ad judicem pertineat justum proferre judicium, oportebit eum diligenter examinare si dicta juratorum in se veritatem contineant, et si eorum justum sit judicium vel fatuum; ne contingat judicem eorum dicta sequi et eorum judicium, ita falsum faciat judicium et fatuum* (Bracton, lib. iv. cap. 19).

VOL. II.

high tribunals in way of appeal, who in this case also do not disturb the judgment of the inferior judicatory, as far as that judgment was a decision on facts; they take cognizance only of matters of law.

Law.

In civil controversies tried by juries, it is competent, under certain restrictions, to the unsuccessful party to apply to the full court at Westminster for a *new trial*; but he will not succeed in his application unless he can satisfy the court either that the judge below misdirected the jury in some matter of law, or that the discretion of the jury has not been properly or legally exercised; "for it is a great principle of law that the decision of a jury upon an issue of fact is in general irreversible and conclusive" (Stephens, *Pleading*, 105, 4th edit.). There are other means by which verdicts may be avoided, not necessary to be mentioned here.

As to the form in which civil controversies are submitted to the judicatories, it may be stated, that the litigants exchange statements until one asserts that his adversary's statement is not in the form prescribed by law, or does not, even if true, sustain his complaint (or defence), or that the facts alleged by the adversary are not true. An allegation of one of these kinds having been denied by the adversary, a question—stated, in truth, by the litigants themselves—arises for the determination of the judicatory.

In criminal cases, generally, the proceeding is different. Crimes being committed against the state, it is the state, in the name of the king, who proceeds for the punishment of the offender. The complaint or charge is made by the proper officer, and exhibited, in the first instance, to a jury,* called, from the eminence of their functions, the *grand jury*, who, sitting by themselves, adopt it (in which case twelve of their number must concur), or reject it. If they adopt it, it is preferred before the ordinary tribunal, consisting of a judge or judges and a jury, and this charge, if denied by the culprit, is the matter they have to decide.

We will now consider the constitution of the judicatory more particularly.

The jury are certain persons indifferently selected by a proper officer, being properly qualified, in point of property and age, for the discharge of their duties. No controversy diminishing the personal security, liberty, or property of the subject† can be disposed of without the concurrence of a jury. The manner in which they exercise their functions will be hereafter explained.

The judges are of two kinds; those who are presumed to derive their authority from the king, in fiction of law the supreme judge, and those whose jurisdiction is the result of the common or traditional law or of acts of parliament.

(1.) The king is, in theory of law, presumed to be the King's judges.

* We may also observe, in allusion to what is said in reference to the grand jury, that their interference is sometimes dispensed with, in such cases of misdemeanor affecting the king's crown and dignity as the king's attorney-general thinks fit. He is justified, at his pleasure, of becoming himself the party charging and filing his information *ex officio*, which the common jury, in the ordinary form, have then to try. And in minor offences, informations may be filed on the application of private parties, permission having been previously obtained from the Court of King's Bench.

† This must be understood with the important exception of controversies relating to property entertained by Chancery, and with the other exceptions, scarcely deserving notice, of those cases within the summary jurisdiction of magistrates.

5 K

Law.

supreme judge of the state (Bracton, lib. iii. cap. 9-10), and to be present at one and the same time in all his courts of justice in all parts of the realm. But this is true only in theory of law,* for he "hath committed his power judicial, some to one court and some to another; so as if any would render himself to the judgment of the king in such cases where the king hath committed all his power judicial to others, such render would be of no effect" (3 Inst. 71). *Proprio ore, rex nullus Angliæ judicium proferre usus est* (Fortescue, *De Laud. Leg. Ang.* cap. viii.); and neither the king himself nor his privy council (see p. 809) can question, determine, or dispose of the property, real or personal, of any subject, nor adjudge any criminal case (12 Rep. 64); but all matters of such a kind are to be tried in the ordinary courts of justice, and the ordinary course of law (*Magna Charta*, cap. 29; 16 Car. I. cap. 10). The judges—*consules non unius anni*—are considered as the depositories of the fundamental laws of the realm, and have gained a known and certain jurisdiction, which can be increased, diminished, or amoved by no authority less than that of Parliament (2 Hawk. P. C. 2).

The superior courts of common law.

There are the three superior courts of common law, in each of which sit the king's judges: the Court of King's (or Queen's) Bench, the Court of Common Pleas, and the Court of Exchequer. Each of these courts consists of five judges, those of the latter being styled Barons; and one in each court presiding, those in the two first-named being styled Chief Justices, and that in the latter Chief Baron.

The original distribution of business amongst these courts was as follows: the cognizance of crime and of such matters of controversy as directly concern the king, his rights, dignity, and functions (with the exception of matters relating to his revenue), belonged to

* Anciently the king used personally to decide causes in the aula regis (*Dial. de Scacc.* lib. i. s. 4); and Edward I. frequently sat in the Court of King's Bench (4 Burr. 851), an example in which he was imitated by the celebrated Alice Perrers, or Pierce, the mistress of his redoubtable grandson Edward III. (Barrington, *Observations on 36 Edward III.*; but see Henry, *Hist. G. B.* vol. v. p. 382.) Edward IV. went one circuit with his judges, to try some heinous offenders (*Hist. Croyl. Cont.* 1 Gale, *Script.* 559); but he had previously (in the second year of his reign) sat three days together in the King's Bench, in order to understand the law (Stowe, *Chron.* p. 416. London, 1631). According to his own account, James I. was anxious to acquire this knowledge (*Rym. Fæd.* vii. 19). Municipal jurisprudence appeared to have peculiar charms in his eyes, for we are told that when he was in Denmark, for the purpose of marrying a Danish princess, he spent less time in the society of his destined consort than in the courts of justice. "Happy they," says Fuller, "who have no other business there!" After his accession to the English crown, he was desirous of exercising judicial power in his own person. In this he was opposed by the judges. "A controversy of land was heard by the king, and sentence given, which was repealed for this, that it did belong to the common law." The king, in allusion to this circumstance, observed, that "he thought the law was founded upon reason, and that he and others had reason as well as the judges." To which Coke replied, that "true it was that God had endowed his Majesty with excellent science, and great endowments of nature, but his Majesty was not learned in the laws of his realm of England; and causes which concern the life or inheritance or goods or fortune of his subjects are not to be decided by natural reason, but by the artificial reason and judgment of law, which law is an art which requires long study and experience before that a man can attain to the cognizance of it, and that the law was the golden metwand and measure to try the causes of the subjects, and which protected his Majesty in safety and peace." With this the king was greatly offended, and said that "then he should be under to the law, which was treason to affirm." Coke retorted by quoting Bracton: "*Quod rex non debet esse sub homine, sed sub Deo et lege*" (12 Rep. 65).

the Court of King's Bench; civil suits between subject and subject, involving their private rights (called *communia placita*, or common pleas), to the Court of Common Pleas; and matters relating to the royal revenue, to the Court of Exchequer. By aid of fictions of law, however, the exclusive province of the Court of Common Pleas has been invaded; the Courts of King's Bench and Exchequer, in sequence of the maxim, *boni judicis est ampliare jurisdictionem*, have now extended their jurisdiction to all controversies in which the object of the parties is the recovery of goods or chattels specifically, or recompence (damages, as it is technically phrased), or other redress, for breach of contract, or any injury of a like kind. The jurisdiction of the Court of Common Pleas remains unaltered. It is, then, the appropriate duty of these courts to determine all questions of law involved in any controversy, according to the above numeration.

Anciently, all legal controversies were determined in the courts sitting usually in the king's ancient palace at Westminster;* nowhere else was a judge to be seen. Thither came all the parties; and where the case was one involving disputed points, thither also came witnesses and jury (2 Inst. 422). Now, in controversies simply legal in their character, and the determination of which requires no jury, thither still the parties must resort. And in controversies for the final determination of which, as is ultimately proved, the verdict of a jury is necessary, all questions to be disposed of prior to the consideration of the question of facts, being of course questions of law, the courts of Westminster, or one of them, are still competent to decide. But when the controversy has assumed its final shape, and the question of fact comes to be tried, then, unless they be present in the county where the trial is to occur,† the jurisdiction of these judges passes into certain of their number, and some other persons commissioned by the king twice every year, to travel through the country, to preside over the deliberations of all juries, and to decide all questions of law which may possibly arise.

The judges of the three superior courts of common law, and the three vice-chancellors, are appointed by the king: their salaries are fixed and ascertained; and neither these nor their commissions terminate or are vacated by demise of the king; but they are removable on the address of both Houses of Parliament (12 & 13 Will. & M. cap. 2; 1 Geo. III. cap. 23; 53 Geo. III. cap. 24; 5 Vict. cap. 5). So dearly is the independence of these judges tendered by the law, that they are forbidden to obey any injunction which may

* The sittings of the Common Pleas are fixed at Westminster (*Mag. Chart.* cap. xi.); but the Courts of King's Bench and Exchequer may follow the king, as the first did Edward I. to Roxburgh (3 Comm. 42), and both did to Shrewsbury (Barrington on *Mag. Chart.* cap. 11), and to York.

† Compare 4 Inst. 73, with 2 Hale, P. C. 4, and 2 Hawkins, P. C. 17. For the purpose of punishing the malversations in office committed by civil or military authorities in the colonies, it has been enacted that they may be prosecuted in the Court of King's Bench, in England, the offence to be laid as if it had been committed in Middlesex; and the court may adjudge the officer incapable of serving the king for the future. If the party proceeding institutes civil proceedings, the fact may be laid at Westminster, or in the county where the defendant may be resident (42 Geo. III. cap. 85). All offences committed on the high seas are to be tried by commissioners appointed by the king, of whom the admiral or his deputy, that is, the judge of the Admiralty Court, is always one; or they may be tried by the Central Criminal Court (28 Hen. VIII. cap. 15; 39 Geo. III. cap. 37; and 4 & 5 Will. IV. cap. 36, s. 22).

Law.

The Court of King's Bench, of Common Pleas, of Exchequer.

Sittings at Westminster.

Duties of judges; their independence.

Law. interrupt them in the discharge of their duties. They must, as the king is bound to do (*Magna Charta*, cap. xxix.), neither sell nor delay justice (*per* Lord Nottingham, Duke of Norfolk's case, 3 *Ch. Ca.* 1; *per* Willes C. J., 2 Kenyon, 478), nor stay the delivery of their judgment,* even at the king's command (*per* Coke C. J. and the other judges; *Biog. Brit.* art. Coke), nor deliver their opinion beforehand on any criminal or other case that may come before them judicially (3 *Inst.* 29).†

Local courts.

(2.) Inferior in jurisdiction to these king's judges are certain others, whose jurisdiction is local, and derived either from the traditional or common law or from act of parliament. It was the policy of the ancient law to bring justice to every man's door. Resort could not be had to the royal courts until the suitor had demanded "right at home;" and thrice was the demand to be made.‡ Most of the local tribunals have now fallen into decay, and the renowned County Court, the genuine offspring of Teutonic times,§ is confined in its jurisdiction to certain civil controversies, where the matter in issue is of a value less than twenty pounds (3 & 4 Will. IV. cap. 112). Over this court presides the sheriff, who is appointed by the king, and who is to a certain extent his officer; and all the freeholders of the county are presumed to be present.

County court.

Corporation courts.

In certain corporate towns also courts are holden, where judges, whose jurisdiction is limited, preside: the functions and authorities of these are defined by acts of parliament (5 & 6 Will. IV. cap. 76; 7 Will. IV. cap. 105; 1 Vict. cap. 78). The justices of the peace share also, to a certain extent, in the judicial power, and are empowered singly, and without intervention of a jury, to decide trifling matters, chiefly criminal—violations of the law, to which the sanction annexed is not severe in its character. Also, in their quality of conservators of the peace, they may cause the apprehension of those who have committed, or are suspected of having committed, any breach of the peace or higher offence, even although their jurisdiction does not authorize them in any further proceeding beyond remitting the charge to a superior tribunal (2 Hale, P. C. 108). These justices, besides the jurisdiction belonging to each singly, have a jurisdiction when they sit together in the Court of Quarter

Court of Quarter Sessions.

* Palgrave, *Essay on the King's Council*, p. 125.

† Coke, however, styles the judges the king's council in matters of law (*Co. Litt.* 110 a, 304 a). We may believe this, however, to be simply an allusion to the fiction that the king is judge in all his courts; for, in the famous case of commendams, Coke stoutly denied that the judges were bound to obey a command from the king to stay the hearing or decision of a cause (*Biog. Brit.* art. Coke). "To the end that the trial may be more indifferent, seeing that the safety of the prisoner consisteth in the indifferency of the court, the judges ought not to deliver their opinions beforehand of any criminal case that may come before them judicially" (3 *Inst.* 29). The same reason applies to civil cases. It was, however, a usual practice to obtain extrajudicial opinions from the judges, both in early and almost recent times. It was done in Sir John Fenwicke's case, in Francia's case (*Foster, Cr. Law*, 241), and in the reign of Geo. I., on the question whether the education and marriage of the Prince of Wales' children belonged to the king or to their father (11 *Harg. St. Tr.* 295); and it was also done in Admiral Byng's case (*Fortescue, Rep.* 385; *Harg. note Co. Litt.* 120 a). Mr. Hargrave has printed a very singular conference between the king and the judges previous to his assenting to the Petition of Right (Preface to Sir Matthew Hale's *Jurisdiction of the Lords' House*, p. xlix.).

‡ Palgrave, *Rise and Progress of the English Commonwealth*, part i. p. 283.

§ Meyer, *Esprit Inst. Jud.* t. i. p. 385, et seq. Bouquet, *Droit Pub. de France*, p. 135.

Sessions of the peace—a court competent to decide civil controversies, where the point in dispute is whether against the party complaining an injury has been committed with violence, either actual or implied (trespass, as the law phrases it), and criminal, where any one is charged with a crime amounting to what the law styles felony. Still neither this nor the corporation courts above mentioned can try cases of treason, murder, or any capital felony, nor any felony which, when committed by a person not previously convicted of a felony, is punishable by transportation for life, nor certain other heinous offences (5 & 6 Vict. cap. 38).

Law.

Allusion must also be made to the court of the Coroner—an officer chosen for life by the freeholders of the county, and whose jurisdiction extends chiefly to the inquiring into the manner of the death of any person who is slain, dies suddenly or in prison. The inquiry must be made by a jury, whom the coroner instructs in all matters of law, and generally superintends; and when they find any one guilty of homicide, he is to commit the person so found to prison.

Coroner's court.

The extraordinary judicatories are those of the Court of Chancery and the House of Lords.

Extraordinary judicatories.

The action of the Court of Chancery may be generally described as supplying the wants and correcting the rigour of the positive established law (*Lord Chan. King, Legal Jud. in Chanc.* p. 26). It administers a peculiar branch of law styled *equity*; but although its proceedings are said to be "*secundum discretionem boni viri*," yet "when it is asked, *vir bonus est quis?* the answer is, *qui consulta patrum qui leges juraque servat*,"* and, as is said in Rook's case (5 *Rep.* 99 b), that discretion is a science, not to act arbitrarily according to men's wills and private affections, so the discretion which is executed here is to be governed by the rules of law and equity, which are not to oppose, but each in its turn to be subservient to the other. This discretion in some cases follows the law implicitly; in others assists and advances the remedy; in others, again, it relieves against the abuse, or allays the rigour of it; but in no case does it contradict or overturn the grounds or principles thereof, as has been sometimes ignorantly imputed to this court:† that is a discretionary power which neither this nor any other court, not even the highest, acting in a judicial capacity, is by the constitution entrusted with"‡ (*per* Sir J. Jekyll, M. R.,

The Court of Chancery.

* "The judges of the common law" says Lord Commissioner Whitelocke, "have certain rules to guide them; a keeper of the seals has nothing but his own conscience to direct him, and that is oftentimes deceitful. The proceedings in Chancery are *secundum arbitrium boni viri*, and this *arbitrium* differeth as much in several men as their countenances differ; that which is right in one man's eyes is wrong in another's."

† The allusion is probably made to Selden. "Equity is a roguish thing. For law we have a measure—know what to trust to; equity is according to the conscience of him that is Chancellor; and as that is larger or narrower, so is equity. 'Tis all one as if they should make the standard for the measure a Chancellor's foot. What an uncertain measure would this be! One Chancellor has a long foot, another a short foot, a third an indifferent foot: 'tis the same thing with the Chancellor's conscience" (*Table Talk*, art. Equity). Lord Eldon has made indignant allusion to this diatribe against equity, (*Gee v. Pritchard*, 2 *Swanst.* 414). But see some observations of Vaughan, C. J., Sir Matthew Hale, C. B., and Sir Orlando Bridgman, L. K. (1 *Mod.* 307–9).

‡ The equitable jurisdiction of the Roman prætor was similar. The prætorian law is styled, by Marcianus, *viva vox juris civilis*; according to Irnerius, *quia voluntatem juris civilis eloquitur*; existing, in short, *adjuvandi, supplendi jus civile* (*Dig.* I. 1. 7, 1). On this

Law.

Cowper v. Earl Cowper, 2 P.W. 753). In this court presides the Lord Chancellor, or Lord Keeper,* a principal officer of state, to whom is committed the custody of the Great Seal, and who is in all law matters adviser of the crown. He is also usually a peer, and president of the House of Lords, and, from the eminence of his functions, stands next in rank to the Archbishop of Canterbury.

The defects of the common law, and of the machinery by which it is administered, are the *presumed* foundations of Chancery jurisdiction; and it is usual for the complainant seeking its aid to aver, as a matter of form, that he is remediless at common law. This is taken as justifying his application to Chancery, where is held the maxim, *nullus recedat à Cancellariâ sine remedio*. Often the averment is false, inasmuch as courts of common law may assume a jurisdiction previously entertained exclusively by courts of equity, who are not thereby deprived of their jurisdiction (Toulmin v. Price, 5 Ves. 238). Sometimes courts of equity acknowledge rights of which courts of law can take no cognizance; sometimes the former afford only a more complete and efficient remedy than the latter. For example, matters of account can be better adjusted by a court of equity than at common law; and thus the old action of account has fallen into disuse; questions of dower and tithes more easily determined; and partitions less expensively effected. The whole theory of a court of equity may be summed up in a word—it is to afford *relief*. Like every other court of justice, it declares rights only incidentally (*per* Lord Cottenham, L.C.; K. Bruce, V.C.; 11 Law J. (N.S.) Ch. 406); it acts upon the person, and not directly upon the thing, and in pursuit of its object it will restrain legal rights, undoubtedly “a dangerous jurisdiction” (Hill v. Barclay, 16 Ves. 406). It will relieve against accident, mistake, and fraud, where by so doing it serves the ends of substantial justice; never, however, deserting the straight paths of law and

subject the opinion of Lord Hardwicke is valuable: “Some general rules there ought to be, for otherwise the great inconvenience of *jus vagum et incertum* will follow; and yet the prætor must not be absolutely and invariably bound by them, as the judges are by the rules of the common law. In the construction of trusts, which are one great head of equity jurisdiction, the rules are pretty well ascertained; so they are in cases of redemption of mortgages, which makes another great branch of that business. But as to relief against frauds, no invariable rules can be established. Fraud is infinite, and were a court of equity once to lay down rules how far they would go, and no farther, in extending their relief against it, or to define strictly the species or evidence of it, the jurisdiction would be cramped, and perpetually eluded by new schemes which the fertility of man’s invention would contrive” (Letter to Lord Kames, published in the *Life* of the latter by Lord Woodhouselee, vol. i. p. 244).

* Sir F. Palgrave has shown how the equitable jurisdiction of the Chancellor arose out of that anciently exercised by the council. The nature of this latter may be learnt from a schedule containing, amongst others, the following article for the “good governance of the land.” “Item, that all billes [that shall be putt unto the counsaill] that comprehend maters terminable at the commune lawe that semeth nought fenyed, be remitted there to be determined; but if so be that ye discrecion of the counsaill feele to greet myght on that oo side and unmyght oo that othir” (*Rot. Parl.* 2 Hen. VI. vol. iv. p. 201). Various statutes delegated to the Chancellor portions of the jurisdiction exercised by the council in aid of the common law, and, in these instances, as well as when any special writs were granted, which the Chancery was empowered to grant, if none already existed, provided for the evil complained of (*Stat. Westm.* II., 13 Edw. I. cap. 24), the application was made to the Chancellor, who granted or refused the remedy. Fraud or violence, or doings against good conscience, were the substance of the proceedings in these branches of Chancery jurisdiction (Palgrave, *Essay*, p. 94).

precedent. Its relief is often more effectual than that of a court of law, especially in cases of contract, because it need not, as must the latter, when a contract has been violated, compel a recompence, by the party guilty, to the party suffering the wrong,—but it may compel the actual execution of the contract itself. It regards penalties for the payment of money in a different light to that of courts of law, who treat them as forfeited if the money is not paid when stipulated. A court of equity, on the other hand, considers them as securities only, and to be relieved against on payment, within a reasonable term, of the principal monies and lawful interest. It will relieve also, in cases “of general concern,” upon general principles of mischief to the public (*per* Lord Hardwicke, *Walmesley v. Booth*, 2 Atk. 27), as in cases of “a catching bargain against a necessitous and improvident heir” (*Barnardiston v. Lingood*, 2 Atk. 135); for “devouring an heir,” as Lord Chancellor Cowper called it, is a practice against which the court directs all its influence.

Its mode of proceeding differs from that of courts of law. It is judge of law and facts, although in some cases it may on both crave the opinion of a common law court and a jury respectively. It compels the discovery of facts by the defendant on oath, whom courts of law never examine. Its forms of procedure have been framed according to the course of the civil law in some respects, and analogous to the common law in others (*Davis v. Davis*, 2 Atk. 23). The care of lunatics and idiots, and jurisdiction in cases of bankruptcy, were until lately possessed exclusively by the Lord Chancellor; but his duties in these respects have of late been materially lightened. The Courts of Chancery jurisdiction are, (1) that in which the Chancellor presides; (2) the Court of the Master of the Rolls; and (3) those of the three Vice-Chancellors, the senior of whom at present bears the title of Vice-Chancellor of England.

Such is a brief sketch of a court of equity whose origin is so ably described by one of its most distinguished judges:—

“New discoveries and inventions in commerce,” says Lord Hardwicke, “have given birth to a new species of contracts, and these have been followed by new contrivances to break and elude them, for which the ancient simplicity of the common law had adapted no remedies; and from this cause courts of equity, which admit of a greater latitude, have, under the head *adjuvandi vel supplendi juris civilis*, been obliged to accommodate the wants of mankind. Another source of the increase of business in the courts of equity has been the multiplication and extension of trusts. New methods of settling and incumbering landed property have been suggested by the necessities, extravagance, or real occasions of mankind. But what is more than this, new species of property have been introduced, particularly by the establishment of the public funds, and various transferable stocks, that required to be modified and settled to answer the exigencies of families, to which the rules and methods of conveyancing would not ply or bend. Here the liberality of courts of equity has been forced to step in and lend her hand.”*

The House of Lords is the supreme appellate jurisdiction in the kingdom, and is the court of final resort for all cases which courts of law and equity have en-

Law.

* Letter to Lord Kames, *Memoirs*, vol. i. p. 247.

Law. tertained. In cases also of high treason and felony committed by peers, peeresses by birth or marriage, and queens dowager or consort, it has original and exclusive jurisdiction as a criminal court, provided that it is in session; and if not, its authority in this respect is exercised by certain of its members, specially summoned for the purpose, called, from its president, the Lord High Steward's Court.

Ministerial administration of the law. Having considered the judicial administration of justice, we have now to consider that which may properly be denominated the ministerial. And here we may notice, that, treating of the jurisdiction of the judges, publicists have often regarded that jurisdiction as susceptible of a twofold division—the contentious and the voluntary jurisdiction. In their sense, the jurisdiction of the Court of Chancery in its capacity, anciently important, of *officina brevium*, or workshop of original writs, would be styled its *voluntary* jurisdiction, these writs being due *ex officio et debito justitiæ*, and the Chancellor having no authority to deny them. So also the jurisdiction of the Consistory and Prerogative Courts, in granting probate of wills and administrations of the effects of intestates, is styled *voluntary*; although, seeing that such granting may be disputed, and become the subject of *contentious* jurisdiction, the phrase cannot well be justified. What is called the *voluntary* jurisdiction of judges, is, in point of fact, no more than certain *ministerial* duties executed by them in addition to their judicial duties.

Contentious and voluntary jurisdictions.

Uncontroverted claims and charges.

It has been already observed that the judicial power is conversant only with *controversies*.* If the charge or complaint of the prosecutor or plaintiff be not denied, it is taken as proved; and the judicatory or their officers, exercising no discretion in the case, simply proclaim and direct the execution of the penalty which the law annexes to proved offences of the kind. Here their duties are purely ministerial; and thus no power otherwise than ministerial is evoked when a right has been invaded, and, the fact of its invasion having been properly charged, has not been denied. The inferior ministerial officers of the law which require notice, are—

The sheriff.

(1.) The sheriff, who is bound personally or by his officers to execute the process and obey the commands of the king's courts, and

The constable.

(2.) The constable, whose duty it is to preserve the peace, and who is for that purpose armed with very large powers, of arresting or imprisoning, of breaking open houses, &c., in exercise of which powers he is amenable to the law which has confided them to him. The justices of the peace, and other functionaries, have also their ministerial duties.

273. II. DEFENCE.—The defence of the realm is entrusted by the law to the care of the king, to whom, for that purpose, is confided the government and command of the militia within the realm, and of all forces by sea and land, and of all forts and places of strength (13 Car. II. cap. 6). The law, to enable him to exercise these high functions with efficiency, vests in him the authority of appointing all ports and havens, of

causing the erection of beacons, lighthouses, and sea-marks (these latter privileges being now exercised through the medium of the Corporation of the Trinity House, 8 Eliz. cap. 13), and of prohibiting the exportation of arms or ammunition.

The military force, which exists for the defence of the country, is of three kinds—

(1.) The *Militia*, a citizen soldiery, consisting of a certain number, fixed by the Privy Council every ten years, of the inhabitants of every county, chosen by ballot. They must serve for five years, either in person or by approved substitutes. The officering the militia is intrusted to the lord lieutenants, but the officers must have certain qualifications of property, according to their ranks; the actual enrolment and organization of the regiment is the task of the deputy lieutenants. In ordinary times, the period during which they may be placed on duty is limited to twenty-eight days in every year (55 Geo. III. cap. 65), and the king may suspend their embodiment during any year he pleases (57 Geo. III. cap. 51). In the cases of invasion, or imminent danger thereof, of rebellion and insurrection, the crown can call out the militia, increasing, if it pleases, the quota fixed by the Privy Council for any county, and place them under military command. They need serve only in Great Britain and Ireland, and in the latter country for not more than two successive years. In addition to this, the regular militia, there is the local militia, inferior in its numbers, but more limited in service, and the yeomanry, who are mounted Yeomanry volunteers.

(2.) The Army, a standing military force, unknown to the early days of our constitution, and denounced by the highest authorities in constitutional law as menacing the subversion of our liberties. The jealousy of our legislature, itself once the unwilling servant of military power, has removed the probability of such perils for the future; whilst the fact, that, in these latter days, England has become a great continental power, and possessed of a vast transmarine empire, extending through both hemispheres, sufficiently demonstrates the necessity of an organized and disciplined force, capable of easy transport to any point where danger is anticipated.*

The raising or keeping a standing army within the kingdom in time of peace, unless it be by consent of Parliament, is against law (1 Will. & Mary, st. 2, cap. 2). It has been declared contrary to the laws and franchise of the land for the crown to issue commissions for the punishment of soldiers, mariners, or others, for any crime whatsoever (3 Car. I. cap. 1). It is, then, by permission of Parliament, annually granted, that our standing army is maintained; and it is governed under the authority of an annual Act, "to punish mutiny and desertion, and for the better payment of the army and their quarters."

(3.) The Navy, to whom we owe the vast extent of our dominion, and the exemption so long enjoyed by our country from the miseries of an invasion. The right possessed by the crown, of impressment, or of compelling seafaring men to enter the service of the

* It may, however, be remarked, that sometimes the controversy which calls the judicial power into operation, is a controversy only in name and shape. Of the suits entertained by the Court of Chancery, a large proportion have no other purpose than to transfer the management and administration of property from individuals to the court itself; but although not in reality *contentious*, they are so in appearance and pretence. See Coke's observations respecting the Court of Requests, 4. Inst. 97-8.

* The object of a standing army is thus recited in the annual Mutiny Act: "Whereas it is adjudged necessary by Her Majesty and this present Parliament, that a body of forces should be continued for the safety of the United Kingdom, the defence of the possessions of her Majesty's crown, and the preservation of the balance of power in Europe (5 & 6 Vict. cap. 12).

Law.

navy, is considered as part of the common law (Foster, *Cr. Law*, 154); but though the king can thus claim the service of all his subjects, he is bound to remunerate them (Foster, *Cr. Law*, 160-1). The navy is governed under the authority of an annual Mutiny Act.

Laws re-
specting
social
economy.

274. III. SOCIAL ECONOMY.—Laws which directly consult the health, wealth, convenience or comfort of the public, may properly be referred to this head: and here it may be proper to observe, that the genius of English law differs from that which obtains generally on the Continent, inasmuch as a great deal of what is there ordinarily undertaken by the government, is here relinquished to private enterprise and discretion. In one material point, however, our social legislation possesses a decided advantage over that of the greater part of continental Europe—in respect of the laws which relate to the relief and support of the poor. Although public opinion has been, and still is, much divided as to the policy or impolicy of the late alteration in those laws, this at least is plain, that they stand in advantageous contrast to those of every other state.

History of
the poor
laws.27 Hen.
VIII. cap.
25.

It will be desirable to state briefly the history of legislation with reference to this subject. The early statutes seem to have had no other purpose than to prevent vagrancy; and, in an age of defective police, wandering labourers and "valiant beggars" were objects of terror in the distant village and to the lone husbandman. These Acts regulate begging. In the 27th Henry VIII. cap. 25, we see the first germ of a poor law. It does not authorize, however, the compulsory levy of money for the relief of the poor; but simply commits the charge of the poor to certain officers, who are to expend for their benefit the sums raised by voluntary contribution: the statute further prohibits private alms-giving. Upon the sturdy beggar it inflicts the punishment of whipping for the first offence; cropping of the right ear for the second; and, on a third offence, requires him to be sent to the next gaol till the Quarter Sessions, to be there indicted for wandering, loitering, and idleness; and if convicted, he was to suffer death, as a felon and an enemy of the commonwealth.

1 Edw. VI.
cap. 3.

These "goodly statutes, partly by foolish pity and mercy of them which should have seen the same executed, and partly from the perverse nature and long-accustomed idleness of the persons given to loitering," seem to "have had small effect;" idle and vagabond persons being unprofitable members, or rather enemies of the commonwealth, were suffered to remain and increase. The 1 Edw. VI. cap. 3, which thus recites the evils consequent on the state of vagrancy, provides that an able-bodied man not applying himself to honest labour, nor offering to serve for meat and drink, if he could obtain nothing more, should be taken for a vagabond, branded on the shoulder, and adjudged a slave for two years to any person who shall demand him, to be fed on bread and water and refuse meat; if he run away within that period, he was to be branded on the cheek with the letter S, and adjudged a slave for life. There are many other provisions equally stringent.

Inefficacy
of early
legislation,

The experience of the early acts of parliament made in relation to the relief of the poor, affords a useful lesson on the fruitlessness of ill-advised legislation: all these acts of parliament, all the gentle askings of the collectors, all the exhortations of the ministers, and the charitable ways and means of the

bishop, appear to have been ineffectual in persuading the parishioners to intrust to the collectors the distribution of their alms: and as to the provisions against the vagrants, their inefficiency is shown by the recital of the 14 Eliz. cap. 5, which states, that "all the parts of this realm of England and Wales be presently with rogues, vagabonds and sturdy beggars exceedingly pestered; by means whereof daily happeneth in the same realm horrible murders, thefts, and other great outrages, to the high displeasure of Almighty God, and to the great annoyance of the commonwealth." At length was passed the celebrated 43 Eliz. cap. 2, 43 Eliz. cap. 2; which creates the well-known office of overseer, and commits to it the duty of levying and administering a rate for the benefit of the poor. This rate was to be applied in setting to work and apprenticing the children in each parish, whose parents appeared to the proper officers unable to maintain them; in setting to work all persons having no means of subsistence, or using no trades; and for the necessary relief of such of the poor as were unable to work from age or infirmity.

Law.

The misdirected humanity of subsequent legislatures led them to infringe the principle of this beneficent law; and through a variety of causes, unworthy of specific detail, its wholesome provisions became neglected in practice, and abuses connected with the administration arose, which threatened to destroy at once the value of our property and the character of our labourers. An Act has been lately passed which professes to avert evils so dreadful in their consequences. By this Act (5 Wm. IV. cap. 76) the superintendence of the administration of the relief of the poor is intrusted temporarily to three commissioners, appointed by the crown. They are authorized and required to make all rules, orders and regulations for the management of the poor, for the government of workhouses, for the education of the children therein, and for the management of parish poor children and the supervision and regulation of the houses wherein they are kept, for the apprenticing the children of poor persons, and for the guidance of all officers connected with the management of the relief of the poor, and generally for all matters relating to expenditure in the relief of the poor. A power is reserved to the king in council to disallow the orders of the commissioners, which must be laid before Parliament, and before they have any operation must be submitted to a secretary of state; nor can any general order be rescinded or suspended without permission of such secretary, (5 & 6 Vict. cap. 58, s. 3). The commissioners, however, cannot interfere directly in ordering relief to be given to any individual; indeed, the actual administration of relief is still intrusted to local officers. In exercise of the authority for that purpose given them by the Act, the commissioners have formed unions of parishes, for the more economical and impartial administration of relief; each parish, however, being separately taxed for the relief of its own poor. Each union is governed, subject to the superintendence of the commissioners, by its board of guardians, consisting of persons possessing a certain amount of property, chosen by the rate-payers and landowners of the parishes forming the union, to whom are added certain *ex-officio* members, being the resident justices of the peace. The actual dispensers of relief are paid servants of the board, by whom they are chosen, subject to the approbation of the commissioners.

Poor law
commis-
sioners,

Unions.

Law. Principles and object of the Poor Law Amendment Act.	The Poor Law Amendment Act was passed, not for the purpose of abolishing the necessary relief to the indigent, but for preventing various illegal and dangerous practices which had by degrees grown up in the administration of such relief (<i>First Ann. Rep. App. A. No. 4</i>). By law, the guardians can only give relief to the able-bodied needy in the workhouse, or in some cases out of the workhouse, by setting them to work. It is an essential condition, that when the wants of any destitute able-bodied person are supplied by relief given in return for work, the work or mode of relief should not be such as to raise the condition of the pauper above that of the self-supporting labourer, or induce any one to make the parish the first instead of the last resource in case of need (<i>Fourth Ann. Rep. App. A. No. 3</i>). To carry out this principle—to render as far as possible, without infringing the great laws of humanity, the condition of the pauper inferior to that of the independent labourer—the new system has endeavoured to confine the dispensation of relief to the walls of the workhouse. The relief of the <i>destitute</i>, and not of the <i>poor</i>, is the object of our poor-law system; and although to become permanently chargeable to a parish it is necessary that the individual should have fulfilled the condition imposed by law in the requirement of a settlement, as it is called, yet the destitution of persons, and the fact of their being within the parish or union, entitles them to immediate relief, without any regard to their settlement (<i>Instructional Letter</i>, 5th Feb. 1842). The Poor Law Amendment Act evidently contemplates, under ordinary circumstances, the adoption of the workhouse as the most effectual remedy for the evils inevitable on a state assuring to its subjects subsistence when, from misfortune or <i>misconduct</i>, they are unable to obtain it of themselves. In cases, however, where adequate workhouse accommodation has not been provided, or where large numbers of able-bodied persons are often suddenly thrown out of employment by the fluctuations of manufactures, the Poor Law Commissioners may, if they think fit, prescribe other conditions for the relief of the able bodied than admission into the workhouse (<i>Instructional Letter</i>, 30th April, 1842). The protection of the public health is provided for by various wholesome statutes, as well as by the common law. Although imperfectly administered, there is no doubt that these laws are eminently calculated to secure their purposes. Drainage, so essential to the preservation of health, is placed under the superintendence of commissioners of sewers, by the 23 Hen. VIII. cap. 5, while the highway laws, including the act 5 Eliz. cap. 13, provide for the cleansing of the public roads and ditches. The common law denounces all nuisances, public, common and private; a public nuisance being that "which is a nuisance to the whole realm; private, that which is to a house or mill," &c. (2 <i>Inst.</i> 406); a common nuisance being an offence against the public, "either by doing a thing which tends to the annoyance of all the king's subjects, or by neglecting to do a thing which the common good requires" (Hawkins, <i>P. C.</i> cap. 107). It is held to be a common nuisance, and indictable, to divide a messuage in a town, for poor people to inhabit, by which it will be more dangerous in time of infection (2 <i>Roll. Abr.</i> 139). The stopping of wholesome air is held to be a nuisance, as well as the stopping of light (Aldred's case, 9 <i>Rep.</i> 57). Similar in policy to these laws is that which treats persons who, labouring under infectious diseases,	go abroad and expose others to the infection, as liable to punishment for a misdemeanor (<i>R. v. Vantandillo</i>, 4 M. & S. 73). It is to be regretted that our modern legislation in reference to this subject—whether it is in the form of sewer acts, acts of municipal police, building acts, or the like—should be local in its operation and spirit, and so imperfect in its machinery. In the possession of good and available means of communication, a chief source of national wealth consists; but here the state, except under certain circumstances, commits to private exertion, stimulated by personal interest, what in other countries is undertaken by itself. The highways and turnpike roads, although under different regulations, are both superintended by individuals authorized thereto by the legislature, from whom they receive certain powers and general instructions. Water communications, canals and rivers rendered navigable by artificial means, are either under the control of persons acting on behalf of the public or of enterprising companies seeking their own profit, but so far contributing to the public good as to justify the state in intrusting them, though upon specified conditions, having reference to the public advantage, with large privileges and extensive authority. In this country such companies are numerous, applying their resources, derived from the combination of great wealth with disciplined talent, to various purposes, and privileged by the state, whose interests they so largely serve. It would be obviously impossible to detail the various laws by which the supreme power has endeavoured to secure the social happiness of the community. Guided by general principles, it leaves for the most part to individual prudence, and to the operation of instinctive self-love, all arrangements connected with the pursuits of industry. It has been taught by history that arrangements of this kind depend upon principles beyond its reach, and that legislative interference does little more than disturb and perplex relations which will inevitably establish themselves. Political sovereignty, conscious of its own weakness, and distrustful of its own wisdom, will not seek to derange these relations, because it is conscious that their establishment is inevitable. Between the buyer and the seller, whether of labour or its products, political sovereignty is a mediator without authority; * nor in our state does it interfere, except in special cases, and for special reasons. The usury laws, which invalidate contracts for the loan of money on exorbitant interest—the laws for the protection of shipping, manufactures, and agriculture, home and colonial—the factory laws (3 & 4 Will. IV. cap. 103), regulating the employment of labourers in the great hives of industry, and abolishing certain forms of remunerating them (1 & 2 Wm. IV. cap. 36)—and the law forbidding the employment of women and girls, and regulating that of boys, in mines and collieries (5 & 6 Vict. cap. 99), are exceptions to the general principles, and exceptions pleading in their behalf special circumstances as entitling them to exemption from a policy wise and beneficial in the main. 275. IV. FOREIGN RELATIONS. — With regard to foreign matters, the king is considered as the delegate or representative of the state (1 <i>Comm.</i> 252), and has,
Work-house. Title to relief.		Roads. Canals.
Out-door relief.		
Laws relating to public health;		Principles of legislation on social economy.
		Foreign relations.

* See Lord Ashburton's *Inquiry into the Causes and Consequences of the Orders in Council*, p. 133, first edition; and some remarks by that eminent publicist of Prussia, M. Schmalz, *Econom. Politiq.* tome ii. p. 144.

Law. as such, the sole right of sending and receiving ambassadors or other ministers from foreign states. To him also is committed the duty of superintending all negotiations, and his acts are binding on the whole community.

Finance. 276. V. FINANCE.—No man is compellable to make or yield any gift, loan, benevolence, tax, or such like charge, without common consent by act of parliament, according to the ancient laws of this realm (3 Car. I. cap. 1, s. 10); and all levying of money for or to the use of the crown, by pretence of prerogative, without grant of parliament, and for longer time or in other manner than the same is or shall be granted, is equally illegal (1 Wm. & Mary, stat. 2, cap. 2). In the reign of Charles II., but not for the first time, the Commons asserted those exclusive rights in regard to the raising and appropriation of the national revenues which have ever since been conceded to them. They then resolved—(1) that all aids given to the king are the sole gift of the Commons; (2) that money bills ought to originate in the House of Commons (4 *Inst.* 29); and (3) that it was the sole right of the Commons to direct, limit and appoint, in such bills, the ends, purposes, considerations, &c. of the grants they contain* (*Comm. Journ.* 3 July, 1678). The form observed in making these grants is shortly this: the House determines, by a resolution, what sum it will vote, and then “goes,” as it is technically phrased, “into a committee of the whole House,” when every member has the privilege of speaking more than once, which is not permitted to him (except in explanation) in an ordinary sitting of the House. The means of raising the sum is then agreed on, from which circumstance the committee is called “a committee of ways and means.” The resolutions of this committee are then solemnly affirmed by a vote of the House, and a bill to effectuate the resolutions is ordered to be brought in, which is read a first, second, and third time, and sent up, like an ordi-

Exclusive privilege of the Commons.

Money bills.

* The right of Parliament to indicate the application of their supplies was familiar to the early periods of our history, a remarkable instance having occurred in the reign of Richard II. (3 *Rot. Parl.* 5, 6). Cromwell would never suffer the insertion of any clauses of appropriation in the money bills passed during his Protectorate (Clarend. *Life* and contin. vol. ii. p. 10. Oxf. 1827. Scobell, *Acts and Ordin.* vol. ii. p. 311-12, fol.). He was aware, from the example of Charles I., how much they tended to cripple the exercise of the executive power (Clarend. *Hist. Rebell.* vol. i. p. 321-2. Oxf. 1807). When such a clause was inserted in a bill passed in 1665, for raising a supply for the purposes of the Dutch war, Clarendon and Lord Ashley (afterwards Earl of Shaftesbury) vainly attempted to dissuade Charles II. from giving it his assent (Clarend. *Life*, vol. iii. p. 10-15).

This command of the public purse by the representatives of the people is undoubtedly one of the reasons that we have been exempted from that oppressive taxation which ground the yeomanry of France down to the very dust. In allusion to the French peasants, Aylmer says, “that, amongst other grievances, they pay till their bones rattle in their skin” (*Harbrowe for faithful Subjects*). Sir Dudley Carleton, in a speech delivered in the House of Commons in 1626, said, plainly in reference to France, that there the common people looked “not like our nation, with store of flesh on their backs, but like so many ghosts, and not men, being nothing but skin and bones, with some thin covering to their nakedness, and wearing only wooden shoes on their feet; so that they could not eat meat or wear good clothes, but they must be taxed unto the king for it” (2 *Parl. Hist.* 120-1). Fortescue draws a similar picture of their condition in his time, and dwells on what had long been, and was long considered, a heavy grievance. *Præterea non patitur rex quenquam regni sui salem edere, quem non eruat ab ipso rege, pretio, ejus solum arbitrio, assesso* (*De Laud.* cap. xxxv.). Philippe le Bel was very active in enforcing this obnoxious duty on salt, so much so that Edward III. called him, “son frere de la loi salique” (Barrington on 25 Edw. I.).

nary law, for the approval of the House of Lords. The Commons, however, have determined that, “in all aids given to the king by the Commons, the rate or tax ought not to be altered by the Lords” (*Comm. Journ.* 13 April, 1671)*. The bill having passed the Lords, is then submitted to the crown for its assent. Previous to the House of Commons agreeing to its votes of supply, a statement of the expenditure and receipts of the last year is laid before it, and an estimate of the probable expenses of the year, whose exigencies are then to be provided for, is also exhibited.

For a full account of the manner in which the revenue of the country is raised and applied, reference should be made to the article TAX in the Fourth Division of this *Encyclopædia* (vol. xxviii. p. 470).

277. VI. RELIGION.—Sovereignty has its moral duties as well as its legal rights. The true province of a government—what it should undertake of itself—what relinquish to the free choice of its subjects, is one of those questions hardly worth discussing, because hardly susceptible of satisfactory solution. Still we need not go very far into the philosophy of morals, where broad and deep are laid the foundations of municipal law, and whence spring the obligations to obey it, to learn that whatever is involved in the duty of a government, more at least is involved than securing to every man the quiet enjoyment of his own; that a government has hardly done its part when it has plucked the sword from the hand of civil outrage, and surrounded hall and cottage with the strong barriers of judicial protection. If man have a twofold nature, his wants, results of that nature, must be twofold in their kind; and has a government fulfilled its mission when it has consulted but one class of those wants, and that the least in moment? Surely, then, government has something to do with religion; and surely this, if for no other purpose than to sanctify and consecrate its own operations, and to enable it, in its corporate capacity, to share the blessings to be sought by its members! The state is bound up with the church, says a great lawyer, not to make the church political, but to make the state religious.

On such intelligible principles does the state concern itself with religion; and it is to us a matter of deep thankfulness that, instead of seeking for itself, in the *hortus siccus* of dissent, our state should have abided by that ancient church whose foundations were laid in the land by primitive, if not apostolic, hands. Of the Church of England, lawyers† have so written as would imply their concurrence in the calumny, that she is a parliamentary church, the offspring of troubled times. Far earlier was her date; and, in her reformation, she, in the words of Bishop Bull, “as far as it was

Duties of a government.

Connexion of church and state.

The Church of England:

catholic in spirit;

* Roger North illustrates by an anecdote which shows in a strong light the jealousy with which the Commons regarded the slightest change made by the Lords in a money bill. A money bill had passed the Commons, in which, by a mistake of the engrossing clerk, the first instalment was made payable in 1673, a year already past. When, in the first reading, in the Lords, the error was discovered, the greatest consternation ensued: to alter the bill would be to secure its rejection by the Commons; to suffer the figures to stand would be equally bad: the penknife of a clerk cut the knot; and when the bill was read a second time, the date appeared as it was originally intended to have done (*Examen*, p. 460).

† Blackstone, in his allusions to this subject, is very unsatisfactory; and a scholar should have had more toleration than to have styled the learned Anselm, the most profound and accomplished doctor of his age, “that obstinate and arrogant prelate” (1 *Comm.* 379).

Law. practicable, and the age allowed, conformed to the example of the ancient catholic church.* The position allotted her by the constitution of the country she challenges as hers of right. "Therefore," says Dr. South, "when men were led by the written word, and not by the *ignis fatuus* of a bold fancy, styling itself divine revelation,† the church was always recognised as Christ's court here upon earth, fully empowered and commissioned from him to decide all emergent controversies, to interpret doubtful commands, and to make wholesome sanctions and institutions as particular occasions and the circumstances of affairs should require; that so it might be that the assistance of the Spirit promised to the church was not a vain thing, or a mere verb."‡ It must, then, be understood, that the creation of the church was no act of the state.§ If Parliament were, as once did two of its branches usurping its authority, to vote the Church of England abolished, she would be a church no less, she would forfeit nothing but her power to command that aid of the civil power which she now possesses, and which she rewards so amply by teaching that honour is the due of the king, and that of dignities no evil is to be spoken.

her authority; not created by the state. The united church of England and Ireland. By the Act of Union with Ireland (39 & 40 Geo. III. cap. 67), the churches of England and Ireland, as then by law established, are united into one Protestant episcopal church, to be called the United Church of England and Ireland; the doctrine, worship, discipline, and government of which is to be and remain the same as already established for the Church of England, and its continuance as the Church of England and Ireland is a fundamental part of the Union.

Supreme head on earth. The king is recognised by law as the only supreme head on earth of the Church of England (26 Hen. VIII. cap. 1; 1 Eliz. cap. 1); and the independence of that church of all foreign prelates and persons whatsoever is, in affirmance of the common law, secured by a solemn act of the legislature (25 Hen. VIII. cap. 21). Right of making canons; The clergy, "one of the greatest states of the realm" (8 Eliz. cap. 1; 4 Inst. 321), as represented in the houses of convocation, have the right, if so licensed by the king, of making canons, which, when confirmed by him, operate upon and bind the clergy, so far as

* Sermons.

† Similar language was used by a great man, whose death attested his devotion to his principles. "So," says Laud, "there shall be no public service of God; but some *ignis fatuus* or other, under the name of light in the conscience, shall except against it" (*On the Liturgy*, p. 99. Oxford, 1840).

‡ Sermon xxxiv. vol. vi. p. 208. Oxford, 1823.

§ It is not to be denied that it is the popular, say rather, vulgar opinion, that the church owes its possessions to the benevolence of the state, that it is in fact a church *endowed* by the state. The fact, unhappily, is far otherwise. It is to be lamented that the state has rarely concerned itself with church temporalities any further than to appropriate them to its own purposes, or to others less holy. The spoliation of the church, under the pretence of reformation, were ably exposed by contemporaneous eloquence. "My lords and masters," says Bishop Latimer, "I say that all such proceedings, as far as I can perceive, do intend plainly to make the yeomanry *slavery*, and the clergy *shavery*. We of the clergy had too much; but this is taken away, and we have now too little" (*Sermons*, vol. i. p. 78. Lond. 1758). The poet is not less forcible:—

"We rob the church * * * * *
Men seek not to impropriate a part
Unto themselves, but they can find in heart
To engross up all: which vile presumption
Hath brought church livings to a strange consumption;
And if this strong disease do not abate,
'Twill be the poorest member in the state."
(Wither, *Abuses Stript and Whipt*, book ii. sat. 4.)

they are not repugnant to the royal prerogative, or the laws, customs and statutes of the realm (25 Hen. VIII. cap. 19; 4 Inst. 322-3). Their right to make these canons results from the king's licence, which he is empowered to give. These canons are binding on the clergy, (*More v. More*, 2 N. R. 157); but they do not, *proprio vigore*, bind the laity, unless they be enacted by Parliament (*Middleton v. Crofts*, 2 Atk. 650); for "no new laws can be made to bind the whole people of this land but by the king with the advice and consent of both houses of parliament, and by their united authority" (*per Hardwicke C. J., Ib.*)* The Convocation, to which so large an authority is intrusted, consists of two houses, the upper and lower house. In England there are two provinces, Canterbury and York, over each of which an archbishop presides, and each province has its convocation. To the upper house come the archbishop and all the bishops of the province; and to the lower house, the representatives of the clergy of the dioceses and of each chapter therein (4 Inst. 322; 1 Comm. 280). To convene, prorogue, restrain, regulate, and dissolve the convocation is the right of the king; but this once important body exercises, at present, an authority only in name, and assembles only for the sake of form.

Law. Law. binding on the clergy. Canon law binds the laity: The canon law, which, administered by the ecclesiastical courts, or, in the language of our earlier textbooks, the courts Christian, is still binding on the laity, as well as the clergy; but is so binding because, solely, it has been received and admitted either by consent of Parliament or by immemorial usage and custom in some particular cases and courts. From hence is its authority derived (*Hale, Hist. C. L.* chap. ii.).

Besides the two provinces already noticed, England is divided into parishes and dioceses, the diocese being further divided into archdeaconries and rural deaneries. This has reference to a *local* classification of the clergy, and is also *legal* in its character. The clergy may also be considered in their *spiritual* capacities according to their order in the church, as deacons, priests, and bishops. Of this classification, whose antiquity is coeval with the origin of the Catholic Church, the law takes notice. It being our part to consider them in their *legal* character, and to inquire with what rights they are clothed by law, and what duties the law expects them to perform, the first arrangement will be the most proper for our purpose.

The spiritual care of the *parish* is the business of Parish. the legally appointed minister, who is, according to circumstances, styled *parson*, or, more usually, *rector* or *vicar*. He is styled parson or rector when he is entitled to *all* the ecclesiastical dues within the parish, and vicar when entitled only to some, the remainder having been alienated. The right of nomination to a parish is in various hands. Sometimes this right belongs to the bishop, sometimes to the crown, and sometimes to private persons. By whomever exercised, it must not be exercised for money or any other corrupt consideration; for this is the offence of *simony*, so called in reference to a well-known circumstance (*Acts*, viii. 18), but one which practically defies all legislative prohibition (31 Eliz. cap. 6). To be legally nominated or presented, as the phrase is, the clergyman must be

* It has been doubted how far the canons of 1603, which, in the above case, Lord Hardwicke decided bound the clergy only, were binding in the province of York (*Rex v. Archbishop of York*, 6 T. R. 491-2. *arg.*).

Law. a priest, having been ordained in the manner indicated by the canons and liturgy of the Church of England (13 & 14 Car. II. cap. 4)—must therefore be twenty-four years of age, as that is the earliest age at which priest's orders can be obtained; but deacon's orders at twenty-three—must be presented by the patron of the advowson (that is, the person lawfully entitled to nominate)—must be instituted, as it is called, by the bishop, and properly inducted.

Institution. The institution by the bishop enables the clerk (as every duly ordained clergyman is called in law) to enter into his parsonage-house and take his tithes or ecclesiastical dues; but previous to induction he cannot lease them. Institution may be refused by the bishop—(1) if the patron have been excommunicated; (2) if the clerk be a bastard, an outlaw, an excommunicate, an alien, or similarly disqualified; or, (3) if he hold heretical opinions; (4) be of vicious life and conversation; or, (5) insufficient in respect of his learning. If the objection be one of a spiritual kind, as heresy or the like, and the patron be a layman, the bishop is bound to state his objection, or cannot, as he otherwise may, present to the living himself: but when the objection be one of another kind, as alienage, &c., or when the patron is a spiritual person or corporation (as a dean and chapter), no objection need be stated. If the bishop be patron, the acts of presentation and institution are one, and called *collation*.

Induction. Induction is the ceremony of delivering corporeal possession of the church, and endows a parson or vicar with his full rights. Vicars and curates are bound to reside in their parishes, unless their absence be licensed by their ordinary, who is in a variety of cases empowered to do so (1 & 2 Vict. cap. 106).

By this Act also, various provisions are made respecting the residence of the clergy, and former Acts are repealed by it. It forbids any spiritual person who holds a benefice from accepting another, or any cathedral preferment to hold with such benefice; and also forbids persons holding any cathedral preferment from holding with it any preferment in another cathedral. This regulation is subject to exceptions, very important and numerous, and to the power of the archbishop to grant licences of dispensation.

Perpetual curate. A parson or vicar is a corporation, his rights being continued in perpetuity to his successors. Inferior to him is the curate, who (with the exception of the *perpetual curate*, appointed by the owners of the tithes,) is named by him, to assist him in his clerical duties, and of whom the law takes properly no notice, except in the regulation, in some special cases, of his salary.

The bishop. The power and authority of the *bishop** in temporal matters consists principally in *overseeing* and correcting the manners of his clergy, whom he may punish by ecclesiastical censures through his court. He may himself be deprived by his archbishop, if guilty of any gross or notorious crime (Bishop of St. David's v. Lucy, 1 Salk. 134). The bishop presides over a diocese.

Dean and chapter. The *dean and chapter* are the council of the bishop, to assist him with their advice in matters of religion, and also with the temporal concerns of the see. The dean is sometimes nominated by the king, and some-

* It has been considered that, at common law, a bishop is a mere temporal magistrate, and is not properly to be designated bishop until the king have given him his temporalities (8 Rep. 137; 12 Rep. 9; marginal notes).

times elected by the chapter; the canons or prebendaries, sometimes by the king, sometimes by the bishop, and sometimes by each other. The *archbishop* is the chief of the bishops and clergy in his province, in which there is a diocese where he acts as a bishop. He decides all ecclesiastical controversies within his province, by way of appeal from the bishop. During the vacancy of an archiepiscopal see, the dean and chapter are the spiritual guardians; and during the vacancy of an episcopal see, the archbishop acts as such. He can resign only to the king, by whom he may be deprived for any gross or notorious crime. It is the privilege of the Archbishop of Canterbury to crown the kings and queens of England; and when he resigns his see, it must be into the hands of the king. A portion of the power usurped by the pope in this country is conferred on this archbishop by an act of parliament (25 Hen. VIII. cap. 21), empowering him to grant dispensations in any case where it may contravene neither the Holy Scriptures nor the law of God (Gibson, *Codex*, 413); and it is by virtue of this authority that he grants special marriage licences, dispensing with the necessity of observing the canonical hours, &c.

In the king, as head of the church, is the right of nomination to all bishoprics and archbishoprics vested. In the case of bishoprics created by or since the reign of Henry VIII. the nomination is direct; in other cases it assumes the form of an election by the chapter of the cathedral.

The duties of the *archdeacon* and *rural dean* have reference chiefly to the repair and condition of ecclesiastical edifices.

It will not be necessary to consider at any length the constitution of the church courts, an extensive reform in which may shortly be anticipated. "The manner of proceeding in causes for the correction of clerks" has been the subject of a late Act, "for better enforcing church discipline" (3 & 4 Vict. cap. 86).

Formerly, to borrow the expression of the great Hooker, "the Church of England and the people of England were the same people;" but, for causes not necessary to be discussed here, this is not the case at present; and therefore it is necessary to observe that, while associating itself with a church of fixed doctrines and settled formularies, the state *tolerates* other creeds and modes of worship.

Having described the objects with which the sovereign power concerns itself in this country, and the regulations which it has established in respect thereof, it becomes desirable that we should ascertain the mode in which the sovereign power itself operates, and the character of the body in which it resides.

It has been already observed, that the sovereign power of the English state is possessed by the Parliament itself, composed of King, Lords, and Commons.

278. (I.) The King.*—The illustrious person who

* Under this term is to be understood queens regnant (Mary, st. 3, cap. 1). Shortly after the accession of Mary, a person, who had been employed by Cromwell, and lately released from prison, where he had been confined for high treason, sent to the Queen, through the Chancellor of Milan, a little book, counselling her to declare that the statutes which were restrictions upon the kings of England did not extend to the queens, and that she held herself not bound by them. The Queen showed the book to Gardiner, who declared it "naught, and most horrible to be thought on;" in which opinion the Queen concurred, and threw it into the fire (Burnet, *Hist. Ref.* vol. ii. p. 558-9. Oxford, 1829).

Law. shall sit on the throne of this kingdom must be—(1) heir to the king last possessed; (2) a descendant of Sophia of Hanover, grand-daughter of James I. of England; (3) not reconciled to or holding communion with the see or church of Rome, nor professing the popish religion, nor married to a papist; but (4) joining in communion with the Church of England as by law established (1 W. & Mary, st. 2, cap. 2; 12 & 13 Will. III. cap. 2).

his attributes;

To the king our law-books ascribe the attributes which can belong in reality to none except the great Sovereign of the universe. They style him, what he is not, supreme (1 Comm. 241), and ascribe to him qualities to which never pretended those heroic and magnanimous princes who surrounded the throne of England with the imperishable halo of historic renown.

In the *idea* of the law of England, the king possesses absolute perfection (1 Comm. 426), absolute immortality (Ib. 249), and legal ubiquity (Ib. 270). In him there is no folly or weakness; he can neither think nor do wrong (Ib. 240). He never dies (Ib. 246), and is as invisible as he is immortal (2 St. Tr. 598). His legal authority is absolute and irresistible; the minister and deputy of Heaven, there is none above him but God (1 Comm. 951). He is the universal lord and original proprietor of all lands in his kingdom,—the whole soil being vested in him; nor can any subject hold lands in the kingdom but of him, directly or remotely (2 Comm. 51–9, 60; Y. B. 24 Edw. III. f. 65 b). He is the fountain of honour and dignity; all titles of nobility or otherwise are derived solely from him (1 Comm. 396). He is “in name the state itself. All the powers of the state, legislative and executive, are nominally in him” (24 St. Tr. 1183). These attributes and rights, so high and extensive, are practically subjected to the modifying operation of the common and statute law. Unhappily, instead of being considered, as great lawyers have styled them, a portion of law, they have been considered as prior in their origin, and evidences of the intrinsic authority of the king, which was called his *prerogative*† (Co. Litt. 90 b).

prerogative;

“The common law,” to quote the judicial sentiment so often cited by Sir Edward Coke, “has so admeasured the prerogatives of the king, that they can neither take away nor prejudice the inheritance of any one”

* The ancient common or traditional law considers that the king can never be subject to the disabilities of minority (Co. Litt., 43 a; Plowd. 212). It has, however, been usual, since the time of George II., when the heir to the throne has been under the age of eighteen, to pass an act of parliament providing or requiring a regency for the administration of the government in event of the crown devolving on an heir not having attained that age, until it be so attained (24 Geo. II. cap. 24; 5 Geo. III. cap. 27; 1 Will. IV. cap. 2; 3 & 4 Vict. cap. 2).

† “The king’s prerogative,” says Mr. Amos, “is nothing more than the king’s law” (Notes, Fortescue, *De Laud. Leg. Ang.* p. 44). “If you ask,” says Selden, “whether a patron may present to a living after six months by law, I answer no; if you ask whether the king may, I answer, he may by his prerogative, that is, by the law that concerns himself in that case” (Table Talk, art. Prerogative). “The law,” says Bracton, “gives to the king dominion and power” (lib. i. cap. 8). “The king’s prerogative,” says Sir Henry Finch, “stretcheth not to the doing of any wrong, for it groweth wholly from the reason of the common law, and is as it were a finger of that hand, although so much differing in fashion, that if you set them in parallels together, you shall find it to be law almost in every case of the king that is no law in the case of a subject. And yet, for all that, they are not two laws, but one law; only the common law is the *primum mobile* which draws all the planets in their contrary course” (Law, 85). See also a debate in the House of Commons (1 Rushw. 561–2), and the case of proclamation (12 Rep. 74).

(3 Inst. 84). “In governing the people,” says Fleta, “the king has none above him, except the law, which made him king, and his court, that is, the barons” (lib. i. cap. 5, s. 4). If these views be correct, and Cicero* has rightly defined what a king is, our ruler cannot properly be termed a king; nor, if Blackstone’s conception of a monarchy be just, can our state be called a monarchy (1 Comm. 50).†

Law.

“A king of England,” as Fortescue assured a Lancastrian prince, “does not bear such a sway over his subjects as a king merely, but in a mixed political capacity: he is obliged by his coronation oath‡ to the observance of the laws” (*De Laud.* cap. xxxiv.); and, as he observes in another part of his work, “a king whose government is political, can neither make any alteration or changes in the laws of his realm without the consent of his subjects, nor burden them against their wills with strange impositions” (Ib. cap. ix.). It is, however, a very singular circumstance, that Charles V. of Germany considered that the prerogatives of a king of England were considerably larger than those of a king of France; and so he told Sir Philip Hobbs, our ambassador.

Although the king, unlike the Roman emperors, is irresponsible, not *solutus legibus*, no legal remedy is available against him—an immunity he shares with his colleagues in sovereignty; yet courts of justice, as the ministers of the law, would lend him no aid to enforce any breach of the law, but would punish those that did so. It is in this way that the limitations on royal power act; and this will be sufficient to show that the king is not supreme, and is improperly denominated sovereign.

Long usage has established the maxim, that the king can act only through his servants. Considering the vast extent of our empire,—the necessity of securing, at almost every step, in almost every proceeding, the concurrence of the two houses,—the wisdom and propriety of conciliating public opinion, and allying to the acts of the crown the sympathies and approbation of the people,—it is clear that the king, in the exercise of his prerogatives, must avail himself of the

* *Cum penes unum est omnium summa rerum, regem illum unum vocamus* (See *Re. Pub.* lib. i. cap. 26).

† But see Gibbon, *Decl. and Fall*, chap. iii., and some judicious observations by Mr. Cornwall Lewis, *On the Use and Abuse of Political Terms*, p. 59–73.

‡ The Saxon kings had a coronation oath; and even the Conqueror swore, on two occasions, to observe the ancient laws. Their enemies charged Charles I. and Laud with having altered the oath, by striking out the words *quas vulgus elegerit*, and inserting “agreeable to the king’s prerogative.” By the present coronation oath the king “solemnly promises to govern the people of this kingdom of England, and the dominions thereto belonging,” to cause, as far as he can, “law and justice in mercy to be executed in all his judgments; to maintain the laws of God, the true profession of the Gospel, and the Protestant reformed religion established by the law; to preserve unto the bishops and clergy of this realm, and to the churches committed to their charge, all such rights and privileges as by law do or shall appertain unto them or any of them.” Then, laying his hand upon the holy Gospels, he is to say, “The things which I have herebefore promised I will perform and keep, so help me God;” and he is then to kiss the book. The Act of Union with Scotland (5 Ann. cap. 8) requires the king, at his accession, to take and subscribe an oath to preserve the Protestant religion and the Presbyterian church government of Scotland; and at his coronation to take and subscribe a similar oath to preserve the settlement of the Church of England within England, Ireland, Wales, and Berwick, and the territories thereunto belonging. Who can assent to the doctrine that the “coronation is but an ornament, or solemnity, or honour” (3 Inst. 7), so indignantly repudiated by Foster (*Cr. Law*, 189)?

Law.

advice, and act through the agency of others. This is a necessity, not indeed resulting from law, but the natural and inevitable sequence of circumstances and events, necessitating a course of action on which length of time and uniformity of practice—*fons omnis publici privatique juris**—have insensibly conferred the character of law, and to which they have annexed all its attendant obligations and rights.

Up to the period when this doctrine became settled, the polity of England, to borrow a judicial metaphor (Chudleigh's case, 1 Rep. 120), was half buried "in a sea of troubles, and endangered with utter drowning." The authority of the crown, when fully exercised, was constantly checked by the counter-authority of the House of Commons—*sævos compescuit ignibus ignes*,—who, that the government should perform its functions unimpeded, were required to be either subservient or supreme. It is only since the doctrine has been fully established, that the king, in using his powers, is to act in accordance with the rest of the Parliament, that the English constitution has rested on a firm and steady basis.†

Responsibility of ministers.

It has been understood from the earliest times that a minister of the king is responsible to the houses of parliament, both for the advice he gives his master and for his own conduct in his office. A learned historian considers that it was the proceedings taken by the Commons against Michael de la Pole and others, in 1386, accused of having illegally affixed the great seal to pardons, sold their influence with the judges to suitors, and dissipated the royal treasures in useless expenses and for their own benefit, which formed the first† conspicuous exercise of the formidable power of the Commons to proceed against public officers for in-

* *De Oratore*, lib. ii.

† Observations so sweeping must not rest on assertion; but, too sweeping they will not be thought by any one who studies our history from the Restoration to the accession of the House of Hanover. Read the record of the struggles between Charles II. and his Commons, each using their constitutional powers to the uttermost; and what was the result?—a tyrannical king, a rebellious assembly, and a people divided between fear of a despotism and dread of a republic; foreign money lavished to perpetuate a struggle so favourable to foreign ambition, and wise men lamenting the exile of Clarendon and the neglected counsels of his true-hearted friend Southampton, who in the sunny days of royal popularity saw the first tokens of the storm glooming on the horizon (Clarend. *Life*, and contin. vol. iii. p. 229. Oxf. 1829; Burnet, 168). The advice of Temple was slighted by the court, and, one after another, moderate men withdrew from the conflict, and left the two extremes to their common battle. The court long dispensed with a parliament, despised the one it afterwards summoned, and St. Germain's was at last the refuge of its members. And the reign of William III. affords an instance not less instructive. The House of Commons, proud of the overthrow of an ancient dynasty, was not more chary than its predecessors in the use of its extreme powers; and, at a period of disunion at home, and of danger from abroad, they thwarted the king in every exercise of his prerogative, insulted his person, alienated his affections, impeached his wisest and honestest ministers, brought the name of England into contempt, and upon themselves the just censure of one of their members, that "no one can know one day what the House of Commons will do the next" (Dalrymple, *Memoirs*, App. ii. 240).

‡ An earlier and more remarkable case was the proceedings against Robert Baldocke, bishop of Norwich, chancellor to Edward II., and a distinguished member of the Gascon party. He was charged with having advised the king to seize and waste the possessions of several churches as well as the temporalities of his own see and that of Lincoln (*Rot. Parl.* 1 Edw. III. n. 3; 2 Edward III. n. 21; 28 Edward IV. n. 4; and App. vol. i. p. 440). In this reign the Great Charter was confirmed, with a remarkable addition: "Forasmuch as many people be aggrieved by the king's ministers against right, in respect to which grievance no one can

jurious mal-administration or evil counsel (Mackintosh, 1 *Hist. Eng.* 323; *Rot. Parl.* iii. 216). Certain judges—who, in the same reign, assured the king that as he might at his pleasure remove or punish any of his officers, the Lords and Commons could not, without his licence, impeach them—were attainted; and a useful precedent was thus established, which our ancestors, in subsequent ages, jealous of their liberties, were not slow to follow. A chancellor guilty of bribery (Lord Bacon's case, 2 *St. Tr.* 1087; Lord Macclesfield's case, 16 *St. Tr.* 767), or of abusing the functions of his office (Lord Clarendon's case, 6 *St. Tr.* 331; Lord Somers' case, 14 *St. Tr.* 250); a judge advising the king contrary to law (Tresilian's case, 1 *St. Tr.* 89); or other magistrate attempting to subvert the fundamental laws or introduce arbitrary power (Lord Strafford's case, 3 *St. Tr.* 1381; Lord Clarendon's case, *ut sup. cit.*); a chancellor thought to have affixed the seal to an ignominious treaty (*Ib.*; Lord Somers's case, *ut sup. cit.*), or to a blank treaty (Burnet, 677–80); a lord admiral neglecting the safeguard of the sea (the Duke of Buckingham's case, 2 *St. Tr.* 1326; Lord Orford's case, 14 *St. Tr.* 242); an ambassador betraying his trust (Card. Wolsey's case, 4 *Inst.* 89; the Earl of Suffolk's case, 1 *St. Tr.* 91; the Earl of Bristol's case, 2 *St. Tr.* 1267); a privy councillor propounding or supporting pernicious and dishonourable measures (Suffolk's, Bristol's, and Strafford's cases, *ut sup. cit.*); or a confidential adviser of the king obtaining exorbitant grants or incompatible employments (Clarendon's, Orford's, and Somers' cases, *ut sup. cit.*; Lord Halifax's case, 14 *St. Tr.* 295–6);—in all these instances the Commons have asserted their right of impeachment.

The Parliament of 1661 exercised so frequently their privilege of impeaching the king's advisers, that it has been questioned whether they were actuated by any other motive than simply asserting their rights (Hallam, 2 *Const. Hist.* 237). James II. considered this impeaching privilege as one inconsistent with royal authority;* and his predecessor told Lord Essex that though he did not wish to be, like the Grand Seigneur, surrounded by mutes and bags of bowstrings, to strangle men at his leisure, yet "he did not think that he was a king so long as a company of fellows were looking into all his actions, and examining all his ministers as well as all his accounts" (Burnet, *Hist. own Times*, 838).

The case of the famous Earl of Danby deserves especial notice. He, being then Lord Treasurer, instructed

recover without a common parliament, we do ordain that the king shall hold a parliament once in the year, or twice if need be" (5 Edward III. cap. 29). This has been ignorantly construed to mean that every year a new parliament shall be assembled. The purpose of convoking this assembly, as stated, appears so singular as to excite surprise that it has not been adverted to by Blackstone (*Discourse on the Great Charter*, 2 *Law Tracts*, 112. Oxford, 1762).

* The observations of James are too important to be omitted. Speaking of his brother, he says: "The most fatal blow the king gave himself to his power and prerogative was when he sought aid from the House of Commons to destroy the Earl of Clarendon. By that he put that House again in mind of their impeaching privilege, which had been wrested from them by the Restoration; and when ministers found that they were like to be left to the censure of the Parliament, it made them have a greater attention to court an interest there, than to pursue that of their princes, from whom they hoped not for so sure a support" (*Life*, by himself, vol. ii. p. 593). Lord Mansfield prophesied, after Mr. Pitt's concurrence in the impeachment of Warren Hastings, that the instrument he had used against others would in time be turned against himself or his friends—a prediction fulfilled by the impeachment of Lord Melville.

Law.

Law.

our ambassador at Paris, by letter, to negotiate a peace with France, conditional on the king receiving an annuity from that country, of a certain amount and for a certain time. Our secretary of state was not to be informed of this condition. The king added a postscript to the letter, implying that it was written by *his own order*. But this was not considered as a justification of the minister (Burnet, 293-301). When the Earl of Nottingham, then chancellor, was examined touching his share in the pardon which Danby received from the king, pending his impeachment, he replied that "he neither advised it, drew it, nor altered one word of it;" and that the seal was affixed, at the king's command, by the person who usually carried it; adding, that he did not consider it in his custody at the time (7 Grey's *Debates*, 55). This was considered a sufficient justification.

Choice of ministers.

In his choice of ministers, although the king exercises his own discretion, it is understood to be, and in fact must be, a discretion governed by considerations both of the public service and of public opinion.* These indicate the necessity of his making his selection of members of both houses of parliament who are capable of there justifying their conduct, and of obtaining the concurrence of the houses in the policy they advise the king to pursue. Thus may Lords and Commons be considered, in a certain way, and to a certain extent, to share in the exercise of the royal functions, and even to specify to royalty the men it should call to its councils.

George II.

The glorious reign of George II. would be remarkable if it were only for the firm establishment of these doctrines, which have imparted security alike to the throne and the liberties of the people. George II. was disposed to plunge the nation into a war which neither concerned its interests nor could add to its glory, and dismissed (April, 1757) the ministers who refused to concur in his plans. Two months (June) afterwards, yielding, like a prudent and constitutional prince, to the unanimous voice of his people, he restored them to their employments, and added another to the obligations conferred by the wisdom and patriotism of his illustrious house on their English subjects.

Privy Council.

Formerly, in the most important functions of government, the king was advised by his Privy Council; and this is directed by law to transact all matters and things relating to the well governing of the kingdom, of which, according to the laws and customs of this realm, it has properly cognizance. All its resolutions are to be signed by such of its members as advise and consent to the same (12 & 13 Will. III. cap. 2 & 3). Any natural born subject taking the proper oaths is eligible as a member; but the last-mentioned statute disqualifies all persons born out of the king's dominions, although naturalized by parliament. The 1 Geo. I. stat. 2, cap. 4, s. 2, enacts that no bill of naturalization shall be received without a clause to this effect: but when any eminent foreigner is naturalized, it is usual to pass an Act repealing these Acts in his favour, and then to pass an Act to naturalize him. Such was the course pursued in the case of Prince Albert (3 Vic. cap. 1 & 2).

* It is well observed, in language better than ours, that it is understood that all the great powers of royalty, whether they relate to "the execution of the laws, the nomination to magistracy and office, to the conducting the affairs of peace and war, or to the ordering of the revenue, should all be exercised on national principles and national grounds" (Burke, *Thoughts on the Cause of the present Discontents*, Works, vol. ii. p. 260. 1801).

At present, the chief powers of the council have devolved upon a small committee of their number, consisting usually of the great officers of state, who alone are summoned, and are styled, in common parlance, the Cabinet Council, or more briefly, the Cabinet. This change took place when the crown first saw the policy of conciliating the various influential members of parliament by admitting them to a share in its councils. Sir William Temple, to whose advice this great change may be ascribed as a proximate cause, could hardly have foreseen the revolution in the character of our government which it has slowly and insensibly achieved.*

Law. Cabinet Council.

279. (II.) The second member of parliament is the House of Lords, composed—(1) of the English archbishops and bishops, with the exception of the Bishop of Sodor and Man; (2) of four of the Irish prelates, sitting according to a rota; (3) of the peers of England and of the United Kingdom; (4) of the sixteen Scottish peers and the twenty-eight Irish peers, elected, the first every parliament, the second for life, by the peers of Scotland and Ireland respectively as their respective representatives. To the king belongs the right of creating peers of the United Kingdom.

280. (III.) The third member of parliament is the House of Commons, the elected representatives of the people, and chosen by certain qualified persons.

Duties of Parliament.

According to the ancient form of the writ, parliament, using the term in its vulgar sense, which intends the two houses, assembled on account of "certain arduous and urgent matters touching the king and the state, and defence of the state and church of England" (4 *Inst.* 9). With such purposes it is difficult to reconcile the haughty and intemperate language of Elizabeth † and James I. The former, however, was prudent enough on some occasions to comply with the advice of her Commons (4 *Parl. Hist.* 452-82); whilst the latter provoked their famous declaration (due chiefly to the counsels of Coke, Noy, Glanville, and other great lawyers), "that the arduous and urgent affairs concerning the king, state, and the defence of the realm and of the Church of England, and the making and maintenance of laws, and redress of mischiefs and grievances which daily happen within this realm, are proper subjects and matter of counsel and debate in parliament" (Hatsell, *Privilege Ca.* 79). In earlier times, indeed, the kings were in the habit of seeking the advice, well knowing that they should need the

* On this interesting subject, so little noticed by historians or publicists, consult Temple's *Memoirs*, Works, p. 474, London, 1757; and Burnet, p. 302. Temple advised the king to admit to the Privy Council the chief leaders of the popular party, "who were of the most appearing credit in both houses;" and Gonovelt, the French ambassador, told him, "that a king of England who will be the man of his people, is the greatest king in the world; but if he would be anything more, he is nothing at all!" See a curious conversation between Henry III. and the Prior of the Hospitallers of St. John's, in Matthew Paris (p. 736).

† The queen forbade the parliament from discoursing of affairs of state, thinking that they ought not to deal, to judge, or to meddle with her Majesty's prerogative royal (Dewes, 479-645); see also *ib.* 410-70; Hatsell, *Priv. Ca.* 83; 4 *Parl. Hist.* 452-82). James I. considered that "as it is atheism and blasphemy in a creature to dispute what the Deity may do, so is it presumption and sedition in a subject to dispute what the king may do in the height of his power: good Christians will be content with God's will, revealed in his word, and good subjects will rest in the king's will, revealed in his law" (Works, 557-531). This profane comparison was familiar to the servile lawyers of the day (see Finch, *Law*, 81-3).

Law.

assistance, of the great council of the realm;* and in more recent periods the right of remonstrance has been exercised by both houses with great effect.† The dependence of the crown,‡ and of every branch of administration, on the annual supplies voted by the houses, gives to their remonstrances a practical authority obviously of the highest importance, and which prevents, in the most effectual manner, the abuse of the powers intrusted to the executive officers of the realm. It is held the first duty of these houses "to refuse to support [a] government [as the king's servants or ministers are called] until power was in the hands of persons who were acceptable to the people, or while factions predominated in the court, in which the nation had no confidence." (Burke, *Thoughts on the present Discontents*, Works, vol. ii. 263).

Declara-
tion of
Rights.

The two houses of parliament have, on two memorable occasions in our history, exercised an extraordinary power, justified by extraordinary circumstances, but establishing an important precedent. From the Declaration of Rights, afterwards embodied in the Bill of Rights (1 Will. & M. stat. 2, cap. 2) we may gather the principles on which "the glorious revolution of 1688" was undertaken.§ The declaration states that James II. had endeavoured to subvert and extirpate the Protestant religion, and the laws and liberties of the kingdom; it enumerates the overt acts evidencing such endeavour, all which it stigmatizes as contrary "to the known laws and statutes and freedom of this realm." It declares that James had abdicated the throne, which was thereby vacant. After reciting fur-

* See *Rot. Parl.*, 13 Edw. III. n. 6; 21 Edw. III. n. 58; 28 Edw. III. n. 59; 6 Rich. II. n. 14-5; 7 Rich. II. n. 16; and Brown's *Miscellanea Aulica*, 205. These are amongst the earliest instances of a king seeking the advice of his parliament in relation to foreign affairs. It has been contended that in negotiations with foreign powers, the king can act *ex mero motu*, and without communication with even his privy council; and when he does consult them, he is not bound to follow their advice (Burnet, 677-8).

† Sometimes the remonstrance has been against the counsellors with whom the king has surrounded himself, and sometimes against the policy he has pursued. In the reign of Henry VI. (1451), the commons prayed the king to dismiss certain peers and other persons from his person and councils for ever (Hume, chap. xxi.). The most memorable instances of the influence of the Commons in thwarting the ambitious designs of a king, were, perhaps, those of the addresses of the house which forced Charles to abandon his pretended prerogative of dispensing with the penalties (*Com. Journ.* 27 Feb. 1672), and suspending the operation of laws relating to the church (Burnet, 205-32). The remonstrances of the Commons with Charles II. on the marriage of his brother with Mary d'Este, and their spirited conduct, which broke up the cabal ministry, are described in 2 Grey's *Debates*, 190, 214-22-39.

‡ Edward I., in the twenty-seventh year of his reign, plainly stated that the large grants and supplies he had received formed the sole reason of his having agreed to the confirmation of the great charters. It must not, however, be supposed that this dependence was without its advantage to the crown, for thereby its revenue became much more easily collected. The difficulties which beset the royal commissioners appointed to collect, or rather solicit, the pecuniary aid of the people, ere the privileges of the Commons were established, are amusingly illustrated in a letter addressed to Hubert de Burgh, "the justiciar and king's counsellor," by the royal chamberlain. Mr. Lyson's abstract of this letter is to be found in Sir Francis Palgrave's *Report* for 1831-4, and which may be found in the Appendix (XVII.) to Mr. C. P. Cooper's (privately printed) *Statement respecting the new edition of Parliamentary Writs* (p. xi.).

§ The absurd doctrine of an original contract between king and people—so false in fact, so idle in theory—is not noticed in this great declaration. It is to be found in a resolution of the House of Commons (*Journals*, 28 Jan. 1688, O. S.), and may be characterized as a vain attempt to incorporate into the institutes of English public law a decision, doubtful in its value, of one of the most vexed and least determinable of ethical questions

ther, that the Prince of Orange, by the advice of the house and others, had summoned the convention from which this declaration issued, and which asserted itself to be assembled "in a full and free representative of the nation," it solemnly denounced all the illegal acts of James it had previously mentioned, and, on behalf of the nation, claimed the premises as their ancient and undoubted rights. In the hope that the prince would perfect the deliverance he had begun, they stated their resolution that he and the Princess of Orange should be, and they thereby declared them to be, king and queen of England, France, and Ireland. It is easy to distinguish in this great act the constitutional from the revolutionary principles on which it proceeded. So far were those principles revolutionary as they concerned the right of the houses to judge how far James had abdicated; for if abdication involved the idea of intention (*voluntas*), who would affirm that James had abdicated, seeing that he had himself signified his intention of returning? And could his have been considered as an abdication which would be binding on his posterity? The necessity of the case forced them into this construction of his conduct, for they did not desire to establish the doctrine asserted, with less prudence, but with greater boldness, by the Scottish Estates, that by violating the fundamental laws of the realm, he "had forefaulted the right to the crown" (*Account of the Proceedings of the Estates of Scotland*, &c. p. 35), nor to give any authority to doctrines falsely ascribed to them since, which assert the rights of a people to "cashier kings for misconduct." Still, whatever their language may have been, so far their conduct was revolutionary. In what, then, was it constitutional? So far as it asserted their right, the throne being vacant, to fill it; their right to provide for the defect of royal power, by clothing another with its functions; their right to restore a portion of the supreme authority, which had, in some way or other, become wanting to the state. And this right, in less troubled days, they asserted with success. In 1789, the unhappy malady of the king produced a temporary interruption to the exercise of the royal authority. Mr. Fox, prompted, it has been said, by Lord Loughborough, declared that the heir apparent, being of full age, "had as clear, as express a right to assume the reins of government, and exercise the power of sovereignty, during the continuance of his Majesty's incapacity, as if his Majesty had a natural demise" (27 *Parl. Hist.* 707). Mr. Pitt, however, stood forward as the champion of constitutional rights, and asserted "that in case of an interruption of the royal authority without any previous lawful provision having been made for carrying on the government, it belonged to the other branches of the legislature, on the part of the people at large—the body they represented—to provide, according to their discretion, for the temporary exercise of the royal authority, in the name and on the behalf

Law.

Principles
of the Re-
volution of
1688.Right to
supply the
deficiency
of royal
authority.

* It is impossible to deny, that in the conference between the houses on the word "abdicated," for which the lords would have substituted "deserted," their lordships had the best of the argument. But then the results of their success! what mischief would it not have caused! It was the exigency of the occasion that justified what was done, and against this, of what value were all the arguments of the publicist, though he might

"— quote Puffendorf and Grotius,
And prove by Vattel,
Exceedingly well,
That the deed was quite atrocious?"

Law. of the sovereign, in such manner as they should judge requisite" (*Ib.* 709.) Again, "the Lords and the Commons represented the whole body of the people, and with them it rested as a right—a legal and constitutional right—to provide for the deficiency of the third branch of the legislature, whenever a deficiency arose. They were the legal organs of speech for the people, and such he (Mr. Pitt) conceived to be the true doctrine of the constitution As the power of filling the throne rested with the people at the Revolution, so at the present moment, on the same principle of liberty—on the same right of Parliament—did the providing for the deficiency" (*Ib.* 741-2). At the present day, and in the mind of no constitutional lawyer, can a doubt exist as to the soundness of Mr. Pitt's positions, which are worthy of being thus stated in full, not only from the character of the statesman, and the importance of the occasion, but because they espoused a principle which lies at the very root of the constitution—the sole source of our public law.

Such is the Parliament, whose harmonious action is necessary for the sovereign power to act advantageously for the public, whose interests it is presumed to guard.

Forms of
legislation.

The forms observed in legislation are regulated by the custom of Parliament, to be observed or omitted at its will. They are as follows. Every law, in its inception condition, is termed a *bill*. With the exception of bills of grace or pardon, which usually emanate from the crown,—of bills relating to the rights of the peerage, to the election and qualification of members of the House of Commons, and to the raising or disposition of monies to the public use (the first of which usually originate in the House of Lords, and the latter uniformly in the House of Commons), either house may (which the king, with the exception above stated, cannot) initiate any bill. The bill is in writing, and submitted in succession to both houses, the opinion of which is taken in reference to it, as a whole, four times, on the first, second and third readings, as the stages are technically called, and on the motion "that this bill do now pass;" and between the second and third times the bill is proposed clause by clause, when any alteration or addition may be made in any of the clauses. Clauses may be added also on the third reading, or previously, when the committee report the bill to the house; but all additions and alterations must be approved by both houses before the bill can be considered to have been assented to by them. It is proper to add, that in the case of bills originating in the House of Commons, leave must be obtained from the house before the bill can be "brought in;" nor can any bill be introduced affecting the rights of the crown without its permission. At any stage of the bill, it may be referred to the consideration of a select committee, named by the house under whose consideration it is, out of their members; and the House of Lords is in the habit of referring certain bills to the opinion of the learned judges. When all the above forms have been complied with, the bill is submitted to the king, who gives or withholds his consent.

The king cannot suspend or dispense* with these laws of Parliament, or any other laws, on any pretence

* The practice of dispensing with statutory penalties is supposed to have been commenced by Henry III., in imitation of the popes, and was recognised as lawful by many eminent lawyers (*Luder's Law Tracts*, vol. ii. p. 327). It did not extend to the common law, nor to statutes relating to things necessary for the good of the church or commonwealth, and in which all the king's subjects have an interest (*3 Inst* 154). Such a practice as this is plainly inconsistent

whatever (1 Will. & M. stat. 2, cap. 2). But he* has the power, sometimes, in virtue of his prerogative or right of pardoning offenders, of suspending their full operation as to particular persons (27 Hen. VIII. cap. 24). But this is subject to one important exception (31 Car. II. cap. 2). And the right of dispensing with or suspending laws has been also by one house of parliament abjured, for themselves and their successors (*Lords' Journ.* 4 Feb., *Commons' Journ.* 16 Feb. 1784).†

with the declaration of Fortescue, that without consent of parliament, the king cannot alter the laws (*De Laud.* cap. xxxvi.); but is quite compatible with the opinions of Herbert, Chief Justice to James II., who said "that the laws of England are the king's laws, and that therefore it is the inseparable prerogative of the kings of England to dispense with them."

* Even a king *de facto* (*Bro. Abr.* tit. *Charter de Pardon*, 22).

† One word must be permitted in reference to the privileges of the two houses. Upon this subject very confused notions evidently pervade the minds of the public, or a late controversy would have never occurred. Undoubtedly, the Houses of Lords and Commons are, like the king, directly, and, in themselves, irresponsible. "The freedom of speech, and debates and proceedings in parliament, ought not to be impeached or questioned in any court or place out of parliament" (1 Will. & M. stat. 2, cap. 2). "The independence of parliament is the corner-stone of our free constitution," and "the judges who invaded it in the reign of James I. and his son, have justly shared with those who betrayed the rights of the people in the case of ship money, the abhorrence of all enlightened men" (*per Denman C. J.*, *Stockdale v. Hansard*, 9 A. & E. 110).

The judges are, however, judges of the law; and if either house were to transcend its privileges, "and the subject comes judicially before a court of law, a court of law will not swerve from its duty, but decide according to law" (*per Kenyon, C. J.*, *R. v. Wright*, cit. 6 St. Tr. 1124). In reference to a question respecting a parliamentary election, a learned judge said, "For myself, I will never be bound by any determination of the House of Commons against bringing an action at common law against a false or double return; and a party injured may proceed in Westminster Hall, notwithstanding any order of the house" (*per Wilmot C. J.*, *Wynne v. Middleton*, 1 Wils. 128).

It is true that the judges cannot reach the house, nor could the judges reach the king, were he to exceed the lawful bounds of his prerogative; but, as the fact is, from the necessity of the case, neither king nor house can act, except through agents, and these agents the judges can reach. It is true that the house could retort by committing the judges; but this would be to suspend the constitution, for the independence of the judges is as much guaranteed by the constitution as the independence of the House of Commons. If it be asked, then, whether privilege is bounded by law? it is answered, yes; privilege is bounded by law, as prerogative is bounded by law; as the authority of the courts, the powers of magistracy, the rights of the people, are bounded by law, by law defended, and by law defined.

The question has been involved in great mystery. Strange language has been used, and hints and insinuations thrown out, that these privileges are original and essential attributes of the Commons House, not to be reached by legislation, not defined nor to be defined, and not understood nor to be understood. Such also was prerogative said to be before the great revolution. But parliamentary privilege is not of such recondite learning: "experience hath made" it "well known to parliament men" (4 *Inst.* 4). We have the "means of knowing what are the privileges of the House of Commons" by consulting "the records of that house," and by searching "into the history of parliament for those cases in which a claim of privilege has been made, and examine whether it has been admitted or refused" (1 *Hatsell, Preced.* 1-2). "The law of parliament," says Mr. Hallam, "as determined by regular custom, is incorporated with our constitution, but not so as to warrant an indefinite uncontrollable assumption of power in any case" (*Mid. Ages*, ch. viii.) "No resolution of either house of parliament can make that a legal privilege which was not so before; the law of parliament may be expounded from time to time, but cannot be altered without the authority of the whole legislature" (*Dwarris, On Stat.* 68).

No one is disposed to deny that the House of Commons has privileges; and, according to the high authority of Mr. Hatsell, these appear to have been defined by a resolution agreed to by the house, on the motion of Sir Edw. Coke, in 1621 (*Priv. Ca.* 160). But "power, and especially the power of invading the rights of others,

Law.

Law.
Crimes.

In concluding the consideration of public law, we shall consider the chief of those wrongs which are presumed to be perpetrated to the injury of the public, and are called *crimes*.

281. We shall first treat of those crimes only which are properly to be considered as committed against the state, and in which the injury to individuals is only accidental.

High
treason.

(1.) *High treason*, the most heinous crime acknowledged by the law, is defined by the statute 25 Edw. III., the provisions of which have been extended by the 36 Geo. III. cap. 7 (a temporary Act, made perpetual by the 59 Geo. III. cap. 6). Of high treason there are several species: (1.) Compassing or imagining the death of the king, queen consort, or their eldest son and heir. (2.) Violating the queen consort, the king's eldest daughter unmarried, or the wife of his eldest son and heir. (3.) Compassing, imagining, inventing, devising, or intending death, destruction, or any bodily harm tending to the death, destruction, maiming, wounding, imprisonment or restraint of the person of the king; or, (4.) to deprive or depose him from the style, honour, or kingly name of the imperial crown of this realm, or other his dominions. (5.) Levying war against the king in the realm, and compassing, imagining, inventing, &c. such levying, in order by force or constraint to compel the king to change his measures or councils; (6.) or in order to put any force or constraint upon, or to intimidate or overawe either or both houses of parliament. (7.) Being adherent to the king's enemies in the realm, and aiding them there or elsewhere. (8.) Moving or stirring any foreigner or stranger with force to invade the realm or other the king's dominions. (9.) Counterfeiting certain of the king's seals (11 Geo. IV., & 1 Will. IV. cap. 66). (10.) Slaying the chancellor and certain other judicial officers, being in their place doing their offices. All such compassings, imaginations, inventions, &c. must be expressed, uttered, or declared by publishing any printing or writing, or by any overt act or deed.

In cases of treasonable attempts on the life or person of the king, the culprit is to be tried as if he were charged with murder (39 & 40 Geo. III. cap. 93; 5 & 6 Vict. cap. 51). The *punishment* remains as in other cases. The chief distinction in the proceedings at a trial for high treason, from all other trials, is the necessity of having (except in the cases above mentioned) the overt acts, which evidence each treason, proved by

is a very different thing from privilege: it is to be regarded, not with tenderness, but with jealousy; and unless the legality of it be most clearly established, those who act under it must be answerable for the consequences" (*per* Patteson J., *Stockdale v. Hansard*, 9 A. & E. 214).

"In one sense," says Coleridge J., "the house alone is to judge of its own privileges. In the case of a recognised privilege, the house alone can judge whether it has been infringed, and how the breach is to be punished;" but he repudiated the doctrine "that the house is alone and exclusively the judge of its own privileges, in the sense that it alone is competent to declare their number and extent; and that whatever the house shall resolve to be a privilege, is by such resolution conclusively demonstrated to have been so immemorially" (*Id.* 218). Hear a high authority on this head: "Parliament men," says Selden, "are as great princes as any in the world, when whatsoever they please is privilege of parliament; no man must know the number of their privileges; and whatsoever they dislike is breach of privilege. The Duke of Venice is no more than the Speaker of the House of Commons, but the senate at Venice are not so much as our parliament men, nor have they that power over the people, who yet exercise the greatest tyranny that is anywhere" (*Table Talk*, art. Parliament).

two witnesses (7 Will. III. cap. 3, s. 2, 4). On this subject, see the article TREASON in the Fourth Division (vol. xxviii. p. 739).

(2.) *Misprision of treason*, which consists in the Misprision bare knowledge and concealment of treason, without of treason. any assent thereto; punished by loss of the profits of lands for life, forfeiture of goods, and imprisonment for life (1 Hale, P.C. 374).

(3.) *Sedition*.—Fairly, and with temperance, decency, and respect, to discuss the policy of the king and ministers, without imputing to them corrupt motives, is without question lawful (*R. v. Lambert et al.*, 2 Camp. 398). Nor would this be a free country if it were not so. But cursing the king, wishing him ill, spreading tales against him, or doing whatever tends to lessen him in public esteem, and to embarrass his government, by lessening him in the eyes of his people (*R. v. Tuchin*, Holt, 424), and by alienating their affections from him, are all so many seditions, and this whether any ill effects had been intended or not, and whether the expedients used were ridicule or obloquy. (*R. v. Cobbett*, Holt, *On Lib.* 529; *R. v. Burdett*, 14 East, 150; *R. v. Harvey*, 2 B. & C. 257). The punishment for sedition is fine and imprisonment.

(4.) *Præmunire*.—A class of offences which derive *Præmunire* their name from the initial word of the writ or process framed for the execution of certain statutes made against the exorbitant pretensions of the see of Rome.* The 16 Ric. II. stat. 5 (*execrabile illud statutum*, as it is, naturally enough, styled by Martin V.), usually styled the Statute of *Præmunire*, enacts, that whoever should procure, at Rome or elsewhere, any translations, processes, excommunications, bulles, instruments, or other things which touch the king, against him, his crown and realm, and all persons aiding and assisting therein, shall be put out of the king's protection, their lands and goods forfeited to the king's use, and they shall be attached by their bodies to answer to the king and his

* It has become the fashion to ascribe the Acts against Roman Catholicism, or, more properly, popery, to the intolerance of the Reformers, who sought to fetter conscience by laws. But if we read the early history of the Anglican church, we shall find that the excesses of Roman ambition were then guarded against with a zeal for her spiritual independence not one whit less affectionately jealous than they have been since. The Anglo-Saxon clergy were distinguished by their unfriendliness to transmontane doctrines and precepts (*Kemble, Cod. Dip. Æv. Ang.-Sax.* pref.); and even in our darker ages there were not wanting wise princes and holy prelates who asserted the liberties and free rights of the Anglican church (*See* Mr. Churton's admirable work on the *Early English Church*). Politically speaking, what could have been done with Rome when one of her bishops (Pius V.), after declaring, in a bull dated March, 1569, that, "as successor of St. Peter, he was constituted by Him who ruleth on high over all nations and all kingdoms, that he might pluck up, destroy, dissipate, ruin, plant and build," went on—"We deprive her [Queen Elizabeth] of her pretended right to the kingdom, and of all dominion, dignity, and privilege whatsoever, and absolve all the nobles, subjects, and people of the kingdom, and whoever else have sworn to her, from their oath, and all duty whatsoever, in regard of dominion, fidelity, and obedience." (*Camden, Annales, sub anno 1570. See also* Caudrey's case, 5 Rep. xl.) The oath which was taken by the pope's collector, before the rupture of the Anglican church with that of Rome, is worth preserving, as a proof that the first clause of Magna Charta—*Ecclesia Anglicana libera sit et habeat omnia jura sua integra et libertates suas illæsas*—was not suffered to be wholly a dead letter. "*Nullam executionem literarum vel mandatorum Domini Papæ per me vel alium, faciam nec fieri permittam, quæ poterit esse præjudicialis Regiæ Majestati dicti Domini nostri Regis, aut Regaliâ, legibus vel juribus suis vel eidem Regno. Nullas literas Papales nec alias recipiam, nisi eas sitiis, quo potero, deliberavero concilio dicti Domini nostri Regis antè quam publicentur seu deliberentur alicui personæ viventi.*"

Law.

council, or process of *præmunire facias* shall be made out against them. Those are guilty of *præmunire*—(1.) Who appeal to Rome from any of the king's courts, sue there for any licence or dispensation, or obey processes issuing thence (24 Hen. VIII. cap. 12; 25 Hen. VIII. cap. 19 & 21). (2.) Who, being a dean and chapter, refuse to elect, or an archbishop or bishop, refuse to consecrate, any one named by the king to any see (25 Hen. VIII. cap. 20). (3.) Who assert maliciously and advisedly, by speaking or writing, that either or both the houses of parliament have a legislative authority without the king (13 Car. II. cap. 1). (4.) Who send any subject of this realm a prisoner beyond sea—an offence involving heavier penalties, and which the king himself cannot pardon (31 Car. II. cap. 2). (5.) Who assert maliciously and directly, by printing, writing, and advisedly speaking, that the Pretender or any person, other than according to the Acts of Settlement and Union, is king, or that the king and parliament cannot make laws to limit the descent of the crown (6 Ann. cap. 7). (6.) Who knowingly and wilfully solemnize, assist, or are present at any forbidden marriage of any descendant of George II. prohibited from contracting matrimony without the consent of the crown (12 Geo. III. cap. 11). These are the chief offences of *præmunire*. The punishment is thus stated:—"That, from the conviction, the defendant shall be out of the king's protection, and his lands, tenements, goods and chattels forfeited to the king; and that his body shall remain in prison at the king's pleasure, or (as other authorities have it) during life" (*Co. Litt.* 129 b).

Disobedi-
ence to
royal com-
mand.

(5.) The king, as the representative of the state, is entitled to demand, for the purposes of the state, the services of his subjects. To refuse, therefore, to assist by advice in the councils of the king, to aid in the defence of the realm, to yield assistance to the civil power by joining in the *posse comitatus*, to attend the privy council, being thereto convoked by royal summons, to return from abroad at the royal command, or to go thither when in like manner forbidden, or to serve an office to which one was duly elected, is punishable with fine or imprisonment, or both.

Unlawful
oaths.

(6.) *Unlawful oaths*.—To administer, or aid, abet or consent to the administering of oaths binding the persons taking the same to engage in mutinous or seditious purposes, or disturb the public peace, &c., is felony in those who administer and those who take the same, and punishable with transportation for not more than seven years (37 Geo. III. cap. 173). The oaths must purport to bind those taking them—(1) to engage in some mutinous or seditious undertaking; (2) to disturb the public peace; (3) to be of some association formed for such purpose; (4) to obey orders of a committee or body not lawfully constituted, or of a leader or other person having no lawful authority for that purpose; (5) to abstain from informing or giving evidence against any associate or other person; (6) and from revealing any unlawful combination, (7) illegal act done or to be done, or (8) any illegal oath administered, tendered to, or taken by such person, or to or by any other person, or the import of any such oath. It is felony to administer, cause to be administered, or aid or abet in administering unlawful oaths, purporting to bind the party taking the same to the commission of treason or felony. He who administers or aids &c. in administering such oaths is punishable by transportation for life, or for not less than fifteen

years, or by imprisonment for not less than three years (7 Will. IV. and 1 Vict. cap. 91); and he who takes them is punishable with transportation for life, or a shorter period, if the court thinks fit (52 Geo. III. cap. 104).

Law.

(7.) *Inciting to mutiny*.—To seduce any soldier, sailor or marine from his duty or allegiance, to incite or stir him up to an act of mutiny, to make or endeavour to make a mutinous assembly, or commit any traitorous or mutinous practice whatsoever, is felony, punishable, under the 7 Will. IV. and 1 Vict. cap. 91, as above mentioned.

(8.) *Stealing or embezzling naval or military stores* is punishable with transportation for life, or not less than seven years, or by imprisonment, with or without hard labour, for not less than seven years (4 Geo. IV. cap. 53). If the value of the stores be under twenty shillings, the offender may be fined, at the discretion of a certain officer connected with the department to which the stores belonged (1 Geo. I. stat. 2, cap. 25). The security of the public stores is likewise provided for by other acts of parliament.

Stealing or
embezzling
naval or
military
stores.

We have now to treat of offences against public justice. Judicial

(1.) *Extortion of officers of justice*, who, under pre-tence of authority, extort illegal fees from prisoners, is punishable with fine or imprisonment, or both.

(2.) As also is *bribery*, or attempting to bribe such an officer (3 Inst. 147); and an officer, judicial or ministerial, taking a bribe, is punishable in the same way.

(3.) *Misconduct of officers of justice*, being what the law terms malfeasance, that is, wrong-doing,—or culpable non-feasance, that is, neglect of an officer of justice in regard to his office,—is a misdemeanor, punishable with fine or imprisonment, or both.

When magistrates are guilty of oppression or other wilful malfeasance in the execution of their duties, they are generally proceeded against by information in the King's Bench. The punishment is fine or imprisonment, or both.

(4.) He who negligently suffers the escape of one lawfully in his custody in respect of some criminal charge, is punishable by fine; and he who so escapes is punishable by fine and imprisonment. A voluntary escape amounts to the same offence, and is punishable in the same manner as the offence of which the prisoner is guilty, and for which he is in custody. The officer, however, cannot be thus punished until after the original delinquent has been found guilty and convicted.

Escape.

(5.) *Prison-breach*, or breaking out of prison, is felony, if the defendant were in custody for treason or felony (1 Edw. II. stat. 2), punishable by transportation for seven years, or imprisonment for not exceeding two, with or without hard labour, or solitary confinement for the whole or any part of the same (7 & 8 Geo. IV. cap. 28); and, if a male, to be once, twice, or thrice publicly or privately whipped, in addition to the imprisonment, if the court thinks fit (7 & 8 Geo. IV. cap. 28). If the defendant were in custody for any other offence, he may be punished by fine or imprisonment. Aiding and assisting prisoners escaping from prisons (certain prisons excepted, for which special provision is made,) is felony, punishable by transportation not exceeding fourteen years (4 Geo. IV. cap. 69). To aid them in escaping from constables or officers conveying them under warrant to prison, is felony, punishable by transportation for seven years (16 Geo. II. cap. 31). Assisting prisoners of war to

Prison
breach

Law. escape is felony, punishable by transportation for life, fourteen or seven years (52 Geo. III., cap. 156).

Return from trans-
portation. (6.) A transported convict, returning before the expiry of his sentence, is punishable with transportation for life, or for not less than seven years, or imprisonment for not less than three years (4 & 5 Vict. cap. 56), with or without hard labour, for not more than four years.

Perjury. (7.) *Perjury* may be defined to be a crime committed when a *lawful oath* is administered in some *judicial* proceeding, entertained by a competent jurisdiction, to a person who swears wilfully, absolutely, and falsely, with deliberation and intention, on a point material to the matter in question (3 *Inst.* 164; *R. v. Aylett*, 1 *T. R.* 69, 1 *B. & A.* 21). Perjury is punishable at common law with fine, imprisonment, and hard labour (3 Geo. IV. cap. 114), at the discretion of the court; and by the 2 Geo. II. cap. 25, the judge may order the party to be transported, or imprisoned and kept to hard labour, for not more than seven years. False affirmations by Quakers, Moravians, and other separatists, are punishable in the same manner, under divers acts of parliament. Subornation of perjury is punished in the same manner as perjury.

Com-
pounding
of felony. (8.) *Compounding of felony* is when one, for consideration of reward, agrees to forbear the prosecution of a person who has committed a felony against him. It is punishable with fine and imprisonment. And compounding of informations upon penal statutes, without the permission of the court, is punishable by a fine of ten pounds, and incapacity for the future to sue on any popular or penal statute (18 Eliz. cap. 5). Taking a reward for the recovery of stolen goods is felony, punishable with transportation for life or not less than seven years, or imprisonment for not more than four years, with or without hard labour or solitary confinement for the whole or any part of the same; and, if a male, to be whipped, once, twice or thrice, if the court thinks fit (7 & 8 Geo. IV. cap. 29; and see 5 & 6 Will. IV. cap. 38).

Conspiracy (9.) A *conspiracy* is an agreement between two or more persons, (1) falsely to charge another with a crime, either from malice or to extort money; (2) wrongfully to injure any person or persons in any other manner; (3) to commit any offence punishable by law; (4) to do any act with an intent to pervert the course of justice; (5) to effect a legal purpose with a corrupt intent, or by improper means; (6) to which we may add, conspiracies or combinations of journeymen to raise their wages, &c. (Archbold, *Crim. Plead.* 579-80). Conspiracy is, at common law, a misdemeanor, punishable by fine or imprisonment, or both.

Offences against social economy are now to be considered.

Coining. (1.) *Coining*.—The right of coining money, and fixing its denomination, is reposed by law in the king; and anciently (25 Edw. III.) its violation, by counterfeiting coin, was punishable as high treason.* The present law, which, it would seem, has impliedly repealed its predecessor, is very properly milder in its provisions. The making or counterfeiting of gold or silver is felony, punishable, at the discretion of the

Law. court, by transportation for life, or not less than seven years, or by imprisonment for not more than four years, with or without hard labour, or solitary confinement during the whole or any part thereof (2 Will. IV. cap. 34). To impair, diminish, or lighten gold or silver coin, with intent to make the same pass current, is a felony, punishable by transportation for not more than fourteen or less than seven years, or imprisonment, with or without the aggravations above mentioned, not exceeding three years (*Ib.*). Disposing of, or receiving, or offering so to do, counterfeit gold and silver coin at less than their pretended denominations, is punishable in the same manner as counterfeiting the same. The counterfeiting of copper coin is less severely punished. The statute, however, contains a variety of provisions well calculated to ensure the purity and authenticity of our circulating medium.

(2.) *Nuisance*.—Anything that is injurious to the health, personal safety, morals, convenience, or comfort of the public, is held to be a common nuisance. The offences included under this head have each their appropriate punishment assigned by law. They are, however, too numerous to be detailed here; nor will it be expected that we should treat of smuggling, or offences against fiscal laws, which, from their nature, are continually fluctuating.

Offences against the public peace now claim our notice, of which the most important is—

(1.) *Riot*, which is where three or more assemble together—their assemblage being accompanied with circumstances of either actual force or violence, or of apparent tendency thereto, as if they are armed, use threatening language, turbulent gestures, &c., calculated to inspire terror—and then proceed to execute their design, and actually do execute it (*R. v. Hughes*, 4 C & P. 372; *R. v. Birt*, 5 C. & P. 154; 1 Hawkins, c. 65, s. 1-5). It is immaterial whether the act done be unlawful or not, if so done as to inspire people with terror. Riot is punishable by fine or imprisonment, and by hard labour (3 Geo. IV. cap. 114). Where twelve or more persons are engaged in a riot, and they remain together one hour after proclamation made, according to 1 Geo. I. stat. 2. cap. 5, they are punishable by transportation for life, or for not less than fifteen years, or to be imprisoned for not exceeding three years (7 Will. IV. and 1 Vict. cap. 91). Even to oppose the making of the proclamation is punishable in the same way.

If persons assemble together on a lawful occasion, as for innocent diversion, &c., and, suddenly quarrelling, fight, it is but an *affray*, and no riot, unless indeed, previous to fighting, they form themselves into parties with promise of mutual assistance. Affray is punished by fine or imprisonment, or both; but to constitute an affray, it must have taken place in a public street or highway. Riot is also to be distinguished from what is called a *route*, which is, when such an assembly as is first described proceeds to execute its design, but does not effect the same; and if it does not so proceed, it can be dealt with only as an unlawful assembly.

(2.) *Threatening letters*, which we have treated so fully in another place as to require no mention here (see Division IV. vol. xxviii. p. 611).

We have postponed to this place the consideration of another class of crimes, some of them more heinous in their character than those we have been discussing; and our arrangement has had reference simply to the

* *Sicut legem figere signum est summi imperii, ita et nummum cudere, nam νόμισμα, ut docet Aristoteles, et nomen suum et vim habet ἀπὸ τοῦ νόμου. Hinc majestatis criminibus accensetur nummos corrumpere.* (Grotius, *Annotationes in Novum Testamentum*, Matr. xxii. 20, p. 85. London, 1727).

Law. convenience of treating them in a body, while some of them are cognizable only by peculiar tribunals. The crimes to which we allude are those committed against the ordinances of the church.* Of these, the chief are:

Apostasy. (1.) *Apostasy*, which is a total renunciation of Christianity, is an offence properly to be punished in the ecclesiastical courts, which correct the offender *pro salute animæ*. The temporal law interferes in the matter no further than to incapacitate any person educated in, or having made profession of Christianity, who, by writing, printing, teaching, or advised speaking, shall deny its truth and the authority of the Holy Scriptures, on the first offence, for holding any office or place of trust; and on the second, for bringing any action, and for being guardian, executor, legatee, or purchaser of lands; and it subjects him, moreover, to three years' imprisonment, without bail (9 & 10 Will. III. cap. 32). This does not extend to those denying the divinity of one of the persons of the Holy Trinity (53 Geo. III. cap. 60). The delinquent, who within four months after the first conviction publicly recants his errors, is discharged from all his disabilities.

Heresy. (2.) *Heresy*† is the public and obstinate denial of some essential doctrine or doctrines of Christianity, and is defined by Sir Matthew Hale as *sententia rerum divinarum humano sensu electa, palam docta, pertinaciter defensa* (1 P. C. 384).

It is a crime to be punished by spiritual censures only, and one of which the ecclesiastical judge solely has direct cognizance, and his jurisdiction is not assisted by the temporal law (29 Car. II. cap. 9), though he ought not to declare any opinion to be heresy which has not been so declared‡—(1) by the Canonical Scriptures; (2) by the four first general councils, or such others as have used the words of the Holy Scriptures; (3) or which shall be hereafter declared heresy by the Parliament, with the assent of the clergy in convocation (1 Eliz. cap. 1). Excommunication, the most powerful engine which the spiritual judge could use, is now attended with no civil disabilities whatever (53 Geo. III. cap. 127, an act which extends to England only). Incidentally, also, the question of heresy may be determined by the temporal courts, as in an action at law against a bishop for non-institution of a clerk, where, if he defends himself on the ground that the clerk held heretical opinions, the court will be bound to ascertain the truth of the defence (1 Hawkins, P. C. cap. 2).

* Blackstone styles them offences "against God and religion," a title so general, that it would embrace every species of crime.

† The first statute against heresy is the 5 Rich. II. st. 2, cap. 5, by which sheriffs of counties are empowered to arrest public preachers of heresy. This was properly an *ordinance*, and not a statute, but improperly inscribed in the parliament roll. This violation of the constitution awakened the ire of the Commons, who made a declaration, in which they recite the ordinance, and add, "that it never was assented to or granted by them; but that what had been proposed in this matter was without their concurrence, and they pray that the statute be annulled, for it was never their intent to bind themselves or their descendants to the bishops more than their ancestors had been bound in time past" (*Rot. Parl.*, 6 Rich. II). The 2 Hen. IV. cap. 15, crept into the statute-book in a similar way.

‡ Properly, these limitations on the authority of the ecclesiastical judge, in determining what is heresy, had reference only to the high commission court, since abolished (1 Will. & M. stat. 2, cap. 2): "yet seeing in the high commission there be [were] so many bishops and other divines, and learned men, it may serve for a good direction to others, especially to the diocesan, being a sole judge in so weighty a cause" (3 *Inst.* 40).

Law. (3.) *Reviling the ordinances of the Church*.—Whoever reviles the sacrament of the Lord's Supper may be punished by fine and imprisonment (1 Edw. VI. cap. 1; 1 Eliz. cap. 1). Any minister speaking in derogation of the Book of Common Prayer, shall, if unbeneficed, be imprisoned for one year for the first offence, and for life for the second; and, if beneficed, shall, for the first offence, be imprisoned six months, and forfeit a year's value of his benefice; for the second, be deprived and imprisoned one year; and for the third, be also deprived and imprisoned for life; while those who in plays, songs, or open words shall speak in derogation, depraving or despising of the said book, forcibly preventing its reading, or causing any other service to be used in its stead, shall forfeit for the first offence one hundred, and for the second, four hundred marks, and for the third, all his goods and chattels, and suffer imprisonment for life (1 Eliz. cap. 2). Respecting laws framed at the time when Rome and Geneva had combined against our liturgy, nothing need be added, except an expression of regret that they should have been suffered so long to remain unexecuted.

(4.) *Blasphemy against God* is an offence against our Blas-laws; for however well the Roman emperor said to phemy. the senate, "*Deorum injurias diis curæ*" (*Tac. Ann.* lib. i. s. 73), to deny the existence or insult the majesty of the Almighty is not only an offence to God and religion, but a crime against the laws, state and government * * * To say that religion is a cheat, is to dissolve all those obligations whereby the civil societies are preserved; and Christianity is part of the laws of England; and therefore to reproach Christianity is to speak in subversion of the law (*per Hale C. J.*, *R. v. Taylor*, 1 Vent. 293). Rightly, therefore, was it a part of the punishment of the defendant in this case, that he should stand twice in the pillory, with a paper on his head, on which was inscribed, "*For blasphemous words tending to the subversion of all government.*" The punishment of the pillory is, however, now abolished (1 Vict. cap. 23). Blasphemy is an offence by the statute (9 & 10 Will. III. cap. 32), and at common law. If the blasphemy be contained in a libel, the blasphemer may, on a second conviction, be banished for life from the king's dominions, for any term the court shall direct: if he do not, thirty days after sentence pronounced, go into banishment, the king is empowered, with the advice of his privy council, to send him to any place out of his dominions; and his return before the period of banishment have expired, subjects him to transportation for fourteen years (60 Geo. III. and 1 Geo. IV. cap. 8).

We have now to consider those offences which, being directed against individuals, their persons or their property, may properly be styled *private crimes*.

(1.) *Larceny, or theft*, is the felonious taking or carry-Larceny. ing away of the personal goods of another, and is either simple, which is where the taking is unaccompanied by additional atrocity, or mixed or compound, which includes the aggravation of taking from a house or the person. The punishments for larceny are various.

(2.) *Receiving stolen goods*, whether chattels, money, Receiving valuable security, or other property, is felony in all cases stolen where the stealing or taking the same is felony, and goods. punishable by transportation for not more than fourteen or less than seven years, or imprisonment for not more than three years, with or without hard labour, or with

Law.

or without solitary confinement, for the whole or any part of the time (7 & 8 Geo. IV. cap. 29, s. 4); and if a male, to be once, twice or thrice publicly or privately whipped, in addition to the imprisonment, if the court shall think fit (7 & 8 Geo. IV. cap. 29, s. 54).

Where the stealing, taking, obtaining, or converting of the goods amounted to a misdemeanor only, the receiving the same is also a misdemeanor, and punishable with transportation for seven years, or imprisonment for not more than two years; and if a male, to be once, twice, or thrice publicly or privately whipped (if the court shall so think fit), in addition to such imprisonment (7 & 8 Geo. IV. cap. 29, s. 55).

Embezzlement.

(3.) *Embezzlement* is the felonious appropriation by one, to his own use, of chattels, money, &c. whereof he is possessed for the use of another. Embezzlement by clerks or other servants is larceny, punishable by transportation for not more than fourteen or less than seven years, or by imprisonment, with or without hard labour, and with or without solitary confinement, for the whole or any part of the imprisonment (7 & 8 Geo. IV. cap. 29, s. 4), not exceeding three years; and if a male, to be once, twice or thrice publicly or privately whipped, in addition to the imprisonment, if the court shall think fit (7 & 8 Geo. IV. cap. 29, s. 47); embezzlement by bankers is punishable with transportation for not more than fourteen or less than seven years, or by fine or imprisonment, with or without hard labour, and with or without solitary confinement, for the whole or any part thereof (7 & 8 Geo. IV. cap. 29, s. 4), or both (7 & 8 Geo. IV. cap. 29, s. 49; see also 4 & 5 Vict. cap. 56).

Cheating.

(4.) *Cheating* is the obtaining from any one, under false pretences, any chattels, money or valuable security, with intent to cheat or defraud him of the same: it is punishable by transportation for seven years, or fine or imprisonment, with or without hard labour, and with or without solitary confinement, for the whole or any part of the time (7 & 8 Geo. IV. cap. 29, s. 4), or both.

Burglary.

(5.) *Burglary* is the breaking and entering the dwelling-house of another in the night time, that is from nine o'clock in the evening to six o'clock the next morning, with intent to commit a felony therein. The entering with intent to commit a felony, or, being there, the committing any felony, and in either case breaking out of the said dwelling-house in the night time, is also a burglary, which is punishable by death, if attended by violence; and if not, by transportation for life, or not less than ten years, or by imprisonment for not exceeding three years (7 Will. IV. and 1 Vict. cap. 86).

Any permanent building in which the renter or owner or his family dwell, or a set of chambers in an inn of court or college, or any out-houses communicating with the dwelling-house, either immediately or by means of a covered and enclosed passage, is held to be included in the term dwelling-house.

Arson.

(6.) *Arson* is the wilful and malicious setting fire to churches, chapels, houses, shops, warehouses, offices, outhouses, or buildings for trade or manufactures, and is, in the case of dwelling-houses, any person being therein, a felony punishable with death; if with intent to rob, it is punishable as felony, with transportation for life, or not less than fifteen years, or imprisonment not exceeding three years (7 Will. IV. and 1 Vict. cap. 87). Setting fire to a church, chapel, warehouse, &c., with intent to injure or defraud any person, is

felony, punishable with transportation for life, or not less than fifteen years, or with imprisonment not exceeding three years. To set fire to ships, with intent to commit murder, is the same offence, punishable with death; or if with intent to defraud owners or insurers, it is felony, punishable by transportation for life, or not more than fifteen years, or with imprisonment for not more than three years. Setting fire to agricultural produce or coal mines is a felony, punishable by transportation for life, or not less than fifteen years, or imprisonment not exceeding three years (7 Will. IV. and 1 Vict. cap. 89).

(7.) *Forgery* may be defined the unlawful making or alteration of a writing with intent to defraud another. It is a felony, punishable by transportation for life, or not less than seven years, or by imprisonment for not more than four nor less than two years (7 Will. IV. and 1 Vict. cap. 84).

(8.) *Murder* is thus defined by Lord Coke:—Where Murder. a person of sound memory and discretion unlawfully killeth any reasonable creature in being, and under the king's peace, with malice aforethought, either expressed or implied, this crime is felony, punishable by death; as also are injuries with intent to murder; but if the injury be not effected, the punishment is transportation for life, or fifteen years, or imprisonment for not more than three years (7 Will. IV. and 1 Vict. cap. 85).

(9.) *Manslaughter* is the unlawful and felonious killing of another without any malice expressed or implied. (1.) Involuntary, where a man doing an unlawful act, not amounting to felony, kills another. (2.) Voluntary, where upon a sudden quarrel two persons fight, and one of them kills the other, and where a man greatly provokes another by some personal violence, and the other immediately kills him. Manslaughter is felony, punishable by transportation for life, or for not less than seven years, or with imprisonment, with or without hard labour, not exceeding four years, or with fine (9 Geo. IV. cap. 31).

Rape.—For a definition of this crime, reference should be had to the Fourth Division of this work. It will be sufficient to observe, that by the 4 & 5 Vict. cap. 56, the punishment is diminished to transportation for life.

(10.) *Assault*, which is a forcible attempt to commit a crime against the person of another, is a misdemeanor, punishable by fine or imprisonment, or both.

Of private offences, these may be considered the chief.

PART II.

The Private Law of England.

In discussing the private law of England, we shall consider—(1) its several species, and their respective sources; (2) the extent of its authority; (3) its general doctrines; and (4) its procedure.

Sect. 1.—Species and Sources of English Private Law.

282. The private law of England may fitly be considered as composed of various systems of law, different in their origin, character, and, in some cases, obligation. It will be convenient to consider these systems separately, and as they are *general* or *particular* in their obligation. Of the first kind are the traditional or common law and the statute law; of the second, are the customary law and the laws ecclesiastical and maritime.*

* Although the division of the law of England into the written and unwritten law is authorized by the examples of Sir Matthew

Law.

Manslaughter.

The private law.

Its species.

Law.
The Com-
mon Law.

(I.) The common law, as it is emphatically called, is the traditional jurisprudence of the land, "the universal and immemorial usage throughout the country" (3 D. & E. 261), and is in England what the Roman civil law is in some countries—a law which speaks when other laws are silent; and, from the flexible and subtle qualities it possesses as a law of principle and reason, it supplies all the deficiencies of legislative foresight.*

Hale (*Hist. C. L.* ch. ii.), and Blackstone (1 *Comm.* 63), and sanctioned by the example of Justinian (*Inst.* lib. i. tit. ii. s. 3), and of the Greeks (*Dig.* I. 1, 6), it is not easy to apprehend the principle upon which it rests, seeing that all law in England is reduced to writing, and to be found in books. And besides, the circumstance of a law being in writing, is an accident, having no relation to the authority or nature of the law—to its historical character or scientific capabilities. Classifications of this kind have a direct tendency to impede the progress of the law as a science. Furthermore, the classification censured is one not recognised by the law itself; and it is a wholesome maxim—*Ubi lex non distinguitur neque vos distinguere debemus* (see Lord Stair's *Inst.* I. 13, 1).

* Of the origin of this system so little is known, that we must regret its earlier monuments have not been subjected to a more critical and intelligent examination than they have hitherto been. Indeed the history of English law has yet to be written; and it may be permitted to the author of this note, "much revolving these things," to advise the student that the historical information in both Blackstone and Mr. Reeve's learned work is to be received with great caution. Those works were written at a time when historical criticism was less profound in its researches, and less intellectual in its appreciation than it has been since.

A variety of opinions has been broached regarding the source of the common law. Fortescue believes that the common law of his time (and, with trifling exceptions, it is that of ours) was known to the Britons, and survived all the various revolutions which have since swept over the island, (*De Laud.* cap. xvii.); others ascribe it to the wisdom of the Danes, improved by few additions from the Saxon, and fewer from the Norman laws. (Mr. Hakewill's *Disc.* 1 Hearne's *Coll.* 6). Burke conceives the Conquest to be the important era in our legal history, (*Works*, vol. v. p. 595. 1813); while Sir William Temple (*Works*, vol. iii. p. 130-1) and Sir Matthew Hale (*Hist. C. L.* chap. v.) concur in the modern opinion, that Saxon laws and Saxon institutions long survived the defeat at Hastings, and that the Conquest did not disturb the civil polity of the country in any material degree (1 Ellis, *Introd. Domesd.* 196; Palgrave, *R. and P.* i. 53-5).

Perhaps, from a rigorous investigation into the grounds and reason of the common law, it might appear that for certain of its peculiarities it is indebted to the Roman law. This system of jurisprudence was largely studied in England in very early times. Claudius Caesar, we are told by Tacitus, sent over a colony to instruct the natives in Roman law (*Ann.* lib. xii. cap. 32). Papinian, assisted, it has been thought, by Paulus and Ulpian (Selden, *Diss. ad Fletam*, cap. iv. s. 3.), sat on the bench at York, (*Duck. De Usu Jur. Civ.* lib. ii. cap. viii. s. 66), as did the Emperor Severus and his son Caracalla (Drake, *Hist. York*, p. 10), one of their judgments, there delivered, being preserved in the Code (III. 6, 8); and at the same place, in the days of Alcuin, as he himself testifies, there was a famous Roman law school, (*Poema de Pontif. et Sanct. Ebor. Opera*, i. 256. Ratisb. 1777). St. Aldhelm, the learned bishop of Sherborne, in the seventh century, is supposed to have compiled a treatise on Roman law (Palgrave, *R. and P.* i. 648). He alludes to the ardour with which that system was studied in England in his times, (*St. Aldelmi Epist.* 2 Wharton *Ang. Sac.* 6). The results of these studies are apparent in what we call the Anglo-Saxon laws (Savigny, *Geschichte*, b. 2, kap. x.); and the father of English history informs us that Ethelbert, who died in 613, established a system of laws in imitation of those of Rome (Beda, *Hist. Ecc.* lib. ii. cap. 5). Our early law writers Bracton and Fleta, quote the Roman law repeatedly (2 Burr. 1079); and the proem to Glanville's famous treatise is almost a paraphrase of that to the *Institutes*. The great Vacarius expounded Roman law at Oxford in 1129 (Wenck, *Magister Vacarius*, 65), as shortly afterwards did Francesco Accorso or Accursius, the younger of the two famous Bolognese glossators of that name (Tiraboschi, *Dell. Lit. Ital.* tom. iv. lib. ii. s. 21. Milan, 1823); and our great bishop Grosseteste wrote a treatise in its favour. (Selden, *Diss. ad Fletam*, cap. ix. p. 538, 1685). Theobald, the Norman archbishop of Canterbury, founded at Oxford a professorship of civil law, and used to

The sources of the common law are twofold—judicial opinions, which are the recorded opinions of the judges, and certain text writers.

Judicial opinions are recorded—

(1.) In what are styled reports, which are histories of the several cases, containing a summary of the proceedings, of the arguments of counsel, and the decisions of the judge, with his reasons, as stated by him. From Edward II. to Henry VIII. we have a series of these reports, taken by the prothonotaries, or chief scribes, of the court, authorized by the judges, and published at the expense of the crown (1 *Comm.* 71). They have been subsequently the work of "private and contemporary hands" (*Ib.* 72); and their authority depends upon the sense the court entertains of their fidelity. A manuscript report of a case is also authority (Willes, 239), although new to the court or some of the parties, or both (1 P. W. 680).

(2.) In what is styled the record of the court (*Co. Records. Litt.* 24 a; 1 *Ca. temp. Hard.* 362-74), which is "a memorial or remembrance, in rolls of parchment, of proceedings and acts of a court of justice which hath power

retain in his house, according to Peter of Blois, his chaplain (*Epist.* vi. 3. Paris, 1519), several learned persons, famous for their knowledge of law, who spent the hours between prayers and dinner in lecturing, disputing, and debating causes. Very nearly, in truth, was fulfilled the prediction of Juvenal—

Gallia causidicos docuit facunda Britannos.

In early times the clergy were, in England, both civil and common lawyers, and *nullus clericus nisi causidicus* became a proverb. "The person most skilled in the ancient laws and customs of the land," in the time of the Conqueror, was Egelric, bishop of Chichester (*Spic. ad Eadm.* p. 197); and in the days of his successor flourished one Ranulph, "a clerk," or priest, whom William Malmesbury (123) styles "*invictus causidicus*." The monks of Croyland (Ingulph. 36) and of Spalding (Nicholls, *Literary Anec.* vi. 38) were distinguished as lawyers; and when we remember the attachment of the clergy to the civil law, we can hardly doubt that that system was not without its influence in giving at least form and coherency to the doctrines of our law. On this subject, see 12 *Mod.* 482, and some observations by Sir William Jones, in his *Life* by Lord Teignmouth, vol. ii. p. 168.

Our knowledge of Anglo-Saxon jurisprudence is in reality less extensive and precise than is generally believed; for the collections which profess to contain the Anglo-Saxon laws are composed only of the records of decisions, operating as precedents, or the scanty enactments of the *witenagemote* (Thorpe, *Anc. Laws and Inst. of Eng.* pref. iii.). It must also be remembered that Anglo-Saxon England cannot properly be considered as one state, governed by one law; it was a congeries of states, each state having its own laws.

These laws, however, were similar in their character; however they differed in the peculiarities of their provisions: and in this character, without doubt, shares the common law of England, which seems instinct with the free spirit of Teutonic antiquity, and in its genius wholly Anglo-Saxon. It was refined and subtilized by the acute jurists of Normandy, *quibus nulla gens magis litigiosa atque in controversiis machinandis et proferendis fallacior reperiri potest* (Honorius, *Praxis Prud.* Pol. 212. Frankf. 1616); and by its application to new but analogous cases—an analogy extending from age to age—an immense fabric of jurisprudence was at length built, on somewhat rude foundations (Mackintosh, 1 *Hist. Eng.* 274). It has been said that the common law and statute law flow originally from the same fountain—the legislature; the statute law being the will of the legislature remaining on record in writing; the common law nothing else but statutes anciently written, but which have been worn out by time (*per* Wilmot C. J., *Collins v. Blantern*, 2 Wils. 348). Whatever may be the legal value of this theory, it has none in history, as it is not only not true, but the precise converse of what is true; the greater number of the early statutes having been, and often professing to be, only declarations of the common law. Without doubt, Parliament was, in its origin, a court of justice, and derived its modern power of making, from its ancient power of declaring, law. It was for "the maintenance and execution," and not the creation, "of the laws," that Parliament was to assemble every year (14 Edw. III. cap. 14; 36 Edw. III. cap. 10).

Law.
Sources.

Law. to hold plea according to the course of the common law" (*Co. Litt.* 260 a). With the exception, in some cases, of the register of decrees in the Court of Chancery (*Lane v. Williams*, 2 Vern. 277-92), it usually contains only the judgment, and not the reasons on which it is founded (3 *Rep.* pref. iii.),—which makes records of inferior value to the reports, for "our bookcases are the best proof of what the law actually is" (*Co. Litt.* 254 a).

Judicial opinions are of two kinds, and may be distinguished as (1) *decisions*, or (2) *dicta*.

Judicial decisions.

(1.) *Judicial decisions*.—It is to be understood that the appropriate duty of the judge is to decide the particular case submitted to him. He does not sit to determine general questions of law; and, despite the "great deal of learning and judgment" which distinguished the brief "interlocutions between the judge and the pleaders", so remarkable in the reign of Edward I.* (*Hale, Hist. C. L.* chap. vii.), it has been thought more prudent in the judge to confine himself to the question before him (2 Durnf. & E. 73). Sometimes a case may involve two or more questions, and the determination of one, and that of the least general interest, may decide the case; and courts have refused to go beyond the latter, or have decided upon the special circumstances, without entertaining the general question at all (8 Taunton, 55)—a practice which is not invariable (3 Durnf. & E.

697), nor deserves to be so. A case decided† becomes a precedent, and governs similar cases for the future; and this in order to secure fixity in the law (2 C. & M. 64), uniformity in the decisions of the courts (6 B. & C. 533), and to obviate the inconvenience resulting from upsetting a decision on the strength of which transactions may have occurred (6 Bing. 24). The value of a precedent is enhanced by its having been long held law (*Jacob*, 461), supported by many decisions (3 Atk. 763), or followed in practice (2 B. & Ad. 943). But it is sometimes overruled, when contrary to reason or common experience (*Willes*, 182; 11 East, 570), or, in cases relating to the construction of deeds, to the rules of language and criticism (1 B. & Ald. 330), or when founded on a mistake of the law (7 B. & C. 476), or even when the attention of the court has not been directed to a strong authority on the point (2 M. & S. 277). These objections do not always weigh with the court, and rarely in reference to a case that has been considered to have settled the law upon any point (2 Bing. 451). "In all usages

and precedents, the times [are to] be considered where—in they first began, which, if they were weak or ignorant, it derogateth from the authority of the usage, and leaveth it for suspect" (*Bacon, Adv. of Learning* book ii.).

A decision operates on an undecided case—

First, by furnishing a rule applicable to the case Rule. a rule being a point decided or laid down by competent authority, and is in the nature of a maxim; or,

Secondly, the undecided case may be within the Reason of reason of the decision; wherefore it has been said, a case. "that the reason of a resolution is more to be considered than the resolution itself" (*per Holt C. J.*, 12 Mod. 294); but still a judge is not bound to give the reasons of his decision (*Gregory v. Molesworth*, 3 Atk. 626).

Often the judges have been disposed to overrule a decision which they considered objectionable; and, in objectionable precedents. forbearing to do so, have often prevented other cases from coming under its influence, by instituting between them distinctions of circumstances (3 Atk. 68; 7 B. & C. 195-6). Such distinctions, unless for such a purpose, or unless previously recognised and established (*Harwood v. Oglander*, 8 Ves. 125), the courts usually discountenance, and sometimes positively refuse to acknowledge (2 Atk. 41-314).

In deciding a new case,* "to which there is nothing New cases. parallel in the books" (1 Lord Raym. 692), the judge is sometimes influenced by considerations of "reason and justice" (3 Bing. 266), or reasons from the analogy of other cases, if he can (*per Best C. J.*, 3 Bing. 265), remote as the analogy may be; or perhaps he rests his decision on certain rules of construction of deeds, on general practice, on the books of entries and the returns of writs, and on convenience, if the new case be one of practice (*Ram, Science of Legal Judgment*, 155).

Where two decisions of equal value, in relation to the Anti-nomie. courts from which they emanate, are inconsistent with each other, as a rule, the second usually prevails when a fresh case has to be decided (4 B. & C. 589); but this rule is not invariable (9 East, 157). A series of recent uniform decisions prevails over decisions of prior date (1 Taunton, 234); but where there is no uniformity in the decisions, and an election has to be made amongst them, it would appear that the judge would be controlled in that election by considerations of reason and justice (6 Durnf. & E. 655; 9 Bing. 203).

(2.) *Judicial dicta* are opinions expressed by judges Judicial dicta. unnecessarily, as not containing the grounds of the judgment pronounced on the case before the court. They have a certain value, but never that of a judicial decision (5 M. & S. 185; 1 Burr. 153; 2 East, 17),

* Speaking of the year-books, Roger North says, "it is obvious what deference ought to be had to them, more than to the modern reports; for, passing by their short and material rendering the sense of the pleaders and of the court, it must be observed, that the whole cause, as well the special pleadings as the debates of the law thereupon, were transacted or alleged at the bar; and the prothonotaries, *ex officio*, afterwards made up the records in Latin; and the court often condescended to discourse with the serjeants about the discretion of their pleas and the consequences with respect to their clients. And the court did all they could to prevent errors and oversights" (*Life of Lord Keeper North*, vol. i. p. 29, 1826). In *Seymour v. Barker* (2 Taunton, 201), the judges appeared to discountenance a reference to the year-books for an authority; but *Vyryan v. Arthur* (1 B. & C. 410) was decided chiefly on the authority of a case in those reports.

† The value of a decision depends, in some measure, on the court in which it is pronounced. Thus a judgment of the House of Lords, the highest appellate jurisdiction, is of greater value than one of the Court of King's Bench, sitting *in banco*, which is itself preferable to a mere ruling of a judge at *Nisi Prius*. The value, indeed, of a *nisi prius* ruling is very low (2 Bing. 90). Unquestionably the character of a judge enhances the value of his decisions.

* "The case is in some sense new, as many others are, which continually occur; but we have no right to consider it, because it is new, as one for which the law has not provided at all, and because it has not yet been decided, to decide it for ourselves, according to our own judgment of what is just and expedient. Our common law system consists in applying to new combinations of circumstances those rules of law which we derive from legal principles and judicial precedents; and for the sake of attaining uniformity, consistency, and certainty, we must apply those rules where they are not plainly unreasonable and inconvenient, to all cases which arise; and we are not at liberty to reject them, and to abandon all analogy to them, in those to which they have not yet been judicially applied, because we think that the rules are not as convenient and reasonable as we ourselves could have devised" (*per Parke J.*, *Mirehouse v. Bishop of Lincoln*, 3 Bing. 515). "*Le juge qui refusera de juger, sous prétexte du silence, de l'obscurité, ou l'insuffisance de la loi, pourra être poursuivi comme coupable de déni de justice*" (*Tit. Prélim. de la Pub. des Lois; Premier Proj. de Code Civil*).

Law. not being confined to the particular circumstances in reference to which they were uttered (*per* Lord Kenyon, 7 Durnf. & E. 287); but the degree of their value depends on a variety of circumstances.

Text-books. Certain *text-books*, written by eminent lawyers, have also an authority* in Westminster Hall, where they may be cited in proof of the law (*Co. Litt.* 11 a). Amongst these, the writings of Sir Edward Coke hold a deserved pre-eminence; and an observation, undoubtedly just, made in reference to these, will explain the value of our text-books in evidencing the law. In reference to a particular position of Coke's, Lord Wynford observed, that "Lord Coke had no authority for what he states; but I am afraid," he adds "that we should get rid of a great deal of what is considered law in Westminster Hall, if what Lord Coke says without authority is not law. He was one of the most eminent lawyers that ever presided as a judge in any court of justice, and what is said by such a person is good evidence of what the law is" (2 Bing. 296-7). In reference to a position in the *Digest* of Chief Baron Comyns, the same learned judge remarks of the author, "this he lays down on his own authority, without referring to any case, and I am warranted in saying we cannot have a better authority than this learned writer" (5 Bing. 387-8). Similar language was used by Lord Kenyon (3 Durnf. & East, 64-631).

Statute aw. Public Acts. (II.) The *statute law* consists of the statutes, or acts of parliament, which are written laws made by Parliament. Of these there are two kinds—(1.) Public Acts, which are those that relate to the whole kingdom—to the king, queen, or heir apparent; all great officers; all officers of any kind, as all sheriffs, all lords of manors, the whole spirituality, and to trade in general, and this though in relation to a special or particular thing. Of these the courts are bound *ex officio* to take notice. (2.) Private Acts concern only a particular species, thing, or person, or relate only to certain places. These Acts the courts are not bound to know, and they must formally be stated to them in all controversies, by the parties who rely on such Acts. A private statute is said not to bind or include strangers in interest to its provisions; and it has been held, that though every man be so far a party to such a statute as not to gainsay it, yet he is not so far a party as to give up his interest (*per* Hale C. B., *Lucy v. Levington*, 1 Vent. 175).

Nature of Acts. According to their objects, statutes may introduce a new law, or declare (declaratory statutes) the old, or supply its defects (remedial statutes), and may affix penalties for transgression of their provisions (penal statutes). A statute can be repealed or altered only by a statute.

* Mr. Ram (*Science of Leg. Judg.* 82) very properly applies to the authority of these works the observations which Wilmot J. has made in reference to the authority of courts. He considers "authority" to have the same signification which it bears in Livy's character of Evander (*R. v. Almon*, *Wilmot's Notes*, 256 *et seq.*) *Evander tum ea auctoritate magis, quam imperio regebat loca* (Liv. i. 7). "A learned writer," says Harrington, "may have authority, though he has no power" (*Oceana*, *Prel. Works*, p. 39). In our laws holds the maxim, *che un opinione di Carpsivio, un uso antico accenato da Claro, un tormento con iracunda compiacenza suggerito da Furinaccio, sieno le leggi a cui con sicurezza ubbidiscono coloro che tremando dovrebbero reggere le vite e le fortune degli uomini* (Beccaria, *Dei Delitti e delle Pene*, pref. p. 6. Vienna, 1798). Something similar was the reputation enjoyed by Pothier, in France: *Les ouvrages de Pothier n'ont pas été reçus comme lois; mais ils ont obtenu un honneur semblable: car plus des trois quarts du Code Civil ont été littéralement extraits de ses traités* (Dupin, *Dissert. sur la Vie, &c. de Pothier*, p. cxiv.).

Law. In construing statutes, the great principle is to ascertain and give effect to the intentions of the legislature.

The principal rules of construction are as follows:—

(1.) Statutes are to be construed according to their intent, which is to be collected, first, from the words employed, and, when the sense of these is ambiguous, from the occasion and necessity of the statute, provided that this construction is not inconsistent with their language. The intent, when it can be learnt with certainty, will always prevail over the literal meaning of the terms (11 Rep. 73).

(2.) They are to be construed one part with another, so that the statute may operate as a whole—*ut res magis valeat quam pereat*.

(3.) When its language be of doubtful import, a statute is to be construed in reference to others, *in pari materia*, relating to the same subject (*Rex v. Loxdale*, 1 Burr. 447).

(4.) Remedial statutes are to be construed liberally,* so as to meet the end in view, and to prevent a failure of the remedy.

(5.) Penal statutes (with the exception of those against frauds) are to be construed strictly, as are—

(6.) Statutes taking away trial by jury, and unfriendly to the liberties of the subject, and

(7.) Statutes imposing charges upon the subject, and

(8.) Statutes conferring powers derogatory to the rights of private property, and

(9.) Private acts of parliament, conferring upon individuals or a company new and extraordinary powers of a special kind, affecting private property or exempting it from public burdens, and

(10.) Statutes creating new jurisdictions; for the superior courts shall not be ousted of their jurisdiction but by express words or by necessary implication.

(11.) Statutes are not to be construed so as to admit of absurd consequences.†

Our statutes commence with the reign of Henry III., and those passed down to the end of the reign of Edward II., with some *incerti temporis*, are styled *vetera statuta*; all subsequent in date are called *nova statuta*. The earliest statutes are in Latin, the 51 Henry III. being the first in French. After this time, Latin and French were indiscriminately used, up to the 33 Hen. VI., when was passed the last statute wholly in Latin. The publication of the statutes in English is supposed to have begun at the commencement of the reign of Henry VII., after the fourth year of whose reign English has been, without doubt, the language uniformly used.

In the construction‡ of statutes, our judges in former

* To interpret a statute *strictly*, is to adhere precisely to the words or letter of the law, which includes, of course, fewer particulars than a freer construction. To interpret it *liberally*, largely, and comprehensively, is to carry the meaning of the lawgiver into more complete effect than a confined interpretation would allow (*Rutherford, Inst. Nat. Law*, book i. chap. vii. s. 7-11).

† There is another rule relating to the construction of statutes, which, as containing a general principle, is worth detailing at length: "When an act of parliament doth alter the service, tenure of the land, or other thing in prejudice of the land, or of the custom of the manor, or in prejudice of the tenant, then the general words of such act of parliament shall not extend to copyholds; but when an act of parliament is generally made for the good of the weal public, and no prejudice can accrue by reason of alteration of any interest, service, tenure, or custom of the manor, then many times copyholds and customary estates are within the general provision of such Acts" (*Heydon's case*, 3 Rep. 8).

‡ "We cannot say of the legislature," Lord Tenterden once observed, "that it is *inops consilii*, but we may say that it is *magnus*."

Law. Construc-
tion of
statutes.

Sources of
statute
law.

Language
of statutes.

Law.

times assumed an authority which made it difficult to determine the boundaries of the legislative and judicial powers. They did not seem to have coincided with the maxim of Lord Bacon, "*Optima est lex quæ minimum relinquit arbitrio judicis; optimus judex qui minimum sibi*" (*De Aug. Scient.* lib. viii. cap. iii. aph. 46).

inter opes inops" (*Carr v. Stephens*, 9 B. & C. 758). An opinion of this kind appears to have prevailed amongst the judges both in early and comparatively recent times, although few have avowed so broadly, as did Chief Baron Hobart, their belief in the "liberty and authority that judges have over laws, especially over statute laws, according to reason and best convenience, to mould them to the best and truest use" (*Lord Sheffield v. Ratcliffe*, Hobart 347). Acting, however, on his principles, judges have defeated the palpably expressed intentions of the legislature, sometimes by a dexterous invention of fictions, sometimes by a strict, and sometimes by a lax construction of the statute. The history of estates tail illustrates the point. The nature of the estate is explained subsequently (p. 828, *post*), where it is clearly shown that the intention of the legislature, in making the law to which estates tail owe their existence, was to preserve, in the line of descent indicated by the donor, all lands entailed: but the judges very early lent their aid to the tenant in tail anxious to alienate. Straining the language of the Statute of Gloucester (6 Edw. I. cap. 3), they held, in the first place, that a tenant in tail could safely alien and defeat the rights of his issue to whom it was after-limited, provided an estate in fee simple, of equal value, should devolve upon his heir. This heir was considered to have thus received a recompence (*Octavian Lombard's case*, Y. B. 44 Edw. III). To evade this, Richel, a judge in the reign of Richard II., invented a form of limitation, which he fondly hoped all the *astutia* of his brethren would not enable them to defeat; but it was set aside as void (*Litt. lib. iii. s. 720*); as also was a settlement by Thirning, the chief justice (21 Hen. VI., H. T. 33 b). The next step was to apply to estates tail the fiction of common recoveries, by which the operation of the statutes of mortmain had been already successfully eluded. A common recovery was in form an action at law, and in reality a mode of conveyance. It may be explained thus. The party purchasing pretended to be entitled to the lands in question, sued out his writ against the seller, who came into court, and instead of defending his title, called upon some person, who it was feigned had originally warranted the title to him, to appear and defend the title attacked, or to give him land of equal value. On this the person (usually named the vouchee) appeared to the action, and the seller (demandant) intreated the permission of the court for a private conference with the grantee, which was never denied. Shortly after this the demandant returned into court, while the vouchee disappeared; on which the court, presuming the title of the demandant good, immediately gave judgment in his favour. (*Pigott's Treatise of Common Recoveries*). This was the form latterly adopted, and of its validity to bar the issue in tail no doubt seems to have been entertained; but in the reigns of Henry IV. and V. it seems to have been doubtful how far the default of the tenant (there being no vouchee) would avail. At length it was decided that such a recovery was good against the issue (3 Hen. VI. 556). The celebrated Taltarem's case (12 Edw. IV., E. T. 2 b) is said to have been "brought on the stage" by Edward IV., in order to obtain, as was actually done, a solemn decision affirming the efficacy of common recoveries. The apparent object of the king was to destroy the power of the overgrown feudatories, whose vast possessions and wealth threatened the stability of the throne. Thus did the *astutia* of the judges most effectually defeat the operation of the famous statute of *De Donis*, the repeal of which had been frequently sought without effect from the legislature (6 Rep. 40 b).

The result has been, that it is now law that no tenant in tail can, by any condition, limitation, custom, recognizance, statute, or covenant whatever, be restrained from aliening (*see the cases collected by Mr. Butler; Co. Litt. 379 b. n. 1*).

A particular form of actions, known as *real actions*—now, with certain exceptions, abolished by act of parliament (3 & 4 Will. IV. cap. 27)—were previously practically repealed by the decisions of the judges. If, from the ignorance or negligence of counsel or attorney, there was any slip in the proceedings, the court would not allow of any amendment, and thus assumed in fact a power "indirectly to defeat a remedy tolerated by the legislature" (*Real Prop. Commissioners' First Report*, p. 42).

It has been said of the Court of Chancery, that its acts have been rather legislative than judicial (*Humphreys, Observ. Law Real Prop.* p. 94); and we certainly find that it has not evinced what Beccaria styles *l'errante instabilità delle interpretazioni*, but has boldly contravened the obvious and expressed will of the

The operation of a statute as law commences from the day the royal assent is given, which is indorsed on the Act by the clerk of the Parliament (33 Geo. III. cap. 13); but its operation, by a clause in its body, may be suspended for a certain period; or the statute may have a retrospective operation, if it contain an express provision to that effect (*Churchill v. Crease*, 5 Bing. 178; and *see Ib.* 492).*

Of the laws local and particular in their operation are—(I.) The customary law, composed of certain immemorial and local customs. Amongst such are gavelkind,† borough English,‡ and the customs belonging to various manors.§ For a custom to be good in law, it must have existed "so long that the memory of man knoweth not to the contrary," must have been continued, must have been acquiesced in, must be reasonable and certain, compulsory, and not left to every man's option to obey or not; and, lastly, customs must be consistent with each other.

(II.) The civil and canon laws operate in England, so far as they have been admitted and received by immemorial usage and custom, in some particular cases and some particular courts, or so far as they may have been introduced by act of parliament (*Hale, Hist. C. L.* ch. ii.).

As to the canon law. The 25 Hen. VIII. cap. 19, confirms all canons, constitutions, ordinances, and

legislature. The Statute of Frauds (29 Car. II. cap. 3) required certain contracts to be reduced to writing. Still a court of equity will enforce a verbal contract of the description, on the alleged ground that it had been agreed that its terms should be reduced to writing (2 Cha. Ca. 135). A defendant was required by the court to confess or deny, in his answer on the oath, the fact of having entered into such a contract, in order, in the latter case, of rendering his answer equivalent to writing (1 Dick. 39). An agreement partly performed, courts of equity held not within the statute (2 Sch. & Lef. 5), whose intentions they again defeated, by giving effect to a verbal agreement for a mortgage, accompanied by a deposit of the title-deeds. These deeds were held a security for subsequent advances, if the court were satisfied, and verbal evidence was sufficient, that these advances were made on the terms of being so secured. This fact they would ascertain themselves, if the parties differed in respect of it (17 Ves. 230; 14 Ves. 60).

That great chancellor Lord Nottingham, in reference to some cases which had been cited, declared, that "he would alter the law upon that point" (*Freeman v. Goodham*, Ch. Rep. 295); but Lord Talbot, when this declaration was quoted, remarked, "I do not see how anything less than an act of parliament can alter the law" (*Heard v. Stamford*, 3 P. W. 411).

Other instances of the judges assuming to themselves legislative power might be mentioned. One is too remarkable to be omitted, relating to the conclusion of the oath taken in judicial proceedings. In the reign of Edward I. the oath concluded, "*Si Dieu m'aide et les Seintz*" (*Stat. Exon.* 14 Edw. I.); and, as late as 1553, the oath administered in the House of Lords concluded, "So help me God, and all Saints." The alteration ought to have been made by the legislature (*see Alderson B., Frost's case*, 2 Moody, C. C. 150). The following "instances of a wide and spirited departure from the express language of the statutes" are cited by Sir Fortunatus Dwaris (*On Statutes*, 708):—5 Eliz. cap. 4, s. 41: *R. v. St. Nicholas in Ipswich*, Burr. S. C. 91;—3 Will. & M. cap. 11: *Anthony v. Cardigan*, 2 Bott. 172;—8 & 9 Will. III. cap. 30: *R. v. Clayhydon*, 4 T. R. 100; *R. v. St. Mary, Lambeth*, 8 R. 236; *R. v. King's Pyon*, 4 East, 351.

*The principle is, that "*Nova constitutio futuris formam debet imponere, non præteritis*," a principle not peculiar to our law. *En général, les lois n'ont point d'effet rétroactif. Le principe est incontestable* (*Disc. Prélim. du Prem. Proj. du Code Civil*). *La loi ne dispose que pour l'avenir; elle n'a point d'effet rétroactif* (Art. 2, tit. Prélim. de la Pub. des Lois. *See also* *Gilmore v. Shuter*, 2 Lev. 227; *Att. Gen. v. Lloyd*, 3 Atk. 541; *Ib. v. Andrews*, 1 Ves. sen. 225).

† Div. IV. vol. xxii p. 491. ‡ *Ib.* vol. xviii. p. 721.

§ *See* COPYHOLD, Division IV. vol. xx. p. 239. The enfranchisement of copyholds and customary freeholds is facilitated by the 4 & 5 Vict. cap. 35.

Law.

Operation.

Qualities of a good custom.

Canon law.

Law.

synodals provincial, not repugnant to the law or the royal prerogative, which had been previously made "by the clergy of this realm;" and all "other ecclesiastical laws or jurisdictions spiritual" are also confirmed by the 35 Hen. VIII. cap. 16. A complete review of the canon law being purposed at this time, and which never took place, these laws were intended only to subsist temporarily. The two Acts were repealed, the first in terms, the second impliedly, by the 1 Ph. & M. cap. 8, which was itself in terms repealed, as far as it repeals the first Act, which is thereby revived. The second Act has never been revived, and therefore it would seem that no part of the canon law, except such as emanated from "the clergy of this realm," has, by law, any operation here; a fact which appears hitherto to have escaped the consideration of the learned judges of both the common law and ecclesiastical courts (*see Edes v. the Bishop of Oxford, Vaughan, 21-4*).

Civil law.

A portion of the civil law is received in our ecclesiastical and admiralty courts, and, in some instances, in the courts of law, especially in those instances in which the ecclesiastical courts follow it; then, in order that there may be an uniformity of judgments in the different courts, the temporal courts follow it also. In cases of legacies given out of personal estate, the temporal courts often follow the civil law (1 P. W. 1; 3 Atk. 346). From the civil law "we borrow all or at least the greatest part of our rules upon legacies" (*per Sir W. Grant, M.R., 6 Ves. 243*). The civil law maxim, *ex nudo pacto non oritur actio* (Cod. II. 2, 3, 10), is adopted by our law (3 Burr. 1670-1); and sometimes we follow the rules it lays down respecting the application of indefinite payments made by a debtor to his creditor.

Sect. 2.—The Authority of the Private Law of England.

Operation of law of England:

283. In considering the authority of the private law of England, we will consider the extent of its operation, first, as to *places*, and, secondly, as to *persons*.

in England.

I. First, then, it operates throughout that part of the united kingdom of Great Britain and Ireland called England, within which is included the town of Berwick upon Tweed* (20 Geo. II. cap. 42), through Wales (27 Hen. VIII. cap. 26; 34 & 35 Hen. VIII. cap. 26), and through Ireland—first, as to the common law (*Co. Litt. 141*); secondly, as to the whole law existing in the tenth year of the reign of Henry VII. (Irish Stat. 10 Hen. VII. cap. 22); thirdly, as to all laws passed since the Union (39 & 40 Geo. III. cap. 67), unless Ireland be excluded expressly or by implication. The 6 Geo. I. cap. 5, declared that the English parliament could bind the Irish people by its laws; but this Act was repealed by the 22 Geo. III. cap. 53; and the exclusive right of the Irish parliament to legislate for Ireland was declared by the 23 Geo. III. cap. 28. It would appear, therefore, that statutes of the English parliament passed between the 10th Hen. VII. and the 22 Geo. III. have no operation in Ireland.†

* On the high seas, beyond low-water mark, a portion of which is within the realm of England, the common law has no operation, that is within the High Admiral's jurisdiction; but it does operate between high and low-water mark when the shore is dry (*Finch, Law, 77*).

† Before the Union, Ireland was for many purposes considered as a foreign country. It was beyond the seas, within the meaning of

Except as respects certain fiscal regulations, the English private law does not extend to the islands of Man, Guernsey, Jersey, Sark, Alderney, and their appendages, nor to Scotland (5 Ann. cap. 8).

As to the colonies, a distinction must be taken between such as were settled by our subjects, and such as were ceded to or conquered by us. In colonies of the first kind (of which Australia is a familiar example), the colonists are understood to carry to their new abode, first, the common law of England, and secondly, such statutes passed previous to the settlement of the colony as are applicable to their own circumstances; so that laws relating to bankruptcy, the maintenance of the poor, police, tithes, &c., have in such a colony no operation. Statutes passed subsequent to the settlement do not operate in the colony, unless it be named or virtually included therein; for the general run of laws enacted by the superior state are supposed to be calculated for its own internal government, and not to extend to its remote dependencies. But these are bound by statutes relating to navigation, trade, revenue, and shipping, although not specially named (2 P. W. 75). The laws of a conquered colony continue until altered by the conqueror, that is, in England, the crown (*per Lord Mansfield, Wall v. Campbell, Cowper, 204*).

Our laws, however, like those of every other country, have, by the comity of nations, an extra-territorial operation. Generally speaking, such of them as define the *status* or civil condition and capacity of our subjects travel with them wherever they go, and their obligation it is not possible to remove. Such are laws which may properly be styled *personal*, determining whether a man be legitimate or illegitimate, under or of full age, insane, an idiot, a bankrupt, married or divorced, and the like. As to laws governing the succession to an intestate's property, it is doubtful whether our law operates extra-territorially; but it is clear, if an Englishman changes his domicile to Scotland, and dying intestate, that the law of Scotland would control the disposition of his personal estate (*Curling v. Thornton, 2 Addams, 17*). The incidents to these qualities may depend on the law of the country to which he goes; but, as qualities, they will, by the general comity of nations, be generally recognised by the foreign tribunals. These will also recognise our laws respecting the validity of contracts, unless indeed inconsistent with the rights and interests of the citizens of their state, or with its public policy and fundamental laws. The king, his ambassadors (but not consuls), fleets, and armies, while residing in, or travelling to, a friendly state, are still under the entire dominion of our laws (3 Rob. 13-27; 4 *Ib.* 26).

Our law acknowledges the validity of contracts entered into in a foreign country, and there intended to be performed, if they are valid according to the law of that country (*The King of Spain v. Machado, 4 Russ. 225; Potter v. Brown, 5 East, 130*); for, if an obligation is valid where it is professed to originate, it is valid every where else. Not only does our law recognise a marriage esteemed valid where it was

Law.

Not in Man or the Channel Islands, nor Scotland. The colonies.

Extra-territorial operation of English law.

Foreign

contracts.

Foreign

marriages.

the 21 Jac. I. cap. 16, and with respect to the exceptions of disabilities under the statute relating to fines (*see 18 Edw. I., Mod. lev. Fin.*, compared with 4 Hen. VII. cap. 24), and the old rule regarding illegitimacy from impossibility of access (*Jenk. Cent. 10*). And since the Union, Ireland is in some respects considered as a foreign country (*Battersby v. Kirk, 3 Scott, 11; Whyte v. Rose, 11 Law J. (N. S.) Exch. 457*).

Law.

contracted (*Lacon v. Higgins*, 1 Dow. & Ry. 38), but it does not generally recognise marriages celebrated abroad unless there considered valid (*Butler v. Freeman*, Ambl. 303; but see *Ruding v. Smith*, 2 Hagg. 385). A court of equity, with that liberality which is its proper characteristic, has so far taken cognizance of a foreign law as to have forborne to compel a husband to make a settlement upon his wife entitled to a share of his personal property under the Statute of Distributions, on its being proved that the wife was resident in Prussia, and that, by the law there, a moiety of the husband's effects must have come to her (*Sawyer v. Shute*, 1 Ans. 63; and see also 3 Ves. 323).

Process on foreign contracts.

But although the authority of the law of England does not extend to determine the validity of foreign contracts, still if their enforcement is craved of our courts, its authority does extend to the definition of the forms and means by which the enforcement shall be induced. The process must be according to our law: so a foreigner may be here arrested for a debt, or in equity upon a writ of *ne exeat regno*, although, by the law of the place where the debt was contracted, he could not have been imprisoned (*De la Vega v. Vianna*, 1 B. & Ad. 284; *Flack v. Holm*, 1 Jac. & W. 405); a doctrine which has been carried so far as to justify proceedings against a person as partner, because he was jointly concerned in trade with another in Holland, although partnership does not there arise from such a community of business (*Shaw v. Harvey*, Mood. & M. 526). Further, as to the time of commencing proceedings on foreign contracts, the English courts will have regard, not to the *lex loci*, but to the English statutes of limitation (*The British Linen Company v. Drummond*, 10 B. & C. 903).

Interpretation of deeds executed here.

Our law, also, it would appear, claims to determine the interpretation and effect of every deed or will executed here (*Trotter v. Trotter*, 4 Bligh (N. S.), 502; *Anstruther v. Chalmers*, 2 Sim. 1); although it is doubtful whether, in cases of devises of lands, the tribunals of the country where the lands lay would accord them this jurisdiction (*Vattel*, liv. ii. ch. viii. s. 3; 1 Wheaton, *Elem.* 136).

Foreign revenue laws, and laws opposed to our policy or the law of nature.

"No country ever takes notice of the revenue laws of another (*per Lord Mansfield*, *Holman v. Johnson*, Cowper, 343; and see further 2 Peake, 81, and 1 Esp. 389); nor will ours admit the operation within its limits of any foreign municipal law, contrary either to its public policy or to the law of nature. A slave, a native of East Florida, where the law countenances slavery, escaped to a British man of war; and it was held, that coming under the protection of our colours, he relieved himself from the operation of a local law of East Florida, which was contrary to the natural law of eternal justice (*Forbes v. Cochrane*, 2 B. & C. 448; and see the case of Somerset the negro, and Hargrave's note, *Co. Litt.* 79 b).

II. The persons subject to the authority of the English private law are now to be considered; and first, as they are subjects or aliens.

Natural-born subjects.

(1.) Natural-born subjects are such as are born within the dominions of the crown of England, or, as it is said, within the allegiance of the king. This in-

* It must be observed, with reference to the proof of foreign law, that it must be established. The court cannot take notice of it judicially, (1 P. W. 431; 3 Ves. & B. 99). As to how this proof is to be made, at law, see *Lacon v. Higgins* (*ut sup. cit.*); General Picton's case, 30 St. Tr. 514; 1 Dow. & Ry. 42, n. a; in equity see *Hill v. Reardon*, Jacob, 90; *Flack v. Holm* (*ut sup. cit.*).

cludes the children of our ambassadors, wherever born, and children whose fathers (or grandfathers by the father's side) were born within the allegiance of the king, unless their ancestors were attainted or banished for high treason, or, at the birth of such children, were in the service of a hostile foreign power. These are properly subject to all the laws of England, entitled to the rights it confers, subject to the disabilities it contains—a principle subject to certain important modifications. It is not in the power of any private subject to shake off his allegiance, and to transfer it to a foreign prince (*Macdonald's case*, Foster, Cr. L. 59), unless, as it would seem, by the permission of his own prince (*Doe dem. Thomas v. Acklam*, 2 B. & C. 796; see also *Bright's lessee v. Rochester*, 7 Wheaton, *American Rep.* 535). *Nemo potest exuere patriam* is the fundamental maxim of our constitutional polity; and therefore a natural-born subject who enters into a foreign service, and engages in hostilities against his native country, if taken, may be here punished as a traitor (Foster, Cr. L. 183-4).

(2.) Aliens, or persons born out of the king's allegiance, are still, to a certain extent, while resident here, subject to our laws. Although he cannot hold land or other real property to his own use, an alien may, if a subject of a friendly state, acquire a property in money, goods, or other personal property, may hire a house to live in, may bring actions concerning personal property, make a will, and dispose of his personal estate. During an alien's residence here, he owns a local or temporary allegiance to the crown, and is therefore subject to our criminal jurisdiction; and if, having a family and effects here, leaving them, he goes to his own country, and adheres to his prince in his hostilities against us, he may, if taken, be punished as a traitor by our laws (Foster, Cr. Law, 185). An alien may be made a denizen by royal letters patent, or naturalized by act of parliament. In the last case he is admitted to all the rights of a natural born subject, except the right of becoming a privy councillor, member of parliament, holding offices, grants, &c. A denizen may acquire real estate by purchase or devise, but cannot inherit; but his issue born after denization may inherit from him. He is subject to the same disqualifications as a naturalized subject.

Law.

Aliens:

their local allegiance.

Denizen.

Sect. 3.—The Doctrines of the English Private Law.

284. It is of the essence of a law that it creates rights, and it ought for a violation of those rights to provide an adequate remedy. In considering the general doctrines of English private law, it will be necessary to ascertain what are the rights they originate, for what wrongs they provide, and what are the remedies so provided.

The rights of which a man may be possessed must have reference to persons or to property.* Rights which have reference to persons are those arising out of the various relations, natural or artificial, common in all civilized communities. Of the natural relations to which we allude, the principal result from marriage, from whence are derived the rights of husband and wife; from parentage, whence the rights of father and child; from guardianship, whence the rights of guardian

Rights in respect of persons.

Natural relations

* Blackstone's division of rights, into rights of persons and rights of things, however familiar to our ears, is, as far as the expressions go, so exceedingly objectionable that it has not been, in terms at least, pursued in this chapter.

Law. and ward; and from *service*, whence the rights of master and servants. First, then, as relates to matrimony.

Marriage. (1.) "In civil society, marriage becomes a civil contract, regulated and prescribed by law, and endowed with civil consequences" (*per* Lord Stowell, *Dalrymple v. Dalrymple*, 2 Hagg. Rep. 63). Formerly, in order that the contract should be valid, the marriage must, unless the archbishop had granted a dispensation, or special licence, as it is called, have been celebrated in some parish church, public chapel wherein marriages were usually solemnized prior to the 26 Geo. II. or which has been since licensed for the purpose by the bishop (4 Geo. IV. cap. 76), or place licensed for the purpose and for the performance of divine service during the repair of the church or chapel (*Ibid.* 5 Geo. IV. cap. 32). Now, however, those who prefer it may have their marriage solemnized in the presence of the superintendent registrar of the district, in a licensed place of worship or not, without any ceremony other than a declaration by both parties of their intention in the act (6 & 7 Wm. IV. cap. 85). Otherwise it must be celebrated by a person whom the parties believe to be in holy orders (4 Geo. IV. cap. 76); and, however the marriage be solemnized, the parties must not be subject to the canonical disabilities of precontract (but see *Harg. Co. Litt.* 79 b), consanguinity, affinity,* or impotency, nor under the legal disabilities arising from a subsisting marriage, want of age, want of consent of parents or guardians, which in cases of minors is necessary, and want of reason (see 15 Geo. II. cap. 30). Parties under legal disabilities cannot contract a marriage—the contract is *void ab initio*. The marriage of parties under canonical disabilities, not being those of affinity and consanguinity, is merely *voidable*, that is, can be avoided by proceedings in the ecclesiastical courts, but subsists until a sentence is there obtained. The marriage of

* The 32 Hen. VIII. cap. 38, declares that all persons not prohibited by God's law may marry, and that all marriages are indissoluble contracted by lawful persons in the face of the church, and afterwards consummated. Nothing is to impeach a marriage (God's law excepted) not within the Levitical degrees, the remotest of which is that of uncle and niece. All impediments arising from precontracts (except when they had been consummated) were abolished. But this latter clause was repealed by 2 & 3 Edw. VI. cap. 23 (see 26 Geo. II. cap. 33). Before the Marriage Act (26 Geo. II. 33), it is questionable whether any ecclesiastical ceremony was necessary to a marriage (*Jesson v. Collins*, 2 Salk. 437; *R. v. the Inhabitants of Brampton*, 10 East, 288; *sed vide Haydon v. Gould*, 1 Salk. 119). Before the laws of Innocent III., who became pope in 1198, we are told that the ceremony of marriage was not in use in the church; that is, as we suppose, to be valid at common law: the man simply went to the woman's house, and took her away to his own (*Moore*, 170). "Marriage," says Pope Evaristus, "ought not to be contracted in a clandestine manner; for marriage is no otherwise lawful, but when the wife is demanded of such persons as seem to have the government and dominion over the woman's person, and who have the care and guardianship of her; and unless she be espoused by the consent of her parents and nearest relations, and be endowed according to law, and likewise, at the time of her nuptials, receive the sacerdotal benediction, according to custom, by means of prayers and oblation, and also be given in marriage by her *paranymphe*, such marriage is deemed unlawful by the canon law. This was the ancient way of celebrating legal marriages, for otherwise they are only styled *conjugia præsumpta*, and not lawful marriages; and by law are rather termed *adulteria*, *strupra*, and *contubernia*, than lawful marriages. Let no believer or Christian, of what degree soever he be, presume to celebrate wedlock in private, but let him publicly marry in the Lord, receiving the benediction of the priest" (*Ayliffe, Parergon*, 364; see *Fleury, Mœurs des Chrétiens*, p. 228; *Selden, Uxor Hebraica*, lib. i. cap. 28). Chief Justice Pemberton was inclined to think that "words of contract *de presenti* repeated after a person in orders" was a good marriage (*Wild v. Chamberlain*, 2 Show. 300).

parties within the prohibited degrees of affinity and consanguinity is absolutely void. (5 & 6 Will. IV. cap. 54). The ecclesiastical courts are not permitted to dissolve a marriage after the death of either of the parties (*Salk.* 548). When the ground of the dissolution existed *previous* to the marriage, such as consanguinity, and did not arise afterwards, as might be in the case of impotency, the effect of such a dissolution is a total abrogation of the marriage, and all its incidents, from the beginning, improperly called a divorce *à vinculo matrimonii*. In such cases, of course, the issue are bastardized, and each party is capable of contracting marriage with another.*

There is another kind of divorce, a divorce *à mensâ et thoro* (from bed and board), which, without dissolving the marriage, separates the parties. It is decreed by the ecclesiastical courts, when either party has committed adultery, or in cases of extreme cruelty apprehended by one party from the other. The courts, however, in all cases require the fact of adultery or cruelty to be clearly established before they will consent to grant a divorce (*Evans v. Evans*, 1 Hagg. 36). A wife divorced *à mensâ et thoro* is entitled to an allowance from her husband, settled by the ecclesiastical judge, provided she has not eloped from her husband, and is living in a state of adultery (2 Hagg. 199-203).

The law considers husband and wife as one person. The husband is bound to provide his wife with necessities, and must pay her debts for them, unless she has eloped from him and is living in adultery, and he is liable for all her debts contracted previous to marriage. The wife cannot sue or be sued without her husband's concurrence, and in his name, even although she be divorced *à mensâ et thoro*, and have a competent alimony (*Lewis v. Lee*, 3 B. & C. 291).

Husband and wife cannot, in criminal prosecution, give evidence for or against each other; and, except in cases of treason and murder, and in offences of the lowest class, the command or authority of a husband will privilege a wife from punishment. The wife, being presumed to be under the coercion of her husband, can execute no valid deed, nor even, except under special circumstances, can she devise her lands to her husband.

(2.) Children may be either legitimate or illegitimate. A child is called legitimate if born in lawful wedlock, or within a competent time afterwards, the maxim being, *pater est quem nuptiæ demonstrant*; unless, indeed, it can be shown that there was a natural impossibility that the husband of the mother could be the father, or even a probability strong enough, in the view of the judicatory, to rebut the contrary presumption (*R. v. Luffe*, 8 East, 193; *Goodright v. Saul*, 4 T. R. 356; *Le Marchant's Gardner Peerage case*). A parent, being himself competent thereto, is bound to support such of his children as, from infancy, disease, or accident, are incapable of supporting themselves, and is entitled to the benefit of the labour of such as live with him; and he who marries a woman with children is bound to maintain them, being so incapable, until such woman's death (4 & 5 Wm. IV. cap. 76).

* The practice of dissolving marriages by act of parliament, on the ground of adultery, commenced with the case of Lord de Roos, in 1692: only five Acts of the kind passed up to the accession of Geo. I.: there were between 1715 and 1775 sixty; from that time to 1800, seventy-four; and thence to 1830, about ninety (1 Burge, *Comm.* 655).

- Law.** A parent may maintain his children in their lawsuits, and is justified in committing assault and battery in their defence. While they are minors, he may correct them with moderation, and may delegate this authority to their tutor or schoolmaster; he may appoint their guardian by his will, and, during his life, act as their trustee, and receive the profits of their property, for which he must account to them when they reach their majority. The child is equally bound to support and assist the parent, their duties being reciprocal. By a recent statute, judges in equity are empowered to afford mothers, against whom adultery has not been established, access to their infant children, and, if these latter be under the age of seven years, to give them into the custody of such mothers (2 & 3 Vict. cap. 54).
- Rights of the parent.**
- Duties of the child.**
- Illegitimate children.** The mother is bound to support her illegitimate children; but if they become chargeable to the parish, the father, if he can be ascertained (the mother's oath not being taken as conclusive of the fact), is liable (4 & 5 Wm. IV. cap. 76).
- An illegitimate child is called *filius nullius*, and, although he may obtain a surname by reputation, can neither inherit nor have heirs, except of his own body.
- Guardianship.** (3.) Previous to the statute of Charles II., hereafter noticed, the father might, by deed or will, assign a guardian to any woman child under the age of sixteen; and in default of such assignment, the mother was to be guardian (4 & 5 Ph. & M. cap. 8; 3 Rep. 39).
- Guardians in socage.** The guardianship of a minor inheriting an estate in lands of the tenure of socage, devolves on the next of kin, on whom the inheritance cannot possibly descend, as it would otherwise be "*quasi agnum committere lupo ad devorandum*" (Co. Litt. 88 b; Fortescue, *De Laud.* cap. 44; 1 P. Wms. 260; Harg. Co. Litt. 88 b). From this guardianship the infant is relieved when he reaches his fourteenth year. There are also testamentary guardians, who are appointed by virtue of the 12 Car. II. cap. 24,* which authorizes any father by deed or will to dispose of the custody of his child, born or unborn (7 Ves. 348), to any person, not being a popish recusant, either in possession or reversion, until such child attains the age of twenty-one years, or, in the case of a female, marries (Mendes v. Mendes, 3 Atk. 625). The testamentary guardian supersedes the guardian in socage, and has the custody and management of the real and personal property of the infant, to whom he is obliged to account when the infant comes of age.
- Testamentary guardians.**
- Guardians appointed by Chancery.** If the father appoints no guardian, the Court of Chancery will do so, on the petition of the infant (*Ex parte Salter*, 2 Dick. 769); and the same court will appoint another guardian, where the first refuses to act (1 Sch. & Lef. 106); and it will take care that these, as well as testamentary guardians, act rightly by the ward (Duke of Beaufort v. Berty, 1 P. W. 702).
- Guardians to natural children.** Strictly speaking, a father cannot appoint testamentary guardians to a natural child; but, if he names in his will persons to act as such, the court, on petition, will appoint those persons (Chatteris v. Young, 1 Jac. & W. 106). In order for their own protection, therefore, guardians are in the habit of applying to the court, and acting under its direction (1 Comm. 463). Its jurisdiction in the case of wards has been traced no higher than the year 1696 (Harg. note, Co. Litt.
- 88 b). An infant may be sued in his own name; but he defends himself by his guardian; and one, if he has none, will be appointed for this express purpose (*ad litem*) by the court in which the action is brought. An infant cannot alien his estate; but if a trustee or mortgagee, he may convey his interest as such under the direction of a court of equity. He may purchase lands, but disown the purchase, if he will, when he becomes of full age; and he can also bind himself by contracts for necessaries, but not beyond.
- Law.**
- (4.) Under the head of *service*, it will be sufficient to remark that, in the cases of *menial servants*, unless any special contract be made, it is understood that the hiring is for one year, terminable at any period by the giving of a month's notice or a month's wages. *Apprentices*, who are one class of servants, are those who are bound by deed to serve their masters for seven years, to be maintained and instructed by them. They may be discharged from their service, on reasonable cause, by a justice of the peace, or by the justices at quarter sessions. They nominally have the exclusive right of exercising their trade, but practically this does not exist (5 Eliz. cap. 4, s. 31; 54 Geo. III. cap. 96). To trades not in existence when the first-named Act passed, and to trading in country villages, this law never applied. A master may give his labourer reasonable correction; and any servant, &c. assaulting his master or mistress, may be imprisoned one year, or undergo any other corporeal punishment, not extending to life or limb.
- Authority of the master.**
- A master may maintain his servant in his actions at law, and himself bring an action for the beating of his servant, assigning as his reason his damage by the loss of his servant's service, a loss that must be proved on the trial. He may justify an assault in defence of his servant, as may a servant in defence of his master (Tickell v. Read, Lofft. 215). An action can be maintained by a master against any one who induces his servant to leave his service, in which way it is that a parent can punish the seducer of his daughter. By some evidence, however slight, he must, to sustain the action, prove that the daughter was in the habit of performing acts of service to him indispensable; but the action cannot be maintained if the daughter have left the parental roof with no intention of returning. Such an action may be brought even by an adopted father (11 East, 23).
- Rights of the master.**
- A master is generally answerable for his servant's conduct, if acting in obedience to his commands—and whatever a servant is permitted to do in the usual course of his business is equivalent to a command—and, in some cases, for his servant's negligence.
- Liabilities of the master.**
- The only artificial relation of which any notice need here be taken is that of *partnership*, which is an union of individuals for certain undertakings. Partnership may be constituted either by the spontaneous act and *merus motus* of the parties, or by the operation of law.
- Partner ship.**
- First, then, as to voluntary partnership:
- It is, in general, constituted by an agreement among individuals, to share in the profits of their joint undertaking in some concern. Communion in profits, in proportions however unequal, constitutes a partnership as far as the public are concerned, and presumptively as between the parties; a presumption capable, however, of rebuttal, on the establishment of the fact that no partnership was intended.
- Voluntary partnership.**

* This celebrated statute was drawn by Lord Hale. *Eyre v. Countess of Shrewsbury*, 2 P. W. 125. But see Harg. Co. Litt. 108 a, n. (1).

Law.

The act or agreement of one partner, in relation to and in the course of the partnership business, is esteemed the act of the firm, and binding on all the partners, even although the fact of their being partners be unknown, and whether they be *dormant*,* as the partner is called who does not visibly appear in the management of the partnership, or *nominal*, as he is called who, without having any interest in the profits, allows his name to be used, and deceives the public by appearing as a partner. But, of itself, a mere admission of partnership, to one who has contracted with the firm, fixes an individual with no liability on account of the contract, if the admission was made *subsequently* to the contract. Nor can any one sue another as partner, if he has been informed of a stipulation that the latter should not participate in the profits or losses of the partnership. It must, also, be distinctly understood that an agent, such as a traveller, clerk, factor, broker, &c., in receiving or agreeing for a commission or similar profit on the sale of goods, does not *quoad hoc* constitute himself a partner.

A partner cannot sue his co-partner *at law* in respect of anything connected with or involving the consideration of the partnership accounts: these, with all partnership affairs, are properly the peculiar province of a court of equity.

Secondly, as to those partnerships which are created by law, and called corporations. These are considered artificial persons, to whom the law ascribes the quality of immortality. A corporation may be sole, when its privileges belong only to one natural person and his successors; or it may be aggregate, when it includes a number of persons and their successors. Thus a bishop and a parson are instances of corporations sole, and their successors do not properly inherit from them, but take the property they possessed, in virtue of their character as their successors. Of a corporation aggregate, a dean and chapter is a familiar example. These corporations are in fact metaphysical entities, creations of law (10 Rep. 32); and so far so, that their possessions are not understood to belong to their members, but to that abstract being which they constitute in their united existence. In reference to their purposes, corporations are either *ecclesiastical* or *lay*; ecclesiastical corporations being such as are composed wholly of spiritual persons, as in the instances above given; lay corporations, the members of which include laymen. The first exist for spiritual purposes: of the latter there are three kinds—civil, eleemosynary, and trading. Civil corporations exist for certain political purposes, for the government of towns and cities; for the promotion of education, *e.g.* the universities and their colleges; for the protection of the public from incompetent practitioners of medicine and surgery; and for the prosecution of inquiries into the arts and sciences, and generally for assisting in the advancement of knowledge. Eleemosynary corporations are those of a charitable kind, and intended for the distribution of alms, and the relief of the sick and needy. Trading corporations are those partnerships on which corporate privileges are conferred, in consideration of the vastness of their operations, and the benefit consequent to the country

at large. Of such a kind are the East India and South Sea Companies, and the Bank of England, and many others, similar in kind, however inferior in importance.

Corporations must be created in one of two ways, either by the king's prerogative, or by act of parliament. The consent of the king to the erection of a corporation may be expressed or implied. It is implied in all cases of corporations at the common law, as the corporation of a parson, dean and chapter, and the like; and in corporations existing by prescription, as the City of London, which has been a corporation time whereof the memory of man runneth not to the contrary. It is expressed, when he creates the corporation by his royal charter. This charter is often confirmed by act of parliament; and this is done when privileges transcending the existing laws are to be granted, as the right, for example, of taxing the subject. The crown is also empowered to incorporate certain associations with certain limited privileges. In the case of municipal corporations, the crown is empowered by act of parliament (7 Will. IV. and 1 Vict. cap. 78), on the petition of the inhabitant householders of any borough for a charter, if, advised by the Privy Council, it should think fit to grant the same, to extend to such new corporation all the powers and provisions of the general Municipal Corporations Act (5 & 6 Will. IV. cap. 76), which confers on these bodies many very important privileges. The crown, in granting a charter to a class, may authorize a person or persons to ascertain the individuals composing that class (*per* Denman C. J., and Williams J., *Rutter v. Chapman*, 8 M. & S. 106, *et seq.*), and, it has been said, may even delegate to an individual the power of appointing the first members of a corporation (*per* Tindal C. J., and Patteson J., *ib.*; *Sutton's Hospital case*, 10 Rep. 23; *Bro. Ab. tit. Prerogative*, 53; *Jenkins, Cent.* 87, p. 270). The parties ascertained or appointed, however, derive their qualities and rights from the charter, and not from the individual naming or ascertaining them (*per* Coleridge J., *Rutter v. Chapman, ut sup. cit.*).

A corporation must have a name. All corporations have—(1) perpetual succession; (2) are sued and sue, implead and are impleaded, grant and take, in their corporate name, in which they do all the acts of natural persons compatible with their corporate existence; (3) may purchase lands, and hold them for the benefit of themselves and successors;* (4) usually have a common seal, the affixing of which binds the corporation; (5) usually make bye-laws, which, so far as they are not repugnant to the king's prerogative, or to the common profit of the people, are binding on all the members of the corporation. A corporation, being an invisible person, (1) must appear by attorney; being an artificial person, (2) can commit no crime, nor be (3) subject to personal injuries; it cannot (4) act as an

* Subject, however, to the statutes of mortmain, which make void the alienation of lands in perpetuity to any corporation, without licence from the crown, with certain exceptions,—one in favour of poor vicarages and of Queen Anne's Bounty, and another in favour of gifts of land, or money to be laid out in land, for charitable uses, which are good if made by deed executed in the presence of two witnesses, twelve months before the donor's death, and enrolled six months before his death in the Court of Chancery (2 Comm. 268-74; and see 43 Geo. III. cap. 107, and 45 Geo. III. cap. 101). Being excepted out of the Statute of Wills (34 Hen. VIII. cap. 5), corporations cannot take lands by devise, except in some cases for charitable purposes (43 Eliz. cap. 4; 9 Geo. II. cap. 36).

* A dormant partner is not, however, bound by an express contract with the known and acting partners; and as respects contracts impliedly entered into with the partnership by persons ignorant of his being a partner, his liability ceases *ipso facto* on his ceasing to be a partner.

Law.

Creation
of corpo-
rations.Qualities
of a cor-
poration.

Law.

executor, administrator, or trustee;* and, having no soul, (5) cannot be subjected to ecclesiastical censures. A corporation aggregate may take goods and chattels for the benefit of themselves and their successors; and so may a sole corporation, when it be the representative of a number of persons, as the master of a hospital, who is a corporation for the benefit of the poor brethren; but not so sole corporations, representing none but themselves, as bishops, parsons, &c. (1 *Comm* 477; 2 *Comm* 431-2), with exceptions of a local and peculiar kind. As a rule, the majority in an aggregate corporation binds the minority (see 27 Hen. VIII. cap. 27; *R. v. Monday*, Cowper, 538; 1 Kyd. *Corp.* 400, &c.).

Visitation.

To prevent a corporation from deviating from its end, it (with the exception of such as are in their nature merely trading corporations) is properly subjected to visitation. Of all ecclesiastical corporations, the ordinary is usually the visitor; of all lay corporations which are eleemosynary, the founder, his heirs or assigns; and of the remainder, the king, either through the medium of the Court of King's Bench (1 *Comm* 481-2), or rather, perhaps, through the medium of the Chancellor (see *R. v. St. Catherine's Hall*, 4 *T. R.* 233). In this last case it was held, that in default of the founder's heirs, and of a special visitor appointed by the founder or charter, the king, in the person of his chancellor, had the right of visitation. The principle which governed this case, one of an eleemosynary lay corporation, one might suppose would extend to all such cases in which the king could act as visitor.

Dissolu-
tion of cor-
porations.

Corporations may be dissolved by act of parliament, by the natural death of all of its members, by a surrender of its franchises into the hands of the king, or by forfeiture of its charter if it have abused those franchises. On its dissolution, its lands revert to its founder, donors, or their heirs.

Rights to
property.

We now come to consider the rights which relate to property. These rights differ according as the property is, in the view of the law of England, *real* or *personal*. Real property is said to consist of *lands*, *tenements* and *hereditaments*; lands including every species of soil, with its produce, as woods, mines, &c., with edifices that stand on it, and even lakes or ponds. Tenement signifies properly that which, in law phrase, may be *holden*, that is, possessed, subject to the superior right of another; and hereditament, that which may be inherited; and this properly would be applicable to personal as well as real property (*Co. Litt.* 6).

Real pro-
perty:corporeal,
and

Real property or hereditaments, as is the usual phrase, is of two kinds; *corporeal*, which includes every species of property comprised under that *nomen generalissimum* (*Co. Litt.* 4 a), land; *incorporeal*,† which imports certain rights issuing out of a thing corporeal, or concerning, or annexed to, or exerciseable within the same (1 *Comm.* 20). Of incorporeal hereditaments Blackstone enumerates ten sorts: (1.) Advowson, which is the right of presentation to a church or ecclesiastical

incor-
poreal.

benefice (see Div. IV. vol. xvii. p. 140). (2.) Tithes (see Div. IV. vol. xxviii. p. 654). (3.) Common, or right of common, whether common of pasture, which is a right of feeding one's beast on another's land; or of piscary, which is a right of fishing in another man's water; or of turbary, which is a liberty of digging turf on another man's ground; or of estovers, which is a liberty of taking necessary wood from another's estate (see Div. IV. vol. xx. p. 67). (4.) Right of way, which is the right one may have of going over another man's ground. (5.) Offices, which are a right to exercise a public or private employment, and to take the fees and emoluments thereto belonging. (6.) Dignities, which are titles of honour and precedence. (7.) Franchises, which are royal privileges granted to a subject (see Div. IV. vol. xxii. p. 314). (8.) Corodies, which are rights of sustenance, or of receiving a certain allotment of provisions for one's maintenance (see Div. IV. vol. xx. p. 261). (9.) Annuities are similar in their nature, only they arise from temporal, not spiritual, persons (see Div. IV. vol. xvii. p. 616). (10.) Rents, which are certain profits issuing yearly out of certain lands and tenements corporeal (see Div. IV. vol. xxvii. p. 17).

Law.

As to that species of real property included in the term corporeal hereditaments, we will consider the modes, first of its enjoyment, and secondly of its acquisition. The various modes of enjoyment are called estates, which may be considered in reference (1) to their quality, (2) quantity, (3) certainty, (4) time, and (5) number of persons participating.

By the law of England all land is supposed to be *Tenures*. holden mediately or immediately of the king,* a doctrine which may be referred to the early periods of our legal history. The various species of tenure are so fully described in other parts of this work,† that it is unnecessary to do more than notice a few striking incidents to their character.

Tenures, by their very nature, required that every transfer of the land should be made by public delivery of possession. Ulphus, the son of Toraldus, turned aside into York, and filled the horn he was wont to drink out of with wine, and before the altar, on his bended knees, drinking it, gave away to God, and to St. Peter, the prince of the apostles, all his lands and revenues. The delivery being thus public, the ownership was always patent and notorious, a favourite object with our ancient common law (*Taylor v. Horde*, 1 Burr. 108). Tenures imposed on the tenant the necessity of rendering certain fruits and performing certain services to his lord; and if he committed any act in derogation of his lord's rights, he forfeited his

Notoriety
of transfer
and posses-
sion.

* Corporations, however, are considered by the Court of Chancery as trustees for charitable purposes (*Lydiat v. Sir John Foach*, 2 Vern. 412); but for no others, (see *Mayor of Colchester v. Lowton*, 1 Ves. & Bea. 244). Under some circumstances the court will even remove them from their trust (*Mayor, &c. of Coventry and Att. Gen.*, 7 Bro. P. C. 235).

† "*Res incorporales*," says the civilian, "*autem sunt, quæ tangi non possunt: qualia sunt ea quæ in jure consistunt, sicut hæreditas, usus-fructus, usus, et obligationes quoquo modo contractæ*" (*Inst.* II. 2). The definition is much too large for our purpose, but conveys a distinct notion of the subject.

* The source of this royal right is explained by an eminent antiquary, who tells us that "King William I. was seized of the whole kingdom of England in demesne; that he retained part of it in his own seisin, and other parts thereof he granted and transferred to others." (*Madox, Bar. Ang.* book i. c. 2, p. 25) — a statement utterly irreconcilable with history, for *Domesday* teaches us that many persons held, in the days of the Conqueror the same land of which they were possessed in the time of the Confessor, and in the same way; and besides, William swore at Westminster to maintain his new subjects in their rights, and to govern them by their ancient laws. There is no question that the true origin of the doctrine is to be found in the ancient Teutonic polity, which considered the ownership of all lands as vested in the state or people (*Cæsar, De B. G.* l. iv. s. 1., l. vi. s. 22; *Tacitus, De Mor. Germ.* s. 26), of which the king was afterwards held the representative (*Allen, Rise and Growth of Royal Prerog.* 131-64).

† Division IV. vol. xxviii. p. 502

Law.

interest in the lands. A system like this, bears visible impress of its military origin, and to military considerations is the maxim clearly ascribable, that the tenancy of the land should always be filled, and that the tenant could make no disposition of his interest likely to involve a vacancy in the same tenancy, although but for a moment. The system which gave birth to tenures, and which has imported so many doctrines into our law, has long passed away: its effects, however, still remain, and correctly to apprehend the rationale of common-law estates, reference may be still had with advantage to the *Assises de Jerusalem*, and to the Lombardy feudalists. In process of time, the cunning of the monastic conveyancers, ever fond of subtilizing the law, invented a distinction between the right to property and the right to enjoy the use of property. In this way were the statutes of mortmain (p. 825 n. *antè*) evaded; for the gift was made, not to the convent, but to the use of the convent. This use being an unsubstantial thing, could be subjected in no way to the rules of tenure.* It could not publicly be delivered, nor could it be forfeited; it could yield no fruits, discharge no duties. He to whom it belongs, so long as the legal estate was vested in loyal hands, might commit treason with impunity; and it enabled the poor tenant to defy the avarice of his lord. An use could be devised, which a legal estate could not, except in certain places, until the 34 Hen. VIII. cap. 5. To the perfection of this system of uses, the Court of Chancery, where, for many years, an ecclesiastic presided, lent its ready aid; and Sir John de Waltham, in the reign of Edward III., an inferior officer therein, by the invention of a process commonly known as the writ of subpœna, enabled the court to compel the legal or apparent owner of the land† to perform the uses subject to which the land had been conveyed to him. "It would," as Lord Coke very properly observes, "be absurd to say that confidence and trust can be reposed in land, which, in regard to sense, is inferior to brute beasts; and it would be less absurd to say that beasts may be trusted, who have sense, and want reason, than that land, which wants sense and reason also, may be trusted" (1 Rep. 127).

Legal estates.

Beneficial estates.

Statute of Uses;

There are, then, two qualities of estates—legal estates, and beneficial or equitable estates. The first are such as derive their existence from the common or the statute law; and the second from the introduction of uses, and which are called equitable, because they have been fostered and protected by the Court of Chancery, the author and expounder of equity law; while the courts of common law refused (4 Edw. IV.), and still refuse, to recognise them. For the purpose either of destroying these beneficial estates (*Co. Litt.* 272; Chudleigh's case, 1 Rep. 131; Gilbert *On Uses*, 74–182), or of removing the abuses to which they had become subject (Bacon, *Reading on Stat. of Uses, Law Tracts*, 332; 2 Sanders, *On Uses*, 89), was enacted the Statute of Uses (27 Hen. VIII. cap. 10), which provides, that where one is seised of lands, &c. to the use of another, he who has the use shall become seised of the lands,

* See Spence's *Inquiry into the Origin of the Laws, &c. of Modern Europe*, p. 9.

† At first, the beneficial owner had nothing to rely on but the honesty of the legal owner; and we are told that the confidence reposed was rarely abused. "The reason why no mention is made in our ancient books of uses is, because men were then of better consciences than now they are; so as the feoffees did not give occasion to their feoffors to bring subpœnas to compel them to perform the trusts reposed in them" (*per* Manwood J., Brent's case, 2 Leon. 15).

&c., and have the same estate, title, right, and possession therein as he previously had in the use. This Act contemplates only the case where there are two persons concerned, one (a feoffee to uses) having the legal estate, and another (the *cestuique trust*) having the beneficial or usufructuary estate; but it did not provide for a case in which land should be conveyed to one for the use of another for the use of a third. It was held, therefore, that the statute operated upon such a transaction as this, simply, in transferring the legal estate to the person to whom the first use was declared, and stopped there, without affecting the third person, in whom the use would consequently still remain. To prevent, then, the statute from destroying uses, all that was necessary was to create, instead of one, two beneficial estates, the first of which is immediately destroyed by the inevitable operation of the statute. "By this means," says Lord Hardwicke, "a statute, made upon great consideration, introduced in a solemn pompous manner, hath had no other effect than to add, at most, three words to a conveyance" (*Davenport v. Oloys*, 1 Atk. 591). This is hardly an accurate representation* of the effect of the Statute of Uses, which not only added three words to a conveyance, but has served to convey to the "legal ownership all the flexible and popular qualities of the use" (Hayes, *Introd. Conv.* ch. ii.); for it has been "the course of the statute to bring the possession to the use" (*per* Walmesley J., Bacon, *Reading*, 324), and not to impose on the use the fetters to which the common law subjected the possession.

defeated in construction.

It may be proper here to observe, that, formerly, the trustee attainer or conviction for high treason of a trustee, or person holding the legal estate, while the beneficial estate is in the hands of another, was followed by the same consequences as if he were the beneficial owner; but the rule is now done away with, except as to any beneficial interest which the trustee might have in the trust property.

The *quantity* of the interest has reference both to the nature of the quality absolutely, and perhaps to the number of the tenants.

(I.) Estates are either freehold or less than freehold. Freehold estates are of inheritance or not of inheritance. Freehold estates of inheritance† are, (1) estates in fee-simple, (2) in fee base, and (3) in fee tail, or conditional. Freehold estates not of inheritance are, (1) estates tail after possibility of issue extinct; (2) for life; (3) by the courtesy of England; and (4) dower. Estates less than freehold are, (1) estates for years, (2) at will, (3) year by year, and (4) by sufferance.

These and their incidents must be briefly described. First, then, of freehold estates of inheritance.

(1.) Tenant in *fee simple* (or in fee, as he is usually

Feesimple.

* Vaughan C. J. describes the principal effect of the statute as having been "to introduce a general form of conveyance, by which persons may execute their intents and purposes at pleasure, either by transferring their estates to strangers, by enlarging, diminishing, or altering them, to and among themselves at their pleasure, without observing that rigour and strictness of law as was requisite before the statute" (Vaughan, 50).

† It must be borne in mind that the estates of inheritance hereafter described, are those which the law will alone recognise; nor is it competent to any one to make a new inheritable estate at his will (*Co. Litt.* 13 a, 27 b); so that if a gift were made to one and his heirs *male*, the law would reject the word *male*, and the estate descend to the heirs generally; but if the limitation appeared in a will, the gift would be construed as importing an estate tail (*Litt. s.* 31; *Co. Litt.* 27 a; see also the famous Prince's case, 8 Rep. 32-43).

Law.

called) is he that hath lands, tenements, or hereditaments to himself and his heirs for ever, which is the largest and most beneficial estate recognised by the law of England, and is an unqualified inheritance in lands, unlimitable in its duration as to extent (Litt. sec. 1). He is called *tenant* in virtue of the doctrine, just mentioned, which treats the king as the universal landlord—a doctrine so far recognised by our law, that in corporeal inheritances (for to incorporeal the doctrine does not extend) the tenant in fee simple is formally styled as being seised in *his demesne as of fee*.

An estate in fee simple, if undisposed of by the living owner, devolves in inheritance to his heirs in the course indicated by law. To the creation of an estate in fee-simple, or any other inheritable estate, the use of the word "heirs" is generally essential, except in the cases of devises (which are gifts by will), and when the gift is to the king, in whom a fee-simple will vest, although the word "heirs" be not used.

Fee base.

(2.) A *fee base* is such a fee as hath a qualification subjoined thereto, and is determinable as that qualification terminates; as if lands are given to a man and his heirs, being lords of a particular manor (Preston, *On Estates*, 431-2). The owner, while his estate lasts, has all the rights and privileges of a tenant in fee simple (1 Cruise, *Dig.* 79).

Fee tail.

Statute *De Donis*.

(3.) A *fee tail* is a conditional fee, restrained to some particular heirs, in exclusion of the others. It derives its existence (Litt. s. 3) from the famous statute *De Donis* (13 Edw. I. cap. 1),* styled, either from its purposes or framers, the Statute of Great Men,† which enacts, that where lands were given to a man and the heirs of his body, they should go to the issue; and if there be no issue, then should revert to the donor; and declares that an alienation by the first taker was *contra formam doni*. On the construction of this statute, the judges divided the estate given into two parts; one which is the fee tail, and the other called the *reversion*, which is the ultimate fee simple expectant on the failure of issue. All corporeal hereditaments, and incorporeal concerning, annexed to, issuing out of, or which may be exercised within corporeal, as rents, commons, and the like, may be entailed. An estate tail (as it was first called in Westminster II. 13 Edw. I. cap. 4) may be general or special, male or female. It is *general*, when it is to the heirs of the body generally; *special*, when to certain heirs, as heirs by a particular wife, &c.; *male* or *female*, when to the heirs male or to the heirs female of the body respectively. The principal incidents to a tenancy in estate tail are, (1) the right of the tenant to commit what is called *waste*; he may pull down houses, fell timber, and do like acts,

The reversion.

Species of estate tail.

Incidents of the estate.

which in the case of inferior estates would induce forfeiture; (2) his widow shall have her dower out of the estate; (3) the husband of a female tenant-in-tail may be tenant by the courtesy of the estate; (4) and the estate itself may be destroyed by the tenant executing a deed, which must be enrolled in the Court of Chancery six months after execution, and which must purport to be a conveyance in fee of the land entailed, and must have the consent of a certain party named in the act, if the estate tail be expectant on a particular estate or estates of a certain description (3 & 4 Will. IV. cap. 74).

As to freehold estates not of inheritance; and first of—

(1.) *Estate tail after possibility of issue extinct*—a denomination applied (to alter the definition of Dr. Wooddeson) to the estate of a male or female donee* in tail, whose husband or wife, as the case may be, from whom the inheritable issue was to spring, is deceased, leaving no issue living, so that there remains no possibility that the estate should be transmitted according to the form of the gift; a possibility which is equally absent when the parties have been divorced for some cause which makes the marriage void *ab initio*; but then they have not this estate, which must proceed from the act of God, but are simply tenants for life (Woodd. *Lect.* 19; *Co. Litt.* 28 a). That one should have this estate, he must once have had a descendible estate; and he is distinguished chiefly by his non-accountability for waste (that is, for pulling down houses, felling trees, and so despoiling the land†) from a tenant for life. If he aliens his interest, his alienee takes, according to Coke (*Co. Litt.* 28 a), only an estate for life—a doctrine which has been impugned by Lord Hale (Note on *Co. Litt.*, *ut sup. cit.*, Harg. edit. n. 6).

(2.) *Estate for life*, which is where a lease is made of lands or tenements to a man to hold for his own life or for the life of another, or for the lives of certain others. In these last cases he is called tenant *pur autre vie* (Litt. s. 56). A grant made without expressing the quantity of interest intended to be passed, is construed to convey an estate for the life of the person to whom it is made (*Co. Litt.* 42 a). Except for necessary repairs, and at seasonable times, a tenant for life may not fell timber‡ (*Co. Litt.* 53), nor commit other waste, but is entitled to reasonable estovers or botes, and his executors to emblements—advantages which belong, even in a greater degree, to his under-tenant.

(3.) *Tenant by the courtesy of England*, is the life estate to which the husband is entitled, after the death of his wife, in those lands in which she was actually possessed of an estate of inheritance, when he has had issue by her born alive, and capable of inheriting her estate.

(4.) *Tenant in dower*.—For an explanation of this

* It rather re-established than created entails (Taylor v. Horde, 1 Burr. 115; see Barrington's commentary on the statute).

† "Il y fueront sages gents queux fieront c'est statut," said Sir William Herle, Chief Justice of the Common Pleas; "and I may say," observed Coke, "qu'ils fueront sages gents interpretent c'est act" (2 Inst. 335). It certainly had not the effect intended. Of the skilful lawyer, whose character he paints in such vivid colours, Chaucer tells us that—

"All was fee simple to him in effect"

(Prol. v. 321, Works, 1798): so soon did the wisdom of those who interpreted get the better of the wisdom of those who made the statute. One of its objects was to relieve the estates of the nobles from the penalty of forfeiture in cases of high treason (Bacon, *Use of the Law*; *Law Tracts*, 143. 1737); but to this, estates in tail were afterwards subjected (26 Hen. VIII. cap. 15, s. 5; see Div. IV. vol. xxviii. p. 417).

* In a work intended for general readers exclusively, it may not be thought unnecessary to state that the relation of persons to a conveyance is, in technical phraseology, marked by the conclusion of the legal appellations. Thus, the donor gives; the donee is he to whom the gift is made; mortgagor, he who mortgages; mortgagee, he to whom the mortgage is made.

† It was said that, although punishable for waste, the tenant after possibility had no property in the timber (Herlakenden's case, 4 Rep. 63); but this has been denied; but it is established that wilful devastation of the property will be restrained by injunction from the Court of Chancery (Williams v. Williams, 15 Ves. jun. 427).

‡ It is *equitable* waste in a tenant for life to cut timber "planted or growing for ornament," as distinguished from timber which is merely ornamental in fact (see Strathmore v. Bowes, and other cases quoted, Hayes, *Conv. App.* 22-3, *ut sup. cit.*).

Law.

Law.	estate, reference may be made to the Fourth Division of this work (vol. xxi. p. 417).	year's notice to quit, expiring at the period of the year when the tenancy commenced—a rule which does not apply to the case of a weekly tenancy of furnished apartments, in the absence of any agreement to that effect (Chitty, 340). The notice may be verbal or in writing, unless it have been agreed that it should be in writing, certain as to the premises given up, and extending to the whole of the premises. Great facilities have been given to landlords by a late act of parliament (1 & 2 Vict. cap. 74) in recovering possession of premises after the determination of certain tenancies.	Law.
Estate for years.	(1.) <i>An estate for years</i> , which is any estate, by whatever words created (<i>Co. Litt.</i> 45), that must expire at a period, however remote, certain and prefixed; from which circumstance it is often called a <i>term</i> . The tenant for years is entitled to the same estovers, and, with a slight exception, his executors have the same rights as to emblements, unless he should have renounced them by special contract, which a tenant for life has.	(4.) <i>An estate at sufferance</i> is where one comes into possession of land by lawful title, but keeps it afterwards without any title at all. A common instance of this is a tenant holding premises unmolested, over the time for which they were let to him.	Estate at sufferance.
Estate at will.	(2) <i>An estate at will</i> , as its name imports, is where lands or tenements are let to be held at the will of him who lets them, and no longer. A general letting of land, <i>i. e.</i> , a demise without limit as to the period of holding, was, up to the reign of Henry VIII., held to create this estate, which is now scarcely known, as the courts no longer consider that a tenancy at will arises <i>by implication</i> , when the tenancy is not expressly agreed to be a tenancy from year to year. But a person who holds rent-free by the permission of the owner is still considered a tenant at will (<i>R. v. Collett</i> , Russ. & R. C. C. 498).	By the <i>certainty</i> of interest is properly to be understood estates either <i>conditional</i> or <i>absolute</i> ; conditional estates being those whose existence depends upon the happening or not happening of some uncertain event, whereby the estate may be either originally created, enlarged, or finally defeated (<i>2 Comm.</i> 152). Absolute estates are such as depend upon no contingency whatever. Conditions may be implied or expressed: for example, where a grant is made to a man of an office, the law implies the condition that he shall hold it only so long as he duly executes the same; and failing so to do, he who granted it, or his heirs, may lawfully deprive him thereof.	Conditional estates.
Estate from year to year.	(3.) <i>An estate from year to year</i> may arise not only from express stipulation, but even from that general letting heretofore held to constitute an estate at will, with the exception, it would seem, of lodgings (<i>per Lord Mansfield</i> , <i>Right v. Darby</i> , 1 T. R. 159). It may arise from a variety of circumstances, not compatible with our design to detail; but we may remark a singular example of the interpretation of a statute, that although the legislature provided that an agreement in words (a parol agreement, as it is termed) for a term for more than three years, not reduced to writing and signed, should operate as a tenancy at will, it has been held that an agreement without such formalities operates as a tenancy from year to year (29 Car. II. cap. 3; <i>Clayton v. Blakey</i> , 8 T. R. 3). The mere payment of rent affords a presumption, which may be rebutted, of a general letting from year to year (<i>Doe dem. Pritchard v. Dodd</i> , 5 B. & Ad. 689); and the person who pays the rent may show on whose behalf it was received (<i>Doe dem. Harvey v. Francis</i> , 2 M. & Rob. 57). In the absence of an express agreement, the tenant is liable only when an injury happens to the premises through voluntary negligence, and not from injuries arising from accidental fire, wear and tear of time, or the like (<i>Woodfall, Landl. and Ten.</i> by Harr. 3rd edit. 398). In the case of a farm, a promise is implied on the part of a yearly tenant, that he will use it in an husbandlike manner, and cultivate the lands according to the custom of the country where they are situate (<i>Chitty, On Contracts</i> , 337, 3rd edit.). A landlord is not liable to an action for not repairing, unless he has agreed to repair; but if the premises, through want of repair, should become uninhabitable and profitless to the tenant, he will, under some circumstances, be at liberty to quit without further notice, and discharged of any liability to future rent.	Conditions impossible when created, becoming afterwards impossible by the act of God, or the act of their maker, contrary to law, or repugnant to the nature of the estate, are void. If such a condition is to be performed <i>after</i> the estate vests (as in the case of a gift to A, on condition that he shall marry B by a certain day), the estate at once becomes vested absolutely in him to whom it is given, discharged of the condition. If such a condition is to be performed <i>before</i> the estate vests (as in the case of a grant to A, that he shall have a certain estate if he goes to Rome in one day), the condition being void, so also is the estate, which was to arise only upon its performance. The most ordinary kinds of estates upon condition are mortgages (<i>Div. IV.</i> vol. xxv. p. 297), and estates by statute merchant, statute staple (<i>Ib.</i> vol. xxviii. p. 37), and <i>elegit</i> (<i>Ib.</i> vol. xxi. p. 456), which have been too fully noticed elsewhere to require any comment in this place.	Conditions impossible.
Liability of tenant.		Estates considered in reference to the <i>time</i> of their enjoyment—the time when the rents and other advantages can lawfully be taken—are either in <i>possession</i> , <i>remainder</i> , or <i>reversion</i> . When the interest actually resides in the tenant or owner, he is said to have an estate in possession; when an estate is limited to be taken after another estate has determined, according to the very nature and extent of its original limitation, it is an estate in remainder.	Possession.
Liability of landlord.		Remainders are either <i>vested</i> or <i>contingent</i> . A remainder is <i>vested</i> where one has an immediate fixed right of future enjoyment, an estate <i>in presenti</i> , although it is only to take effect in possession of the profits at a future period; and this is such an estate as may be transferred, aliened, and charged much in the same manner as an estate in possession (<i>Cruise, Dig. tit. Remainder</i> , ch. i. s. 9).	Remainder.
Termination of the tenancy.	If there be no express agreement on the subject, and no immemorial usage prevailing in the place to the contrary, one party, or if dead, his personal representative, must, in order to terminate the tenancy, give to the other, or his representative in a like case, half a	A remainder is <i>contingent</i> so long as the future estate has no capacity of taking effect in possession on the determination of the particular estate. Into the	contingent.

Law. profound and complicated learning relating to contingent remainders, it is here unnecessary to enter, nor would a brief consideration of their qualities possess any practical value. The incomparable treatise of Mr. Fearne is too well known to render it necessary to observe, that there may be found all that can be required on the subject.

Reversion. An estate in reversion is the residue of an estate left in the grantor, to commence in possession after the determination of some particular estate granted out by him (2 *Comm.* 175); or it may be perhaps better defined as the returning of the lands in possession to the donor or his heirs, after the grant is ended (*Co. Litt.* 142 b).

Estates considered in reference to the *number of persons participating* in their enjoyment, are estates in joint tenancy, tenancy in common, and in coparcenary; but these have been treated at sufficient length in the Fourth Division of this work (vol. xxviii. p. 492), so that it is unnecessary to allude further to them in this place.

Title by possession. Having now discussed, as completely as our limits would allow, the modes of enjoyment of real property, we now come to consider the manner in which it may be *acquired*. It may be acquired by merely obtaining *possession*, that is, the actual occupation of the estate; and if this be accomplished in opposition to the rights of its legal owner, it will lie with him, by the assertion of those rights, and by an appeal to the tribunals of his country, to eject the intruder. Until this is done, the actual possessor has, in right of his possession, *prima facie* evidence of a legal title; and this possession may, by length of time, and negligence of the lawful owner, ripen into a perfect and indefeasible title. No title can be completely good without actual possession. This, however, is a title capable of being defeated. The other modes of acquiring property, recognised by the law of England, are of two descriptions: property may be acquired by either *descent* or *purchase*.

Descent is where the title to property vests in a man by the single operation of law; and *purchase*, where the title vests in him by his own act or agreement, conjointly with such operation, or independently thereof (2 *Comm.* 201).

Title by descent. (I.) *Descent*.—When a man, on the death of his ancestor, acquires his estate in right of being his representative, he is said to take by descent. The law, in order to prevent the confusion which would naturally ensue on the death of any one possessed of property, were the future right to that property undetermined, has wisely, by a variety of rules, defined and ascertained the person who shall be so entitled, and who, in reference to the deceased, is termed his heir at law. The rules regulating descent are thus stated by Blackstone:—

Canons of descent. (1.) Inheritances shall lineally descend to the issue of the persons who last died actually seised, *in infinitum*, but shall never lineally ascend.

(2.) The male issue shall be admitted before the female.

(3.) Where there are two or more males in equal degree, the eldest only shall inherit; but the females all together.

(4.) The lineal descendants *in infinitum* of any person deceased shall represent their ancestor; that is, shall stand in the same place as the person himself would have done, had he been living.

(5.) On failure of lineal descendants or issue of the

person last seised, the inheritance shall descend to his collateral relations, being of the blood of the first purchaser, subject to the three preceding rules.

(6.) The collateral heir of the person last seised must be his next collateral kinsman of the whole blood.

(7.) In collateral inheritances, the male stock shall be preferred to the female (that is, kindred derived from the blood of the male ancestors, however remote, shall be admitted before those from the blood of the female, however near), unless where the lands have in fact descended from a female (1 *Comm.* 208–234).

These rules now exist—subject, however, to certain 3 & 4 provisions contained in a late act of parliament (3 & 4 Will. IV. cap. 106), which was passed in accordance with the recommendations of the commissioners for inquiring into the law of real property (*First Report*, pp. 10–16).^{*} This Act provides, (1) that the descent shall always be traced from the purchaser, that is, the person who last acquired the land otherwise than by descent, escheat, partition, or enclosure; but that the last owner shall be considered as the purchaser, unless the contrary be proved (s. 2); (2) that the heir entitled under a will shall take as devisee, and that the limitation to the grantor or his heirs shall create an estate by purchase (s. 3); (3) and that where heirs take by purchase under limitations to the heirs of their ancestor, the lands shall descend as if the ancestor had been the purchaser (s. 4); (4) that brothers or sisters shall trace descent through their parent (s. 5); (5) that every lineal ancestor is capable of being heir to his issue, and is heir in preference to collateral persons claiming through him (s. 6); (6) that the male line of inheritance is to be preferred (s. 7); (7) that the mother of more remote male ancestors be preferred to the mother of less remote male ancestors (s. 8); hereby recognising the propriety of Mr. Justice Blackstone's opinion (2 *Comm.* 238), so long contested, and by such respectable authorities. The only other provision in this act deserving of notice, is the abolition of the rule excluding the half blood. Now, the half-blood, if on the part of a male ancestor, inherits after the whole blood of the same degree; if on the part of a female ancestor, after her (s. 9). This Act, it should be observed, does not extend to any descent taking place before the 1st day of January, 1834 (s. 11); nor to limitations, made before the same date, to the heirs of a person then living; these take effect as if the Act had not been made (s. 12).

(II.) *Purchase*.—Acquisition by purchase may be effected either partially by operation of law, or wholly by an act of the parties. (1.) Of the first mode of acquisition there are several species, which we shall notice in their turn. First, acquisition by *escheat*. Where one dies without heirs, his lands are said to escheat, or fall to the lord of whom they were holden. With the exception of lands of copyhold or customary tenure, this lord is the king. This defect of heirs may arise in a variety of ways; and, amongst others, from the doctrine of corruption of blood, as it is called, by which, when one is attainted of treason or convicted of murder, he loses his capacity of transmitting his property by way of inheritance. In the case of attainder for treason, the land, unless it be copyhold, is forfeited to the crown absolutely; but,

^{*} The value of some of these changes is attested by Lord Loughborough's observations in *Brydges v. the Duchess of Chandos*, 2 Ves. jun. 426.

- Law** in the case of conviction for murder, the crown becomes possessed only for a year and a day, and afterwards the lord takes by way of escheat. Secondly, there is acquisition by *forfeiture*, which is a punishment annexed by law to an illegal act or negligence committed by the owner of lands or hereditaments, whereby he loses all his interest therein, and they go to the party receiving the injury, as a recompence for the same. The illegal acts upon which forfeiture is consequent, are chiefly—(1) crimes and misdemeanors; (2) alienation contrary to law; (3) nonperformance of conditions; (4) waste; (5) breach of copyhold customs; (6) bankruptcy.
- Forfeiture:** (1.) Crimes and misdemeanors. He who commits treason or misprision thereof, felony, *præmunire*, draws a weapon on a judge, or strikes any one in the presence of one of the king's principal courts of justice, forfeits his lands and tenements to the king. In the case of persons committing treason, the forfeiture is absolute, and vests in the king for ever all such lands, tenements, and hereditaments of inheritance, in fee simple or fee tail, as the criminal was possessed of when the treason was committed, and discharged of all incumbrances induced subsequently to such commission. The king also takes the profits of all lands and tenements which the criminal had in his own right for life or years, so long as such interest shall subsist. In felony and the other crimes above mentioned, the offender forfeits all his lands and tenements in fee simple to the crown for a short period; for the king shall have them for a year and a day, and commit on them what waste he pleases. These forfeitures are usually compounded for. Forfeitures of this description also relate back to the time when the offence was committed.
- for crimes and misdemeanors;** (2.) Alienation contrary to law. A gift contrary to the statutes of mortmain, and alienation to an alien, involve forfeiture to the crown of the lands purposed to be so given or alienated. Further, alienations by particular tenants, of estates greater than the law entitles them to convey, being in contravention of the rights of those entitled to the remainder or reversion, are forfeitures to him whose rights are sought to be infringed: as for example, if one who has a life estate should pretend to convey the fee simple, which is an estate larger than his own, he is, in intendment of law, considered to have by his own act determined his original interest.
- treason;** (3.) Nonperformance of conditions. We have already spoken of conditional estates, and it is sufficient here to observe, that the breach or nonperformance of lawful conditions annexed to these estates produces forfeiture of the same.
- felony.** (4.) Unlawful waste is punished by forfeiture of "the *thing* (construed to extend to the *place*; 2 *Inst.* 303) wasted" (Stat. Glouc., 6 Edw. I.). If waste be done at one end only of a wood, or perhaps in one room of a house, if capable of convenient separation, that part only shall be forfeited (2 *Comm.* 284).
- Alienation contrary to law.** (5.) Breach of the customs of copyhold manors in many cases subjects the tenant to a forfeiture of his possession. As it is probable that these customs will speedily disappear under the influence of the new enfranchisement Act, already alluded to, this is a subject hardly worth dwelling upon.
- Non-performance of conditions.** (6.) Bankruptcy: hereafter is defined what constitutes an act of bankruptcy: it is sufficient here to observe, that a bankrupt's real and personal estate becomes, under a process hereafter to be described, vested in certain persons, who will deal with the same, under certain restrictions, for the benefit of the bankrupt's creditors.
- Bankruptcy.** Thirdly, title by *prescription*, which is only to incorporeal hereditaments, such as rights of common, rights of way, and the like, arises where one can show that he and those under whom he claims have, for a certain period, been accustomed to enjoy the same as of right. This enjoyment must be had, not secretly, by stealth, by tacit sufferance, or by permission asked from time to time on each occasion, or even on many occasions, of exercising it; it must be had openly, notoriously, without particular leave at the time, by a person claiming to use it without danger of being treated as a trespasser, as a matter of right (*Tickle v. Brown*, 6 N. & M. 239). Claims to rights of common, and other profits, to be taken and enjoyed upon the lands of the king, or any ecclesiastical or lay person or body corporate, cannot be defeated after thirty years' uninterrupted enjoyment, by showing their commencement prior to such period; and after sixty years' enjoyment, such rights become absolute and indefeasible, unless it should appear that the same were taken and enjoyed by some consent or agreement expressly made or given by deed or writing. Claims to rights of way, and other easements, when such have been actually enjoyed without interruption for twenty years, are defeasible by showing their commencement prior to such period; and after an enjoyment for forty years, the right thereto becomes absolute and indefeasible, unless it could be shown to have been taken and enjoyed by some consent or agreement expressly made or given by deed or writing. Right to the use of light enjoyed for twenty years, unless it appears that the same was enjoyed by consent or agreement, is indefeasible (2 & 3 Will. IV. cap. 71).
- Acquisition by act of the parties.** Having thus shown the various modes of acquisition by operation of law, wholly and partially, we have now to consider the modes of acquisition by act of the parties.
- Deed of conveyance:** And first, then, of acquisition by deed of conveyance, which is a writing sealed and delivered by the parties. In treating this subject it will be proper to consider—(1) who may convey, and to whom; (2) what may be conveyed; (3) what are the requisites of a deed of conveyance; and, (4) what are its kinds, and their operation.
- Convey;** (1.) Conveyances to and by idiots, lunatics, infants, who may and persons under duress, are not void, but voidable; that is, should the idiot or lunatic recover his reason, the infant attain his majority, the person under duress be relieved thereof, they may affirm or avoid these transactions, according to their pleasure. The conveyance, or other contract of a married woman, during her marriage, is absolutely void: she may, however, be a purchaser without consent of her husband, and the purchase is good until he avoids it by some act of dissent; but if he does not, she herself may do so, after his death. Those attainted of treason, felony and *præmunire*, are incapable of conveying from the time of the commission of their offences.
- what may be conveyed.** (2.) The mere right by the possession of the property cannot be the subject of a conveyance; nor can be contingencies or mere possibilities be assigned to a stranger, unless coupled with some present interest; but a reversion or a remainder vested may be granted, because the possession of a particular tenant is the

- Law.** possession of him in reversion or remainder: it is the possession that is the proper object of every conveyance.
- Deed must be written.** (3.) To constitute a valid deed, it must be written or printed on paper or parchment, sealed and delivered: notwithstanding the Statute of Frauds, signature is not necessary.
- Species of deeds.** (4.) The various species of deeds may be properly considered under two heads: first, those which are effectual at common law; and, secondly, those which derive their operation from the Statute of Uses.
- Feoffment.** Of the first kind is the feoffment, which is a conveyance of corporeal hereditaments from one person to another, by delivery of the possession upon or within view of the hereditaments so conveyed. To complete the operation of the feoffment, must be performed the ceremony called "livery of seisin," which was the delivering, upon the land, in the presence of witnesses, of a detached portion of the soil. It is this livery which is considered to pass the land: the charter of feoffment operates merely as the record or written evidence thereof.
- Gift.** A gift is properly a conveyance in fee tail, and, equally with feoffments, requires livery of seisin. A grant is the proper method at common law of transferring the property in incorporeal hereditaments, or such things whereof no livery can be had. A lease is a conveyance of lands or tenements made for life, years, or at will, but always for less interest than he who makes it hath in the premises: it is usually made in consideration of rent or other annual recompence, and the general words of operation are, "demise, grant, and to farm let." As to what is capable of being demised, it is obvious that whatever may be granted or parted with for ever, may be granted or parted with for a time; and thus we hear of leases of live cattle for a year (*Billingsley v. Hersey*, 2 Bulst. 7); and also of household goods for certain periods, and on certain considerations. Incorporeal hereditaments may generally be the subjects of a lease: this, of course, does not include judicial offices, nor peerages.
- What can be leased.** As to who may lease—such as are of full age, and seised in fee, may of course grant leases of any duration. Infants making leases can avoid the same on attaining their majority. A tenant in tail may make a lease to endure for not more than three lives, or twenty-one years, and such a lease binds his issue in tail, but neither remainder-men nor reversioners. A husband, seised in right of his wife in fee-simple or tail, may make a similar lease, provided the wife joins therein, and which will bind her and her heirs. All persons seised of an estate in fee simple, in right of their churches, not including parsons and vicars, may, without the concurrence of any other, make such a lease as will bind their successors. That the lease should be valid, it must comply with the numerous requisitions of the statute (32 Hen. VIII. cap. 28). The leases of ecclesiastical corporations are subject to various restrictions (5 & 6 Vict. cap. 108).
- Who may lease.** An assignment is a transfer or making over to another of the interest which one has in the unexpired residue of a term or estate for years. A release is a discharge or conveyance of a man's rights in lands or tenements to another, that hath some former estate in actual possession.
- Assign-ment.** Of conveyances which derive their operation from the Statute of Uses, the lease and release is the instrument in the most frequent use. Its origin may be thus explained: before the passing of the Statute of Uses, a mere contract for sale, made in consideration of money or other valuables, paid to the seller, was sufficient to transfer the use; for the law not recognising the use, none of the solemnities necessary for the transfer of property were required. After the passing of the Act, which brought the possession to the use, it was easy, by a mere contract transferring the use, to transfer the possession as completely as if all the legal requisitions had been complied with. To prevent this, an Act was passed (27 Hen. VIII. cap. 16) by which every such contract, bargain, and sale, as it was called, was declared void, if embracing freehold property, unless made by a writing, sealed, and enrolled, within a certain period after its execution, in the Court of Chancery. This Act did not, however, extend to terms for years; and contracts for the sale of beneficial interests in such were still valid, although not enrolled. It became then usual to contract, for a nominal consideration, to sell the use of land for one year; and the purchaser had immediately the possession transferred to him by the operation of the statute. Having thus acquired the possession, the seller, in whom the inheritance remained, executed to him a release, relinquishing all his interest in the land, and thus was the transaction completed. The lease and release was, we are told by *Fabian Philips*, at first purposely contrived by *Francis Moore*, at the request of the Lord Norris, to the end that such of his kindred or relations should not take notice, by any search of public records, what conveyance or settlement he should make of his estate (*Treatise on Capias*, p. 115). The necessity for a lease or bargain and sale for a year is removed by the 4 Vict. cap. 21. There are several other deeds deriving their operation from the Statute of Uses, which it does not fall within our plan to notice.
- Release.** Freedom of alienation was unknown to early times; and we owe the power of transferring by will, or, as it is technically called, devising, to a statute of Henry VIII. Previous to that time, no greater interest would pass by a will than a term for years. Wills are denominated, by Lord Dyer, the laws that private men are allowed to make, and they have effectually served to prevent an accumulation of property, which would have materially impeded the social and political improvement of the country. Before the Statute of 32 H. VIII. cap. 1, devises were indirect, and had their operation in consequence of some feoffment to the use of a will, which the Court of Chancery would enforce: the Statute of Wills, however, corrected by the 34 Hen. VIII. cap. 5, authorized all persons seised in fee simple, not being a wife, an infant, an idiot, or a person of not sane memory, by will an testament in writing, to devise to any other person except bodies corporate, two-thirds of their lands, tenements, and hereditaments held in chivalry, and the whole of those held in socage, which now in fact amounts to the whole of their landed property not being copyhold.
- Lease and release.** The law relating to devises is now regulated by the 7 Will. IV. and 1 Vict. cap. 26, which has abolished the whole of the former statute law relating to wills. Of its provisions we have now to give some account.
- First, as to who may make a will. A will may be made by any one not under twenty-one years of age, not an idiot, a lunatic or person of unsound mind, or a married woman. This last disqualification is subject to certain exceptions; for a married woman may make

Law.

Wills.

32 H. VIII. cap. 1.

34 H. VIII. cap. 5.

7 Will. IV. and 1 Vict. cap. 26.

Who may make a will.

Law. a will if her husband be banished for life by act of parliament; or also, it would seem, if he be transported for a fixed term of years as a felon. A married woman may also make a will of property settled to her separate use, or an appointment by will in pursuance of a power to be executed notwithstanding the subsistence of her marriage.

How the will is to be made. Secondly, as to how the will is to be made. In order to be valid, the will must be in writing, and signed at the foot by the testator, as the person making the will is called, or by some other person in his presence, and by his direction; and such signature must be made or acknowledged by the testator in the presence of two or more witnesses present at the same time; and such witnesses must attest and subscribe the will in the presence of the testator. Any interest whatever in real or personal estate, given or appointed to any person, or to the wife or husband of any person, who shall attest the execution of the will, is void. Such person is nevertheless to be admitted a witness to prove the execution, validity, or invalidity thereof. Creditors, however, may be attesting witnesses without forfeiting the benefit of any charge contained in the will for the payment of debts.

Revocation of wills. Thirdly, as to the mode of revocation. The revocation of a will can be effected by another will or codicil, duly executed as aforesaid, or by some writing declaring an intention to revoke the same, and executed as an original will must be; or by the burning, tearing, or otherwise destroying the instrument by the testator, or by some person in his presence and by his direction, with the intention of revoking the same. The marriage of the testator is a revocation of any will made by him previously.

What may pass by a will. Fourthly, as to what may pass by will. Every person may devise, bequeath, or dispose of, by his will (or codicil) duly executed, all estates, real and personal, to which he shall be entitled at law or in equity at the time of his death, and which, if not so devised, bequeathed, or disposed of, would devolve upon his heir at law or customary heir, or, if he became entitled by descent, upon the heir at law or customary heir of his ancestor, or upon his own executor or administrator. This includes estates of customary freehold or tenant right, or customary or copyhold, although the ancestor may not have surrendered, or, being entitled, as heir, devisee, or otherwise, to be admitted thereto, he shall not have been so admitted. It extends to estates for the life of another (*pur autre vie*), whether the same be freehold, customary freehold, tenant right, customary or copyhold, or of any other tenure, and whether the same be a corporeal or incorporeal hereditament. It includes all contingent, executory, or other future interest in any estate real or personal, whether or not the testator be ascertained as the person, or one of the persons, in whom the same may become vested, or whether he be entitled thereto under the instrument by which the same respectively were created, or under any disposition thereof by deed or will. It extends also to all rights of entry, and embraces all such of the same estates, interests, and rights respectively, and all other estates real and personal, to which the testator may be entitled at the time of his death, although, perhaps, subsequent to the execution of his will. By the law as it stood previous to the Wills Act, a devise to A, not naming his heirs, and being unaccompanied by any words from which the intention to devise the fee could

be collected, was held to pass only an estate for life. The new Wills Act has altered this rule, by enacting that if real estate be devised to any person without any words of limitation, such devise will be construed to pass the fee simple, or other the whole estate or interest which the testator had power to dispose of by will in such real estate, unless a contrary intention appear by the will.*

We have now to treat of personal property, or, in legal phrase, goods and chattels; which comprehends all species of possessions not freehold, and not capable of descending to the heir at law, but which, on the owner's death, vest in his executors or administrators.

Chattels are considered by the law to be of two kinds, real and personal. (1). Chattels real, are such as concern or savour of real property, such as, terms for years, the next presentation to a church, and the like, and are called real, as being interests issuing out of, or annexed to, real estates. (2). Chattels personal, are properly things moveable, such as animals, household stuff, money, corn, and so forth.

(I.) The most usual mode of acquiring personal property is marriage. All chattels which belonged formerly to the wife are, by act of law, vested in the husband, with the same degree of property, and the same powers as the wife, when unmarried, had over them.

Chattels real belonging to the wife cannot be disposed of by the husband by will, if he dies before his wife, in case of his not having dealt with them during his lifetime. So with *choses in action*—as personal chattels are called, whereof a man hath not the occupation, but the bare right to occupy, such as debts upon bond, contracts, and the like,—if the husband reduces them into possession by receiving or recovering them at law, they become wholly and entirely his own; but if at his death they are still *choses in action*, they survive to the wife. All chattels which the wife has in her own right, become the property of the husband on his marriage. If, however, her personal property is so circumstanced that the husband is forced to crave aid from a court of equity in reducing it into possession, the court will grant its assistance only on the terms that the husband makes a competent settlement in favour of his wife,

(II.) *Grants or gifts* of chattels personal are defined as the right of transferring the right and possession of them, whereby one renounces, and another immediately acquires, all title and interest therein. Properly to constitute a gift or grant, it should be accompanied with delivery of possession; for then it is out of the giver's power to retract it, even although he gave it without any consideration or recompence. It will be void, however, if made in prejudice of creditors, or if the giver were under legal incapacity.

(III.) Contract is defined by Blackstone to be an agreement, upon sufficient consideration, to do or not to do a particular thing. That a contract should be binding, there must be a promise by one party, and an accept-

* In reference to this provision in the Act it may be observed, that previously it was thought that if a man devised particular lands, which he had not at the time, but afterwards purchased, or devised all lands which he should die seised of, that such devises would be valid. Lord Chief Justice Saunders, under this impression, devised all lands which he had, or afterwards should have, in Fulham. His executors were Holt and Pollexfen, Chief Justices, and Mr. Sergeant Maynard. The latter thought the devise invalid, contrary to the opinion of the Chief Justices (*Laurence v. Dodwell*, 1 Lord Raym. 438). Holt subsequently altered his opinion.

Law. **Express contract.** **Implied contract.** **Consideration.** **Contract of sale or exchange.**

ance thereof by another (Pothier, *Oblig. P. I. c. 1, s. 1, art. 2*). The agreement must be, in general, obligatory upon both parties, or neither will be bound. A contract may be *express* or *implied*. An *express* contract is where the terms of the agreement are openly uttered and avowed at the time of the making, such as to pay a certain price for certain goods. An *implied* contract is such as reason and justice dictate, and the law therefore presumes every man undertakes to perform. If I take up wares from a tradesman without any agreement of price, the law concludes that I contracted to pay their real value (2 *Comm.* 443). A valid and sufficient consideration or recompence is of the very essence of a contract: without it, the promise is void, and creates no legal responsibility.

The most usual contracts by which the right to chattels personal may be acquired are, (1) contracts for the sale or exchange of goods; (2) bailments; (3) contracts of hiring and borrowing; (4) contract of debt.

(1.) A sale or exchange is a transmutation of property from one to another, in consideration of some price or recompence in value. If the transfer be for money, it is a sale; if for goods by way of barter, it is an exchange. In the case of a sale of goods generally, as distinguished from the sale of a specific chattel, no property in them passes until they are delivered. Up to that time they remain the property of the seller. But if, by the contract or bargain, the seller appropriates to the buyer a specific chattel, and which the latter agrees to take and pay for at a stipulated price, the effect of the contract is to vest the property in the buyer at once.

Market overt. The general rule is, that one who has no authority to sell, cannot, by making a sale, transfer the property to another. This rule is subject to an exception. A sale of goods by one not rightfully entitled to them, does transfer the property, if it be a *bond fide* sale, and effected in market overt. Market overt, or open market, is a fair or market held at stated intervals, by virtue of a charter or prescription; but in the city of London, every shop is, except on Sunday, market overt, in respect of the goods usually and publicly sold therein, and appropriate to the trade of the occupier. This exception must be taken subject to the provisions of the 7 & 8 Geo. IV. cap. 29, which entitles the owner of any chattel, &c., or other property stolen, to restitution, although such chattel may have been *bond fide* sold in the interval, provided the owner or his executor prosecute the thief or receiver to conviction, and the court award restitution.

Solemnities of a contract of sale. No contract for the sale of any goods, wares, or merchandises, for the price of 10*l.* or upwards, shall be allowed to be good, except the buyer shall accept part of the goods so sold, and actually receive the same, or shall give something in earnest to bind the bargain, or in part of payment, or unless some memorandum or note in writing of the same bargain be made and signed by the parties to be charged with such contract, or their agent thereunto lawfully authorized (29 Car. II. cap. 3). This enactment is extended to all contracts for the sale of goods of the value of 10*l.* and upwards, notwithstanding the goods may be intended to be delivered at some future time, or may not at the time of such contract be actually made, procured, or provided, or fit or ready for delivery, or some act may be requisite for the making or completing thereof, or rendering the same fit for delivery (9 Geo. IV. cap. 14, s. 7).

Law. **Contract of bailment.** (2.) Bailment is a delivery of goods in trust, upon a contract expressed or implied, that the trust shall be faithfully executed on the part of the bailee; that is, the person to whom such delivery is made. Thus, if goods be delivered to an innkeeper, and, whilst in the inn, they be damaged or stolen, either by the servants of the innkeeper or by strangers, the innkeeper is liable. So is a carrier, who, for hire, undertakes the carriage of goods for any person, either by water or land, considered to be under a contract in law to convey them safely. If one delivers anything to his friend to keep for him, the receiver is bound to restore it on demand, and is in the meantime answerable for its damage or loss, if occasioned by his gross neglect. With the possession, a special qualified property is transferred from the bailor to the bailee, enabling this latter to vindicate in his own right his possessory interest against any stranger.

(3.) Hiring and borrowing are contracts, whereby the possession of transient property is transferred for a particular time or use, on condition to restore what is so hired or borrowed on the expiry of the time, or the performance of the use, and in case of hiring (for borrowing is merely gratuitous), paying the price agreed on by the parties, or implied by law from the value of the service. The hirer or borrower obtains a temporary property in the thing, subject to an implied condition to use it with moderation. The owner or lender retains his reversionary interest, and acquires a new property in the price or reward.

(4.) Debt, which includes every contract whereby a determinate sum of money becomes due to any one, and is not paid, but remains in action merely. A debt of record is a sum of money appearing to be due by the evidence of a court of record; debts by specialty are when a sum of money becomes or is admitted to be due by instrument under seal; debts by simple contract are such where the contract is either oral or contained in writing not sealed.

(IV.) *Bankruptcy.*

Acquisition by bankruptcy. (1.) As to who may be a bankrupt. The various classes of traders capable of becoming bankrupts are enumerated in the 6 Geo. IV. cap. 16, s. 2, and the 5 & 6 Vict. cap. 122, s. 10. Aliens, denizens, women, persons residing even in Scotland, Ireland, the Isle of Man, the British colonies, or in any foreign country trading to or from this country, may be made bankrupt here. It would seem that a married woman is subject to be made a bankrupt, if she carries on trade free from the control of her husband, and if he be not answerable for the debts contracted by her in it. Peers, members of the House of Commons, servants of foreign ambassadors, clergymen, and public officers may be bankrupts, if they be traders.

(2.) As to what acts constitute bankruptcy. These are minutely detailed in the Bankrupt Consolidation Act, of Geo. IV. To leave the country, to remain abroad, to suffer or procure an undue arrest, fraudulently to convey lands or other property, are considered so many acts of bankruptcy, if done with intent to defeat or delay creditors.

(3.) How one may be made a bankrupt. Any creditor (whose debt amounts to a hundred pounds or upwards), if a trader within the Statute of Bankrupts, may apply to the Lord Chancellor by petition, stating that his debtor is a trader, the nature and amount of his debt, that he has become bankrupt, and praying

Law. his lordship to issue his fiat, authorizing the petitioner, as such creditor, to prosecute his complaint in her Majesty's Court of Bankruptcy. This fiat being granted, it is opened by one of the commissioners of the Court of Bankruptcy; and proof being given before them of the petitioning creditor's debt, of the fact of the trading, and of the act of bankruptcy, the trader is declared a bankrupt. It is then usual for the creditors to choose certain persons to act as trustees on their behalf, and who are called assignees. These having been chosen, all the personal effects and estate of the bankrupt become vested in them, and all his lands, tenements and hereditaments (except estates tail, for which special provision is made by the 3 & 4 Will. IV. cap. 74, ss. 55-6) in the United Kingdom, or any of the dominions, plantations, or colonies belonging to the king. The assignees or commissioners dispose of the bankrupt's property for the benefit of the creditors. In order to facilitate the examination of the various claims of creditors against the bankrupt's estate, it is desirable that the bankrupt should be examined; and if he fails therefore to surrender himself, and submit to be examined before the court, or upon examination does not discover all his real and personal estate, and how he has dealt with it, and surrender his books and papers, within a certain period after he has been declared a bankrupt, he is punishable with transportation for life, or not less than seven years, or with imprisonment, with or without hard labour, for not exceeding seven years. If he do surrender himself, submits to a full examination, and makes a true discovery of his estate and effects, he may reasonably expect his certificate, which is an instrument reciting the whole proceedings, and contains a consent, by a certain proportion of his creditors, that he shall have such allowance and benefit as are given to him by statute, and be discharged from his debts. The property of the bankrupt is, of course, divided amongst his creditors.

Acquisi- by insol- vency; (V.) *Insolvency*.—The probability that a sweeping change will speedily be effected in the laws relating to insolvent debtors renders a bare reference to the recent Acts (1 & 2 Vict. cap. 110, and 5 & 6 Vict. cap. 116) all that is necessary.

by will and admi- nistration. (VI.) *Will*.—Personal property is acquired by will in the same manner as real property; and in the case of persons possessed of personal property dying intestate, the ordinary grants administration to one of the intestate's kindred, by whom the property is divided amongst his next of kin, in conformity with the Statute of Distributions (22 & 23 Car. II. cap. 10; explained by 29 Car. II. cap. 3).

In concluding this rapid review of the doctrines of English property law, a few remarks may be permitted having reference, not to the details of the system, but to its spirit and characteristic features.

Observa- tions on English property law. Whoever studies this system with intelligent attention can scarcely fail to recognise in it a system admirably adapted to the requirements of an ancient and wealthy people, whose commercial pursuits have not abated their pride of ancestry or their attachment to aristocratic institutions. If without manners laws are nothing, laws not congenial to manners are worse than nothing; and the highest praise that can be accorded to any legal system is, that it harmonizes with the genius and spirit of the people. It was well remarked by a great man, *Angustia prudentiæ humanæ casus omnes quos tempus reperit non potest capere* (*De Augm.*

lib. viii. cap. iiii. aph. 10). But this is a poverty the legislator has been slow to perceive; and thus we find him in many countries, instead of relinquishing to individual prudence the task of arranging for the future enjoyment of property, undertaking of himself arbitrarily to define to whom, and how, that enjoyment should be secured, as though there were given to him foresight to anticipate all possible exigencies, and wisdom to provide arrangements adequate to their satisfaction. Thus, in the *Code Civil*,—that great work by which the modern Caesar of France hoped to realize the glory coveted by his predecessor of old,—while instead of the so-called illiberal law of primogeniture, equal partibility was made the rule, freedom of alienation was greatly shackled by the numerous restraints upon voluntary gifts, whether testamentary or made *inter vivos*. Freedom of alienation is one of the most striking characteristics of our own law, which in all cases leaves the disposition of property as free as is consistent with public policy. Not to interfere with the individual—to relinquish to him, as far as may be, the full control over his own—is the rule of English law. It does interfere, indeed, to prevent private caprice from being a source of public mischief; and thus, as we have already seen, it will not suffer the creation of a new inheritable estate; nor will it permit an estate tail to be so limited as to be inalienable; nor will it suffer the creation of any estate inalienable beyond a certain period (Sugden, note, Gilbert *On Uses*, 260). These and various other things it forbids, as contravening some principles of public policy.

One of its provisions has been frequently attacked, Law of and often defended with more zeal than discretion. primoge- niture. This is the doctrine of primogeniture. The extent of its operation has been greatly overrated, or we might inquire on what ground it has been made the subject of such hot dispute. The law of primogeniture does not hold when the heirs are females; it does not hold in the case of personal property, the aggregate of which far exceeds descendible land in produce and value. In all settlements, the widow's jointure and the younger children's portions (which are personal property) are charged on the estate, and form large deductions—the one from its income, and the other from its capital (Humphreys, *Observations*, 201). "Intestacy"—and it is only in cases of intestacy that the law of primogeniture can have any operation—"but rarely happens where there is property to give, and the owner of it has a family or relations for whom he is anxious to provide; and it would be impossible, by a general law, to make a proper distribution applicable to all the different circumstances arising from the state, ages, and rank of the family, and the nature and value of the property. The general law is well adapted to the great classes of society engaged in professions and trade, where an intestate who has real property has usually personal property of much greater value, which is divided among all the children" (Mr. Tyrrell's *Suggestions* [24]; *First Rep. Real Prop. Com.* p. 480). We may remark, in the first place, that "it is not so material that these arbitrary rules should be the one way or the other, as that they should be fixed" (*per Sir W. Grant, M. R., Lord Oxford v. Lady Rodney*, 14 Ves. 425). And looking to the feelings and wishes of the great body of the larger number of landed proprietors in this country, "the rule" could scarcely be "fixed" any "way" more congenial to those feelings than by

Law.

Law.

devolving the real property on the representative wholly and undivided, instead—for this is the alternative—of dividing it into fragments, which will assuredly lower the position and dignity of the family, and possibly contribute nothing to the advantage of its members.

Settle-
ments.

The system of "settlements" adopted in this country appears admirably calculated to reconcile the maintenance of a family, by the preservation of its landed property, with the claims of the wife and the younger children. "The extent of the power of settling estates, or creating different interests, in this country," says Mr. Tyrrell, "is generally approved of as a proper medium between too great a restraint upon alienation or an insufficient protection to the aristocracy. Estates may be settled by deed or will, so as to secure the possession of them to the family as long as the policy of encouraging a free commerce in land can reasonably permit; at the same time every privilege may be given to the person in possession, consistent with the preservation of the property, which he could exercise if he were the absolute owner. The tenant for life may be empowered to provide out of the estate for his wife and his younger children, and to grant leases, work mines, cut timber, and sell or exchange the estate. When he has a son of the age of twenty-one years, he can, with his concurrence, re-settle or alien the property; and it is usually settled, on the marriage of the son, for the benefit of the next generation; and if no such settlement be made, the son, during the life of the father, cannot absolutely dispose of the inheritance" (*First Rep. Real Prop. Com.* p. 484 [41-2]). Most effectually in this way are accomplished all the advantages which a system of equal partibility would secure, and without the evils it necessarily occasions.

Legal and
beneficial
estates.

Here also we may remark, what the reader can scarcely have failed to notice of himself, the singular peculiarity in English law, which distinguishes between the right to property and the right to enjoy the use of property; this latter right being recognised and enforced by a distinct judicatory, guiding itself by rules, both of law and evidence, unknown to the ordinary tribunals of the land; seeking, it is probable, in the first instance, to do little more than relieve individuals from the heavy pressure of laws imperfect and often morally unjust.* Courts of equity insensibly progressed, until they emancipated property from the trammels of feudality,

* *Jeo aie oie mon Seigneur Coke a citer, two verses pur ceo hors de Sir Thomas More—*

"Three things are to be helpt in conscience,
Fraud, accident, and things of confidence."

(Rolle, *Ab., Chancery* (N.), 1). Mr. Fonblanque (*Treat. on Equity*, p. 9, 5th edit.) expounds the theory of courts of equity after this fashion. He considers them as in some cases assistant to, in others concurrent, and in others again exclusive of the jurisdiction of courts of common law.

It assists the jurisdiction of these—(1) by removing legal impediments to the fair decision of a question depending in a court of law; as in the case of an ejectment brought to try a title, equity restrains the party in possession (unless he have an equal claim to its protection) from setting up such a defence as will prevent a trial of the right; (2) by compelling a discovery which may enable it to decide; (3) by perpetuating testimony when in danger of being lost before it has been used. It is also assistant, by rendering the judgments of law courts effective—(1) by providing for the safety of property in dispute, *pendente lite*; (2) by counteracting fraudulent judgments; and (3) by putting a bound to vexatious and oppressive litigation. It exercises a concurrent jurisdiction with courts of law in many cases of fraud, accident, mistake, account, partition and dower, but claims an exclusive jurisdiction in most matters of trust and confidence.

and established a system of rights and remedies adapted to the necessities of an improving state of civilization. Such a spectacle as this embodies a striking warning against rash speculations on the effects of untried laws and imagined institutions. What would have been anticipated from the existence in the same country of two systems of law—distinct in their origin—dissimilar in many of their doctrines—administered in different courts, differing materially in the process by which they severally investigate submitted controversies, and enforce their respective judgments? Nothing but confusion and uncertainty in the law, to terminate only with the absorption or annihilation of one jurisdiction or the other. Lord Bacon spoke of the Court of Chancery to James I. as "the court of your absolute power;" and so it was originally considered. Speculating, then, upon the probable issue of the contest, the judgment would scarcely hesitate as to which would secure the victory. This appears to have been Lord Bacon's apprehension: *Maxime omnium interest certitudinis legum, ne curiæ prætoris intumescant et exundent in tantum, ut prætextu rigoris legum mitigandi etiam robur et nervos iis incidant aut laxent, omnia trahendo ad arbitrium* (*De Augm. lib. viii. cap. iii. aph. 43*). We know; because we have seen the event, that these apprehensions were groundless: we detect in the progress of equity the steps to perfecting a system adapted perfectly to our wants, and capable of indefinite expansion to satisfy the requirements of an altered society.

If it be thought these striking improvements, which we owe to courts of equity, were properly the work of the legislature, we cannot deny it; but still may be permitted to rejoice, looking at what our statute law is, that the legislature did not make the attempt. Innumerable would have been the questions raised—lengthened the litigation—doubtful for long the decisions with which a statute, disturbing to such an extent ancient and familiar rules, would have tasked the patience of the suitors and the wisdom of the judges. It is the nature of a statute law, let its provisions be ever so skilfully framed, to import uncertainty into the law, and to affect injuriously its scientific capabilities. "Sure am I," says Lord Bacon, "there are more doubts that rise upon our statutes, which are a text law, than upon the common law, which is no text law" (*Proposal for amending the Laws*).

The refinements with which our property law abounds are ascribable clearly, first, to the fact that the foundations of our law, being laid in times remote from the present, not only in years, but in habits of thought, in social character, in political necessities, doctrines and principles, still remain, inconsistent with modern circumstances, but whose removal would involve consequences highly injurious to the law, as impairing its systematic and scientific character (*see some observations of the Real Property Commissioners, Third Rep.* p. 4). To prevent, however, the unrestrained operation of these principles and doctrines, the courts have invented distinctions, subtle, but intelligible enough, which, preserving these principles to maintain the harmony and connexion of the system, still avert the mischief which would ensue on their active operation. Again, these refinements are called for by the minute and ramifying relations of society, which demand modifications of property too various and complicated for a legislature to provide. Citing testimony to this truth, we must conclude. Hear Julianus.

Law.

Refine-
ments in
property
law.

Law. *Neque leges, neque senatus consulta, ita scribi possunt, ut omnes casus qui quandoque inciderint, comprehendantur, sed sufficit et ea, quæ plerunque accidunt, contineri* (Dig. I. 3, 11). *Les besoins de la société sont si variés, la communication des hommes est si active, leurs intérêts si multipliés, et leurs rapports si étendus, qu'il est impossible au législateur de pourvoir à tout. Dans les matières qui fixent particulièrement son attention, il est une foule de détails qui lui échappent, ou qui sont trop contentieux et trop mobiles pour pouvoir devenir l'objet d'un texte de loi. D'ailleurs, comment enchaîner l'action du temps? Comment s'opposer au cours des événemens; ou à la pente insensible des mœurs?* (Disc. Prel. sur Code Civil).

Sect. 4.—Procedure.

Procedure at common law. 285. We have now to treat of civil wrongs and their remedies; civil wrongs being personal injuries, not criminal in their character. It will be desirable to treat of the former under the technical but convenient classification of the latter; a classification which has reference to the nature of the subject of the action.

Actions: real; persons. mixed. Actions, then, are styled real, personal, and mixed. When the action addresses itself to things directly (*actio in rem*), seeking the recovery of real property only, it is a real action; while personal actions are *in personam*, and seek the recovery of specific chattels, or some pecuniary satisfaction or recompence. Mixed actions partake of the nature of both these descriptions. By a late act of parliament, recovery of real property can be obtained at law, with certain exceptions, only in one way;* and the great number of real actions with which our ancestors were conversant are now abolished (3 & 4 Will. IV. cap. 27). Of personal actions, the most common are—(1.) Debt, which is brought where one claims the recovery of a debt, that is, a certain sum of money alleged to be due to him. Generally it is founded on some contract alleged to have taken place between the parties, or on some matter of fact from which the law will imply such contract. (2.) Covenant, which lies where one claims damages for a breach of promise under seal. (3.) Detinue, where a party seeks the specific recovery of goods and chattels, or deeds and writings, detained from him. (4.) Trespass, where one seeks damages for a trespass committed against him, the trespass being an injury committed by violence, actual or implied; the law implying violence where the injury is of a direct and immediate kind, and committed on the person or tangible and corporeal property of the plaintiff. Trespass upon the case is where a party seeks for damages for any wrong or cause of complaint, to which covenant or trespass will not apply. Of this action there are several kinds, two of which are well known.

Assumpsit. Of these, the first is, assumpsit, where one seeks damages for the breach of a promise not under seal. The promise may be implied or expressed, the law always implying a promise to do that which the party is legally bound to perform. Trover is the action usually adopted to try disputed questions of property as goods and chattels. Replevin is an action to try the legality of a distress for rent; and in form the party seeks damages for the illegal taking and detaining of his goods and chattels.

Debt. Covenant. Detinue. Trespass. Case. Assumpsit. Trover. Replevin.

* For the action of ejectment, here alluded to, see Div. IV. vol. xxi. p. 441.

The following is a statement of the more important stages in personal actions.

Law. Writ of summons. All actions of this description commence with a writ of summons directed to the defendant, and issued from a court of law, at the instance of the plaintiff. This writ apprizes the defendant that an action of a certain specified description has been commenced against him by an individual named, and desires him to appear to the action, warning him that, in neglecting to do so, the court will proceed to give judgment and award execution against him. If the action is to proceed controversially, the defendant enters an appearance, as it is called, a mere formal proceeding, which is a proper preliminary to his defending the action. The next step taken is by the plaintiff, who declares the grounds of his complaint in a formal document, which is communicated to the defendant, and the preparation of which requires considerable skill and professional knowledge.*

To this the defendant either *pleads* or *demurs*. In demurring, he objects to the declaration of the plaintiff, as insufficient on legal grounds; being either informal in its statement, or alleging facts which, however true, do not prove that the plaintiff has received any injury. By demurring, the defendant precludes himself from denying the truth of such facts as have been sufficiently pleaded. The law requires the opposite party to deny the allegation of him demurring, and thus the parties are said to be at issue; and this issue being an issue of law, is triable by the judges.

Instead of demurring, however, the defendant may plead, or put in a plea—pleas being of various kinds. They may consist merely of exceptions to the *remedy* to the sought, deny the jurisdiction of the court, or the sufficiency of the plaintiff, as being an alien, &c., or aver other facts, which, without affecting the real merits of the action, impugn simply its conformity to law. They may, on the other hand, be to the *action*, and either deny the truth of the plaintiff's allegations, or avoid their legal effect by setting out other facts, which give them another complexion. The defendant, in short, may either traverse or confess and avoid his opponent's statements (see TRAVERSE, Div. IV. vol. xxviii. p. 735). If he traverses, then must the plaintiff join Traverse.

* It is particularly to be remarked that the practice of pleaders (1 East, 210), as well as the practice of conveyancers (1 Turn. & Russell, 86), "amounts to a very considerable authority; and I am justified in that assertion by some of the greatest men who have sat in Westminster Hall, who, I am persuaded, in many cases, if matters had been *res integræ*, would have pronounced decisions very different from those which they thought proper to adopt" (*per* Lord Eldon, *Smith v. Earl of Jersey*, 2 Brod. & B. 598). Such being the case, it is obviously desirable that this practice should have some standard.

"From which to reason, or to which refer;"

and this is obtained by the spontaneous recognition of the authority of certain text-books. The importance of such works, it is difficult to overrate, and their value is correctly explained by Best C. J., who, in alluding to Phillipps's *Treatise on the Law of Evidence*, says, "I refer to [it] not as authority, but as proof of the understanding of Westminster Hall on the subject" (4 Bingham, 614). Lord Kenyon, in like manner, speaks with great respect of Marius' *Advice concerning Bills of Exchange*, on the ground that "it was written in early times, by a person conversant with the custom of merchants respecting bills of exchange;" and that "the rules laid down by him have been since received by the mercantile world" (6 D. & E. 212). See also the observations on Preston's *Treatise on Conveyancing*, by Bayley J. (8 B. & C. 518), and on Mitford's *Treatise on Chancery Pleadings* (2 Younge & J. 41; 1 Sim. 369-71).

Law. issue; and this issue is to be tried by a jury. It is, however, competent to the plaintiff to prolong the contest, by either demurring to the defendant's plea, or pleading thereto; such plea being called a replication; to which, in turn, the defendant may demur, or put in his rejoinder. The following table shows the titles of the various statements which each party may put forth:—

THE PLAINTIFF.	THE DEFENDANT.
The declaration	The plea
The replication	The rejoinder
The sur-rejoinder	The sur-rebutter.
The rebutter.	

Trial and judgment. When, at length, one or other party traverses, and issue is joined, the trial commences. The various kinds of trial and judgments are enumerated in an article on the subject in the Fourth Division of this work.

Observations on English pleading. The most striking peculiarity in this system is, that the question for the determination of the judicatory is, in effect, stated by the parties themselves; and this, if it enhances the cost, and prolongs the period of the suit, there is little doubt promotes the ends of substantial justice. "It is difficult not to admire the logical skill with which fact is separated from law; and the whole subject of litigation is reduced to one or a few points, on which the decision must hinge" (2 Mackintosh, *Hist. of Eng.* 275).

The late alterations which have been accomplished in our system of pleading, under the authority of the 3 & 4 Will. IV. cap. 42, have served to correct many of the evils which had previously deformed it. "Know, my son," says the great master of our law, "that it is one of the most honourable, laudable, and profitable things in our law, to have the science of well pleading in actions reals and personals; and therefore I counsel thee especially to employ thy courage* and care to learn this" (Litt. s. 534). Coke derives the word plea à *placendo, quia omnibus placet* (Co. Litt. 303); but it is very doubtful if, in his day, the general approbation of the mode of legal procedure warranted such a derivation. There is nothing in the present state of public opinion, most certainly, to warrant it. Still, if forms of proceeding are to exist, and if the discretion of the judge is not to determine the controversies submitted to him, it is difficult to see how the evils complained of can be obviated. "It is easy," says one who will not be suspected of any undue leaning to law conservatism, "to form a conception of justice in the abstract, such as it would appear to the eyes of an infallible judge, well acquainted with all the circumstances of the case; but to distinguish this abstract justice from legal justice would be a vain and dangerous pretension, which would give the judge up to the phantoms of his own imagination, and make him lose sight of the only true guide—the law" (Bentham, *Rationale of Judicial Evidence*). Lord Mansfield observed, that "the rules of pleading are founded in strong good sense," admitting, however, that, "by being misunderstood and misapplied, they are often made use of as instruments of chicane" (Robinson v. Raley, 1 Burr. 214); nor is it possible, in a system necessarily technical, to avoid evils of this kind.†

* That is, ardour. *Corage*, properly, however, signifies anger; and in this sense it is used in the preamble of 1 Edw. III. stat. 1, "Continuant leur mauveste, moverent le corage de dit Roy Edouard."

† The rash confidence with which extensive alterations in our legal institutions have been advised and attempted, indicates,

The procedure in criminal cases is sufficiently explained by various articles in the Fourth Division, and does not appear to demand any further explanation.

The procedure in Chancery has been already sketched in the description given of the jurisdiction of the court; nor is it capable of a more elaborate discussion without involving the consideration of details that would extend this chapter beyond its proper limits.

We have not pretended, of course, to enumerate all civil wrongs, which are as numerous as the rights of which they are infractions; but it is sufficient for our purpose to indicate the more usual modes of legal redress, and the more usual injuries.

Such is a faithful outline of the great doctrines of English law. And now, if the expounder might assume the critic's office, would he express those feelings of admiration and gratitude which the contemplation of such a system—so wise in principle, so beneficent in action—naturally evoke. He will, however, content himself with the statement of one fact, which evidences most conclusively the worth of the law of England. It is eminently adaptive to varying circumstances—not rigidly opposed to change: by the very facility of its own nature, it becomes accommodated to the alterations which changing times and changing manners effect in social relations and their resulting necessities. Owing its existence, not to the skill and sagacity of an individual or an age, but the accretion of the skill and the sagacity of many ages, it has grown with our growth, and strengthened with our strength. Unlike the mushroom constitutions and unsubstantial polities whose seeming merits have deluded our neighbours,—

"'Twas not the hasty product of a day,
But the well-ripened fruit of wise delay."

Ours may be the boast of the Roman statesman—
"Nostra respublica non unius esset ingenio sed multorum; nec una hominis vita sed aliquot constituta seculis et ætatibus; neque cuncta ingenia conlata in unum tantum posse uno tempore providere, ut omnia complecterentur sine rerum usu et vetustate."*

with tolerable clearness, the extent to which jurisprudence has been studied as a science amongst us. The historical school of law, which has become so illustrious on the Continent, has done much towards arresting that wild desire for new laws which spread in France and Germany quite as extensively as it did here. Were the history of our law better known, we should not see amongst the ranks of would-be innovators so many fitted for better things. The history of our judicial procedure tells us plainly that the more you relax strict rules—the less regard you pay to forms, incalculably the more uncertain do you make the administration of the law, and proportionably do you increase the cost and dissatisfaction of the suitor. It is to be regretted that the charge of lawyers' hostility to innovation is not better founded than it is. Cromwell ascribed to their avarice the failure of his endeavours to effect useful law reforms. "The sons of Zeruah," he said, "are too strong for us; and we cannot mention the reformation of the law, but they presently cry out we 'design to destroy property;' whereas the law, as it is now constituted, serves only to maintain the lawyers, and to encourage the rich to oppress the poor" (Ludlow, *Mem.* p. 123). The remarks of Hobbes apply to these bold legislators with peculiar force. "If," he says, "on those matters, on which we speculate for the sake of exercising our genius, any error is introduced, no loss but of our time ensues; but in our meditations which relate to the purposes of life, not only from our error, but our ignorance, necessarily must arise offences, quarrels, and violent deaths" (*Philosophical Elements concerning a Citizen*).

* Cicero, *De Republicâ*, lib. ii. 1.

Law. Procedure in criminal matters. Procedure in Chancery.

Concluding observations.

L A W.

CHAPTER VI.

THE LAW OF SCOTLAND.

Law. Preliminary observations. 286. ON many grounds the Law of Scotland is a system deserving a more detailed and prolonged examination than is consistent with our space. It requires, however, but slight acquaintance with its provisions to discover that it has retained longer than any other system of European law the tokens of its origin, and that it sheds considerable light on the early history of legal institutions, both in England and elsewhere. The importance of the study of laws as a source of historical information has been noticed by one especially qualified to judge in the matter: *Versatur infelicitas quædam inter historicos vel optimos, ut legibus et actis judicialibus non talis commorentur aut si forte diligentiam quandam adhibuerint, tamen ob authenticis longe varient* (*De Augm.* lib. viii. cap. iii. aph. 29). The law of Scotland has a value other than an historical value; and we cannot but admire the wisdom and genius of those venerable magistrates who have successively presided over its administration, and, by their profound knowledge of jurisprudence, have rendered it a system well adapted to the wants and character of a brave and ancient people. There have not been wanting in this age of innovation those who have advised the abolition of this system, as cumbrous and ill adapted to modern times; but, trying such projects by the test which the experience of the last half-century affords us, we may reasonably question the wisdom of any change which should go the length of substituting one system for another, and thus rendering for a period, whose termination is doubtful, the administration of the law perplexing to the judge and uncertain to the suitor. Topical changes—the removal of specific evils—may be easily and safely accomplished by the legislature; and we may leave to the sagacity and experience of the judges, by discountenancing unnecessary refinements, by the power they possess of interpreting doubtful doctrines and perplexed statutes, and by their prætorian functions—their *nobile officium*, to borrow the phrase of Scottish jurists, the task of rendering the theory or scientific *rationale* of the law in every way worthy the progress of the times, and fitted to the varying necessities of the people for whom it exists.

Thus have systems of law been moulded into conformity with successive ages, and such seems the course dictated by experience—an experience fruitful of warning against the rashness of those speculators who, in seeking impracticable objects, would expose to a century of uncertainty and dismay the wealth and property of the country,—

— Tu, nisi ventis
Debes ludibrium, cave.

“Concerning this part of the kingdom,” says Coke, in allusion to Scotland, “there are many things worthy of observation;” amongst which he mentions “that

these two mighty, famous, and ancient kingdoms, viz. England and Scotland, were anciently one” (3 *Inst.* 345),—a fact which is also affirmed in the 1 Jac. I. cap. 1.* The other points of agreement noticed by Coke (*ut sup. cit.*) are the similarity of the religion “and service of God” in both countries, which exists no longer, and the unity of language, government,† and laws. This agreement in the laws of England and Scotland is affirmed by James I., in one of his speeches to the English parliament, in which he observes that “all the law of Scotland for tenures, wards, liveries, and seignores is drawn from England; and James I. of Scotland, being educated in England, brought the English laws in a written hand” (*Works*, p. 523).

* The historical statements contained in laws, and the general truthfulness of their preambles, are usually suspicious enough. See a statement in Alfred's laws (*Anc. Laws and Inst.* p. 26, and Palgrave, *R. and P. Eng. Com.* i. 59; Tac. *De Mor. Germ.* s. xii.); see also 25 Edw. III. st. 1; 3 Hen. VI. cap. 2; 24 Hen. VIII. cap. 12; 31 Hen. VIII. cap. 13; 13 Car. II., printed 1 Madd. *Chan. Prac.* 3; and see also an ordinance of John, king of France, alluded to by Robertson, *Charles V.* vol. i. p. 250. Coke observes “that it cannot be thought that a statute which is made by the authority of the whole realm, as well as of the king and of the lords temporal and spiritual as of all the commons, will recite a thing against the truth” (4 *Inst.* 343). But see the title of the kings of England to Ireland deduced in the Irish Act, 11 Eliz. st. 3, cap. 1.

† This is hardly consistent with the difference, which could scarcely be unknown to Coke, between the manner of legislating in England and Scotland. At the Union, the very constitution of the two parliaments appears to have been different; for in Scotland the parliament consisted of only two estates, although formerly it consisted of three: 1st, the bishops, abbots, and other ecclesiastical persons sitting in right of their sees or benefices, or perhaps in right of their lands held directly from the crown; 2nd, the barons, and commissioners of shires, who were representatives of the smaller barons, free-tenants of the king; 3rd, the burgesses, representatives of the royal burghs. These sat together in one house, having a common president, and all sharing as well in the judicial as in the legislative functions of the parliament. At the commencement of the session a committee was appointed, styled the Lords of the Articles, by whom all bills were prepared; and no bill not approved by them could be discussed in parliament. At first this committee, whose origin dates about the year 1367, was named by the parliament itself; but in course of time it became in fact nominated by the crown, which thus obtained the initiative in all legislative measures. In allusion to this circumstance, and referring to the bills which parliament had to consider, our James I. observed to the English parliament, that in Scotland only such bills as he allowed “are put into the chancellor's hands to be proposed to parliament; and after this, before I put my sceptre to a law, I order what I please to be erased” (*Works*, p. 523). This was felt to be so injurious to the liberties of the people, that of the fifteen articles of grievance presented to William III. the constitution of the Lords of Articles was the first (Burnet, 589), and the committee was accordingly suppressed in 1690. Sir Thomas Craig (*De Feudis*, lib. i. cap. 7, s. 11) considers that in order that a bill should become law, the three estates must have assented to it, the dissent of one estate proving fatal.

Law.

The concordance between the laws is also noticed by Bulstrode Whitelocke (*Commentary on the Writ for choosing Members of Parliament*, vol. ii. p. 272). An opinion seems to have been entertained in early days, that Scotland was indebted to foreign countries for its laws, as appears from the preamble which introduces several old systems of law, and which is so curious in itself that it is worthy of a place here:—

“Because thar wes in that tyme sa many sundry maner of men in Scotland, of divers naciouns and divers conditions, the quilks held ilkan sundry maner of lauwis and consuetuds, that na man stud aw off an uther; and that tyme wes the noble abbay of Dunfermlyn be the said Saint Davy fundit and byggit in the honour of St. Mergarite his moder; in the quilk tym he mad mony castell and abbays and chanounryis, with sundry nouneris, and mayson Dieuwis, that is to say, almis housses, in the honour of God and our Lady suet Saint Mary, and began throu a visoun he met in his slep at Dunfermlyn; than send he after his baronnis and chesit out xxiiii. of the wissast and most hable men to . . . and gert tham be sworn, that they suld passe twa and twa in company togidder over all Cristyyn kynriks and wryt up all the lauwis of ilke land, bath in burcht and in land, and geff tham twa yers space to do this and cum again til hym; and quhen they come agayn thay fand the kyng at the New-castale upon Tyne, makand and biggand the castalle and uthir syndry abbays of blak monks and of quhyt, and of quhyt chanounis and nownerys and maison Diewes, and that he mad and stablest all his lauwis of Scotland, bath in bouch and in land, be hale assent of all his prelatis, lordis, baronis, burgisses, and commouns.”*

This is probably a fiction, copied from the account given by Livy of the embassy sent by the Romans to the Greek cities to procure information respecting their laws (Pomponius, in *Dig.* I. 2, 2, 4). It has, however, been generally thought that the Scotch are much indebted to the English and French education of their king, James I., for the constitution of their tribunals and their forms of process. Their College of Justice they owe to James V., who had seen the merits of a similar institution in France. In the same manner is it highly probable that many of the improvements introduced into our own law in the reign of Henry II. arose from the familiarity with the details of judicial policy the king may be supposed to possess from his office of Grand Seneschal of France and from his general conversance with historical literature (Girald., *Cam. Hib. Exp.* 784).†

The present authority of the law of Scotland rests upon the Act of Union of the two kingdoms. The Union was effected in 1707, and the articles are ratified and confirmed by the 5 Ann. cap. 8. The principal articles of the Union are thus stated by Blackstone:—

1. That on the first of May, 1707, and for ever after,

* Lord Hailes' *Examination of some of the Arguments for the high Antiquity of the Regiam Majestatem*, p. 31 (Edinb. 1769).

† A history of the law of Scotland would be a valuable contribution to European literature. I must be permitted to express my concurrence in the wish stated by Sir Francis Palgrave (*R. and P. Eng. Comm.* i. 480), that the same pen that has already done such justice to the political and social history of Scotland should undertake this task; a task not unworthy the

“Revered defender of beauteous Stuart,”
whose name and reputation his descendants have prolonged.

the kingdoms of England and Scotland shall be united into one kingdom by the name of Great Britain.

Law.

2. That the succession to the monarchy of Great Britain shall be the same as was before settled with regard to that of England.

3. That the United Kingdom shall be represented with one parliament.

4. That there shall be a communication of all rights and privileges between the subjects of both kingdoms, except where it is otherwise agreed.

9. That when England raises 2,000,000*l.* by a land-tax, Scotland shall raise 48,000*l.*

16, 17. That the standards of the coin, of weights, and of measures shall be reduced to those of England throughout the United Kingdom.

18. The laws relating to trade, customs, and the excise shall be the same in Scotland as in England. But all the other laws of Scotland shall remain in force, although alterable by the parliament of Great Britain; yet with this caution, that laws relating to public policy are alterable at the discretion of the parliament; laws relating to private rights are not to be altered, but for the evident utility of the people of Scotland.

22. Sixteen peers are to be chosen to represent the peerage of Scotland in parliament, and forty-five members to sit in the House of Commons. This article is so prefaced as to prevent the crown creating a Scotch peerage with an elective right.

23. The sixteen peers of Scotland are to have all the privileges of parliament; and all peers of Scotland shall be peers of Great Britain, and rank next after those of the same degree in England at the time of the Union, and shall have all the privileges of peers, except sitting in the House of Lords and voting on the trial of a peer (1 *Comm.* 96-7).

The statute 5 Ann. cap. 8, recites two acts of parliament: one of Scotland, whereby the Church of Scotland, and also the four universities of that kingdom, are established for ever, and all succeeding sovereigns at their accession are to take an oath inviolably to maintain the same; the other of England, 5 Ann. cap. 5, whereby the Acts of Uniformity of 13 Eliz. and 13 Car. II. (except as far as the same had been altered by parliament at that time), and all other Acts then in force for the preservation of the Church of England, are declared perpetual; and it is stipulated that every subsequent king or queen shall, at their coronation, take an oath inviolably to maintain the same within England, Ireland, Wales, and the town of Berwick-upon-Tweed. And it is further enacted, that these two Acts “shall for ever be observed as fundamental and essential conditions of the Union.”

Sect. 1.—Sources of the Law of Scotland.

287. The law of Scotland is, like our own, made up of customs, parliamentary and adopted laws, to be studied in reports of judicial decisions, in the records of parliaments, and in authorized text-books. Upon the authority of the Roman or civil law, the Scottish jurists do not quite agree. “Our customs, as they have arisen mainly from equity, so they are also from the civil, canon, and feudal laws, from which the terms, tenors, and forms of them are much borrowed: and therefore these (*especially the civil law*) have great weight with us, namely, in cases where a custom is not yet formed. But none of these have with us the

Sources of
Scottish
law.

Civil law.

Law.

authority of law, and therefore are only received according to their equity and expediency, *secundum bonum et æquum*. * * * But, for the full evidence of the contrary, there is an express and special statute declaring this kingdom subject only to the king's laws, and to no other sovereign's laws" (Lord Stair, *Institutions*, I. 1, 15). An eminent Scottish lawyer, writing subsequently, considers the authority of the Roman law to have been more extensive in its operation. "Although we have collected and adopted a law of our own, which is partly written and partly consuetudinary, yet, as in the infinite variety of cases which daily emerge, some questions will often occur, which by no rule in our law can well be decided, our judges therefore have recourse to the civil law, and are thereby so far governed as they find the rule of decision therein set forth nowise disconform and inconsistent with our own law and practick; and therein the judges are well justified by several acts of parliament, whereby it is declared that the lieges shall be governed by the king's own laws (that is, by our own peculiar laws and customs), and by the common laws of the realm (that is, by the civil law); so that we consider the Roman laws which are not disconform to our own fixed laws and customs, to be our law." He proceeds to give some instances, and concludes—"From all which it manifestly appears that the civil law has all along been considered as our law, and it is justly made the rule of judgment in all cases wherein our law is silent, and when such decisions prove nowise derogatory to our own proper laws and customs" (Bayne's *Disc. R. & P. Laws of Scotland*, p. 166, 1726).*

Abrogation
of statutes
by desue-
tude.

It is a maxim of the civil law, that laws may be abrogated by desuetude. *Ea vero, quæ ipsa sibi quæque civitas constituit, sæpe mutari solent, vel tacito consensu populi, vel alia postea lege lata* (*Inst.* I. 2, 11). The law of Scotland† differs with the law of England, in considering statutes as repealed by desuetude; so that "if a statute after long standing has never been in observance, or, having been, has run into desuetude,

* Dr. Irving, in his *Introduction to the Study of the Civil Law* (p. 108, *et passim*), alludes to the defective knowledge of this system possessed by English lawyers. He has not, however, informed us in what degree matters are better in Scotland, nor favoured us with the names of any Scottish jurists who have contributed to the literature of the civil law anything equalling the celebrated chapter of Gibbon (which has been translated and commented on by Hugo and Warnkönig); the *De Usu et Autoritate Juris Civilis Romanorum* of Dr. Duck, whom Giannone and other civilians styled their Coryphæus, and whose work was reprinted at Leyden and Leipzig; and the *History of the Legal Polity of the Roman State*, by Dr. Bever. "With us now-a-days," observes Professor Wilde, "the authority of this whole law seems in Scotland to be referred to Voet. It is not the *Corpus Juris* itself that is held in estimation; and the authority of this Dutchman is now far beyond any authority of Paulus or Ulpian, or any decision of a Roman emperor, strengthened by all the advice of all the lawyers in his states. In short, the commentaries of Voet are made our Roman law" (*Preliminary Lect. to the Course of Lect. on Inst. of Justinian*, p. 53, edit. 1794).

† The civil law professorships in Scotland appear to have been either more recently established than we might have naturally supposed, or to have been held as sinecures; for Craig, who died in 1608, laments that no public lectures of the civil law were read (*Jus Feud.* p. 14, 1732). James Melville gives the following account of his attending law lectures at St. Andrew's: "In the third and fourth yeirs of my course, at the direction of my father, I hard the commissar Mr. Wilyeam Skein, teatche Cicero *De Legibus*, and diuers partes of the *Institutiones* of Justinian. I was burdet in the housse of a man of law, a verie guid honest man, Andro Greine by nam, who louit me exceiding weill, whase wiff also was an of my mothers [friends]; I am sure

consuetude prevails over the statute till it be renewed either by a succeeding parliament or by a proclamation from the council" (Mackenzie, *Inst.* I. 1, 10). "This power in custom to derogate from prior statutes has been confined by most writers to those concerning private right; and it hath been adjudged once and again, that laws which relate to the public policy cannot fall into disuse by the longest contrary usage" (Erskine, *Inst.* I. 1, 45). "No statute can, however, be repealed by mere non-usage or neglect of the law, though for the greatest length of time; non-usage is but a negative, which cannot constitute custom; there must be some positive act that may discover the intention of the community to repeal it" (*Ib.*). As to what constitutes this state of desuetude, some difference appears to exist among Scottish jurists, some concluding that sixty, and others that a hundred years is necessary thereto. As to the statutes relating to the administration of justice, Lord Stair considers that the authority of the Lords of Session extends "not only to the interpretation of acts of parliament, but to the *derogation*; therefore especially so far as concerns the administration of justice, which is specially committed to them, whereby all old acts of parliament concerning the form of process are in desuetude, and in several points more recent statutes" (*Inst.* I. 1, 16). As, however, from decisions of the Lords of Session, contrary to Lord Stair's opinion, an appeal lies to the House of Lords, this method of abrogating at the mere will of the court exists no longer.

Law.

Sect. 2.—Rights relating to Persons.

288. We will first consider the rights resulting from Marriage; marriage.

Marriage, by the law of Scotland, is constituted by the natural consent of the parties to take each other, from that time forward, for man and wife, but not from a promise to do so at some future time. When, however, such a promise have been followed by actual intercourse, this raises a presumption that, at or immediately previous to such intercourse, a consent, constituting marriage was exchanged. But this is a presumption capable of rebuttal. Parties living together as man and wife are considered as married, unless it be conclusively proved that they never intended marriage, but lived together only to save appearances. This is called a marriage by *habit and repute*. The regular form of celebrating a marriage in Scotland is by the parties appearing before a clergyman, after due proclamation of banns, and notifying their consent to the union, upon which he then declares

how con-
stituted.

sche had nocht sone nor bern sche loued better. This lawier tuk me to the Consistorie with him, whar the commissar wald tak pleasour to schaw ws the practice in the judgment of that quhilk he teatched in the scholles. He was a man of skill and guid conscience in his calling, lernit and diligent in his profession, and tuk delyt in na thing more nor to repeat ower and ower again, to anie schollar that wald ask him, the things he haid bein teatching. Likways my ost Andro acquaintit me with the formes of summons and lybelling of contracts, obligationnes, actes, &c., but my heart was nocht sett that way (*Diary of Mr. James Melville*, p. 23, printed for the Bannatyne Club, 1829). The Scottish lawyers, however, chiefly studied on the Continent, at first in the French, and afterwards the Dutch universities. So general was the practice in early times, that Spalding, alluding to Sandilands, professor first of the canon and afterwards of the civil law in King's College, Aberdeen, observes, that it was not a little "strange to see one man admitted to teach the lawes who was never out of the countrie studeing and learning the lawes" (*Hist. Troubles*, vol. i. p. 179, edit. 1828-9).

Law.

them to be husband and wife.* To constitute a lawful marriage, the consent must be freely given by both parties for the purposes of marriage with each other; and these parties must not be within the forbidden degrees of consanguinity or affinity (which are the same in Scotland and England); and must be of sound mind, capable body, and consenting age, which, according to the Scotch law, is the age of puberty, defined to be fourteen in males, and twelve in females (Stair, *Inst.* I. 4, 6, More's note; 1 Burge, *Comm.* 173).

Dissolu-
tion of a
marriage.

A marriage may be declared void *ab initio* for some cause which subsisted previous to the period it was sought to be contracted (Stair, III. 3, 42), or it may be dissolved by a judicial sentence. This power, unknown to the judicatories of England, is exercised by the Court of Session in Scotland (2 Geo. IV. and 1 Will. IV. cap. 69). The grounds on which it proceeds are either wilful desertion or adultery. The Scotch statute 1573, cap. 55, enacts, "that where any of the spouses shall divert frae the other without sufficient grounds, and shall remain in his or her malicious obstinacy for four years," it shall be competent to the injured party to take proceedings, which, by virtue of the statute, are made the foundations of a divorce. Practice has diminished the four years to one (Lothian, *Consist. Law*, p. 97). In cases of divorce on the ground of adultery, marriage between the committers of the adultery is forbidden (Stair, I. 47). Divorces *à mensâ et thoro* are also granted in cases of cruelty and maltreatment (1 Burge, *Comm.* 635).

Marital
rights.

The effect of a marriage in Scotland is to confer on the husband a power over both the person and the estate of the wife, a power which may be limited by convention, but which, in default of convention, deprives her acts, in dealing even with her own property, of any validity, and nullifies the obligations she might contract if she were unmarried. This is a principle subject to an exception, which includes obligations arising from the wife's delict; but even these obligations have no such operation as would prejudice the husband, unless he have been a party to the delict. Take the example of a pecuniary fine: for this the wife's person cannot be attached, as her coverture pre-

vents her discharging it, nor generally can her property be attached, because her property is equally that of her husband. Unless, therefore, she have property which is exclusively her own, all proceedings against her must be suspended during the subsistence of the marriage. If, however, she has a separate estate, not affected by the *jus mariti*, that may be attached; and that she can herself, by her own acts, charge and burden. She is also liable, though not in her person, for debts contracted after a legal, or even after a voluntary separation, if the husband allows her a proper provision. As a rule, a wife cannot sue or be sued without joining her husband.

The rights arising out of the relation of father and child now claim our attention. Parent
and child.

Children born in wedlock are properly legitimate. The Scotch law appears to be more rigid in its exceptions from the rule already (ch. v. p. 2, s. 3) referred to, *pater est quem nuptiae demonstrant*, as it appears to require proof that access on the part of the husband was physically impossible (1 Burge, *Comm.* 64; Stair, III. 3, 42); and it adopts ten months as the *ultimum tempus gestationis* (*Ib.*). A child born out of wedlock may be legitimated by the subsequent marriage of its parents, it being a presumption of law that the marriage, in fact, preceded the birth. A marriage contracted with another, by either of the parents, subsequently to the birth of the child, or subsisting at the time of its birth, is clearly inconsistent with this legal presumption; and it is the better opinion that, in such a case, legitimation *per subsequens matrimonium* would not take place.* A child so legitimated possesses the same legal rights as one born in lawful wedlock. Legitimacy.

The law of Scotland considers that a father is bound to supply his lawful children with all necessaries, not only during their infancy, or while incapable of earning their livelihood, but also until they are put in some way doing for themselves. It seems, however, from *Maule v. Maule* (June 1, 1825), to be doubtful how far a son, arrived at majority, and bred to a profession, can claim beyond a bare subsistence from his father. Daughters, of such a rank or education as are not qualified to earn their own livelihood, have a claim for necessaries until their marriage. In default of the father, his heir or representative is bound to afford aliment to the children, unless his estate be very narrow, when the mother is bound to contribute. After the father and mother, the obligation devolves on the paternal grandfather, even although the father should be alive, supposing he is unable to discharge this duty. The claim of a son's wife or widow on his father is involved in some difficulty. Parents are not legally bound to educate their children. Rights of
children.

As in England, the father is guardian and administrator in law to his children, being minors (*i. e.*, under twenty-one), and during pupilarity (*i. e.*, under the age of fourteen, if males, of twelve, if females), is entitled to their custody, a right to which, in defect of the appointment of tutors or guardians, the mother succeeds during her widowhood. Children are bound to support their parents when they are incapable of supporting themselves. The father of an illegitimate child (ascertained to be father by presumptive evidence, fortified by the oath of the mother,) is bound to support Rights of
parents.

* "There must be clear evidence that the parties intended to marry each other. Hence the most formal acknowledgment of marriage, even although made *in facie ecclesiae*, will be of no avail, if it shall appear that such was not the true intention of the parties. It may be difficult, often impossible, to show that the parties did not mean to be bound by what they have apparently done, and they will not be permitted lightly to disclaim their own solemn and deliberate proceedings; but still there is no form or ceremony which affords such conclusive evidence of marriage as to make it impossible to show that marriage was not intended by the parties. Such evidence should be received with jealousy, and sifted with rigour; but it is, nevertheless, competent. Accordingly, the most explicit oral or written declarations of marriage, and even the celebration of marriage by a clergyman, have been disregarded, when other circumstances have shown that the parties did not intend to bind themselves by marriage. But where one party has led the other to believe that marriage was intended, such party will not be permitted to plead any secret or reserved intention of keeping free of this contract" (More, notes to Stair's *Inst.* vol. i. p. xiv. ed. 1832). It must be clearly understood that the consent constituting marriage must be a consent *de presenti*, and not *de futuro*, which constitutes only espousals, for the non-completion of which an action of breach of promise lies. In *Graham and Erskine v. Burn*, 2 Jan. 1685, the court awarded to the woman indemnification of actual expenses she had incurred, and also a small sum "for the loss of the market," as Lord Fountainhall expressed it.

* It has been thought that any impediment to the parent's lawful marriage at the time of the child's conception would prevent his legitimation (1 Burge, *Comm.* 95).

Law.

him till he be able to support himself; but the mother is entitled to his custody. A bastard has no power of bequeathing his personal property except he has lawful issue, and can neither nominate executors nor leave legacies; nor can his agnates succeed to him, unless he has obtained *letters of legitimation* from the king, who is otherwise entitled to take. He may, however, convey all his property, real and personal, whether he has lawful issue or not, by a general disposition, provided it be not of a testamentary nature. A bastard, arrived at the age of puberty, may also appoint curators and tutors to his lawful children, &c.

Artificial
relation-
ships:

Tutors,

their spe-
cies.Factor *loco*
tutoris.Office of
tutors.

Curators.

Of the artificial relationships recognised by the law of Scotland, we shall notice that of tutors and curators. "Tutors and curators," says Lord Stair, "succeed in the place of parents, and their obligations have a near resemblance" (I. 6, 1). The tutor acts for the pupil, as the ward is called; and it is his business to protect his pupil, to support and educate him, out of and according to the means of the pupil, and when he attains the age of discretion, to restore to him his own, and yield him an account of his dealings. Tutors are either testamentary, of law, or dative. No one can be a tutor of law who is not a male, nor less than twenty-five years of age; but a tutor testamentary may be appointed and act, if a major, and although a female. Tutors testamentary can be appointed only by the father, who at the time of the appointment was in *liege poustie* (see p. 850, *post*). The tutor of law is the nearest kinsman of the father's side; but if he does not claim the post within a certain time, a tutor dative is appointed by the king in Exchequer; and until such appointment—which, if I do not greatly err, is rarely made in modern times—the Court of Session supplies his place with a factor *loco tutoris*.

The office of a tutor is wholly gratuitous, and his powers extensive. He can direct the education and place of residence of his pupil; subject, however, to the control of the Court of Session; but cannot sell his property, nor alter its nature, nor grant leases beyond the subsistence of his office: he cannot generally allow more for the support of the pupil than the produce of his stock, except where this is insufficient. He is liable for defect of diligence in the management of the pupil's property; and his office expires by his outlawry or death, or by the death of his pupil, by his own removal for malversation, or "as suspect," by the Court of Session (which in some cases interferes to prevent the appointment of a tutor in law), or, in the case of a mother acting as tutrix-testamentary, if she have conducted herself immorally.

Curators are either appointed by the father, in a deed executed in *liege poustie*, or by the minor himself. They differ also from tutors in their office, "in this, that tutors are given chiefly for the pupil's person, but curators are given for the right managing of the goods and affairs; and tutors act for and in the name of their pupils, who in their pupilarity have no discretion; but curators cannot, and are only obliged by their office to authorize their minors, and act with them, by consenting to their deeds" (Stair, I. 6, 33).

Sect. 3.—The Rights relating to Immoveable Property.

289. The rights relating to immoveable property have now to be noticed.

In Scotland, "the property of all lands and immoveables, or hereditaments, is either allodial or feudal"

(Stair, II. 3, 4). Allodial lands are those enjoyed in full possession, and the owner of which has no superior: feudal lands, on the contrary, are those held under a lord. In England we have no allodial lands, since the crown has parted with its patrimonial possessions; and in Scotland all lands are feudal, except as to the king's rights, churches, churchyards, manors, and some lands in Orkney and Zetland.*

In feudal lands there is the *dominium directum*, or superiority; and there is also the *dominium utile*, or right of enjoyment and occupation, which is the ownership of the vassal.† The estate, in respect both of the *dominium directum* and the *dominium utile*, is called

Law.

* These lands are called *udal* lands, and the heritors are subject to a payment styled *skat*. It was a question whether this *skat* was merely a tax or tribute, or whether it was duty payable in *agnitionem dominii*. The Lords of Session, when the matter was before them, considered these *udal* lands to be allodial; and Lord Hailes derived the word from *all* and *od*, signifying *plenum vel absolutum dominium* (Dundas v. Orkney and Shetland heritors, 24 Jan. 1777, 5 Suppl. 609). In the Islandic, *édal* signifies, not only allodial property, but also *res derelicta* (Haldorsonius, *Lexicon Islandicum*, vol. ii. p. 121). Mr. Burge includes, amongst the allodial lands in Scotland, glebes (2 Comm. 237); but glebes and mansees, although they have no express holding, appear to be holden of the king in mortification, as it is called, because they are mortified, "having no other *reddendo* than prayers and supplications" (Stair, II. 3, 39).

By the statute 1633, cap. 10, the superiority or lordship of all church lands is annexed to the crown; and by 1661, cap. 53, the right of vassals of such lands to hold under the lords of erection (as the plunderers of the ancient Scottish church were designated) was asserted, and an option given them of recurring to their position as tenants of the crown, unless they had expressly renounced it. It was also decided that manors and glebes were held of the crown (Fletcher v. Bishop of St. Andrews, 28th March, 1628, 1 Suppl. 258).

† The feudal system, if it ever existed in England, except in legal theory, has now so long disappeared, that it will be interesting to examine slightly its practical details as still subsisting in Scottish law. There the duties of lord and vassal are still held reciprocal. The superior is bound to enter his vassals, when called upon to do so; and if he fail, the statute 1474, cap. 57, renders him liable to lose all the casualties of superiority during the life of the tenant, and who may, for his own security, apply to the over lord for a charter to supply the neglect of his immediate superior. But this applies only to the superior and vassal, and their heirs, and not to their singular successors (assignees or purchasers). These latter can, however, compel the superior to enter them under 20 Geo. II. cap. 50, which specifies in what manner the entering is to be claimed. The superior is also bound to enter the adjudger (one in whose favour has been made an adjudication, "by which all rights of the ground are conveyed from debtors to their creditors, by decree of the Lords" of Session. Stair, IV. 51, 1), or to pay the debt, and come in their place as to the lands; but when the lands have been sold by judicial sale, the superior is bound to enter the purchaser, and cannot insist upon getting the lands at the price agreed to be paid for them. It is competent to the superior to sell or convey his superiority to a third person; but he cannot convey or divide so as either to increase the number of superiors or to interpose a mid-superior between himself and the vassal. The superior is generally entitled to certain feu or blench duties (trifling rents), and to certain advantages on the entering of heirs, purchasers, and creditors. Superiors are often entitled also to personal services; but these have been much affected by the 1 Geo. I. cap. 54, which, in order to carry out the policy of the government of the day, who strove to break down, as far as possible, the system of vassalage, enacted, that the annual sale of the services called personal attendance, hosting, hauling, watching and warding, due by any charter, &c., should be paid in money only, the rate to be ascertained by the Court of Session. The 20 Geo. II. cap. 50, further enacts, that no tacksman (tenant under a lease) should be obliged to perform any services except those expressly enumerated and ascertained by a writing signed by the parties, except mill services. The same act abridges the rights of superiors in many other important instances.

Law. a fee, fief, or feud, and sometimes, though more rarely, a feu. The donee is called the fiar. The fee is simple or taillie.

Fee simple. The fee simple is the estate in virtue of which the fiar is properly proprietor of the lands, which descend to his heirs, and which he may alienate by conveyance or deed in his lifetime. In the grant of such estate, the grantor may burden it with the payment, to him or to another, of a certain sum, which, if in the deed there be clearly stated both the amount of the sum and the name of the creditor, will continue a real charge on the land. The burden, to be real, must be imposed by a clause, either charging the right itself, or making the right null if payment be not made on the day fixed. It is otherwise only personal, and constitutes no real incumbrance on the land, although obligatory on the acceptor of the grant and his heirs. The grantor may also reserve for himself or another a power or faculty to burden the land with debt, provided it be defined in amount, and appears on the record.

Conditions. The fee simple may be burdened with provisions or conditions. If these are impossible or unlawful, "and be conceived as suspensive clauses annexed to the disposition, they annul the same" (Stair, II. 3, 56); but if otherwise, they are merely considered themselves null, as is the case with a provision inconsistent with the right. The estate may be subjected to a condition or clause of pre-emption, giving to the superior the first refusal of the land and to the prohibition to sub-feu. "Its object is not to preserve the superior's influence, but to maintain and render effectual a valuable and pecuniary interest. It is a stipulation that the vassal shall not have it in his power to alienate the fee otherwise than by a conveyance with the proper warrants for holding only of the superior" (2 Burge, *Comm.* 241-2).*

Fee tail; An entail,† tailzied feu, or tailzie, is an estate where

* All clauses *de non alienando sine consensu superiorum* were abolished, in consequence of the rebellion of 1745, by the 20 Geo. II. cap. 50.

† Upon the origin of strict entails, Scottish lawyers appear to be divided. Some held that valid and effectual strict entails could have been made at common law, and that the statute 1685, cap. 22, had no other intention than to regulate entails subsequent to its date. Subsequently it was held, and both these opinions are the bases of judicial decisions, that no entail, whether executed before or after the statute, will be effectual unless it were recorded, and all the transmissions and conveyances thereof regulated by that statute. Thence an inference that at common law no entail was good; an inference broadly stated by Lord Meadowbank, in *Hamilton v. Macdowall* (*Fac. Coll.* p. 326), who held it to be a matter perfectly clear, and declared that arguments urged in favour of the existence of entails at common law, independent of the statute, could not impose, "I do not say on any lawyer, but on any person even moderately skilled in the rudiments of law. I am not speaking," he added, "from my own authority; I speak the unanimous judgment of the court while Lord Justice-Clerk Miller and Lord Braxfield sat on the bench. It was the unanimous opinion of the court, in the case of Agnew of Sheucan, that the case of Stormont was wrong decided, and that entails had not a foot to stand upon, but the statute 1685. They are the mere creatures of that statute." It was the Stormont case, above alluded to, which first affirmed the effect of prohibitory and resolute clauses in so fettering the heir in possession as to prevent any of his deeds affecting the estate (Sandford, *On Entails*, p. 42). This last decision, Lord Kames declared, was obtained by a prevailing attachment to entails, which at that time had the grace of novelty, and were not seen in their proper light (*Law Tracts*, p. 137). Sir George Mackenzie, who was Viscount Stormont's counsel, observes, that because such clauses prejudice creditors and commerce very much, and seem to be inconsistent with the nature of property and dominion, therefore an act of parliament was

the course of descent is according to a special provision—as, to the heirs male, or heirs of male of the fiar's own body, or to the heirs of such a marriage, or to the heirs of Titius, whilks failing, to the heirs of Seius, &c. Of these tailzies there are many several ways, as the fiar pleaseth to invent, and ordinarily in them all the last member or termination is, to heirs whatsoever of the last branch, or person substituted, or the disponent's heirs; and when that takes effect by succession, the fee, which was before tailzied, becomes simple (Stair, II. 3, 43).

The first person taking under the entail is called the institute; the remainder, heirs of entail, tailzie, or substitutes.

Entails may be of four kinds.

(1.) The entail with a simple destination, in which the heirs, as they succeed, are not prohibited from aliening or charging the subject, and which merely amounts to a direction of the course in which the subject is to descend, unless any of the possessors should interfere.

(2.) The entail with prohibitory clauses, prohibits the heir succeeding, according to the form of the prohibitory clause, either generally from doing any act or deed by which the course of descent should be altered, or the lands affected, or, declaring more specifically, that it shall not be lawful to any of the heirs of entail to charge with debts or alienate the property, or to alter the destination. Such prohibitions avail only *inter heredes*; they do not affect third parties. The proprietor cannot convey such an estate gratuitously, but he may do so for onerous causes, and the subject is attachable by creditors. But such an estate raises a *jus crediti* on the part of the substitutes, who, by a process called *inhibition*, may protect themselves against future debts or contracts.

(3.) The third species of entail is an entail guarded with a resolute clause; a resolute clause being a clause declaring that the contravener's right shall be null, by force of which his right to the subject will be forfeited; but his onerous deeds, &c., prior to the forfeiture, are still effectual upon the estate.

(4.) The fourth and strictest entail is an entail with irritant, resolute, and prohibitive clauses, and complying in its constitution with all the requirements of the statute 1685, cap. 22. This statute, as to all entails conformable to its directions, declares the clauses "to be real and effectual, not only against the contraveners and their heirs, but also against the creditors, comprizers,

necessary to secure them (*Inst.* III. 8, 17); and the famous statute 1685, cap. 22, was drawn by him. Mr. More, in his learned notes to Lord Stair's *Institutions*, has clearly shown, that in spite of the assertions already mentioned, of the statutory origin of entails, the courts have still evinced a practical scepticism on the subject. I must be allowed in *alienas segetes falcem inserere*, and to state the substance of his remarks. The statute 1685 imposes certain conditions on the creation of entails as essential to their validity. Clearly an entail not conformable to the requirements of the statute could not operate under the statute; so that where the entail, by reason of its omitting some clauses required by the statute, has been held incomplete, the heir in possession is not barred from aliening, nor can he be restrained in any way from doing so. He is treated, so far, as holding his estate in fee simple; so far, but no farther; for the courts have also decided that he is liable in damages to the substitute heirs in tail, to the degree, at least, of being bound to reinvest for their benefit the price he had received. This appears to rest upon the doctrine that the entail was binding at common law *inter heredes*; but if binding, why were the heirs incapacitated from interdicting, as in ordinary cases, the sale?

Law. adjudgers, and other singular successors whatsoever, whether by legal or conventional titles." The nature of the *resolutive* clause has been already explained; the *irritant* clause goes a good deal farther, inasmuch as it avoids all deeds executed in contravention of the prohibitions of the entail. Both clauses must, however, be inserted, that the benefits of the statute may be enjoyed.

Stat. 1685, The statute requires certain solemnities to be observed as essential to the validity of an entail entitled to the benefit of its provisions. Amongst these are two important requisites: (1) that the authority of the Court of Session should be given to the entail; and, (2) that it be recorded in the register appointed for the purpose.

cap. 22. It has been held that an entailer cannot make a gratuitous settlement of the estate so as to bind himself and exclude his creditors. Lands entailed may be sold to pay the entailer's debts (*see* 6 & 7 Will. IV. cap. 42; 4 Vic. cap. 24).

If a substitute neglects to protect his rights, as by suffering acts of contravention to pass forty years without declaring an irritancy upon them, or if he permit the estate to be enjoyed upon an unlimited title, or upon a title different from the deed of entail, he forfeits his own rights.

That the entailed estate may vest in the substitute, he must serve himself heir to him in whom the fee last vested, unless he succeed under a disposition, which embraces him *nominatim*; and the effect of the general service is to convey every personal right that was in the deceased to the heir.

Construction of strict entails. The prohibitory, resolutive, and irritant clauses, as impeding the free use and circulation of property, are construed very strictly by the common law. The judges, instead of endeavouring to carry out the entailer's intentions, seek for some loophole whereby they may defeat them. "Unless the entailer has by the most apt and explicit terms prohibited alienation, contraction of debt, and alteration of the order of succession, and guarded his prohibitions by the proper resolutive and irritant clauses, no effect whatsoever will be given to his obvious meaning, and no inference, however plain, will be permitted to supply the place of the direct words which have been held essential for this purpose; and hence, though there should be the most express prohibition against altering or changing the order of succession appointed by the deed, this will not prevent the heir in possession from selling the estate, and conveying it to a stranger, and thus defeating the entail by carrying it off to a new set of successors. Nor will a prohibition against selling the estate, so that it may descend in perpetuity to the heirs appointed by the deed, bar the heir in possession from gratuitously conveying it to strangers, and altering the order of succession. Nor will a prohibition to contract debt upon the estate, whereby it may be adjudged [taken upon process] by creditors, and carried off from the appointed series of heirs, be inferred by the most express and cautious clauses for preserving the estate under the entail, and for its descent to the substitutes called by the deed: and so far has this rigid species of construction been carried, that even though the prohibitory clauses should be expressed in the most apt and correct terms, yet, if there be any defect in the resolutive or irritant clauses, the entail will have no more effect against creditors and third parties than it would have had if the prohibitory clause which is not

properly guarded by the resolutive and irritant clauses had been omitted" (More, note, Stair, *Inst.* 183).

Law. An entail is a fee, although limited and restrained; and the heir of entail is a *fiar* unlimited, except as to particular points. It is then clear that all the incidents of an estate in fee simple belong to an entail, not inconsistent with its nature as an entail. Thus, upon an entail, unless it have been specially provided otherwise, attach the widow's *terce*, and the widower's *courtesy*. The tenant is entitled to cut down timber, but probably not young trees, when the cutting is manifestly injurious to the estate. If a mansion be specially included in the deed of entail, it is not competent to the heir to pull it down in order to sell the materials. A heir of entail is *personally* liable for debts due by the entailer; but if he pays them, he is entitled to keep them up as charges upon the estate entailed. By the 10 Geo. III. cap. 51, succeeding heirs of entail are liable to the extent of three-fourths of the sum laid out by a preceding heir in buildings and improvements, proper notice having been given of the same. The 20 Geo. II. cap. 50, empowers heirs of entail to sell the superiorities of entailed estates to their vassals, under conditions therein prescribed; and, by various subsequent statutes, powers are conferred, or burdens imposed, upon such heirs, who are enabled, it should be further remarked, to grant provisions of a certain amount to their wives, husbands, or children, payable out of the rents of the entailed estates, notwithstanding the terms of the deed of entail (5 Geo. IV. cap. 87).

Another species of estate is that of a conjunct fee and life-rent—a very singular modification of interest in property, but which may be thus explained. If the grantees are strangers, the conjunct fee would be considered as giving to the survivor the whole benefit of the subject; and after his death, the fee divides equally between the heirs of both. If the right be taken to two jointly, and their heirs, without any mention of life-rent, the conjunct heirs enjoy the subject equally while both alive; but on the death of one, then his share of fee and life-rent descends to his heirs, instead of accruing to the survivor. If the grantees be husband and wife, unless the infestment expresses it to be to the longest liver, and unless the right be originally derived from the wife, and a life-rent only was intended to be reserved to the husband, he is sole *fiar*, and she possesses nothing but a life-rent.

We have treated of rights to the dominion; we shall now briefly consider those real rights which exist apart from the dominion, which are burdens on land; and first, of servitudes, personal and real.

Personal servitudes are those whereby the property of one is subservient to the person of another than the *fiar*. Servitudes personal for time of life are called *life-rents*, of which we shall now speak. This life-rent is a usufructuary interest, and to it is applicable the definition of usufruct given in the *Institutions*—*jus alienis rebus utendi fruendi, salva rerum substantiâ* (II. 4, 1). It is called a personal servitude, in reference to the usufructuary, he retaining his lien on the property by whomever enjoyed, and it ceases on his own death; but it may be considered as a real servitude, as binding, not the person, but the land, by whomever possessed. Life-rents are *legal* or *conventional*. Legal life-rents are the results of the operation of law. The *terce* of the widow, and the *courtesy* of the widower, are examples.

- Law.** I. Unless a special contract between the parties exist to the contrary, the wife is entitled to a life-rent of a third of the heritable subjects in which her husband dies infest. This is called her *terce*, and she the *tercer*. If a special provision have been granted by the husband to the wife, she is excluded from the *terce*, unless it appears to have been intended that she should have both.
- Terce.**
- Service.** In order to receive her third of the rents by virtue of the *terce*, the widow must be *served* to it. A brief or writ is obtained from the Chancery, directed to the sheriff of the county where the lands lie, who summonses a jury of fifteen men, who are sworn to inquire—
- Kenning.** (1) whether the widow was lawful wife to the deceased; (2) whether the husband died seised of the lands in question. When served, the widow may sue tenants for the third of the rents, and possess the land *pro indiviso* jointly with the heir; but she can exercise no other authority until the sheriff *ken* her to her *terce* by dividing the land between the heir and her. The *terce* is excluded—(1) by a decree declaring the nullity of the marriage; (2) by the dissolution of marriage before a year and a day, without issue; (3) by the wife's adultery or wilful desertion.
- Courtesy of Scotland.** II. By the courtesy of Scotland, the husband is entitled, after his wife's death, to a life-rent of all real property inherited by her, and whereof she was infest, if there was a child of the marriage born alive, capable of having been heir to the wife. The theory of law considers him as entitled less as husband of the heiress than as father of the heir. The liabilities of *tercer* and tenant by the courtesy materially differ: the former is in no degree affected by her husband's debts; but the latter, as he enjoys the life-rent of his wife's whole heritage under a lucrative title, is considered as her temporary representative, and is liable to the payment, not only of all the yearly burdens charged on the subject, but of the current interest even of personal debts, while his right subsists, to the extent of the benefit he enjoys by the courtesy (1 Burge, *Comm.* 429).
- Conventional life-rent.** Conventional life-rent arises from reservation or grant. In the former case, the owner, in transferring the fee to another, reserves the use and enjoyment of it to himself; in the latter case, it is constituted by deed of conveyance, and with all the solemnities observable in passing the fee.
- Real servitudes;** Real servitudes are of less importance, and may, agreeably to the classification of the *Digest* (L. 16, 198), be considered as *urban*, which include those constituted in favour of houses or buildings, whether situate in city or country, or as *rural* or *prædial*, which include those having relation to farms, fields, or gardens.
- Support.** (1.) The servitude of *support* "whereby the servient tenement is liable to bear any burden for the use of the dominant" (Stair, II. 7, 6). This combines the *onus ferendi* (whereby the wall or pillar of one house is bound to sustain the weight of the buildings of the neighbour), and the *tigni immittendi* (which is the right of inserting a joist or beam from the wall of one house to another, that the latter may bear the weight), known to the Roman law.
- Stillicidium.** (2.) *Stillicidium* is the rain-water running from the roof or eaves of a house by drops, and which, when collected in a spout, is called *flumen*. Except by a servitude of this kind, no one can build so as to expose his neighbour to this annoyance.
- Light or prospect.** (3.) Servitudes of *light* or *prospect* are restrictions upon natural liberty, to the effect of preventing the servient proprietor from raising obstructions to the light or prospect of the dominant tenement.
- Law.** Such are the principal urban servitudes. The *rural* are chiefly as follow:—
- (1.) Ways or roads, being horse-roads, foot-roads, cart ways or coach roads or ways, or loanings, whereby cattle roads may be driven from one field to another.
- (2.) The servitude of *aquæductus* confers the right of *Aquæduc-* guiding water through, and by dams gathering water *tus* upon, another's ground, for one's own use, care being taken to keep the waters from injuring the servient tenement.
- (3.) *Aquæhaustus* enables a landholder to water his *Aquæhaus-* estate at any river, brook, well, or pond, in his neigh- *tus* bour's grounds. The law upon this subject contains a variety of provisions for the protection of the various parties interested, which we shall not attempt to detail.
- (4.) The servitude of *feal* and *divot*, which authorizes *Feal and* one to take feals or divots (green turf used for hedging *divot* or thatching) from the surface of the servient tenement, for the purpose of thatching.
- (5.) The servitude of *feal* is a right of raising turfs or *Feal* peats for the purposes of the dominant tenement.
- (6.) The servitude of *thirlage* restricts the inhabi- *Thirlage* tants of certain lands to the use of a particular mill. There are various duties involved in this servitude, to which a detailed allusion is not necessary.
- We shall now consider those real rights which result *Heritable securities.* from heritable securities.
- Heritable securities* are various in their kinds. There *Wadset:* is the *wadset*, by which the debtor conveys the land to the creditor, to be enjoyed by him until the debt be repaid; the form of which is an absolute conveyance, and a separate instrument contains the clause of *reversion*, on payment at a certain time and place; and also a clause of *requisition*, enabling the debtor to quit possession, and recover his debt. A *wadset* may be *proper* or *improper*, the chief distinction *its species.* between the two being, that, under the first, the *wad-* setter (mortgagee in possession) enjoys the rents and profits of the lands, as his recompence for the loan; while in the second place he receives them, and, if deficient, looks to the reverser or debtor to make up the deficiency; but if they exceed his interest, is bound to account for them in discharge of the principal; and when his receipts equal that and the interest, he is bound to divest himself in favour of the reverser. The debt may be discharged by either party; by the one under the clause of *reversion*, by the other under the clause of *requisition*. The *reversion* is a real right, descending to the heir.
- The chief heritable security for debt, however, was *Heritable bonds.* originally in form a rent-charge; it is now what is called a *heritable bond*, which obliges the debtor to infest the creditor* in the lands (that is, to convey them to him in fee), for securing the principal, interest, and penalty; and he assigns the whole rents during the non-redemption. The creditor may enter into immediate possession, and his rights are terminated by his recovering the debt or by its being paid to him.
- Scottish conveyancers prefer to this, the *bond and* *Bond and disposition in security.* disposition in security.
- * This contemplates the case of a creditor only. But a heritable bond may be granted to a cautioner (one who is surety for another), as a security to him, when it is styled an infestment of relief. In this case the infestment is conditional, and does not operate until the cautioner pay the debt, or be proceeded against for its payment.

Law.

Law.
Absolute
disposition
and back
bond.

disposition in security, which besides giving the creditor a right to the lands, empowers him to sell them in discharge of his advances. The last species of heritable security is an *absolute disposition*, with a *back bond*; the first of which appears to constitute the creditor, owner of the lands absolutely; whilst he declares, in the latter, that they had been disposed him in security, and promises to divest himself of all right to them on payment. He is, at the same time, protected by the clause, empowering him to sell, and in the meantime receives the rents in payment of his debt and interest. This security requires no re-conveyance of the lands.

Bonds carrying interest.

Bonds conditioned for the payment of money, although not secured on land, provided they carry interest, may be either heritable or moveable, according to circumstances. They are heritable always, if in express terms they exclude the executors of the creditors from taking, or, if they contain a clause of infestment, because such discovers "the creditor's intention to constitute with his money a feudal right on land" (2 Burge, *Comm.* 37). Otherwise such bonds are moveable as to the succession,—that is, descend to the children and next of kin; but as to the reciprocal rights of husband and wife, see sect. 4, *post*; and the right of the crown or fisc (see p. 848, *post*) are heritable.

Rights having a tract of future time.

Rights having a tract of future time, as it is expressed, that is, rights to payments future and periodical, by their essential constitution are considered as heritable, and descend to the heir. These rights are not the less heritable, because affecting property of a personal nature, and not charged upon land. It must, however, be understood, that a debt is not considered as having a tract of future time, merely because the term of payment is postponed, or because it is payable in instalments, even although interest be payable in the interim; and the reason of this is, that the debt vests absolutely from the first, and there is "a mere delay of payment granted to the debtor; and it makes no difference, as to this question, whether such delay be granted to one term only, or to several terms" (More, note, *Stair, Inst.* 142).

Annuities.

A right of annuity is an example of a right having a tract of future time, for the right to receive the successive annual sums vests only as the time for payment elapses. Upon the heir, therefore, descends this right, its profits or its burdens. A right to *rents* is another example. *Offices*, granted to continue after the death of the patentee, are heritable, descending to the heir.

Rents.
Offices

Trust estates.

Another burden upon real property arises when an estate is conveyed, and the deed, and the formal documents passing the estate, express it to be a disposition in trust, and these are recorded: then is a *jus crediti* created on behalf of the person to whose use the trusts exist, which is a real burden on the estate.

Feu-farm.

Another species of real right is that arising from lands in *feu-farm*, which are lands, granted upon the terms of paying or delivering to the superior a fixed yearly rent, in money or grain, or by the performance of certain services proper to a farm, as ploughing, sowing, &c. The grant in feu-farm resembles the emphyteusis of the civil law, from which it is supposed to be derived, and gives to the tenant a *jus dominio proximum*, and full right to cultivate the land as he pleases, and to receive its profits, and subject it to the burden of his debts. The restrictions upon alienation, are, practically, very slight.

Tacks.

By the jurisprudence of most countries, a lease is

considered as a personal contract between the landlord and tenant, and the rights devolving on the tenant descend in the same manner as all other personal property; and such, we are told, was originally the case in the law of Scotland; but such has been the effect of a statute (1449, cap. 18), that tacks, as leases are called, have become real rights, descendible to the heirs. Tacks must properly be in writing, if to subsist for more than a year; but if the tacksman, on the faith of his lease, have made *extensive* improvements, or similar *rei interventus* have taken place, the rule is relaxed. An obligation in writing to grant a tack, if followed by possession, is equivalent to a tack, and is binding on the obligor and his singular successors. It is the *possession* which constitutes the tack a real right, and seisin is not necessary.

Having considered the various modifications of Titles to interest in real property recognised by the law of estates. Scotland, we now inquire in what manner it may be transferred and acquired. This may be done either by simple operation of law or by the act of parties.

(I.) *Succession ab intestato*, or descent, as it is called in the law of England, is regulated by rules which vary according as the fee has been inherited or acquired—is, to use the technical phrase, a fee of *heritage* or a fee of *conquest*. Those acquiring by conquest are called heirs of conquest; although they may be heirs of line; but the heirs of line retain their name, in distinction to heirs of conquest.

First, then, as to heirs of line, who are—(1.) The Heirs of lawful issue of him who died last seised, the males* line. being preferred, and each taking in succession according to seniority, and the children, representing deceased parents, taking in their room. Failing males and their issue, the female issue inherit *pro indiviso* as heirs portioners, the issue of such females, as predeceased, taking their mother's portion. The eldest heir portioner by legal succession has a right to the mansion-house and the grounds on which it stands, to such peerages, dignities and titles as are not limited otherwise, and to subjects indivisible, such as superiorities. (2.) Failing children, grandchildren, and their descendants, resort is had to collaterals. On the death of a middle brother, his younger brothers (and their issue) succeed in the order of seniority, and elder brothers succeeding to younger are, with their issue, preferred in an inverse order—from younger to elder upwards. The brothers and their issue succeed to a sister, in preference to her sisters. (3.) The whole blood is preferred to the half-blood consanguineous; but the half-blood uterine† is wholly excluded. If the half-blood taking be the issue of a former marriage, the youngest brother is preferred; and after him (failing his issue) the next youngest, the opposite rule holding if the half-blood be of a subsequent marriage. Failing collaterals, the father and his relations inherit; but the succession never ascends to the mother, nor (even although the estate have been her own, if it has vested in her son or daughter,) do her relations ever take.

In failure of the three lines of succession enume-

* Lord Stair mentions John Campbell of Skeldon succeeding to his father in a ward-fee, being the seventeenth son (II. 4, 44).

† Relations are consanguineous when born or descended from the same father; uterine, when born or descended from the same mother. On the subject of collateral consanguinity, see an excellent tract by Blackstone, in his *Tracts*, vol. i. p. 11. Oxford, 1761.

Law. rated—the descending, collateral, and ascending lines—the king takes by a caducary right, and not a right of succession, although he is styled *ultimus hæres*.

Heirs of conquest. Secondly, as to heirs of conquest. "Succession to conquest, as contradistinguished from succession to heritage, takes place only where a middle brother or sister (or their issue) dies, leaving younger or elder brothers or uncles. The younger brother (or uncle) and his issue take the heritages; the elder and his issue, the conquest" (4 Burge, *Comm.* 89–90).

Prescription positive; (II.) By prescription, positive or negative. Positive prescription arises when one has possessed, peaceably and without lawful interruption, his lands, annual rents or other heritages, heritable rights and privileges, servitudes or other real burdens, by himself or others, in virtue of feftment for the space of forty years subsequent to their date, or so far back as memory can reach; he shall not be troubled or disquieted. Negative prescription is the result of a forfeiture of a right, through the proprietor's neglect to exercise it for forty years. Rights, thus prescribed against, are not only moveable rights, and arise out of simple obligations, but rights of action on heritable bonds, reversions, &c. The effect of this prescription is to extinguish all claims that have been allowed to lie over without being made or insisted on for forty years after they were enforceable; and it is founded on the presumption, common to other systems of jurisprudence, either that such claims have been satisfied, or that the creditor has relinquished them.

negative. We shall now consider the method of acquisition by the act of the parties.

Charters. To the constitution of a valid conveyance of a fee there are two requisites—a writing containing a present dispositive act, with a direction that seisin should be delivered, and a delivery of seisin as directed. This writing is usually styled a charter; but no precise form is necessary, and it may be a charter only in fiction of law: thus a precept of seisin, as the direction above mentioned is technically called, may be only a disposition with a clause obliging the grantor to sign a charter. So long, in fact, as the disposing will is evinced, the first requisite of the law is complied with. Charters granted by subjects are either *à me* or *de me*: in the first, the holding is, not of the grantor, but of his superior, whose confirmation is necessary to make the holding valid; the second reserves the superiority to the disponent. Charters are also either *original* or *in progress*, *voluntary* or *necessary*.

original; A charter is said to be *original* when it is creative of a fee *de novo*; all its provisions are understood in subsequent charters, all whose ambiguous provisions are interpreted by it. A charter *in progress* disposes of a fee already in existence; as, for example, a charter from a superior to the heir of his vassal; or by a vassal to another, to be holden of his superior.

voluntary; A *voluntary* charter is granted by a superior *ex mero motu*.

necessary. A *necessary* charter he grants in obedience to law; as when a creditor by process has obtained a right to his debtor's lands, the superior is bound to give him charter and seisin.

A charter contains—

(1.) The name and titles of the party granting.

(2.) A recital of the causes of the grant, which may be either *onerous*, or *gratuitous*, if made for love and favour. "We follow not," says Lord Stair, "that sub-

Law. tilty of annulling deeds because they are *sine causâ*; but do esteem them as gratuitous donations; and therefore narratives expressing the cause of the disposition are never inquired into" (II. 3, 14).

(3.) The dispositive clause of a charter describes the nature and boundaries of the subjects conveyed, and expresses the order of succession and limitation of the fee.

(4.) The clause of *tenendas* (the *tenendum* of English lawyers) states the tenure by which the lands are to be holden.

(5.) The clause of *reddendo* expresses the duties or services to be yielded, and which, from this circumstance, are themselves called the *reddendo*.

(6.) The clause of warrandice, or warranty, whereby the grantor binds himself to the validity of the right conveyed.

(7.) The precept, or letter of seisin, being a direction from the superior to his baillie to give seisin or possession to the vassal or his attorney, by delivering to him the proper symbols.

The precept is executed by what is termed an *instrument of seisin*, which is in effect an attestation of a notary that possession was truly given by the superior or his baillie, to the vassal or his attorney, by the delivery of the proper symbols, according to the precept. These symbols are—for lands, earth and stone; for annual rent out of land, the same, with the addition of a penny-money; for parsonage tithes, a sheaf of corn of the church; for jurisdictions, the book of the court; for patronages, a psalm-book and the keys of the church; for fishings, net and coble; for mills, clap and hopper, &c. This seisin ought to be taken on the lands contained in the precept—a rule which, under special circumstances, may be dispensed with.* The legislature

* This was done in the case of the grant of Canada and Nova Scotia, by James I. to Alexander the Earl of Stirling, whom he styled "my philosophical poet." The seisin, by the king's direction, was taken at the gate of the Castle of Edinburgh, and afterwards confirmed by act of parliament, 1653, cap. 28. The usual form in which seisin was delivered is thus stated by Mr. Burge:—

"First, either the vassal or his attorney appears on the lands of which seisin is to be given, holding in his hand the precept, which he delivers to the superior, or in his absence to the baillie. The baillie immediately delivers it, in the presence of witnesses, to the notary, who is to extend the instrument. Then the notary reads the precept; after which the superior, or his baillie, delivers to the vassal, or his attorney, earth and stone of the lands, or the other proper symbols, if the right granted be not a right of property in lands. Lastly, the vassal, by himself or his attorney, takes the instruments in the hands of the notary before witnesses that he hath received state and seisin of the lands in due form. On the whole of this transaction an instrument is extended in writing by the notary, who, after reciting in it the precept of seisin, and as much of the charter as is necessary for understanding the precept, subjoins his own attestation, that he, having been specially called for the purpose, knew, saw, and heard that these facts were so done as they are recited in the instrument, to which he and the witnesses adhibit their subscriptions" (2 *Com.* 729).

In ancient times it was the delivery of seisin that constituted the transfer, and the charter was only the evidence of the seisin. "Amongst savage nations, the want of letters is imperfectly supplied by the use of visible signs, which awaken attention and perpetuate the remembrance of any public or private transaction" (Gibbon, *Decl. and Fall*, ch. 44). Under the Roman law, the absolute right of property was called *mancipium*, *jus Quiritium*, probably from the *mancipatio*, which was one of those modes (*modi adquirendi civiles*) pointed out for its acquisition. This *mancipatio* was nothing but the solemn delivering over of the thing in the presence of a determinate number of witnesses and a public officer (*Cod. Theod.* VIII. 12, 1, 2, & 8; and Warkönig's note on Gibbon, chap. 44, Milman's edit. vol. viii. pp. 82–3).

Law.

has interfered most materially with the feudal character of Scottish law, rendering the vassal, to all intents and purposes, the absolute proprietor of the fee. At first, it was competent to the superior to object to receive any vassal other than the heir of the vassal named in the investiture; but as the law now stands, he must receive creditors, appraisers or adjudgers (acting under legal process), the purchasers of a bankrupt's estate, upon payment of one year's rent, and all singular successors (as those are called who appear in any other character than heirs)—they having obtained from the vassal a disposition containing a procuratory of resignation, as it is called, and also yielding to the superior the fees or casualties he is entitled to on the vassal's entry—a year's rent (Stat. 1469, cap. 37; 1672, cap. 19; 1681, cap. 17; 1690, cap. 20; 20 Geo. II. cap. 50, s. 12).

Confirmation of the superior.

Where the lands are disposed to be held of the disponent, the confirmation of the superior is not necessary, except as a matter of prudence to protect the disponent from the disponent's defaults in respect of his superior.

Where the disposition is to be holden of the disponent's superior, it may be perfected by either confirmation or resignation.

(1.) The first case, as the superior admits the disponent (who has already taken seisin on a precept contained in the disposition) by a confirmation, written on a separate charter, reciting the charter confirmed.

Resignation.

(2.) By resignation, the vassal surrenders his fee to the superior, in whom (if the resignation be *ad perpetuam remanentiam*) the property, as well as the superiority, in consequence, vests, or (if it be *in favorem*) he receives it to deliver it again to a third party specified. The form of resignation is properly this: the vassal surrendering, on bended knee, delivers the symbol of staff and baton into the hands of the superior, who, in the case of a resignation *in favorem*, immediately delivers it to the disponent, and a notarial act, recording the ceremony, completes the transfer. In practice, a *pen* is the symbol actually delivered.

The *dominium utile*, or right of enjoyment, distinguished from the proprietorship, passed by the delivery of a symbol on the spot (Spence, *Inquiry*, 32).

Sir Francis Palgrave has collected some interesting examples of this conveyance of property by means of symbols. St. Birlanda refused to associate with Oidelardus, her leprous father, who disinherited her, and conveyed all his domains and villans to St. Gertrude, by placing on the shrine of that saint a turf, a twig, and a knife. This transaction took place in the reign of Dagobert. When Tassilo, the duke of the Bavarians, submitted to Charlemagne, and renewed his vassalage, he surrendered his dominions by delivering a staff, at the head of which was the image of a man. According to Kofod Ancher, the ceremony called "Skjodning" was, amongst the Danes, the only mode of transferring property. A turf or clod was cast, or "shot," by the grantor into the cloak or hood of the grantee; and it could only be done in full court or a place of equal publicity. It was by delivering a small silver sword that Culen, the Scoto-Pictish king, gave investiture to Gillespie Moir (A. D. 965) (*R. and P. Eng. Comm.* i. 141; ii. 227). It is related that when William the Conqueror landed at Pevensey Bay, he stumbled and fell on his hands. His superstitious troops loudly exclaimed, "*Mal signe est çui!*" "No," replied the duke, recovering himself, and raising one of his hands, which held a clod of earth, "I have taken seisin of the country." Similar presence of mind has been ascribed to Camillus (Plut. *Vit.* t. i. p. 214, edit. Schæfer, 1826). It is singular to observe that the distinction observed by the Romans, between the mode of passing the dominion and the usufructuary estate, should have been observed by the barbarians. This latter was conveyed in England by the delivery of a rod or branch of the tree; in Scotland, of a straw; and in the Belgic districts, of a staff.

Every obligation relating to heritable rights must be—(1) in writing; (2) signed (personally, or, in the case of illiterates, by a notary) and witnessed; and (3) in obligations of great importance (understood to be obligations granted for a consideration exceeding in value 100*l.* Scots), subscribed or sealed by the grantor (or, in the event of his being illiterate, by two notaries), before four witnesses particularly described. All original charters, contracts, and others whatever, not mentioning the name, dwelling-place, or other denomination of the writer, are null. All deeds destitute of the required solemnities "shall bear no faith in judgment, or shall be null, and not suppleable by any condescendence;" and cannot therefore be produced in evidence against the grantor, nor be supported by any proof whatever. Exceptions to this principle are too trifling to require notice.

Law. Requisites of transfers, &c.

Another method of transferring property is by act *Apprizings*.

of the parties uniting with the operation of the law, as when lands are taken in execution by a creditor. By a very ancient statute (Stat. Alex. II. cap. 24) it was provided, that if the debtor had no moveables, the sheriff should give him notice to sell his lands, which if he did not do, they were transferred to the creditor in discharge of his debt. The statute 1469, cap. 36, gave the debtor a power of redemption within seven years. The creditor taking is called the *apprizer*; the transference was accomplished by *letters of apprizing*, and which were founded upon a judgment of the Court of Session.

The excess of the rents, after payment of the interest of his debt, the apprizer received in part discharge of that debt; and all apprizers' apprizings, *led* within a year of the first effectual apprizing, placed the subsequent apprizers on the same footing as if the first apprizer had *led* for them all. Apprizings were formerly the chief executorial, or processes chiefly used to enforce judgments, whereby all rights of the ground were conveyed; but now *adjudications* are the executorial *Adjudications* chiefly in use; and by them all rights of the ground are conveyed from debtors to their creditors by decret of the Lords of the Session, and which pass as effectually and fully as they could do by voluntary disposition; except in the former there are no clauses of warrandice, which are always contained in the latter. There are their forms; two forms of adjudication under the statute 1672; one is called the *special*, the other the *general* adjudication. The special adjudication was where debtor special; and creditor concurred in an amicable action; in which the former showed his title, concurred in proceedings ascertaining the value of the lands, and in setting off a portion to be adjudged to the creditor equivalent to his debt, and one-fifth more, subject to redemption within five years. The general adjudication embraced all the debtor's general lands and real rights, which it passed to his creditor, subject to redemption in ten years. If the debtor did not concur, as required in the special adjudication (and this is the usual practice), the general adjudication proceeded. The alternative is offered to him. The debtor is, in practice, able to redeem after the term fixed by law, until "a decree of declarator of expiry of the legal" has been obtained, which is the nature of our decrees of foreclosure pronounced by the Court of Chancery. To authorize an adjudication, the debt or obligation, on account of which it is used, must be regularly constituted, either by the decree of a competent court, or by a signed document, such as a bond, bill, or other obligation in writing, determining the

Law.

precise amount and character of the debt. An adjudication followed by charter and seisin, and forty years' possession after the expiry of the legal time for redemption, constitutes an indefeasible title to land.

Judicial sales.

The 54 Geo. III. cap. 137, enacts that a judicial sale, at the instance of creditors, may in all cases proceed where the interest of the debts and the other annual burdens exceed the yearly income of the subjects under sale, or where a sequestration shall have taken place under this statute, without any other proof of bankruptcy or insolvency. A sale at the instance of creditors is authorized by the action of judicial sale and ranking, which enables a creditor holding a real security on an insolvent estate, to sell the same, and to have the price; and an apparent heir (that is, an heir who had not perfected his title by entry) might bring it to sale, whether the estate was insolvent or not. It is only after having adjudged, that a creditor can raise an action of ranking and sale, and the creditor selling must be in possession of the estate. The title of the purchaser under such sale is protected against the claims of the debtor, and those deriving right from him, against all claims of his creditors, but not against the claims of third persons whose rights are not derived from the bankrupt, but may prove to be preferable to his.

Judicial sale and ranking.

In concluding this rapid survey of the law of Scotland as respects *real property*, it will be proper to allude to some of its more remarkable characteristics.

Testament. (I.) It may be observed, in the first place, that heritage or real property is not capable of disposition by testament. If the instrument in which such a disposition is attempted is clearly intended to be testamentary, then, so far as it disposes of heritable property, is it held void. The principle of law goes so far as to invalidate all dispositions transferring or burdening heritable property to the prejudice of the heir, if executed by one during a sickness whereof he dies, unless he survive the execution sixty days, or after the execution go to kirk or market unsupported. Hence the distinction, so well known to Scotch lawyers, of *death-bed* and *liege poustie**—the technical terms indicating two states of competency and incompetency to burden or dispose of an estate to the prejudice of the lawful heir. "When an overture was made in the parliament (1672) to allow heritors to burden their estates with three or four years' rents on death-bed, providing for younger children, the motion was rejected, as tending to destroy the ancient families of the nation" (Lord Fountainhall, II. 325).

Death-bed and liege poustie.

* A man is in *liege poustie* when he is in the posture of a liege able to serve his lord in the war, like Chaucer's knight, of whom he says,

"Ful worthy was he in his lord's werre."

The decisions on this subject are very curious. It would appear that, formerly, deeds executed by a person while labouring under a disease of which he ultimately died, were void, unless he had given proof of his convalescence by going to kirk or market unsupported. Nor would this avail to rebut the presumption that the deeds were made *ex capite lecti*, as it was called, unless he went to kirk or market in the daytime; if to the kirk or kirkyard, then at a time when people are gathered together there for any public meeting, civil or ecclesiastic; or if to the market, then at market time. It appears doubtful if going to a dissenting chapel would equally avail. The ground of this doctrine of avoiding does not appear to be, that the party executing was afflicted with a *morbis santicus*, or disease affecting the whole body and brain; for it did not appear sufficient, in the case of a disposition of Lord Balmerino in favour of Lady Couper, to show that after it was made "my lord made bargains and counts, seemed to be merry and laughed,

(II.) For an heir to perfect his title to his predecessor's estate, he must (except in a few cases) enter, either by precept of *clare constat*, or by service and retour. The first method of entry is so called from the initial words of the precept, *Quia mihi clare constat*, &c., by which the superior acknowledges that the defunct died last vest and seised in such lands and annual rents, that the same are holden of him by such a tenure, and that the obtainer of the precept is nearest and lawful heir to the defunct in the said lands; that he is of lawful age for entering thereto, and therefore commands his bailiff to infest him therein. The doctrine of entry by service and retour is very complicated, and gives rise to many questions, which are unnecessary to be discussed here. The form is simply this. A brieve or writ is obtained from the Chancery, addressed to the judge ordinary where the land or annual rent lies, such as the sheriff, steward, baillie of royalty or regality, or baillie of burgh-royal, directing him to inquire by a jury or inquest (consisting usually of fifteen persons), publicly summoned, "who died last vest and seised as of fee in such lands or annual rents, and if in the faith or peace of our sovereign lord, and who is his nearest and lawful heir therein, of whom it is holden in chief, by what service, and what the value of it is now and in time of peace, and if the said heir be of lawful age; in whose hands the same now is, from what time, how, and what service, by whom, and through what cause" (Stair, III. 5, 28). The service is the sentence or decret of the inquest, signed by their chancellor, and also authorized by the signature of the presiding judge, and so returned into Chancery, where it is registered, whence it is called a *retour*. The Court of Session may, however, annul it, if its correctness be impeached.

Law.

Clare constat.

Service and retour.

(III.) We must also notice the system of registering deeds which obtains in Scotland. By an Act, 1617, cap. 16, all reversions, regresses, and bonds for making reversions and regresses, assignations and discharges thereof, renunciations of wadsets, grants of redemption, and instruments of seisin, were required to be registered within sixty days after their dates. The registration may be effected either in certain local registries or in the general register office in Edinburgh. If seisins are not recorded, they shall make no faith in judgment to the prejudice of those who have acquired right to the lands contained in them; but the non-registration does not avoid the seisin.

Registration.

It will readily be seen how anxiously the Scotch law provides for the publicity of the rights and the transfer of the rights to property. The advantages of a system so wholly foreign to the policy of modern English law have not been deficient in advocates. Blackstone, after describing the forms of English conveyancing, observes: "In these there is certainly one palpable defect, the want of sufficient notoriety; so that purchasers or creditors

kept on his clothes, kept the table, came from his chamber to the hall, whistled to himself and danced." In this case, "no particular disease, but only sickness, was proved." In going to kirk or market, the grantor must not be supported; but when he goes not for the express purpose of validating his deed, "but principally to hear sermon or for devotion, or about affairs to the market," then, "taking the party by one hand, or helping him at a rugged ground, would not infer supportation;" and "if a gentlewoman accustomed to be led by the hand, should go so led to kirk or market, by reiterated acts it might be sufficient for inferring health or convalescence;" but it may afford a presumption of continuance of sickness, and "therefore in that case ladies behaved to forbear being led from honour" (Stair, III. 4, 28; IV. 20, 46). The law has since been altered as above stated.

Law. cannot know with any absolute certainty what the estate and title to it really are, upon which they are to lay out or to lend their money. In the ancient feudal method of conveyance (by giving corporeal seisin of the lands) this notoriety was in some measure answered; but all the advantages resulting from them are now totally defeated by the introduction of death-bed devises and secret conveyances" (2 Comm. 342).*

Sect. 4.—Rights in respect of Moveable Property.

Moveables. 290. The transfer of and title to moveables will not detain us long. Perhaps the most striking distinction between this branch of the Scotch law and the English law of personal property, is the fact that the former permits such a limitation of moveable property as amounts, in short, to an entail. The heirs to whom it is limited acquire a *jus crediti*, and in that right can prevent alienation, and cast in damages the heir contravening the prohibition against it.

acquired by marriage. (1.) Moveables may be acquired by *marriage*, which is, in truth, a legal assignation, by the wife to the husband, of her whole moveable estate; but the husband cannot, by any testamentary provision, donation *mortis causâ*, or gratuitous deed on death-bed, prejudice either the *jus relictæ*, or widow's part, or the *bairns'* part. The former, where there are *bairns*, is a third; and where there are none, half. The *bairns'* part, if the widow live, is a third; if she be dead, a half. Property may be excluded from the *jus mariti* by the direction of the person conveying; and the *jus* never extends to paraphernalia, which include the lady's clothes and proper ornaments—the whole *vestitus* and *mundus muliebris*. This right of the husband is nullified by the dissolution of the marriage, by death within a year and a day of its contraction, unless there have been a child of the marriage born alive.

Legitime. (2.) Succession by *legitime* and *jus relictæ*.—Children are entitled, as already observed, after their father's death, to a share of his moveable property, which is called their *legitime*, or portion natural, or *bairns'* part of gear. This is a right belonging only to *children*, and affecting only the *father's* estate. The *jus relictæ* has been already explained.

(3.) Succession *ab intestato*.—If the intestate die unmarried, then his whole moveable property, and if married, then the deceased's part (that is so much of his moveables as the *widow* and *bairns*, either or both, are not entitled to), descends equally to the nearest of

* The propriety of establishing a general registry of deeds in England has been a subject frequently mooted in Parliament. The difficulty of devising adequate machinery appears to offer a serious impediment to the accomplishment of the design; and the experience of the local registries of Yorkshire and Middlesex, and the Bedford Level (15 Car. II. cap. 7; 2 & 3 Anne, cap. 4; 6 Anne, cap. 35; 7 Anne, cap. 20; 8 Geo. II. cap. 6), is fruitful of warnings against hasty legislation on the subject. In 1652 a commission of twenty-one members, appointed in the preceding year by the House of Commons, to "inquire of the inconveniences of the law," prepared a bill for a general register, upon a plan recommended by Sir Matthew Hale (see his tract, entitled, *A Treatise, showing how useful, safe, reasonable, and beneficial the enrolling and registering of all Conveyances of Lands may be to Inhabitants of this Kingdom*; Lord Somers' Tracts, by Sir W. Scott; vol. vi. p. 191). A similar bill was brought in, in 1657, 1663, 1664, 1670, 1677, 1685, 1693, 1694, 1697 and the two following years, in 1734, 1758, and in 1816. Roger North tells us that his brother, the Lord Keeper, highly approved of the plan (*Life*, vol. i. p. 224), as did Sheppard, the ostensible, though not real author (see Buller J., 4 D. & E. 649) of the celebrated *Touchstone* (England's Balme, p. 115).

kin, the whole blood excluding the half, and the lines of succession following in the same order as in heritable succession. If the heir of the intestate, in his heritable estate, comes in with the other next of kin, and will not share the heritable estate with them, he will have no part of the moveable.

(4.) Succession by *testament*.—Testamentary dispositions of the dead's part are valid by the law of Scotland, although executed on death-bed, provided they be in writing (or authenticated by a notary or a clergyman and two witnesses), if they express, in whatever form, the testator's disposing intentions.

(5.) *Assignations* are conveyances, whereby one who is in the right of any subject may convey the property of it to another, who is thereupon entitled to perfect his right by legal proceedings. *Intimation*, as the formal document is called, or some notification equivalent, given to the debtor, is essential to the completion of the conveyance; for although an assignation not intimated is good as against the grantor, yet a second assignation intimated will be preferred to the first assignation not intimated. Payment of interest by a debtor to a creditor is equivalent to intimation.

(6.) *Cessio bonorum*.—An insolvent debtor having been imprisoned for one month, whose failure has not been fraudulent, who has disclosed all the circumstances of his bankruptcy, calls together his creditors, and the Court of Session will grant him a *cessio*, by which he is liberated on making a conveyance *omnium bonorum* (with certain necessary exceptions) for the benefit of all his creditors, according to their respective interests. This *cessio* does not apply to property subsequently acquired by the debtor: that is, notwithstanding, attachable by creditors.*

Sect. 5.—The Court of Session.

291. Without venturing to describe the system of local tribunals or the system in which criminal law is administered in Scotland, we shall confine ourselves to a brief statement of the functions and character of the Court of Session, the supreme tribunal *in civilibus*, and whose constitution, from the importance of its functions, is naturally a subject of curiosity.

* To prevent the anticipated evil effects of a system, which, by liberating the debtor, should enable him again to abuse the credulity or deceive the caution of mankind, "the law hath also appointed that magistrates, where dyvours are incarcerated, must not set them out until they have the habit appointed to be worn by dyvours as their outmost garment; and they ought to set the dyvour for some time, in the most public time of the day, upon a place appointed for that purpose, commonly called the dyvour-stone. And these decreets bear, that the dyvour shall be no longer free from personal execution than while that habit is upon him, when his clothes are on, and so visible that all who converse with him may see it" (Stair, IV. 52, 34). But the improved humanity of modern times dispensed with the dyvour's habit; it would, as Lord Meadowbank observes, "according to the present state of the public feeling, be a disgrace to the public administration of justice. It would shock the innocent, it would render the guilty miserably profligate" (Smith v. Likely, 6 Feb. 1813). The case of Rutherford, 5 Dec. 1676 (*Dirl. Dec.* 193), appears to have been one of the first in which the habit was dispensed with; and although the party appeared to have contracted the debt as cautioner for his father, several of the lords hesitated, being "of opinion that the act of *sederunt* made no distinction, and being made upon good consideration, and conform to the practice of all other nations, that bankrupts might be known by a habit to be persons that deserved no trust, and that others may be affrighted from contracting or undergoing debts which they are not able to pay." It is abolished by the 6 & 7 Will. IV. cap. 56, s. 18.

Law.
5 Ann.
cap. 8.

The Articles of Union (5 Ann. cap. 8, art. xix.) provide that the Court of Session,* or College of Justice, should, after the Union, and notwithstanding thereof, remain, in all time coming, within Scotland, as it was then constituted by the laws of that kingdom, and with the same authority and privileges as before the Union; subject, nevertheless, to such regulations for the better administration of justice as should be made by the parliament of Great Britain; and that thereafter none should be named by her Majesty, or her royal successors, to be ordinary lords of session, but such as had been in the College of Justice as advocates or principal clerks of session for the space of five years, or as writers to the signet for the space of ten years; with this provision, that no writer to the signet should be capable of being admitted a lord of session, unless he had undergone a private and public trial in the civil law before the Faculty of Advocates, and be found by them qualified for the said office, two years before he had been named to be a lord of session; yet so as the qualification made or to be made for capacitating persons to be named ordinary lords of session should be altered by the parliament of Great Britain.

The privileges granted to the judges of this high tribunal by the kings of Scotland were very considerable, and evince a great solicitude to render the administration of justice pure and unsuspected.† No lord can sit and vote in any case where the pursuer or defender is his father or father-in-law, brother or brother-in-law, son or son-in-law, or uncle or nephew.

Actions: Lord Stair considers the actions known to Scottish law as divisible under four heads.

declaratory; (1.) *Declaratory* actions, wherein the right of the pursuer is craved to be declared, but nothing is claimed to be done by the defender. Any defence that he might have used in the declaratory action, the defender is precluded from using in the petitory or possessory actions.

* "This court was called the Session, because it was to sit at such places as the king appointed, but was not to follow his court, as his council did before, nor to go through the kingdom as the Justice-General and Chamberlain did. By that title it was also distinguished from ecclesiastic courts, which were called consistories, where the judges did stand in administering justice" (Stair, IV. 1, 8).

† Thus, by the statute 1537, cap. 68, entitled *The King's good Mind anent the Lords of Session*, the king promises, "that he shall not by any private writing charge, command, or desire the Lords to do anything otherwise, in any matter that shall come before them, but as justice requires, or to do any thing that may break statutes made by them; and that he shall authorise, maintain, and defend their persons, lands, and goods from all harm, wrong, and hurt and injury, to be done to them by any manner of person; and who does in the contrary shall be punished with all rigour. And because they present his person, and bear his authority, in doing of justice, that he shall have them in special honour and maintenance, and shall give no credit to any man that will murmur them or any of them, by doing wrong or dishonesty; but they shall be called before him, and gif they be found culpable, to be punished therefor, after the quality and fault of the demerit; and if they be found clean and innocent, the person complaining shall be punished with all rigour, and never have credit with him again. Likeas, he exeems them frae all paying of taxes, contributions, and other extraordinary charges to be uplifted in any time thereafter. And grants to them, that gif any person dishonours and lightlies them, or any of them, any manner of way, that they command and charge and put the person or persons in ward in any of his castles they will, till they have made satisfaction of the fault at the Lords' consideration, if the fault be small and injurious; and gif it be great, till the king be advertised, that he may cause the same to be amended, and punishment made thereof as effeirs."

(2.) *Petitory* actions, so called because something is claimed to be done or suffered by the defenders, seek to establish a perfect right in the pursuer, whether it be real or personal, such as the restitution or delivery of moveables, and all poindings of the ground.

(3.) *Possessory* actions, or judgments, are such as claim the transfer, retention, or recovery of possession, and do not insist upon an absolute right. Of this kind are actions of spulzie, which are sustainable if the pursuer has a title only in possession. Spulzie is the unlawful taking away of moveables without consent of the owner, and the action seeks restitution with all possible profits,* or reparation. Wrongous intromission with moveables is another example. In neither of these cases is other than a lawful possession necessary as the foundation of the action.

Actions may also be subject to another division, familiar to European jurisprudence: they may be considered as either *real* or *personal*.

(1.) A *real action*, *actio in rem*, is founded on a right in the thing itself—*jus in re*, founded either on the right of property, *jus dominii*—or in the rights of servitude, hypothec, pledge (*jura servitutis, hypothecae, pignoris*), &c.

(2.) A *personal action* arises from a right of obligation against a person, or a *jus ad rem*. In the first case it can be directed only against the obligee or his heirs; in the second, against whomever possesses the subject in which the right exists, although they may not represent the obligee.

There appears to be another and technical division of actions, into ordinary and extraordinary; which, being founded on forms of process, we shall not further consider.

It will be well to describe briefly the various stages of an ordinary action; in which Mr. Swinton's clear and accurate sketch will be a useful and safe guide (see Spence's *Inquiry*, App. No. II. p. 569).

An ordinary action is instituted by an original writ called a *summons*, prepared and signed by a writer to the signet, and passed her Majesty's signet. The summons is of three parts:—

(1.) The *libel*, setting out the ground or circumstances on which the pursuer lays his action, *e. g.* for payment of an alleged debt.

(2.) The *decerniture*, or *conclusion*, stating the terms on which the pursuer desires a decree or judgment to be pronounced in his favour.

(3.) The *citation*, or *will*, is a warrant summoning the defender to appear before the court on a certain day, or to allege a reasonable cause to the contrary. The defender is served with a copy of the summons, enters an appearance, and having seen the summons and writs founded on, puts in written *defences*, containing simply a statement of the facts and pleas of law, and the documents evidencing them, upon which his defence rests. The pursuer is allowed to see the defences and documents, and then is the action debated.

The hearing takes place, in the first instance, before the *Lord Ordinary*, sitting in the *Outer House*, as it is called, who may pronounce either a definitive or interlocutory judgment, according to the nature of the case; or, if the case involve a question of difficulty, he may

* It has been held, however, that a tailor could not seek damages for the loss of his trade through the illegal taking away of his thimble and shears. The damage must be direct, and not consequential.

Law. appoint the parties to put in argumentative pleadings in writing, called *memorials*.

Condescend- If the parties differ as to the truth of facts, and a proof by parole testimony becomes necessary, two courses are open to the Lord Ordinary; he may judge of the relevancy of the libel, and allow a proof, after advising special pleadings to be put in for that purpose, which are called a *condescendence* and *answers*. In the one, the pursuer sets out backed by the facts he avers and offers to prove in support of his libel; in the other, the defender admits or denies the pursuer's averment, and makes counter-averments, if he has any. On the other hand, without determining the relevancy of the libel in *illo statu*, the Lord Ordinary may allow a proof before answer.

Proof. The *proof* is taken in the presence of a commissioner appointed by the Lord Ordinary, who examines the witnesses, commits their depositions to writing, and reports the whole, either to the Lord Ordinary or to the court, according to his directions.*

Representation. Whatever judgment the Lord Ordinary pronounces, the losing party may submit it to his review within a certain period, by an argumentative pleading in writing, called a *representation*. This representation the Lord Ordinary may appoint to be answered, the *answers* being also argumentative pleadings on the other side, sustaining the judgment. Upon this contest the Lord Ordinary pronounces an interlocutor, adhering to or contradicting his former judgment. If he adheres, a second representation is competent; and if it be appointed to be, and be answered, and the judge still

adheres to his decision, no further proceedings in respect thereof can take place in the Outer House. Law. If the Lord Ordinary should alter his opinion, two representations are also available.

The cause thus disposed of, may be brought under Reclaiming the notice of one of the two divisions of the Inner House petitions. by presenting a *reclaiming petition*, which is printed, and distributed among the judges, by whom it is either appointed to be answered or *simpliciter* refused. When the answers are put in, the case is enrolled for advice. The petition and answers having been considered by the judges at home, are decided upon in open court; and the decision can be again submitted to them by a second reclaiming petition; and if, on advising on this, their lordships adhere to their first judgment, the cause is final in this court. It is to be understood that the two judgments, either of the Lord Ordinary or of the Division, must be in favour of the same party, and in the same terms; for should any alteration be made in the judgment either of the Lord Ordinary or of the Division, it is competent for the losing party twice to represent or reclaim.

The Court of Session has an original and privative Jurisdiction in all actions of declarator for ascertaining rights, and in all actions of reduction for setting aside false or improbative writings. It has also the exclusive Court of Session. jurisdiction in processes of *cessio bonorum* and many other cases. It has large powers as a court of review, and, in addition to its ordinary jurisdiction, appears to have exercised a species of prætorian authority, called its *nobile officium*.

Lord Stair observes, that "new cases require new *Nobile cures*," and that the strict rules of law should often *officium*. be relieved by the equitable interposition of the judge. He adds, that "every sovereign court must have this power, unless there be a distinct court for equity from that for law, as in England." The Privy Council of Scotland formerly administered equity; and it appears to be thought that the Court of Session has succeeded to its powers in this respect. A few instances of the exercise of this power, as claimed, illustrate the subject. The Court of Session annexed a penalty for not observing certain regulations prescribed by a statute, asserting that it had the right thus to make additions to the statute, to render it effectual; but this decision was reversed by the House of Lords on appeal. It has appointed judges to fill a temporary vacancy; it has appointed trustees, those named having died or declined to act; it appoints factors, or guardians, to take charge of the affairs of those who by nonage, incapacity, absence, or other cause, might suffer loss, if their affairs were neglected.

From the decisions of the Court of Session an appeal lies to the House of Lords.

It cannot escape the most cursory reader, that in this hasty review of the law of Scotland, the criminal and ecclesiastical law, and correlative matters, have not been alluded to. This omission has arisen from a desire to consider more fully than would otherwise be practicable those doctrines of Scottish law which afford the greatest interest to the English reader.*

* For the materials of this chapter the writer is indebted chiefly to Lord Stair's *Institutions*, edited by Mr. More (1832), and Mr. Burge's *Commentaries on Colonial and Foreign Law*.

* Mr. More, in his learned notes to Lord Stair, has discussed some very interesting points relating to the Scottish law of evidence. The subject is one of such obvious value and general interest, that I must be permitted to state the substance of his remarks. In his opinion, the law of Scotland, as it existed no very long time back, treated all witnesses as incompetent, to whose evidence full and entire credit might not be given. The first consequence of this doctrine, and that not the least remarkable, was, that, as to such witnesses as the law recognised, full credence was to be given, the judge being relieved of the onerous duty he had in every court in Europe to perform, the duty of weighing well, and setting the true value upon, the testimony offered before him. The law itself had determined who was, and who was not, trustworthy; and more attention appeared to be paid to the number of witnesses than to the quality of their evidence. It excluded domestic servants, tenants, paupers, and all others who were supposed to be more or less subject to influence. It even excluded women. This rule, slightly relaxed, subsisted up to a late period; for in 1675 the Court of Session determined that women could not be adduced to prove certain facts respecting the loan of a bed; it being pleaded, "that albeit women be admitted witnesses at the bearing of a child, where men cannot be present, they were never admitted in any other civil process." "The lords found the reply of lending the bed relevant to be proved by habile witnesses, but refused to admit women witnesses." The rule became more and more relaxed, until women were held admissible as witnesses *ubi est penuria testium*. In 1736, women were not admitted to prove a grantor crazy, because there was no penury of witnesses. The rule is now considered as abolished; but the custom is, that men, and never women, should attest deeds.

Further, the rule of Scotch law which excluded so large a class of persons, must also have often prevented the only evidence capable of reflecting light on many transactions. The old rule still subsists to a certain extent. The father or son of a party, his uncle and nephew, his mother, brother and sister, and all other relations, uterine and consanguinean, and even his mother-in-law, stepmother, son-in-law, and brothers-in-law, are held to be generally incompetent witnesses for him. The whole of Mr. More's note is highly interesting and suggestive (Stair, vol. ii. p. 409).

L A W.

CHAPTER VII.

THE LAWS OF THE BRITISH COLONIES AND DEPENDENCIES.

Law. 292. THE Privy Council, previous to 1722, came to the following resolutions:—

Settled colonies. (1.) That if there be a new and uninhabited country found out by English subjects, as the law is the birthright of every subject, so, wheresoever they go, they carry their laws with them; and therefore such new found country is governed by the law of England; though after such country is inhabited by the English, acts of parliament made in England without naming the foreign plantations, will not bind them.

Conquered colonies, (2.) Where the king of England conquers a country, there the conqueror gains a right of property in such people, in consequence of which he may impose on them what laws he pleases. But,

and their laws. (3.) Until such laws are given by the conquering prince, the laws and customs of the conquered country shall hold place, unless where these are anything contrary to our religion, or enact anything that is *malum in se*, or are silent, for in such cases the laws of the conquering country shall prevail (2 P. W. 75).

The British possessions may be considered as acquired, (1) by settlement, and (2) by conquest; while there are others—the Channel Islands and the Isle of Man—the sovereignty of which has been otherwise obtained.

British colonies; settled, (1.) The colonies *settled* by English subjects are—New Brunswick, Nova Scotia, Cape Breton, Prince Edward's Island (about 1497), Barbadoes (1605), the Bermudas (1609), St. Christopher's (1623), Nevis, (1628), the Bahamas (1629), Gambia (1631), Montserrat and Antigua (1632), Jamaica (1655), Tortola and Anguilla (1666), New South Wales and Sierra Leone (1787), Van Diemen's Land (1803), and New Zealand (1839).

and conquered. (2.) The colonies acquired by *conquest*, or rather by capitulation, cession, or treaty, are—Dominica, Grenada, St. Vincent, Tobago, and Upper Canada (the possession of which was assured to this country by the treaty of 1763), Jamaica, the Falkland Islands, and we may add Gibraltar, though not properly a colony. The bounty of the crown, or the enactment of local legislatures, has conferred on these possessions the right to be governed by the law of England, as far as is suited to their condition. The remainder of our foreign possessions obtained by conquest are—Demerara, Essequibo, and Berbice, now united under one government, that of (1) British Guiana, (2) Cape of Good Hope, (3) Ceylon, (4) Lower Canada, (5) St. Lucia, (6) Mauritius, (7) Trinidad, (8) Malta, and (9) the vast possessions governed by the East India Company in Hindostan. The enjoyment of the respective laws of the dependencies has been assured to them.

Law. In those colonies in which the English law is said to operate, it operates so far only as it is suitable to the condition and circumstances of the inhabitants. It is, in short, rather the basis of their jurisprudence, inas- Operation of English law in the colonies. much as, in many cases, its provisions are modified, and its policy widely departed from by the enactments of the local legislatures and orders in council.

We must content ourselves, in this place, with pointing out the systems of law and their sources, which, Systems of colonial law. subject to the alterations introduced by local legislation, operate in the colonies whose native jurisprudence has not been obliterated by conquest.

(1.) In the Channel Islands there is the *coutume* of Normandy; (2) in the Isle of Man, a law peculiar to that island; (3) in Malta, the Roman or civil law; (4) in Lower Canada and St. Lucia, the *coutume* of Paris; (5) in British Guiana, the Cape of Good Hope, and Ceylon, the Roman Dutch law, *Rooms Hollands-regt*; (6) in Trinidad, the law of Spain; (7) in the Mauritius, the *Code Civil*, promulgated by a decree dated the 15 *Vendemiaire*, an 14, and the Code of Civil Procedure, adopted in July, 1808; (8) in British India, the Hindoo and Mahomedan laws.

What reflections are not suggested by this bare enumeration of distinct and dissimilar systems of law, governing states in habits and location so far removed from each other, and all owing their present existence to that respect for the natural rights of the conquered which has ever with us moderated the ardour of conquest! When we also remember that the authority of these laws is permitted to extend not only to the conquered people, but to the conquerors; that they are not *personal** laws, compulsory only on those whose allegiance to them was native and original, but that they define the rights and regulate the conduct of our natural-born subjects, we cannot fail to be struck by the contrast our colonial policy offers to that of the great colonizing people of antiquity, who, while they permitted the subjugated race to enjoy their own institutions, never degraded the dignity of a Roman citizen by compelling his obedience to a foreign jurisprudence.

* When the Goths, Burgundians, Franks and Lombards founded kingdoms in the countries formerly subject to Rome, they left to the Romans the enjoyment of their own laws; the conquering people preserving theirs. Thus, in the same city, the Lombard lived under the Lombardic, and the Roman under the Roman law, and this gave rise to that condition of civil rights denominated *personal rights* or *personal laws*, in opposition to territorial laws. "It often happens," observes Bishop Agobardus, in an epistle to Louis le Debonnaire, "that five men, each subject to a different law, may be found walking or sitting together." Savigny, *Geschichte des Römischen Rechts im Mittelalter*, b. 1. k. iii. s. 90-1.

Law. *In jure Romano non est mirum nihil hæc de re extare, cum populi Romani per omnes orbis partes diffusum et æquabili jure gubernatorum imperium conflictui diversarum legum non æque potuerit esse subjectum* (2 Huber, lib. i. tit. 3, p. 538). The policy of England has been very different, and we see its fruits in the increasing importance of our colonial possessions, and their undiminished affection to the parent state.

Another reflection this enumeration of the sources English colonial law naturally suggests to the mind, is the wide-spread influence of Roman legislation, surviving the subversion of Roman greatness, and extending to countries of which Rome knew not the very existence. "*Servatur ubique jus*," epigrammatically enough, says a great jurist, "*non ratione imperii, sed rationis imperio*."*

Coutumes of Normandy and Paris. The *coutumes* of Normandy and Paris, the basis of the law operating in the Channel Islands, in Lower Canada, and St. Lucia, are in writing, and published, like the rest of the customary law of old France—a work commanded by Charles VII., after he had expelled the English, and which languished until the reign of Louis XII., and was not finally completed until 1620. The *coutume* of Normandy is best known by the edition and commentary of Basnage. Of the *coutume* of Paris, made the law of the French West India colonies by the 33rd article of the *Arrêt* of the *Conseil d'Etat du Roi* of May, 1664, a brilliant but superficial writer has said, that it is interpreted by twenty-four commentaries, agreeing in nothing but the proof of how ill the law was devised.† In Lower Canada some of the ordinances of the French kings also have effect.

Roman Dutch law. The Roman Dutch law consists of the civil law and the ordinances and edicts issued by the supreme power in Holland.

Spanish law. The Spanish law consists of—(1) the *Fuero Real*, the work of Alonzo the Wise, in 1255, but designed by his father, Ferdinand III., and intended as a precursor to (2) the *Partidas*, according to a writer quoted by Mr. Burge, "a complete body of law (*el cuerpo completo*), which combines the public with the private law, all digested and precompared in a most scientific, just, solid, Christian, and equitable manner, and which has not, perhaps, its equal in all Europe;" (3) the *Recopi-*

lacion, the first part of which was published in 1567, by order of Philip II., and which includes *Las Leyes de Estilo*, promulgated in 1310, as altered by the *Pragmaticas*, the *Ordeniamento Real*, authorized and published by Ferdinand and Isabella in 1496, the *Ordeniamento del Alcala*, made and authorized in 1348, the law of the Cortes of Toro in 1505, and other laws, and the *Autos Accordatos*, published in 1745; and as to the colonies, (4) the *Cedulas Reales*, or instructions to the governors of Spanish colonies; and (5) the *Recopilacion de las Indias*, a collection of the laws of the Spanish settlements.

Such, with the *Code Civil*, are the chief sources of English colonial law: and we have now to allude briefly to the sources of the laws of British India.

In allusion to a king who has effected a recent conquest, "let him," says Menu, the great Hindoo legislator, "establish the laws of the conquered nation as declared in their books" (ch. vii. 203). A British act of parliament has declared, in reference to the dominions of the East India Company, that "it is expedient that the inhabitants should be maintained and protected in the enjoyment of their ancient laws, usages, rights, and privileges" (21 Geo. III. cap. 70). To a great extent the wish of the legislature has been complied with; and while criminal justice is administered according to a code promulgated by British authority, and while all that relates to the conduct of suits, the admissibility of evidence, and the competency of witnesses is defined by British law, still in all matters relating to the two great heads of inheritance and contract, the king's courts in India are to accord to the natives the benefit of their own law.* Thus, in these respects, the Hindoo is governed by his own law, the Mahomedan by that to which he had anciently owed allegiance; but the Parsee (Persian emigrants, settled for above a thousand years in Bombay), having properly no law of his own, is governed in a great measure by that of England. The Hindoos exceeding by far in numbers either of the other races, it will be Hindoo law alone which will be here considered.

The Hindoo claims for his law a divine origin. The their books to which he ascribes this character contain either sources; that revelation which he calls *sruti* (*audition*), or revealed law, and which he believes to have been wholly

* The grounds of the authority of the civil law in Holland, as recited by Groenewegen, illustrates the source of its influence in modern Europe. *In Belgio autem multo majorem auctoritatem obtinet jus Romanum: cum enim passim summâ ratione atque æquitate abundet, olim pro justitiæ et æquitatis exemplo in judiciis allegatum, tandem vero etiam pro legibus receptum est atque etiam num regulariter legum vim atque auctoritatem apud nos obtinet* (Ad. Inst. lib. i. Proem. n. 3). So says Dumoulin in reference to France. *A jure scripto Romano mutuamur quod et æquitati consonum et negotio de quo agitur, aptum congruumque invenitur, quod non unquam fuerimus subditi Justiniano magno aut successoribus ejus sed quia jus illo auctore à sapientissimis viris ordinatum, tam æquam est tam rationabile et unde quaque absolutum ut omnium ferè Christianarum gentium usu et approbatione commune sit effectum* (Gloss. n. 150).

† Voltaire, *Dict. Phil.*, art. *Lois*. Mr. Park gives a list of thirty-seven commentaries and gloss on the *coutume* of Paris, and twenty-seven on that of Normandy (*Contre Projêt to the Humph. Code*, pp. 249-52). *Il n'étoit possible en France*, says a learned English jurist, *même aux jurisconsultes les plus profonds d'apprendre à connoître cette multitude d'usages alors en vigueur dans nos provinces*. And he adds, in a note, *c'est par rapport aux nombreuses coutumes, si souvent opposées, qui étoient alors en vigueur dans l'ancienne France, que Montaigne a dit, qu'il y avoit en France plus de lois que dans tout le reste du monde ensemble, et plus qu'il n'en faudroit pour régler tous les mondes d'Epicurus* (*Lettres sur la Cour de la Chancellerie*, p. 297). Of old France, Napoleon said, that it was *plutôt une réunion de vingt royaumes, qu'un état*.

* To have described the judicial system of India, would have required more space than we can afford. It may be mentioned, that in the north-west provinces of Bengal, which now constitute the Agra government, civil suits, to which Europeans are not parties, and where the amount in litigation does not exceed 30*l.*, are decided by a native judge, or *moonsif*. If the *res litis* is under 100*l.*, the suit is decided in the *Sudder Ameen*; and above this is the principal *Sudder Ameen*, unlimited in its jurisdiction, as far as the amount of the subject litigated is concerned, and is a court of appeal from the other courts. Above these are the district judges (Europeans), who preside over the *Zillah* courts, from which an appeal lies to the appellate court, the *Sudder Dewannee Adawlut*, who also receive appeals from the principal *Sudder Ameens*. The average duration of a suit in these provinces is four months and ten days. The proportion of suits to population, in Bengal, is about one suit annually to 381 persons (*Paper* by Colonel Sykes, Stat. Soc. London, 20th March, 1843). In 1793, the *Zillah* courts were established in Bengal, as also the court of *Sudder Dewannee Adawlut*, and the provincial courts of appeal of Calcutta, Patna, Dacca, and Moorshedabad. It may, perhaps, be proper to observe, that in doubtful matters the European courts take the opinion of Pundits or Hindoo jurists. Sir William Jones, declared, however, "that he could not with an easy conscience concur in a decision merely on the written opinion of native lawyers, in any case in which they could have the remotest interest in misleading the court."

Law. dictated by divinity, or *smriti* (recollection), or remembered law, preserved by inspired writers, who had revelation present to their memory, and who have recorded holy precepts for which a divine sanction is presumed. Forensic law, as distinguished from religious doctrine and ritual law, is contained chiefly in the *Smriti* or *Dharma Sastra*. This is embodied in the writings of ancient authors, such as Menu, with whom, in most points, most of them concur; but the number of authors entitled to this rank in Hindoo jurisprudence is a matter of doubt.

their species; Law has been cultivated as a science by this singular people, and we find among them divers law-schools, each of which acknowledge the authority of different commentators upon their ancient books. Indeed, the authority of these commentators appears, in courts of law, to have superseded in a great measure the authority of the writers they profess to interpret; and as these commentators differ among themselves, and our law-courts in a great measure follow the commentator approved by the school prevailing in the district in which they exist, it is obvious that there is properly no one system, but in truth many systems, of Hindoo law, operating simultaneously in various parts of the country.

and schools of interpretation. The written law, whether *sruti* or *smriti*, is subject to the same rules of interpretation. Those rules are collected in the *Mimansa*, which is a disquisition on proof and authority of precepts. It is considered as a branch of philosophy, and is properly the logic of the law. In the eastern part of India, *i. e.* Bengal and Bahar, where the *Mimansa* is less studied than in the south, the dialectic philosophy, or *Nyaya*, is more consulted, and is there relied on for rules of reasoning and interpretation upon questions of law as well as upon metaphysical topics. Thence result two schools or sects, which, by a various construction of the same text, deduce different inferences; and each of these schools is again divided into other schools, each adopting for its chief guide a particular author, to whose doctrines they give currency in particular districts. The works of these authors are *Digests* or *Commentaries*; and it appears to have been the policy of each successive dynasty in India to have issued some work of these descriptions, containing the rules by which they proposed to carry on the government.

Colonial judicial system. 293. Without referring to the complex judicial system of British India, it may be remarked, that justice is administered in our colonies by various courts of justice, empowered by royal charter or acts of the colonial legislatures. In Demerara, Berbice, the Cape of Good Hope, Ceylon, Trinidad, St. Lucia, Lower Canada, Newfoundland, the Mauritius, New South Wales, Van Diemen's Land, Western and Southern Australia, Jersey, and Guernsey, the supreme courts are courts of equitable as well as legal jurisdiction. They grant probate of wills and letters of administration, and have the custody of lunatics and the appointment of the guardians of minors. In the other colonies the equitable jurisdiction is administered by the governor, who,

Law. as ordinary, grants probate and administration. In Upper Canada is a Court of Probate, where the governor presides; and the civil list of Canada, settled by the 3 & 4 Vict. cap. 35, contemplates the appointment of a vice-chancellor—an office lately created in Jamaica. In most of the colonies there are courts of appeal (which probably in the West Indies will merge in those contemplated by the 6 & 7 Will. IV. cap. 17), the Judicial Committee of the Privy Council, established by the 3 & 4 Will. IV. cap. 41, being the ultimate appellate jurisdiction, when the *res litis* is of a certain amount. It receives appeals, also, from the *Sudder Dewannee Adawlut* courts, or supreme native courts of British India.

294. Legislative authority in the colonies is exercised, Colonial legislation. in some, by the king in council, and the laws are issued in the shape of orders in council; in others, and in the major number, by a body consisting of persons nominated by the crown, and of representatives chosen by qualified persons, each class (except in the case of the legislature of Newfoundland; 5 & 6 Vict. cap. 120) sitting apart. That the measures they pass should become law, they must first receive the assent of the governor. They may, however, be afterwards disallowed by the king. All laws, bye-laws, usages, and customs which shall be in practice in any of the plantations, being repugnant to any law made or to be made in this kingdom relative to the said plantations, shall be utterly void and of none effect (7 & 8 Will. III. cap. 22). The governors of colonies also, by their instructions, are forbidden to assent to certain laws which are supposed to be inconsistent with our public policy. Thus they are forbidden to assent to any act of their legislatures for the dissolution of a marriage. It is not to be understood that the establishment of a colonial legislature limits or affects in any way the rights of the Imperial Parliament to legislate for the colonies—a right asserted as far back as the time of the Protectorate (Scobell, *Ordin.* part ii. p. 132). But except on an exigency, it rarely exercises its extreme rights, leaving to the colonial legislature the duty of framing the laws suitable to the condition of its constituents.

It may be remarked in conclusion, that in British India the right of legislation is vested generally in the Governor-General, provided that in its exercise he does nothing to affect the mutiny acts, the prerogative of the crown, the authority of parliament, the rights of the East India Company, or any part of the unwritten laws or constitution of the United Kingdom, whereon may depend the allegiance of any person to the crown of the United Kingdom, or the sovereignty or dominion of the crown over any part of India. The Directors of the East India Company may disallow the laws made by the Governor in council; and it is to be remembered that the king's Commissioners for the Affairs of India control all acts of the Company concerning that country, and all official communications are to be approved by them; so that in fact, the crown, through its servants the Board of Control, has the right of disallowing all laws made in India (3 & 4 Will. IV. cap. 85).

OUTLINES OF THEOLOGY.

INTRODUCTION.

Theology. (1.) UNDER the general term *Theology* we comprehend all the knowledge which man can obtain respecting God, whether concerning His nature and attributes, or concerning the relation in which man stands to Him. This subject we shall divide into two portions, making the sources of our knowledge the basis of our distinction. These portions are *Natural Theology* and *Revealed or Scriptural Theology*. Under the first we propose to treat of that knowledge of God which the contemplation of His works and His providential care, as seen in the natural world, is calculated to instil into the mind of man. Under the latter we propose to ascertain what revelation God has been pleased to make of Himself to mankind, and what the Holy Scriptures teach us concerning the nature and attributes of the Deity, and the relation in which man stands to Him.

The science of Theology is very differently divided by different authors, but without any very great difference as to the subjects of which they treat under this name. Thus a common division (justly blamed by Limborch, *Theologia Christiana*, p. 1; comp. also Buddei *Institutiones Theologiæ Dogmaticæ*, lib. i. ch. i. sec. 39,) used to be that into, 1, *Archetypal Theology*, or that knowledge which God himself has, concerning Himself and divine things; and, 2, *Ectypal*, or that knowledge which, being derived from the former, is communicated from God in three different ways: 1, by the *Hypostatical Union*, to Jesus Christ. 2, by *Vision*, as to the Angels. 3, by *Revelation*, as to men. Turretin calls the former *Infinite Theology*, the latter *Finite*, and together they form his subdivision of *True Theology*, the whole subject being divided in *True* and *False Theology*, the latter including both *heathen* and *heretical Theology*. In Thomas Aquinas the first sections of the first part of his *Summa Theologiæ* treat of the nature and objects of Theology (under the name of *Sacra Doctrina*), as, for example, whether it be a science, &c. (Comp. also P. a Joseph *Idea Theologiæ Speculativæ*.)

Again, Turretin (p. 16 of his *Institutiones Theologiæ Elencticæ*) affirms that God is the *object* of Theology, which is denied by Limborch, who considers that the *worship* of God is the proper *object* of Theology; but this difference arises rather from the different senses in which they have used the word *object*. Thomas Aquinas very beautifully says Theologia "*a Deo docetur, Deum docet, ad Deum ducit*." (Comp. Buddeus, *ubi supra*, sec. 40-46.)

Again Buddeus defines *Theology*, a "knowledge of divine things, necessary to be known by sinful man in order to his salvation, joined to a power of teaching and defending the same." He divides it into *Theoretical* and *Practical*; lib. i. sec. 42. (Comp. also his *Institutiones Theologiæ Moralis Prolegom. passim*.) Another common division in Germany is into *Theology*, *Anthropology*, *Christology*, and *Eschatology*, the latter word meaning the doctrines concerning the *last things*, i. e. the Judgment, &c.

Some writers use the words *Theology* and *Religion* in the same sense, others distinguish between them. See some very able remarks in Dr. Turton's *Natural Theology* (p. 1-33) on this point. Bishop Marsh (*Lectures*, part ii. sec. ii.) has some useful suggestions relative to a systematic course of Theology.

With these references to common works on Theology we dismiss the consideration of the division of the subject, as it does not appear of much practical use, and certainly does not admit of proper discussion within our limits. (Comp. Buddeus' *Theol. Dogm.* lib. i. sec. 38.)

(2.) The subject here proposed for our consideration being of the deepest importance to man, has been treated of at greater length than most other depart-

ments of human knowledge. The inquiries, whether there be another state of existence after the present, and if there be, on what terms happiness is attainable there; whether man be an object of care to the Creator of the universe, and if he be, by what means he may become acceptable to this great Being; whether God has revealed himself to man, and if he has, what is the substance of that communication; these are questions of overwhelming interest to a rational and accountable being, and we cannot wonder, therefore, if every circumstance relating to the evidences for Natural and Revealed Religion, and to the interpretation of Scripture, has been examined and discussed with a minuteness and to an extent unequalled in any other science. We cannot, therefore, within the brief limits to which we must confine ourselves, profess to give more than a mere outline of the great truths which this science discloses to us. But our sincere and earnest desire is, that this portion of our work may prove of *practical* utility, by bringing forward, as far as our limits allow, the chief results which a just estimate of evidence, and a true interpretation of Scripture, present to man, and by indicating, where it shall appear desirable, the best sources for pursuing solid inquiries, and forming a right judgment on many questions of surpassing interest and importance.

1. For a general *Bibliography* on this subject, we may refer to Bishop Wilkins' *Ecclesiastes*; Bray's *Bibliotheca Parochialis*; Wotton's *Thoughts on a proper Method of studying Divinity*, 1734 (and reprinted 1818); and to the very valuable and copious bibliographical volume of Horne's *Critical Introduction to the Study of the Scriptures*. Among foreign works on Theological Bibliography, we may specify Walch, *Bibliotheca Theologica*; and in more recent days, the very useful systematic catalogues of Ersch, *Literatur der Theologie* (by Böckel, 1822); Winer's *Handbuch der Theologischen Literatur*, 8vo. 2 vol. 1838; and Clarisse, *Encyclopædia Theologica Epitome*.

2. On referring to the Miscellaneous Department, in BAPTISM, BISHOPS, CANON, BIBLE, &c., and to the Historical Division, under the titles APOSTOLIC AGE, MINISTRY OF CHRIST, MIRACLES, ECCLES. HISTORY, *passim*; HERESIES of the 1st, 2nd, 3rd, and 4th Centuries, &c., the reader will see that much more space has already been devoted to many Theological questions than they can usually obtain in works of this general nature. A repetition of those statements would be neither just nor satisfactory. Many branches of inquiry, therefore, which usually belong to Treatises on Theology having been anticipated in this work, we shall pass over these with brevity, while we supply at greater length what has not yet been touched upon, and thus endeavour to consolidate our Theological articles into a systematic form.

(3.) Having, therefore, considered Natural Religion, and having adduced some evidence to show that the Bible is a Revelation from God, we shall endeavour to point out the main substance of that revelation, and briefly to show what Scripture teaches—

1. Concerning God;
2. Concerning man as an individual;
3. Concerning the Church;
4. Concerning the Judgment, and the final condition of man.

CHAPTER. I.—NATURAL THEOLOGY.

The Power, Wisdom, Goodness, and Justice of God, &c.—His Government.—Man's Duties.—His Failure in them, and consequent Need of a Revelation.

Theology. (1.) In a former portion of this work (see ATHEIST) we took occasion to show that all who adopt the Atheistic Hypothesis necessarily involve themselves in endless difficulties and contradictions. Among other improbabilities, the Atheist has to assume that the present state of things must have existed from eternity, and to regard all that takes place in the universe as the result of accident or of blind necessity. For if it be admitted that there ever was a period in which the universe did not exist, it is not easy to avoid the acknowledgment that there must have been a creation, and, as a necessary consequence, that there must be a Creator, who is independent of all matter, and possessed of such attributes or qualities as are adequate to produce all creation out of nothing. It is this Creator whom we call God. Assuming, therefore, the existence of God, the main object of THEOLOGY is to ascertain the attributes and perfections of this First Great Cause, from whatever sources information respecting Him can be derived; and then to develop, as far as may be, the relation in which the Creator stands to his works. We shall thus, in some degree, be brought acquainted with God; with His designs respecting His intelligent creatures; and with our consequent duties towards Him, ourselves, and our fellow-men.

Observation 1. We do not notice the speculations of the German metaphysicians respecting the arguments for the existence of the Deity, because their views are but little relevant to our subject, and, besides, belong more properly to the department of Metaphysics. In the prosecution of the object we have in view, we are as little concerned with the speculations of Kant, and others of the German school, as the writer of a practical treatise on mathematics would be with a metaphysical disquisition on the nature of mathematical demonstration. We content ourselves, therefore, with referring to Kant, *Kritik der reinen Vernunft*, div. iii. sec. 3-7, p. 611-658, edit. 1794; and for some remarks on Kant's views, to a former part of this volume (*Moral and Metaphys. Philosophy*, p. 665-667). See also the Essay, "Ueber Begriff und Möglichkeit der Philosophie," by Professor Schmidt of Rostock, especially sec. 121-124.

Obs. 2. In demonstrating the existence of the Deity, English writers have adopted two modes of argument, the one called *a priori*, the other the argument *a posteriori*.

The argument *a priori*, as it is improperly termed, (See Turton's *Nat. Theol.* p. 236-276, but especially p. 242-245,) is that used by Dr. Clarke. It is not an argument *a priori* to reason that there must be such a being as God, when we argue back from the simple fact that something now exists, to the conclusion that something must always have existed which cannot be a contingent being. This is not reasoning *a priori*, it is only prior to the examination of particular phenomena.

On this argument, see Waterland's Dissertation on the argument *a priori*, &c., and Gretton's Review of the argument *a priori*. Dr. Turton states that he has materials for a complete history of this argument. The publication of such a work would be a great benefit to Theological literature.

The argument *a posteriori* is that which is otherwise called the argument from design.

Against this reasoning it has been contended that the very word *design* implies a *designer*, and its use is therefore a *petitio principii*.

The sufficient answer to this objection is, that no one wishes to use the word *design* till he has proved the existence of that which is necessarily called by this name. He states phenomena, which must result, he contends, from design. Here is the strength of the proof, and its first step, which is to infer design.

The next step is almost a self-evident proposition; if there is a design, there must be a designer. We would here merely refer to the strong expression of Sir J. F. W. Herschell, (*Discourse of the Study of Nat. Philosophy*, p. 7,) that the inquiries of Natural Philosophy, considered in reference to the existence and principal attributes of the Deity, have rendered "doubt absurd, and Atheism ridiculous." The pretended proof of La Place, that the law of gravitation (the only law which would, as far as we know, hold our system together) is a necessary deduction from the properties of matter, has frequently been considered and answered. See Robison's *Mechanical Philosophy* and Whewell's *Bridgewater Treatise*.

The books which have been consulted for the compendium of Natural Theology given in this chapter, are chiefly these:—

Bishop Wilkins's *Principles of Natural Religion*. Dr. Samuel Parker's *Demonstration of the Divine Authority of the Law of Nature*. Cudworth's *Intellectual System*. R. Baxter's *Reasonableness of the Christian Religion*. Archbishop Tillotson's *Discourses upon the Attributes of God*. Bishop Cumberland's *Treatise on the Laws of Nature*. Wollaston's *Religion of Nature Delineated*. S. Clarke, *on the Being and Attributes of God*. Archbishop King's *Origin of Evil*. Bishop Conybeare's *Defence of Revealed Religion*. Bishop Butler's *Analogy*, &c. J. Bowen's *Series of Discourses on the Principles and Evidences of Natural Religion*. Tunstall's *Lectures on Natural and Revealed Religion*. Tucker's *Light of Nature Pursued*. Abernethy's *Discourses on the Being and Natural Perfections of God*. Leland's *Deistical Writers*. Leland's *Advantages and Necessity of Christian Revelations*. Knight, *On the Being of a God*. Brown's *Essay on the Existence of a Creator*. Knowles, *The Scripture Doctrine of the Existence and Attributes of God as manifested by the Works of Creation*. Paley's *Natural Theology*. The *Bridgewater Treatises*. J. Foster's *Discourses on the Principal Branches of Natural Religion*. Crombie's *Natural Theology*. M'Culloch's *Proofs and Illustrations of the Attributes of God*. Lord Brougham's *Discourses of Natural Theology*. Dr. Turton's *Natural Theology*, considered in reference to Lord Brougham's *Discourse on that subject*.

We may also mention Bentley's *Sermons*, Barrow's *Works*, Ray's *Wisdom of God*, &c., and Derham's *Physico-Theology and Astro-Theology*, as being useful works for these points; though the two latter are superseded by Paley and other similar works. Fabricius, in his *Delectus Argumentorum et Syllabus Scriptorum qui veritatem Religionis Christianæ, &c. asseruerunt* (Hamburg, 1725), gives a full account of ancient and foreign writers who have handled this argument.

(2.) Now when we come to examine from whence our knowledge of God may be derived, it is evident that sources of information may be either natural or supernatural; either that light which is reflected so strongly from the world around us and within us, or direct revelation from God himself. We thus arrive at that two-fold division of Theology which is popularly expressed by the terms NATURAL and REVEALED RELIGION; and our endeavour shall, therefore, be to discuss the subjects according to this order. Let it be premised, however, that in endeavouring to ascertain, apart from revelation, some of the attributes of the Deity, our aim will be rather to direct attention to what may be gathered respecting the Supreme Being from the works of creation than from metaphysical considerations or abstract reasonings upon the necessary perfections of an eternal self-existent Creator. We have decided upon this course, partly because, abstractedly considered, most of the divine attributes are, from the very nature of the case, placed beyond the possible range of man's inquiries; but chiefly because

Theology. the visible creation, which is open to the inspection of every man, bears such marks of the Godhead, that all who do not read there palpable traces of his eternal power and goodness, must be regarded as without excuse.* Thus, also, as we are better able to gaze upon the sun through some intervening medium than with the naked eye, so may we more readily discern somewhat of the Creator's attributes and perfections, as He has revealed himself by means of his works, than by contemplating him as He exists in his essential majesty and glory.†

The power
of God.

(3.) The most cursory and unlearned view of the creation, then, suggests abundant evidence of the Divine Power. There is no perfection of the Deity so often marked and attended to by mankind as this. Nations who entertain but faint and imperfect ideas of any other of the Divine attributes, have yet vivid apprehensions of a supreme invisible Power. This perfection of God is to be read most plainly in the storms and convulsions of nature; yet it may be traced also in the structure of man's body, and in the world around us, in all its extent, variety, and teeming animation; whilst the vault of heaven above, studded with light and glory, bears indisputable testimony to the same attribute of the Creator. Or if the works of creation be subjected to the scrutiny of science, the result only reveals more fully the Maker's power. There is seen, that though the earth be peopled by countless myriads of living creatures, yet all were formed with surpassing care; all made accurately to observe their generic and specific distinctions; all depending for their subsistence on Him who created them. Then the firmament above, which at first might seem to be but a vast assemblage of heavenly bodies, wonderful indeed in appearance, but grouped together in indiscriminate splendour, is found to be made up of orderly systems, each subject to its own laws, all combining in one harmonious whole; some occupying positions that man's skill can define; others stretching out into an infinity of space, from the exploring of which the mind shrinks in conscious feebleness. We thus become impressed with the conviction that the power of the Creator is beyond comparison greater than any which we feel ourselves to be capable of exerting, or observe any other agents with which we are acquainted to possess. For whilst man finds his ability to act, continually and effectually controlled, our knowledge of creation, limited though it be, gives us no reason for supposing that any thing is beyond the possibilities of the power of the Deity: every thing that falls under our observation being the work of His hands; all the materials of which the universe is framed having been called by Him out of their previous nothing into their present existence. So far, therefore, as any judgment at all can be formed of the power of the Creator from the contemplation of the universe, the wise of all ages have considered themselves warranted to conclude that *omnipotence* is an attribute of the Deity.

The wis-
dom of
God.

(4.) The belief in an Almighty Creator requires, also, that we regard Him as possessed of knowledge in every respect adequate to the work of creation. Whatever the extent of creation may be, known to Him who made all things must have been the whole system of the universe, in all the details of its arrangements, and

in the effects which can result from all possible combinations. There can, in fact, be no limits assigned to the intelligence of the Deity. It is upon this boundless knowledge, or *omniscience*, of the Supreme Being that the Divine *wisdom* is based, which is so conspicuously displayed in the visible creation. Let but a man, for instance, consider his own bodily and mental constitution, and he cannot fail to be struck with wonder at the thought that every one of his members is constructed in a manner best adapted to the discharge of the functions it is destined to perform; each part so arranged as best to receive the nutriment proper unto it, and to minister at the same time most effectually to the general purposes of human existence. Then, in mental endowments, how admirably adapted is man for his position in the creation! With powers of mind that qualify him for accomplishing the great results, by the mutual aid and co-operation of each with the other; with sympathies and affections, endowed with hopes and fears, and passions, without which mankind would not have been fitted for the various relations of social life, it is impossible for us, in all these adaptations, to overlook the wisdom of God! And if we turn our attention to the brute creation, we find the same wisdom displayed in those instincts and habits which tend to secure the irrational creatures from injury, and lead to the sustentation of life, and the continuance of their species. And how wonderfully has the material universe been formed, by climate, by seasons, and by all its treasures of natural productions, to minister to the well-being and enjoyment of every living creature on the face of the earth, whether man or beast! The deeper and more accurate our researches, the more boundless and sure do the evidence of the Creator's wisdom appear. Every thing is seen in its exact position; so that, in its individual order and operation, nothing interferes with the design and use of other things. Nothing is redundant, nothing defective, nothing disproportionate; but all is symmetry and perfection! The most complicated movements seem to minister only more effectually to the simplicity of others; counteractions are met by some antagonist influences; uniformity is diversified by infinite variety.* In a word, on whatever portion of the universe we look, there are presented to us such manifestations of order, arrangement, and design, for the accomplishment of certain well-recognised purposes, that the intellect of man fails to discover any other means by which the same results could be so effectually secured. The contemplation, therefore, of all this exquisite, superhuman contrivance, and matchless adjustment of every thing to its own end, cannot but lead to the conviction that, as the Lord is great, and great is His power, so His *wisdom* is *infinite*. (*Psalm cxlvi. 5, Prayer-book Translation*)

(5.) And whilst we thus infer the wisdom of the Deity from the manifest order of the universe, and from the adaptation of every thing to its individual purpose in the general system, we may not obscurely gather, from the uniformity of design and harmony of operation which pervade all nature, that the whole proceeded from One designing Power. So far as our researches are capable of extending, we perceive the same laws every where to obtain, the same effects following the same causes. Wherever, for example, we

Theology.

The unity
of God.

* *Acts*, xiv. 17; xxvii. 24; and *Rom.* i. 19, 20.

† Tucker, *Light of Nature*, ii. 31.

* McCulloch, *On the Attributes*.

Theology. can discover motion, the laws of motion are every where the same. The day and night, summer and winter, seed-time and harvest every where alike succeed each other with the same unbroken regularity. Provision for the sustenance and propagation of organized and animated being is made, throughout the universe, with the same manifest regard to those analogies and proportions which are the natural characteristics of one harmonious design. It may be granted, indeed, that the unity of design and operation which we thus trace in the natural world does not conclusively prove the *unity of God*, yet it may be legitimately maintained, that so far as any illustration of the Divine attributes are to be collected from the universe of mind and matter, it is but natural to infer unity of agency from unity of design. Even Mr. Hume, therefore, acknowledges, that "were men led into the apprehension of invisible intelligent power by a contemplation of the works of nature, they could never possibly entertain any conception but of one single Being, who bestowed existence and order on this vast machine, and adjusted all its parts according to one regular plan or connected system."*

Omnipresence.

(6.) Then, again, as the universe is a work of the Creator's power, made up of systems countless in number, and related to beings infinite in multitude, there is no region to which our knowledge can extend, or of which our imagination can conjecture, that is not subject to the Creator's notice. For wherever creation exists, there is necessarily found the operation of such laws as are calculated to secure the results toward which each portion of creation is destined to contribute. "But a law cannot administer itself: all law, on the contrary, always supposes an agent; and without the presence of an agent that is conscious of the relations in which the law depends, and competent to produce the effects which the law prescribes, the law must be without efficacy. Hence we infer that the intelligence by which the law was at first ordained, and the power by which it is put in action, must be present at all times, and in all places where the effects of the law occur; that the agency of the Divine Being pervades every portion of the universe, producing all action and passion, all permanence and change."† In other words, every portion of organized or unorganized matter, of corporal or spiritual substance, is superintended by an *omnipresent* God with the same vigilance as if it were the sole object of the Divine attention.

The goodness of God.

(7.) But if God be thus a Being of infinite power, boundless in wisdom, and who is every where present, it becomes a matter of great importance to ascertain whether or not these Divine attributes are exerted to a benevolent end. And that *goodness* is an attribute characteristic of the Deity, may partly be inferred from the fact, that all the inanimate portions of the creation, which are themselves incapable of either pleasure or pain, are yet manifestly formed to minister to the happiness of sentient and intelligent beings. It has not less truly than eloquently been observed, that "every sun-beam that lightens man's brow, and cheers his spirit; every flower of the field that opens its blossom grateful to the smell, and delighting the eye; every shower that falls to refresh the earth; the genial atmosphere; the verdant meadow; the perennial water-spring; all ministering to the pleasure or neces-

sities of the animate creation, prove the benevolence of the great Creator."* Then, there is the "aptitude of every sentient being to the circumstances in which it is placed; the provision made for its support, its defence, and its enjoyments; the sources of gratification with which man and all other sentient beings are furnished:"† all these separately and collectively testify that God is good, for all things might have been differently ordered. The very extensiveness of the Divine bounty may render our perception of it less vivid; and the common benefits of our nature may escape our observation, because they are common: nevertheless, there is no age or nation on which the Deity has not left visible traces of His fatherly care and goodness towards the works of His hands. It need not be denied that innumerable evils present themselves to our notice, in those wars, and famines, and pestilences, and individual distresses that visit the world, in the crimes of mankind, and in the sufferings of the brute creation. But whilst imperfection is inherent in every thing that is not Divine, and moral evil, and therefore disorder, seems to be the possible attendant upon the permission of free agency in intelligent beings, it may safely be maintained that good is, by the appointment of God, made greatly to preponderate over evil; and that the very sufferings which are permitted to exist, only prove the more powerfully illustrative of the benevolence of the Almighty. As, moreover, the human mind is too limited in its range to be able fully to comprehend, or even trace out, the various connexions and dependencies of the universe, we are not in circumstances to decide with certainty upon the reasons and results of the Divine operations. Those things, therefore, in creation, or in the order of Divine Providence, which sometimes appear to us to be imperfect or evil, may really, though we discern not how, be working some beneficial result; nay, may be absolutely necessary for securing the completeness and happiness of all God's works.

(8.) To regard, however, the Deity as a Being of essential goodness, notwithstanding the existence and prevalence of evils, implies that we are impressed with the conviction, that of all God's creatures, each has assigned to it that exact measure of good and evil which leaves no room for suspicion that the Deity has acted with partiality towards any. In other words, we are accustomed to regard the Deity as a Being of unimpeachable equity. Indeed, when we see, in the general frame of nature, such importance assigned to individual operations, as that whilst every thing contributes to the harmony of the universe, that harmony can be maintained only as the influence of each thing by itself is preserved in its integrity; we trace in the wisdom that thus contrived all things, the equity, also, which provided that nothing in creation, however apparently insignificant, should be without its due advantage and honour in the system. Or, again, as it cannot but be admitted that the Creator stands in the same relation to all His works, He having, for his own good pleasure alone, alike created all, there is no apparent reason why the Deity should, from partiality, prefer one of his creatures to another. Independently, therefore, of that general perfection which is usually ascribed to God, and which excludes the idea of such defects as partiality would imply, it may be assumed that God is alike good to

* *Natural History of Religion*, sec. ii.

† Whewell, *Bridgewater Treatise*, p. 361.

* Crombie's *Natural Theology*, vol. ii. p. 179.

† *Ibid.* p. 181.

Theology. all the works of his hands. For although apparent inequalities are to be observed in the course of the world, yet the more accurately these are examined, the less they are found to be, and the fuller the evidence that there can be no "respect of persons," or of creatures, "with God."

(9.) Such being the chief attributes and perfections which considerate minds have almost universally assigned to the Deity, it cannot but be a subject of great interest to ascertain what the moral relationship is in which man stands to this God. Indeed, the contemplation of those things in creation which daily challenge our observation, seems almost unavoidably to force upon us an inquiry of this nature. If all that variety and skill which are written on the works of God lead to the conclusion that all and each of those works were made for purposes peculiar to each; and if we are confirmed in that conclusion by further observing the world, animate and inanimate, subject to those fixed laws which secure the fulfilment of certain known and beneficent purposes, can man form an exception to the rest of God's works, in having been created for no purpose whatever? And if man were made to sustain some part in the world, is it likely that his Maker should leave him destitute of those attributes of body and mind which should best serve to secure the end of his creation? Is it not, on the contrary, reasonable to suppose, that as an examination of the natural world sufficiently indicates the purposes for which irrational creatures were formed, so a careful survey of man's endowments in body and mind will afford intimations that the Deity has not "made all men for naught"?

But on entering upon this investigation, we cannot but notice the distinction that exists between man and the rest of animated nature. Whilst others of God's creatures are formed to be obedient to instinct, or to bodily appetite, it is man's high prerogative to be endowed with reason and conscience; with the power to cast his thoughts forward to the future, as well as backward on the past; possessed, in a word, of faculties and susceptibilities totally distinct from any of those known to mere animal nature. And whilst these attributes of mind place man in that lordly position which he holds in the universe, the moral and religious apprehensions of which his intellectual nature is capable would seem to require a correspondent elevation of his thoughts toward God. When, for instance, we regard God as creating all things by the word of His power, the very thought ought surely to carry with it a feeling of solemnity. There can be no natural or rational ideas of any thing, if the consideration, that we are never beyond the power and influence of the self-existent, Almighty Creator and Sustainer of all things, do not call forth impressions of profound reverence and awe. For, what good or evil cannot so great and powerful a Being as the Maker and Upholder of the universe do for, or inflict upon, so feeble and helpless a creature as man? In this respect, therefore, the words of inspiration may be appropriately used to express the suggestions of Natural Religion:—"Let all the earth fear the Lord; let all the inhabitants of the world stand in awe of Him; for He spake, and it was done; He commanded, and it stood fast." (*Psal.* xxxiii. 8, 9.)

As in every thing created, also, there are found so many indications of Divine wisdom, and traces of Divine goodness, our reason teaches, that in proportion

Theology. to the excellency and glory of the works should be our admiration of the Maker. Considering, moreover, the extent and magnitude of the Creator's works, and those manifestations of His unwearied providence by which we are surrounded, the Psalmist seems but to give utterance to the feelings of every thoughtful and pious observer of the universe:—"When I consider Thy heavens the work of Thy fingers, the moon and the stars which Thou hast ordained, what is man that Thou are mindful of him?" (*Psal.* viii. 3, 4.) If, therefore, a contemplation of the works of creation do not produce in the mind a feeling of grateful humility, the only inference that it seems reasonable to deduce from the existence of such insensibility is, that it must be to man's reproach if the wonders of creation have been spread out before him in vain!

(10.) If, however, we were to rest satisfied merely with an expression of such devout feelings as those which have been alluded to, we shall certainly fall short of much that right reason would suggest to us. For when we consider that it is God "who hath made us, and not we ourselves," there is suggested to us God's right of ownership as it respects man, as well as regards the rest of the works of creation. But what a train of consequences does this thought bring with it! When we can claim any thing as the fruit of our own invention, care and industry, all men admit that we may justly call that our own; that we may, without prejudice to the rights of others, claim the enjoyment and disposal of that in which we have thus so indisputable a property. Yet God's right of ownership in creation may be asserted in an infinitely higher degree than that of man in any of his works. For as it was by an original, uncommunicated, independent power that all things were produced out of nothing, and as creation subsists only by the will of the Creator, it is the high prerogative of the Great Owner of all to place His creatures exactly as seems best to His divine wisdom; apply them to such uses as He deems to be most fitting for them; transfer or take away any thing, or all, which He only originally gave!

Now, if the law of an intelligent nature be to do whatever is agreeable to the relationship in which it is placed, there follows, from the notion that we are "God's workmanship," that we should at once deliberately resign ourselves to His disposal, without hesitation or reserve; that our wills being thus subordinated to that of our Maker, we should be content to abide by His allotment of our condition in this life, in whatever station or circumstances we may be placed: for as we can claim nothing as our right, we ought to receive every thing as a free gift. Man also being the property of the Creator, it follows that the right of self-disposal can extend no further than it has been conceded to us by our Maker; and that we may not employ the faculties and endowments of our mind, or the members of the body, in any manner that God has not sanctioned. As respects our worldly possessions, also, we are not entitled to use them except with reference to the purpose for which they were committed to our custody. And, finally, none can rightly presume to dispose of other men's goods, but as the Maker and Owner of all has been pleased to assign to each their several measures of authority and subjection.

(11.) We are thus, further, led to regard the Deity as a Supreme Ruler and Legislator, whose dominion is without bounds, and whose sovereign authority is beyond control; whilst the whole human race must be con-

Theology. sidered as members of one vast community, the characteristic of which is subordination; all rational intelligences being subject to one and the same Lord, and, under Him, to each other as right reason may suggest. The duties also that arise out of this relationship, though manifold in act, will yet be, in principle, analogous to those which are admitted to be inseparable from all well-governed communities: only, as the economy of the Divine government is conducted on far higher principles of truth and equity than is possible in the most perfect of human sovereignties, and as the object of every government is to mould the character of the subject into the spirit of its laws, it follows that our duty to our Maker and to our fellow-men must be measured by the standard of God's dealings with His creatures, and not by the rule of human selfishness.

It is almost unnecessary to remark, that there is implied in the ideas which have been thus briefly developed, a desire on the part of man to acquaint himself with the perfections and will of this great Lord of the universe. We cannot intelligently revere a sovereign of whom we know nothing; nor can we yield a rational obedience to laws of which we are altogether ignorant. It is the part of a reasonable being, therefore, to trace out manifestations of the Deity wherever they are to be found, and to collect and treasure up every intimation of God's will, in what manner soever it may have been conveyed.

Duty of gratitude.

(12.) It is not, however, so much the consideration of that absolute dominion which the Deity exercises over the works of His hands, in virtue of creative power, that challenges the obedience and services of His rational creatures, as the conviction that we are the object of His boundless goodness. Wherever we turn our thoughts we find ourselves still encircled with Divine beneficence. For man's sake the earth is decked with varied garniture—affords him gratification to every sense, and provision for every want. The heavens above, and the waters under the earth, minister of their kindly influences, and contribute liberally of their stores. The whole course of nature is, in fact, but a transcript of the Almighty's regard for His rational creatures. Now, although many selfish considerations may intervene to blind man to those obligations which arise out of ownership and sovereignty, yet the claims of a benefactor appeal so directly to what may be called the better feelings of the human heart, that there is no relationship so vividly apprehended, and the neglect of the duties arising out of it more generally condemned. It is the observation of St. Augustine,* that if after fashioning the marble into a human figure, the sculptor could inspire it with life and feeling, and give it motion and understanding and speech, we cannot imagine but that the first act of the figure thus made instinct with all the attributes of the living man, would be to prostrate itself in thankful subjection at the sculptor's feet, and offer to him whatever it might be or could do, as but its bounden obligation to him who fashioned and imparted to it all it possessed. And if the promptings of nature thus be to move us to love those who have manifested regard for us, the power of gratitude may reasonably be expected to call forth in us a sense of the obligation we owe to Him who "breathed into our nostrils the breath of life, and is now daily pouring His benefits upon us." On the other hand, to undervalue God's benefits, or to demean ourselves as if indifferent to His

goodness, would argue a sottishness of character utterly unworthy a rational soul. For whatever difference of opinion may obtain as to what those acts of worship are which Natural Religion teaches man to offer as likely to be well pleasing unto God, there can be no question but that the same law which demands that we should study how we may best gratify our benefactors, calls upon us continually to aim at doing the good pleasure of our Maker.

(13.) It may, perhaps, be objected that Natural Religion has been made to teach more plainly the relationship in which we stand to God and to each other, and our resulting duties, than is really the case; and that man's responsibilities, regarded as a merely rational creature, have been proportionately exaggerated. Nor need it be doubted but that, in the discussions of questions of a moral nature, we are far more influenced by our familiarity with revelation than we are oftentimes conscious of. Whilst, however, it seems to be preferable thus to allow to Natural Religion every advantage to which, under the most favourable circumstances, it can fairly lay claim, it might confidently be maintained, that even if estimated by the lenient demands of heathen codes of morality, the responsibilities of man would be found sufficiently numerous. The only question is, to what extent have his moral obligations been regarded, and his recognised duties performed?

(14.) And here we are confronted by the fact, from whatever cause it may proceed, that, coupled with our acknowledgment of duty, there is in all, more or less, a backwardness to attend to it. Hence it is universally found, that however busily the human intellect may exercise itself about other matters, its inquiries after the Most High are but few and limited; that the sacrifices of praise and thanksgiving are therefore offered with reluctance or dislike, if not supplanted by deeds of impiety; that the sovereign authority of the Almighty is treated with negligent disregard, if not resisted by overt acts of rebellion; that the manifestations of God's providential dispensations and fatherly care do not call forth in man's heart correspondent emotions of gratitude, if they be not made the subjects of murmuring and discontent. So, again, however persevering man may be in following out purposes which accord with the suggestions of the lower propensities of his nature, he is habitually unstable as regards his moral resolutions. If, in fact, a judgment were to be formed of the state of man's soul by inward experience, as illustrated by those outward acts which meet the eye, it would seem as if reason had either laid aside its dominion, or else were unable to enforce its commands. There is, in a word, a perpetual conflict going on between reason and sense; and this not merely under the disadvantages of ignorance, but in spite of the best moral instruction, and the fullest conviction of the importance of obedience to just laws. Man being thus at war with himself, it is the less to be wondered at, that he should not only neglect, but become injurious to his neighbour. And, therefore, we read in the Statute Book of every age and nation the history of that "malice and envy, that hatefulness and hating of one another,"* which has characterized man's social life from the beginning unto this day. The result of all is, that under the most favourable point of view, it will have to be acknowledged that all have more or less failed to

Serm. de Verb. Domin.

* Tit. iii.

Theology. perform their acknowledged duty to God and to man. In some respect or other, all have also violated those rules of right which reason and conscience urge us to observe. In other words, every man, in having done what he knew or might have known to be wrong, and in having neglected to do what he knew or might have known to be his duty, stands CONVICTED OF SIN.

(15.) But as "we see the whole world and each part thereof so compacted that as long as each thing performed only that work which is natural unto it, it thereby preserveth both other things and also itself; and, contrariwise, that if any principal thing, as the sun, the moon, any one of the heavens or elements, but once cease, or fail or swerve, we easily conceive that the sequel thereof would be ruin, both to itself and whatsoever dependeth on it; is it possible that man, being not only the noblest creature in the world, but even a very world in himself, his transgressing the law of his nature should draw no manner of harm after it?" (Hooker, *Eccl. Pol.* i. 9.) An answer to this all-important inquiry is supplied by the fact, that the experience of mankind has so surely connected difficulties and sufferings with the neglect or violation of man's duties to himself and to society at large, that motives to practise general rectitude and integrity, on the score even of common prudence, are to be found embodied in the proverbs of all nations. And if punishment be thus palpably the admitted result of offences against those natural laws which regulate man in his individual and social capacities, there can be no obvious reason assigned why man's duty to God may be neglected with impunity. Indeed, if it be allowed that we owe any duty to God at all, the neglect of that duty being a neglect to fulfil some of those purposes for which man was created, is so far to introduce disorder into the universe. The same reasons, therefore, which would assign retribution to the violator of those laws on the observance of which the course of this world is made to depend for its well-being, would lead to the conclusion that offences against God also will not pass unrequited.

Retribution.

These, and other considerations which might be adduced, clearly warrant the inference, that none can depart from rectitude, or neglect the performance of duties, without exposing himself to the righteous displeasure of that Almighty Being who, by the very act of prescribing laws to the creatures of His hand, declared it to be His will that those laws should be obeyed. The tendencies of man's moral nature, also, all bear witness to this truth; for the shudderings of fear, and the forebodings of conscience respecting the probable misery consequent on misdoing, are but so many indications that there is some judgment in store for every one that doeth evil, whether that evil have reference to the Creator or the creature.* Then follows the appalling thought, that in whatever obscurity and difficulty the subject may be involved, yet the contemplation of man's intellectual nature discovers to us unquestionable intimations that there will be a future state of existence.† But as our habitual consciousness assures us that, amid every external or mental transition from state to state, the soul, as to all that constitutes its essence and responsibility, remains always the same, what reason is there for believing that our destination to a future state can become to us any thing better than

Theology. a continuation of that displeasure of God which in this life we have such just cause to apprehend?

And yet it does not appear, on principles of human reason, how man can rid himself of this well-grounded apprehension of the Divine judgment. For if we should hope for reconciliation with God simply on ground of His mercy and goodness, our reason would remind us, that offences, merely as offences, can have no claim upon the Divine forgiveness, except at the sacrifice of those great principles on which the stability of all government depends. If we should hope for pardon on account of any thing we could do, or for any sorrow and amendment that we might manifest, who can tell us the exact amount of service to be performed? or that it is an established rule of the Divine government to free the evil-doer from punishment whenever he may repent and amend his life? On the contrary, those superstitious rites which heathen nations have practised shew, by their very number and variety, how little competent the human mind is, when left to itself, to decide upon what are, and what are not, fitting means to propitiate the Deity; whilst it is an every-day observation, that penal consequences continue to pursue the offender long after his crimes have been repented of and forsaken.

(16.) Although, therefore, Natural Religion may serve the important purposes of bringing us acquainted with many of those duties which arise out of the relationship in which we stand to the Creator and to our fellow-creatures, it fails to disclose how that demerit which is consequent upon a neglect of those duties can certainly be removed; and how man may be restored to the lost favour of his Maker.

Natural Religion being thus defective as regards a subject of all others the most important to man as an accountable, yet sinful and immortal being, the necessity for some direct revelation from God has, in all ages, been felt;* that by this means the mind of man might be authoritatively assured on what terms the Creator would pardon His offending creatures, and how far that forgiveness would extend.

(17.) It becomes, therefore, the duty of every rational man to inquire whether any such revelation has been made?—by what proofs man may attain to a certain conviction on this important question?—and how he may ascertain the sum and substance of this communication from God? These inquiries will form the subject of the next chapter.

* See the remarkable indication of such a desire in the celebrated passage of Plato towards the end of the second *Alcibiades*.

We may state here that, if space had permitted, we should gladly have referred to the writers of antiquity for the illustration of many sentiments contained in the preceding pages. We must, however, recommend Cicero's *Treatises, De Natura Deorum*, lib. ii., and *De Legibus*; and, also, Xenophon's *Memorabilia*, lib. i. 4, and lib. iv. 3. The authors who, like Seneca, lived after the incarnation of our Saviour, contain many things worthy of notice on the great truths of Natural Religion; but we believe they are more indebted to the light reflected by the Christian Revelation than is usually allowed. In the quotations referring to a future life, adduced by Leland, &c. to enforce the necessity for a Divine Revelation, the uncertainty of the heathen, in speaking of a future state, seems rather to express their misgivings as to what might be their condition in that state than any doubt as to the state itself. Our impression is, that the idea of an hereafter is inseparable from the human soul, whatever external disadvantages it may labour under as to instruction. (S. Tertullian. *De Testim. Animæ*, ch. iv.) We cannot follow out the subject here, but we must regard it as deserving of more attention than it has yet received.

* See the testimony to this power of conscience in Juvenal, *Sat.* xiii. 2 and 192, *et seq.* † Turton, *Nat. Theol.* p. 188.

OUTLINES OF THEOLOGY.

CHAPTER II.—EVIDENCES.

Evidences.—Revelation.—Scripture.—Tradition.—Fanaticism.—The Rule of Faith.

Theology. (1.) THE first inquiry which we now propose to institute is, to ascertain whether any such Revelation as that alluded to in the close of the last Chapter has been given to man; and whether it is contained in the *Bible*; and, secondly, whether it is contained *only* in the *Bible*.

Natural government of God.

The first point is chiefly a question of evidence; but before we inquire into the particular evidence for any revelation whatever, some few preliminary observations are requisite. If we have before us so large a book from the hand of the Almighty as the Natural World, it is reasonable to endeavour to gather from it some hints as to God's method of dealing with man. It is at least likely to preserve us from the error of forming unreasonable expectations, as to what revelation may be presumed to do for us; and it may furnish us with reasons for not rejecting a revelation as coming from God, on account of difficulties similar to those with which we are surrounded in the natural government of God.* Assuming, then, that the course of nature is a system under the government of God, we are enabled to discern some great principles which characterize it. It will hardly be denied by any who agree to this statement, that there is some degree of probability, that our existence does not terminate with our present life.† But if it be granted that a future state is in any degree probable, a very important question arises, how we may best secure our interest in that state? We observe, that the constitution of nature admits of misery as well as happiness; and that both these are the consequences of our own actions, which consequences we are enabled to foresee. Should rewards, therefore, and punishments in another world depend on our good or ill behaviour here, it would only be an appointment of the same kind as that which we observe in the Divine government according to the regular course of nature.‡

Moral government of God.

But if we carry our observations a little further, we

* The very brief summary of Bishop Butler's argument, which follows, is chiefly a condensation of Bishop Halifax's Preface, compared with the *Analogy* itself.

† The brevity with which we are compelled to treat this subject prevents our entering into the proof of the second proposition. We must refer to Bishop Butler's *Analogy*, ch. i. for a very luminous statement of the question. It must be remembered that he is not demonstrating the *certainly* of a future life, but arguing that it has some degree of probability. This he does from natural considerations only, which suits the course of his argument; but if we take in, as in this state of our argument we may, moral considerations also, from the nature of the Deity and of man, the probability is placed upon a much higher footing.

This is not, however, the point in the argument which we are chiefly concerned to demonstrate. Our object is simply to call attention to the analogy between the system of Nature and that of Revelation. (See Clarke's *Evidences of Natural and Revealed Religion*, vol. ii. prop. iv.)

‡ See Butler's *Analogy*, part i. ch. ii. Butler particularly insists on the government of God by *punishments* in the natural world, as that is the most frequent subject of objection.

may perceive that this natural government of God is a moral government also, in which rewards and punishments are the consequences of actions, considered as virtuous and vicious. It is indeed imperfectly so, as the essential tendencies of virtue and vice to produce happiness and the contrary are often hindered from taking effect; but it appears to contain the rudiments of a righteous administration, under which, in a more perfect state, the rules of retributive justice will more completely obtain.*

This moral government also implies the notion of some sort of trial, or a moral possibility of acting wrong, as well as right, in those who are the subjects of it. By this the doctrine of religion, that our present life is a state of probation, is rendered credible, as being analogous to this moral government; prudence being required to secure our interest here, as virtue is requisite to secure our eternal happiness. But this is not all; our present life is also a state of discipline and improvement, each part of our life being a state of discipline for that which succeeds it. By this means, the formation of habits of piety and virtue is promoted, which are, besides conscience, of great use to creatures in a state of moral imperfection, as a safeguard against the dangers to which we are exposed.†

(2.) Objections will, however, arise against the wisdom and goodness of the Divine government, to which analogy, as shewing only the truth or credibility of facts, affords no complete answer; but it suggests to us that the Divine government is a scheme or system, not a series of unconnected facts. The government of the natural world appears to be such a system, and seems to bring about *ends* by *means*, which we might *a priori* have judged likely to have a contrary effect: it is carried on by *general laws*, whose utility we are enabled to discern, but are utterly incompetent judges of the *whole*, because we see so small a part of it. If, therefore, the moral world be a similar system, and still more, if the two together form one great scheme, we shall be obliged to acknowledge ourselves unable to judge of the whole, from our incompetent knowledge of the whole, or even of the mutual bearings of the several parts of it. All objections, therefore, to the goodness and wisdom of the Divine government may be founded merely on our ignorance, and to such objections our ignorance is the proper and satisfactory answer.

(3.) These considerations apply primarily to Natural Religion, but they are of great service towards forming right views of Revealed Religion also. The importance of Christianity to man cannot be over-estimated, when we consider that it is,—

* See Butler's *Analogy*, part i. ch. iii.; comp. Clarke's *Evidences*, &c. *ubi supra*.

† See Butler's *Analogy*, part i. ch. iv. and v.

Theology.

1st. An authoritative republication of Natural Religion, confirming its most important doctrines, and,

2ndly. The Revelation of a new dispensation of Providence, originating from the pure love of God, and conducted by the mediation of His Son, and the guidance of His Spirit, for the recovery and salvation of mankind represented as in a state of ruin.

Presumption
against Re-
velation in
general.

(4.) Now the presumptions against Revelation in general are, that it is not discoverable by reason, is unlike what is so discovered, and that it is introduced and supported by miracles. It is far from being true, that it is so wholly unlike what we may observe in the constitution of nature; but even if it were, our knowledge of the whole is insufficient to entitle us to make this an objection to so wide a scheme. Nor is there anything incredible in Revelation, as being miraculous, whether miracles be supposed performed at the beginning of the world, or after a course of nature is established. Not at the *beginning of the world*,* for something of a miraculous kind *must* have happened then. Not *after the settlement of a course of nature* on account of miracles being contrary to that course, as being contrary to experience. We have no parallel case to compare with this, and ascertain whether God would attest a revelation by miracles.

Particular
objections
to Christi-
anity.

Presumptions against revelation in general being disposed of, the foregoing principles enable us to answer the usual objections against Christianity in particular, as distinguished from objections against its evidences.

1. Because it contains many things which beforehand appear to us liable to objections, and is different from what we might have expected; as, for example, its late appearance in the world, and its want of universality.

Ans.—This part of the system has its counterpart in the natural world. We might have expected remedies for all bodily diseases, but they were long unknown, and are now known but to few.

2. Christianity is a scheme imperfectly comprehended by us,—a consideration which answers objections from any seeming want of wisdom or goodness in it, just as the same objections were answered in regard to the constitution of nature.

3. The restoration of mankind, as shewn in the Gospel, required a long series of means and contrivances, whereas it ought to have been brought about at once.

Ans.—Everything we see in the course of Providence shews us the folly of this objection, for all ends are there brought about, not directly, but by a succession of means.

4. The mediatorial scheme and the vicarious punishment, which the scheme of Christianity presents to us, are sometimes objected to.

Bishop Butler, in answer to this, points out instances in which the visible government of God is administered upon analogous principles. A sinner is not always pardoned and freed from inconvenience on simple repentance, but provisions are made, even in nature, by which the miseries men have entailed on themselves are removed or mitigated, sometimes by extraordinary exertions on their own part, but more frequently by the intervention of others, who voluntarily submit

to much inconvenience and suffering, as the price of rescuing their neighbour, and thus furnish a natural instance of vicarious punishment being available to good.

5. The want of universality and the want of intuitively clear evidence for Christianity, are lastly made grounds of objection.

Ans. 1. The gifts of Providence are dispensed unequally, and even if Christianity had been universal, the diversity of men's abilities, &c. would soon have rendered it unequally distributed; and men will be judged according to their light.

2. As we are obliged, in life, to act upon probability, we have no right to look for an intuitive knowledge of the truth of religion. The proof we have is amply sufficient in reason to induce us to accept it, and dissatisfaction with it may be our own fault.

The above hints will enable us to see how much use may be made of the observation of God's natural government of the world, in obviating objections to the Christian Revelation. It affords somewhat of a presumption in its favour, that it really provides the very remedy which Natural Religion (*see the close of the last Chapter*) impels us to desire, but fails entirely to point out, a means of restoration to God's favour and of holiness; it holds out the hope of justification and sanctification to man exactly where his inquiries into Natural Religion leave him without comfort or remedy.

These considerations are entirely preliminary; but they are sufficient, if fairly carried out,* to justify us at once in turning our attention to the particular evidence for Christianity; and in concluding, with Bishop Butler, that objections against Christianity, as distinguished from objections against its evidence, are frivolous. We proceed, therefore, now to sketch out a slight view of some of the evidences for the truth of the *Bible*: in so doing, as we must *select* from a large variety of arguments with which God has mercifully supported his religion, we shall endeavour to use those which will give us the widest views of his dealings with mankind, and of the preparation which he made of old for Christianity.

β. Particular Evidence for Christianity.

(5.) In reasoning on the evidence of Christianity, we have great reason to be thankful both for the variety of the proofs by which the Almighty has been pleased to support it, and for the simple and obvious facts and principles on which they are grounded. The first matters on which we are about to found our chain of reasoning are easy of proof, and admit of very little hesitation. The foundation of Christianity was laid long before its Great Author came into the world, and the *great facts* of the Jewish dispensation—and our argument will require only these to be established—are supported by an evidence wholly irresistible.

It cannot be disputed that at the time in which Christianity arose, a nation was in existence, called the Jewish nation, which claimed to be the depository of a revelation from God; that this revelation was written in the *Old Testament*, and contained what were then

* There was then no *course of nature* to be contradicted by miracles. See Campbell on *Miracles*, on these two points. He shews that a miracle *must* have taken place at some period, if miracle means deviation from the course of nature.

* As they are in Bishop Butler's *Analogy*, to which these outlines form only, as it were, a table of contents. That work is one which is indispensable to every one who wishes to have an accurate, as well as comprehensive, view of God's dealings with mankind.

Theology. believed to be a series of prophecies, and that these prophecies had awakened in that nation a general expectation of some great person, who was to arise as a deliverer to the nation. It is equally indisputable that this nation professed obedience to a ceremonial law, which they attributed to Moses, and which they considered him to have received from God. There are facts so certainly known and universally acknowledged as to require no proof. And if we pursue this subject further, we shall find that this law, as contained in the *Pentateuch*, must have been in existence for many centuries, and that the whole body of the *Old Testament* had been translated into the Greek language nearly 300 years before the coming of our Saviour. We will not now argue from these statements to the Divine teaching of this people. The superiority of the Jewish people, as regards their religious views and their moral code, compared with their inferiority to the great nations of antiquity (as the Greeks and the Romans) in all besides—in the elegancies of literature and the arts of life, is a phenomenon which admits of easy explanation, when we deduce their moral and religious code from a Divine revelation, but presents most formidable difficulties on any other hypothesis.

Antiquity of the Old Testament. But leaving these considerations, we proceed to point out some other extraordinary circumstances relative to the *Old Testament*. It is written in a language which was ceasing to be a living language, nearly about the time when historical writing began in other countries. The *History of Herodotus* was about contemporary with the latest Hebrew book; and the language in which Malachi was written, was just then beginning to give place to a kindred dialect, if it had not been superseded by it.* Historical considerations shew us that the *Pentateuch* (certainly the oldest† part of the *Bible*, unless *Job* be thought of the same age, though that is not likely,) must have an antiquity far transcending this epoch.

The Pentateuch.

Without laying any stress upon the existence of the Samaritan *Pentateuch*,‡ and the reception of that portion only of the sacred volume by the Samaritans, no person acquainted with the facts of History and Philosophy could suppose the *Pentateuch* to have an origin posterior to the time of Ezra; nor can the law then have been a new series of obligations. The facts of History

* Gesenius (*Geschichte der Hebräischen Sprache und Schrift*, part i. sec. 13) denies the usual interpretation of Nehemiah, ch. viii., v. 8, and considers that מְפָרֵשׁ does not mean translating into a different dialect; but simply explaining difficult passages. But Gesenius is not an authority to be implicitly followed in that work. He there attempts to shew that the writer of the *Book of Chronicles* misunderstood the Hebrew of the *Book of Kings*; but his attempt is most rash and unsound. See Rose's *Hulsean Lectures*, Appendix.

† Philological considerations shew this also. Gesenius acknowledges it with regard to all but *Deuteronomy*; but see the observations at the end of this paragraph.

‡ Dr. Graves has made this the first step in his argument for the antiquity of the *Pentateuch*, as proving its existence prior to the breach between the two parties. But Hengstenberg, in his *Die Authentie des Pentateuchs Erwiesen*, &c. 2 vols. 8vo. (the latest and most elaborate defence of the *Pentateuch*), considers this an unsound and useless argument.

Professor Hengstenberg quotes most of the previous writers who have treated these questions, so that a reference to his work is sufficient.

There are some useful arguments in Professor Blunt's *Veracity of the Books of Moses argued from the Undesigned Coincidences to be found in them*. Also in the *Lettres de quelques Juifs*, &c.

Theology. compel us to admit that it must have been recognised long before the Captivity, as the law of at least the kingdom of Judah. Professor Hengstenberg has lately investigated the question of its reception in the kingdom of Israel, as shewn by the books of Kings, and by indications in the books Hosea and Amos, the two prophets who chiefly addressed themselves to Israel. To his researches, and to Dr. Graves, we must refer those who wish to trace the *Pentateuch* up to the time of Moses.

Observation.—We have (after an examination of most of the celebrated German writers on the subject) very little confidence in the dicta of modern philological scholars as to the difference in dialect and epoch between various portions of the *Old Testament*, as shewn by their style and diction. Gesenius (*Geschichte*, &c., ut supra) divides the *Bible* into books written before and books written after the Captivity, and contends that there is a clear distinction in their style and language. Other philologists bring down the *Pentateuch* to the Captivity. Now these marks of difference either exist or they do not: if they do, the latter philologists are worth nothing; if they do not, the other mischievous views of Gesenius, which rest on this foundation, are worth little attention. There are one or two peculiarities in the *Pentateuch*, which seem decidedly archaisms, but great care is needful in arguing from internal evidence of style.

For further details on this subject, see the Appendix to Rose's *Hulsean Lectures*, and Professor Hengstenberg's work above cited.*

If now, after shewing the antiquity of this volume of the *Old Testament*, we examine its contents more accurately, we find that it contains a large number of passages, claiming to be delivered as prophecies by men inspired by God, and commissioned to deliver these messages. We find also two classes of these prophecies, one of them relating to the destinies of neighbouring nations, the other to the condition of the Jewish church and its relations with the heathen world; in which last class we reckon, of course, the prophecies relative to the Messiah.

We may safely assert, that when we compare these prophecies with their fulfilment, no human composition can rival them. The Jews were much connected with some of the most remarkable nations of antiquity, e. g. with Tyre, with Babylon, and with Egypt. Now of these nations the fate was predicted at least 2400 years, in terms which describe their present condition as accurately as if written by a traveller at the present day. For Tyre, see *Isaiah* xxiii., *Ezek.* xxvi. xxvii.; for Egypt, *Ezek.* xxix. (especially v. 15) xxx. xxxii.; for Babylon, *Isaiah* xlii., especially v. 19, 22, &c.

This list of prophecies, the actual fulfilment of which challenges our attention, prior to any proof of inspiration, might be considerably increased by those relating to Nineveh, Edom, the Arabians, (*Gen.* xvi. 10, 12,) and the prophecies of Daniel respecting Greece and Rome; but we shall mention only one more, because it is remarkable as the oldest prophecy on record, with regard to the nations of the world: we mean the curse pronounced on Canaan, *Gen.* ix. 25. It is remarkable that no Canaanitish nation has ever yet maintained a pre-eminence in the world, while the descendants of Ham are still the storehouse of slaves for all the nations who have not yet abolished that infamous traffic, the slave-trade.

The prophecies, also, regarding the destruction of Jerusalem and the dispersion of the Jews, the latter of

* See also Bishop Herz's tract, *Sind in den Buchern der Könige, Spuren des Pentateuchs und der Mosaischen Gesetze zu finden?* Altona, 1822. This was written in answer to *De Wette*, and has been used by Hengstenberg.

Theology. which continue to give a wonderful testimony at the present day, are another class of external evidences, which challenges the attention even of the most careless observer.

With regard to the prophecies alluded to in this section, we may just refer to Bishop Newton on the prophecies, from which most subsequent works on the subject have been compiled. There are some very useful observations on this point, also, in Mr. Davison's *Discourses on Prophecy*, especially Discourse xi. His Discourse xii. treats of Daniel's Prophecy of the Four Empires.

The promise of the Messiah.

(6.) Let us now turn our attention to the second class of prophecies, of which we spoke. It is easy to shew that, throughout the *Old Testament*, there is a continued reference to a promise, which was to prove of most extensive benefit to the human race. This promise is said in the first instance to have been made by the Supreme Being to the first parents of mankind, purporting that certain privileges forfeited by them should be restored in the person of one of their descendants, described there as the seed of the woman. (*Gen.* iii. 15.*) This promise is continually renewed throughout the volume, every fresh intimation of it being usually accompanied by some additional circumstance, which serves to limit it, to clear up and define its meaning, until at length it can only be understood to indicate the approach of some great and mysterious Being. (*Gen.* iii. 15; xii. 1, 3; xxvi. 18; xlix. 10, 11; *Deut.* xviii. 15, 18; *Isaiah* xi.; *Malachi* iii. 1; iv. 5, 6,† &c.) The names by which this exalted person is described, besides the peculiar title of Messiah (*i. e.* Christ, or the Anointed), are very remarkable: 'Thy King cometh,' 'The Son of God,' 'The Lord our Righteousness!' (*Zech.* ix. 9; *Psalms* ii. 7; *Jer.* xxiii. 5, 6.)

On a still closer examination, we find the nature of the Revelation, of which this messenger was to be the bearer, disclosed with much clearness as to its leading points. It was to be a dispensation, in which an atonement and reconciliation was to be effected between man and his Maker.

Again, the false religions of the world were then to disappear, the idols to be abolished, the kingdoms of the earth were to become the kingdoms of the Lord, and the earth was to be covered with the knowledge of the Lord.

Again, the dignity of the messenger was to be even greater than that which we have heretofore expressed. He is called 'Wonderful;' 'the Mighty God;' 'the Everlasting Father;' 'the Prince of Peace.'

Isaiah liii.

But although the nations were to be blessed in him, though his dominion were to be so wide, yet he was to have no earthly glory at his appearance, "no beauty that we should desire him:" he was to grow up silently like a tender plant; he was to be a man of sorrows and acquainted with grief; he was to be led as a lamb to the slaughter, and to make intercession for the transgressors, and he was to yield his soul an offering for sin. Beside all this, without reference to the

* This promise was given in the curse pronounced upon the serpent. The mistranslation of the Vulgate "*ipsa*," by which it is transferred to the *woman* instead of her seed, is well known. The Hebrew text entirely rejects this corruption, though the *pronoun* (by an archaism found in the Pentateuch) is common, the *verb* is in the third person masculine, which settles the question. The *ejus* of the vulgate is indefinite, but the affix in Hebrew is masculine also.

† It must be remembered that the references here given are only sufficient to point out the great features on which the argument depends. An adequate collection of prophetic passages from Scripture would occupy some pages.

Theology. promise itself, there are many circumstantial prophecies relative to the time at which the Mediator of the new covenant was to come into the world. The place of his birth, the time of his appearance, and a multitude of minor circumstances personally characterizing him are added, which it is superfluous to enumerate, but which designate him very definitely.*

The above brief indications of the nature of the prophecies relating to the kingdom and person of the Messiah are all which our limits allow us to bring forward; but though these be sufficient, yet the argument rather suffers from the paucity of citations.

When the prophecies are collected fully, and accurately arranged, their force is overwhelming.

The types also, the paschal lamb, the brazen serpent, (acknowledged by the Jews to be typical,) the sacrifices of the law, and the rest of the typical pre-figurations of our Lord, might justly be adduced, but we are desirous of bringing forward only those passages, the prophetic nature of which does not admit of dispute. A good *Christology* of the *Old Testament* would be most valuable. We have not examined Professor Hengstenberg's *Christologie des Alten Testaments* sufficiently to speak of it decidedly, but his other works have given us a very high opinion of his powers. Bishop Kidder's *Demonstration of the Messiah* is a very valuable work. Robinson, on the *Prophecies* relating to the Messiah, is a mere practical application of most of the passages, and is not argumentative or illustrative. To be of real value, however, such a work must be written with very wide knowledge and great judgment. Every English compendium gives us a list of passages arranged under heads relating to the offices, &c. of Christ. These are sufficient for all common purposes, as God has mercifully taken care to write his lessons to mankind very clearly.

(7.) We are now about to apply these observations which we have made respecting the *Old Testament* to the facts recorded in the *New*. We presuppose in our readers a general acquaintance with the history and the contents of the *New Testament* and the early history of Christianity. But we may as well observe, that here our argument does not absolutely require a proof of the genuineness and authenticity of the Gospels. The main outline of the history of our Saviour might satisfactorily be made out from other writers, even if the Gospels had been lost to us. Christianity itself is a phenomenon which requires to be accounted for by a sufficient cause, and certainly none is adequate which does not imply that its first founders believed that the story on which their faith was founded was a miraculous one. And we could show from independent authority that the story which they did believe resembled that given in the Gospels; and the early propagation of the faith is a well-known matter of history.

The proof of the assertion respecting the main facts of the Gospel would occupy too great a space, as its value depends on details. But for a sufficient demonstration of it we have only to refer to Paley, *Evidences*, part i. ch. vii., and to Archdeacon Lyall's *Propædia Prophetica*, lect. p. 136 to 151, to which last work we are much indebted in the course of this argument. These references supersede the necessity of others.

(8.) Assuming then, after the proof referred to, that the Accomplishment leading facts recorded in the Gospel were believed by its first teachers, we may point to the fact of the success of Christianity, as the most remarkable accomplishment of prophecy which the world has ever beheld. This new and spiritual kingdom was set up: it arose at the very time predicted, and before the destruction of the Temple, which took place soon afterwards, and thus intimated that the time had passed for the coming of the Messiah. (See *Malachi*, iv. 1.) It banished idolatry, ere

* The time of his coming is determined by Daniel in the celebrated prophecy of the *twenty-two* times, and by Malachi it is restricted to the times of the Temple.

Theology. long, from most of the accessible quarters of the world, and gave a general religious knowledge to the nations of the world, such as had never even been dreamt of before. These were contingent events quite beyond the control of man, and far beyond his unassisted wisdom to predict. This forms one great branch of the argument, and the personal history of our Saviour forms another. Many of the prophecies accomplished in his life are in some degree obscurely expressed; but had these marks been too plainly set down, the malice of man might have attempted to evade them.* These, which were not absolutely recognised beforehand as prophecies, may be called imperfect prophecies; those which were, perfect prophecies. The perfect prophecies were clearly fulfilled, if the great and leading events of our Lord's life, as believed by his first followers, really took place, and the imperfect ones are so many and extraordinary as to strike our attention very forcibly, even if considered alone. We are here entitled to ask, even before we assume or prove the authenticity of the Gospels, whether all this can be the result of collusion or of chance? Can the prophets, so remote in time from each other, have been engaged in any collusion? Can they have engaged the Apostles to assist them, at the risk of their lives, their fortunes, and their reputation? Can it have been the result of chance? Has chance ever made such a chain of prophecies, extending to such distant generations, and produced their accomplishment? Again we are entitled to ask, can the conduct of our Saviour and his Apostles be accounted for on any principle of enthusiasm or imposture? The very thought is absurd! But even if it could, would this explain the accomplishment of all these prophecies, and the success of our Saviour, who must, on such an hypothesis, be accounted an ordinary peasant of Galilee? Yet these are only a few of the earliest difficulties which meet us, in any attempt to account for the phenomena of Christianity on any hypothesis but the simple one of its truth and Divine origin. We cannot conclude this portion of our argument better than by the words of St Augustine:—

Venit Christus; complentur in ejus ortu, vita, dictis, factis, passionibus, morte, resurrectione, ascensione, omnia praeconia prophetarum.† “Christ came; in his origin, life, sayings, doings, sufferings, death, resurrection and ascension, all the predictions of the prophets are accomplished.”

Truth of the Gospel. (9.) But we are not reduced to such meagre evidence for the principal facts of the Gospel history. The genuineness and authenticity of the *New Testament* rest on evidence which cannot be gainsaid; and the testimony by which the miracles of our Lord are accredited is absolutely and entirely distinguished, by its strength, from that which supports any other miraculous accounts. The work of Paley has completely proved this point; and indeed so confidently

* The multitude of marks which concur to prove our Saviour the intended Messiah, and which agree with no other person, have often been pointed out. This is done with great clearness and ability by Archdeacon Lyall, in his *Propædia Prophetica*, Lecture vii. p. i., who has pointed out such marks as might have been evaded if too prominently set forth before their fulfilment; e. g. the gall and vinegar, the thirty pieces of silver, the time of our Saviour's crucifixion, &c.

Indeed, the whole of the *Propædia Prophetica* is deserving of great attention, as shewing the place prophecy holds in the argument for Christianity.

† Augustin. in *Epistol. Voluntiano*. Opp. tom. ii. p. 6, edition 1616.

Theology. do Christians trust to this evidence, that writers upon evidence in modern days usually consider it sufficient to bring forward the evidence for the facts of Christianity, (both the ordinary and the miraculous events recorded in the *New Testament*), and they conceive the proof to be complete. We may certainly so far accede to their views as to grant that if evidence ever established the title of any facts whatever to our credence, these are supported by the most unexceptionable and overwhelming mass of testimony, and that the arguments adduced by Paley and others completely prove the truth of the facts. But upon this point it is unnecessary to enter here, because the subject has been so copiously treated in another division of this work. (*Encyc. Metr. Hist. Div.* vol. ii. p. 626, 643.)

In Paley's *Evidences*, the arguments for the genuineness of the *Gospels* and the *Acts of the Apostles* are stated with great clearness and force. The great storehouse is, however, Lardner's book, which Paley has admirably abridged in ch. ix. of his *Evidences*.

The *Horæ Paulinæ* of Paley proves St. Paul the author of the Epistles known under his name, with great ingenuity, and in a manner which entirely settles the question for ever. And if these Epistles be proved to come from him, how strong an independent proof we have in his history of the truth of the Gospel. See Lord Lyttleton's *Essay on the Conversion of St. Paul*.

After all that has been written on the subject of miracles, we shall not add more, except to observe that Archdeacon Lyall, in answering Hume, argues that testimony cannot prove a miracle: it can prove the facts (which he contends Paley's arguments do prove), but cannot shew that they proceed from Divine interposition. Perhaps this is to carry the argument too far, when we consider the nature of the miraculous facts of the Gospel. A person, professing to have a Divine message, and to perform miracles in proof of it, after a judicial condemnation and public death, is raised from the dead, and ascends to Heaven. If the two last facts be included in the proof, surely we cannot doubt that, with this testimony after his death, his message came from God! Archdeacon Lyall has beautifully pointed out the reasons why the Apostles argued more from prophecy than from miracles, and to his work we refer our readers.

(10.) Our first argument therefore is, that, without assuming the truth of the facts believed in the present day and in the early days of Christianity, “the correspondence between the present belief of mankind and that promised revelation, which is the subject of almost every page in certain books of the *Old Testament*, is not a verbal coincidence, but a coincidence of facts—a coincidence between an established belief, about which there can be no doubt, and a previous expectation not less certain, founded on the faith of prophecy.” (*Propædia Prophetica*, p. 145.) “If this were a single prediction, the coincidence might be accidental; but the coincidences here, are not of a kind, and if they were, they are too numerous to admit of this supposition.” (*Id. ibid.*) And yet what middle ground is there between affirming them to be mere chance, and admitting the Divine origin of the religion?

Prophecy, therefore, we contend, is a sufficient proof of the truth of Christianity, and would be so were we deprived of the other great arguments by which its Divine origin is supported.

Again, we contend that the miracles of the *New Testament* are accredited by such an overpowering weight of testimony, that they challenge the belief of every impartial inquirer. Either proof, that from miracles or that from prophecy, is ample and convincing.

But as some minds do not admit so easily the conclusion that miracles of themselves are a sufficient evidence of God's sanction and authority to a religion, we

Theology. further observe, that even to this class of minds the miracles, combined with a previous system of prophecy, offer a proof in every possible respect entirely unexceptionable.

We forbear to touch upon other topics, and to press another class of evidence. We mean the internal evidence for the truth of Christianity from the character of its Divine Author, and the nature of His doctrines. Our purpose was not to give a summary of all the arguments which can be used, but simply to give a sufficient answer to the inquiry, "whether any revelation, such as that alluded to in the close of the preceding chapter has been given to man; and whether it is contained in the *Bible*?" and, as we conceive a sufficient proof has been already afforded that the *Bible* does contain such a revelation, we hasten on to the next portion of our inquiries, by which we propose to ascertain whether this revelation is contained *only* in the *Bible*?

The line of argument which we have taken is very much like that suggested by a work we have frequently quoted, Archdeacon Lyall's *Propædia Prophetica*. We ought, however, to observe, that it suffers much by abridgment, and to point out one or two remarkable features in this discussion. That writer lays much stress on the fact that the prophecies applied in the *New Testament* to our Saviour, were acknowledged by the Jews previous to our Saviour's appearance to be prophetic of the Messiah. He has given a list of the oldest Jewish authorities (none of whom are, however, older than about the second century after Christ) in confirmation of this statement. (p. 100-107.) It must be supposed that these interpretations are probably the traditional teaching of a still more ancient date, and thus they may be brought to the times of our Saviour.

We may perhaps be asked, how then do you answer the arguments of a German neologist, who contends that the *New Testament* is only an accommodation? The first answer is, that it is absolutely incompatible with the facts of Christianity and the character of its first teachers to suppose any such fraud. And, secondly, if men had conspired to meet the expectations of their own days, they would surely have tried the more obvious and more alluring views of earthly glory; and certainly, after the death of their leader, a few uninspired fishermen could never have imposed a religion on the world. They could not have invented the scheme; and even if they could, their success would remain unaccounted for, except by a Divine origin of their religion. Bertholdt's *Christologia Judaica* is the work where this view is most offensively brought forward. Professor Lee, in answering him, denies his premises, denies that he has any proof that the Jewish opinions he produces are as ancient as the times of our Saviour. We must add that, granting his premises, his conclusion is utterly untenable.

We should also wish to call attention to the ingenious supposition made in the *Propædia Prophetica*, of a sealed document which was to accredit an ambassador whose proposals were to be of an extraordinary nature, the sealed document being sent long before his arrival, but not opened till he required its testimony. This, of course, is prophecy, the meaning of which was not fully unfolded till it was fulfilled. But the work itself should be read and well considered by all who desire to understand the nature of the evidence given by prophecy.

We may now mention a few works on the Evidences of Christianity, which are highly valuable, although the course of our argument has not led us to make much use of them.

1. We may refer those who wish for an answer to Deistical objections, to Leland's *View of the Principal Deistical Writers*, &c., and to Bishop Butler's *Analogy*. The Deists generally attacked the *Bible* itself, rather than its evidence; but Butler has shewn (see Section 1 of this Chapter) that such objections are frivolous.

In answer to Gibbon and T. Paine, Bishop Watson's *Two Apologies* are of value; and in answer to Voltaire, the *Lettres de quelques Juifs*, &c., by the Abbé Guenée, ought to be read.

2. Stillingfleet's *Origines Sacrae* (full of matter relative to ancient profane history) will be found very valuable; as also Jenkins' *Reasonableness and Certainty of the Christian Religion*.

VOL. II.

Theology. Leland's *Advantage and Necessity of Revelation*. R. Millar's *History of the Propagation of Christianity*. Arthur Young's *Historical Dissertation on Idolatrous Corruptions in Religion*. Dr. Ireland's *Christianity and Paganism compared*, &c. Rose's *Christianity always Progressive*.

3. On the evidences of prophecy: Bishop Kidder's *Demonstration of the Messiah*. Bishop Chandler's *Defence of Christianity from the Prophecies*, and *Vindication of the Defence*. Bishop Newton, *On the Prophecies*. Hurd, ditto. Mr. Davison's *Discourses on Prophecy*.

4. Internal evidences: Soame Jenyns' *View of the Internal Evidence for Christianity*. Bishop Sumner's *Evidences of Christianity*.

The above are a few of the principal works on the evidences for Christianity. It is needless to mention Paley's *Evidences*, and his *Horæ Paulinæ*, which have already been quoted: they are in the hands of every one who pretends to take any interest in these subjects.

γ Is this Revelation contained ONLY in the Bible?

Having obtained satisfactory evidence that the *Bible* contains a revelation from God, we proceed to inquire whether this is the only authorized and authentic channel through which God has made known His will to man.

The consideration of this question leads directly to an investigation of the claims of tradition.

Christianity is evidently an exclusive religion, admitting of no compromise or co-existence with any other, such as any of the forms of Paganism or Mohammedanism. The question is therefore limited to this point, whether, concurrently with the Holy Scriptures, we have any *other* authentic source of revelation from God within the Jewish and the Christian Church? The Jews and the Roman Catholics both assert that there is, besides the *written* word of God, another, *unwritten* word of God, of equal authority with Scripture, and independent of it. Some bodies of Christians also maintain the necessity of an inward revelation; and others have appealed to miracles and prophecies in modern days, as accrediting special revelations from God. These positions must therefore be severally examined.

In the *Encyc. Metr. Hist. Div.* vol. ii. p. 580, some account is given of the Gentile religions. It has been deemed unnecessary to occupy any space with arguments against Heathenism or Mohammedanism. The latter is treated of historically in the same division of this work, vol. iii. p. 348-362. For a general account of Mohammedan doctrines, Mr. Forster's *Mohometanism Unveiled*, Dr. W. C. Taylor's *History of Mohammedanism*, Sale's *Koran*, and White's *Bampton Lectures*, may be sufficient: but these works, for the most part, give only a popular view of the subject. For a knowledge of the religion of India, we may refer also to *Encyc. Metr. Misc. Div.* article INDIA. Carwithen's *Bampton Lectures*, and Ward's *Hindoos*, are also useful; but for accurate information on this subject, Oriental sources must be consulted. The *Theological Catalogue* of Talboys (p. 314-324) gives an useful list for these purposes. Much information on heathenism generally is found in the works of Stillingfleet, Jenkin, Leland, &c. referred to under the head of Evidences.

1. Jewish Tradition.

The doctrine of the Jews relative to tradition, as Jewish stated in their writers of authority, such as Maimonides, &c., is, that Moses, while he was on the Mount, received both the *written* Law of the *Pentateuch* and also the *oral* Law, or the *proper interpretation* of this written Law, which was subsequently embodied in the *Mishna* and the *Talmud*.* This latter, the body of decisions respecting the interpretation of the written Law, was, according to them, handed down from him

* See the quotation from *Shemoth Rabba* in Eisenmenger's *Entdecktes Judenthum*, vol. i. p. 299, &c.

Theology. orally, till in process of time it became necessary to write it down, lest it should become lost or corrupted.*

As the written Law contains both *precepts* and the *history* of their delivery, so the *Talmud* contains both explanations of the precepts, each explanation being called an *Halacha*, and historical matter connected with them, each narrative being termed an *Agada*.† The extravagant assertions of the Jews with regard to the *Talmud* having been communicated to Moses and orally propagated from his time, need scarcely any argument to expose them. Those who are best acquainted with the *Talmud* know well how contradictory a large portion of it is to the plainest dictates of morality and religion, as found in the *Bible*, and how inconceivably childish, incredible, and foolish is the matter with which it often abounds. But the decisive point against it, is the utter want of any evidence to establish these pretensions.

In the *Pirke Aboth*, sec. 1, the succession from Joshua to Ezra is thus disposed of:—"Moses received the Law from Mount Sinai, and delivered it to Joshua; Joshua delivered it to the elders, the elders to the prophets, and the prophets to the men of the Great Synagogue."

The account given by Maimonides is just as unsatisfactory, and yet this is the sort of evidence for the early existence of this oral Law, on which we are required to believe that all the decisions of the *Talmud*, collected nearly one thousand years after the days of Ezra, were received by Moses from God Himself on Mount Sinai!

The account given by Maimonides (preface to the *Yad Hachazakah*) is this: "Although the oral Law was not written, Moses, our master, taught it all in his council to the 70 elders: Eleazar also, and Phinehas and Joshua, all three received it from Moses. But to Joshua, who was the disciple of Moses, our master, he delivered the oral Law, and gave a charge concerning it. In like manner Joshua taught it by word of mouth all the days of his life; and many elders received it from Joshua, and Eli received it from the elders, and from Phinehas." (Maim. ap. M'Caul, *Old Paths*, p. 177.) No successor is named to Joshua, and no proof of the existence of a *Sanhedrim* for the 200 years between Joshua and Eli can be given. Indeed the *Bible* would shew that there was then no *Sanhedrim*. See Dr. M'Caul's *Observations* in *loc. cit.*

An acquaintance with the views of modern Jews is absolutely requisite to those who enter into the Jewish controversy. A few treatises of the *Talmud* have been translated, but Chiarini's translation of the *Berachoth* is invaluable. Dr. Pinner's *Extracts, translation, and edition of the Talmud*, in Hebrew and German (*Compendium des Hierosolymitanischen und Babylonischen Talmuds*. Berlin, 1832), appears not to have advanced, till the close of last year (1842), beyond the Introduction. Besides the edition of the *Mishna* by Surenhusius, an useful edition of it, with points, and a German translation in Hebrew letters, has lately been published at Berlin. Lightfoot, Schættgen, Eisenmenger, Chiarini, M'Caul, &c., as quoted above, will be found of much service. Dr. Pinner and Jost (the chief editor of the Berlin *Mishna*) are both Jews; the latter is, however, rationalistic in his views. See his *Geschichte des Israelitischen Volks*.

2. On Tradition in the Christian Church.

Romish doctrine. Tradition.

In considering the question of Christian Tradition as an independent Divine authority, the claims of the Church of Rome alone come under examination. It is, therefore, necessary briefly to state the doctrines of that church on this important subject, in the language of her public formularies and her accredited

writers. Among her writers the principal authority is Theology. Cardinal Bellarmine, who has devoted a considerable portion of his controversial works to a treatise *De Verbo Dei*. Of this treatise, the three first books "relate to the written word of God (*verbum Dei scriptum*) as contained in the *Old* and *New Testament*; But the fourth book, which relates wholly to Tradition, is entitled 'Of the *unwritten* word of God' (*de verbo Dei non scripto*), and therefore declares by its very title the quality of that Tradition which is the subject of that book."* Bellarmine there observes, that although the name of Tradition† is a general term implying any doctrine, whether written or unwritten, which is communicated by one to another, yet Theologians properly restrict it to mean only *unwritten* doctrines.‡ He further states, that these are called *unwritten* doctrines, not because they are *nowhere* written, but because they were not written by the authors themselves.

These traditions are divided into three classes, of which we have here to consider only the two first; as the third (*Ecclesiasticæ traditiones*) is confessedly of human origin, and relates to customs, and not to doctrines. The two classes with which we are concerned are called, 1. *Divinæ*, and, 2. *Apostolicæ*, and are thus defined:—

1. *Divinæ [traditiones] dicuntur, quæ acceptæ sunt ab ipso Christo, Apostolos docente, et nusquam in divinis literis inveniuntur.* (Bell. *De Verbo Dei*, iv. 2.)

"*Divine [traditions]* are those which were received from Christ himself, teaching [them to] the Apostles, and are nowhere found in Scripture."

2. *Apostolicæ traditiones propriæ dicuntur illæ, quæ ab Apostolis institutæ sunt, non tamen sine assistentia Spiritus divini Sancti, et nihilominus non extant Scriptæ in eorum epistolis.* (Id. *ibid.*)

"*Apostolical traditions* are properly those which were taught by the Apostles, not, however, without the assistance of the Holy Spirit, and which, notwithstanding, are nowhere written in their Epistles."

Bellarmino (iv. 3.) proceeds to observe, that the doctrine of the Romish Church is, that the necessary doctrine, whether relating to *faith* or *morals*, is not all expressly contained in Scripture, and that besides the *written* word of God, the *unwritten* word, i. e. *divine* and *apostolical* traditions, are necessary; while Protestants contend that all things *necessary* to *faith* and *morals* are contained in the Scriptures. He states, that the total rule of faith is the word of God, or the revelation of God made to the Church, which is divided into two *partial* rules, *Scripture* and *Tradition*.

The Council of Trent, although it does not make use of the words *divine* and *apostolical* traditions, yet distinguishes the two classes of doctrines, as those taught by our Lord's *own mouth* and *by his Apostles*: it states that the whole truth and instruction so taught is to be found in the *written* books, and the *unwritten* traditions derived from Christ's own mouth, or from

* Bishop Marsh's *Comparative View of the Churches of England and Rome*, p. 6, last edition, 1841.

† Tradition is sometimes used for the *vehicle* by which a doctrine is transmitted, sometimes for the '*doctrina tradita*,' or the doctrine so transmitted. It is in the latter sense we are chiefly concerned with it here. Tradition is also used sometimes in the plural, denoting *Apostolical* doctrines, each of which is a tradition; but when used in the singular without the article, it usually denotes the *sum* total of these doctrines.

‡ Bellarmine, *De Verbo Dei*, lib. iv. cap. 2.

* See Maimonides, preface to the *Yad Hachazakah* and to the *Mishna*; the *Pirke Aboth*, sec. 1, with the notes in Leusden and Surenhusius; Chiarini's *Theorie du Judaïsme*, vol. i. p. 1-63; Ditto, *Le Talmud de Babylone traduit*, &c., *Prolegomènes passim*; Dr. M'Caul, the *Old Paths*, No. 45; Lightfoot's Works, *passim*.

† Chiarini's *Theorie du Judaïsme*, p. 62.

Theology. the Apostles, or from the teaching of the Holy Spirit; and it determines that the *Scriptures* and these *traditions* are to be received with equal regard and veneration (*pari pietatis affectu ac reverentia*).—*Concil. Trident. Canones et Decreta. Sess. iv. Decretum de Canonicis Scripturis.*

It will be seen, by the above statement, that *Scripture* and *Tradition* are placed by the Church of Rome upon an equal footing, and rendered independent of each other.* Every article delivered by the one must be equally valuable, equally necessary, with those of the other.

Now this view is entirely opposed by the Church of England, in her Sixth Article, which asserts, that "Holy Scripture containeth *all* things necessary to salvation, so that whatsoever is not read *therein*, or may be proved *thereby*, is not to be required of any man that it should be believed as an article of faith, or be thought requisite or necessary to salvation."

The rule of the Romish Church is therefore rejected by the Church of England; and on an examination of their respective doctrines, it will be seen that the doctrines received by Romanists on the sole evidence of tradition are rejected by the Church of England.

This is a brief statement of the principal questions at issue respecting tradition. It now only remains to inquire, as briefly as possible, on what grounds we may justly form an opinion on this point.

Bossuet, in the 18th chapter of his *Exposition de la Doctrine de l'Eglise Catholique*, says that Jesus Christ, having founded his church on preaching, the unwritten word was the *first* rule of Christianity; and when the writings of the *New Testament* were added to it, its authority was not forfeited on that account; which makes us receive with equal veneration all that has been taught by the Apostles, whether by writing or by word of mouth, which St. Paul expressly recommends to the Thessalonians. (2 *Thess.* ii. 14.) The fallacy in this argument is quite apparent; it lies in the word *added*, which is a *petitio principii*, for the very term implies such a difference between the things themselves as tacitly affords a foundation for the future† superstructure of the Romanists. It artfully implies that at least a *part* of the word of God, as delivered by Christ and his Apostles, was *not* recorded in the *New Testament*, a point which the Romanists cannot fairly take for granted, when arguing with those who deny it. With regard to the passage from St. Paul, it seems rather to refer to directions given to the Thessalonians as to conduct than to doctrines; and even if it referred to doctrines, cannot prove the point required, unless it can be shewn that these doctrines are not *now* a part of the written word of God, either in the Apostolical Epistles or the Gospels, as a part of the teaching of our Lord.‡

* According to the Romish view, the chief depository of traditions are the works of the Fathers, from which, however, the *doctrina tradita* is supposed to be transferred to the decrees of general councils; but still, as no such council was held till the IVth Century, the works of the Fathers must still be regarded as the *primary* repository.

† See Bishop Marsh's *Comp. View*, &c., p. 69.

‡ See Irenæus, who in speaking of the Apostles, (iii. 1.) says, *per quos Evangelium pervenit ad nos, quod quidem tunc præconaverunt postea vero per Dei voluntatem in Scripturis nobis tradiderunt, fundamentum et columnam fidei nostræ futurum.* The above passage of Irenæus ought to be remembered well by all who enter into this question.

Theology. The argument of the Romanists fails in the very first step, that of proving the *mere existence* of any such traditions; but if we examine it further, we shall find every succeeding step liable to some fatal objection or resting on some feeble argument. If there were any such traditions as these, how are we to *know* them,* and separate them, as found in the works of the fathers, from the less authoritative matter by which they are surrounded? When the Romanists appeal to the Fathers as speaking of traditions which came from the Apostles, the argument proves valueless, when we remember that under that word were understood the doctrines *written* in the Scriptures, or *customs* derived from the Apostles; and therefore such passages would give no sanction to the views propounded by the Church of Rome. The last weak argument we shall notice is that of Bossuet, who observes, "a most certain mark that a doctrine comes from the Apostles is, when all Christian churches embrace it, without its being in the power of any one to shew were it had a beginning." (Comp. Bell. *De Verbo Dei*, iv. 9.) It is ingenious to convert the *uncertainty* of the origin of a doctrine† (or our ignorance of *where* it had a beginning) into a *certain* mark that it comes from the Apostles; but it would seem to most rational minds a better argument to say, that of the doctrines recorded in the *New Testament* we *do know* the origin, and this very knowledge is an imperative reason for our receiving them. And it is a sufficient reason to establish at once a difference between those doctrines and mere traditional ones, which the Romish Church places on exactly the same level.

The question of *Tradition* is discussed in every controversy between the Anglican and the Romish Church. But, in general, in the older writers, two other questions are so mixed up with it, that they can hardly be separated, the *doctrine of Fundamentals* and the *final Resolution of Faith*. The Church of Rome holds all points defined by the Church to be fundamental. The Protestants give no catalogue of *Fundamentals*. Archbishop Laud said, that "all the points of the Creed were fundamental." (Laud's *Conference with Fisher*, § 2.) See also White's *Conference with Fisher*, § 2, 5, and 6. See also Stillingfleet's *Grounds of the Protestant Religion*. Mr. Palmer's *Treatise on the Church*, vol. i. p. 122, has some very useful observations on the *Doctrine of Fundamentals*, in which he objects to the use of the word.

On Tradition and Scripture, the treatise of Whittaker, *De Sacra Scriptura*, ought by all means to be consulted, as well as Bishop Marsh's *Comparative View*, &c. The *Downside Discussion* is also useful, although some of the Protestant statements about the Church, &c. are unsound. Stillingfleet's *Tradition and Scripture compared*, and his *Council of Trent disproved by Tradition*, are very valuable. Archbishop Tillotson's book, *On the Rule of Faith*, is a sufficient answer to the weak book against which it was written, but does not give a general view of the whole question.

3. Inward Light and Special Revelations.

There is yet one other source of knowledge which men have invented beyond the word of God; that is to say, false and pretended revelations. In many ages

* The particular Romish Church setting herself in the place of the whole Catholic Church, makes her acceptance of them a sufficient proof that they are traditions of the whole church from the first times.

† It would, indeed, be a strong argument in favour of any doctrine, to shew that it had been universally received at all times; but we may confidently ask which of the doctrines received by the Church of Rome, and rejected by the Church of England, such as *Transubstantiation*, the *Worshipping of the Host*, the *Propitiatory Sacrifice of the Mass*, the *Invocation of Saints*, the *Communion under one kind*, &c., can be supported by any such evidence from the first four centuries?

Theology.
Internal
light.
Special re-
velations.

of the Church these have existed, but have usually died away after a short season. Some of the most remarkable have been those of the IInd Century, viz., the heresy of Montanus. Of this a sufficient account is given in another portion of this work, Hist. Div. iii., 150.* Subsequently, Ecclesiastical History gives us several instances of similar claims to prophecy and divine illumination, as for instance, Munzer and Stork in Germany, just at the rise of the Reformation, and subsequently Kotter, Poniatowsky and Drabicius, whose prophecies are found collected in a volume called *Lux e Tenebris*, published in 1665,—these prophets having uttered prophecies which were altogether falsified by the events, e. g. the success and glorious subsequent career of Gustavus Adolphus just before the battle of Lutzen, in which he perished.† Again, about the beginning of the last Century occurred the case of the French prophets, of which some account will be found elsewhere. (Hist. Div. vol. v. p. 1139.) Their views were revived, and some of their publications were reprinted, by Mr. Irving, a few years ago. In the preface to his reprint of *The General Delusion of Christians* (an answer to *The Spirit of Enthusiasm Exorcised* of Dr. Hickes), Mr. Irving maintains that the predictions of these prophets were partly given by God's spirit, but were affected and imitated in other cases by the devices of Satan; and this so effectually that it is impossible now to separate the gold from the dross, or scarcely to recognise any of the precious metal. It will hardly be necessary to do more than to mention these unhappy attempts to claim those gifts of the spirit which are miraculous, instead of endeavouring to profit by the influence of the Holy Spirit of God upon the heart of man, in leading it to holy thoughts and righteous conduct. For another form under which this delusion has affected man, we must refer to our Miscellaneous Division, under SWEDENBORG.

There is one more circumstance which claims our notice, and that is the necessity of an Inward Revelation, as maintained by Barclay in his *Apology for the Quakers*. (Prop. i.—iii.) It is unnecessary to do more than to refer the reader to this portion of the author, who is acknowledged to give us such a system as Quakers profess to abide by (Wotton's *Thoughts on the Study of Divinity*, p. 50); for it seems to us that a treatise which gives only the second place to Scripture (Barclay, Prop. iii. sec. v.), proved to come from God by prophecies and miracles, and sets above it the inward light which a man believes to come from God, cannot produce much effect on any person acquainted with the word of God. Nor does there seem to us anything in his three propositions but very illogical reasoning and very erroneous conclusions. To those, however, who wish for an answer to this treatise, we may mention the *Confutation of Quakerism*, by T. Bennet, D.D. London, 1733, third edition.

No serious and religious mind can for a moment doubt the necessity of the guidance of God's Spirit to man, to lead him to all things profitable to his salvation; but this influence is a moving influence on his will, and not an immediate revelation to his intellect. That God,

Theology.
as we continue to do His will by this influence more perfectly, will vouchsafe to us a further knowledge of His doctrine, follows from our Lord's own words, and forms the hope of every devout Christian. As he becomes more influenced by that Spirit, his views of God's dealings with man, we cannot doubt, will be clearer and more satisfactory, and the Scriptures will bring greater knowledge of these to his mind, and further comfort to his heart; but we know of no principle either of reason or Scripture which will admit of our placing any such influences in the light of inward revelations, or allow them to supersede the appointed human means of obtaining a knowledge of Divine things.

This is, we must remark, only one of the common forms which mysticism assumes. When a man is persuaded that he has immediate revelations from God, and can yet produce no evidence but that internal impression (which it is clear can be no evidence whatever to another man's mind), it is of course nearly useless to reason with him. We can only point out to him that he gives no evidence of the fact which is trustworthy to another, and point out the difference between his case and that of Scripture revelation. When we have full evidence that God has interposed miraculously to instruct the world in the case of certain individuals, we do not require external evidence in every special case: they are proved already to be living under a different condition, and in a state of immediate communication with God, and that is sufficient; but the proof that they are so is in the first instance external. Scripture gives no instances of an internal call from God with special revelation, unaccompanied by external proofs.

Of the Rule of Faith.

Having stated the Roman Catholic doctrine of Tradition, and shewn that it is unwarranted by any principle which can be judged trustworthy, we now proceed very briefly to consider what is usually called "The Rule of Faith." We must, however, premise that this is a term which we should prefer to see entirely disused. The principle for which we contend is, that the Inspired Scriptures are sufficient for salvation; so that those things which are not read therein, nor can be proved by them, are not to be required to be believed as necessary to salvation. We can allow no place to any unwritten tradition to stand beside this depository of God's word, and challenge our belief to any matters of faith not contained in that book, or proved by it, with equal claim and independent authority. This, it will have been seen by the preceding section, is the claim set up by the Church of Rome for unwritten tradition. Nothing short of this claim satisfies either the language of the Council of Trent or the authorized writers of that church. We are not aware that the authorized formularies of either church have defined this expression. It has, however, been used by controversialists on both sides; and the usual statement has been, that Scripture and Tradition is the "Rule of Faith" to the Church of Rome; Scripture the "Rule of Faith" to the Church of England. This is the position very commonly assumed by the writers on this controversy. We conceive that this statement is unnecessary on either side, after the explicit statements of the 6th, 7th, and 8th Articles of the Church of England, and the equally explicit declarations of the Church of Rome in the Fourth Session of the Council of Trent, and the Creed of Pius IV. We think the controversy between the two churches is uselessly encumbered by the use of this term, and that it only serves to engender greater strife, and makes it less easy to discern clearly the principles by which both are guided. But on the other hand the Church of England is ready to shew that she

* In addition to the sources of information there quoted, we may refer also to Neander, *Allgem. Geschichte der Christl. Kirche*, vol. i. sec. v. (In the translation, vol. ii. p. 165-195.)

† For a good account of these German prophets, see *The Modern Claims to the Possession of the Extraordinary Gifts of the Spirit, Stated and Examined, &c.*, by the Rev. W. Goode, 1833.

Theology. has rejected nothing in her articles which has any real claim to universality and primitive antiquity, and at the same time to shew that those points for which the Church of Rome appeals to tradition, even if legitimately tried by that test, have no real claim to our acceptance.* The words "Rule of Faith" (*regula fidei*) are found in Tertullian (*De Virg. Vel.* p. 173, ed. Rigalt: compare *Adv. Praxeam*, p. 501), and there applied to the Creed. Irenæus uses the words *veritatis regula*, but we believe that the phrase *regula fidei* first occurs in Tertullian. We are aware that Irenæus appeals to tradition; but, on the other hand, when we examine the articles of faith for which he makes this appeal—an appeal, be it remembered, in answer to an appeal to tradition also on the part of the heretics—we find that it is for those very propositions which are found in the Creed.† But as tradition has been as fully considered as our limits allow, in a preceding section, we need not repeat these arguments. We have already observed that we prefer not to use the words "Rule of Faith;" but while we do this, we cannot allow that the *Bible*, interpreted according to the private judgment of individuals, is a sufficient *test of doctrine*, or the test by which the Church of England meant to abide in setting forth the great doctrines of the Scripture as taught in the Church. The Church of England, in its Articles, expressly declares the three Creeds to be certain, as being able to be demonstrably proved from Scripture; and the greatest writers of that church have not scrupled to acknowledge that the *universal testimony* of the Primitive Church to the doctrines of Scripture is a guide not to be departed from in ascertaining the doctrines of our faith; but in so doing they utterly repudiate the *particular* tradition of the Church of Rome.

We have, then, in conclusion, to state the following general results:—

1. That the Scripture containeth all things necessary to salvation.
2. That the Creeds, though not drawn originally from the Scriptures, contain a doctrine identical with that of Scripture, may be proved from Scripture, and cannot be gainsaid.
3. That the Church of England rejects no doctrine attested by the universal consent of the Catholic Church for the first four centuries.

* See Stillingfleet's *Tradition and Scripture compared*; his *Council of Trent disproved by Catholic Tradition*; Whitaker's *De Sacra Scriptura*; Bishop Marsh's *Comparative View*.

† Irenæus, *Contra Hæreses*, lib. iii. in initio, cap. i.-iii. Massuet, in his *Dissertation* on this subject, makes a great handle of this passage against Grabe, whose notes, however, he does not venture to quote.

4. That she allows no other independent source of Divine authority, and that the articles on which she differs from the Church of Rome can be shewn to be late introductions into the Church, and many of them absolutely opposed to primitive tradition and to Scripture.

For the use which individuals are to make of Scripture, we need not enter into much detail. That the fullest and most free use of the Holy Scriptures ought not only to be allowed, but recommended and enjoined by the Church, we cannot hesitate to affirm; but not in order to draw from them a system of doctrine. We believe that no discoveries of great religious truths remain to be made; we believe that the early Christian Church was in possession of the great fundamental points of doctrine, and that he is in the safest and most healthy condition of mind who, submitting his private judgment on these points to the universal Church of Christ, reads the Scripture in the spirit of prayer, chiefly for the edification of his soul and for instruction in righteousness; to attain a further acquaintance with the infinite goodness and mercy of God; to attain to a purer love of God's holy truth, and a deeper gratitude for His overwhelming love; to attain to a more humbling conviction of sin, a more penitent heart, a more ready acquiescence in God's will, and a more perfect obedience to His law. He that desires thus to know God, as revealed to us in Scripture as the Father who created, the Son who redeemed, and the Holy Ghost who sanctifies us; he who desires from his perusal of Scripture to rise with a more enlightened faith in Jesus Christ, and a brighter charity to man, we believe turns Scripture to its chief purposes as destined for individuals, and he will find it a lamp to his feet and a comfort to his heart!

Archdeacon Manning, in his Visitation Sermon entitled *The Rule of Faith*, has treated many of these points well. He maintains that the "Rule of Faith" is the *test of doctrine*, the rule by which we may ascertain what are the doctrines of the Gospel; and he thus briefly expresses what he considers the "Rule of Faith" of the English Church: "Scripture and Antiquity, or Scripture and the Creed attested by Universal Tradition." And he shews by quotations that such a rule is recognised by the great writers of the Primitive Church and the Church of England. We have contended against the phrase; but in the practical application of such a rule we have certainly a perfect reliance that it would establish nothing rejected by our church, nor impugn anything which it maintains. And there is good reason for this. One principle by which the reformers were guided, was deference to the universal consent of antiquity in extracting doctrines from Scripture. This is a different thing from the *interpretation* of particular passages of Scripture. The latter is the province of *Hermeneutic* and *Exegetical* Divinity. To this work, and the others mentioned in preceding sections, we must refer those who wish to enter further into this question.

APPENDIX TO CHAPTER II.

On the Argument from Prophecy.

Theology. It may, perhaps, be desirable to sketch out the course which the argument from prophecy here brought forward takes, and to point out the only difficulty to which it seems liable.

It is first pointed out that the *Bible* is the most ancient historical record existing; that several portions of it claim to be prophetic; and that when we apply the test of history to many of the predictions contained in it, the truth of this claim is wonderfully confirmed by the agreement of these predictions with subsequent experience and history.

These are mere preliminary steps to a consideration of the grand argument arising from the fulfilment of the prophecies of the *Old Testament* by the events recorded in the *New*, and by the results consequent on the preaching of the Gospel. They are merely meant to challenge our attention to the *Bible* as being *prima facie* distinguished from all other books.

The prophecies of the *Old Testament* are then examined which relate to the promise given to man in the person of his first parents, and continued from time to time throughout the age to which the *Bible* belongs.

These prophecies may be divided into perfect and imperfect. (See Archdeacon Lyall's *Propædia Prophetica*.)

1. The *perfect* prophecies are those which were understood and acknowledged to be prophecies of the events of the Messianic age.

2. The *imperfect* prophecies are those the meaning of which was not ascertained and acknowledged till after the events had taken place in which their fulfilment is traced. Such are many of those which relate to minute points in our Saviour's history, which are yet far too numerous and remarkable to admit of any explanation but that which supposes them Divine predictions. (See above.)

Archdeacon Lyall, in the valuable work to which we have so often referred, has treated this question with great clearness and force. He argues that the prophecies were intended to educate the Jewish people (to be a *propædia* to them) as to a future revelation, and enable them to recognise it when it pleased God to give it; and that, as we might expect from these circumstances, the main object of the Apostles, in most of their arguments, is to prove that Jesus was the Messiah whom the Jews had been taught to expect. He has given the opinions of the oldest Jewish writers of authority (as we have before stated), and he believes that they represent, from traditional sources, opinions which we may consider contemporary with, and previous to, the times of the Apostles.

The question is one of great importance, and it requires great judgment and discrimination to discuss it properly. Professor Lee, it will have been seen, takes an entirely different ground, and denies that these opinions can be traced to so early a period. This would, of course, give a complete answer to Bertholdt; but

the question still arises, which view is founded in *Theology*. truth? and we must not adopt this opinion merely because it affords a ready answer to a neological—we might almost say an infidel—work, which, after all, is of no great weight or importance. The collections of Bertholdt are useful; his reasoning upon them of little value; and we think the considerations offered in our note upon the subject (p. 869), if pursued legitimately, would furnish an overwhelming argument against him. We may be fully assured, that whatever the Almighty has done in the matter of evidence, will be liable to no real objections, although our ignorance may occasionally raise up apparent difficulties.

Commending, therefore, the whole subject to the strict investigation of the learned, we shall close these remarks by adducing the summary given by Archdeacon Lyall, which is almost indispensable to the complete development of our argument.

Extract from Archdeacon Lyall.

"In order to keep the proofs which I shall bring forward within a reasonable compass, I shall confine them to a fixed part of the Old Testament. By far the largest number of the passages alleged by the writers of the New Testament are found in two books, viz. Isaiah and the Psalms. Let us then take these, and examine, one by one, every passage quoted from them by the Apostles, as applicable to Christ. Next let us turn to the two works just mentioned, to see whether the same passages were referred by the ancient synagogue to the Messiah. Whenever this shall appear to have been the case, it will be evident that the sense put upon them by the Apostles was not of their own 'private interpretation,' but was that which the nation at large had been instructed to receive.

"Ps. ii. 1, 2, 6, 8, quoted Acts iv. 25, 28; xiii. 33. Referred to the Messiah in Melchita, fol. 3, 3. Sohar. Gen. Midrash. Tehillim.

"Ps. viii. 4, 6, quoted Heb. ii. 6, 9. Referred to the Messiah in Tikkune Sohar. c. 70.

"Ps. xvi. 8, 11, quoted Acts ii. 25, 32. Referred to the Messiah in Bereschith Rabba, sec. 88.

"Ps. xxii. 1, 8, 16, 18, quoted Matt. xxvii. 46. Referred to the Messiah in Midrash Tehillim. Pesikta Rabbathi in Talkut Simeoni, fol. 56, 4. Sohar. Numer. fol. 100.

"Ps. xl. 6, 8, quoted Heb. x. 5, 10. Referred to the Messiah in Midrash. Ruth, fol. 43, 3, 4.

"Ps. xlv. 1, 7, quoted Heb. i. 8, 9; Rom. ix. 5. Referred to the Messiah in Targum. Sohar.; and also by the modern Jewish commentators.

"Ps. lxviii. 18, 19, quoted Ephes. iv. 8. Referred to the Messiah by R. Obadja Haggaon, cited by Cartwright. Schemoth Rabba, sec. 35.

"Ps. lxix. 21, quoted Matt. xxvii. 34, 48.—Gall and vinegar given to Christ to drink. I have found no Jewish authority for the application of this particular fact to the Messiah, either in Schoettgenius or the *Pugio Fidei*; but the Psalm itself is applied to him generally by several writers quoted by Martini.

"Ps. cx. 1, 4, quoted Heb. v. 5, 6; vi. 19, 20. Compare Sohar. Gen. fol. 35; Sohar. Num. fol. 99; Midrash. Tehillim ad loc. Targum. Sohar. Chadash, fol. 42; Gen. fol. 42, 29.

"Ps. cxviii. 22, 23. Compare Sohar. Gen. fol. 118; Idem Numer. fol. 86, *et passim*.

"The above are the only Psalms to which I can find any plain allusion in the New Testament; and if we may trust to the references given by Schoettgen and Raymundus Martini, they are all of them, either generally or particularly, applied to the times of the Messiah by the old Rabbinical writers. We are not, however, to suppose that those here quoted are the only Psalms which

Theology. the ancient Jewish Church so explained. On the contrary, many not adduced in the New Testament might be added. The principle of interpretation adopted by the Jews would appear to have been very simple. It was, that whenever any expressions were found in the prophetic writings conveying a meaning too high and comprehensive to admit of an historical application to known persons or events, such expressions should be referred either to the Messiah himself or to his promised kingdom. As to double senses of the prophecies, of which Grotius and Warburton talk, and other writers after them, it may be doubted whether such a notion ever entered into the minds of the ancient Jews. Their rule seems to have been founded on the opposite supposition, that no prophecy could have two senses; and, therefore, that when the literal sense of the inspired writer afforded no intelligible meaning, the words were to be understood prophetically. In fact, if it be once allowed that a prophecy is capable of more than one true interpretation, where are we to fix the limit? The danger of such a principle needs not to be pointed out; and except it be founded on stronger reasons than are given by Grotius in his Commentary on St. Matthew, chap. i., it is as unfounded as it is dangerous. But to return to our subject.

Of the sixty-six chapters which compose the Book of Isaiah, all, except fifteen, are referred, by one Jewish writer or another, to the times of the Messiah; but in the New Testament, I think that there are not quotations from more than sixteen or seventeen.

"Is. ii. 1. 5. Conversion of the Gentiles. John x. 16; Acts xxviii. 28. These passages are applied to the Messiah in the Targum, and generally by Jewish commentators, both ancient and recent.

"Is. vii. 14. The miraculous birth of Christ. '*Hoc caput*,' says Schoettgen, '*Judæi antiquiores, ex inscitâ, juniores vero, ex malitiâ neglexerunt.*' I shall take occasion, in my next lecture, to offer some remarks upon this important prophecy, which will, I hope, both explain the ignorance of the ancient Jews, and vindicate the present from the charge here preferred by Schoettgen: but in the meantime it is sufficient to say that this prophecy stands out almost singly, as one which the Apostles have applied to Christ on their own authority.

"Is. viii. 13, 14. Christ a stone of stumbling. Rom. ix. 33; 1 Peter ii. 7, 8. Applied to the Messiah in Sanhedrim, fol. 38. Breschith Rabba, sec. 42, fol. 40.

"Is. ix. 'Unto us a child is born.' This very important

Theology. prophecy is referred to the Messiah in the Targum, and it is generally so understood by Christians. Nevertheless, I cannot satisfy myself that any allusion to it is to be found in the New Testament.

"Is. xviii. 16. Christ the chief corner stone. 1 Pet. ii. 3, 6. Applied to the Messiah in Sanhedrim, fol. 98, 1. Talkut Simeoni, i. fol. 49, 3. Breschith Kezara, citante Raymundo Martini in *Pug. Fid.* ii. 4, p. 313.

"Is. xxx. 3, 4, 15. Miracles of the Gospel, and effusion of the Spirit. Acts ii. 4; Rom. xi. 18. Compare Janchuma, fol. 1, 2; Debarim Rabba, sec. 6, fol. 258, 2; Sohar, Chadash, fol. 89, 3.

"Is. xxxi. Times of the Messiah. New Testament *passim*. See Pesikta Rabbathi, fol. 29, 3. Tanchuma, Talkut Simeoni, i. fol. 157, 1. Sohar. Exod. fol. 34, col. 134.

"Is. xl. John the forerunner of Christ. This chapter is referred to the Messiah by the present Jews as well as by the ancient. See Kimchi; Aben Esra; Pesikta in Talkut Simeoni, ii. fol. 49, 1, as quoted by Schoettgenius *in loco*.

"Is. xlii. 1, 7, 16. New Testament *passim*. Applied to Christ in the Targum, and by all the present Jews.

"Is. liii. The whole chapter is referred to the Messiah in the New Testament, as it is also in the Targum, and in the Sohar *passim*.

"Is. lv. 1, 5. Christ the living water. John iv. 10, 14. Schoettgen quotes from Galatinus, Breschith Rabba ad Genes. xlix. 14; but the passage is not found, he tells us, in the editions which he has consulted.

"Is. lx. Glory of Christ's kingdom. New Testament *passim*. So applied in the Targum, and by the ancient Jewish Church, *passim*.

"Is. lxi. Christ anointed by the Spirit. Luke iv. 16; Matt. iii. 16, 17. This chapter is referred to the Messiah by the modern Jewish commentators as well as the ancient.

"If it would not be tedious, it would be a task of no difficulty to go through the remaining passages quoted from the Old Testament by the Apostles, in confirmation of Christ's divine commission. They are, I believe, not more than between twenty and thirty; and with the single exception of Job xix. 25 (about which the Jews both of the present and of former times are silent), in every instance the authority of the ancient synagogue may be produced in confirmation of the interpretation the Apostles affixed."—*Propædia Prophetica*, p. 101—106.

OUTLINES OF THEOLOGY.

CHAPTER III.—SCRIPTURE DOCTRINES.

The Doctrine of Holy Scripture, (1) respecting God, and (2) respecting Man.

Theology. (1.) ALTHOUGH, as we have seen, the indications of a God are so many "in the things which are made," that all must be regarded as "without excuse" who do not recognise there the "power and Godhead" of the Supreme Lord, yet the idea of what God is, can only be communicated by God himself. On this subject, of all others, we require the teaching of Divine revelation. The Psalmist, therefore, after asserting that the heavens declare the glory of God, and the firmament His handy-work, and that the regular successions of day and night proclaim continually the knowledge of the Most High, refers at length to the revelation of God's law as the source from whence, after all, the most perfect instruction respecting the Deity will have to be collected.*

We may not, however, suppose that revelation is intended to make us so familiar with the nature and attributes of God, as that we should, by searching it, "find out God to perfection." He who dwells in the light that is inaccessible to the mortal gaze, must, in the nature of things, be so far exalted above the region of man's finite apprehension, that in our most advanced stage of information respecting God, we shall have to admit that "such knowledge is too wonderful and excellent" to be attained by us; and that even with the assistance of revelation we shall be conscious that, as yet, "we see through a glass darkly." Nevertheless, so far as God has been pleased to reveal himself in His word, we not only may, but ought, reverently to study what revelation teaches respecting the Divine attributes and perfections, that by this means we may catch some rays of the ineffable majesty and glory of Him "whom to know is everlasting life."

(2.) On opening the *Bible*, then, we find that we are no longer left to collect the *unity* of God from probable inference, for we perceive that cardinal verity to be proclaimed from one end of the *Old Testament* to the other. There is no truth, in fact, which God's ancient people were more strictly charged to remember, than that "the Lord is God in heaven above and in the earth beneath; there is none else:" that "the Lord our God is one Lord."† Therefore, we find an acknowledgment of this prime article of revelation in almost all the recorded addresses to God that believers of old uttered:‡ whilst the Almighty calls His chosen nation to bear witness that "before Him there was no God formed; that besides Him there is no God."§ The same important doctrine pervades the *New Testament* Scriptures also. Our Lord himself, for instance,

represents the unity of God to be the first and great Theology. commandment.* Then the Apostle of the Gentiles, in speaking of the conversion of the Thessalonians to the Christian faith, represents that important event as a turning "to God from idols, to serve the living and true God:"† and in accordance, therefore, with this sentiment, he asserts on other occasions that there is to Christians "but one God, the Father, of whom are all things: the one Lawgiver who is able to save and to destroy; the blessed and only Potentate; the King of kings, and Lord of lords."‡

(3.) Yet this one Lord is not a Being that can be represented to the bodily senses. Nothing, indeed, can be more striking than the contrast between the idea of God which, in this respect, Revelation inculcates, and that notion of Deity which originates in the inventions of the human mind. Whilst those attempts to exhibit the "King, immortal, invisible," by sensible images, which mark the history of superstition in every age, shew how low and unworthy are the conceptions of the Divine Nature that man can entertain, the Scripture representations of God are, that He "is a Spirit," whom no man hath seen, or can see; who can be likened to nothing, and compared to no likeness; and that, therefore, it is as useless as it is sinful to picture the great God by the image of anything that is "in heaven above, or in the earth beneath, or in the water under the earth."§ Although, therefore, the Scriptures may sometimes speak of the 'face,' the 'eye,' or the 'arm' of the Lord, as if the Deity were possessed of the different members of the human body; yet the incorporeity of God is so plainly assumed throughout the *Bible*, that representations of this nature cannot be understood in any literal acceptance of the terms used. But as we ourselves are frequently obliged to refer to sensible objects in order more clearly to express our conceptions of spiritual things, so the Almighty, in condescension to our limited faculties, has been pleased to speak of Himself by references to things with which we are familiar, that thus we might be led more vividly to apprehend, not the form and essence of His invisible nature, but the real existence of the Divine attributes and perfections as manifested in visible operation.|| The same observation applies also to

* *Psalm* xix.; See also *Psalm* xxix. xcii.

† *Deut.* iv. 35, 39; vi. 4; xxxii. 39.

‡ *1 Sam.* ii. 2; *Isaiah* xxxvii. 16, et alibi.

§ *Isaiah* xliii. 10; xlv. 6, 8.

* *Mark* xii. 29, 30.

† *1 Thess.* i. 9.

‡ *1 Cor.* viii. 4, 6; *1 Tim.* ii. 5.

§ *John* iv. 24; *Exod.* xx. 4; *Isaiah*, xl. 18.

|| Tertullian (*Adv. Praxeas*, cap. vii.) makes use of an expression ("quis negabit Deum corpus esse, &c.") which has caused him to be charged with maintaining the corporeity of the Deity. It is probable, however, that by *corpus* he meant substance generally, as expressive of individuality and real existence, and as opposed to *inanimatus*. We are still inclined to suggest this explanation, although St. Augustine seems hardly willing to give him the benefit of it. (*Aug. Ep.* 157.) There were others, also,

Theology. such portions of Scripture as represent the Deity as being actuated by love or pity, anger or jealousy, or other human affections,—these various terms being employed to convey to us some idea of those various workings of Divine providence and grace which are observable in the universe.

(4.) The God who is thus purely spiritual, “without body, parts, and passions,”* is also the king “eternal,” “the number of whose years cannot be searched out. He is the first and the last; whose existence was before the world began, and who continueth and liveth for ever: for before the mountains were brought forth, or even the earth and the world were made, He is God from everlasting and world without end. He only hath immortality: He was, and is, and is to come.”†

(5.) In intimate connexion with this doctrine, the *Bible* declares to us the *immutability* of God. With Him “one day is as a thousand years, and a thousand years as one day. His counsel standeth fast for ever, and the thoughts of his heart to all generations. With Him is no variableness, neither shadow of turning. The heavens and the earth shall perish, but He endureth: they all shall wax old as doth a garment, and as a vesture shall they be changed; but He is the same, and His years shall have no end.”‡

(6.) The Scriptures, therefore, as might be expected, display to us the Divine *power* and majesty in a much more striking light than those attributes are reflected even in the works of creation. We may be at no loss to conclude, from what nature reveals, that the power of God surpasses all that we can conceive of; but still all that creation teaches falls far short of such declarations of that Divine attribute as the *Bible* contains. We there learn that “by the word of the Lord were the heavens made; and all the host of them by the breath of His mouth: that He spake and it was done; He commanded and it stood fast. He alone spreadeth out the heavens and treadeth on the waves of the sea: He maketh Arcturus, Orion, and Pleiades, and the chambers of the south. He stretcheth out the north over the empty place, and hangeth the earth upon nothing. He gathereth the waters of the sea together as an heap: He layeth up the deep in store-houses: He measureth the waters in the hollow of His hand, and meteth out the heavens with the span; and comprehendeth the dust of the earth in a measure, and weigheth the mountains in scales, and the hills in a balance. Behold he taketh up the isles as a very little thing—all nations before Him are as nothing. He doeth according to his will in the army of heaven, and among the inhabitants of the earth; and none can stay His hand, or say unto Him, what doest Thou?”§

(7.) So, again, that *wisdom* which is manifested in the adaptation of every thing to its proper place in the universe, and which implies, also, a perfect knowledge, on the part of the Deity, of all things that are the subjects of creation, is represented in Scripture as

Theology. extending without limit of time, place, or circumstance. We are told, that “known unto God are all His works, from the beginning of the world: that He declareth the end from the beginning, and from ancient times the things that are not yet done.” We are assured that the “eye of the Lord is in every place; that He looketh unto the ends of the earth, and seeth under the whole heaven: neither is there any creature that is not manifest in his sight: that He filleth all in all.”*

(8.) We feel, moreover, to be surrounded by an *ever-present* God, when we read, “O Lord thou hast searched me, and known me; Thou knowest my down-sitting and my up-rising; Thou understandest my thoughts afar off. There is not a word in my tongue, but lo! Thou, O Lord, knowest it altogether. If I ascend up into heaven, Thou art there; if I make my bed in hell, behold, Thou art there. If I dwell in the uttermost parts of the sea, even there shall Thy hand lead me, and Thy right hand shall hold me. If I say, surely the darkness shall cover me; even the night shall be light about me. Yea, the darkness hideth not from Thee, but the night shineth as the day: the darkness and light are both alike to Thee. None can hide himself in secret places that the Lord Jehovah cannot see him. Wherever we may be, God is not far from every one of us:” to Him “all hearts be open, all desires known, and from Him no secrets are hid.”†

(9.) And if amid the apparent inequalities of this life, and the conflict of good and evil, natural religion can spell out with difficulty that the maker of all is *good* unto all; the sure word of God teaches that “the Lord is loving to every man, and that his tender mercies are over all His works; that the earth is full of His goodness: that He opens His hand and satisfieth the desire of every living thing:” nay, that “God is love.” Therefore, when the Most High condescended to make himself most familiarly known to Moses, it was as “the Lord God, merciful and gracious, long-suffering and abundant in goodness:—as keeping mercy for thousands.”‡ And that the Divine goodness may be more effectually brought home to the mind, the Scriptures repeatedly direct attention to those manifestations of benevolence in all that God does, and hath done, to bring mankind to happiness and salvation; and to the fact that “he causes his rain to descend on the bad as well as on the good; on the unjust equally with the just; and is continually doing good to the unthankful and the evil.”§

(10.) Yet is this benevolent God himself removed from all possible sympathy with evil: for we find ourselves called upon by Revelation to reverence “the Lord our God as *holy*: righteous in all His ways, and holy in all his works:” nay, “glorious in holiness.” That, as possessed of all moral perfection, He consequently is a God who can have no complacency in wickedness, and with whom no evil can dwell: that He regards all workers of iniquity with displeasure, and that “the sacrifice, the way, the thoughts of the wicked, are an abomination unto Him.”||

of the early Christian writers who, in their controversies with those Heretics that denied the doctrines of the Trinity and Incarnation, sometimes used terms respecting the mode of the Divine existence which are liable to misinterpretation.

* *Acts* i.

† *Job* xxxvi. 26; *Psalms* xc. 3, et seq.; cii. 24, 28; 1 *Tim.* vi. 16; *Rev.* i. 4; iv. 9.

‡ 2 *Pet.* iii. 8, 9; *Psalms* xxxiii. 11; *James* i. 17; *Psalms* cii. 27, 29.

§ *Job* ix. 8, 9; xxvi. 7; *Psalms* xxxiii. 6, 7, 9; *Isaiah* xl. 12, 17; *Dan.* iv. 35.

VOL. II.

* *Acts* xv. 18; *Isaiah* xlv. 10; *Prov.* xv. 3; *Job* xxviii. 24; *Heb.* iv. 13.

† *Psalms* cxxxix. 1, 12; *Jer.* xxiii. 24; *Acts* xvii. 27: *Communion Service*.

‡ *Psalms* ciii.; civ.; cxix. 68; clv. 9, 15, 16; *Ezod.* xxxiii. 19; xxxiv. 6; 1 *John* iv. 8.

§ *John* iii. 16; *Rom.* v. 6—11; *Tit.* ii. 11—14; *Matt.* v. 45.

|| *Ezod.* xv. 11; *Josh.* xxiv. 14—16; *Psalms* v. 4; cxlv. 17; *Prov.* xvi. 8, 9, 26.

Theology.

(11.) For inseparably connected with the revelation of the Divine holiness is the awful attribute of the Divine justice. The Deity is declared to be a "just God;" righteous in all his ways. The Lord Jehovah is thus presented to our faith as prescribing most righteous laws; as administering those laws with the most perfect impartiality; as manifesting, in His providential control of the world, His approbation of good and His displeasure against evil; whilst yet we are taught to expect the full development of the workings of this awe-inspiring attribute only in a future life. The Bible thus clears up those difficulties which otherwise arise out of the apparent affliction of the righteous and prosperity of the wicked, because we are assured, however much appearances may be to the contrary, that the "Judge of all the earth" does "right;"* for that the very habitation of His throne is justice: that hence, as no one good thing which He has promised can fail to come to pass, so not one punishment which He has threatened against the evil-doer will remain uninflicted.

1. Our object being, as we have before observed, to treat this subject practically, we have here touched only upon some of the Divine attributes; and that in the order which appeared best adapted to connect what we had to say with what precedes under the head of Natural Religion. It may, however, not be out of place to mention some of those divisions under which different theologians have arranged the Divine attributes.

a. Some, for instance, divide the attributes of the Deity into *positive* and *negative*. Under the former division are included those attributes an analogy for which we can find (of course in a vastly inferior degree) in ourselves, such as wisdom, benevolence, &c. By *negative* attributes are understood those which imply the absence of all imperfection, such as infinity, omnipresence, &c.

b. Other divines, again, regard the attributes of Deity as being either *active*, *i. e.* those which in any way imply action or operation, such as omnipotence, &c.; or *passive*, those, namely, which imply inaction or repose, such as eternity, &c.

It is manifest, however, that this division of the attributes of God is in the main identical with that which has just been mentioned; whilst the terminology employed is open to objection, since this idea of rest can scarcely be considered as applicable to the God of the Bible.

c. The Divine attributes have further been distinguished by the terms *moral* and *physical*, or *natural*; by *moral* being meant such perfection of the Deity as admit (in some feeble degree) of imitation, such as goodness, holiness, &c.; whilst by the term *physical* or *natural* are to be understood those attributes which find no resemblance in anything created, such as omnipotence, independence, &c.

These perfections have, therefore, been again distinguished, one as being *communicable*, the other as *incommunicable*; as *imitable* and *inimitable*.

In this third division of the Divine attributes it is that theologians now generally acquiesce, as being the least liable to objection.

See Wilkins on *Natural Religion*; Limborg, lib. ii. c. 2; Turretin, disc. iii. quest. 6; Buddei's *Instit. Theol.*

* Psalm xix. 8-12; Rom. ii. 6-10; Gen. xviii. 25; Psalm lviii. 10, 11; lxxiii; Josh. xxiii. 14-16.

Dogm. lib. ii. c. 1, s. 2; Knapp, *Vorlesungen über die Christliche Glaubenslehre*, art. ii. s. 18. Theology.

2. Lord Bolingbroke, among other rejectors of Revelation, censures divines for making any distinction between the moral and physical attributes of the Deity. He insists that the only attributes we can know of, are the Divine power and wisdom, as manifested in the moral and physical world, assuming, at the same time, that in that world no traces of God's moral perfections are to be perceived. As believers in the Bible we should, of course, be unwilling to accept Bolingbroke's theory in exchange for a single line of all which that sacred volume contains of the revelation which God has been pleased to make of His glorious perfections. But what is more, we should also feel called upon to object to any system of religion, like his lordship's, which, so contrary to true theism, would get rid of all knowledge of the moral attributes of God, in order that man may be set free from all obligations to imitate the Deity in those attributes. It might further be objected, that to deny that we can possess any true idea of the moral attributes of the Deity, is to weaken the argument in favour of a future state of retribution. So long as we admit that God is just and good, we cannot but conclude that some time or other He will place a marked distinction between the good and the bad; and, since we do not uniformly see that distinction made in this life, we believe it probable that there will be a future state of rewards and punishments. If, however, we have no knowledge of the justice and goodness of the Deity, this argument must be abandoned. See Leland's *View of the Deistical Writers*, letter xxiii.

3. Whilst upon this subject, we may take occasion further to observe, that those attributes of Deity which are denominated *moral*, are regarded by theologians as belonging to, or connected with, the will of God. Then, the will of God that anything exterior to Himself should take place is technically known as the *Divine decree*. It is scarcely necessary to observe, that it is from the consideration of this attribute of God that those thorny discussions have arisen respecting the *doctrine of the Divine decrees*, and which, at intervals, have agitated the Church. These discussions began during the controversy of Augustine with the Pelagians, were revived in the IXth Century by Gottschalk, and were continued downward, from the schoolmen, in the disputes of the Dominicans and Franciscans, of the Calvinists and Arminians, until the extermination of the Jansenists by the Pope and Jesuits.

Even if space admitted of it, we have no inclination to enter upon the merits of the arguments, on both sides, that bear upon the questions in dispute; for when we reflect that the accuracy of all reasonings connected with the doctrine of the Divine decrees depends on what man can know and comprehend of the liberty of the Divine will, as expressed in the actings of infinite wisdom, justice, and goodness from eternal ages, we feel that we have to do with subjects in the consideration of which the most acute polemic may soon be "in wandering mazes lost."

(12.) But besides that clear development of the Divine attributes which the Scriptures thus afford above all that Natural Religion can supply, no attentive reader of the *New Testament* can help observing the very peculiar manner in which Three personal subjects are repeatedly spoken of in connexion with the

Theology. Godhead. When, for example, the Evangelist* relates the circumstances attendant on the baptism of Christ, we find mentioned the voice of the Father, the Spirit of God, and Jesus, the well-beloved Son. So, again, we are told that in the last charge which Christ gave to His disciples, He bid them baptize all nations in the name of the Father, and of the Son, and of the Holy Ghost;† as in the conversation He had with them before His crucifixion, He had spoken of the Comforter, the Holy Ghost, whom the Father would send in the name of Christ.‡ Then, when the apostles were fulfilling the mission thus assigned to them, we find them writing of “access to the Father through Christ, by one Spirit;”§ and comprehending the sum and substance of all spiritual blessings in “the grace of our Lord Jesus Christ, the love of God, and the communion of the Holy Ghost.”||

(13.) Now, although the texts which have been cited may not, as taken by themselves, prove that the Three personal subjects mentioned necessarily belong to the Divine Nature, and possess equal honour, yet these Scriptures are amply sufficient to shew that Three personal subjects are actually named in connexion with the Godhead; that those subjects are distinguished from each other; and, moreover, that Christ regarded the doctrine of Father, Son, and Holy Ghost to be a fundamental truth in His religion. It remains, therefore, that we examine such Scriptures as mention these three subjects separately, in order that we may thus learn all that has been revealed respecting their personal nature and mutual relationship. As, however, it is generally agreed that when the term “Father” is applied to God, it is intended to designate the whole Divine Nature, we may confine our attention to such portions of Revelation as separately relate to the Son, and to the Holy Ghost.

(14.) And here, in the prosecution of our inquiry, we find all the attributes of Deity to be interchangeably and equally assigned to each of the sacred persons. Thus of the Son it is written, that in Him “dwells all the fulness of the Godhead” (Col. ii. 9); that He is eternal, being before all things (John i. 1, 3; Col. i. 17; John xvii. 5; Heb. xiii. 8); omnipresent (Matt. xviii. 20); omniscient (John xxi. 17; Heb. iv. 12, 13); omnipotent (Phil. iii. 21; Heb. vii. 25; Rev. i. 8). There are also ascribed to Him the works of God, such as creation and providence (Col. i. 15, 16; Heb. i. 3); and the performance generally of such works as are peculiar to God the Father (John vi. 19). Therefore, the Son is consistently addressed by the Divine name, as Jehovah (Numb. xxi. 5, 8, comp. with 1 Cor. x. 9); God (Matt. i. 23; John i. 1; 1 Tim. iii. 16; Tit. ii. 13); Lord (John xx. 28; 1 Cor. ii. 8). And, finally, the Divine worship and honour is required to be yielded to Him equally with God the Father (John v. 23; xiv. 13, 14; Phil. ii. 9, 10). In like manner do we find the attributes and works of Deity ascribed to the Holy Ghost. Being before all things, He is eternal (Gen. i. 2; Heb. ix. 14). He is omnipresent (Rom. viii. 9; 1 Cor. iii. 16); omniscient (1 Cor. ii. 10); omnipotent (1 Cor. xii. 11); and as being called the “power of the Highest,” and the “Finger of God” (Luke i. 35; xi. 20); whilst it is the

Third Person who is said to create the world (Gen. i. 2; **Theology.** Job. xxxiii. 4; Acts iv. 24, 25, comp. with Acts i. 16), to inspire the Prophets, and to enable men to do all those wonderful works which bespoke the present Deity (Acts ii. 4; 2 Pet. i. 21). For these reasons, to the Holy Ghost is ascribed the Divine name, as the Lord, Jehovah (Exod. xvii. 7, comp. with Heb. iii. 7, 9; Numb. xii. 6, comp. with 2 Pet. i. 21; Isaiah vi. 9, comp. with Acts xxviii. 25, 26; 2 Cor. iii. 15–17); God (Acts v. 3, 4; 1 Cor. iii. 16, comp. with vi. 19, 20).

As, therefore, we thus find, in sacred Scripture, the same perfections of Deity ascribed equally to each of the three Divine Persons, it follows that all three are very God, co-equal, co-eternal, co-essential in perfection and glory; in other words, it is an undoubted article of revealed truth, that the object of worship is one God in Trinity, and Trinity in Unity. A few of the passages from which it has been justly gathered that the *Old Testament* teaches the same doctrine, may here briefly be adverted to. The plural form of the word *Elohim*, joined as it is with a singular verb, although it may not of itself prove the doctrine, at least harmonizes beautifully with it. Again, the use of the plural number in describing the counsel of the Divine Being respecting the creation of man,—“Let us make man in our image” (Gen. i. 26), indicates the same great truth. To these indications may be added the *trisagion* used in Isaiah vi. 3.

Again, an unequivocal testimony to the Divine nature of the Messiah is afforded by many of the prophetic declarations concerning Him. He is called “the Mighty God,” “the Everlasting Father,” “the Son of God,” and “God” (Isaiah ix. 6; Psalm ii. 7, xlv. 6; Psalm xc. 1, comp. Heb. i. 5–13). With these and similar indications may be taken the remarkable declaration of Malachi (iii. 1), that “the Lord” should come to his temple,—a prediction which the Jews themselves applied to the Messiah.*

The Spirit of God is also spoken of in many passages of the *Old Testament*, which are justly conceived to apply to the third person in the Divine Trinity (e. g. Gen. i. 2; Job xxvi. 13; Psalm civ. 30; Isaiah xi. 2, lxi. 1). So interwoven, indeed, is this fundamental article of belief with the very frame and texture of revelation, that no man can explain the *Bible* in its unimpaired integrity without taking this doctrine for granted.

Nor is the information which revelation contains respecting man less important than all it teaches respecting the Deity. Every conjecture which human reason can suggest as to the degenerate state of man is authoritatively verified; and much more is told us respecting the human race, in their relation to God, than without revelation we could possibly have known.† Thus, we read that man was originally created in the image of God; and that he was upright (Gen. i. 26; Eccles. vii. 29). And, if we ask why then so little of this godlike original is found in the world as it now is, revelation teaches that disobedience to the command of God brought sin into the world, “with all our woe:” that it was in Adam that we all died; that from our

* Matt. iii. 16, 17.

† Matt. xxviii. 19.

‡ John xiv. 26.

§ Ephes. ii. 18.

|| 2 Cor. xiii. 14; see also 1 Pet. i. 2.

* We may also mention, as a very prevalent opinion of divines, that all the *Theophanies* recorded in the *Old Testament* relate to the Son.

† Horn on the *Misery of Man*.

Theology. first parents are derived, to their latest posterity, all that blindness of heart, all that unholiness of affections, those corrupt imaginations and desires, and those evil workings of the mind, which issue in overt acts of sin (*Rom. v. 12, 14, 17-19; 1 Cor. xv. 22; Job xiv. 4; Psalm i. 5*). We thus become acquainted with the important truth, that every man who is naturally engendered of Adam is very far gone from original righteousness; is of his own nature inclined to evil; and is, from his birth, obnoxious to God's righteous judgment. If now we take into account the actual turnings aside from God,* the cursing and bitterness, the swiftness to shed blood, the destruction and misery, and those many depraved works of the flesh which afflict human society at large, and degrade individuals, we are prepared to understand that the moral condition of man with reference to his Maker is that of wretchedness and condemnation.

The true extent of the misery of this condition is not, however, felt until we perceive that, in ourselves, we are not in circumstances to deliver our own souls from impending judgment. Our experience scarcely needs the testimony of revelation to convince us that we are not able to change the natural ungodliness of the heart and affections into "faith and calling upon God;" that we are not competent to remove from ourselves the guilt contracted by sin; nor to regain those paths of righteousness from which all by nature stray. As, also, our natural tendencies are not Godward, and indulgence in sinful habits more effectually blunts the moral feelings and perceptions, each successive day spent in forgetfulness of God renders our natural condition only the more hopeless. In the mean while the *Bible* reveals to us that "the wicked shall be turned into hell, and all the nations that forget God:" that "indignation and wrath, tribulation and anguish," must be the inevitable portion of "every soul of man that doeth evil," without respect to person, age, or nation. The undefined forebodings of the natural conscience are thus realized in the fearful certainty of a revelation "from heaven" of God's wrath "against all ungodliness and unrighteousness of men;" so that the most prosperous career of vice is nothing better than a "treasuring up of wrath against the day of wrath" (*Psalm ix.; Rom. i. 18; ii. 8, 9; iii. 19*).

It is at this juncture that revelation brings us acquainted with the forgiving love of God through a *Mediator*. We find the promise of a Redeemer recorded contemporaneously with the history of that sin and death which entered the world by our first parents' disobedience. We see this promise to be from time to time renewed, with more or less clearness and detail of circumstances, until the actual appearance of the Mediator in the flesh. But that appearance was so little after the usual fashion of mankind, that the testimony of revelation can alone unfold the wondrous history of God's well-beloved Son. Begotten from everlasting of the eternal Father; of one substance with the Father; the very and eternal God; He took man's nature in the womb of a virgin mother, and of that mother was born into the world, yet without sin. It was thus provided that in this Redeemer's person the Divine and human nature should be inseparably united, that as the reasonable soul and body is one man, so God and man might be one Christ. Thus constituted to be the propitiation for the sins of a fallen world, this perfect God

Theology. and perfect man died upon the cross, that the "chastisement of our peace" might be "upon Him," and that "by His stripes we might be healed." He died and was buried; rose again from the dead; ascended into heaven; and now appears in the presence of the Almighty Father as our everliving intercessor. Thus, as transgression came by man, so satisfaction was made by man; as the offence of sin is against God, so reconciliation was purchased by the merits of the Son of God. (*Psalm ii. 7; John i. 1-14; Luke ii. 40; 1 Tim. ii. 5, 6; Phil. ii. 6, et seq.; 1 John ii. 1, 2, et alib.*)

The Scripture revelation of a Mediator solves therefore the difficulty which Natural Religion cannot explain. The Deity is here revealed to us as a "just God, and yet a Saviour;" as so tempering His justice and mercy together that He neither condemns sinful man to remediless misery, nor by His mercy delivers us without a just ransom. That ransom, however, having been fully provided in the sufferings and death of Christ,* the justice of God and His mercy embrace together and fulfil the mystery of our redemption. Yet as redemption, like every other blessing, can profit those only who are personally the subjects of it, it remains for us to ascertain how "this great salvation" becomes available to all who desire to partake of it. On this point the Scriptures teach that, "As Moses lifted up the serpent in the wilderness, even so must the Son of Man be lifted up; that whosoever believeth in him should not perish, but have eternal life. For God so loved the world that he gave His only-begotten Son, that that whosoever believeth in Him should not perish, but have everlasting life: He that believeth on him is not condemned; but he that believeth not is condemned already." (*John iii. 14-18; comp. John xx. 31; Acts xx. 31; Rom. iii. 26, v. 1, x. 4, et al.*) We thus learn that whilst "we are accounted righteous before God only, for the merit of our Lord Jesus Christ, and not for our works and deservings;"† the "mean and instrument of our salvation required on our parts is Faith; that is to say, a sure trust and confidence in the mercies of God,‡ whereby we persuade ourselves that God both hath and will forgive our sins; that He hath accepted us again into His favour; that He hath released us from the bonds of damnation, and received us again into the number of His elect people; not for our merits or deserts, but only and solely for the merits of Christ's death and passion, who became man for our sakes, and humbled himself to sustain the reproach of the cross, that we thereby might be saved, and made inheritors of the kingdom of Heaven."

When, however, we find so much written in commendation of faith, that by it alone we are said to be justified, it is not meant that he who exercises such a faith can at any period be without true repentance,§ hope, charity, and the reverent fear of God. Nor when the Scriptures declare that we are justified freely by God's grace, is it to be concluded that nothing is required on our part afterwards. Nor is it intended to exclude or supersede the necessity of union with our Saviour in his church and in his sacraments, through which the benefits of our Saviour's passion are made available to man.||

* *Homily on the Nativity.*

† *Acts xi.*

‡ *Homily on the Passion.*

§ *Homily on Salvation.*

|| It may be well to call attention to the fact that the *Homily*

* *Rom. iii.*

Theology.

The Scriptures ascribe the merit and deserving of our justification wholly and solely to the abundant grace of God in Christ Jesus, apprehended by faith; yet nothing is more evident than the declarations which speak of this faith as a principle which cannot lie dormant, but which is always bringing forth some good work; and inseparably connected with an earnest endeavour to obey God's commandments in all holiness of life; a fear to offend so merciful a Redeemer; a readiness to do good to every man; and "above all things, and in all things, to advance the glory of that God of whom only cometh our salvation." Thus is it that by good works the faith which justifies may be as "evidently known as a tree is discerned by its fruits;" and thus every Christian man is diligently working out his "own salvation with fear and trembling at the very time that his faith leads him to depend solely on God to work in him both to will and to do of His pleasure."

This inworking of faith and holiness the Scriptures represent to be effected by the power of the Holy Ghost. It is that sacred Person of the blessed Trinity who so acts upon the hearts of men that they become possessed of convictions respecting themselves and their need of salvation, which, without His sacred influence, they would never have had. From Him proceed those fruits of faith which "are pleasing and acceptable to God in Christ."* He, in a word, "is the only worker of our sanctification, and makes us new men in Christ Jesus."

Thus does the Scripture revelation of God, as Father, Son, and Holy Ghost, appear not as a merely speculative doctrine, but as a fundamentally practical truth. For as salvation, with all its present concomitant blessings, originates in the grace and favour of God the Father, so is it bestowed and comes to man only for the sake of God the Son, nor by the Son but through God the Holy Ghost.† Yet by this threefold Divine agency it has been provided that those who by nature are sons of Adam, and who, as Adam's offspring, partake of the sinfulness of that nature from which they derive their origin, become, in the second Adam, God the Son, children of the Most High, by a spiritual and heavenly birth; and are separated from the world as a holy people unto the Lord, by the indwelling power of the Holy Spirit.

Such being the wonderful provision made for the rescue of mankind from sin and death eternal, and to bring them to holiness and everlasting life, it were a dishonour to the mercy and love of God to suppose that the Divine counsel had not secured that a knowledge of the great mystery of redemption should be perpetuated to all generations. We are thus brought to consider the means provided for the preaching of the Gospel, or man regarded as a member of the Church.

Obs. 1.—The sects against which the Church has had to maintain the great fundamental doctrines of the Trinity, the divinity of our Saviour and of the Holy Spirit, are the Arians and Socinians. The latter deny also the doctrine of the Atonement and

on Salvation (referred to in the eleventh article) uses the word baptized as equivalent to justified in the following passage: "Our office is not to pass the time of this present life unfruitfully and idly, after that we are baptized or justified, not caring how few goods works we do to the glory of God and profit of our neighbours," &c. *Homily on Salvation*, part iii.

We must be careful, in all our considerations on the subject of justification, always to distinguish between justification, sanctification, and final salvation, the two former being essential to the last.

* *Homily for Whitsunday.*

† Hooker, v. c. 56.

the personality of the Holy Ghost; while the Arians, who admit the pre-existence of our Lord and the personality of the Holy Ghost, recognise the doctrine of the Atonement, and acknowledge an inferior divinity both in the Son and Holy Ghost. It was chiefly against Arianism that the early Church had to contend; and for a history of the controversies in early ages, we must refer to the historical portions of this *Encyclopædia*, especially vol. iii. chap. liv. p. 326. The works of Bishop Bull, Bishop Horsley, and Dr. Waterland, will be found very useful in all that relates to the Trinity. Dr. Burton's *Testimonies of the Ante-Nicene Fathers to the Doctrines of the Trinity and the Divinity of our Lord*, are a very admirable supplement to those of Bishop Bull; and Mr. Newman's *History of the Arians of the Fourth Century* gives a very lucid summary of the points, as discussed at that season.

Our object has been to adhere as closely as possible to the very language of our Church on the subject of man's sinful state, and his recovery by Jesus Christ, it being our desire to recognise no other human authority in regard to a doctrine of so much importance. To understand, however, fully the language of our articles on these points, an historical knowledge of the opinions held respecting them previous to the Reformation is an admirable assistance, if not rather an indispensable requisite. In this respect the *Bampton Lectures* of the late Archbishop Laurence will be found extremely useful. The writers who have treated on the various subjects included under what are usually termed "The Five Points," are by far too numerous to be specified; and as most of the common works on Divinity give references to them, it seems superfluous to attempt it here.

Obs. 2. We may here conveniently state some of the peculiarities of the Socinian views in regard to the state of man. It must, however, be premised, that modern Unitarians' views differ very widely from those of the older Socinians, and include many shades of opinion; and some, indeed, who call themselves Unitarians, differ from moral Deists only in believing that the miracles and resurrection of our Saviour prove the truth of a future state.

The Socinians hold that man's condition now is similar to that of Adam, and that consequently there is no such thing as original sin—that the death denounced against Adam was only a denunciation of future punishment to sin—that man does not require the grace of God to begin good, but to bring it to perfection—that man's will is free to good (while many modern Unitarians hold the doctrine of philosophical necessity)—that moral virtue is the only ground of salvation, which salvation is nevertheless a free gift of God.

The older Socinians acknowledged a prophetic, priestly, and regal office in Christ; the first in preaching the Gospel; the second in his office in heaven, which is nearly identical with (3) his regal office. But modern Unitarians are far from attributing that dignity to Christ after his resurrection which the older Socinians did not deny. Justifying faith, according to them, is only trusting to the promises of God, and a consequent obedience to them.

These opinions are merely stated here; but it does not seem requisite to discuss them, as the opinion of the mere humanity of our Saviour, upon which the whole system rests, is utterly irreconcilable with the whole scheme and language of Scripture, and is entirely rejected by the concurrent testimony of Christian antiquity.

The following extracts from Dr. Berriman's *Moyer's Lectures* for 1723 and 1724 (the subject of which is an historical account of the Trinitarian Controversies), may be found useful in illustrating some portion of the preceding chapter.

Extracts from Dr. Berriman.

"The doctrines of Socinus were by some of his followers methodized and digested into regular systems, and by others defended against the various objections, whether of *Romanists* or *Protestants*. A scheme it was which did entirely change the whole nature and design of Christianity. It not only took in that grand point, in which the *Sabellians* and the *Arians* agreed, that the Supreme Deity is personally but one, concurred also with the latter, that our blessed Saviour is not *God over all*; and with the former, that the Holy Spirit is only a divine influence, without any personal subsistence; but it went on with *Artemon* and others to deny that Jesus Christ had any real existence before his birth of the Virgin; and its patrons having set up private judgment as their supreme rule, concluded from the whole, more impiously, indeed, but still more consistently than former heretics, that whatever is said of the merit and satisfaction of Christ, his sacrifice for sin, and his redemption of sinners, his unchange-

Theology.

Theology. able priesthood, and intercession for us at God's right hand, has altogether a metaphorical or figurative meaning, widely different from that in which the Church had always understood and made use of those expressions. To these, if we add the many other errors of this newfangled scheme, concerning the constitution of the Christian Church and the appointment of its *ministry*, the efficacy of its *sacraments*, and the secret *operations* of Divine *grace*, the interpretation of *Scripture*, and the *rules* of Christian *obedience*, the state of the *soul* after death, the *resurrection* of the body, and the future *judgment*, we shall have cause to say that there was never any heresy that did so artfully disguise so great a number of impieties as this *hydra* of *Socinianism*; which made so low an account of the unfathomable mystery of our *redemption*, that there can be little ground to wonder if, besides the *judaizing* errors already mentioned, there should be some who apostatized (as *Socinus* himself could not entirely disown) into Mahometism, or into downright *Atheism*; nay, even if some of those who did not openly apostatize, should yet boast of their agreement with the followers of *Mahomet* in their notions of the Divine unity, and their little difference from them in respect of Christ." (Berriman, p. 410.)

* * * * *

"It was not long after this [the publication of a translation of Biddle's *Catechism*, about 1665] that *Sandius* published his *Ecclesiastical History*, manifestly calculated for the service of the Arian cause, and to persuade his readers that till the time of the *Nicene* Council the Catholics had those very sentiments which were then embraced by *Arius* and his associates, and all who differed from them in these points had been esteemed as hereticks. This groundless calumny (which had been but too much countenanced by the writings of *Petavius*, though with a different view) gave

occasion to that admirable *Defence of the Nicene Faith* which was drawn up by our incomparably learned Bishop *Bull*, in opposition at once to the Arian and the *Jesuit*; and which was afterwards followed by his other treatise of the *Judgment of the Catholic Church concerning the Necessity of believing Christ's Divinity*, in opposition to *Episcopius* and his *Remonstrant* brethren. Meanwhile the controversy which prevailed chiefly among us was not upon the *Arian*, but *Socinian* scheme; though as *Sandius* had plainly showed his opinion, that there was nothing which should hinder those two parties from communicating with each other, so the *Socinians* were generally of the same mind, and content to join with such as advanced somewhat higher than themselves, provided they denied the Son's proper and essential divinity. Some of them adhered to Biddle's scheme already mentioned, but the greater part seem to have embraced the grossest sort of *Socinianism*, as well by disowning the personality of the Holy Ghost as disclaiming likewise all worship or invocation of Christ, for which the Polish *Socinians* would doubtless have rejected their communion. (Berriman, p. 424.)

Dr. Berriman then proceeds to mention Dr. *Sherlock's Vindication of the Doctrine of the Trinity*, which some persons charged with Tritheism, and Dr. *South's Animadversions upon Dr. Sherlock*, which was again charged by others with Sabellianism, and to intimate the advantage taken of this dispute by the Unitarians of those days. Some of the Unitarians then acknowledged themselves ready to conform and use the expressions of the Church of England, provided they might by persons understand only *modes* or *names* of relations. This, of course, was only a form of Sabellianism.

For a short account of the controversy in later days, see the Historical Division of this *Encyclopædia*, vol. v. p. 1139-40.

Theology.

OUTLINES OF THEOLOGY.

CHAPTER IV.

The Doctrine of Scripture respecting the Church, and respecting a Future Life.

Theology. **HAVING** now considered the chief substance of the revelation of God respecting the nature and attributes of the Deity, and the relation between God and man as a fallen and sinful being, needing restoration to God's favour, and assistance from on high,—needing pardon, justification, and sanctification,—we proceed further to view the means by which these gracious purposes of God are applied and brought home to individuals. In the Scriptures is found the substance of that communication from God through which each man is to become capable of salvation; but we ask at once—was this communication left in the world to make its own unaided way to the heart of each individual, or was the propagation and maintenance of the truths contained in it enjoined on a distinct body of men, and all who receive that truth gathered together into one body? * The answer to this inquiry is obvious. The doctrine of redemption, as preached to man, clearly implies a society; and Scripture, speaking of this society, calls it the Church. We therefore now proceed to inquire what Scripture teaches us concerning the Church.

The Church.

In commenting upon that portion of the creed in which we profess our belief in “the holy Catholic Church,” Bishop Pearson observes, that our Lord, in first speaking of this Church, spoke of it as that which as yet was not.† Thus he said to St. Peter, “And upon this rock I will build my Church” (*Matt. xvi. 18*). But when he had ascended into heaven, and the Holy Ghost had come down—when Peter had converted three thousand souls (*Acts ii. 41*), which were added to the hundred and twenty disciples (*Acts i. 15*), then there was a Church (and that, too, built upon St. Peter,‡ according to our Saviour's promise); for after that we read “the Lord added to the Church daily

such as should be saved”* (*Acts ii. 57*). “A Church, then, our Saviour promised should be built, and by a promise made before his death. After his ascension, and upon the preaching of St. Peter, we find a Church built or constituted, and that of a nature capable of daily increase.” (Bishop Pearson, *On the Creed*, p. 542, Dobson's edition.)

Without entering into the questions, to which we must hereafter recur, as to the catholicity and external unity of this Church, we may observe that it is seldom denied that such a body as the Church exists and is supposed in Scripture; but the great differences which exist among men in regard to it relate to its nature and constitution.

If we inquire from Scripture what is the meaning of the term (*ἐκκλησία*) rendered *Church*, we shall perceive that it is a word which in Greek means an assembly or congregation, with the notion of being “called forth;” and in the *New Testament* it always signifies a company of persons professing the Christian religion. “Sometimes,” observes Bishop Pearson, “the Churches of God are diversified as many; sometimes, as many as they are, they are all comprehended in one.” Thus “the Churches” are mentioned *Acts xvi. 5*; *1 Cor. xiv. 34*, &c.; “the Churches of God,” *2 Thess. i. 4*; *1 Cor. 15, 16*; “the Churches of the Gentiles,” *Rom. xvi. 4*; “the Churches of the saints,” *1 Cor. xiv. 33*. Sometimes we read of a church as designating a few believers gathered together in one house, as the Church in the house of Priscilla and Aquila (*1 Cor. xvi. 19*). So also we read of the Churches of Samaria and the Churches of Asia, meaning the Christian congregations in those districts. But then we observe, that even when the Scriptures speak of many such congregations in great cities, they speak of them under the notion of one Church. Thus St. Paul writes to the Corinthians, “Let your women keep silence in your Churches” (*1 Cor. xiv. 34*). But yet the dedication of the epistle is “To the Church of God which is at Corinth” (*1 Cor. i. 2*).

Now just as these many *Congregations* formed one *Church*, so these many *Churches* formed one *Church* under Christ, the great bishop of our souls; and in this sense we speak of THE CHURCH as the great society founded by our Lord.

We find, therefore, that the Church was considered as a society built upon the foundation of Jesus Christ; and from other parts of the Scriptures, especially the Epistles of St. Paul, we gather that this society was furnished with its own officers and ministers, and that it had its own ceremonial of worship and sacramental

* The Bishop of London, in his *Three Sermons on the Church* (London, 1842), has cited a very striking sentence on this point from Archbishop Bramhall (*Works*, p. 615): “If he think that Christ left the Catholic Church, as the ostrich doth her eggs in the sand, without any care or provision for the governing thereof in future ages, he erreth grossly.” To which Bishop Blomfield adds the following remark: “It is expressly asserted by Clement of Rome, a companion of the Apostles, that such provision was made by them.” (*Three Sermons*, &c., p. 32.)

† By this assertion we need not understand the bishop to speak as if the Jewish Church had nothing to do with the Christian. We may understand it of the Church as peculiarly a Christian Church, founded not merely, as the Jewish Church was, on the belief of a future Messiah, but on the belief of that Messiah having come in the person of our Lord and Saviour Jesus Christ. The Church may be viewed as existing from the earliest ages, and continuing to the present day, as the Patriarchal, the Jewish, and the Christian Church. The assertion of Bishop Pearson will apply to it, as the *Christian Church*, without considering it a new creation, independent of the former developments of the Church.

‡ See Tertull. *De Pudicitia*, c. 21; and Basil. *Adv. Eunom.* l. ii. sec. 4.

* Some useful remarks on the meaning of the phrase will be found in the Bishop of London's *Three Sermons on the Church*.

Theology. observances. The two last particulars are indicated in the *Acts of the Apostles*, where it is said of the disciples that they continued stedfastly in the Apostles' doctrine and fellowship, "and in breaking of bread and in prayers." (*Acts* ii.)

Reserving to a later portion of this chapter some notice of the nature of the connexion between various branches of this great body, and the means of distinguishing the true Church of Christ amid contending bodies, who claim, each for themselves, that holy name, we shall now briefly consider the polity, the sacraments, and the ceremonial of the Church of Christ.

Sect. 1.—Of the Polity of the Church of Christ.

Polity of the Church.

The general character of the Christian Church in the Apostolic age and the time of the Apostolic Fathers has been so fully treated in the Historical Division of this work, and under the word **BISHOP** in the Miscellaneous portion, that it is needless to repeat in this place the statements there made; and we merely request that the following observations may be considered as supplementary to those articles.

Ministers.

On inquiring into the nature and constitution of the Christian Church, as laid before us in Scripture, we find that certain offices are there mentioned as being already appointed. If we examine the *Epistle of St. Paul*, we shall find (besides some orders of men possessing the extraordinary gifts of the Spirit, as, for example, *prophecy* and the gift of *tongues*, &c., as mentioned *Ephes.* iv. 11, and *1 Cor.* xii. & xiv., that three orders of men are mentioned as bearing a share in the ministry of the Church; viz., Bishops, Elders or Priests, and Deacons; of which names the two first are used in Scripture with some latitude and ambiguity.

Ordination.

It is shown, by arguments which admit of no answer, that the Apostles did ordain ministers by the imposition of hands, and did give them authority to ordain others; as, for example, Titus in Crete, and Timothy in Ephesus (*See 1 Tim.* iv. 14; *Tit.* i. 5). Nor is there, in the reason of the thing, any cause why this ordinance should last for one or two generations, and then be discontinued; but, on the contrary, every reason would lead us to consider it a permanent and perpetual appointment. And accordingly we find that in the Universal Church this was taken as a noted and fundamental point. No one was considered capable of transmitting the grace of the Holy Spirit, which gives men authority and power to minister in things pertaining unto God, except such as had received it themselves by spiritual succession from the Apostles. Nor is it denied by the more learned of the opponents of episcopacy that the government of the Church and the ordination of its ministers was restricted to bishops as early as in the age immediately succeeding the Apostles. The acute Chillingworth has drawn from this a conclusive argument for the Apostolic origin of the Episcopate.

"Episcopal government," he says, "is acknowledged to have been universally received in the Church presently after the Apostles' times. Between the Apostles' times and this presently, there was not time enough, nor possibility of so great an alteration" [as that from Presbyterial to Episcopal government], "and therefore there was no such alteration as is pretended; and therefore Episcopacy, being confessed to be so ancient and catholic, must be granted also to be Apos-

tolie—*quod erat demonstrandum.*" (Chillingworth, *The Theology. Apostolical Institution of Episcopacy Demonstrated.* Works, vol. ii. p. 531—537.)

As the offices of Bishop and Deacon have been fully treated of in other parts of this work (*see* Hist. Div. vol. ii., and Miscell. Div. vols. ii. and iv., articles **BISHOP** and **DEACON**), we do not enlarge upon this subject, but simply observe, that the British Bishops may be proved to derive their orders in unbroken succession from Apostolic times; and their authority in this country may be traced to its earliest conversion. Hence every authority due to an Apostolic Church may be lawfully claimed by them.

Obs. 1. The subject of Episcopacy has been so frequently handled by our most eminent divines, that it would be almost impossible to enumerate the names of their treatises upon it. Bishop Taylor, Bishop Hall, and Archbishop Bramhall, ably maintained the cause of Episcopacy; and Dr. Hammond has answered the arguments of the Presbyterian divines of the XVIIth Century. In later times, Leslie, in his *Essay on the Qualifications to administer the Sacraments*, and Bennet, in his *Rights of the Clergy*, and his *Discourse of Schism*, also have ably maintained the cause. For more specific references to these great storehouses of argument, and for a very lucid and admirable summary of their substance, we must refer to the *Sermons on the Commission and consequent Duties of the Clergy*, by the late Rev. Hugh James Rose; to the Rev. W. Palmer's *Treatise on the Church*; to Bishop Blomfield's *Three Sermons on the Church*; and Bishop Onderdonk's *Episcopacy tested by Scripture*. Amongst other older works, however, we must just mention Field, *Of the Church*, as deserving especial attention.

If, on this subject, any individuals prefer their own views to the universal and uninterrupted practice of the Church for nearly 1600 years, we do not mean to argue the question with them. With those on whom such an argument has no effect, we should not expect any reasoning we could offer, to prevail. There is one point, however, which, not having been previously considered, requires a passing notice. It is the story known by the name of *The Nag's Head Fable*, by which the less respectable of the Roman Catholic writers attempt to discredit the consecration of Archbishop Parker. We may add also that some of the more respectable writers of that communion, from a want of information on the subject, have given currency to this disreputable fabrication. *See* especially Massuet, in his *Notes on Irenæus*, vol. ii. p. 105.

The following remarks are from the notes to the *Sermons on the Commission and consequent Duties of the Clergy*, above referred to.

"It is curious that they who attacked the English ordinations did so inconsistently.

"First,—they alleged that King Edward's ritual for ordination was bad. Next, they said that bishops consecrated by men who were separate from the Church, and had no authority or jurisdiction, were no bishops; confounding the difference between the powers of ordination and of jurisdiction. Then marriage, they said in their ignorance, made the consecration of bishops void. Ashamed of all these stories, they next said that the new bishops had not been consecrated at all by imposition of hands. Finally, they brought up the Nag's Head story. This story was first told by a Jesuit, called Sacro Basco, or Holywood, in 1604, forty-five years after Parker's consecration! Kellison, a Romanist writer, knew nothing of this story in his first book, but published it in his second!"—Rose's *Sermons*, p. 227.

We now add Archbishop Bramhall's account of the Nag's Head fable (*Works*, p. 436), quoted in the same book as the shortest and fullest refutation of the story. It is as follows:—

"They say that Archbishop Parker and the rest of the Protestant bishops in the beginning of Queen Elizabeth's reign, or at least sundry of them, were consecrated at the Nag's Head in Cheapside, together, by Bishop Scory alone, or by him and Bishop Barlow, without sermon, without sacrament, without solemnity, in the year 1559 (but they know not what day nor before what public notaries), by a new fantastic form. And all this they, on the supposed voluntary report of Mr. Neale (a single malicious spy), in private to his own party, long after the business pretended to be done.

"We say that Archbishop Parker was consecrated alone at Lambeth, in the Church, by four bishops, authorized thereunto by Commission under the Great Seal of England, with sermon,

Theology. with sacrament, with due solemnities, on the 17th day of December, anno 1559, before four of the most eminent public notaries in England, and particularly the same public notary was principal actuary both at Cardinal Pole's consecration and Archbishop Parker's. And that all the rest of the bishops were consecrated at other times, some in the same month, but not upon the same day; some in the same year, but not the same month; and some the year following. And to prove the truth of our relation, and falsehood of theirs, we produce the register of the see of Canterbury, as authentic as the world hath any; the registers of the other fourteen sees then vacant, all as carefully kept by sworn officers as the records of the Vatican itself. We produce all the commissions under the Privy Seal and Great Seal of England. We produce the rolls and records of the Chancery; and if the records of the Signet Office had not been unfortunately burned in King James's time, it might have been verified by them also. We produce an Act of Parliament, express to the point, within seven years after the consecration: we produce all the controverted consecrations published to the world in print, anno 1572, three years before Archbishop Parker's death, whilst all things were fresh in men's memories."

Obs. 2. We must not omit simply to state that some men, especially the Puritans (see Hooker, book v. sec. 78, note, where he quotes Cartwright), have objected to the use of the word Priest in respect to a Christian minister, as implying "a sacrificer." Hooker, to avoid disputes, agrees in some degree to forego the use of the word, and says, "I rather term the one sort presbyters than priests, because in a matter of so small moment I would not willingly offend their ears to whom the name of priesthood is odious, though without cause." But it must be remembered, that although we repudiate the Roman Catholic sacrifice of the Mass, in which it is declared that the priest offers Christ upon the altar a propitiatory sacrifice for the living and the dead, we acknowledge in another sense a sacrifice in the Eucharist. The oblation of the gifts is a sacrifice in one sense; and, at all events, the prayer and praise there offered to God may most justly be called a spiritual sacrifice; and in this sense it is difficult to see how any offence can be engendered by the use of the term Priest. But this controversy is too wide to enter upon here. We refer for further information to Dr. Waterland on the *Eucharist*; to Mr. Palmer, *Treatise on the Church*, pt. ii. ch. viii.; Bishop Burnet and other writers on Art. 31, &c.

Sect. 2.—*The Ceremonial, Liturgy, and Sacraments of the Christian Church.*

Ceremonial of the Church.

As the worship of God is one of the principal objects of the Christian Church, it is clear that some external ceremonial must be prescribed in it upon competent authority. The notices contained in Scripture with regard to this point are undoubtedly rather scanty, but they are sufficient to guide us in the main objects of Christian worship, without fettering particular churches as to the minute details of the manner in which the great object is to be effected.

The public worship of God is particularly specified in Scripture, and "the breaking of bread and prayers," of which we read (*Acts* ii. 42), may be considered to include both the ordinary provision for this purpose, and the celebration of the Lord's Supper.

Forms of Prayer.

As the subject of Liturgies has been treated at considerable length in the Miscellaneous Division of this work, under the article LITURGIES, we shall not here dwell upon the various topics connected with the mode of conducting Divine worship. We may, however, refer to one or two works not mentioned in that article. In the able treatise of T. Bennet, entitled *A brief History of the joint Use of precomposed set Forms of Prayer*, &c., by T. Bennet, M.A. (2nd edit. 1708), will be found a most elaborate discussion, to show that "the ancient Jews, our Saviour, and his Apostles, never joined in any prayers but precomposed set forms only;" that these forms were such as the congregations "were accustomed to, and thoroughly acquainted with;" and that "their practice warrants the the imposition of a national precomposed Liturgy."

VOL. II.

Theology. It is from this valuable work that Wheatley's *Preface to the Illustrations of the Common Prayer* (referred to under LITURGY) is almost entirely drawn, and the work is here mentioned because it well deserves a most attentive perusal, as the results to which it leads us are of great importance.

Another work* of great value, lately published, deserves mention here, viz., Mr. Palmer's *Origines Liturgicæ*. A considerable portion of that work is devoted to the elucidation of the antiquities of the English Liturgy and Ritual, &c., but there is prefixed to it a most valuable dissertation on ancient Liturgies, to which we would direct especial attention. The information there condensed is only to be obtained from many scattered sources (although much of it is to be found in Renaudot's *Liturgiarum Orientalium Collectio*), and is nowhere else so lucidly arranged.

Some important results appear to follow, from the arguments adduced in this dissertation, relative to the condition of the primitive Liturgies. Mr. Palmer observes, in his introduction, "It seems to have been often assumed by the learned, that there was originally some one Apostolic form of Liturgy in the Christian Church, to which all the notices which the Fathers supply might be referred. Were this hypothesis supported by facts, it would be very valuable. But the truth is, there are several different forms of Liturgy now in existence, which, as far as we can perceive, have been different from each other from the most remote antiquity. And with regard to the apparent propriety of the Apostles instituting one Liturgy throughout the world, it may be observed, that it is quite sufficient to suppose all Liturgies originally agreed in containing everything that was necessary for the due celebration of the Eucharist; but that they adopted exactly the same order, or received everywhere the same rites, is a supposition equally unnecessary and groundless."—p. 5, 6.

Mr. Palmer afterwards observes (*Introduction*, p. 8), "after a careful examination of the primitive Liturgies of the Christian Church, it appears to me that they may all be reduced to four, which have been used in different churches from a period of profound antiquity." These are,—

1. *The Great Oriental Liturgy*, used in all churches from the Euphrates to the Hellespont, and from the Hellespont to the southern extremity of Greece.
2. *The Alexandrian*, used from time immemorial in Egypt, Abyssinia, &c.
3. *The Roman*, used in Italy, Sicily, and the civil diocese of Africa.
4. *The Gallican*, used in Gaul and Spain, and probably in the exarchate of Ephesus, till the IVth Century.

These Mr. Palmer considers the parents of all existing forms of Liturgies; and he adds, that their antiquity was so very remote, their use so extensive in those ages when bishops were most independent, that it seems difficult to fix their origin at a lower period than the Apostolic age.

* The mention of Mr. Palmer's work renders it superfluous to enumerate other writers on the subject besides those referred to under LITURGIES. There are such ample references to *Assemani*, *Renaudot*, *Goar*, *Gavanti*, *Bona*, &c., and other writers, both of the Romish communion and others, that any person reading Mr. Palmer's work will be able to pursue the subject as far as he may be inclined.

Theology. The elementary members* of which these Liturgies consisted were for the most part the same, but they appear to have varied chiefly in the order in which these parts were arranged. Those belonging to each of these great families resembled the others used in the same ecclesiastical districts in their arrangement, but differed from those of the other families. Thus, in all *Alexandrian* Liturgies, the general prayers for men and things occurred in the middle of the Eucharistia or thanksgiving, and before the hymn *Ter sanctus*; while in the *Oriental* Liturgies the general prayers are deferred till after the end of the benediction of the gifts. (Palmer, p. 99.)

This substantial uniformity, under circumstantial variation, is certainly a remarkable phenomenon; and to those who would study it in all its bearings we recommend the work of Mr. Palmer. Another very valuable portion of his book is the answer contained in the second volume to the objections of Dr. Brett and others (especially non-jurors), against our office for the Holy Communion.

Having made these few observations on the subject of Liturgies, we proceed to consider next the Sacraments of the Church.

It is perhaps needless to refer more particularly to writers upon the *Book of Common Prayer*. The *Appendix* to Dr. Nicholls's *Paraphrase*, consisting of *Notes* from Bishop Andrewes, Bishop Cosins, &c., is very useful. H. L'Estrange's *The Alliance of Divine Offices*, Dr. Comber's *Companion to the Temple*, Bishop Sparrow's *Rationale of the Common Prayer*, Wheatley, and the more modern work of Shepherd, will all be found of value in the elucidation of the *Book of Common Prayer*, as well as the edition of it with notes by Bishop Mant, the *Liturgy compared with the Bible*, by Mr. Bailey, Keeling's *Liturgiæ Britannicæ*, and Clay's *Book of Common Prayer*.

Sect. 3.—Of the Sacraments of the Church.

In the definition of a sacrament given in the *Catechism of the Church of England*, it is stated to be "an outward and visible sign of an inward and spiritual grace, given unto us, ordained by Christ himself as a means whereby we receive the same, and a pledge to assure us thereof." The previous question is, "How many Sacraments hath Christ ordained in his Church?" and the answer is, "Two only, as *generally necessary to salvation!* that is to say, Baptism and the Supper of the Lord."

By these answers, we learn what our Church considers requisite to form a Sacrament, properly so called.

1. It must have been appointed or ordained (as to its outward sign) by Christ himself.
2. It must be *generally necessary* to salvation, and thus needful, under ordinary circumstances, for all men.
3. The sign, of which it partly consists, must be an outward and visible sign.
4. This sign must be a sign of an inward and spiritual grace.
5. It must be also the means of conveying that grace, as well as a token that we receive it.

By these conditions, or by some of them, the other five ordinances called Sacraments by the Romanists, viz.,—*Confirmation*, *Penance*, *Orders*, *Matrimony*, and *Extreme Unction*, are excluded from the rank of Sacraments, according to the views of our Church.

* By *Liturgy*, it will be seen that we here mean what we call *Communion Service*; and the elementary members of which we speak are, the prayers for all conditions of men, the oblation of the gifts, the *Sursum Corda*, the *Ter sanctus*, &c.

Theology. The Roman Catholic Church (*Concil. Trid.*, Sess. vii. 1), pronounces an anathema against all who affirm that the number of sacraments is more or less than *seven*, and joins with this declaration the same anathema against every one who denies that all of these were instituted by Christ. It is needless to enter much at length into this discussion, as the number of sacraments which men allow must clearly depend on their *definition* of a sacrament. The language of the articles and homilies sometimes seems to admit that other rites may be called sacraments, but in *another sense*; and much to the same purpose is the language of Melancthon, in his *Loci Theologici* (*De Numero Sacramentorum*). The subject had been much discussed in the time of Henry VIII. See the answers of the Bishops to the queries sent to them, as recorded in Burnet's *Reformation*, vol. i. pt. ii. record xxi.

Stillington has shown that there is no Catholic tradition for *seven* sacraments, that this number was not acknowledged for many centuries before the Council of Trent, and disproves Bellarmine's bold assertions. P. Lombard is the first writer who fixes on seven as the number of the sacraments. Concerning the agreement of the Greek Church in the number seven, which Bellarmine and De la Hogue (*De Sacramentis*, p. 7-8) appeal to, as proving the antiquity of this determinate number of seven, it is very summarily disposed of. 1st,—There was no such consent even among the Latin fathers, and the Greek fathers who are quoted are very *late*. For more on the number of sacraments, see Hooker, b. v. sec. 57, and the writers on our Articles.

We proceed now to consider the two sacraments, Baptism and the Supper of the Lord.

Baptism.—As the subject of baptism has already been treated of at some length (*Miscel. Div. vol. ii.* article BAPTISM and BAPTISTS), we have here chiefly to supply what is wanting in that article.

It will be seen from that article that baptism was not a new ceremony, but had been long familiar to the minds of the Jews as an initiatory rite. This very circumstance may serve to account in some degree for the little detail with which it is spoken of in Scripture, and for the omission of some of those particulars at which men have occasionally stumbled in modern days. Thus, the baptism of infants is not specified as being a thing then understood and implied. But as this holy rite is of the utmost importance, as being that whereby we are brought into covenant with God, and made members of Christ's Church,—as it is that to which the holy Church attributes the heavenly grace of regeneration,—it will be requisite to show some of the grounds on which the baptism of infants has been most justly retained in our Church.

It has been said that the Church of England is obliged to call in tradition to defend infant baptism, because it cannot be proved by Scripture. This assertion is by no means just; but were it so, infant baptism is a *custom*, not a *doctrine*, and the *practice* of the Church is the best means which we have of ascertaining in what sense she understood the commands of Scripture. But we can show for the custom such warrant from Scripture that we should fear, if we ever gave it up, that we were abandoning a plain commandment of our Saviour. We shall, therefore, in the very brief defence of it which we shall here make, first adduce some passages from Scripture, and then state what Christian antiquity enables us to infer.

In the first place, however, we must request attention to the baptism of proselytes, as practised by the ancient Jews.* It is known that the existing children of proselytes were baptized, however tender their age. Those born after the parents' reception into the Jewish Church were treated as children of Jews; the males

* See the Introduction to Wall's *History of Infant Baptism*, abridged in *Ency. Met. Misc. Div.* article BAPTISM.

Theology. were circumcised, and a sacrifice was offered for the females; as is proved at large in Wall's *History of Infant Baptism*.

This being the case when baptism was enjoined, we should not expect any particular specification of infants as capable of baptism, because by the custom of those days when baptism was spoken of this was implied. Let us now attempt to ascertain the doctrine of Scripture and the practice of antiquity on this point.

1. *Scripture*.—The rite of circumcision at once prepares us to admit that infants are capable of being received into covenant with God. Nor does it make much difference that the one is an outward ceremony and a less spiritual covenant. We learn from St. Paul, *Rom. ii. 28, 29*, that circumcision was a sign of a spiritual covenant also.

Let us then apply these two facts to the explanation of our Lord's command to his apostles,—“*Go ye therefore and teach all nations, baptizing them in the name of the Father, and of the Son, and of the Holy Ghost*,” *Mat. xxviii. 19*; observing that the word (*μαθητεύσατε*) translated ‘*teach*,’ more properly means ‘*disciple*,’ or ‘*make disciples of*,’ which is much the same as ‘*making proselytes*.’ Had the command been ‘*make disciples of all nations, circumcising them*,’ could there have been a doubt that infants would have been included in this command? Considering that the infants of proselytes were also usually baptized, can we think that anything else was intended or understood by the command, as it stands, than that infants as well as adults should be baptized. The proof rather lies on those of the opposite opinion. It being usual to baptize infants, if our Saviour had intended to exclude them, he would have specified it.*

Dr. Gale† labours very hard to prove that *μαθητεύσατε* must mean *teach*, and not ‘*disciple*,’ but is not very successful in his observations on Wall; who, in his *Defence of the History of Infant Baptism*, has replied to them.

Again, our Saviour says, *John iii. 15*, *Except a man be born of water and of the Spirit, he cannot enter into the kingdom of God*; from which the necessity of baptism for all may be inferred. The Greek word is *τις*. The antipædo-baptists endeavour to wrest this text from us by arguing that infants are not capable of the grace of the Holy Spirit, and therefore cannot have this saving baptism, but are saved without baptism. Of some of the graces of the Holy Spirit we may suppose them

incapable, but of the precise nature and mode of the operations of God's Spirit on the heart we are so profoundly ignorant that we can draw no conclusion from such a statement. (See Wall and Bishop Bethell.) Again, our Lord (*Mark x. 14*) says, “*Suffer the little children to come unto me, and forbid them not, for of such is the kingdom of God*.” This passage needs only to be combined with the last cited to show how strong a testimony it gives to these deductions from Scripture.

With regard to the expression, “*he and all his*,” used of the gaoler who was baptized at Philippi, and the “*household of Lydia*,” who were baptized with her, it is perhaps unwise to rest much on such passages (*Acts xvi. 15, 38*; *xi. 14*). In *Acts viii. 12*, only men and women are mentioned.

Lastly, we may refer to *1 Cor. vi. 14*, the sense of which (*holy and saints* being often used for *baptized persons*) would be paraphrased thus,—

“*For it has often come to pass that an unbelieving husband has been brought to the faith, and so to baptism, by his wife, and vice versa; if it were not so, and if the wickedness of the infidel (whether wife or husband) prevailed, their children would be kept unbaptized, and so be unclean; but we see the contrary, for they are generally baptized, and so become holy or sanctified.*”

For a defence of this interpretation, which coincides with that of Gregory of Nazianzum, St. Augustine, and Pelagius, we must refer to Wall's *History of Infant Baptism*, pt. 1, chaps. xi. xv. xviii.

2. *The Practice of Antiquity*.—In arguing the question of infant baptism from the practice of antiquity, we must observe that those passages by *inference* support our views which maintain the doctrine of original sin, and the absolute necessity of baptism to salvation. An infant, who is liable to condemnation on account of original sin, clearly requires baptism to obtain remission of that sin; and if baptism is unconditionally necessary for salvation, it must be necessary for infants as well as others. The passages adduced from Clement of Rome maintain the doctrine of original sin, and state that it affects even a child who lives but one day.

There is a passage also in the Pastor of Hermas which makes for infant baptism, but it requires too long an examination to be adduced here; it will be found well discussed both in Wall and Bingham. The passages which are of most importance are those from Justin Martyr, Irenæus, and Tertullian.

Justin Martyr (*Dial. cum Tryphone Judæo*) maintains the doctrine of original sin, and in the same dialogue makes baptism correspond to circumcision. He also speaks of persons 60 and 70 years of age, who had been made disciples to Christ from their childhood (*οἱ ἐκ παιδῶν ἐμαθητεύθησαν τῷ χριστῷ*); their being “*discipled*” would have been within a few years after the writing of St. Matthew's gospel, and they were made so “*from their childhood*!” So that this shows the practice of the *Apostolic* age.

The following extract from Bingham's *Antiquities of the Christian Church*, b. xi. ch. iv. § 9 and 10, will give us the substance of what is to be gathered from Irenæus and Tertullian.

“*Not long after the time of Justin Martyr and the author last mentioned*” (the author of the *Recognitions*, falsely ascribed to Clemens Romanus), “*lived Irenæus, bishop of Lyons, who, as Mr. Dodwell evidently shows,*

* See Wall; and Lightfoot, *Harmony of the Gospels*, and *Hor. Heb.*, *Matt. iii. 6*. The late Mr. Coleridge, in his valuable work entitled *Aids to Spiritual Reflection*, p. 358—378, while arguing for the practice of infant baptism, yet denies that there is any sufficient proof that it was in use in the age of the Apostles. He does not allude to the above passage, and seems to assume (p. 359) that the baptism of infants cannot be shown to be a known practice of that age; but takes no notice of the argument from the baptism of infant children of proselytes. But he afterwards contends most earnestly (p. 361) that nothing conclusive against the practice can be drawn from the silence of the Apostles; and he declares, in a note (p. 363), that the assertions of Robinson, in his *History of Baptism*, that *infant baptism did not commence till the time of Cyprian, &c.*, are rash, and not only unwarranted by the authorities he cites, but *unanswerably confuted* by the arguments of Wall and others. He does not deny that it was the practice of the Apostles; he only denies that we have sufficient proof of it. Mr. Coleridge, however, in both cases, only gives the conclusions at which he has arrived; with one exception, he answers no argument derived from Scripture.

† Gale's *Reflections on Wall's History of Infant Baptism*, letter vii. p. 244—289.

Theology. and Mr. Cave from him, was born in the latter end of the 1st Century, about the year 97, and was a disciple of Polycarp, who was a disciple of St. John. About the year 176 he wrote his book against heresies, being then about 80 years old, and died not many years after. So that he must needs be a competent witness of the Church's sense and practice upon this point during the IInd Century. Now there are three things related on this matter, which appear very evident from him: (1) that the Church then believed the doctrine of original sin; (2) that the ordinary means of purging away this sin was baptism; (3) that children as well as others were then actually baptized to obtain remission of sins, and apply the redemption of Christ to them. For the doctrine of original sin he sometimes calls it the sin of our first parents,* which was done away in Christ by his loosing the bonds wherein we were held and bound over unto death; the sin whereby we offended God† in the first Adam by disobeying his commands, but were reconciled to God in the second Adam by obedience unto death. So that infants as well as others were under the guilt of this sin, and had need of a redeemer, with the rest of mankind, to deliver them from it. Now the ordinary way of being freed from this original guilt, he says, is baptism, which is our regeneration‡ or new birth unto God. And this he expressly affirms to be administered to children as well as adult persons. 'For,' says he, 'Christ § came to save all persons by himself; all, I say, who by him are regenerated unto God, infants and little ones, and children, and youths, and elder persons. Therefore he went through the several ages, being made an infant for infants, that he might sanctify infants; and for little ones he was made a little one, to sanctify them of that age also.' No art can elude this passage, so long as it is owned that regeneration means baptism. And for this, we have the explication of Irenæus himself, who calls baptism by the name of regeneration; and so all the ancients commonly do, as Suicerus (against whom I am now disputing) scruples not to own, alleging Justin Martyr,|| Chrysostom, and Gregory Nyssen to this purpose; which fully evinces infant baptism in the age of Irenæus, that is, in the IInd Century, to have been the common practice of the Church.

"In the latter end of the IInd Century, and beginning of the IIIrd, lived Tertullian, presbyter of the Church of Carthage; who, though he had some singular notions about this matter, yet he sufficiently testifies the Church's practice. In his own private opinion he was for deferring the baptism of infants, especially where there was no danger of death, till they came to years of discretion; but he so argues for this, as to show us that the practice of the Church was otherwise. 'For,' says he, 'according to every one's condition¶ and disposition, and also their age, the delaying of baptism is more advantageous, especially in the case of little children. For what need is there that the godfathers should be brought into danger? because they may either fail of their promises by death, or they may be deceived by a child's proving of wicked disposition. Our Lord says, indeed, 'Do not forbid them to come unto me.' Let them come, therefore, when they are grown up; let them come when they can learn; when they can be

taught whither it is they come; let them be made Christians when they can know Christ. What need Theology. their innocent age make such haste to the forgiveness of sins? Men proceed more cautiously in worldly things; and he that is not trusted with earthly goods, shall he be trusted with divine? Let them know how to ask salvation, that you may appear to give it to one that asketh. For no less reason unmarried persons ought to be delayed, because they are exposed to temptations, as well virgins that are come to maturity as those that are in widowhood by the loss of a consort, until they either marry or be confirmed in continence.' The way of Tertullian's arguing upon this point shows plainly he was for introducing a new practice; that therefore it was the custom of the Church in his time to give baptism to infants as well adult persons; and his arguments tend not only to exclude infants, but all persons that are unmarried or in widowhood, for fear of temptation; which are rules that no one beside himself ever thought of, much less were they confirmed by any Church's practice. But even this advice of Tertullian, so singular as it was, seems only calculated for cases where there was no danger or apprehension of death; for otherwise he pleads as much for the necessity of baptism as any other, both from those words of our Saviour,* 'Except one be born again of water and the Spirit, he cannot enter into the kingdom of heaven;' as also from the general corruption of original sin, which renders every son of Adam unclean till he be made a Christian, which is only done in baptism; for men are not born Christians, but made so: and therefore, in case of necessity, he thought every Christian had power to give baptism rather than any person should die without it; which seems to imply that his opinion for delaying baptism, whether of infants or others, respected only such cases where there was no danger of death; but even in those cases the practice of the Church was otherwise; for she baptized infants as soon as they were born, though without any imminent danger of death, as appears from Tertullian's discourse itself, who laboured to make an innovation, but without any success; for the same practice continued in the Church in the following ages."

The above arguments and extracts will be sufficient, we trust, to prove that infant baptism was the practice of the early Church, and to explain the course which Scripture takes concerning this great question. They are to be considered, not as a full exposition of the views of the Church in regard to baptism, but as supplementary to the article to which we have above referred. (*Ency. Met. Misc. Div.*, article BAPTISM and BAPTISTS.) We now proceed to the other Sacrament of the Christian Church—the Supper of the Lord.

It will be observed that we have not here entered at all into the question of regeneration, as connected with baptism. There can be no doubt, from the offices of the Church of England, that she considers baptism to be the means by which the grace of regeneration is received; and this, we humbly conceive, is the true sense of Scripture. But to engage in the discussion of this point would carry us far beyond the limits to which we are necessarily confined. Some who differ widely on the subject will probably find that their difference chiefly arises from a difference in their definition of the term Regeneration. The following extracts from Bishop Bethell's very valuable work, entitled *A General View of the Doctrine of Regeneration in Baptism* (2nd edit. 1836), will, however, be acceptable to most readers.

* Irenæus, lib. v. c. 19. † Id. v. 16. ‡ Id. i. 18.
§ Id. ii. 39. || Suicer. *Thesaur. Eccles.* voc. 'Αναγεννησις.
¶ Tertull. *De Baptismo*, c. 18.

* Tertullian, *De Anima*, c. 40; *De Baptismo*, c. 13.

Theology.

In that book, Bishop Bethell observes, in speaking of the word Regeneration,—“No reasonable doubt can be entertained that it was appropriated to that grace, whatever may be its nature, which is bestowed upon us in the sacrament of baptism (including, perhaps, occasionally, by a common figure of speech, its proper and legitimate effects, considered in conjunction with it), from the beginning of Christianity to no very distant period of ecclesiastical history. In those few passages of the ancient Christian writers where it evidently bears another signification, it is evidently used in a figurative manner, to express such a change as seemed to bear some analogy in magnitude and importance to the change effected in baptism. At the time of the Reformation the word was commonly used in a more loose and popular way, to signify sometimes justification, sometimes conversion, or the turning from sinful courses, sometimes repentance, or that gradual change of heart which is likewise styled renovation. Hence, in popular language, it came to signify a great and general reformation of habits and character, and the words ‘regenerate’ and ‘unregenerate’ were substituted for the words converted and unconverted, renewed and unrenewed, righteous and wicked. But in the hands of the systematic Calvinists, the word passed from the popular to a strict and determinate meaning; and they pronounced regeneration to be an infusion of a habit of grace, or a radical change of all the parts and faculties of the soul, taking place at the moment of the effectual call. From hence the transition to a sensible change was easy and natural; and what was a theological speculation in the system of the scholastic divines, became, in the hands of less subdued and less calculating spirits, the stronghold of enthusiasm.”—Bishop Bethell, *On Baptismal Regeneration*, p. 6-8.

The facts stated in this extract might suggest to those persons who have actually wished to alter the baptismal office of our ritual, because they cannot reconcile it with their notions of regeneration, whether it would not be wiser and more modest to reform their own ecclesiastical vocabulary, than to lay rash hands on aught so holy as our *Book of Common Prayer*. The reference to Bishop Bethell’s book supersedes the necessity for more numerous references, as the most valuable writers will be found quoted by him. Bishop Pearson, *On the Creed*, art. x., and Hooker, book v. § 59, *et seq.*, ought not, however, to be omitted.

We must add one more extract from the same volume:—

“But the truth* is, that the doctrine of baptismal regeneration, or salvation, though it has been called in question by those Protestant Churches (usually called the Reformed Churches) which have adopted the sentiments of Zuinglius and Calvin, and their off-shoots, the Arminians and Socinians, is the doctrine not of the Church of Rome only, and of all the Eastern Churches, but of the Protestant Lutheran Churches, and the Protestant Church of England and Ireland. In common with the Church of Rome and the Lutheran Churches, we hold that regeneration, or the new birth, is the spiritual grace of baptism conveyed over to the soul in the due administration of that sacrament. We hold, in common with those churches, that in adults, duly qualified by repentance and faith, the guilt of sin, both original and actual, is cancelled in baptism; that in infants, who have committed no actual or wilful sin, and can possess no such qualifications, the guilt of original sin is done away; and that infants, no less than adults, are made, in baptism, children of God, members of Christ, inheritors of the kingdom of Heaven, and partakers of the privileges, and blessings, and promises of the Gospel covenant. But the Church of Rome contends,† that not only the guilt, but the very essence and being of original sin, is removed by baptism; the Church of England declares that this ‘corruption of nature remains even in the regenerate.’ The Church of Rome has decreed that that ‘concupiscence [or fuel (*fomes*), as it is called] which remains after baptism has not, properly speaking, the nature of sin;’ whereas we affirm that ‘concupiscence has the nature of sin,’ and allege the authority of an apostle in support of our opinion. We believe that sacraments are not only signs of grace, but means or instruments through which God consigns over to the soul the grace which they signify. The followers of Zuinglius and Calvin, with whom the author of this pamphlet agrees, contend that this is the very error of the *opus operatum*—the opinion, that is—that where there is no positive bar, sacraments produce a saving effect without suitable affections on the

* This passage is in reply to the statement of an anonymous author of a pamphlet, who says that “transubstantiation and baptismal salvation,” which he classes together as branches of the same corrupt doctrine, are “the two great subjects of dispute between Protestants and [Roman] Catholics.”

† See the *Canones et Decreta Concil. Trident.* sessio v.; and the *Cat. Concil. Trident.* pars ii. ch. ii. 43.

Theology. part of the recipient. We, however, affirm, as a general truth, that such affections of mind are indispensable, and that where they are wanting, sacraments produce no beneficial effects. But as we are convinced that the baptism of infants is a part of our Saviour’s institution, we do not conceive that, in their case, the unavoidable want of these qualifications is any impediment to the saving grace of the sacrament.”—Bishop Bethell, Preface, p. xvii.-xix.

The question of *lay-baptism* has already been treated of, under the article BAPTISM, in Vol. I. of the Miscellaneous Division of this *Encyclopædia*. We may just add, that the non-jurors generally considered lay-baptism invalid; and that Dr. Bingham, the learned author of the *Antiquities of the Christian Church*, wrote a defence of lay-baptism, which he entitled, *A Scholastical History of Lay-baptism*, to which we must further refer those who wish to investigate this point, as well as to the *Lay-baptism Invalid* of Mr. Lawrence, with whom Bingham was in controversy on this point. Bingham maintained that in the primitive Church the rule was, that laymen were not permitted to baptize in ordinary cases; but when a case of necessity arose, that their baptism was considered valid. The Roman Catholic Church goes far beyond this,—allowing *Jews, Turks, and Pagans* to baptize in cases of necessity (see the *Rituale Romanorum*, tit. *De Baptismo Ministro*).

The ecclesiastical courts of this country have laid it down that the baptism of dissenters is sufficient for the purpose of entitling a person to burial (*Kemp v. Wickes*, and more recently *Escott v. Mastin*); but they suppose the baptism administered in the name of the Holy Trinity. The decision is one much controverted. The case has not yet been tried of an Unitarian baptism in the name of Jesus Christ only.

The Sacrament of the Lord’s Supper.

The account of the institution of the Sacrament of The Eucharist, as recorded in the most solemn and affecting simplicity of the Evangelists, and repeated in the independent testimony of St. Paul, must be familiar to the mind and heart of every Christian. It is impossible for any one who has a reverential spirit to meditate on these passages of Scripture without feeling the extreme solemnity of the season chosen by our Lord for the institution of this Holy Sacrament. Without presuming to search out the motives which induced our Lord, in his infinite wisdom and mercy, to select this most affecting season for this purpose, we may nevertheless observe, that it is admirably calculated to subdue all irreverence of speculation on this mysterious rite, and to lead us to connect its celebration most clearly and closely with the atoning Sacrifice of which it is the commemoration. It is, indeed, a subject on which the reverential believer would desire to speak only in the tones of gratitude, of faith, and of love. But, unhappily, such appalling errors have been maintained with respect to it, and have led to such grievous superstitions, that it is impossible to treat even of this soul-subduing subject without entering upon controversial discussions, if we desire that the simple truth, as propounded to us in the Word of God, and as received by the primitive Church, may be maintained.

Article xxxviii. of the Church of England declares that—

“The Supper of the Lord is not only a sign of the love that Christians ought to have among themselves one to another, but rather is a sacrament of our redemption by Christ’s death, insomuch that to such as rightly, worthily, and with faith receive the same, the bread which we break is a partaking of the Body of Christ, and likewise the Cup of Blessing is a partaking of the Blood of Christ.

“Transubstantiation (or the change of the substance of bread and wine) in the Supper of the Lord cannot be proved by Holy Writ; but is repugnant to the plain words of Scripture, overthroweth the nature of a sacrament and hath given rise to many superstitions.

Theology.

"The Body of Christ is given, taken, and eaten in the Supper, only after an heavenly and spiritual manner. And the mean whereby the Body of Christ is received and eaten in the Supper, is Faith.

"The Sacrament of the Lord's Supper was not by Christ's ordinance reserved, carried about, lifted up, or worshipped."

The Romish doctrine.

The Romish doctrine of Transubstantiation is plainly expressed by the Council of Trent, in its 13th session. It is there stated that the Church has always maintained that *per conversionem panis et vini conversionem fieri totius substantiæ panis in substantiam corporis Christi Domini nostri, et totius substantia vini in substantiam sanguinis sui*; and it is added, that this change of substance has properly been called Transubstantiation by the Church. It is, therefore, the doctrine of the Romish Church that the accidents of bread and wine remain, but not its substance.

It will be seen at once that this is a doctrine of immense importance; and it is a matter of notoriety that the very consequences which might naturally be expected to flow from an admission of its truth, have been pushed by the Church of Rome to their fullest extent.

In that Church the consecrated wafer,* being considered to be changed into the very substance of Christ's Body, is ordered to be adored with the same reverence as that with which the Deity himself is worshipped. (*Concil. Trident. sess. xiii. cap. v. and canon vi.; and Catechism. ad Parochos, pt. ii.; De Euchar. Sacram. viii.*)

The oblation of this sacrament is also by that Church considered a propitiatory sacrifice, available not only for those who partake of it, but for others also, whether living or dead, to obtain remission of sins and other benefits of Christ's passion,—his sacrifice being thus, as it were, repeated.

These two consequences, which have flowed from the admission of this doctrine of transubstantiation, attest its great importance, and would at once suggest to any thoughtful mind the necessity of extreme care and caution before he would accept it. If it be not true, the first of these must be an act of idolatry, and the second a delusion. The chief Scriptural authority on which our faith in this doctrine is challenged by its advocates, is the passage of Scripture in which our Lord says, "This is my body,"—an expression which the Romanist understands *literally*, and the Protestant *figuratively*.

Extreme Protestant view.

There is, besides this extreme, into which the Church of Rome has fallen, another also, of an opposite character. This sacrament is by some conceived to be a *mere commemorative rite*,—a doctrine which is as far from reaching to the truth of the matter, as laid down in Scripture, as the doctrine of transubstantiation goes beyond it.

The Church of England holds a middle course.

Between these two extremes the Church of England has held a middle course, not defining the mode by which the faithful receiver of the holy symbols becomes a partaker of the body and blood of Christ, but absolutely denying the gross and carnal mode in which the Church of Rome has defined it; and at the same time,

* The words *sacramentum* and *Eucharistia* are used in the canons and decrees of the Council of Trent for the wafer, as some passages absolutely require this interpretation: *e. g.* sess. xiii. cap. vi. (See the *Catechismus ad Parochos*, pt. ii., *De Euch. Sacram. viii.*)

by adhering to Scripture truth and primitive belief, securing this Holy Sacrament from every degree of irreverence.

The Church of England (in her *Catechism*, and in Art. xxviii.) acknowledges that the Body and Blood of Christ are verily and indeed taken and received by the faithful in the Lord's Supper; but at the same time declares that it is taken and received only after a heavenly and spiritual manner.* The Church of England, with the primitive Church, gladly acknowledges, that after Consecration the bread and wine are no longer common bread and wine; that though their substance remains unchanged, yet that they are changed *in their property as in their use*; that from being mere food for the body, they become *sacramentally* the Body and Blood of our Saviour to the soul, as being in their use ordained means by which the faithful believer becomes a partaker of the benefits of Christ's body and blood, and by which he is made one with Christ, and Christ with him. As long as we keep to language like this, we maintain nothing but what Scripture sanctions, and the primitive Church received.

Obs.—It will be observed, that while our authorized formularies only so far define the mode in which the Body of Christ is received in Sacrament, that they assert it to be received only after a spiritual and heavenly manner, they do not use the term "*real presence*." This phrase is, however, used by the great writers of our Church, who have expressly treated of these points, *e. g.* Andrewes, Jeremy Taylor, Wake, &c. And the great truth expressed in the authorized formularies is equally strong in asserting the mystery of the Sacrament; but it may be worth considering whether the use of the term *presence* introduces any difficulty which is avoided by the other mode of expression. When used in its proper sense, the term "*real presence*" is only tantamount to the expressions of the authorized formularies of our Church; and the great primitive truth and mystery is asserted by both modes of expression. But without venturing, in a matter of so great depth, to object to a course followed by so great men as those we have referred to, and others of as noble name in our Church, we may, with reverence, examine the effect which the introduction of the term may possibly have, and inquire humbly and modestly still, whether it has a tendency to *define* the mysterious truth involved in our sacramental doctrine, or even to advance a step towards *defining* it. Without asserting that it does so, we still think that it is a question deserving of consideration; if it does, we ought at least to be aware of it, whatever course we take; if it does not, then the phrase is cleared from all objection.†

The sacred symbols are, therefore, more than common bread and wine after consecration; not by any change in their substance, but by a change in their properties and use. *How* they are the means of conveying the benefits of the passion of Christ to us, we presume not to inquire. It is enough for us to believe, as St. Paul has expressed it, in 1 Cor. x. 16—"The cup of blessing which we bless, is it not the communion of the

* On the subject of the Sacrament we may refer to the following works by English divines, which we have found, among many others, very useful:—Moreton, *On the Mass*; White's *Conference with Fisher*; Laud's *Conference with Fisher*; Wake, *On the Eucharist*; Waterland, *On the Eucharist*; Cosins, *Historica Transubstantiationis*; Usher's *Answer to a Jesuit's Challenge*; and Turton's *Answer to Wiseman*. In Bellarmine's controversial works will be found the chief arguments used by the Roman Catholics. The work of Arnauld and Nicole, *La Perpétuité de la Foi*, is also an elaborate work to prove the primitive doctrines the same as the present Roman Catholic; a view to which the works above cited will give a sufficient contradiction. See, against this, Albertinus, *De Eucharistia*, and L'Arroque, *Histoire de l'Eucharistie*.

† We may observe that Archbishop Wake says that "our Church utterly denies our Saviour's body to be *so really present* in the Blessed Sacrament as to leave Heaven, or to exist in *several places at the same time*;" and Bishop Taylor says, "By spiritually, they (the Papists) mean, present after the manner of a spirit; by *spiritually* we mean, present to our spirits only."

Theology. blood of Christ? The bread which we break, is it not the communion of the body of Christ?"

Our Church, in her xxviiith Article, asserts that transubstantiation is repugnant to the plain language of Scripture; and we proceed, therefore, to compare that doctrine with Scripture, and with the faith of the early Church.

1. *Scripture.*—The declaration of our Lord, "Take eat, this is my body," we contend, against the Romanists, in the account of the institution of the Lord's Supper, must be considered as figurative, just as the language of Scripture is found to be in many analogous cases. Let us take the phrase, "that Rock was Christ," 1 Cor. x. 4;—will it be contended that this was not figurative language? Again, look at our Lord's own language, on the same evening as that on which he instituted this sacrament, "I am the true vine." Is it possible to take this in any other than a figurative sense?

These are instances of figurative language which suggest at once to us the propriety of receiving the above declaration of our Lord in a figurative sense also. There are other circumstances also connected with the occasion, viz., the paschal feast, on which our Saviour made this declaration, which strengthen the argument for its figurative sense.* The language of the Jews at that solemn season was of a similar nature with our Lord's words. The master of the house was accustomed to take bread, break it, and give it to the guests, saying, *This is the bread of affliction which our fathers ate in Egypt*; and the paschal lamb they called the *body of the Passover*. Again, the master of the house blessed the cup, gave thanks, and then delivered it to the rest saying, *This is the fruit of the vine, and the blood of the grape*; and we cannot but suppose that our Lord made his declarations with a tacit reference to these customs.†

Archbishop Wake further observes, in reference to this passage of Scripture:

"It is the acknowledgment of their greatest writers" (*i. e.* the Romanists) "at this day,‡ that if the pronoun *this*, in that proposition, '*This is my body*,' be referred to the bread which our Saviour held in his hand, which he blessed, which he brake, and gave to his disciples, and of which, therefore, certainly, if of anything, he said '*This is my body*,' the natural repugnancy that there is between the two things affirmed of one another, bread and Christ's body, will force them to be taken in a figurative interpretation; forasmuch as 'tis impossible that bread should be Christ's body otherwise than by a figure. And, however, to avoid so dangerous a consequence, they will rather apply it to anything, nay, to nothing at all, than to the bread; yet they would do

* See Archbishop Wake, *On the Eucharist*, Introduction, p. 3., who quotes authorities for this statement.

† For further illustrations we must refer to Archbishop Wake, *ubi supra*.

‡ The Archbishop places the following note in the margin of this passage: "See Bellarmine's words in the *Defence of the Exposition of the Doctrines of the Church of England*, pp. 56, 57. To which may be added Salmer, tom. ix. tr. 20; Suarez, *Disput.* 58, § 7; Vasquez, *Disput.* 201, c. 1, &c."

In Bellarmine's controversial works, tom. iii., will be found his four books *De Sacramento Eucharistiae*, in the first of which (chapters 10 and 11) is found his discussion of the words *Hoc est corpus meum*. We think few readers could peruse this portion of the treatise without observing the great ingenuity of the Cardinal, and, at the same time, the very unsatisfactory nature of his arguments.

Theology. well to consider whether they do not thereby fall into as great a danger on the other side; since, if the *relative* *this* do's not determine these words to the bread, 'tis evident that nothing in that whole proposition do's; and then, how those words shall work so great a change in a *subject* to which they have no manner of relation, will, I believe, be as difficult to show as the change itself is incomprehensible to conceive." (Wake, *On the Eucharist*, p. 20.)

These considerations, to which many more might be added (as, indeed, many more are, in the very treatise just quoted), will, we trust, be sufficient to support the statement of Article xxviii. of our Church, with regard to the language of Scripture. We proceed now to compare transubstantiation with the faith of the early Church.

Among the Roman Catholics, the Jansenists maintain that the sixth chapter of St. John's Gospel is *not* to be understood of the Sacrament of the Lord's Supper. Others maintain that it is to be referred to this subject. The latest attempt which has been made to support transubstantiation by that portion of Scripture is to be found in Dr. Wiseman's *Lectures on the Eucharist*, which have been so ably answered by Dr. Turton. (See also Waterland, *On the Eucharist*, ch. vi.) Wake also has touched upon this chapter, and observes, that, with regard to v. 51, our opponents confess that the two first assertions in it are to be taken figuratively, but insist, at the same time, on taking the third literally.

There is no question that the Fathers occasionally use language, in speaking of the Sacrament, which is by no means inconsistent with transubstantiation, and which might seem apparently to favour it, because they speak of a change in the elements, and of the bread and wine as the body and blood of Christ. But the quotations of this sort are usually more than neutralized by other passages from the same Fathers, declaring the nature of this change, and precluding the notion of any change of substance.

And besides, the very passages themselves, which are adduced to support transubstantiation, will, in fact, be found to fall far short of this object.* They amount, in general, to a declaration that the bread and wine are no longer, after consecration, mere bread and wine, but that they are the body and blood of Christ; but indications sufficiently strong are given to show that this implied no change in the substance of the sacramental elements, but a change in their use and destination; while expressions are used which also show that the gross and carnal understanding of such passages would be highly abhorrent from the views and notions of the authors themselves.

There is a very remarkable passage in a fragment of Irenæus, which Dr. Waterland has referred to. The fragment itself is found in Massuet's *Irenæus*, p. 343, and is to this effect:—The slaves of some Christians having been seized, were put to the torture, to make them bring some accusation against their masters, from the nature of their mysteries; and these slaves [ὅτι ἔχοντες πῶς τὸ τοῖς ἀναγκάζουσι καὶ ἡδονὴν ἔχειν, παρὰ τὸν ἥκουον τῶν δι-σποτῶν, τὴν θέλαν μεταλλάξιν αἶμα καὶ σῶμα εἶναι Χριστοῦ αὐτοὶ νομίσαντες τῷ ὄντι αἶμα καὶ σαρκὰ εἶναι, τοῦτο ἐξῆπτον τοῖς ἐκζητοῦσιν], not being able to gratify those who made this search, inasmuch as they

Cardinal Bellarmine observes, that *hoc* can neither mean this bread nor the body of Christ, but that which is contained under the accidents of the bread. *Hoc non demonstrat panem, nec corpus Christi, sed contentum sub speciebus.* (lib. i. *De Sacr. Euch.* c. xi.) In another passage (c. x.) he observes, that if our Lord spoke of the bread, it would be absurd to say, *Hoc est corpus meum, cum debuisset dicere, Hic panis est corpus meum.* The language concerning the cup seems plainly, by analogy, to point out the bread as the *subject* of this proposition; and the very text which the Cardinal requires, is supplied by St. Paul, who says, "As often as ye eat this bread," &c.

* We observe that Cardinal Perron gives up the treatise of St. Athanasius, usually quoted for this purpose, as being spurious.

Theology.

heard from their masters that the divine communion was the body and blood of Christ, *thinking that it was in reality flesh and blood*, told this to the inquisitors. On which Blandina, one of the Christians tortured on this accusation, replied most nobly and freely, by saying, *How could those bear such a thing, whose discipline restrains them from enjoying even the flesh which is permitted?* [πῶς ἂν ὑποῦσα, τούτων ἀνάσχοιτο οἱ μηδὲ τῶν ἐφικμένων κρεῖων δι' ἄσκησιν ἀπολαύοντις;]*

St. Cyril of Jerusalem.

Let us now proceed to quote one of the most remarkable and strongest passages usually adduced in favour of Transubstantiation, and see to what it amounts. This is the celebrated passage of St. Cyril of Jerusalem, in his fourth catechetical lecture on the Mysteries (*Cat. Mystog.* iv. 3), beginning thus, ὥστε μετα πάσης πληροφορίας, ὡς σώματος καὶ αἵματος, &c. The whole passage runs thus:—

“Therefore, with full assurance of faith, let us partake, as of the body and blood of Christ, for in the symbol (τύπῳ) of bread the body is given to thee, and in the symbol (τύπῳ) of wine the blood is given to thee, that thou, partaking of the body and blood of Christ, mayest be (σύσσωμος καὶ σύναιμος αὐτῷ) of the same body and blood with Christ. For thus we become *Christophori*, his body and blood being distributed into our members, and thus also, according to St. Peter, we become partakers of the Divine nature.”

We here observe that *τύπος* is used—not *εἶδος*. St. Cyril afterwards desires us not to trust our own taste, but to believe the word of Christ; but the words which follow the passage just cited sufficiently explain how they are to be understood. St. Cyril immediately adds a caution, lest we should take this passage too literally. He adds, “Christ, once in conversation with the Jews, said, ‘Unless ye eat my body and drink my blood, ye have not life in yourselves;’ and they, not understanding† spiritually what was said, being scandalized [at it], went away back, thinking that he was inciting them to the eating of flesh (*σαρκοφαγία*).” Here St. Cyril evidently blames the Jews for not understanding our Lord’s words in a spiritual sense, which is a strong argument that he both understood our Lord’s words, and used the same himself, in regard to the Eucharist, in a spiritual, not a carnal sense.

St. Augustine.

In the same manner, very strong language is used by St. Augustine, in some instances, relative to the Eucharist. But if we look at a passage in his *Enarratio in Ps. XCVIII.*, we shall find a plain intimation of the manner in which he used such language, and would have it to be understood: “*Non hoc corpus quod videtis manducaturi estis, nec bibituri illum sanguinem quem fusuri sunt, qui me crucifigent; Sacramentum aliquod vobis commendavi, spiritualiter intellectum vivificabit vos.*”

St. Chrysostom and others.

In the same manner the passages which Romanists may adduce from St. Chrysostom may be met with others, by which it is clear that he spoke of these things spiritually, and not carnally, and that he understood that the body and blood of our Saviour were

spiritually, though really, received in the Eucharist. Theology. For these passages we may refer to Bishop Cosins, *Histor. Transubstantiationis*, ch. v. s. 18, pp. 73, 74.

And in the same manner also the language used by other fathers of the Church, which appears sometimes to approach this doctrine, is explained by passages which show us that the words there used must be interpreted spiritually.

It may perhaps be well to state, that those who wish to pursue the subject to a greater extent will find great advantage from the works already quoted, especially the *Historia Transubstantiationis*, by Bishop Cosins, and the treatise of Archbishop Wake. They will be able there to test the above assertion with regard to the fathers, and they will find the classification of passages on this point suggested by Bishop Cosins, in chap. vi. s. 5, of considerable use. Let them compare also the following statement of Archbishop Wake:—

“For the expressions of the Holy Fathers; it is not deny’d but that in their popular discourses they have spared no words* (except that of *Transubstantiation*, which not one of them ever used,) to set off so great a mystery; and I believe that, were the *Sermons* and *Devotional Treatises* of our own Divines alone, since the *Reformation*, searcht into, one might find expressions among them as much overstrained.† And doubtless there would be as strong an argument to prove *Transubstantiation* now the doctrine of the Church of England, as those to argue it to have been the opinion of those primitive ages.”

“But now let us consult these men in their more exact compositions, when they come to teach, and not to declaim, and we shall find they will then tell us—that these elements are, for their‡ *substance*, what they were before, bread and wine; that they retain the true *properties* of their nature, to *nourish* and *feed* the body; that they are things *inanimate* and void of *sense*; that, with reference to the Holy Sacrament, they are *images, figures, signs, symbols, memorials, types* and *antitypes* of the body and blood of Christ; that, in their *use* and *benefit*, they are indeed the very *body* and *blood* of Christ to every faithful receiver, but in a *spiritual* and *heavenly* manner, as we confess; that, in propriety of speech, the *wicked* receive not, in this Holy Sacrament the *body* and *blood* of Christ, although they do outwardly press with their teeth the Holy Elements; but rather eat and drink the Sacrament of his *body* and *blood* to their damnation.”—Wake, *On the Eucharist*, pp. 24, 25.

We must now commend these statements to the judgment of the reader. He will find in the works already cited abundant means of ascertaining the value and justness of the results which we have here given. The line of argument by which these results are obtained has been indicated, and, as far as our limits allow, illustrated and supported by quotations. Little further appears practicable without a thorough and lengthened investigation into the whole question.

But the result to which we are now brought is unquestionably this—that the Tridentine Council has not only been unwarrantable in defining what Scripture does not enable us to define, but has erred most grievously in defining it in a manner entirely inadmis-

* “Such are μεταποίησις, μεταβολή, μεταρρύθμισις, μεταλήψις, μεταστοιχείωσις, but never μεταουσίωσις. And note, there is hardly one of these words which they have applied to the bread and wine in the Eucharist, but they have attributed the same to the water in Baptism.

† “See Treatise first, *Of the Adoration*, &c., printed lately at Oxford; which would make the world believe that we hold, I know not what imaginary *real presence* on this account; just as truly as the Fathers did Transubstantiation.”

‡ “It is not necessary to transcribe the particulars here, that have been so often and fully alleged. Most of these expressions will be found in the Treatise of *Transubstantiation* lately published” (1687). “The rest may be seen in Blondel, *Eclaircissements Familiers de la Controverse de l’Eucharistie*, cap. iv. vii. viii.; Claude, *Rep. au 2 Traité de la Perpétuité*, part. i. cap. iv. v. Forbesius, *Instructiones Historico-Theolog.* lib. xi. cap. ix.-xiii. xv.; Larroque, *Histoire de l’Eucharistie*, liv. 2, cap. ii.”

To the above authorities of the Archbishop, among which the works of Forbes will be found very valuable, may be added the

* In Euseb. v. 1, Blandina, one of the martyrs of Lyons, makes a similar reply to the accusation, that Christians ate infants at their secret meetings. But this makes no difference as to the argument from Irenæus.

† Touttée (the Benedictine editor) rebukes Mills for translating this thus: “not understanding what was spiritually said.” Touttée appears to be right as to the Greek; but it makes very little difference as to the sense of the passage. If the Jews were wrong in not understanding it spiritually, as Touttée’s own translation implies, then it was said spiritually, i. e., intended to be taken so.

Theology. sible, and opposed to the real statements of Scripture and the true views of the best ages of the Church.

We may here note some of the errors with which the doctrine of Transubstantiation has a close connexion:—

1. The adoration of the host, as remarked upon before.

2. The notion that the mass is a truly propitiatory sacrifice. We must, in speaking upon this point, be careful to define clearly what is meant by the word *propitiatory*. If it merely means *making* or *rendering propitious*, as prayer may be said to render God *propitious*, then the assertion is altogether different from that which is objected to in the Romish doctrine, which asserts that the sacrifice of the mass must be offered as a propitiatory sacrifice *pro vivis et defunctis, pro peccatis, pœnis satisfactionibus et aliis necessitatibus* (*Can. et Decret. Concil. Trid. sess. xxii. can. iii.*; compare cap. ii. of the same session); by which words we may understand that it contains a propitiatory offering capable of taking away sin. It is easy to see the abuses to which such a view must almost necessarily lead. And the imagination will not easily outstep the reality, if we inquire into the practices connected with this doctrine. See the strong language used by our Church, in Article xxxi., upon these practices.

3. The refusal of the cup to the laity. This, though not a necessary consequence of the doctrine of transubstantiation, apparently flowed from it. The argument by which this practice is supported is, that, under the bread, Christ is received fully, and it would have little meaning except as connected with transubstantiation. See Burnet on Article xxx., on the doctrine of *Concomitance*.

The Communion a sacrifice?

We must, however, before we dismiss this most important subject, observe, that although we repudiate entirely the unscriptural and unprimitive doctrine of transubstantiation, we consider the sacred elements, after consecration, more than mere signs or symbols, although they do not lose their nature of bread and wine. They are the ordained means whereby the faithful are made partakers of the body and blood of Christ, and become one with Him and He with them. And this, we trust, may prove to the Romanist, that though we reject the doctrine, which according to their notions elevates the Sacrament,* but according to ours only degrades and carnalizes it, we hold the Sacrament in no less reverence than they do; and if we can only reach to perfect purity of faith, and to a full spiritual apprehension of the mystery of this Sacrament, human language has no

work of Bishop Cosins, and that of Bishop Moreton, above referred to. A brief but convenient collection of passages is also found in Bishop Burnet, *On the Articles*, as well as a short history of the doctrine.

* This is a common argument of Roman Catholic writers. It is one which both parties ought to use only with extreme caution. Our desire ought to be, to know what God has revealed. We are sure, if we are not wanting to ourselves, to find that His way is the most calculated to secure our reverence and love. As an instance of this argument, we may cite several passages of the viii. chapter of the preface of the learned Muratori to the *Roman Sacramentary* published by him. In speaking of Bingham, he says, that he has treated all the Mass with much learning, but has left out or condemned the most excellent (*potiora*) parts of the Catholic doctrine: *Neque enim quidquam habet de incruento Missæ sacrificio; presentiam realem Corporis et Sanguinis Domini in Eucharistia, et Donorum Transubstantiationem negat, Adorationem verò debitam Christo præsentem improbat. Tolle hæc e Missa fidelium: immensa ejus dignitas mirum in modum decrescit, immo perit, &c.* The denial of the real presence is not a fair statement, and the worship that is refused and disapproved, is not worship of Christ, but of the wafer. But to let this pass, we may here mention with approbation the tone assumed by Muratori, in these passages, as being so much more moderate than that of some of his contemporaries and compatriots; e. g. *De Sala*, the editor of the magnificent edition of *Cardinal Bona de Rebus Liturgicis*, published at Turin, 1747-1753, 3 vols. folio. This book, in which one might expect the matter to be treated in a decent style, as being addressed to the learned, and not to the populace, is constantly defaced with coarse and vulgar abuse—*nebulones, deblaterare, impii, blasphemæ*, and similar terms, being amongst the most civil expressions applied to Protestant writers.

VOL. II

adequate means to express our reverence for that institution which confers such inestimable benefits upon us!

Theology.

But before we leave the doctrine of the Sacrament of the Lord's Supper, there is one more point on which we must briefly touch.

The nature of the Romish error concerning the sacrifice of the Mass has been already stated; but the corruption of a legitimate doctrine is not to prevent the expression of its truth. As the error of transubstantiation may not withhold us from acknowledging the primitive truth of the *reality* of the communion of the body and blood of Christ, so the erroneous doctrine of the sacrifice of the Mass need not restrain us from acknowledging that kind of sacrifice which has been so often and ably vindicated by the great writers of the Church of England. The following quotation from Archbishop Laud, on this point, will be found valuable: "And since here's mention happened of *sacrifice*, my *third* instance shall be in the sacrifice which is offered up to God in that *great* and *high* mystery of our *redemption* by the death of *Christ*; for as Christ offered up himself, once for all, a full and all-sufficient sacrifice for the sin of the *whole world*,—so did he institute and command a memory of this *sacrifice* in a Sacrament, even till his coming again. For at and in the *Eucharist* we offer up to God *three* sacrifices: One by the *priest* only; that's the *commemorative* sacrifice of Christ's death, represented in bread broken, and wine poured out. Another by the *priest* and the *people* jointly; and that is the *sacrifice of praise and thanksgiving* for all the benefits and graces we receive by the precious death of Christ. The *third* by every *particular man* for himself only; and that is the sacrifice of every man's *body* and *soul*, to serve him in both, all the rest of his life, for the blessing thus bestowed upon him. Now thus far the dissenting Churches (*i. e.* the Roman and English) agree, that in the *Eucharist* there is a sacrifice of *duty* and a sacrifice of *praise*, and a sacrifice of *commemoration* of Christ. Therefore, according to the former rule, (and here in truth too,) 'tis safest for a man to believe the commemorative, the praising, and the performing sacrifice, and to offer them duly to God, and leave the Church of Rome in this particular to her superstitions, that I may say no more." And would the Church of Rome stand to Archbishop Laud's rule, and "believe dissenting parties where they agree, were it but in this, and that before, of the *real presence*, it would work far toward the peace of *Christendom*. But the truth is, they pretend the *peace of Christendom*, but care no more for it, than as it may uphold at least, if not increase, their own greatness."—Laud, *Conference with Fisher*, s. 35, Num. 7, punct. 3.

In the notes in this passage (among many references in support of the text), the Archbishop says, "the ministration of the Holy Communion is sometimes of the ancient Fathers called an *unbloody sacrifice*; not in respect of any corporal or fleshly presence that is imagined to be there without blood shedding, but for that it representeth and reporteth to our minds that one and everlasting sacrifice that Christ made in his body upon the Cross. This Bishop Jewel disliketh not, in his answer to Harding, Art. 17, Div. 14. . . . And Dr. Fulke also acknowledges a sacrifice in the Eucharist. In *St. Matthew*, xxvi. 26."

The very writers, however, who maintain most strongly the doctrine of the Eucharist sacrifice, protest at the same time most strongly against the mischievous delu-

Theology. sion of the doctrine of the Mass by which it is said that a propitiatory sacrifice is there offered for the living and the dead. (See above, in the observations on the last paragraph, observation 2.)

This subject may be well closed in the words of one of our Homilies. In recommending simplicity and exact following of our Lord's commands in the celebration of our Lord's Supper, it is said,—

"For (as that worthy man St. Ambrose saith) he is unworthy of the Lord, that otherwise doth celebrate that mystery than it was delivered by Him. Neither can he be devout, that otherwise doth presume than it was given by the Author. We must then take heed, lest of the memory, it be made a sacrifice; lest of a communion, it be made a private eating; lest of two parts, we have but one; lest applying it for the dead, we lose the fruit, that be alive." (*First Part of the Sermon concerning the Sacrament.*)

And again in the same Homily,—

"It is well known that the meat we seek for in this supper is spiritual food, the nourishment of the soul, a heavenly reflection, and not earthly; an invisible meat, and not bodily; a ghostly substance, and not carnal; so that to think that without faith we may enjoy the eating or drinking thereof, or that that is the fruition of it, is but to dream a gross carnal feeding, basely objecting and binding ourselves to the elements and creatures. . . . Take then this lesson, O thou that art desirous of this table, of Emissenus, a godly Father, that when thou goest up to the reverend communion to be satisfied with spiritual meats, thou look up with faith upon the Holy body and blood of thy God, thou marvel with reverence, thou touch it with the mind, thou receive it with the hand of thy heart, and thou take it fully with thy inward man."

We may refer those who wish to enter further into this subject to Dr. Mede's *Christian Sacrifice*; Dr. Waterland, *On the Eucharist*; Johnson's (the Nonjuror) *Unbloody Sacrifice*; Hickes, *On the Christian Priesthood*; Bishop Bull's *Answer to Bossuet*; Bishop Patrick's *Mensa Mystica*.

It has, we must remark, been a subject of keen contention at various times, within the pale of our Church, whether the terms *sacrifice* and *altar* ought to be applied to the Eucharist and the Holy Table. Those who object to these terms have not so much fault to find with the doctrines of their opponents, when fairly stated, as with these names, because of the mischievous doctrines which have been mixed up with these names in the Roman Catholic Church. The dispute was very hot between Dr. Heylin (in his *Coale from the Altar*, 1636) and Bishop Williams (*Holy Table; Name and Thing*, &c. 1637) and others.

The Nonjurors sometimes defined a sacrifice to be, some material thing offered and dedicated to God, and made the oblation of the elements the chief point in the sacrifice. But Mr. Johnson, in his *Unbloody Sacrifice*, points out five particulars by which a sacrifice may be characterized: 1. An oblation of something material; 2, that this must be in acknowledgment of God's dominion, &c., and for the remission of sins; 3, that it should be offered on an altar; 4, by proper officers, and with proper rites; and, 5, that it must be consumed as God has appointed. It must be remembered that, among those who advocate the doctrine of a sacrifice in the Eucharist, there are different opinions; and some do not admit the offering of the elements to be a sacrificial act, &c. (About this difference of opinion Dr. Mede's *Christian Sacrifice* may be consulted, chap. i.) But the most objectionable portion of Mr. Johnson's view, as explained by its author, appears to be the second division. He is far removed from supporting the Romish view of "a propitiatory offering for the living and the dead;" but the second section of chapter ii. we think liable to much objection. We acknowledge that, by the whole sacrifice of prayer and praise, the faithful hope, in the language of our Liturgy, to receive all the benefits of our Saviour's passion; and we pray that our sinful bodies may thus be made clean by his body, and our souls washed by his blood; but *beyond this point* we feel that every step we take is

on ground fraught with great danger. Thus far our Church Theology. leads us; but it not only does not authorize us to make a single step beyond this statement, but it has warned us of the errors which the Romish doctrines have introduced by venturing further, and justly declared their pernicious nature in the strongest language used in our Articles. (Art. xxxviii.)

Bishop Davenant, in his *Determinationes* (Quæst. xiii., *Missa Pontificia non est sacrificium propitiatorium pro vivis et defunctis*), has some very able arguments against the Romish doctrine. In Melancthon's *Loci Theologici* (loc. xiii. p. 352, ed. 1557) there is a discussion on the nature of Sacrifices, which he divides into two classes:—1. *Propitiatory*, where a satisfaction is made for sins, which is available to others; but he says the only really propitiatory sacrifice in this sense is that upon the Cross, of which the Levitical were types, &c. 2. *Eucharistical* or *Thanksgiving Sacrifices*, as praise and prayer offered by those who are reconciled to God.

Before we leave this subject we must just call attention to the passage of *Malachi*, i. 11, usually cited by the Romanists as referring to their Sacrifice of the Mass. If the passage be referred to the Eucharist, as in the early ages of the Church it seems usually to have been, it will not serve to defend the views of the Romanists, as the reader of Dr. Mede's work may easily see. It was referred to the whole action of the Sacrament, not to any offering of our Lord's body, as carnally present. Others have denied its reference to the Eucharist. (See the *Downside Discussion*, p. 394-397.) We must also mention that, besides the view of Dr. Mede, another view was also maintained by Dr. Cudworth, who considers the Eucharist a *feast upon a sacrifice*. (See Cudworth's *Works*; and Waterland, *On the Eucharist*; Palmer, *On the Church*, pt. vi. ch. x. &c.)

Sect. 4.—On the Constitution of the Church, and on the Notes of the Church.

Having now considered the chief points relating to the Ministers of the Church, its Liturgy, and its Sacraments, we proceed to point out a few circumstances relating to its constitution, which could not have been conveniently introduced under any of these topics.

There is one subject which might have received notice in the above portions of this essay, that is, the celibacy of the Clergy: but on this point we shall only give a few references to those authors who have shown the untenableness of the Roman Catholic rule in regard to this point, both on Scriptural and ecclesiastical grounds. For Scripture we may appeal to 1 Tim. iii. Indeed Romanists themselves confess that it is not by any law of God, but by the statutes of the Church, that the marriage of persons in Holy Orders is forbidden. (See the references in Field, *Of the Church*, book v. chap. 57.) We may here subjoin an historical extract from Field. After mentioning the case of Novatus the Presbyter of Carthage, about whose wife Cyprian speaks, without blaming his marriage (but blaming his ill treatment of his wife), and that of Tertullian, whose marriage appears from his works, as well as the circumstance that he did not renounce his wife's society, Dr. Field proceeds,—

"Neither have we these examples only, but many more; for we read in Gratian (*Dist.* 56,) of the sons of Presbyters and Bishops that were promoted to the Papal dignity. So was Bonifacius the Pope, the son of Jucundus the Presbyter; Felix the Pope, the son of Fælix the Presbyter; Agapetus the Pope, son of Gordianus the Presbyter; Theodorus the Pope, son of Theodorus the Bishop; and many more he saith there were, who, being the sons of Bishops or Presbyters, were advanced to sit on the Apostolical Throne. And addeth, that when the sons of Presbyters and Bishops are said to have been advanced and promoted to be Popes, we are not to understand them to be such as have been born of fornication, out of lawful marriages, which were lawful to priests before the prohibition; and in the Oriental Church are proved to be lawful unto them even unto this day. Socrates saith (lib. iii. c. 21) that in *Thessalia* there was a particular custom grown in, that if a clergyman, after he became a clergyman, accompanied with his wife, while he married while he was yet a layman, he should be put out of the ministry of the Church. Whereas all the most famous Presbyters, and Bishops also, in the East, might if they pleased, but were no way by any law constrained to refrain from the company of their wives: so that many of them, even when they were Bishops, did beget children of their lawful wives. A particular and most approved example of which we have in the father of Gregory Nazianzen, who, being a bishop, not only lived with his wife till death divided them, but became the father also of Gregory Nazianzen (as

Theology.

worthy and renowned a man as any the Greek Church ever had), after he was entered into the priestly office, as appeareth by his own words, reported by Gregory Nazianzen (in *Carm. de Vita sua*). For after many motives used by him to Gregory Nazianzen, his son, to persuade him to assist him in the work of his Bishoply Ministry, the last that he most insisteth on, is taken from the consideration of his old age disenableing him to bear that burden, and perform that work, any longer that hitherto he had done. And therefore entreating him to put to his helping hand, he breaketh out into these words:—*Thou hast not lived so long a time as I have spent in the Priestly Office; therefore yield thus much unto me, and help me in that little time of my life that is yet behind; or else thou shalt not have the honour to bury me, but I will give charge to another to do it.* Here we see Gregory Nazianzen's father was employed in the Priestly function before he was born; and that therefore he became the father of so worthy a son after he was a Bishop, or at least a Presbyter. Neither was the father of Gregory Nazianzen singular in this behalf." . . . Chrysostome accordeth with Athanasius and Clemens Alexandrinus, and saith (in 1 Tim. iii.) "that marriage is in so high a degree honourable, that men with it may ascend into the Episcopal chairs; even such as yet live with their wives. For though it be a hard thing, yet it is possible so to perform the duties of a Bishop; whereunto Zozomen (lib. i. c. 11,) agreeth, saying of Spiridion, that though he had wife and children, yet he was not therefore any the whit more negligent in performing the duties of his calling; and of Gregorie Nyssene it is reported, that though he were married, yet he was no way inferior to his worthy brother that lived single."—Field, *Of the Church*, book v. ch. 57.

The reader will find a great deal more in this chapter to the same purpose, and a summary account of the manner in which a stricter practice began to prevail, and more stringent canons were enacted.

The present rule of the Eastern Church is, that deacons may marry and live with their wives after receiving priest's orders. In the Apostolical canons, it is forbidden to bishops and priests to leave their wives (Canon 6). And in the Council of Nice, when some were minded to order the priesthood to leave their wives, Paphnutius restrained them, and the assembly assented to his views. But while there is no question that the forcible annexation of celibacy to the priesthood was of gradual growth in the Church, we may remark that Bellarmine (*De Clericis*, lib. i. c. 18) seems to consider it certain, that if it be lawful to live with a wife married before ordination, there can be no unlawfulness in marriage after ordination. For more we may refer the reader to Field, *ubi supra*; to Palmer, *On the Church*; and to many other works which treat expressly on this point. Some curious testimonies from Romanists are adduced in the *Scheme for Union* proposed by Wicelius, *apud* Brown, *Fasciculus Rerum Expensarum*, &c., vol. ii. The credibility of the history of Paphnutius, impugned by Bellarmine, is defended by Field, *ubi supra*. We may also observe that Mr. Palmer (*Treatise of the Church*, part vi. ch. ix.) argues, that "as the Eastern Church held that there was nothing unlawful in continuing in the state of matrimony after ordination, while the Western held that there was no greater fault in contracting marriage after ordination, then we may fairly draw the conclusion that the Universal Church never condemned marriage after ordination." He remarks also, that "the Greek custom of allowing married clergy has never formed any obstacle to their union with the Roman Church."—Towneley, *De Ordinatio*, p. 649.—Palmer, *ubi supra*.

Unity of the Church

We have now briefly to consider the constitution of the Catholic Church in regard to its unity and universality. It has been common, in the controversies between the Church of England and of Rome, for each party to adduce what they term "Notes of the Church."

There are many different definitions of the Church; but that which the Church of England has adopted is the following:—

"The visible Church of Christ is a congregation of faithful men, in the which the pure Word of God is preached, and the Sacraments be duly administered according to Christ's ordinance in all those things that of necessity are requisite to the same" (Art. xix.); which definition is further explained by the 55th canon, "Christ's Holy Catholic Church, that is, the whole congregation of Christian people dispersed throughout

the world;" from which it would appear that this Article regards, not any one congregation, but the whole Church of Christ.

Theology.

That this whole body is spoken of in Scripture as one in some relations and respects, is a matter which admits of no denial; and the creeds enounce, as an article of the Christian faith, that we believe in the Holy Catholic Church: in Scripture, the Church is spoken of as being the Body of Christ, and Christ as being its Head (*Ephes. i. 22, 23; iv. 15; v. 23-32; Col. i. 18; 1 Cor. xii. 4-30, &c.*); a mode of speaking which clearly implies unity. It has been observed before, that while Scripture speaks of single congregations as churches, and also of national churches, it sometimes collects them all under one general term, as "the Church."

But, as in the present condition of the world, many different societies claim to be considered as members of this Body, differing widely from each other in most important points, men require some rules to which we can bring their several pretensions, in order to determine how far they are valid. The marks by which any society may be ascertained to belong to the true Church of Christ, are called the "Notes of the Church." But as these different societies, and especially Romanists and Protestants, are at issue in respect to many great and leading features of the Church, and as they cannot agree in the same conclusion, it is easy to see that each party must necessarily assign different marks as the distinguishing characteristic of the Church. The Notes appealed to, and insisted upon by the Romanists, are generally the following: *Antiquity, Succession, Unity, Universality, and Catholicity*. These are the chief Notes fixed upon by Bellarmine;* while others assign *Unity, Holiness, Catholicity, and Apostolicity*, as flowing from the Nicene Creed. (See De la Hogue, *De Ecclesia*, p. 11.) The Protestants have sometimes maintained the Notes of the Church to consist in the profession of the true doctrines of the Gospel, the right administration of the Holy Sacraments appointed by our Saviour, and an union under lawful and authorized ministers.

Every point on which these Notes of the Church touch, it is clear, involves a discussion in itself, and in its application to each particular case. Before we proceed more particularly to examine briefly some of the questions relating to the Notes of the Church, we must just allude to the promises contained in Scripture respecting its perpetuity. It will be sufficient to adduce the following passages:—

"I will make an everlasting covenant with them; and their seed shall be known among the Gentiles, and their offspring among the people; all that see them shall acknowledge them, and that they are the seed which the Lord hath blessed. (*Isaiah, lxi. 8, 9.*)

See also the prophecy of Daniel: "In the days of these kings shall the God of Heaven set up a kingdom which shall never be destroyed . . . and it shall stand for ever." (*Dan. ii. 44; compare Ps. xlv. 6, xlviii. 8, cxlv. 13, Is. ix. 7, Heb. i. 8, &c.*)

To these prophetic declarations we may annex the positive promises of our Lord, delivered on several different occasions; as, for example, the following passages:—"On this rock I will build my Church, and the gates of Hell shall not prevail against it" (*Matt.*

* It will be seen, in the extract from Mr. Palmer, given below, that Bellarmine adds other notes.

Theology. xvi. 18); "I will pray unto the Father, and he shall give you another Comforter, that he may abide with you for ever; even the Spirit of truth. (*John* xiv. 16, 17.) To these we shall merely add the last words of our Lord before his ascension into heaven: "Lo, I am with you always, even unto the end of the world." (*Matt.* xxviii. 20.)

These testimonies of Scripture are clear and decisive; and it may easily be shown, that from very early times it has been the belief of Christians that perpetuity is an essential attribute of the Christian Church. For the proof of this assertion, we may appeal to the creed itself, and to abundance of passages in the Fathers. It would be idle to place in the creed an acknowledgment of a belief in "the Holy Catholic Church," if that were a body liable to perish. With regard to the Fathers, we need only refer to the well-known passages of Athanasius, Jerome, and Augustine. "The word is faithful," says Athanasius, "the promise is unshaken, and the Church is invincible, though the gates of hell should come, though hell itself, and the rulers of the darkness of the world therein be set in motion."* (*Athan. Oratio, quod unus sit Christus*, tom. ii. p. 51, Benedictine edition.) "The Church may be assailed by persecutions," says Jerome, "to the end of the world, but cannot be subverted; . . . and this will be, because the Lord God Almighty, the Lord God of the Church, has promised that he will do so, whose promise is the law of nature." (Jerome, *Comm. in Amos*, towards the end.) And lastly, Augustine observes: "The Church shall not be overcome, it shall not be rooted up; nor shall it yield to any temptations, until the end of the world shall come." (*Enar. in Ps. LX.*)

The passages here quoted will be sufficient for our present purpose, especially as this is a doctrine not much controverted, and acknowledged in the confessions or formularies of most existing churches.†

It is needless to point out how consoling a truth this ought to prove to every Christian heart. The knowledge that the principles which we have maintained, however their prevalence may be clouded for a season, can never finally be overthrown, must prove to every high-hearted Churchman the best consolation under every passing affliction. The knowledge that the presence of Christ is promised for ever to his Church, is a safeguard against feelings of despondency, and an unfailing source of courage to the Christian warrior under every conflict for the faith.

For much of the matter in the above section, and in some few which follow, we are indebted to the *Treatise* of Mr. Palmer, and to the work of Dr. Field, *Of the Church*, but especially to the former. This general acknowledgment will suffice, instead of more specific reference.

With regard to the notes of the Church, selected by different classes of divines, the following extract from Mr. Palmer (part i. ch. ii.) will illustrate the different views entertained by various communities and theologians.

"The necessity of devising some general notes of the Church, and of not entering at once on controversial debates concerning

* We have availed ourselves of Mr. Palmer's translation of these passages.—Palmer, *On the Church*, pt. i. ch. i. s. 2.

† It may perhaps be worth while to adduce one more passage from St. Augustine. After quoting St. Paul's wish to depart and be with Christ, he adds: "*Dixit hoc quidem, sed quamdiu hic manere potuit? Num quid usque ad hoc tempus? Num quid usque in posterum? Ergo illorum abscessu deserta est Ecclesia? Absit! Pro patribus tuis nati sunt tibi filii. Quid est, Pro patribus, &c. Patres missi sunt Apostoli: pro Apostolis nati sunt ibi [tibi] filii, constituti sunt Episcopi. Hodie enim Episcopi, &c.*" Aug. *Enarrat. in Ps. XLIV.* vol. viii. p. 150, ed. 1616.

all points of doctrine and discipline, was early perceived by Christian Theologians. Tertullian appeals, in refutation of the heresies of his age, to the antiquity of the Church derived from the Apostles, and its priority to all heretical communities. (*Præscript. adversus Hæreticos.*) Irenæus refers to the unity of the Church doctrines, and the succession of her bishops from the Apostles. (*Adv. Hæreses*, lib. i. c. 10, lib. iii. c. 3.) The universality of the Church was more especially urged in the controversy with the Donatists. St. Augustine reckons among those things which attached him to the Church, the consent of nations, authority founded on miracles, sanctity of morals, antiquity of origin, succession of bishops from St. Peter to the present Episcopate, and the very name of the Catholic Church. (*Contra Epistolam Manichæi Fundamenti*, c. 45.) St. Jerome mentions the continual duration of the Church from the Apostles, and the very appellation of the Christian name. (*Dialogus c. Luciferianos.*) In modern times, Bellarmine, of the Roman school, added several other notes; such as, agreement with the primitive Church in doctrine; union of members among themselves, and with their head; sanctity of doctrine and of founders; efficacy of doctrine; continuance of miracles and prophecy, confessions of adversaries, the unhappy end of those who opposed the Church, and the temporal felicity conferred on it. (*De Ecclesia*, lib. iv. c. 3, &c.) Luther assigned, as notes of the true Church, the true and uncorrupted preaching of the Gospel; administration of baptism, of the Eucharist, and of the keys; a legitimate ministry, public service in a known language, and tribulations, internally and externally. (*De Ecclesia et quæ sint Notæ, &c.*) Calvin reckons only truth of doctrine, and right administration of the sacraments; and seems to reject succession. (*Institutiones*, lib. iv. c. 1, s. 7—9.) Our learned theologians adopt a different view in some respects. Dr. Field admits the following notes of the Church; truth of doctrine, use of sacraments, and means instituted by Christ; union, under lawful ministers; antiquity, without change of doctrine; lawful succession, i.e. with true doctrine; and universality in the successive sense, i.e. the prevalence of the Church successively in all nations. (*Of the Church*, b. ii. c. 1, 2, 5, &c.) Bishop Taylor admits as notes of the Church, antiquity, duration, succession of bishops, union of members among themselves and with Christ, sanctity of doctrine, &c."

This list might easily be increased, but it is unnecessary, especially since we are not obliged, as Mr. Palmer justly remarks, to follow implicitly the judgment of particular theologians, in selecting notes of the Church. Some of those of Bellarmine were soon rejected by many of the Romish theologians, (as may be seen from White's *Conference with Fisher*, 2, 9, in the case of miracles, p. 86, *et alibi*.) and are generally rejected by modern Romanists. In the same manner, then, we are at liberty to reject the opinion of private theologians on this subject. The Church of England has not formally enumerated the Notes of the Church, although in its Articles it has defined the Church, and made certain statements in other formularies concerning its constitution. These statements enable us to obtain an authoritative guide on some parts of this question, although they may not entitle us to fix finally all those points which are to be considered "Notes of the Church."

It must also be observed, that some Protestant writers maintain that the Church consists of the elect only. (See the notes on the following section.)

§ Of the Visibility of the Church.

Having shown that, according to Christ's promise, the Church is never to fail, but will endure to the end of the world, we next proceed very briefly to consider the visibility of the Church. By the visibility of the Church is meant the manifest public known existence of congregations, or churches, professing Christianity, and joining in external acts of public worship. (Palmer, *On the Church*, pt. i. ch. iii.) When it is maintained, in this sense, that the Church is, and always will be, visible, it is in opposition to a notion that Christianity is ever to be reduced to obscurity, or to be found only in the secret possession of a few scattered individuals. It may easily be shown, from several passages of Scripture, that the Church was to be thus visible. Thus Isaiah predicts (ii. 2) that it shall come to pass, in the last days, that the mountain of the Lord's house shall be established in the top of the mountains, and shall be

Visibility of the Church.

Theology. exalted above the hills; and all nations shall flow unto it." (Compare with this prophecy *Dan. ii. 35 & 44.**) The words also of our Saviour (*Matt. v. 14*), "Ye are the light of the world; a city that is set on an hill cannot be hid;" and his direction in the case of an offending brother, "tell it to the Church,"—both prove the same point. The latter direction would be unintelligible, if there were no visible Church. And again, from the Epistles of St. Paul, it is manifest that Christians were formed into visible societies by the Apostles; nor do we hear of individual believers, unconnected with their brethren in the possession of Christianity. The testimony of the early Fathers is also clear and decisive to the same great truth. The following quotations will suffice, as they are only specimens which may be supported by other similar statements. Irenæus, in his first book,† having spoken of the Church as having received the office of preaching the word of God and the true faith, goes on to argue, that though it is dispersed throughout the world, the difference of the nations and the languages does not interfere with the unity of the teaching, and then adds: "But as the sun is one and the same in every part of the world, so also this preaching of the truth shines everywhere, and illuminates every man who wishes to come to a knowledge of the truth." The Church has here the gift of preaching the Gospel, and the Gospel is like the sun in enlightening all the world. The Church that has this gift must surely be visible, or its teaching could not be so full of illumination. But in the passages quoted by Mr. Palmer (lib. v. ch. xx.), Irenæus maintains the same truth in more express words.

Cyprian says: "The Church of the Lord, full of light, diffuses her rays throughout the whole world. Yet the light, which is everywhere, is one, nor is the unity of the body separated."‡

In fact, all the Fathers considered the Church as visible throughout the world, in all its particular congregations or churches. The agreement of Christians in modern times upon this point is shown by Mr. Palmer,§ from the various confessions of the Protestant churches on the Continent; and he remarks, that the doctrine of the visibility of the Church was generally acknowledged by the Reformed, until some of them deemed it necessary, in their controversy with the Romanists, who asked them where their Church was before Luther, to maintain that the Church might sometimes be invisible.|| The Articles of our own Church speak plainly upon the subject, and its other authorized formularies are entirely in accordance with them, all concurring in representing the Church as a visible society.

While, however, we thus state the concurrence of

* See Augustin. *Enarrat. in Ps. XLIV.*, towards the end. After a long passage on the Church, he quotes the passage from Daniel, and connects it with that of St. Matthew, thus:—"Voce meâ (*Ps. iii.*) ad Dominum clamavi et exaudivit me de monte sancto suo. De quo monte? De quo dictum est, Non potest civitas abscondi supra montem constituta. Quem vidit Daniel ex parvo lapide crevisse, &c. . . . Ibi adoret qui vult occipere, ibi petat qui vult exaudiri, ibi confiteatur, qui vult sibi ignosci."

† *Contra. Hæres.* lib. i. cap. x. p. 49, ed. Massuet.

‡ Cyprian, *De Unitate*, p. 254, ed. Pamel.

§ Palmer, *On the Church*, part i. ch. iii.

|| This has been shown by Jurieu, in his *Système de l'Eglise*, who endeavours there to show that the Church is essentially visible, and that it never remained obscured, without ministry and sacraments, even in the time of persecution. See Palmer, *ubi supra*; Hooker, *Eccl. Pol.*, book iii. ch. i. s. 10; Field, *Of the Church*, book iii. ch. 8.

Theology. all authorities in this representation of the Christian Church, we must also briefly indicate the views of those who speak of the Invisible Church.

It is maintained by some of the divines of the Reformed churches, that the Church of Christ consists of the elect; but it is observed by Dr. Field, that they who use this language, and "define the Church to be the multitude of the elect, do so, not for that they think them only to pertain to the Church, and no others, but because they pertain unto it principally, fully, effectually, and finally; and in them only is found that which the calling of grace (whence the Church hath all her being) intendeth—to wit, such a conversion to God as is joined with final perseverance; whereof others failing and coming short, they are only in an inferior and more imperfect sort said to be of the Church."* In this sense, they who maintain that the Church is an invisible society, still acknowledge a visible Church, not as a separate body, but as that which containeth the former.† Hooker observes (book iii. ch. i. s. 9): "For lack of diligent observing the difference, first between the Church of God, mystical and visible, then between the visible, sound and corrupted—sometimes more, sometimes less—the oversights are neither few nor light that have been committed;" and he proceeds to name many of those which have occurred, but which we have not space to enumerate. We shall merely remark, that while we make no objection to the use of the phrase, the "Invisible Church," when thus understood, we always mean, when we name the Church in this treatise, the Visible Church of Christ.

The principal objections to this view have been well answered in Mr. Palmer's treatise, in the Appendix to chap. iii. of part iii. We may also refer to Bishop White's *Conference with Fisher the Jesuit*, p. 49—64; Hooker, &c. &c.

The perpetuity and visibility of the Church having been thus established, we proceed briefly to consider some of the Notes of the Church separately. The most convenient arrangement will be that derived from the articles of the Creed commonly called the Nicene. "I believe *One Catholic and Apostolic Church*;" and the corresponding article of the Apostles' Creed, the *Holy Catholic Church*: from which two expressions we may derive the four following Notes—*Unity, Holiness, Catholicity, and Apostolicity.*

Unity.

It has before been observed, that although in Scripture the word Church is often used of a single congregation, yet it is also used of one vast society, comprehending all such particular churches; and that this society is called the Body of Christ, who is said to be the Head of the Church. (*Ephes. i. 22, 23*; v. 23, *et alibi.*) These passages, therefore, indicate that this vast body is in some respects to be considered as one; and we ask at once in what this unity consists. There are two general heads under which the subject may conveniently be considered—Unity in *Communion*, and Unity in *Faith*.‡

* Field, book i. ch. 8.

† It must be added, however, that all the members of the Invisible Church cannot, in this view, be said to be contained in the Visible Church; because some, who will hereafter be saints, may not be, at present, converted. But it may be observed, in reply, that they are rather to be called *potentially* members of the Invisible Church, and not actually so, until their conversion and their reception into the Visible Church.

‡ The subject of Unity is viewed under different divisions in

Notes of the Church.

Unity of the Church

Theology.

1. *Unity in Communion.*—It is easy to show, from a multitude of passages in Scripture, the general duty of religious communion among Christians. It is to be inferred, in the first place, from the relation which is established between Christians: they are *brethren*; and nothing can present a stronger argument for unity and intercommunion than the analogy of a brotherhood. Again, Christians form one *body*, of which Christ is the head; and they are said to be "every one members one of another." (*Rom. xii. 5.*) Compare also with these passages those wherein our Lord commands us to *love* one another; and that remarkable prayer which he uttered "for all them that should believe in him," namely, that they might be *one*, even as He and the Father are *one*. (*John xvii.*) But we are not left to *infer* this duty; we are constantly warned in Scripture against *divisions*. Thus, St. Paul writes to the Romans, "Mark them which cause *divisions* and offences, contrary to the doctrine which ye have learned; and *avoid them*." (*Rom. xvi. 17.*) With this passage compare 1 *Cor. i. 10-12*; *Ephes. iv. 2, 3*; *Phil. ii. 2, &c.* We find still further, that these exhortations produced in the Apostolic age their due effect. We find that the churches were in the habit of saluting each other, in the Epistles of the Apostles; of aiding each other with mutual assistance; and of sending letters of recommendation to other churches with those of their brethren who were on their travels.

These circumstances, and many more which might be adduced, fully justify the conclusion, that "external communion between all Christians, in matters of religion, was instituted and commanded by God."*

The Scripture foundation being thus laid, we might proceed to show that this truth has been maintained by the Church from the earliest times; but our limits forbid our dwelling on that portion of the subject. We only remark, that if this be the command of God, they who wilfully violate this unity cannot escape the guilt of having broken one of the Divine commandments.

It only remains to point out the manner in which this duty of Christian communion may be violated. It is clear that as the Church was destined to be universal, the same teachers could not ordinarily instruct all nations. But then, the institution of particular churches, which was intended, not to *divide*, but to *organize* the universal Church, could not impair the fraternity which resulted from the mutual communion of these churches with God, and the duty of external communion with all Christians. From this view we perceive that the unity of communion in church might be violated in two different ways, *viz.*, by dividing the communion of a particular church, or that of the universal Church. The first occurs when professing Christians divide, or refuse to communicate with the particular church of which they are members; the other, when particular churches refuse to communicate with the universal Church—that is, with the great body of Christians.

The offence thus committed is the heinous sin of schism, against which we are frequently warned in Scripture (see 1 *Cor. i. 10*; *Rom. xvi. 17, 18*; *Heb. xiii. 17*; 1 *John ii. 19, 20*; *Jude 19*); and the mischievous and destructive nature of which is attested

Field, *Of the Church*, book ii. ch. 7. The above division is that adopted by Mr. Palmer.

* Palmer, *Treatise on the Church*, part i. ch. iv. § 1.

by the ecclesiastical authorities of every age of the Church.*

2. *The Unity of Faith.*—As our Lord, during his sojourn on earth, and afterwards by his Apostles and Evangelists, made a revelation of truths salutary and necessary to be believed, it is clear that all Christians are bound to believe all the doctrines thus revealed. Our Saviour promised to his disciples that the Spirit of truth should guide them into all truth (*John xvi. 13*); and his Apostles declare that there is but "one faith;" and that for that faith, once delivered to the saints, Christians are bound earnestly to contend. (*Ephes. iv. 5*; *Jude 3.*) The Church is also said by St. Paul to be established for the maintenance of the truth: "The church of the living God, the pillar and ground of the truth." (1 *Tim. iii. 15.*) The obligation of the Church to believe all revealed truth being established by these and similar declarations of Scripture,† it will follow that *heresy*, or the pertinacious denial‡ of some truth certainly revealed, is a sin of great enormity. That an obstinate rejection of revelation subjects us to God's eternal judgment, is plainly declared by our Lord himself: "He that believeth not shall be damned" (*Mark xvi. 16*); and heresy is a partial rejection of that truth, of which infidelity is the total denial. This passage, therefore, inculcates upon us the danger of denying any portion of revealed truth. But we are not left to gather the mischief of heresy by implication, as the Apostles most unequivocally condemn it. They warn those to whom they write that some of their teachers would pervert the Gospel; but that even if an angel from heaven were to preach any other gospel, he should be anathema: they speak of false teachers, who bring in damnable heresies; and they command their disciples neither to receive those who bring another doctrine§ into their house, nor to bid them God speed. In the *Revelations*, false doctrine is expressly stated to be hateful to God; and, lastly, the disciples are commanded, after one or two admonitions, to *reject* a man that is an heretic. (See *Gal. i. 7-9*; 2 *Pet. ii. 1*; 2 *John 7-11*; *Rev. ii. 15*; *Tit. iii. 10.*) The following passages from the earliest records of the Church show the sentiments then entertained of heresy. Ignatius|| writes thus to the Trallians:

"I exhort you, therefore, not I, but the love of Jesus Christ, to use only Christian food, and to abstain from such plants as are foreign to it,¶ which are heresy."

* See this fully proved in Mr. Palmer's *Treatise on the Church*, *ubi supra*.

† It seems hardly necessary to dwell at any length on this point, or the next, as it seems scarcely credible that any one should allow that God has made such a revelation to man as Christianity, and yet leave it a matter of indifference whether he believes it or not.

‡ Mr. Palmer has properly observed, that pertinacity is requisite to constitute heresy. Ignorance, or temporary error in ignorance, is entirely different from heresy. Field (*Of the Church*, book i. ch. 14, *in initio*) says, "Heretics are they that obstinately persist in error, contrary to the Church's faith;" and the language of Hooker is nearly similar.

§ In this passage the allusion is probably to the Gnostics, or to the Docetæ, who denied that the body of Christ was real, and considered it merely a phantom.

|| *Ignat. ad Trallianos*, ch. vi. p. 146, ed. Hefele; p. 332, ed. Jacobson.

¶ *ἡ ἀλλοτρίαν δὲ βοτάνην ἀπέχισθαι ἥτις ἐστὶν αἵρεσις.* Mr. Palmer translates, "from strange pasture, which is heresy." The passage which immediately follows, and concludes the chapter, is still stronger; but the first words of it are doubtful, the text being certainly corrupt.

Theology.

Irenæus says, "The Lord shall judge all that are without the truth, that is, without the Church" (Irenæus *adv. Hæres.* l. iv. c. 33, or 62, according to others). Tertullian declares, if they are heretics they cannot be Christians (*De Præscript.* c. 37); and Origen pronounces, that as the impure and idolaters shall not possess the kingdom of God, "so neither shall heretics."*

These explicit declarations against heresy might be multiplied to almost any extent, if it were needful; but the danger of heresy is almost universally believed, even by those who would differ the most widely in determining in what it consists.

As the mischief of heresy is so great, it is clear that every Christian church must have the power to exclude from its communion those who are guilty of it; a power which is obviously implied in several passages of Scripture, some of which have been already adduced (see *Tit.* iii. 10; *1 Tim.* vi. 5). And this power most Christian churches have exercised in the right, which they claim, of excommunication.†

We have, therefore, thus far established the two following positions:—

1. That the truth revealed by Christ must be believed by all Christians, in order to salvation; and
2. That heresy, or the pertinacious denial or perversion of the truth, is a grievous sin, and excludes from salvation.

These propositions being very general, and apparently very sweeping in their consequences, it is natural to inquire whether they are not subject to certain limitations; and we perceive at once that we should do great injustice if we considered every error to amount to heresy. Although in *matters of faith* (i. e. those distinctly revealed in Scripture, and attested by universal tradition) there is no latitude allowed, yet there are points, which are called *matters of opinion*, which may be controverted among churches without the unity of the faith being broken.‡ Again, even the Romish theologians grant that a doctrine which is apparently taught by Scripture and tradition, even if it prove to be an error, is not heretical, though denounced by particular churches, if the universal Church has not pronounced a judgment concerning it: even they hold that Transubstantiation and Purgatory were not matters of faith till defined by councils. In estimating also the guilt of heresy, we must make a due distinction between those who apostatize from the truth, having been educated in it, and those who, not having had an opportunity of knowing the truth in their early years, from having been brought up beyond the pale of the Church, and have embraced opinions which are in some points erroneous from their instructors, but who are not obstinate in error, but desire to embrace the truth revealed by Christ, whatever it may be. These considerations naturally present to us,

* Origen, *Ap. Pamphil. Apol.* tom. v. p. 225; *Oper. Hieron.* Paris, 1706; quoted by Mr. Palmer.

† Mr. Palmer justly observes, that "there are certain conditions requisite to a valid ecclesiastical judgment, which if plainly violated, render it null and devoid of all spiritual effect."

‡ "The Romish divines distinguish between theological opinions and doctrines *de fide*. Among the former they reckon the points contested between the Thomists and Scotists, the Jesuits and Dominicans, the Ultramontane and the Cisalpine parties; the doctrine of the immaculate conception of the Virgin, &c." (Palmer, part i. ch. v. § 3.) Thus the Romanists themselves allow this degree of latitude.

Theology. not any recommendation of error in itself, but some modifications in our judgments concerning those who hold it.

With regard to unity in faith, as a sign or note of the Church, we must observe, that although Christians are bound to receive all the truth revealed by Christ, and to desire a perfect union among all the brethren in matters of religion, yet that we have no certain promise that *in fact* the whole Church shall be always perfectly agreed in every point, and that no temporary differences shall ever arise in some particular doctrines. Still it is our duty, by all ordinary means, to endeavour to attain to such an unity; because we must infer that Christ has provided the whole Church with some means for preserving or recovering, within itself, unity in this respect; and we must suppose that there is a possibility of obedience to the precepts of believing all truth, and rejecting all false doctrines.

We may now observe, that actual unity in all matters of faith cannot be a sign of the Church, so as to distinguish it from all rival societies. It does not follow, because there is an apparent difference in doctrine, that there is a real difference in matters of faith; but where there is such apparent difference, the case must be a subject of discussion. We contend, then, that in no Christian society can a pretence be set up to an absolute and perfect apparent unity in doctrine. "It is not merely the Lutherans, Calvinists, and Zuinglians who differ. There are disputes in the Eastern churches; and in the Roman obedience (not to mention the differences about Jansenism) the controversies of Jesuits, Dominicans, and Augustinians, of Scotists and Thomists, of Ultramontanes and Cisalpines, are well known." (Palmer, *On the Church*, pt. i. ch. v. § iv.)

Besides, there may be an *unity of error*. Thus, the Nestorians are united in their erroneous views; but this unity does not prove theirs the true church.* We conclude, therefore, that *perfect* unity in faith is not a sign or *note* of the Church, as distinguishing one among rival claimants to the title of the true Church.

The foregoing observations on the Unity of the Church have been abridged from Mr. Palmer. The same subject is treated somewhat differently in Dr. Field, and in Dr. White's *Conference with Fisher*. Dr. Field (*Of the Church*, book ii. ch. 7), after mentioning the various sorts and degrees of unity found in the Church (1, the unity of origin and of calling; 2, in respect of the same end, whereunto all that are of the Church do tend, signified by the penny given to all the labourers, *Matt.* xx.; 3, the unity in respect of the same means of salvation, faith, sacraments, holy laws, &c. (see *Ephes.* iv.); 4, the unity in respect of the same spirit; 5, in respect of the same head,—even Jesus Christ; 6, the unity in respect of the mutual connection between Christians, &c.); and after pointing out the nature of the unity assigned by Romanists as a note of the Church, draws from it the following conclusion: "This is it, then, which they say, that wheresoever any company and society of Christians is found in orderly subjection to their lawful masters, not erring from the rule of faith, nor schismatically rent from the other parts of the Christian world by factious, causeless, and impious divisions,—that society of men is (undoubtedly) the true and not offending Church of God. This note, thus delivered, is the very same with those assigned by us."

It will be seen by this, that Dr. Field and other writers in some sort make unity a note of the Church, which seems at first sight a contradiction to the conclusion drawn in the text: but it is rather apparent than real. The spirit of unity must be found in the true Church—a desire after unity, and an honest endeavour to attain it; but the conclusion we have drawn is rather limited to the matter of fact: it is only denied above, that in endeavouring to ascertain the true church, such an unity can

* Palmer, *ubi supra*. Compare Field, *Of the Church*, book ii. ch. 7.

Theology. be appealed to as a matter of fact. It is not denied that every society which unjustly, *unnecessarily*, violates the unity of Christ's body is to blame, and so far cuts itself off from that body.

The Romanists, indeed, endeavour to persuade us that the churches of the Roman obedience are perfectly united, and are in fact the *one* Catholic Church. But this is an assumption on their part of the most unwarrantable nature. They have violated the unity of faith in imposing articles of belief which the primitive church not only did not impose, but would have rejected altogether (*e. g.* transubstantiation, &c.); but they have never at any time been the whole church of Christ. They conveniently forget the whole of the Greek Church in this argument. Nor are they so united in themselves in all respects, as has been intimated in the text, and as may be seen at greater length in Mr. Palmer's work, so often cited. In regard to the English Church, the Church of Rome is to be charged with the guilt of schism, not the Church of England.

Much useful matter will be found on these points in White's *Conference with Fisher*, p. 49-110; and in the other works already referred to in this essay.

We proceed now very briefly to consider the remaining notes of the Church, *viz.* its—

2. Holiness.
3. Catholicity.
4. Apostolicity.

Holiness.

The Holiness of the Church may be considered in several points of view. The sanctity of its Divine Head and its first founders; the holiness of its doctrines, and the means of holiness offered by its sacraments; the actual holiness of its members.

Holiness
of the
Church.

Now, there is no question that our Divine Head has called us to holiness, and he is himself its first fountain and spring. He gave himself for his Church, "that he might sanctify and cleanse it with the working of water by the word, that he might present it to himself a glorious Church, not having spot, or wrinkle, or any such thing; but that it should be holy, and without blemish."* (*Ephes. v. 25-27.*) The object, then, of the establishment of the Church being the sanctification of man, it is evident that no society of persons professing to be Christians can really belong to the Church of Christ, if it denies the obligation of good works, or if it teaches that its members may indulge in wickedness with impunity. On this point it is scarcely conceivable that any difference of opinion should exist; but great difficulty would arise if the *means* of holiness afforded by any society were to be considered as a *note* of the Church,† as this would include a thorough discussion of various opinions on the sacraments. Nor can the holiness of the individual members of a society properly be called a note of the Church. Some sectaries have indeed asserted that the Church consists only of the elect, or the visible holy. On the contrary, we must observe, that Scripture teaches us that the Church, or the kingdom of heaven, is like a field, where tares and wheat grow together for a time; or a net that is cast into the sea, and gathers of every kind, both the wicked and the just (*Matt. xiii. 24-50*), and that the separation of these two classes will not take place till the last day. In other words, although the

* In this passage it may be objected that the *invisible* Church is spoken of; but it may be replied, that the passage rather indicates the end and object of our Lord's sacrifice; and while it shows that the end is holiness, the expressions of sanctifying and cleansing indicate a process by which the *visible* Church is purified into the invisible. It is thus perfectly consistent with what follows respecting sinners in the communion of the visible Church.

† This is a different question from the lawful administration of the sacraments. We might justly affirm, that no society could be part of the Church of Christ in which the sacraments of baptism and the supper of the Lord were not duly administered.

invisible Church consists only of those who are finally accepted, yet that many will *outwardly* belong to the visible Church, who are devoid of a lively faith.

While, therefore, the true Church must enforce upon all its members the necessity of personal holiness, if they hope to continue *to the end* a portion of Christ's Church, yet the actual and visible holiness of its members is not to be made a test of the true Church. With regard also to the performance of miracles, as divine attestations of sanctity, we may determine that they must be excluded from the notes of the Church. The existence of holiness is entirely independent of them, and other notes would be required to ascertain whether any asserted miracle belonged to an exhibition of the power of God, or to that class of signs and wonders against which the Scripture warns us.*

Mr. Palmer observes, that the Donatists invented a distinction which more modern sects have adopted. They admitted that sinners might exist in communion with the Church, but they denied that manifest sinners could, in any respect, be of the Church. Mr. Palmer replies to this distinction by showing, that the fact that there are known sinners in any church, does not annihilate its character, or render it apostate. (See *1 Cor. iii. 3, vi. 1-7, v. 1-5; Rev. ii. 18, 20, iii. 1, 4*). Such persons cannot, of course, belong to the Church, considered in its invisible character. Against the notion, that the *visible holiness* of life is requisite for admission into the Church, Mr. Palmer argues that baptism is the only mode of admission into that body, and that the requisites for baptism are a profession of repentance, and faith in Christ—of willingness to obey his laws, and believe his words.

Catholicity.

Among the many characteristic features of the Christian, there is scarcely any by which it is more clearly distinguished from the Jewish Church, than by its universal diffusion over the world. This difference is not incidental, but essential.

The Jewish Church, by the very nature of its ordinances, was restrained to Jerusalem and the country in which it was placed; the last command given by our Saviour to his Apostles plainly intimated that his Church was to extend to all the nations of the world. (*Matt. xxviii. 19, 20.*) This characteristic was also predicted from the time of the earliest promises given to the descendants of Abraham (see *Gen. xxii. 18; Is. ii. 2; Ps. ii. 8, &c.*); and as there is no limitation to any particular period, we must understand these predictions to declare the *permanent* condition of the Christian Church. The notion of a mere successive universality is altogether untenable.† It can scarcely be needful to produce any testimonies from the ancient church to this great truth, as the acknowledgment of the Nicene Creed, and of the *Te Deum* (the holy Church throughout all the world doth worship thee), are sufficient vouchers for its general reception. It may be more satisfactory to close these brief observations with an extract from an eminent Protestant prelate.

Archbishop Usher, in his sermon before the king, from *Ephes. iv. 13*, speaks thus:—

"For the Catholic Church is not to be sought for in any one angle or quarter of the world; but among 'all that in every place call upon the name of Jesus Christ our Lord, both theirs and ours.' (*1 Cor. i. 2.*) Therefore to their Lord and ours it was said (*Ps. ii. 8*), 'Desire of me, and I will give thee the heathen for

* See Palmer, part i. ch. vi.

† It must be remarked, however, that Bellarmine and other Romanists, as well as some of our theologians, have countenanced this notion. See the remarks on this subject in Palmer, part i. ch. vii.

Theology. thine inheritance, and the uttermost parts of the earth for thy possession; and to this mystical body, the Catholic Church, accordingly, 'I will bring thy seed from the East, and gather thee from the West; I will say to the North, give up; and to the South, keep not back; bring my sons from afar, and my daughters from the ends of the earth, even every one that is called by my name.' (*Isaiah* xliii. 5.)

"Thus must we conceive of the Catholic Church, as of one entire body, made up by the collection and aggregation of all the faithful unto the unity thereof; from which union there ariseth unto every one of them such a relation to, and dependence upon the Church Catholic, as *parts* used to have in respect of their *whole*. Whereupon it followeth, that neither particular persons, nor particular churches, are to work, as several divided bodies, by themselves (which is the ground of all schism); but are to teach and to be taught, and to do all other Christian duties, as parts conjoined unto the whole, and members of the same commonwealth or corporation; and therefore the bishops of the ancient Church, though they had the government of particular congregations committed unto them, yet in regard of this communion which they held with the Universal, did usually take to themselves the title of *Bishops of the Catholic Church*: which maketh strongly as well against the new *Separatists* as the old *Donatists*; who either hold it a thing not much material,* so they profess the faith of Christ, whether they do it in the Catholic communion or out of it, or else (which is worse) dote so much upon the perfection of their own part, that they refuse to join in fellowship with the rest of the body of Christians, as if they themselves were the only people of God, and all wisdom must live and die with them and their generation." (Usher's *Sermon*, p. 6, 7; ed. 1631.)

This statement of the archbishop places the great dogma of the Catholicity of the Church in a very clear point of view; and it is obviously unnecessary to multiply quotations on a point which will not be much disputed. It only remains to observe, that this Catholicity is rather an *attribute* of the whole Church than a *note* to distinguish one body of Christians from another as the Church of Christ. It serves, however, thus far as a note of the Church, that it condemns those who wilfully and unnecessarily form a body of separatists, and indicates that we are not to look for the Church of Christ among such a society.

In reference to this attribute of the Christian Church, we may refer to Field, *Of the Church*, book ii. ch. 8 & 9; White's *Conference with Fisher*, p. 109-11; who answers the Romish pretence of universality by observing that the Roman Church is only a particular church. Archbishop Usher also, in the sermon quoted above, answers the common, but absurd question addressed by Romanists to Protestants, "Where was your Church before Luther?" "Whereunto," this prelate observes, "an answer may be returned from the grounds of the solution of the former question: that our Church was even there where now it is. In all places of the world, where the ancient foundations were retained, and these common principles of faith, upon the profession whereof men have been wont to be admitted, by baptism, into the Church of Christ: there, we doubt not, but our Lord had his subjects, and we are fellow-servants; for we being in no new faith, nor no new church; that which in the time of the ancient Fathers was accounted to be *truly and properly catholic*, namely, *that which was believed everywhere, always, and by all*;† that in the

succeeding ages has evermore been preserved, and is at this day entirely professed in our Church," &c. Theology.

For other authorities, see Palmer, *On the Church, ubi supra*, from which this and the preceding sections have been chiefly abridged.

Apostolicity of the Church.

It is admitted by all that the Church is Apostolical, or in some sense derived from the Apostles. Now, every society which is derived from them must be so by spiritual propagation, or derivation, or union; not by separation from the Apostles, or from churches planted by them, or derived from their preaching, under pretence of establishing a new system of Apostolical perfection.* But there is another point of view in which the Church may be considered to be Apostolical; which is, in the authority and succession of its ministry. It will readily be acknowledged that the Apostles not only fulfilled the command of Christ in teaching all nations, but that they constituted presbyters in every church, and that their appointment of ministers was sanctioned by the authority of our Saviour himself and by the Holy Spirit of God. (Compare *Ephes.* iv. 8-15; *Acts* xx. 28; *1 Cor.* xii. 28.) The appointment, therefore, of ministers rests on the practice of the Apostles and the authority of the Holy Spirit. And the opinion of all Christians (with, perhaps, the exception of the Quakers) has been, that this institution is permanent in its nature, and absolutely essential to the Christian Church. St. Ignatius, in speaking of the ministry, says, "without these there is no church" (*Ignat. ad Trall.* § 3); and St. Jerome says, "*Ecclesia non est, quæ non habet sacerdotes*" (*Hieron. Ad. Lucifer.*); and ample testimonies to the same sentiments may be produced from every period of the Church.† We may conclude, therefore, that the ministry is essential to the Christian Church. The Scriptures also teach us that a divine vocation is essential to the Christian ministry. The example of the Old Testament, and the explicit declaration of St. Paul, in the *Epistle to the Hebrews* (ch. v. 4, 5, &c.), assure us that the high office of the ministry is not to be assumed at man's pleasure, but to be received from God's appointment,—that no one can be a legitimate minister of God's Church except he is authorized by Him. (Comp. *Heb.* v. 4; *Rom.* x. 15; *2 Cor.* ii. 9; *Is.* lxii. 1.) Now, in considering what constitutes such a vocation, we must observe that an inward call and a popular election are both entirely insufficient alone to constitute a minister of Christ's Church. There is no instance in Scripture of any man being sent forth as a minister of Christ merely on the strength of an inward call, unattested either by miracles or by an external commission from the ministers of God.‡ Nor is there in Scripture any example of a popular election of ministers independently of the sanction of the Apostles. The seven deacons named by the people were afterwards ordained by them. The Apostles *ordained elders in every city* (*Acts* xiv.); and St. Paul appointed Timothy and Titus as chief pastors in Ephesus and Crete, that they *might ordain presbyters in every city*. And it is quite

* Palmer, *On the Church*, part i. ch. viii.; from whom this section is chiefly abridged.

† The great communities of foreign Protestants acknowledged this truth in most of their confessions, e. g. the Augsburg, Helvetic, Belgic, &c.

‡ We may make almost the same observation with regard to God's revelations in the Old Testament. The inward impulse can be no sign to any other individual, as we have remarked in a former chapter—ch. ii., section on special revelations.

* See Augustin. *Ep.* 48.
† *In ipsâ Catholicâ Ecclesiâ magnopere curandum est, ut id teneamus quod ubique, quod semper, quod ab omnibus creditum est: hoc est enim verè proprièque Catholicum.* (Vincent, *Lirin. contr. Heres.* c. 3.)

Theology. uncertain whether the people had any share in these appointments. It is, indeed, acknowledged by dissenters, that "no case occurs in the inspired history where it is mentioned that a church elected its pastor."* The very circumstance of persons *heaping to themselves teachers after their own lusts* is one of the signs marked out of a great apostasy (2 Tim. iv. 3, 4); which shows distinctly that such elections cannot of themselves be sufficient to constitute a legitimate commission. In the primitive Church, indeed, it is acknowledged that sometimes the people elected their pastors; but the ministers of Christ always confirmed and ordained those who were so elected. We come now to the last stage of this inquiry, which points out to us the mode in which it has pleased the Almighty that the Christian ministry should be perpetuated. It has been shown that neither internal vocation nor popular election can prove a legitimate commission from God. Only one mode, therefore, remains by which men can receive a Divine commission for the sacred office; that is, by means of ministers authorized to convey it to others. It is clear that if God authorized the Apostles and their successors to ordain ministers, and transmit to them a Divine commission, there would then be a clear and intelligible mode in which this commission might be perpetuated in the Church. Accordingly, Christ did so; he gave to the Apostles his own mission: "As my Father hath sent me, even so send I you" (John xx. 21); empowering them to give to *others* the mission which, by the very act of conferring it on the Apostles, he showed to be transmissible. They did transmit it to others; and thus alone were the ministers of Christ appointed. Those thus appointed were said to be constituted by the *Holy Ghost* (1 Cor. xii. 28), and were called *pastors and teachers set by God* in his Church. (Acts xx. 28.) That the succession of such ministers was never to fail, is implied in the promise of our Lord to his Church: "Lo, I am with you to the end of the world." (Matt. xxviii. 20.) We infer from these premises, that an Apostolical succession of ordinations is essential to the Christian ministry; and with this proposition we conclude our remarks upon the Notes of the Church.

It will be observed that the greater part of the above observations on the Perpetuity, Visibility, and Notes of the Church, is an abridgment of the first part of Mr. Palmer's *Treatise on the Church*, and sometimes even in his very words. Although other authorities have constantly been used, we have taken this course because no writer has treated the matter more lucidly; and the brevity with which we have been obliged to write required, as a primary element, the utmost clearness of arrangement. For further satisfaction on each of the above topics, and on the other points connected with the Church, we must refer to that work, to the other treatises already quoted, such as Field, White, Hooker, Laud, &c., and to other great writers of our Church, e. g. Barrow (*On the Supremacy of the Pope*), Bishops Taylor, Hall, Jewell, &c. In the *Determinations* of Bishop Davenant many points will be found very ably maintained against the tenets of the Romanists. Some very valuable matter will also be found in a small compass in Archdeacon Fullwood's *Established Church*, London, 1681, and *Leges Angliæ*. We may just observe, that some of the arguments of the Romanists admit of a very easy and obvious answer. When they claim dominion over the whole Church in right of antiquity, and as the mother of all churches, it is sufficient to reply, that the Church of Jerusalem is more properly to be called the mother of all churches. But many of the subjects in controversy between the Churches of England and Rome we are obliged to pass over, as the discussion of them would far exceed our limits. The great question between them (that of Scripture and Tradition) has been considered with much detail in Chap. II., and this forms the

* J. A. James *Church Member's Guide*, p. 12, second edition.

foundation of their difference. If the principles there advanced are founded in truth, as we can have no doubt that they are, the chief weapon of popery is broken, and its main support undermined, and the war need not be carried on in detail. With these observations, referring once more to the great authorities already cited, we take leave of this portion of our subject.

Sect. 5.—On the Doctrine of Scripture relative to a Future State.

In the first chapter of these outlines, some remarks were made on the belief of the heathen world in a future state of existence for man. Although many of the wisest heathens appear to have entertained such a belief, yet all their views were marked with great uncertainty and vacillation concerning the nature of that condition. All this uncertainty, all those doubts and difficulties, vanish at once before the clear light of the revelation of Jesus Christ, who brought life and immortality to light. It is the main doctrine of the Scriptures, that man is destined for another world, compared with the transcendent importance of which, all the concerns of this life fade into utter insignificance. The declarations of Scripture on this subject are so clear and decisive, that we may bring into a very small space the most important points which it has pleased God fully to reveal; and we may afterwards direct attention to some inferences which may fairly be drawn from those intimations which contain less definite information. The great outlines of the teaching of Scripture on this subject are so clearly marked as to leave no room for doubt that our future state will be eternal, that our happiness or misery in that state will be dependant on the manner in which we have passed through our time of probation here, and that the final condition of every human being will be determined at that day when God shall judge the world by Jesus Christ.* (Matt. xxv. 31–46; Acts xvii. 31; Rom. ii. 16, xiv. 9, 10; 2 Cor. v. 10, &c.)

But before the revelation made to us in the New Testament is considered, the notions of the Jews require a very brief examination. At the time of our Saviour it is clear, from many intimations, that although the party of the Sadducees† denied the doctrine of a resurrection, yet that a general belief in it was prevalent. We may gather this from the language of Martha, the sister of Lazarus, when she said to our Lord, "I know that he shall rise again in the resurrection at the last day." (John xi. 24.) But we may ascend far higher for the antiquity of this belief, as Bishop Bull‡ and others have shown, and show that the prophets and patriarchs looked not only for a temporal reward, but for an abiding home. It is readily

* See Barrow, *Sermons on the Creed*, sermon xxxii.; Pearson, *On the Creed*; H. More, *Mystery of Godliness*, book vi., &c.

† Bishop Bull observes, on this point: "In a word, the doctrine of a future life and judgment continued inviolate and unquestioned among the Jews till after their return from the captivity. After which time (exactly how soon, or how long after, seems to me uncertain), there arose the heresy of the Sadducees, who believed neither the immortality of the soul nor the resurrection of the body, nor the judgment to come. But concerning these Josephus (*Ant.* xviii. 2) observes, that "though they were generally rich and great men" (their principles leading them to mind and seek after the riches and honours of this world), "yet they were very few in number, compared to the rest of the Jews." (Bull, sermon viii., *On the Vanity of this Life, and the Eternity of the Next*, vol. i. pp. 207, 208, Burton's edition.)

‡ Sermon viii. *ubi supra*; and sermon xiv., *On the Recompence of the Reward*, in which Bishop Bull shows that the doctrine of a recompence after life was believed by the people of God before the law was given, &c.

Theology. acknowledged that in the Mosaic law future rewards and punishments were not made the principle of the covenant; but our Lord has shown us how the great doctrine of a resurrection was implied in the language of the *Pentateuch*, and ought to have been inferred from it by the Jews. But the belief in a resurrection was prevalent both in the patriarchal and in the Jewish Church; in the last of which it seems to have attained perhaps a more clear and definite form as the day of our Lord's appearance approached. It is not perhaps possible to trace very accurately the first origin, or the subsequent progress, of this belief, although we must consider it to have resulted from a Divine revelation; but its existence is testified by many passages of Holy Writ. The translation of Enoch must have been an evidence to his contemporaries, and to succeeding generations, that man was destined for immortality; and Enoch himself prophesied distinctly of the day of judgment, as the apostle Jude has informed us.* (*Jude*, verses 14 & 15.) Having quoted this antediluvian testimony, Bishop Pearson proceeds thus: "The testimonies which follow, in the Law and in the Prophets, the predictions of Christ and the Apostles, are so many and so known, that both the number and the plainness will excuse the prosecution. The throne hath been already seen, the Judge hath appeared sitting on it, the books have been already opened, the dead, small and great, have been seen standing before him; there is nothing more certain in the Word of God, no doctrine more clear and fundamental, than that of *eternal judgment*." (*Heb.* vi. 2.)

One or two clear testimonies from the Old Testament may here be referred to: Job proclaims his belief that his "*Redeemer shall stand at the latter day upon the earth*," and that "*though worms destroy his body, yet in his flesh shall he see God*" (*Job* xix. 25, 26); and Daniel speaks of a day when "*many of them that sleep in the dust of the earth shall awake, some to everlasting life, and some to shame and everlasting contempt*." (*Dan.* xii. 2.)† To these testimonies to a judgment and a life to come, many others might be added, from other parts of Scripture; but it is scarcely needful, as what has been advanced will be considered sufficient to maintain the statement of the viith Article of the Church of England, "Wherefore they are not to be heard, which feign that the old Fathers did look only for transitory promises."

We proceed now briefly to advert to the information communicated in the New Testament as to the future state of man.

* Bishop Pearson (*On the Creed*, art. vii.) appeals, as a proof of a judgment to come, to the intimation given to Cain, "*If thou doest well, shalt thou not be accepted? and if thou doest not well, sin lieth at the door*" (*Gen.* iv. 7): "which," he says, "by the most ancient interpretation, signifieth a reservation of his sin unto the judgment of the world to come." (See the *Targum* of Jonathan, the *Jerusalem Targum*, and that of Onkelos, as quoted by Bishop Pearson.) This interpretation has, however, been disputed; and others have interpreted the word here translated "*sin*" to mean, as it does sometimes, a *sin-offering*; i. e. you have the means of atoning for your sin, and reconciling God to you again.

† On these two passages see Bishop Pearson, *On the Creed*, art. xi. Some have attempted to give an entirely different explanation to the words of Job, but we think without much success.

For further testimonies see Bishop Bull (sermons viii. & xiv.); Barrow, *On the Creed*, sermon xxxii.; Burnet, *On the Articles*, art. vii. The *Psalms* and the *Book of Ecclesiastes* are especially referred to by most writers on these points.

It is, then, clearly the doctrine of the New Testament that "God hath appointed a day in which he will judge the world in righteousness, by that man whom he hath ordained, whereof he hath given assurance in that he hath raised him from the dead." (*Acts* xvii. 31.) This declaration is more fully made manifest to us in another passage of Scripture, in which we are informed of the grounds on which the decision of the judge will be made: "For we must all appear at the judgment-seat of Christ, that every one may receive the things done in his body, *according to that he hath done, whether it be good or bad*." (*2 Cor.* v. 10.) These two passages establish the great facts, that God hath appointed a day in which the whole world shall be judged, that Christ shall be the judge, and that men shall be judged according to their works done in this world. There is one particular more left to be supplied—the consequence of the sentence then to be delivered by the Judge. Our Saviour himself, who has given an account of this awful day, of which the above passage of St. Paul is a reiteration and summary, has added this particular, in language which it is impossible to read without a feeling of solemnity and awe. He informs us that all nations shall be gathered before him, and a separation having been made between those who have done his will, and those who have not, that HE will then say to the one, "Come, ye blessed of my Father, inherit the kingdom prepared for you from the beginning of the world;" and to the others, "Depart from me, ye cursed, into everlasting fire, prepared for the devil and his angels;" and he closes this fearful statement by adding, "And these shall go away into everlasting punishment; but the righteous into life eternal." (*Matt.* xxv. 31–46.)

From these and other passages it is abundantly manifest that a solemn day of judgment is appointed, at which each man shall receive his final sentence from the Judge, and that this sentence leads to a final state of everlasting happiness to those who are accepted in that day, and of everlasting misery to those who are then to be banished from the presence of God for ever. These declarations are at once so clear, that they need no comment, and so solemn and awful, that nothing which man can add to them would be calculated to heighten the impression they must make on every rightly constituted mind. We therefore leave them in the plainness and simplicity with which Scripture commends them to our most earnest and serious thoughts, and proceed to close this essay with a few considerations concerning some points, which we left so far unexplained, that we must be humble-minded and content with mere inferences; while they are so far revealed, that we seem almost invited and encouraged to make such inferences.

It must not be imagined, from what is here asserted, that our works in Scripture are represented as an efficient cause of our salvation. There can be no doubt that, from the whole tenor as well as the express declarations of Scripture, we have reason to conclude, that although we are judged according to our works, the efficient cause of our salvation is only the righteousness and the merits of our Lord and Saviour Jesus Christ. Bishop Bull, in his viith sermon (*On different Degrees of Bliss in Heaven*), makes the following observations on this subject. In order to answer an objection, that since the future glory of the saints is the purchase of Christ's righteousness, which is imputed alike to all true believers, and therefore they shall all share equally in the future glory, he observes, that the doctrine of the imputed righteousness of Christ, in the manner in which it is too commonly taught, has no foundation in Scripture; and then proceeds thus:—

"But to answer more directly to the objection, there is nothing

Theology.

more certain than that the future glory of the saints is the purchase of Christ's righteousness. But how? By the meritorious obedience of Christ, in his life and death, a covenant of grace, mercy, and life eternal was procured, ratified and established between God and the sinful sons of men: the condition of this covenant is *faith working by love*, or a faith fruitful of good works; and there is also sufficient grace promised to all that shall heartily seek it, for the performance of that condition. It is from this covenant of infinite mercy in Christ Jesus alone that our "good works have any ordination to so excellent a reward as the future glory; and it is the mercy, the rich mercy, the royal bounty and liberality of God, expressed in the same covenant, that assigns to greater degrees of grace here, greater degrees of glory hereafter." (Bull's *Works*, vol. i. p. 185-6, ed. Burton.)

Two questions appear naturally to arise from the truths which have just been set forth:—

1. What is the intermediate state between death and judgment?

2. What is the nature of the happiness to be bestowed upon the blessed?

With regard to the first question, it appears to be the opinion most consonant to Scripture and to the judgment of the primitive Church, that the soul of man at once passes into a state of happiness or misery, but that this condition is imperfect until the resurrection of the body and the final consummation of all things. Under this view the prayer in our burial service appears to be composed, which entreats God to hasten his kingdom, "that we, with all those that are departed in the true faith of thy holy name, may have our perfect consummation and bliss" in his eternal and everlasting glory. This notion is entirely inconsistent with the purgatory of the Romanists; but as that subject has already been discussed in another portion of this work (Misc. Diy., art. PURGATORY), it will not be necessary to enter upon the arguments connected with that opinion. Those who wish to pursue the subject further, will find much assistance from Bishop Bull's *Sermons* on this subject (sermon ii. & iii. of vol. i. in Dr. Burton's edition).

We have not here entered at length into the doctrine of the resurrection of the body, as it is clearly the doctrine of Scripture. The resurrection of our Lord's body, and the declaration of his apostle, that he shall make our bodies like unto his glorious body (*Phil.* iii. 21; compare *1 John* iii. 2), with many other cogent passages from the New Testament, are sufficient to set forth this doctrine; but to enter upon it fully would exceed our limits. We may refer those who wish to pursue the subject to Pearson, *On the Creed*, and to Bishop Butler's *Essay on Personal Identity*.

With regard to the second question, our information is confessedly only scanty. The terrors of the next world to the disobedient are set forth very plainly, and

in language calculated to strike the careless; to *them* Theology. Scripture speaks of the worm that dieth not, and the fire that is not quenched; but as the promises of glory are addressed to those who love God, it is sufficient that God should assure them that their happiness will then be his care. They have that trust in God which persuades them that what he prepares for them that love him must be supremely happy, and thoroughly adapted to their condition. It is probable that we are not capable at present of understanding the nature of our future condition in any detail. We may, however, gather many *practical* precepts from what is revealed to us. We perceive at once that a nearer approach and greater knowledge of God are amongst the privileges of that condition; which immediately suggests to us the necessity of cultivating holy habits here below, as without them heaven itself would be no place of happiness for us. A more immediate approach to God's presence is promised to us: "we shall see him as he is" (*1 John* iii. 2; compare also *Rev.* xxi. 22, 23); and Him whom we must now worship through ordinances, and as unseen, we shall then be allowed to approach immediately to see him face to face. The passage of the *Revelations*, also just alluded to, suggests another great change in our nature. In the state in which we live on earth, rest is necessary for the wearied body and the jaded mind; but in that glorified condition there shall be no night, and, consequently, no satiety of bliss, no desire for rest; for heaven itself is represented to us as one enduring state of rest; a condition which shall need no vicissitudes, for all shall be supremely blessed. These considerations might easily be multiplied; but it is better, perhaps, to close this treatise with a caution against attempting to be wise above that which is written, and to allow our souls, in confidence and joy, to repose on that most encouraging and cheering description of the happiness of the world to come contained in the following words of Holy Writ:

"I heard a great voice out of heaven, saying, Behold, the tabernacle of God is with men, and he will dwell with them, and they shall be his people, and God himself shall be with them, and be their God.

"And God shall wipe away all tears from their eyes, and there shall be no more death, neither sorrow nor crying, neither shall there be any more pain; for the former things are passed away." (*Rev.* xxi. 3, 4.)

END OF VOLUME II.





Smedley, Edw.

Encyclopaedia Metropolitana; or, universal dictionary of knowledge, on an original plan: comprising the twofold advantage of a philosophical and an alphabetical arrangement, with appropriate engravings Edited by Edw. Smedley, Hugh Jam. Rose, and H. John Rose. (Text: voll. XXVI. Plates: voll. Bd.: II

London 1845

4 Enc. 7 m-2

urn:nbn:de:bvb:12-bsb10351549-2